J. Math. Sci. Univ. Tokyo
18 (2011), 139–154.

Contact Embeddings in Standard Contact Spheres via Approximately Holomorphic Geometry

By David Martínez Torres

Abstract. We generalize the work of A. Mori using approximately holomorphic methods to show that any closed co-oriented contact manifold of dimension $2n+1$ admits contact embeddings in the standard contact sphere of dimension $4n+3$. This also recovers results of M. Gromov but without appealing to an h-principle. Our construction also extends Mori's in being compatible with the open book decompositions carrying contact structures as introduced by E. Giroux. Therefore our work provides a unified setting for contact embeddings and open book decompositions for closed co-oriented contact manifolds. We also remark on the properties of the pages of the open book decompositions.

1. Introduction

A contact distribution ξ on a manifold M of dimension $2n+1$ is a maximally non-integrable hyperplane distribution. This means that for a local 1-form α which defines ξ, the top form $\alpha \wedge d\alpha^n$ is a volume form. The co-orientability of the contact distribution is equivalent to the existence of a global 1-form α defining ξ, a contact 1-form.

The most important example of closed co-oriented contact manifold is the so called standard contact sphere $(S^{2n+1}, \xi_{\mathrm{std}})$, where the contact distribution is defined by the complex tangencies of $S^{2n+1} \subset \mathbb{C}^{n+1}$.

A map between contact manifolds (M, ξ), (M', ξ') is called a contact embedding if it is an embedding which pulls back the contact distribution ξ' to ξ (in the terminology introduced by Gromov these maps are called contact isometric embeddings). The contact embedding problem was first addressed by Gromov, who proved in [11], section 3.4.3, that for arbitrary contact manifolds (M, ξ), (M', ξ') contact embeddings of (M, ξ) into (M', ξ') abide by an h-principle in an appropriate dimension range. In particular

2010 *Mathematics Subject Classification*. Primary 53D35; Secondary 57R40, 53C25.
Partially supported by the Fundação para a Ciência e a Tecnologia (FCT / Portugal).

his results imply that any co-oriented contact manifold (M^{2n+1}, ξ) has a contact embedding into $(S^{4n+3}, \xi_{\mathrm{std}})$[1].

Contact embeddings into standard contact spheres have been addressed using tools different from an h-principle. Approximately holomorphic methods were introduced in contact geometry in [12] to show that any closed contact manifold contains contact submanifolds (see section 2 for basic material in approximately holomorphic geometry). Ideas from approximately holomorphic geometry were used by E. Giroux to prove his spectacular result about the existence of open book decompositions carrying a given contact structure [9].

In [15] Mori used approximately holomorphic methods to show that any closed co-oriented contact 3-manifold admits contact embeddings in $(S^7, \xi_{\mathrm{std}})$. Mori's contact embeddings have the additional feature of being compatible with a canonical open book decomposition of the standard contact sphere, as we shall briefly explain.

Roughly, an open book decomposition of M adapted to a contact form α is a very particular stratification into a contact codimension 2 submanifold, the binding, and an S^1-worth family of codimension 1 submanifolds symplectic with respect to $d\alpha$, which are called the pages. Such an open book decomposition is said to carry the contact distribution defined by α (see definition 4 for details). The main example of an open book decomposition carrying a contact structure is the following: Consider the standard contact sphere $(S^{2n+1}, \xi_{\mathrm{std}})$ and take as defining 1-form for ξ_{std}

$$\alpha_{\mathrm{std}} := \sum_{j=0}^{n} \frac{i}{4}(z_j d\bar{z}_j - \bar{z}_j dz_j)_{|S^{2n+1}},$$

where $z_0, \ldots, z_n$ are coordinates on $\mathbb{C}^{n+1}$.

Let H be a complex linear hyperplane of $\mathbb{C}^{n+1}$. The complex linear hyperplane H determines an S^1-worth of real half hyperplanes with boundary H, which gives rise to $\mathcal{B}^H$ an open book decomposition of S^{2n+1} adapted

[1]In [11], page 339, it is stated that contact embeddings exist for target spheres of dimension $2m+1$, if $m \geq 3n+1$. A detailed proof is given in example 6.2 in [4] by appealing to obstruction theory involving real Stiefel-Whitney manifolds. As an anonymous referee pointed out to the author, by using compatible almost complex structures the homotopic obstruction is the existence of an injection from a rank n complex vector bundle into a rank m complex vector bundle. The connectivity of the corresponding complex Stiefel-Whitney manifold gives the sharper bound $m \geq 2n+1$.

to α_{std} as follows: The binding of $\mathcal{B}^H$ is the intersection of H with S^{2n+1}, which is contact diffeomorphic to the standard contact sphere $(S^{2n-1}, \xi_{\text{std}})$; the pages of $\mathcal{B}^H$ are the intersection of the real half hyperplanes with S^{2n+1}, and they are seen to be symplectomorphic to $(\mathbb{CP}^n \backslash \mathbb{CP}^{n-1}, \pi\omega_{FS})$, where ω_{FS} is the integral Fubini-Study symplectic form. Because the unitary group acts by strict contact transformations on $(S^{2m+1}, \alpha_{\text{std}})$, preserves the standard CR structure of the sphere and acts transitively on complex linear hyperplanes, there is no loss of generality in assuming that H is the hyperplane H_0 defined by the equation $z_0 = 0$.

The compatibility property of the 3-dimensional contact embeddings $\varphi\colon (M, \xi) \to (S^7, \xi_{\text{std}})$ constructed by Mori with the open book decomposition $\mathcal{B}^{H_0}$ of S^7, is that $\varphi^*\mathcal{B}^{H_0}$ is an open book decomposition of M which carries ξ.

In this paper, we interpret Mori's work from a more geometric perspective and show that the standard contact spheres $(S^{2n+1}, \xi_{\text{std}})$ together with the open book decompositions $\mathcal{B}^{H_0}$, are universal in an appropriate sense with respect to closed co-oriented contact manifolds endowed with approximately holomorphic open book decompositions (see section 2 for the precise definition).

Theorem 1. *Let (M, ξ) be a closed co-oriented contact manifold of dimension $2n+1$. Then the following holds:*

(i) *Approximately holomorphic geometry provides a contact embedding*

$$\varphi\colon (M, \xi) \to (S^{4n+3}, \xi_{\text{std}}).$$

(ii) *The contact embeddings provided by approximately holomorphic geometry can be chosen so that $\varphi^*\mathcal{B}^{H_0}$ is an open book decomposition of M which carries ξ. We will refer to $\varphi^*\mathcal{B}^{H_0}$ as a spinning approximately holomorphic open book decomposition (see definition 6).*

(iii) *Let $\mathcal{B}_k$ be a sequence of approximately holomorphic open book decompositions of M which carry ξ. Then a sequence $\mathcal{B}'_k$ of spinning approximately holomorphic open book decompositions which carry ξ can be constructed, having the property that for all k sufficiently large $\mathcal{B}_k$ and $\mathcal{B}'_k$ can be connected by a 1-parameter family of open book decompositions which carry ξ.*

Observe that the dimension range in item (i) in Theorem 1 is the same as in Gromov's work.

The content of this paper is as follows: In section 2 we recall how results in Sasakian geometry together with techniques used in proving them, should indicate us how to proceed in proving analogous results in contact geometry. In particular the embedding result for Sasakian structures in [18, 19] points to approximately holomorphic geometry as the tool to be used in the contact setting; the necessary material on approximately holomorphic geometry and on open book decompositions is also introduced. Section 3 is devoted to the proof of Theorem 1, to remarking on properties of the pages of spinning approximately holomorphic open book decompositions and to briefly comparing our construction with Mori's.

The author would like to acknowledge the work of the referees, whose comments helped shape this manuscript.

2. Approximately Holomorphic Geometry

Approximately holomorphic theory for symplectic manifolds was introduced by S. Donaldson in [5]. Roughly speaking the theory provides, for any closed integral symplectic manifold and for any fixed compatible almost complex structure, generic approximately holomorphic sections of suitable bundles. They are generic in the sense of jet theory [2], so that one can produce among others symplectic submanifolds [5, 1], Lefschetz pencils [6] and embeddings in projective spaces [17].

From the point of view of approximately holomorphic theory it can be said that a symplectic manifold with a choice of compatible almost complex structure and associated metric, is the geometry one gets if in Kähler geometry the integrability condition on the almost complex structure is dropped. And what makes this relation useful is that approximately holomorphic geometry often gives a symplectic version of theorems in Kähler geometry based on the existence of generic holomorphic sections.

The same relation between Kähler and symplectic geometry with compatible almost complex structure holds between Sasakian and contact (metric) geometry (see [3] for Sasakian geometry). Approximately holomorphic geometry has been adapted to contact manifolds [12, 13]. Thus by analogy approximately holomorphic geometry can be expected to provide contact versions of theorems in Sasakian geometry which are based on the existence of generic CR sections. Indeed item (i) in Theorem 1 is the contact version of the following result:

THEOREM 2 (Ornea-Verbitsky [18, 19]). *Any closed Sasakian manifold admits CR immersions and embeddings in Sasakian spheres which are contact diffeomorphic to standard contact spheres.*

We use approximately holomorphic sections instead of CR sections to prove Theorem 1. We recall a few concepts from approximately holomorphic geometry which will be used in the following sections.

Let (M, ξ) be a co-oriented contact manifold and fix α a contact 1-form defining ξ. Let $\underline{\mathbb{C}}$ denote the trivialized Hermitian complex line bundle on M and regard its sections as complex valued smooth functions. We consider for all $k \in \mathbb{N}$ the sequence of connections defined by the contact forms $k\alpha$. Namely

$$\nabla_k s = \mathrm{d}s - ik\alpha s$$

for any $s \in \Gamma(\underline{\mathbb{C}})$.

We fix J a compatible almost complex structure on the symplectic vector bundle $(\xi, d\alpha)$. It allows us for any $s \in \Gamma(\underline{\mathbb{C}})$ to split the restriction of the covariant derivative to ξ as

$$\nabla_{k|\xi} s = \partial_{\nabla_k} s + \bar{\partial}_{\nabla_k} s,\ \partial_{\nabla_k} s \in \Gamma(\xi^{*1,0} \otimes \underline{\mathbb{C}}),\ \bar{\partial}_{\nabla_k} s \in \Gamma(\xi^{*0,1} \otimes \underline{\mathbb{C}}).$$

We also consider the contact metric structure associated to (α, J), i.e. we introduce the metric g defined by $d\alpha, J$ along ξ, and so that the Reeb vector field has norm one and it is orthogonal to the contact distribution. We let g_k denote kg, $k \in \mathbb{N}$.

DEFINITION 1. [12] A sequence of sections s_k of $(\underline{\mathbb{C}}, \nabla_k)$ is approximately holomorphic if for all k large enough

$$|\nabla_k^{j-1} \bar{\partial}_{\nabla_k} s_k|_{g_k} \leq C k^{-1/2},\ j = 1, 2, \tag{1}$$

and

$$|\nabla_k^j s_k|_{g_k} \leq C,\ j = 0, 1, 2, \tag{2}$$

where C is some positive constant independent of k.

In particular for any sequence of vectors $u_k \in \xi$ with bounded norm sequence $\{|u_k|_{g_k}\}$ and for the rescaling $R_k = k^{-1/2}R$ of the Reeb vector field R of α with $|R_k|_{g_k} = 1$, we have the estimates

$$\mathrm{d}s_k(Ju_k) = i\mathrm{d}s_k(u_k) + O(k^{-1/2}), \tag{3}$$

$$|\mathrm{d}s_k(u_k)| = O(1) \tag{4}$$

and

$$\mathrm{d}s_k(R_k) = k^{1/2} i s_k + O(1). \tag{5}$$

Definition 2. [12] A sequence of sections s_k of $(\underline{\mathbb{C}}, \nabla_k)$ is uniformly transverse along ξ to $\mathbf{0}$ if there exists $\eta > 0$ independent of k such that for all k large enough and for all $x \in M$, either (i) $|s_k(x)| > \eta$ or (ii) $\nabla_{|\xi} s_k(x)$ is surjective and has a right inverse whose norm is bounded by η^{-1}.

Condition (ii) in definition 2 is equivalent to asking the image by $\nabla_{|\xi} s_k(x)$ of the g_k-unit ball of ξ_x to contain the ball of radius η.

Observe that uniform transversality along ξ to $\mathbf{0}$ is an open C^1-condition with respect to g_k. Let s_k be as in definition 2. Given δ small enough if $|s'_k - s_k|_{C^1, g_k} \leq \delta$ for all k sufficiently large, then s'_k is seen to be uniformly transverse to $\mathbf{0}$ by using $\eta' := \eta/2$ as uniform constant.

2.1. Open book decompositions carrying a contact structure

The standard open book decomposition $\mathcal{B}_{\mathrm{std}}$ of $\mathbb{C} = \mathbb{R}^2$ is defined to be the stratification given by the origin, which is the binding of the open book decomposition, and the half lines, which are the pages of the open book decomposition.

Definition 3. An open book decomposition $\mathcal{B}$ of a manifold M is a stratification for which there exists a function $f\colon M \to \mathbb{C}$ which is transverse to the stratification $\mathcal{B}_{\mathrm{std}}$, and such that $\mathcal{B} = f^*\mathcal{B}_{\mathrm{std}}$. The binding and pages of $\mathcal{B}$ are the pullback of the binding and pages of $\mathcal{B}_{\mathrm{std}}$ respectively.

Definition 4 ([9]). Let (M, ξ) be a closed co-oriented contact manifold. An open book decomposition $\mathcal{B}$ is said to carry ξ if (i) the binding is a contact submanifold, and (ii) for a choice of positive contact form α the pages are symplectic with respect to $d\alpha$ and the vector field along any page defined by $i_Z d\alpha = \alpha$ points towards the binding, which is the boundary of the page. The open book decomposition $\mathcal{B}$ is said to be adapted to the contact form α.

Definition 5. Let (M, ξ) be a closed co-oriented contact manifold. An open book decomposition $\mathcal{B}$ carrying ξ is called spinning, if there exists a contact embedding $\varphi\colon (M, \xi) \to (S^{2m+1}, \xi_{\mathrm{std}})$ such that $\mathcal{B} = \varphi^* \mathcal{B}^{H_0}$.

Mori [15] constructs, for any 3-dimensional closed co-oriented contact manifold, spinning open book decompositions carrying the contact structure.

Giroux [9] shows that any co-oriented contact distribution on a closed manifold is carried by an open book. Such an open book decomposition is one of the few tools in contact geometry above dimension 3. Their use has applications to existence of contact structures and symplectic fillability problems.

More precisely, Giroux proves that for the bundles $(\underline{\mathbb{C}}, \nabla_k)$, any approximately holomorphic section $s_k\colon M \to \mathbb{C}$ which is uniformly transverse along ξ to $\mathbf{0}$ gives rise to an open book decomposition carrying ξ as follows: If k is sufficiently large for points $x \in M$ such that $|s_k(x)| > \eta$, by property (5) s_k is transverse to the page of $\mathcal{B}_{\mathrm{std}}$ through $s_k(x)$ and the tangent space of the pulled back page is symplectic at x with respect to $kd\alpha$. Because s_k is uniformly transverse along ξ to $\mathbf{0}$ the inverse image of $z \in \mathbb{C}$, $|z| \leq \eta$, defines a contact submanifold. Using a normal form for the contact structure in the tubular neighborhood $s_k^{-1}(B(0,\eta))$ of $s_k^{-1}(\mathbf{0})$, Giroux is able to modify $k\alpha$ inside $s_k^{-1}(B(0,\eta))$ into a contact form $\tilde{\alpha}_k$ defining ξ and such that $s_k^* \mathcal{B}_{\mathrm{std}}$ is adapted to $\tilde{\alpha}_k$ [9]. More details about the construction can be found in the lecture notes [10].

Definition 6. Let (M, ξ) be a closed co-oriented contact manifold and let $\mathcal{B}_k$ be a sequence of open book decompositions which carry ξ.

(1) We say that $\mathcal{B}_k$ is a sequence of approximately holomorphic open book decompositions if there exist α a contact form defining ξ and s_k a sequence of approximately holomorphic sections of $(\underline{\mathbb{C}}, \nabla_k)$, such that for all k large enough (i) $\mathcal{B}_k = s_k^* \mathcal{B}_{\mathrm{std}}$ and (ii) $\mathcal{B}_k$ is adapted to

a contact form $\tilde{\alpha}_k$ defining ξ which coincides with $k\alpha$ away from an η-neighborhood of the binding and which is the modification of $k\alpha$ described in [9] in the η-neighborhood of the binding.

(2) We say that $\mathcal{B}_k$ is a sequence of spinning approximately holomorphic open book decompositions if there exists α a contact form defining ξ, such that for all k large enough $\mathcal{B}_k$ are spinning open book decompositions carrying ξ so that (i) the sequence of embeddings $\varphi_k \colon M \to S^{2m+1}$ is of the form $\Phi_k \circ \psi_k$, where ψ_k is the time 1 diffeomorphism associated to an isotopy $\psi_{k,t}$ and $\Phi_k = s_k/|s_k|$ with $s_k \colon M \to \underline{\mathbb{C}}^{m+1}$ an $(m+1)$-tuple of approximately holomorphic sections; (ii) $(\Phi_k \circ \psi_{k,t})^* \mathcal{B}^{H_0}$ are sequences of open book decompositions carrying ξ for all $t \in [0,1]$ and $\Phi_k^* \mathcal{B}^{H_0}$ is a sequence of approximately holomorphic open book decompositions.

3. Proof of Theorem 1

Let (M^{2n+1}, ξ) be a closed co-oriented contact manifold with fixed contact form α. A brief sketch of the proof of the existence of contact embeddings in standard contact spheres via approximately holomorphic geometry is as follows: The starting point is a result of [16] which provides a sequence of approximately holomorphic sections $s_k \colon M \to \underline{\mathbb{C}}^{m+1}$, whose projectivization defines immersions of M along the contact distribution in projective space. The next step is suggested by Sasakian geometry and amounts to considering instead of the projectivization of the section, its projection onto the sphere S^{2m+1}. After a suitable modification this provides an embedding into the sphere which pulls back ξ_{std} to a contact structure on M. The final step is showing that the segment joining the induced contact structure with the original one (rather suitable contact 1-forms representing them) is by contact distributions; it is suggested by the convexity of the space of Kähler forms with respect to a fixed complex structure.

More precisely, let

$$\begin{aligned} s_k \colon M &\longrightarrow \underline{\mathbb{C}}^{m+1} \\ x &\longmapsto (s_{k,0}(x), \dots, s_{k,m}(x)) \end{aligned} \tag{6}$$

be an $(m+1)$-tuple of approximately holomorphic sections and let $q \colon \mathbb{C}^{m+1} \backslash \{0\} \to \mathbb{CP}^m$ be the tautological projection.

According to [16] (see [13] for the jet theory approach) after a perturbation of size at most $O(k^{-1/2})$ in the C^2-norm with respect to g_k, we can assume that an appropriate version of uniform transversality along ξ of

$$(\partial_{\nabla_{k|\xi}} s_k, s_k) \in \Gamma((\xi^{*1,0} \oplus \underline{\mathbb{C}}) \otimes \underline{\mathbb{C}}^{m+1})$$

to certain stratified submanifold $\Sigma \subset (\xi^{*1,0} \oplus \underline{\mathbb{C}}) \otimes \underline{\mathbb{C}}^{m+1}$ holds. The section $(\partial_{\nabla_{k|\xi}} s_k, s_k)$ is the approximately holomorphic analog of the first jet of s_k, and the stratified submanifold Σ detects the failure of the projectivization $\phi_k := q \circ s_k$ to be well defined and the failure of $d\phi_k^{1,0}{}_{|\xi}$ to be injective.

Now let $m \geq 2n$. Then the consequence of the above perturbation is the existence of $\eta > 0$ independent of k such that for all k large enough $|s_k| > \eta$ and the projectivization $\phi_k \colon M \to \mathbb{CP}^m$ is an everywhere defined approximately holomorphic estimated immersion along ξ. This means that the image by $d\phi_k$ of the g_k-unit ball of ξ_x contains the ball of radius η in $d\phi_k(\xi)$ with respect to the Fubini-Study metric.

Define the projection

$$\begin{array}{rccc} p\colon & \mathbb{C}^{m+1}\backslash\{0\} & \longrightarrow & S^{2m+1} \\ & z & \longmapsto & \dfrac{z}{|z|}. \end{array}$$

The first step to proving item (i) in Theorem 1 is the following:

PROPOSITION 1. *For any $m \geq 2n$ and a choice of sequence of approximately holomorphic sections as in (6), the composition $\Phi_k := p \circ s_k \colon M \to S^{2m+1}$ is an immersion provided that k is large enough.*

PROOF. Because s_k avoids the origin the map Φ_k is everywhere defined. The composition $\phi_k = q \circ s_k = q \circ \Phi_k$ is an immersion along ξ, so we deduce that Φ_k is an immersion along ξ.

Because ϕ_k is an estimated immersion, for any sequence of vectors and $u_k \in \xi$ with bounded norm sequence $\{|u_k|_{g_k}\}$ property (4) implies that we can write

$$\Phi_{k*}(u_k) = v_k + C'_k iz = v_k + \frac{C'_k}{2} R_{\mathrm{std}}, \tag{7}$$

where $v_k \in \xi_{\mathrm{std}}$ and $C_k \in \mathbb{R}$ have both norm bounded by $O(1)$. Here R_{std} denotes the Reeb vector field of α_{std}.

On the other hand, because $|s_k| > \eta > 0$, property (5) implies that for all k large enough $\Phi_k = s_k/|s_k|$ satisfies

$$\Phi_{k*}(R_k) = C''_k k^{1/2} R_{\mathrm{std}} + O(1), \tag{8}$$

where $C''_k > 0$ is bounded by below by a positive constant. It follows from (7) and (8) that $\Phi_{k*}(R_k)$ cannot be in $\Phi_{k*}(\xi)$ for all sufficiently large k. □

Next let $m \geq 2n+1$. We can assume without loss of generality that the map Φ_k into the sphere is an embedding. The reason is that the latter property can be achieved by a perturbation of s_k of size $O(k^{-1/2})$ in the C^2-norm with respect to the metric g_k. The resulting section s'_k is still approximately holomorphic and because uniform transversality along ξ of $(\partial_{\nabla_{k|\xi}} s_k, s_k)$ to Σ is a C^2-open condition with respect to the metric g_k, the projectivization of s'_k is everywhere defined and it is an approximately holomorphic uniform immersion along ξ.

Now we are ready to prove item (i) in Theorem 1. According to Gray stability it is enough to prove that the 1-parameter family

$$\alpha_k^t = tk\alpha + (1-t)\Phi_k^* \alpha_{\mathrm{std}},\ t \in [0,1],$$

consists of contact forms. From property (5) it follows that $\xi^t = \ker \alpha_k^t$ is a hyperplane distribution transverse to the Reeb vector field R, provided that k is large enough. Let $p_k^t \colon \xi_k^t \to \xi$ denote the projection along R and $J_k^t = (p_k^t)^* J$ the pull-back almost complex structure on ξ_k^t. To show that α_k is a contact form, it is enough to show that

$$d\alpha_k^t(u_k^t, J_k^t u_k^t) \geq \eta > 0$$

holds for any $u_k^t \in \xi_k^t$ with $|u_k^t|_{g_k} = 1$, where η is independent of k.

From properties (7) and (8) any bounded sequence of vectors $v_k \in \Phi_{k*}(TM) \cap \xi_{\mathrm{std}}$ can be written as

$$v_k = \Phi_{k*}(u_k) + C'''_k k^{-1/2} \Phi_{k*}(R_k),$$

where $u_k \in \xi$ has bounded norm-sequence $\{|u_k|_{g_k}\}$ and $C'''_k \in \mathbb{R}$ is bounded. Then we can see that

$$|u_k^t - p_k^t(u_k^t)|_{g_k} = O(k^{-1/2}),\ \text{and}\ |J_k^t u_k^t - p_k^t(J_k^t u_k^t)|_{g_k} = O(k^{-1/2}). \tag{9}$$

This implies $k\alpha(u_k^t, J_k^t u_k^t) > 1/2$ and therefore it suffices to show that for all k large enough

$$d\Phi_k^*\alpha_{\mathrm{std}}(u_k^t, J_k^t u_k^t) \geq \eta, \tag{10}$$

with $\eta > 0$ independent of k. Equation (10) follows from (9) and from the fact that ϕ_k is an approximately holomorphic immersion along ξ in a uniform estimated sense. This ends the proof of item (i) in Theorem 1.

Let $z_0, \ldots, z_m$ be coordinates on $\mathbb{C}^{m+1}$, $m \geq 2n+1$ and let $\mathcal{B}^{H_0}$ be the open book decomposition of S^{2m+1} associated to the complex linear hyperplane $H_0 \subset \mathbb{C}^{m+1}$ as defined in the introduction.

Let $\Phi_k = p \circ s_k \colon M \to S^{2m+1}$ be an embedding as in the proof of item (i) and let $\psi_{k,t}$, $t \in [0,1]$, be the isotopy of M given by Gray's stability theorem, so that $\psi_{k,t}^*\xi_k^t = \xi$ and therefore $\varphi_k = \Phi_k \circ \psi_{k,1} \colon (M, \xi) \to (S^{2m+1}, \xi_{\mathrm{std}})$ is a contact embedding for all k sufficiently large. The map $\pi_0 \colon S^{2m+1} \to \mathbb{C}$ given by $(z_0, \ldots, z_m) \mapsto z_0$ is transverse to $\mathcal{B}_{\mathrm{std}}$ and pulls it back to $\mathcal{B}^{H_0}$. Therefore Φ_k is transverse to $\mathcal{B}^{H_0}$ if and only if $s_{k,0}$ is transverse to $\mathcal{B}_{\mathrm{std}}$. According to [12] after perhaps a perturbation of s_k of size $O(k^{-1/2})$ in the C^2-norm with respect to the metric g_k, we can assume that $s_{k,0} \in \Gamma(\underline{\mathbb{C}})$ is uniformly transverse along ξ to $\mathbf{0}$. Therefore $\mathcal{B}_k := \Phi_k^*\mathcal{B}^{H_0} = s_{k,0}^*\mathcal{B}_{\mathrm{std}}$ defines a sequence of open book decompositions of M for all k sufficiently large.

Next we will show that $\mathcal{B}_k$ carries the 1-parameter family of contact structures ξ_k^t introduced in the proof of of item (i), by following Giroux' procedure.

Let $x \in M$ such that $|s_{k,0}(x)| > \eta > 0$. Because as we noted in the proof of of item (i) the Reeb vector field R is uniformly transverse to ξ_k^t (meaning that for k sufficiently large the angle between both distributions is bounded by below independently of k), from property (5) we see that the tangent space of the page through x is symplectic with respect to $d\alpha_k^t$ for all $t \in [0,1]$.

Let $y \in B(0,\eta)$. We claim that $s_{k,0}^{-1}(y)$ is a contact submanifold of (M, ξ_k^t) for all $t \in [0,1]$. To see this notice that we know that the restriction of $k\alpha$ to $s_{k,0}^{-1}(y)$ is a contact form. Moreover its Reeb vector field R_k' is of the form $R_k + O(1)$ [12]. Therefore by properties (4) and (5) we have

$$\mathrm{d}s_{k,j}(R_k') = k^{1/2} i s_{k,j} + O(1),\ j = 1, \ldots, m. \tag{11}$$

Thus by arguing as in the proof of item (i) but using property (11) in place of property (5), we conclude that the restriction of α_k^t to $s_{k,0}^{-1}(y)$ is a contact form for all $t \in [0,1]$ and for all k sufficiently large.

Giroux' perturbation needs a choice of $\eta > 0$ small enough (and independent of k) so that both the contact structure $\xi(y)$ of $s_{k,0}^{-1}(y)$ and the plane distribution defined by the symplectic orthogonal to $\xi(y)$ in ξ, have a small enough variation near $s_{k,0}^{-1}(\mathbf{0})$. It is possible to choose $\eta > 0$ such that the two aforementioned conditions hold for all $t \in [0,1]$. Hence we can produce a smooth 1-parameter family of contact forms $\tilde{\alpha}_k^t$ defining ξ_k^t, and so that $\mathcal{B}_k$ is adapted to $\tilde{\alpha}_k^t$, $t \in [0,1]$.

Therefore for all k sufficiently large $\psi_{k,t}^* \mathcal{B}_k$ is a 1-parameter family of open book decompositions adapted to the forms $\psi_{k,t}^* \tilde{\alpha}_k^t$, which are all contact forms defining ξ. By construction $\psi_{k,1}^* \mathcal{B}_k = \varphi_k^* \mathcal{B}^{H_0}$ and $\mathcal{B}_k$ is a sequence of approximately holomorphic open book decompositions. This finishes the proof of item (ii) in Theorem 1.

In order to prove item (iii) let us assume that we have been given a sequence of approximately holomorphic open book decompositions $\mathcal{B}_k$ adapted to the contact forms $\widetilde{k\alpha}$, and let τ_k be the associated sequence of approximately holomorphic sections of $\underline{\mathbb{C}}$.

Consider s_k a sequence of approximately holomorphic sections of $\underline{\mathbb{C}}^{m+1}$ such that $s_{k,0} = \tau_k$. Because τ_k is already uniformly transverse along ξ to $\mathbf{0}$, the perturbation at the beginning of the proof of item (i) which gives transversality along ξ of $(\partial_{\nabla_{k|\xi}} s_k, s_k)$ to an appropriate stratified submanifold Σ, can be assumed not to alter the first component. It is also easy to see that $\Phi_k \colon M \to S^{2m+1}$ can be arranged to be an embedding without altering the first component. Therefore $\Phi_k^* \mathcal{B}^{H_0}$ is the given approximately holomorphic open book decomposition which is already adapted to $\widetilde{k\alpha}$. It can be checked that the 1-parameter family of contact forms $\widetilde{\alpha}_k^t$ constructed above by suitably deforming α_k^t near the binding can be chosen so that $\tilde{\alpha}_k^0 = \widetilde{k\alpha}$. Hence by composing with the isotopy given by Gray's stability we obtain $\mathcal{B}_{k,t}$ a 1-parameter family of open book decompositions carrying ξ, and so that

$$\mathcal{B}_{k,0} = \mathcal{B}_k,\ \mathcal{B}_{k,1} = \varphi_k^* \mathcal{B}^{H_0},$$

and this finishes the proof of Theorem 1.

Remark 1. Let $\mathcal{B}$ be a spinning approximately holomorphic open book decomposition of M^{2n+1} adapted to a contact form $\tilde{\alpha}$ which defines ξ. By definition $\mathcal{B}$ is isotopic to an approximately holomorphic open book decomposition. Therefore according to [9] a page F of $\mathcal{B}$ has the homotopy type of a CW-complex of dimension n. The open symplectic manifold $(F, d\tilde{\alpha})$ is convex at infinity, and what is relevant is its so called completion [8]. If we define $F' = \{x \in F \,|\, |z_0(\Phi(x))| > \eta\}$, then the completions of $(F, d\tilde{\alpha})$ and $(F', \Phi^* d\alpha_{\mathrm{std}})$ are symplectomorphic. The spinning approximately holomorphic open book decomposition immediately provides a symplectic embedding of $(F', \Phi^* d\alpha_{\mathrm{std}})$ into $(\mathbb{C}^{2n+1}, \sum_{j=1}^{2n+1} dz_j \wedge d\bar{z}_j)$: The composition

$$q \circ \Phi \colon (F', \Phi^* d\alpha_{\mathrm{std}}) \to (\mathbb{CP}^{2n+1} \backslash \mathbb{CP}^{2n}, \pi\omega_{FS})$$

is a symplectic embedding, and the latter space is symplectomorphic to

$$B(0,1) \subset (\mathbb{C}^{2n+1}, \sum_{j=1}^{2n+1} dz_j \wedge d\bar{z}_j).$$

Remark 2. Mori's 3-dimensional contact embeddings are constructed starting with an approximately holomorphic open book decomposition carrying the contact structure. This gives a complex coordinate, which is approximately holomorphic. The remaining three complex coordinates are not approximately holomorphic; they are obtained using a symplectic embedding of the page into the unit ball of $\mathbb{R}^4$. The contact embedding φ is such that $\varphi^* \mathcal{B}^{H_0}$ is a spinning open book decomposition. Strictly speaking it is not a spinning approximately holomorphic open book decomposition, but it is isotopic to an approximately holomorphic open book decomposition (the one used as starting point in the construction).

Mori proposes a similar approach to construct contact embeddings and spinning open book decompositions in any dimension. The delicate part is having an embedding of the page $(F, d\alpha)$ in the unit ball of $(\mathbb{C}^{2m+1}, \sum_{j=1}^{2m+1} dz_j \wedge d\bar{z}_j)$ as exact symplectic manifolds convex at infinity (this is exactly the result obtained in remark 1). In general such an embedding exists when the completion of the page is isotopic to a complete Stein structure.

As recalled in remark 1 the page F of an approximately holomorphic open book decomposition has the homotopy type of a CW-complex of dimension n and carries an almost complex structure. This implies that F can be given complete Stein structures in the same homotopy class of almost complex structures [7], but it is not clear that for any of the Stein structures its associated symplectic structure convex at infinity matches the given one. In this respect Maydanskiy and Seidel [14] give examples of complete exact symplectic structures which are not isotopic to the symplectic structure of a (standard) Stein structure in the same homotopy class of almost complex structures. A sufficient condition for the existence of a complete Stein structure such that its unique associated symplectic structure matches the symplectic structure of the completion of F, is that the completion of the page has a Weinstein structure [7, 8], meaning that the Liouville vector field defined by $i_Z d\alpha = \alpha$ is complete and it is gradient-like for an exhausting Morse function.

It is reasonable to expect pages of an approximately holomorphic open book decomposition to be Weinstein manifolds. Giroux has proposed a proof of this fact in [10], but unfortunately it seems to be not valid. The reason is that to show the existence of a Lyapunov function for the Liouville vector field, [10] appeals to approximately holomorphic geometry to produce a perturbation of the sequence $s_k\colon M \to \mathbb{C}$ which defines $\mathcal{B}_k$, so that $(\partial_{\nabla_{k|\xi}} s_k, s_k)$ becomes uniformly transverse along ξ to the submanifold $\Sigma_0 \subset \xi^{*1,0} \oplus \underline{\mathbb{C}}$ which is the pullback of the zero section of $\xi^{*1,0}$ by the first projection. It is not clear to this author that such a perturbation can be constructed with the current results in approximately holomorphic geometry. This theory can provide perturbations to attain estimated transversality along ξ of $(\partial_{\nabla_{k|\xi}} s_k, s_k)$ to suitable (stratified) submanifolds of $(\xi^{*1,0} \oplus \underline{\mathbb{C}}) \otimes \underline{\mathbb{C}}^{m+1}$. Such submanifolds need to be nearly J_{F_∇}-holomorphic, where J_{F_∇} is an almost complex structure on $(\xi^{*1,0} \oplus \underline{\mathbb{C}}) \otimes \underline{\mathbb{C}}^{m+1}$ (actually on the hyperplane distribution which is the pullback of ξ). This is not the almost complex structure J_0 constructed out of the almost complex structures on base and fiber and the connection. It is a modification of J_0 with the property that if $s_k \in \Gamma(\underline{\mathbb{C}}^{m+1})$ is approximately holomorphic, then $(\partial_{\nabla_{k|\xi}} s_k, s_k)$ is approximately holomorphic with respect to J_{F_∇}. The stratified submanifolds which control the genericity properties of the projectivization of s_k, such as the stratified submanifold Σ introduced at the beginning of the proof of Theorem 1, are nearly J_{F_∇}-holomorphic. The submanifold Σ_0, which is obviously J_0-holomorphic, is not nearly J_{F_∇}-holomorphic. The interested reader may find a detailed discussion of this matter centered about the submanifold Σ_0 in [13], Example 6.1.

References

[1] Auroux, D., Asymptotically holomorphic families of symplectic submanifolds, Geom. Funct. Anal. **7** (1997), 971–995.

[2] Auroux, D., Estimated transversality in symplectic geometry and projective maps. Symplectic geometry and mirror symmetry (Seoul, 2000), 1–30, World Sci. Publishing, River Edge, NJ, 2001.

[3] Boyer, C. P. and K. Galicki, Sasakian geometry. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2008.

[4] Datta, M., Homotopy classification of strict contact immersions, Ann. Global Anal. Geom. **15** (1997), no. 3, 211–219.

[5] Donaldson, S. K., Symplectic submanifolds and almost-complex geometry, J. Differential Geom. **44** (1996), 666–705.

[6] Donaldson, S. K., Lefschetz fibrations in symplectic geometry, Doc. Math. Extra Vol. ICM 98, **II** (1998), 309–314.

[7] Eliashberg, Y., Topological characterization of Stein manifolds of dimension > 2, Internat. J. Math. **1** (1990), no. 1, 29–46.

[8] Eliashberg, Y. and M. Gromov, Convex symplectic manifolds. Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), 135–162, Proc. Sympos. Pure Math., 52, Part 2, Amer. Math. Soc., Providence, RI, 1991.

[9] Giroux, E., Géométrie de contact: de la dimension trois vers les dimensions supérieures. (French) [Contact geometry: from dimension three to higher dimensions] Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 405–414, Higher Ed. Press, Beijing, 2002.

[10] Giroux, E. and J.-P. Mohsen, Contact structures and symplectic fibrations over the circle, Lecture notes.

[11] Gromov, M., Partial differential relations. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **9**. Springer-Verlag, Berlin, 1986.

[12] Ibort, A., Martínez Torres, D. and F. Presas, On the construction of contact submanifolds with prescribed topology, J. Differential Geom. **56** (2000), no. 2, 235–283.

[13] Martínez Torres, D., The geometry of 2-calibrated manifolds, Port. Math. **66** (2009), no. 4, 427–512.

[14] Maydanskiy, M. and P. Seidel, Lefschetz fibrations and exotic symplectic structures on cotangent bundles of spheres, J. Topol. **3** (2010), no. 1, 157–180

[15] Mori, A., Global models of contact forms, J. Math. Sci. Univ. Tokyo **11** (2004), no. 4, 447–454.

[16] Muñoz, V., Presas, F. and I. Sols, Asymptotically holomorphic embeddings of contact manifolds in projective spaces. Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000), 386–390, Contemp. Math., **288**, Amer. Math. Soc., Providence, RI, 2001.

[17] Muñoz, V., Presas, F. and I. Sols, Almost holomorphic embeddings in Grassmannians with applications to singular symplectic submanifolds, J. Reine Angew. Math. **547** (2002), 149–189.

[18] Ornea, L. and M. Verbitsky, An immersion theorem for Vaisman manifolds, Math. Ann. **332** (2005), no. 1, 121–143.

[19] Ornea, L. and M. Verbitsky, Embeddings of compact Sasakian manifolds, Math. Res. Lett. **14** (2007), no. 4, 703–710.

(Received September 8, 2010)
(Revised August 22, 2011)

Centro de Análise Matemática
Geometria e Sistemas Dinâmicos
Departamento de Matemática
Instituto Superior Técnico
Av. Rovisco Pais
1049-001 Lisboa, Portugal
E-mail: martinez@math.ist.utl.pt