J. Math. Sci. Univ. Tokyo
18 (2011), 155–180.

*Connections and the Second Main Theorem for Holomorphic Curves**

By Junjiro Noguchi

Abstract. By means of C^∞-connections we will prove a general second main theorem and some special ones for holomorphic curves. The method gives a geometric proof of H. Cartan's second main theorem in 1933. By applying the same method, we will prove some second main theorems in the case of the product space $(\mathbf{P}^1(\mathbf{C}))^2$ of the Riemann sphere.

1. Main Results

In this paper we are going to prove a general second main theorem and some special ones for holomorphic curves. We begin with the general one. One finds an application of this method in Y. Tiba [12].

(a) Let M be a compact complex manifold of dimension n and let $\mathbf{T}(M)$ denote the holomorphic tangent bundle over M. Let ∇ be a C^∞ connection in $\mathbf{T}(M)$. Let U be a domain of the complex plane $\mathbf{C}$. For a holomorphic curve $f : U \to M$ we have the derivative (1-jet lift) $f'(z) \in \mathbf{T}(M)_{f(z)}$, and we set inductively

$$f^{(1)}(z) = f'(z), \quad f^{(k)}(z) = \nabla_{f'(z)} f^{(k-1)}(z), \quad k = 2, 3, \ldots .$$

We define the Wronskian of f with respect to ∇ by

$$W(\nabla, f) = f^{(1)}(z) \wedge \cdots \wedge f^{(n)}(z) \in K_M^*,$$

where K_M^* denotes the dual of the canonical bundle K_M over M. Because of its local nature it makes sense to say that $W(\nabla, f)$ is holomorphic or that $\log |W(\nabla, f)|$ is subharmonic.

We say that f is ∇-(resp. non)degenerate if and only if $W(\nabla, f) \equiv 0$ (resp. $\not\equiv 0$).

*Research supported in part by Grant-in-Aid for Scientific Research (A) 22244008.
2010 *Mathematics Subject Classification*. Primary 32H30; Scondary 30D35.

Cf. §§2&3 for more notation. The first result of this paper is as follows:

THEOREM 1.1. *Let $f : \mathbf{C} \to M$ be a ∇-nondegenerate holomorphic curve and let $D = \sum_i D_i$ be an effective reduced divisor with only simple normal crossings. Assume*

(i) $\log|W(\nabla, f)|$ *is subharmonic;*

(ii) *every D_i is ∇-totally geodesic.*

Then we have

$$T_f(r, L(D)) + T_f(r, K_M) \leq \sum_i N_n(r, f^* D_i) + S_f(r). \tag{1.2}$$

Here $N_n(r, f^* D_i)$ denotes the n-truncated counting function of $f^* D_i$, and $S_f(r)$ a small term in the Nevanlinna theory such as

$$S_f(r) = O(\log r + \log T_f(r))||,$$

with the order function $T_f(r)$ of f with respect an hermitian metric or an ample line bundle over M.

Let ∇ be the Fubini-Study metric connection on the n-dimensional complex projective space $\mathbf{P}^n(\mathbf{C})$. Then ∇-totally geodesic complex submanifolds of $\mathbf{P}^n(\mathbf{C})$ are complex linear subspaces. A holomorphic curve $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ is linearly nondegenerate if and only if $W(\nabla, f) \not\equiv 0$; moreover, the *Wronskian $W(\nabla, f)$ is holomorphic* (see Theorem 4.1).

COROLLARY 1.3. *Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be a linearly nondegenerate holomorphic curve and let $1 \leq n \leq 3$. Then for q hyperplanes $H_i \subset \mathbf{P}^n(\mathbf{C})$, $1 \leq i \leq q$, in general position we have*

$$qT_f(r, O(1)) + T_f(r, K_{\mathbf{P}^n(\mathbf{C})}) \leq \sum_i N_n(r, f^* D_i) + S_f(r), \tag{1.4}$$

where $O(1)$ denote the hyperplane bundle over $\mathbf{P}^n(\mathbf{C})$.

Note that (1.4) is Cartan's Second Main Theorem ([1]), for $K_{\mathbf{P}^n(\mathbf{C})} = O(-n-1)$. Thus, this gives a *geometric proof* of Cartan's Second Main Theorem.

(b) It is our second aim to consider a special case where we deal with a holomorphic curve $f : \mathbf{C} \to \mathbf{P}^1(\mathbf{C})^2$ (§5). We will consider $\mathbf{P}^1(\mathbf{C})^2$ as an equivariant compactification of the semi-abelian variety $G = \mathbf{C}^{*2}$. This is a quite special case, but interesting, while not much study has been done for it in the past.

Let $E = \mathbf{P}^1(\mathbf{C})^2 \setminus G$ be the boundary divisor and let (x, y) be the affine coordinate system of G. Here we denote by ∇ the flat connection with respect to the invariant vector fields, $x\frac{\partial}{\partial x}$, $y\frac{\partial}{\partial y}$ on G.

We will prove two Theorems 5.7 and 5.12 of Nevanlinna's second main theorem type for ∇-nondegenerate f with some additional condition and a divisor D on $\mathbf{P}^1(\mathbf{C})^2$ whose irreducible components are all ∇-totally geodesic.

Remark 1.5. For a more general case without additional condition, see Y. Tiba [12], where he applies the same method as above for a problem to obtain a new second main theorem.

At the end we will give some examples and problems.

Acknowledgment. The author would like to express his sincere gratitude to Professor Jörg Winkelmann for interesting discussions on the present subject. The author is grateful to the referee for pointing an oversight in the original proof of Theorem 4.1, which was minor but certainly necessary to be fixed.

2. Totally Geodesic Divisor and Lemma on Logarithmic Derivative

Let M be a complex n-dimensional manifold, and let ∇ be a C^∞ connection in $\mathbf{T}(M)$; i.e., for C^∞ vector fields X, Y and a C^∞ function α on M we have

(i) $\nabla_X Y$ is a C^∞ vector field in $\mathbf{T}(M)$, and is linear in X and Y over $\mathbf{C}$;

(ii) $\nabla_{\alpha X} Y = \alpha \nabla_X Y$;

(iii) $\nabla_X(\alpha Y) = X(\alpha) \cdot Y + \alpha \nabla_X Y$.

Let N be a locally closed complex submanifold of M. Take C^∞ sections X', Y' in $\mathbf{T}(N)$, and extend them to C^∞ sections, X, Y in $\mathbf{T}(M)$ over a neighborhood of N in M. Then the restriction $(\nabla_X Y)|_N$ is independent of the extensions X, Y, and so denoted by $\nabla_{X'} Y'$, which is a section in $\mathbf{T}(M)|_N$, but *not* in $\mathbf{T}(N)$ generally.

DEFINITION 2.1. A locally closed complex submanifold N of M is said to be ∇-totally geodesic if $\nabla_{X'} Y'$ is valued in $\mathbf{T}(N)$ for all C^∞ sections X', Y' in $\mathbf{T}(N)$.

Let $f : \mathbf{C} \to M$ be a holomorphic curve. Then $W(\nabla, f)(z)$ is valued in the dual $K^*_{M, f(z)}$ of the canonical bundle K_M at $f(z)$. We take a C^∞ volume form Ω on M. Then $|W(\nabla, f)(z)|^2 \cdot \Omega(f(z))$ is a non-negative C^∞ function in $z \in \mathbf{C}$.

Let $D = \sum_i D_i$ be a divisor on M with irreducible components D_i. Let σ_i be a section of the line bundle $L(D_i)$ determined by D_i such that the divisor (σ_i) coincides D_i, and introduce a hermitian metric $\|\cdot\|$ in every $L(D_i)$. We set

$$\xi(z) = \frac{|W(\nabla, f)(z)|^2 \cdot \Omega(f(z))}{\prod_i \|\sigma_i(f(z))\|^2}. \tag{2.2}$$

As usual, we set $\log^+ \xi(z) = \log \max\{1, \xi(z)\}$.

The following is a version of Nevanlinna's lemma on logarithmic derivative (cf. [3], [8]):

LEMMA 2.3. *Let M be a complex algebraic manifold and let $D = \sum_i D_i$ be a divisor with irreducible components D_i. Assume that*

(i) *D has only simple normal crossings;*

(ii) *every D_i is ∇-totally geodesic.*

Then we have

$$\int_{|z|=r} \log^+ \xi(z) \frac{d\theta}{2\pi} = S_f(r). \tag{2.4}$$

PROOF. Let $M = \cup_\alpha U_\alpha$ be a finite affine covering with rational functions $x^i_\alpha, 1 \le i \le n = \dim M$ over M such that

(i) $x^i_\alpha, 1 \leq i \leq n$, are holomorphic on U_α, and give rise to coordinates in a neighborhood of every point of U_α;

(ii) $U_\alpha \cap D = \{x^1_\alpha \cdots x^{k_\alpha}_\alpha = 0\}$.

Let $V_\alpha \Subset U_\alpha$ be relatively compact open subsets such that $M = \cup V_\alpha$. Let $\mathbf{1}_{V_\alpha}$ be the characteristic function of the set V_α. Set $f^j_\alpha(z) = x^j_\alpha(f(z)), 1 \leq j \leq n$. Let $\Gamma^k_{\alpha ij}$ be the Christofell symbols of ∇ with respect to (x^i_α). Since D_i are ∇-totally geodesic, there are C^∞ functions A_α and B_α on U_α such that

$$\Gamma^h_{\alpha ij}(x^1_\alpha, \ldots, x^h_\alpha, \ldots, x^n_\alpha) = A^h_{\alpha ij} \cdot x^h_\alpha + B^h_{\alpha ij} \cdot \bar{x}^h_\alpha, \quad 1 \leq h \leq k_\alpha. \tag{2.5}$$

Therefore there is a constant $C_\alpha > 0$ such that

$$\begin{aligned} \left| \frac{\mathbf{1}_{V_\alpha}(f(z))}{f^h_\alpha(z)} \Gamma^h_{\alpha ij}(f(z)) \right| &= \mathbf{1}_{V_\alpha}(f(z)) \left| A^h_{\alpha ij}(f(z)) + B^h_{\alpha ij}(f(z)) \frac{\bar{f}^h_\alpha(z)}{f^h_\alpha(z)} \right| \\ &\leq \mathbf{1}_{V_\alpha}(f(z)) \left(|A^h_{\alpha ij}(f(z))| + |B^h_{\alpha ij}(f(z))| \right) \\ &\leq C_\alpha, \qquad 1 \leq h \leq k_\alpha. \end{aligned} \tag{2.6}$$

Here we understand "$\frac{0}{0} = 0$", when $\mathbf{1}_{V_\alpha}(f(z)) = 0$ and $f^h_\alpha(z) = 0$, provided that $f(z) \in U_\alpha$; we extend the above function for all $z \in \mathbf{C}$, as zero, when $f(z) \notin U_\alpha$. We set

$$f^{(l)}(z) = f^{(l)k}_\alpha(z) \left(\frac{\partial}{\partial x^k_\alpha} \right)_{f(z)},$$

where Einstein's convention is used for summation.

There is a C^∞ function a_α on U_α such that

$$\xi(z) = \left| \det \begin{pmatrix} f^{(1)1}_\alpha & \cdots & f^{(1)k_\alpha}_\alpha & \cdots & f^{(1)n}_\alpha \\ f^{(2)1}_\alpha & \cdots & f^{(2)k_\alpha}_\alpha & \cdots & f^{(2)n}_\alpha \\ f^{(3)1}_\alpha & \cdots & f^{(3)k_\alpha}_\alpha & \cdots & f^{(3)n}_\alpha \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f^{(n)1}_\alpha & \cdots & f^{(n)k_\alpha}_\alpha & \cdots & f^{(n)n}_\alpha \end{pmatrix} \right|^2 \frac{a_\alpha(f(z))}{|f^1_\alpha|^2 \cdots |f^{k_\alpha}_\alpha|^2} \tag{2.7}$$

$$
= \left\| \begin{matrix} \frac{f_\alpha^{(1)1}}{f_\alpha^1} & \cdots & \frac{f_\alpha^{(1)k_\alpha}}{f_\alpha^{k_\alpha}} & f_\alpha^{(1)k_\alpha+1} & \cdots & f_\alpha^{(1)n} \\ \frac{f_\alpha^{(2)1}}{f_\alpha^1} & \cdots & \frac{f_\alpha^{(2)k_\alpha}}{f_\alpha^{k_\alpha}} & f_\alpha^{(2)k_\alpha+1} & \cdots & f_\alpha^{(2)n} \\ \frac{f_\alpha^{(3)1}}{f_\alpha^1} & \cdots & \frac{f_\alpha^{(3)k_\alpha}}{f_\alpha^{k_\alpha}} & f_\alpha^{(3)k_\alpha+1} & \cdots & f_\alpha^{(3)n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{f_\alpha^{(n)1}}{f_\alpha^1} & \cdots & \frac{f_\alpha^{(n)k_\alpha}}{f_\alpha^{k_\alpha}} & f_\alpha^{(n)k_\alpha+1} & \cdots & f_\alpha^{(n)n} \end{matrix} \right\|^2 a_\alpha(f(z)).
$$

Note that $\mathbf{1}_{V_\alpha}(f(z)) \cdot f_\alpha^i(z), 1 \le i \le n$, and $\mathbf{1}_{V_\alpha}(f(z)) \cdot a_\alpha(f(z))$ are bounded functions. Therefore we have by (2.7)

$$
\tag{2.8} \log^+ \xi(z) = O\left(\sum_\alpha \left(\sum_{1 \le k \le k_\alpha, 1 \le l \le n} \mathbf{1}_{V_\alpha}(f(z)) \cdot \log^+ \left| \frac{f_\alpha^{(l)k}(z)}{f_\alpha^k(z)} \right| + \sum_{1 \le k, l \le n} \mathbf{1}_{V_\alpha}(f(z)) \cdot \log^+ |f_\alpha^{(l)k}(z)| \right) \right) + O(1),
$$

where the estimate "$O(*)$" is uniform in $z \in \mathbf{C}$ (it is used in the same sense from now on). We first compute $f_\alpha^{(l)k}$: For instance, we have

$$
\log^+ |f_\alpha^{(1)k}| = \log^+ |f_\alpha^{k\prime}| = \log^+ |f_\alpha^{k(1)}|.
$$

For $1 \le k \le k_\alpha$ we have

$$
\log^+ \left| \frac{f_\alpha^{(1)k}}{f_\alpha^k} \right| = \log^+ \left| \frac{f_\alpha^{k\prime}}{f_\alpha^k} \right|.
$$

For $l = 2$ we have

$$
f_\alpha^{(2)k} = f_\alpha^{k\prime\prime} + \Gamma_{\alpha i_1 i_2}^k \circ f \cdot f_\alpha^{i_1\prime} f_\alpha^{i_2\prime}.
$$

Since $\mathbf{1}_{V_\alpha} \circ f \cdot \Gamma_{\alpha i_1 i_2} \circ f$ is bounded, we have

$$
\mathbf{1}_{V_\alpha} \circ f \cdot |f_\alpha^{(2)k}| = \mathbf{1}_{V_\alpha} \circ f \cdot O\left(|f_\alpha^{k\prime\prime}| + \left(\sum_{i=1}^n |f_\alpha^{i\prime}| \right)^2 \right),
$$

$$
\mathbf{1}_{V_\alpha} \circ f \cdot \log^+ |f_\alpha^{(2)k}| = \mathbf{1}_{V_\alpha} \circ f \cdot O\left(\log^+ |f_\alpha^{k(2)}| + \sum_{i=1}^n \log^+ |f_\alpha^{i(1)}| \right) + O(1).
$$

For $1 \le k \le k_\alpha$ we have by (2.6)

$$\mathbf{1}_{V_\alpha} \circ f \cdot \left| \frac{f_\alpha^{(2)k}}{f_\alpha^k} \right| = \mathbf{1}_{V_\alpha} \circ f \cdot O\left(\left| \frac{f_\alpha^{k\prime\prime}}{f_\alpha^k} \right| + \left(\sum_{i=1}^n |f_\alpha^{i\prime}| \right)^2 \right),$$

$$\mathbf{1}_{V_\alpha} \circ f \cdot \log^+ |f_\alpha^{(2)k}| = \mathbf{1}_{V_\alpha} \circ f \cdot O\left(\log^+ |f_\alpha^{k(2)}| + \sum_{i=1}^n \log^+ |f_\alpha^{i(1)}| \right) + O(1).$$

Up to here it is easy to obtain the estimate. For $l = 3$, $f_\alpha^{(3)k}$ starts to involve the partial derivatives of $\Gamma^k_{\alpha i_1 i_2}$:

$$\begin{aligned} f_\alpha^{(3)k} &= f_\alpha^{k\prime\prime\prime} + \Gamma^k_{\alpha i_1 i_2} \circ f \cdot f_\alpha^{i_1\prime\prime} f_\alpha^{i_2\prime} + \Gamma^k_{\alpha i_1 i_2} \circ f \cdot f_\alpha^{i_1\prime} f_\alpha^{i_2\prime\prime} \\ &\quad + \frac{\partial \Gamma^k_{\alpha i_1 i_2}}{\partial x_\alpha^{i_3}} \circ f \cdot f_\alpha^{i_1\prime} f_\alpha^{i_2\prime} f_\alpha^{i_3\prime} + \Gamma^k_{\alpha i_1 i_2} \circ f \cdot f_\alpha^{i_1\prime} f_\alpha^{(2)i_2}, \end{aligned}$$

and

$$\Gamma^k_{\alpha i_1 i_2} \circ f \cdot f_\alpha^{i_1\prime} f_\alpha^{(2)i_2} = \Gamma^k_{\alpha i_1 i_2} \circ f \cdot f_\alpha^{i_1\prime} f_\alpha^{i_2\prime\prime} + \Gamma^k_{\alpha i_1 i_2} \circ f \cdot f_\alpha^{i_1\prime} \cdot \Gamma^{i_2}_{\alpha i_3 i_4} \circ f \cdot f_\alpha^{i_3\prime} f_\alpha^{i_4\prime}.$$

It follows that

$$\begin{aligned} \mathbf{1}_{V_\alpha} \circ f \cdot \log^+ |f_\alpha^{(3)k}| = \mathbf{1}_{V_\alpha} \circ f \cdot O\Bigg(&\log^+ |f_\alpha^{k(3)}| + \sum_{i=1}^n \log^+ |f_\alpha^{i(2)}| \\ &+ \sum_{i=1}^n \log^+ |f_\alpha^{i(1)}| \Bigg) + O(1). \end{aligned}$$

For $1 \le k \le k_\alpha$ we estimate $\mathbf{1}_{V_\alpha} \circ f \cdot \left| \frac{f_\alpha^{(3)k}}{f_\alpha^k} \right|$:

$$\begin{aligned} \mathbf{1}_{V_\alpha} \circ f \cdot \left| \frac{f_\alpha^{(3)k}}{f_\alpha^k} \right| \\ \le \mathbf{1}_{V_\alpha} \circ f \cdot \Bigg(&\left| \frac{f_\alpha^{k\prime\prime\prime}}{f_\alpha^k} \right| + \left| \frac{\Gamma^k_{\alpha i_1 i_2} \circ f}{f_\alpha^k} f_\alpha^{i_1\prime\prime} f_\alpha^{i_2\prime} \right| + \left| \frac{\Gamma^k_{\alpha i_1 i_2} \circ f}{f_\alpha^k} f_\alpha^{i_1\prime} f_\alpha^{i_2\prime\prime} \right| \\ &+ \left| \frac{1}{f_\alpha^k} \frac{\partial \Gamma^k_{\alpha i_1 i_2}}{\partial x_\alpha^{i_3}} \circ f \cdot f_\alpha^{i_1\prime} f_\alpha^{i_2\prime} f_\alpha^{i_3\prime} \right| + \left| \frac{\Gamma^k_{\alpha i_1 i_2} \circ f}{f_\alpha^k} f_\alpha^{i_1\prime} f_\alpha^{(2)i_2} \right| \Bigg). \end{aligned} \tag{2.9}$$

Note that $\mathbf{1}_{V_\alpha}\circ f\cdot\left|\frac{\Gamma^k_{\alpha i_1 i_2}\circ f}{f^k_\alpha}\right|$ is bounded. We compute the fourth term in the right hand side of (2.9): For $i_3\neq k$ we have

$$\begin{aligned}
&\mathbf{1}_{V_\alpha}\circ f\cdot\left|\frac{1}{f^k_\alpha}\frac{\partial\Gamma^k_{\alpha i_1 i_2}}{\partial x^{i_3}_\alpha}\circ f\cdot f^{i_1}_\alpha{}' f^{i_2}_\alpha{}' f^{i_3}_\alpha{}'\right|\\
&=\mathbf{1}_{V_\alpha}\circ f\cdot\left|\left(\frac{\partial A^k_{\alpha i_1 i_2}}{\partial x^{i_3}_\alpha}\circ f+\frac{\partial B^k_{\alpha i_1 i_2}}{\partial x^{i_3}_\alpha}\circ f\cdot\frac{\bar f^k_\alpha}{f^k_\alpha}\right) f^{i_1}_\alpha{}' f^{i_2}_\alpha{}' f^{i_3}_\alpha{}'\right|\\
&\leq\mathbf{1}_{V_\alpha}\circ f\cdot\left(\left|\frac{\partial A^k_{\alpha i_1 i_2}}{\partial x^{i_3}_\alpha}\circ f\right|+\left|\frac{\partial B^k_{\alpha i_1 i_2}}{\partial x^{i_3}_\alpha}\circ f\right|\right)|f^{i_1}_\alpha{}' f^{i_2}_\alpha{}' f^{i_3}_\alpha{}'|\\
&=\mathbf{1}_{V_\alpha}\circ f\cdot O\left(\sum_{i=1}^n|f^i_\alpha{}'|\right)^3.
\end{aligned}$$

For $i_3=k$ we obtain

$$\begin{aligned}
&\mathbf{1}_{V_\alpha}\circ f\cdot\left|\frac{1}{f^k_\alpha}\frac{\partial\Gamma^k_{\alpha i_1 i_2}}{\partial x^{k}_\alpha}\circ f\cdot f^{i_1}_\alpha{}' f^{i_2}_\alpha{}' f^{k}_\alpha{}'\right|\\
&=\mathbf{1}_{V_\alpha}\circ f\cdot\left|\frac{\partial\Gamma^k_{\alpha i_1 i_2}}{\partial x^{k}_\alpha}\circ f\cdot f^{i_1}_\alpha{}' f^{i_2}_\alpha{}'\frac{f^{k}_\alpha{}'}{f^k_\alpha}\right|\\
&=\mathbf{1}_{V_\alpha}\circ f\cdot O\left(\left(\sum_{i=1}^n|f^i_\alpha{}'|\right)^2\left|\frac{f^k_\alpha{}'}{f^k_\alpha}\right|\right).
\end{aligned}$$

Therefore we get

$$\begin{aligned}
\mathbf{1}_{V_\alpha}\circ f\cdot\log^+\left|\frac{f^{(3)k}_\alpha}{f^k_\alpha}\right|\leq\mathbf{1}_{V_\alpha}\circ f\cdot O\Bigg(\log^+\left|\frac{f^{k(3)}_\alpha}{f^k_\alpha}\right|&+\sum_{i=1}^n\log^+|f^{i(2)}_\alpha|\\
&+\sum_{i=1}^n\log^+|f^{i(1)}_\alpha|\Bigg)+O(1).
\end{aligned}$$

In this way we have

$$\mathbf{1}_{V_\alpha}\circ f\cdot\log^+|f^{(l)k}_\alpha|=O\left(\sum_{1\leq i\leq n,1\leq j\leq l}\mathbf{1}_{V_\alpha}\circ f\cdot\log^+|f^{i(j)}_\alpha|\right)\tag{2.10}$$

(continued)

$$+O(1), \quad 1 \le k \le n,$$

$$\mathbf{1}_{V_\alpha} \circ f \cdot \log^+ \left| \frac{f_\alpha^{(l)k}}{f_\alpha^k} \right| = O\left(\sum_{1 \le j \le l} \mathbf{1}_{V_\alpha} \circ f \cdot \log^+ \left| \frac{f_\alpha^{k(j)}}{f_\alpha^k} \right| \right.$$
$$\left. + \sum_{1 \le i \le n, 1 \le j \le l} \mathbf{1}_{V_\alpha} \circ f \cdot \log^+ |f_\alpha^{i(j)}| \right)$$
$$+O(1), \quad 1 \le k \le k_\alpha.$$

Notice that for $j \ge 1$

$$\mathbf{1}_{V_\alpha} \circ f \cdot \log^+ |f_\alpha^{i(j)}| \le \mathbf{1}_{V_\alpha} \circ f \cdot \log^+ \left| \frac{f_\alpha^{i(j)}}{f_\alpha^i} \cdot f_\alpha^i \right| \tag{2.11}$$
$$= \mathbf{1}_{V_\alpha} \circ f \cdot \log^+ \left| \frac{f_\alpha^{i(j)}}{f_\alpha^i} \right| + O(1).$$

Combining (2.8), (2.10) and (2.11) with Nevanlinna's lemma on logarithmic derivative (cf., e.g., [8]), we deduce that

$$\int_{|z|=r} \log^+ \xi(z) \frac{d\theta}{2\pi}$$
$$= O\left(\sum_{\alpha, 1 \le k, l \le n} \int_{|z|=r} \mathbf{1}_{V_\alpha}(f(z)) \cdot \log^+ \left| \frac{f_\alpha^{k(l)}(z)}{f_\alpha^k(z)} \right| \frac{d\theta}{2\pi} \right) + O(1)$$
$$= O\left(\sum_{\alpha, 1 \le k, l \le n} \int_{|z|=r} \log^+ \left| \frac{f_\alpha^{k(l)}(z)}{f_\alpha^k(z)} \right| \frac{d\theta}{2\pi} \right) + O(1)$$
$$= S_f(r). \ \square$$

3. Proof of Theorem 1.1

We first note that the current

$$dd^c \log |W(\nabla, f)|^2 = \frac{i}{2\pi} \partial\bar{\partial} \log |W(\nabla, f)|^2$$

is well defined and is a positive measure on $\mathbf{C}$. For the sake of notational simplicity we write $c_1(D)$ for the curvature form of the hermitian line bundle

$L(D)$ defining the Chern class. It follows that

$$dd^c \log \xi = f^* c_1(D) + f^* c_1(K_M) - \sum_i f^* D_i + dd^c \log |W(\nabla, f)|^2. \tag{3.1}$$

Let $Z = \sum_\nu \lambda_\nu \cdot z_\nu$ be a divisor on $\mathbf{C}$ with distinct $z_\nu \in \mathbf{C}$. We set the k-truncated divisor of Z with $k \leq \infty$ by

$$(Z)_k = \sum_\nu \min\{\nu, k\} \cdot z_\nu.$$

Calculating the multiplicity at $f(z) \in \sum D_i$, we see that

$$-\sum_i f^* D_i + dd^c \log |W(\nabla, f)|^2 \geq -\sum (f^* D_i)_n$$

as currents. It follows from this and (3.1) that

$$dd^c \log \xi \geq f^* c_1(D) + f^* c_1(K_M) - \sum_i (f^* D_i)_n. \tag{3.2}$$

We denote by $T_f(r, L(D))$ (resp. $T_f(r, K_M)$) the order function of f with respect to $c_1(D)$ (resp. $c_1(K_M)$); e.g.,

$$T_f(r, L(D)) = \int_1^r \frac{dt}{t} \int_{|z|<t} f^* c_1(D).$$

Using the counting function $N_n(r, f^* D_i)$ truncated to level n defined by

$$N_n(r, f^* D_i) = \int_1^r \frac{dt}{t} \int_{|z|<t} (f^* D_i)_n,$$

we have by Jensen's formula

$$\begin{aligned} & T_f(r, L(D)) + T_f(r, K_M) \\ & \quad \leq \sum_i N_n(r, f^* D_i) + \frac{1}{2} \int_{|z|=r} \log \xi(z) \frac{d\theta}{2\pi} - \frac{1}{2} \int_{|z|=1} \log \xi(z) \frac{d\theta}{2\pi}. \end{aligned} \tag{3.3}$$

By Lemma 2.3 we see that

$$T_f(r, L(D)) + T_f(r, K_M) \leq \sum_i N_n(r, f^* D_i) + S_f(r).$$

This finishes the proof.

4. Geometric Proof of Cartan's Second Main Theorem

The purpose of this section is to give a geometric proof of H. Cartan's second main theorem ([1]), whose key is the following. We let ∇ denote the connection induced by the Fubini-Study metric form ω on $\mathbf{P}^n(\mathbf{C})$ in this section.

THEOREM 4.1. *Let $f : U \to \mathbf{P}^n(\mathbf{C})$ be a holomorphic curve from a domain $U \subset \mathbf{C}$.*

(i) *The holomorphic curve f is ∇-degenerate if and only if f is linearly degenerate; i.e., the image $f(U)$ is contained in a hyperplane.*

(ii) *The Wronskian $W(\nabla, f)(z)$ is holomorphic in $z \in U$.*

REMARK. So far by our knowledge, the above (ii) was first proved by Siu [11] in the case of $n = 2$. Since there is no reference providing a proof for general n, we give a proof making use of the potential of the Kähler form ω. The statement (i) should be known, but since we do not know a reference and its proof consists a part of the proof of (ii), we here give a self-contained proof.

PROOF. (i) We take a point $z_0 \in U$ and set $w_0 = f(z_0) \in \mathbf{P}^n(\mathbf{C})$. We may assume that $z_0 = 0$. Let $(w^1, \dots, w^n)$ be the normalized affine coordinate of $\mathbf{P}^n(\mathbf{C})$ such that $w_0 = (0, \dots, 0)$ and

$$\omega = dd^c \log\left(1 + \|w\|^2\right), \quad \|w\|^2 = \sum_{j=1}^{n} |w^j|^2.$$

We set

$$\partial_z = \frac{\partial}{\partial z}, \qquad \bar{\partial}_z = \frac{\partial}{\partial \bar{z}},$$
$$f(z) = (f^1(z), \dots, f^n(z)).$$

Setting $\phi(w) = \log\left(1 + \|w\|^2\right)$ and $\omega = g_{i\bar{j}} \frac{i}{2\pi} dw^i \wedge d\bar{w}^j$ we have

$$g_{i\bar{j}} = \partial_i \bar{\partial}_j \phi(w),$$

where $\partial_i = \partial/\partial w^i$ and $\bar{\partial}_j = \partial/\partial \bar{w}^j$. Then the Christoffel symbol Γ^k_{ij} of the connection ∇ is given by

$$\begin{aligned} \Gamma^k_{ij} &= (\partial_i g_{j\bar{l}}) \cdot g^{\bar{l}k} = \partial_i g_{j\bar{l}} \cdot g^{\bar{l}k} \; (= \Gamma^k_{ji}), \\ \nabla_{\partial_i} \partial_j &= \Gamma^k_{ij} \partial_k, \end{aligned}$$

where $(g^{\bar{l}k})$ denotes the inverse matrix of $(g_{j\bar{l}})$ with $g_{j\bar{l}} g^{\bar{l}k} = \delta^k_j$ (Kronecker). For later use we note that

$$\begin{aligned} \partial_h g^{\bar{l}k} &= -g^{\bar{l}i} \cdot \partial_h g_{i\bar{j}} \cdot g^{\bar{j}k}, \\ \bar{\partial}_h g^{\bar{l}k} &= -g^{\bar{l}i} \cdot \bar{\partial}_h g_{i\bar{j}} \cdot g^{\bar{j}k}. \end{aligned} \tag{4.2}$$

Now the power series expansion of the potential $\phi(w)$ about the origin 0 is

$$\phi(w) = \sum_{\mu=1}^{\infty} \frac{(-1)^{\mu-1}}{\mu} \left(\sum_i w^i \bar{w}^i \right)^{\mu}. \tag{4.3}$$

From this power series expansion we see that

i) *the partial differentiation of* ϕ *evaluated at* 0*,*

$$\partial_{i_1} \cdots \partial_{i_p} \bar{\partial}_{j_1} \cdots \bar{\partial}_{j_q} \phi(0) \tag{4.4}$$

can be non-zero only when the partial differentiations ∂_j *and* $\bar{\partial}_j$ *appear exactly in pairs in the partial differentiation;*

ii) *in particular, the values of odd order differentiations of* $\phi(w)$ *are all zero at* $w = 0$*.*

iii) *if we rewrite* (4.4) *as*

$$(\partial_{i_1} \bar{\partial}_{i_1})^{\mu_1} \cdots (\partial_{i_p} \bar{\partial}_{i_p})^{\mu_p} \phi(0), \tag{4.5}$$

where $i_1, \ldots, i_p$ *are distinct, then the values of* (4.5) *are the same for all choices of indices* $i_1, \ldots, i_p$*, because of the symmetry in the variables* $w^1, \ldots, w^n$*;*

We will use only the above properties i) and ii). From these we obtain

$$\begin{aligned} &\partial_i g_{j\bar{l}}(0) = \partial_i \partial_j \bar{\partial}_l \phi(0) = 0, \\ &\Gamma^k_{ij}(0) = 0. \end{aligned}$$

We write $f^{(j)}(z) = f^{(j)k}(z)\partial_k$, locally about 0. Note that $f^{(j)k}(z)$ are not holomorphic for $j \geq 2$:

$$\begin{aligned} f^{(2)k} &= \partial_z^2 f^k + \partial_z f^{i_1} \cdot \partial_z f^{i_2} \cdot \Gamma^k_{i_1 i_2} \circ f, \\ f^{(3)k} &= \partial_z f^{(2)k} + \partial_z f^{i_1} \cdot \partial_z f^{(2)i_2} \cdot \Gamma^k_{i_1 i_2} \circ f \\ &= \partial_z^3 f^k + 3\partial_z^2 f^{i_1} \cdot \partial_z f^{i_2} \cdot \Gamma^k_{i_1 i_2} \circ f \\ &\quad + \partial_z f^{i_1} \cdot \partial_z f^{i_3} \cdot \partial_z f^{i_4} \cdot \Gamma^{i_2}_{i_3 i_4} \circ f \cdot \Gamma^k_{i_1 i_2} \circ f \\ &\quad + \partial_z f^{i_1} \cdot \partial_z f^{i_2} \cdot \partial_z f^{\alpha} \cdot \partial_\alpha \Gamma^k_{i_1 i_2} \circ f. \end{aligned} \tag{4.6}$$

Therefore we see that

$$\begin{aligned} f^{(j)k} &= \partial_z f^{(j-1)k} + \partial_z f^{i_1} \cdot f^{(j-1)i_2} \cdot \Gamma^k_{i_1 i_2} \circ f \\ &= \partial_z^j f^k + P^{jk}(\partial_z f^h, \dots, \partial_z^{j-1} f^h, \partial_{\alpha_1} \cdots \partial_{\alpha_\nu} \Gamma^h_{i_1 i_2} \circ f), \\ &\quad (1 \leq h \leq n,\ 0 \leq \nu \leq j-2), \end{aligned} \tag{4.7}$$

where P^{jk} is a polynomial such that every term has a factor of the form $\partial_{\alpha_1} \cdots \partial_{\alpha_\nu} \Gamma^h_{i_1 i_2} \circ f$. It follows from (4.4) and (4.2) that $\partial_{\alpha_1} \cdots \partial_{\alpha_\nu} \Gamma^h_{i_1 i_2}(0) = 0$. Therefore,

$$P^{jk}(\partial_z f^h(0), \dots, \partial_z^{j-1} f^h(0), \partial_{\alpha_1} \cdots \partial_{\alpha_\nu} \Gamma^h_{i_1 i_2}(0)) = 0,$$

so that

$$f^{(j)k}(0) = \partial_z^j f^k(0).$$

It follows that $W(\nabla, f)(0) = \det(\partial_z^j f^k(0))$. Therefore $W(\nabla, f) \equiv 0$ if and only if the standard Wronskian of f, $\det(\partial_z^j f^k) \equiv 0$, and hence if and only if f is linearly degenerate.

(ii) By a unitary transformation of (w^i) we may assume that the matrix $\left(\partial_z^j f^k(0)\right)_{1 \leq j,k, \leq n}$ is of lower triangle:

$$\partial_z^j f^k(0) = 0,\ j < k, \quad \partial_z^k f^k(0) = c_k \qquad (c_k \in \mathbf{C}). \tag{4.8}$$

We prove

$$\bar{\partial}_z f^{(j)k}(0) = 0, \quad j \le k \le n. \tag{4.9}$$

Let $k \ge j$. Then it follows from (4.7) that

$$\begin{aligned}
\bar{\partial}_z f^{(j)k} &= \tilde{P}^{jk}(\partial_z f^h, \dots, \partial_z^{j-1} f^h, \partial_{\alpha_1} \cdots \partial_{\alpha_\nu} \Gamma^h_{i_1 i_2} \circ f, \\
&\qquad \bar{\partial}_z \partial_{\alpha_1} \cdots \partial_{\alpha_\nu} \Gamma^h_{i_1 i_2} \circ f) \\
&= \tilde{P}^{jk}(\partial_z f^h, \dots, \partial_z^{j-1} f^h, \partial_{\alpha_1} \cdots \partial_{\alpha_\nu} \Gamma^h_{i_1 i_2} \circ f, \\
&\qquad \overline{\partial_z f^\beta} \cdot \bar{\partial}_\beta \partial_{\alpha_1} \cdots \partial_{\alpha_\nu} \Gamma^h_{i_1 i_2} \circ f),
\end{aligned}$$

where $\tilde{P}^{jk}$ is a polynomial naturally derived from P^{jk} by differentiations. Therefore we see that

$$\begin{aligned}
\bar{\partial}_z f^{(j)k}(0) = \tilde{P}^{jk}(&\partial_z f^h(0), \dots, \partial_z^{j-1} f^h(0), \partial_{\alpha_1} \cdots \partial_{\alpha_\nu} \Gamma^h_{i_1 i_2}(0), \\
&\overline{\partial_z f^\beta}(0) \cdot \bar{\partial}_\beta \partial_{\alpha_1} \cdots \partial_{\alpha_\nu} \Gamma^h_{i_1 i_2}(0)).
\end{aligned} \tag{4.10}$$

One infers from this, (4.2), (4.4), and (4.8) that the remaining terms in (4.10) are only those involving

$$\partial_z^l f^{i_2}(0) \cdot \overline{\partial_z f^1}(0) \cdot \bar{\partial}_1 \Gamma^k_{1 i_2}(0), \quad i_2 \le l \le j-1 \ (< k). \text{ (Cf. (4.6).)}$$

Since $\bar{\partial}_1 \Gamma^k_{1 i_2}(0) = \bar{\partial}_1 \partial_1 \partial_{i_2} \bar{\partial}_k \phi(0) = 0$ for $i_2 \ne k$, we have proved (4.9).

We finally see that

$$\bar{\partial}_z W(\nabla, f)(0) = \sum_{j=1}^n \begin{vmatrix}
c_1 & 0 & & \cdots & & & 0 \\
* & \ddots & \ddots & & & & \\
 & & c_{j-1} & 0 & & & \\
\vdots & & & 0 & 0 & & \vdots \\
 & & & & c_{j+1} & \ddots & \\
 & & & & & \ddots & 0 \\
* & & & \cdots & & * & c_n
\end{vmatrix} = 0. \square$$

Remark. Let $\mathbf{B} = \{\|x\| < 1\}$ be the unit ball of $\mathbf{C}^n$ with the Bergman metric $(h_{i\bar{j}})$ on $\mathbf{B}$. Then we have

$$\psi = \log(1 - \|x\|^2) = \sum_{\nu=1}^\infty \frac{1}{\nu} \|x\|^{2\nu}, \tag{4.11}$$

$$h_{i\bar{j}} = \partial_i \bar{\partial}_j \psi.$$

Let $\nabla_{\mathbf{B}}$ be the connection induced from the Bergman metric on $\mathbf{B}$. Because of the type of the power expansions (4.3) and (4.11), we have

COROLLARY 4.12. *The Wronskian $W(\nabla_{\mathbf{B}}, f)$ is holomorphic for a holomorphic curve $f : U \to \mathbf{B}$.*

5. Holomorphic Curves into $\mathbf{P}^1(\mathbf{C})^2$

In this section we set

$$G = \mathbf{C}^{*2},$$

which is a two-dimensional semi-abelian variety. We consider $\mathbf{P}^1(\mathbf{C})^2$ as an equivariant compactification of G. We fix an affine coordinate system $(x, y) \in G \subset \mathbf{P}^1(\mathbf{C})^2$. Then there are invariant vector fields on G,

$$X = x\frac{\partial}{\partial x}, \quad Y = y\frac{\partial}{\partial y},$$

which form a frame of the holomorphic tangent bundle $\mathbf{T}(G)$. In this section, we denote by ∇ the flat connection with respect to the frame $\{X, Y\}$ of $\mathbf{T}(G)$; i.e.,

$$\nabla_X Y = \nabla_Y X = 0.$$

Then ∇ is a meromorphic connection with logarithmic poles along the boundary divisor $\partial G = E$, which has only simple normal crossings.

We set locally

$$u = \log x, \qquad v = \log y.$$

A locally closed complex submanifold N of G is ∇-totally geodesic if and only if N is an open subset of an affine linear subspace $\{(u, v) \in \mathbf{C}^2; \lambda u + \mu v = c\}$ with constants λ, μ, and c; in particular, N is an open subset of a translate of an analytic 1-parameter subgroup of G.

Let $D \subset G$ be an algebraic reduced divisor, and denote by the same D the closure in $\mathbf{P}^1(\mathbf{C})^2$. We are going to deal with the Nevanlinna theory for an algebraically nondegenerate holomorphic curve $f : \mathbf{C} \to \mathbf{P}^1(\mathbf{C})^2$ and for $D + E$; in particular, we are interested in the problem of the possible second main theorem.

If $f(\mathbf{C}) \cap E = \emptyset$, then we have $f : \mathbf{C} \to G$. In this case we know the following theorem by [9] and [10].

Theorem 5.1. *Assume that $f : \mathbf{C} \to G$ is algebraically nondegenerate, and let D be an algebraic reduced divisor on G. Then there an equivariant compactification $\hat{G}$ of G such that*

$$T_{\hat{f}}(r, L(\hat{D})) \leq N_1(r, f^*D) + \epsilon T_{\hat{f}}(r, L(\hat{D}))||_\epsilon, \quad \forall \epsilon > 0,$$

where $\hat{f} = f : \mathbf{C} \to \hat{G}$ and $\hat{D}$ is the closure of $D \cap G$ in $\hat{G}$.

As the second main theorem for $f : \mathbf{C} \to G$, Theorem 5.1 is the best possible result. Therefore in the sequel we will be mainly interested in the case where $f(\mathbf{C}) \cap E \neq \emptyset$. We set

$$f(z) = (F(z), G(z))$$

with respect to the coordinate system (x, y), where $F(z)$ and $G(z)$ are meromorphic functions in $\mathbf{C}$. It follows that

$$W(\nabla, f) = \begin{vmatrix} \frac{F'}{F} & \frac{G'}{G} \\ \left(\frac{F'}{F}\right)' & \left(\frac{G'}{G}\right)' \end{vmatrix} \cdot F \cdot G \cdot \left(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}\right)_{f(z)}. \tag{5.2}$$

Proposition 5.3. *If f is ∇-degenerate and $f(\mathbf{C}) \cap E \neq \emptyset$, then $f(\mathbf{C})$ is contained in the closure of a translate of a 1-dimensional algebraic subgroup of G.*

Proof. Suppose that $W(\nabla, f) \equiv 0$. Then there is a non-trivial linear relation with $\lambda, \mu \in \mathbf{C}$:

$$\lambda \frac{F'(z)}{F(z)} + \mu \frac{G'(z)}{G(z)} = 0, \quad z \in \mathbf{C}. \tag{5.4}$$

If one of λ and μ is zero, the conclusion is immediate. Thus we assume that $\lambda\mu \neq 0$. Since $f(\mathbf{C}) \cap E \neq \emptyset$, there is a point $a \in \mathbf{C}$ with $f(a) \in E$. Then $F(a) = 0$, or ∞, or $G(a) = 0$, or ∞. Assume that $F(a) = 0$; the other cases are dealt similarly. Then there are an integer m and a non-vanishing holomorphic function $\tilde{F}$ in a neighborhood of a such that $F(z) = (z-a)^m \tilde{F}(z)$, locally. It follows from (5.4) that

$$\frac{\lambda m}{z-a} + \mu \frac{G'(z)}{G(z)} \tag{5.5}$$

is holomorphic about a. Therefore $\frac{G'(z)}{G(z)}$ must have a pole at a. Hence there are a non-zero integer n and a non-vanishing holomorphic function $\tilde{G}$ in a neighborhood of a with $G(z) = (z-a)^n \tilde{G}(z)$. We infer from (5.5) that

$$\frac{\lambda m}{z-a} + \frac{\mu n}{z-a}$$

is holomorphic about a, so that

$$\lambda m + \mu n = 0.$$

Combining this with (5.4), we have

$$F(z)^m G(z)^n = c, \quad z \in \mathbf{C},$$

where $c \in \mathbf{C}^*$ is a constant. Thus $f(\mathbf{C})$ is contained in the closure of a translate of the algebraic subgroup $\{x^m y^n = 1\}$ of G. $\square$

Let $D = \sum D_i$ be a divisor on G with only simple normal crossings, where D_i are the irreducible components. We assume that *every* D_i *is* ∇-*totally geodesic.* We deal with the value distribution of f for $D + E$ in two ways.

(1) Here we use an equivariant blow-up of the compactification $\mathbf{P}^1(\mathbf{C})^2$ of G. By [10] (or directly in this case) we have the following:

LEMMA 5.6. *Let the notation be as above. Then there is an equivariant blow-up* $\pi : \hat{G} \to \mathbf{P}^1(\mathbf{C})^2$ *such that* $\hat{D} + \hat{E}$ *has only simple normal crossings, where* $\hat{D}$ *is the closure of* D *in* $\hat{G}$ *and* $\hat{E} = \hat{G} \setminus G$. *Moreover, if the stabilizer* $\{a \in G; a + D = D\}$ *of* D *is finite, then* $\hat{D}$ *is ample on* $\hat{G}$.

Let $f : \mathbf{C} \to \mathbf{P}^1(\mathbf{C})^2$ be a holomorphic curve such that $f(\mathbf{C}) \not\subset E$. Then there is a lifting $\hat{f} : \mathbf{C} \to \hat{G}$ with $\hat{G}$ in Lemma 5.6 such that $f = \pi \circ \hat{f}$. Let

$\hat{E} = \sum_j \hat{E}_j$ be the irreducible decomposition, and denote by $\{\hat{\mathrm{P}}_k\}$ all the crossing points of the $\hat{E}_i$'s.

LEMMA 5.7. *Let $\hat{f} : \mathbf{C} \to \hat{G}$, $\hat{D}$, $\hat{E}$ and $\{\hat{\mathrm{P}}_k\}$ be as above. Assume that f is ∇-nondegenerate. Then we have*

$$\begin{aligned} T_{\hat{f}}(r, L(\hat{D})) \leq \sum_i N_2(r, \hat{f}^* \hat{D}_i) + 2 \sum_j N_1(r, \hat{f}^* \hat{E}_j) \\ - \sum_k N_1(r, \hat{f}^* \hat{\mathrm{P}}_k) + S_f(r). \end{aligned}$$

Since $K_{\hat{G}} = -\hat{E}$ and

$$T_{\hat{f}}(r, K_{\hat{G}}) \leq - \sum_j N_1(r, \hat{f}^* \hat{E}_j),$$

we have the following, formulated closer to the fundamental conjecture for holomorphic curves ([6], §2 and [7]).

COROLLARY 5.8. *Let the notation be as in Theorem 5.7. Then*

$$T_{\hat{f}}(r, L(\hat{D})) + 2T_{\hat{f}}(r, K_{\hat{G}}) \leq \sum_i N_2(r, \hat{f}^* \hat{D}_i) + S_f(r). \tag{5.9}$$

REMARK. (i) The coefficient "2" in (5.9) should be "1" by the fundamental conjecture.

(ii) By Proposition 5.3 for f to be ∇-nondegenerate it suffices to assume that f is algebraically nondegenerate and $f(\mathbf{C}) \cap E \neq \emptyset$.

For the proof of Theorem 5.7 we take a holomorphic section $\hat{\sigma} \in H^0(\hat{G}, L(\hat{D}))$ and $\hat{\tau} \in H^0(\hat{G}, \hat{E})$ such that the divisors $(\hat{\sigma}) = \hat{D}$ and $(\hat{\tau}) = \hat{E}$. We introduce a hermitian metric $\|\hat{\sigma}\|$ (resp. $\|\hat{\tau}\|$) in $L(\hat{D})$ (resp. $L(\hat{E})$). We take a C^∞ volume form Ω on $\hat{G}$. Then we set

$$\hat{\xi}(z) = \frac{|W(\nabla, \hat{f})(z)|^2 \cdot \Omega(f(z))}{\|\hat{\sigma}(f(z))\|^2 \cdot \|\hat{\tau}(f(z))\|^2}. \tag{5.10}$$

LEMMA 5.11. *Let $\hat{\xi}$ be as above* (5.10). *Then*

$$\int_{|z|=r} \log^+ \hat{\xi}(z) \frac{d\theta}{2\pi} = S_{\hat{f}}(r).$$

PROOF. Note that the singularities of $\hat{\xi}(z)$ are those coming from the intersections of $\hat{f}$ and $\hat{D} + \hat{E}$. Locally on $\hat{G}$ with local coordinate $\hat{x}, \hat{y}$ such that $\hat{E}$ is written by $\hat{x} = 0$, by $\hat{y} = 0$ or by $\hat{x}\hat{y} = 0$, those singularities are given by

$$\begin{vmatrix} \frac{\frac{d}{dz}\hat{x}(f(z))}{\hat{x}(f(z))} & \frac{\frac{d}{dz}\hat{y}(f(z))}{\hat{y}(f(z))} \\ \frac{d}{dz}\left(\frac{\frac{d}{dz}\hat{x}(f(z))}{\hat{x}(f(z))}\right) & \frac{d}{dz}\left(\frac{\frac{d}{dz}\hat{y}(f(z))}{\hat{y}(f(z))}\right) \end{vmatrix}, \quad \begin{vmatrix} \frac{\frac{d}{dz}\tilde{\hat{\sigma}}(f(z))}{\tilde{\hat{\sigma}}(f(z))} & \frac{\frac{d}{dz}\hat{y}(f(z))}{\hat{y}(f(z))} \\ \frac{\frac{d^2}{dz^2}\tilde{\hat{\sigma}}(f(z))}{\tilde{\hat{\sigma}}(f(z))} & \frac{d}{dz}\left(\frac{\frac{d}{dz}\hat{y}(f(z))}{\hat{y}(f(z))}\right) \end{vmatrix},$$

or by

$$\begin{vmatrix} \frac{\frac{d}{dz}\tilde{\hat{\sigma}}(f(z))}{\tilde{\hat{\sigma}}(f(z))} & \frac{\frac{d}{dz}\hat{x}(f(z))}{\hat{y}(f(z))} \\ \frac{\frac{d^2}{dz^2}\tilde{\hat{\sigma}}(f(z))}{\tilde{\hat{\sigma}}(f(z))} & \frac{d}{dz}\left(\frac{\frac{d}{dz}\hat{x}(f(z))}{\hat{x}(f(z))}\right) \end{vmatrix},$$

where $\tilde{\hat{\sigma}}$ is the local expression of $\hat{\sigma}$. Therefore, from Nevanlinna's Lemma on logarithmic derivative we deduce the required estimate (cf. the proof of Lemma 2.3). □

PROOF OF THEOREM 5.7. By a careful computation of pole orders in $\hat{\xi}$ we have the following current inequality on $\mathbf{C}$:

$$dd^c \log \hat{\xi} \geq \hat{f}^* c_1(\hat{D}) - \sum_i (\hat{f}^* D_i)_2 - 2\sum_j (\hat{f}^* E_j)_1 + \sum_k (\hat{f}^* \hat{\mathrm{P}}_k)_1.$$

By making use of Jensen's formula and Lemma 5.10 we complete the proof of the present theorem. □

(2) The advantage of Theorem 5.7 is that it is applicable for an arbitrary algebraically non-degenerate $f : \mathbf{C} \to \mathbf{P}^1(\mathbf{C})^2$ with $f(\mathbf{C}) \cap E \neq \emptyset$ (cf. Proposition 5.3). On the other hand it is not so easy to compute the order function $T_{\hat{f}}(r, L(\hat{D}))$. The blow-up $\hat{G}$ was used to get a kind of the general position condition with respect $\hat{D}$ and $\hat{f}$. In the present subsection we are going to deal with the problem imposing such a condition for $f : \mathbf{C} \to \mathbf{P}^1(\mathbf{C})^2$ relative to D, without using a blow-up.

Let $E = \mathbf{P}^1(\mathbf{C})^2 \setminus G$ be the boundary divisor of G, which has four components E_j $(1 \leq j \leq 4)$ with only simple normal crossings at four points,

$$\mathrm{P}_1 = (0,0),\ \mathrm{P}_2 = (0,\infty),\ \mathrm{P}_3 = (\infty,0),\ \mathrm{P}_4 = (\infty,\infty).$$

Let $O(m,n)$ denote the line bundle of degree m in the first factor of $\mathbf{P}^1(\mathbf{C})^2$ and of degree n in the second factor of $\mathbf{P}^1(\mathbf{C})^2$. Then every E_j has bidegree $(1,0)$ or $(0,1)$. Let $\sigma \in H^0(\mathbf{P}^1(\mathbf{C})^2, O(m,n))$ such that the divisor $D = (\sigma)$ is reduced and has no common component with E. We introduce the natural metric $\|\sigma\|$. Let $D = \sum_i D_i$ be the irreducible decomposition of D.

THEOREM 5.12. *Let $D = \sum_i D_i$ be as above, and let $f = (F,G) : \mathbf{C} \to \mathbf{P}^1(\mathbf{C})^2$ be a ∇-nondegenerate holomorphic curve. We assume the following conditions for D and f:*

(i) *$D \cap G$ has only simple normal crossings.*

(ii) *Every D_i (strictly speaking, $D_i \cap G$) is ∇-totally geodesic.*

(iii) *There is a neighborhood V of $\{\mathrm{P}_k\}_{k=1}^4$ such that $f(\mathbf{C}) \cap V = \emptyset$.*

Then

$$T_f(r, O(m,n)) \leq \sum_i N_2(r, f^*D_i) + 2\sum_j N_1(r, f^*E_j) + S_f(r). \tag{5.13}$$

COROLLARY 5.14. *Under the same conditions as in Theorem 5.12 we have, in particular,*

$$T_f(r, O(m-4, n-4)) \leq \sum_i N_2(r, f^*D_i) + S_f(r).$$

Let Ω be the volume form associated with the product of the Fubini-Study Kähler form on $\mathbf{P}^1(\mathbf{C})$:

$$\Omega = \frac{\left(\frac{i}{2\pi}\right)^2 dx \wedge d\bar{x} \wedge dy \wedge d\bar{y}}{(1+|x|^2)^2(1+|y|^2)^2}.$$

Let $\tau \in H^0(\mathbf{P}^1(\mathbf{C})^2, O(2,2))$ such that $(\tau) = E$, and set

$$\xi(z) = \frac{|W(\nabla, f)(z)|^2 \cdot \Omega(f(z))}{\|\sigma(f(z))\|^2 \cdot \|\tau(f(z))\|^2}. \tag{5.15}$$

As in (1), Theorem 5.12 follows from the following lemma.

LEMMA 5.16. *Let $\xi(z)$ be as above in* (5.15). *Then we have*

$$\int_{|z|=r} \log^+ \xi(z) \frac{d\theta}{2\pi} = S_f(r).$$

PROOF. Notice that $D \cap E = \{\mathrm{P}_j\}_{j=1}^4$. Let F be the set of intersection points of the irreducible components of $D \cap G$. Setting

$$V_1 = \{\delta < |x| < \delta^{-1}\} \times \{\delta < |x| < \delta^{-1}\}$$

with $0 < \delta < 1$, we take and fix a small δ so that

$$F \subset V_1, \qquad \overline{f(\mathbf{C})} \cap D \subset V_1. \tag{5.17}$$

In a neighborhood $U(\subset V_1)$ of every point of F, $D \cap U$ is defined by

$$x^{m_i} y^{n_i} = c_i, \quad i = 1, 2, \quad |A| \neq 0,$$

where $m_i, n_i \in \mathbf{Z}$, $c_i \in \mathbf{C}^*$ and $A = \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix}$.

We first estimate $\xi(z)$, provided $f(z) \in U$. It follows from (5.15) that with a positive C^∞ function b on U

(5.18)

$$\begin{aligned}
\xi(z) &= \left\| \begin{matrix} \frac{F'}{F} & \frac{G'}{G} \\ \left(\frac{F'}{F}\right)' & \left(\frac{G'}{G}\right)' \end{matrix} \right\|^2 \cdot \frac{b(f(z))}{|F^{m_1}G^{n_1} - c_1|^2 \cdot |F^{m_2}G^{n_2} - c_2|^2} \\
&= ||A||^{-2} \left\| \begin{matrix} m_1\frac{F'}{F} + n_1\frac{G'}{G} & m_2\frac{F'}{F} + n_2\frac{G'}{G} \\ \left(m_1\frac{F'}{F} + n_1\frac{G'}{G}\right)' & \left(m_2\frac{F'}{F} + n_2\frac{G'}{G}\right)' \end{matrix} \right\|^2 \\
&\quad \times \frac{b(f(z))}{|F^{m_1}G^{n_1} - c_1|^2 \cdot |F^{m_2}G^{n_2} - c_2|^2}
\end{aligned}$$

$$= ||A||^{-2} \left\| \begin{matrix} \frac{(F^{m_1}G^{n_1})'}{F^{m_1}G^{n_1}} & \frac{(F^{m_2}G^{n_2})'}{F^{m_2}G^{n_2}} \\ \frac{(F^{m_1}G^{n_1})''}{F^{m_1}G^{n_1}} - \left(\frac{(F^{m_1}G^{n_1})'}{F^{m_1}G^{n_1}}\right)^2 & \frac{(F^{m_2}G^{n_2})''}{F^{m_2}G^{n_2}} - \left(\frac{(F^{m_2}G^{n_2})'}{F^{m_2}G^{n_2}}\right)^2 \end{matrix} \right\|^2$$
$$\times \frac{b(f(z))}{|F^{m_1}G^{n_1} - c_1|^2 \cdot |F^{m_2}G^{n_2} - c_2|^2}$$
$$= ||A||^{-2} \left\| \begin{matrix} \frac{(F^{m_1}G^{n_1}-c_1)'}{F^{m_1}G^{n_1}-c_1} \\ \frac{(F^{m_1}G^{n_1}-c_1)''}{F^{m_1}G^{n_1}-c_1} - \frac{(F^{m_1}G^{n_1})'}{F^{m_1}G^{n_1}} \cdot \frac{(F^{m_1}G^{n_1}-c_1)'}{F^{m_1}G^{n_1}-c_1} \end{matrix} \right.$$
$$\left. \begin{matrix} \frac{(F^{m_2}G^{n_2}-c_2)'}{F^{m_2}G^{n_2}-c_2} \\ \frac{(F^{m_2}G^{n_2}-c_2)''}{F^{m_2}G^{n_2}-c_2} - \frac{(F^{m_2}G^{n_2})'}{F^{m_2}G^{n_2}} \cdot \frac{(F^{m_2}G^{n_2}-c_2)'}{F^{m_2}G^{n_2}-c_2} \end{matrix} \right\|^2$$
$$\times |F|^{-m_1-m_2}|G|^{-n_1-n_2}b(f(z)).$$

Since $|F(z)|^{\pm 1}$ and $|G(z)|^{\pm 1}$ are uniformly bounded from above by δ^{-1}, provided $f(z) \in V_1$, we have by (5.18)

$$\xi(z) \le P_1 \left(\left| \frac{(F^{m_1}G^{n_1} - c_1)'}{F^{m_1}G^{n_1} - c_1} \right|, \left| \frac{(F^{m_2}G^{n_2} - c_2)'}{F^{m_2}G^{n_2} - c_2} \right|, \right. \left. \left| \frac{(F^{m_1}G^{n_1} - c_1)''}{F^{m_1}G^{n_1} - c_1} \right|, \left| \frac{(F^{m_2}G^{n_2} - c_2)''}{F^{m_2}G^{n_2} - c_2} \right| \right), \tag{5.19}$$

provided $f(z) \in U$, where $P_1(\cdots)$ is a polynomial with positive coefficients. Hence, we may assume that (5.19) holds, provided $f(z) \in V$.

In a neighborhood U' of a point of $D \cap (V_1 \setminus V)$, there is only one irreducible component of $D \cap U'$, to say, given by

$$x^{m_1}y^{n_1} = c_1, \quad m_1 \ne 0.$$

Then we have

$$\xi(z) = \frac{1}{m_1^2} \left\| \begin{matrix} \frac{(F^{m_1}G^{n_1}-c_1)'}{F^{m_1}G^{n_1}-c_1} & \frac{G'}{G} \\ \frac{(F^{m_1}G^{n_1}-c_1)''}{F^{m_1}G^{n_1}-c_1} - \frac{(F^{m_1}G^{n_1})'}{F^{m_1}G^{n_1}} \cdot \frac{(F^{m_1}G^{n_1}-c_1)'}{F^{m_1}G^{n_1}-c_1} & \left(\frac{G'}{G}\right)' \end{matrix} \right\|^2$$
$$\cdot |F|^{-2m_1} b(f(z)),$$

provided $f(z) \in U'$. Therefore we see that (5.19) holds for $f(z) \in V_1$.

Set

$$V_2 = \{0 \le |x| \le \delta\} \times \{\delta \le |y| \le \delta^{-1}\} \cup \{\delta \le |x| \le \delta^{-1}\} \times \{0 \le |y| \le \delta\}$$

(continued)

$$\cup \{\delta^{-1} \le |x| \le \infty\} \times \{\delta \le |y| \le \delta^{-1}\}$$
$$\cup \{\delta \le |x| \le \delta^{-1}\} \times \{\delta^{-1} \le |y| \le \infty\}.$$

It follows from the condition that

$$f(\mathbf{C}) \subset V_1 \cup V_2.$$

Note that there exists a positive constant c_3 such that

$$\|\sigma\| \ge c_3 \text{ on } V_2.$$

Suppose that $f(z) \in V_2$. Then we obtain

$$\begin{aligned}\xi(z) &= \left\| \begin{matrix} \frac{F'}{F} & \frac{G'}{G} \\ \left(\frac{F'}{F}\right)' & \left(\frac{G'}{G}\right)' \end{matrix} \right\|^2 \cdot \frac{1}{\|\sigma(f(z))\|^2} \\ &\le \frac{1}{c_3^2} \left\| \begin{matrix} \frac{F'}{F} & \frac{G'}{G} \\ \left(\frac{F'}{F}\right)' & \left(\frac{G'}{G}\right)' \end{matrix} \right\|^2 .\end{aligned}$$

We see by this that there is a polynomial $P_2(\cdots)$ with positive coefficients satisfying

$$\xi(z) \le P_2\left(\left|\frac{F'}{F}\right|, \left|\frac{G'}{G}\right|, \left|\frac{F''}{F}\right|, \left|\frac{G''}{G}\right| \right). \tag{5.20}$$

Set $P = P_1 + P_2$. We see by (5.19) and (5.20) that

$$\begin{aligned}\xi(z) \le P\Bigg(&\left|\frac{(F^{m_1}G^{n_1} - c_1)'}{F^{m_1}G^{n_1} - c_1}\right|, \left|\frac{(F^{m_2}G^{n_2} - c_2)'}{F^{m_2}G^{n_2} - c_2}\right|, \left|\frac{F'}{F}\right|, \left|\frac{G'}{G}\right|, \\ &\left|\frac{(F^{m_1}G^{n_1} - c_1)''}{F^{m_1}G^{n_1} - c_1}\right|, \left|\frac{(F^{m_2}G^{n_2} - c_2)''}{F^{m_2}G^{n_2} - c_2}\right|, \left|\frac{F''}{F}\right|, \left|\frac{G''}{G}\right| \Bigg)\end{aligned} \tag{5.21}$$

for all $z \in \mathbf{C}$. Applying Nevanlinna's lemma on logarithmic derivatives, we infer that

$$\begin{aligned}&\int_{|z|=r} \log^+ \xi(z) \frac{d\theta}{2\pi} \\ &= O\left(m\left(r, \frac{(F^{m_1}G^{n_1} - c_1)'}{F^{m_1}G^{n_1} - c_1}\right) + m\left(r, \frac{(F^{m_2}G^{n_2} - c_2)'}{F^{m_2}G^{n_2} - c_2}\right) \right.\end{aligned}$$

(continued)

$$
\begin{aligned}
&+ m\left(r, \frac{F'}{F}\right) + m\left(r, \frac{G'}{G}\right) \\
&+ m\left(r, \frac{(F^{m_1}G^{n_1} - c_1)''}{F^{m_1}G^{n_1} - c_1}\right) + m\left(r, \frac{(F^{m_2}G^{n_2} - c_2)''}{F^{m_2}G^{n_2} - c_2}\right) \\
&+ m\left(r, \frac{F''}{F}\right) + m\left(r, \frac{G''}{G}\right)\Bigg) \\
= {}& S_f(r). \ \square
\end{aligned}
$$

Example 1. Let $f(z) = (F(z), G(z))$ be defined by

$$F(z) = e^z, \qquad G(z) = \frac{e^z + 1}{e^z - 1}.$$

It is easy to check that f is ∇-nondegenerate, and to see that the image $f(\mathbf{C})$ is contained by a curve $C \subset \mathbf{P}^1(\mathbf{C})^2$ defined by

$$(x - 1)(y - 1) = 2.$$

Since $C \cap \{\mathrm{P}_j\}_{j=1}^4 = \emptyset$, f satisfies the conditions of Theorem 5.12, although f is algebraically degenerate.

Example 2. Here we give an example of an algebraically nondegenerate $f : \mathbf{C} \to \mathbf{P}^1(\mathbf{C})^2$ for Theorem 5.12. By Fatou's example ([2]) there is an injective holomorphic map $\Phi : \mathbf{C}^2 \to \mathbf{C}^2$ with non-empty exterior open subset U ($\subset \mathbf{C}^2$) of the image $\Phi(\mathbf{C}^2)$. By making use of Picard's theorem (or Casorati-Weierstrass' theorem) we take four points, $(a_i, b_j) \in U$ $(i, j = 1, 2)$ such that

$$
\begin{aligned}
&a_1 \neq a_2, \quad b_1 \neq b_2, \\
&\Phi^{-1}(\{a_i\} \times \mathbf{C}) \neq \emptyset \qquad (i = 1, 2), \\
&\Phi^{-1}(\mathbf{C} \times \{b_j\}) \neq \emptyset \qquad (j = 1, 2).
\end{aligned}
$$

Let $\alpha_i \in \Phi^{-1}(\{a_i\} \times \mathbf{C})$ $(i = 1, 2)$ and $\beta_j \in \Phi^{-1}(\mathbf{C} \times \{b_j\}$ $(j = 1, 2)$. By making use of the affine coordinate (x, y) of $\mathbf{C}^2 \subset \mathbf{P}^1(\mathbf{C})^2$ we consider the following biholomorphic transform of $\mathbf{P}^1(\mathbf{C})^2$:

$$\psi(x, y) = \left(\frac{x - a_1}{x - a_2}, \frac{y - b_1}{y - b_2}\right).$$

Set $\Psi = \psi \circ \Phi : \mathbf{C}^2 \to \mathbf{P}^1(\mathbf{C})^2$. Then points P_j $(1 \leq j \leq 4)$ are exterior points of the image $\Psi(\mathbf{C}^2)$. Let $g : \mathbf{C} \to \mathbf{C}^2$ be a holomorphic curve such that the image $g(\mathbf{C})$ is contained by no analytic proper subset of $\mathbf{C}^2$, and that g passes through all four points α_i, β_j $(i, j = 1, 2)$. Set $f = \Psi \circ g : \mathbf{C} \to \mathbf{P}^1(\mathbf{C})^2$. Then f is algebraically nondegenerate and $f(\mathbf{C}) \cap E \neq \emptyset$. By Proposition 5.3 f is ∇-nondegenerate, too.

Problems. (i) It is an interesting problem to find more examples for Theorem 1.1.

(ii) It is naturally interesting to extend the results of §5 to the higher dimensional case and the case of general semi-abelian varieties.

(iii) Is it possible to decrease "2" to "1" in the inequalities obtained by Theorems 5.7 and 5.12.

References

[1] Cartan, H., Sur les zéros des combinaisons linéaires de p fonctions holomorphes données, Mathematica **7** (1933), 5–31.

[2] Fatou, P., Sur les fonctions méromorphes de deux variables, C. R. Acad. Sci. Paris **175** (1922), 862–865.

[3] Noguchi, J., Holomorphic curves in algebraic varieties, Hiroshima Math. J. **7** (1977), 833–853.

[4] Noguchi, J., Lemma on logarithmic derivatives and holomorphic curves in algebraic varieties, Nagoya Math. J. **83** (1981), 213–233.

[5] Noguchi, J., On holomorphic curves in semi-Abelian varieties, Math. Z. **228** (1998), 713–721.

[6] Noguchi, J., Intersection multiplicities of holomorphic and algebraic curves with divisors, Proc. OKA 100 Conference Kyoto/Nara 2001, Advanced Studies in Pure Mathematics **42**, pp. 243–248, Japan Math. Soc. Tokyo, 2004.

[7] Noguchi, J., Value Distribution and Distribution of Rational Points, Talk at Mittag-Leffler Institute, 27 March 2008 (http://nogpc4.ms.u-tokyo.ac.jp/nog/talks/).

[8] Noguchi, J. and T. Ochiai, Geometric Function Theory in Several Complex Variables, Japanese edition, Iwanami, Tokyo, 1984; English Translation, Transl. Math. Mono. **80**, Amer. Math. Soc., Providence, Rhode Island, 1990.

[9] Noguchi, J., Winkelmann, J. and K. Yamanoi, The second main theorem for holomorphic curves into semi-Abelian varieties, Acta Math. **188** no. 1 (2002), 129–161.

[10] Noguchi, J., Winkelmann, J. and K. Yamanoi, The second main theorem for holomorphic curves into semi-Abelian varieties II, Forum Math. **20** (2008), 469–503.
[11] Siu, Y.-T., Defect relations for holomorphic maps between spaces of different dimensions, Duke Math. J. **55** (1987), 213–251.
[12] Tiba, Y., Holomorphic curves into the product space of the Riemann spheres, preprint UTMS 2010-19, 2010.
[13] Yamanoi, K., Holomorphic curves in abelian varieties and intersection with higher codimensional subvarieties, Forum Math. **16** (2004), 749–788.

(Received February 3, 2011)
(Revised March 17, 2011)

Graduate School of Mathematical Sciences
The University of Tokyo
Komaba, Meguro
Tokyo 153-8914, Japan
E-mail: noguchi@ms.u-tokyo.ac.jp