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Gevrey Regularity in Time of Solutions to Nonlinear Partial Differential Equations

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Abstract. The paper considers nonlinear partial differential equations

$$t^{\gamma}(\partial/\partial t)^m u = G\left(t, x, \{(\partial/\partial t)^j(\partial/\partial x)^{\alpha}u\}_{j<m,|\alpha|\le L}\right)$$

(with $\gamma \ge 0$ and $1 \le m \le L$) in Gevrey classes, and gives a sufficient condition for the following assertion to be valid: if a solution $u(t,x)$ is in C^{∞} class with respect to the time variable t and in the Gevrey class $\mathcal{E}^{\{\sigma\}}$ in the space variable x, then it is in the Gevrey class $\mathcal{E}^{\{s\}}$ also with respect to the time variable for a suitable s. The index s of the time regularity is precisely estimated by the data of the equation. In a particular case, the necessity of the condition is also discussed.

1. Introduction

We denote by t the time variable in $\mathbb{R}_t$, and by $x = (x_1, \dots, x_n)$ the space variable in $\mathbb{R}^n_x = \mathbb{R}_{x_1} \times \cdots \times \mathbb{R}_{x_n}$. We use the notations: $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{N}^* = \{1, 2, \dots\}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial_x = (\partial_{x_1}, \dots, \partial_{x_n})$ with $\partial_{x_i} = \partial/\partial x_i$ $(i = 1, \dots, n)$ and $\partial_x^{\alpha} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$.

For $\sigma \ge 1$ and an open subset V of $\mathbb{R}^n_x$ we denote by $\mathcal{E}^{\{\sigma\}}(V)$ the set of all functions $f(x) \in C^{\infty}(V)$ satisfying the following: for any compact subset K of V there are $C > 0$ and $h > 0$ such that

$$\max_{x \in K} |\partial_x^{\alpha} f(x)| \le C h^{|\alpha|} |\alpha|!^{\sigma}, \quad \forall \alpha \in \mathbb{N}^n.$$

A function in the class $\mathcal{E}^{\{\sigma\}}(V)$ is called *a function of the Gevrey class of order* σ.

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If $\sigma = 1$, the class $\mathcal{E}^{\{1\}}(V)$ is nothing but the set of all analytic functions on V and usually it is denoted by $\mathcal{A}(V)$. For convenience, we set $\mathcal{E}^{\{\infty\}}(V) = C^\infty(V)$. If $1 < \sigma_1 < \sigma_2 < \infty$ we have

$$\mathcal{A}(V) \subset \mathcal{E}^{\{\sigma_1\}}(V) \subset \mathcal{E}^{\{\sigma_2\}}(V) \subset C^\infty(V).$$

Thus, functions in the class $\mathcal{E}^{\{\sigma_1\}}(V)$ are closer to analytic functions than those in $\mathcal{E}^{\{\sigma_2\}}(V)$; in this sense, we can say that functions in $\mathcal{E}^{\{\sigma_1\}}(V)$ are more regular than those in $\mathcal{E}^{\{\sigma_2\}}(V)$.

For an interval $[0,T] = \{t \in \mathbb{R}\,;\, 0 \le t \le T\}$ we denote by $C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V))$ the set of all infinitely differentiable functions $u(t,x)$ in $t \in [0,T]$ with values in $\mathcal{E}^{\{\sigma\}}(V)$ equipped with the usual local convex topology (see Komatsu [12], [13]).

Similarly, for $s \ge 1$ and $\sigma \ge 1$ we denote by $\mathcal{E}^{\{s,\sigma\}}([0,T] \times V)$ the set of all functions $u(t,x) \in C^\infty([0,T] \times V)$ satisfying the following: for any compact subset K of V there are $C > 0$ and $h > 0$ such that

$$\max_{(t,x)\in[0,T]\times K} |\partial_t^k \partial_x^\alpha u(t,x)| \le C h^{k+|\alpha|} k!^s |\alpha|!^\sigma, \quad \forall (k,\alpha) \in \mathbb{N} \times \mathbb{N}^n.$$

Obviously, we have

$$\mathcal{E}^{\{s,\sigma\}}([0,T] \times V) \subset C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V)).$$

In the case $s = \sigma$ we write $\mathcal{E}^{\{\sigma\}}([0,T] \times V)$ instead of $\mathcal{E}^{\{\sigma,\sigma\}}([0,T] \times V)$.

In this paper, we will consider the following nonlinear partial differential equation

$$t^\gamma \partial_t^m u = G\big(t, x, \{\partial_t^j \partial_x^\alpha u\}_{j<m, |\alpha| \le L}\big) \tag{1.1}$$

where $\gamma \ge 0$ and $L \ge m \ge 1$ are integers, and $G(t, x, \{z_{j,\alpha}\}_{j<m,|\alpha|\le L})$ is a suitable function in a Gevrey class (for the precise assumptions, see section 2). And, we will consider the following problem:

Problem 1.1. Let $u(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V))$ be a solution of (1.1); can we have the result $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}([0,T] \times V)$ for a suitable $s \ge 1$? If this is true, determine the precise index s of the time regularity.

The motivation comes from the following two examples:

Example 1.2. Let us consider the initial value problem for the Korteweg-de Vries equation (briefly, KdV equation):

$$\partial_t u + \partial_x^3 u + 6u\partial_x u = 0, \quad u(0, x) = \varphi(x) \tag{1.2}$$

where $\varphi(x)$ is an analytic function on the torus $\mathbb{T}$. It is known that this problem is well-posed (Bourgain [1], Kenig-Ponce-Vega [9], Colliander-Keel-Staffilani-Takaoka-Tao [2]), and its solution $u(t, x)$ is analytic in the space variable x (Trubowitz [18], Gorsky-Himonas [5]) and belongs to $\mathcal{E}^{\{3\}}$ in the time variable t (Hannah-Himonas-Petronilho [6]). In [5], the analyticity in the space variable is stated in the form: there are $C > 0$ and $\delta > 0$ such that

$$|\partial_x^\alpha u(t, x)| \le C^{\alpha+1}\alpha! \quad \text{for } \alpha \in \mathbb{N},\ t \in (-\delta, \delta),\ x \in \mathbb{T}, \tag{1.3}$$

and in [6] the Gevrey regularity in the time variable is stated as follows:

$$\begin{aligned} &|\partial_t^k \partial_x^\alpha u(t, x)| \le C^{k+\alpha+1}(3k + \alpha)!(C^2 + C/2)^k \\ &\text{for } (k, \alpha) \in \mathbb{N} \times \mathbb{N},\ t \in (-\delta, \delta),\ x \in \mathbb{T}. \end{aligned} \tag{1.4}$$

In the paper [6], the proof of (1.4) is done by only a calculation, using the relation (1.2), (1.3) and the induction. Note that (1.3) implies $u(t, x) \in C^\infty((-\delta, \delta), \mathcal{E}^{\{1\}}(\mathbb{T}))$ and (1.4) implies $u(t, x) \in \mathcal{E}^{\{3,1\}}((-\delta, \delta) \times \mathbb{T})$.

Example 1.3. Let $a > 0$, $k \in \mathbb{N}^*$ and let us consider the linear partial differential equation:

$$(t\partial_t + a)^2 u - t^k \partial_x^2 u = f(t, x). \tag{1.5}$$

It is known that the equation (1.4) is well-posed in $C^\infty([0, T], \mathcal{E}^{\{\sigma\}}(\mathbb{R}))$ for any $\sigma \ge 1$ (see Tahara [16]). As to the time regularity, under the assumption $f(t, x) \in \mathcal{E}^{\{\sigma\}}([0, T] \times \mathbb{R})$ we have the following result (Tahara [17]):

$$\begin{cases} u(t, x) \in \mathcal{E}^{\{\sigma\}}([0, T] \times \mathbb{R}), & \text{if } k \ge 2, \\ u(t, x) \in \mathcal{E}^{\{2\sigma-1,\sigma\}}([0, T] \times \mathbb{R}), & \text{if } k = 1. \end{cases} \tag{1.6}$$

In this case, (1.5) is also proved by a calculation, using the relation (1.4) and the fact $f(t, x) \in \mathcal{E}^{\{\sigma\}}([0, T] \times \mathbb{R})$.

Similar time regularity problem is discussed also by Kinoshita-Taglialatela [10] for second order linear hyperbolic equations.

In both examples, the regularity in the time variable t is proved independently from the well-posedness of the problem: the condition depends only on the order of ∂_t, the order of ∂_x and the degeneracy of the coefficients at $t = 0$ in the equation, and its mechanism is quite similar to the one of Maillet's type theorem in Gérard-Tahara [3].

Motivated by this consideration, the author has tried to do a systematic study of Problem 1.1 for a general nonlinear partial differential equation (1.1): we note that the KdV equation (1.2) is a particular case with $\gamma = 0$, $m = 1$ and $L = 3$, and the equation (1.5) is a particular case with $\gamma = 2$, $m = 2$ and $L = 2$.

2. Formulation and Main Theorem

Let $\gamma \in \mathbb{N}$, $m \in \mathbb{N}^*$, $L \in \mathbb{N}^*$, Λ be a subset of $\{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^n\,;\, j < m, |\alpha| \le L\}$ and $d = \#\Lambda$ (the cardinal of Λ). In this paper, we will consider the following nonlinear partial differential equation

$$t^\gamma \partial_t^m u = G\left(t, x, \{\partial_t^j \partial_x^\alpha u\}_{(j,\alpha)\in\Lambda}\right). \tag{2.1}$$

For simplicity we will write

$$Du = \{\partial_t^j \partial_x^\alpha u\}_{(j,\alpha)\in\Lambda}.$$

We denote by $z = \{z_{j,\alpha}\}_{(j,\alpha)\in\Lambda}$ the variable in $\mathbb{R}^d$ (which corresponds to $Du = \{\partial_t^j \partial_x^\alpha u\}_{(j,\alpha)\in\Lambda}$). Let Ω be an open subset of $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d$, and let $G(t,x,z)$ be a C^∞ function on Ω. Let $s_1 \ge 1$, $\sigma \ge 1$ and $s_2 \ge 1$, let $I = [0,T]$ (with $T > 0$), and let V be an open subset of $\mathbb{R}^n$. The main assumptions are as follows.

- a$_1$) $\gamma \ge 0$, $L \ge m \ge 1$, $s_1 \ge 1$ and $\sigma \ge s_2 \ge 1$.
- a$_2$) Λ is a subset of $\{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^n\,;\, j < m, |\alpha| \le L\}$.
- a$_3$) $G(t,x,z) \in \mathcal{E}^{\{s_1,\sigma,s_2\}}(\Omega)$.
- a$_4$) $u(t,x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(V))$ is a solution of (2.1) on $I \times V$; this involves the property: $(t,x) \in I \times V \Longrightarrow (t,x,Du(t,x)) \in \Omega$.

In the condition a$_2$) we denoted by $\mathcal{E}^{\{s_1,\sigma,s_2\}}(\Omega)$ the set of all functions $f(t,x,z) \in C^\infty(\Omega)$ satisfying the following: for any compact subset H of Ω

there are $C > 0$ and $h > 0$ such that

$$\max_{(t,x,z)\in H} |\partial_t^p \partial_x^\alpha \partial_z^\nu f(t,x,z)| \le C h^{p+|\alpha|+|\nu|} p!^{s_1} |\alpha|!^{\sigma} |\nu|!^{s_2} \quad \text{for } \forall (p,\alpha,\nu) \in \mathbb{N} \times \mathbb{N}^n \times \mathbb{N}^d.$$

If $\sigma = s_1 = s_2$ holds we write $\mathcal{E}^{\{\sigma\}}(\Omega)$ instead of $\mathcal{E}^{\{\sigma,\sigma,\sigma\}}(\Omega)$.

In order to state our main theorem and the precise formula of the index s of time regularity, we need to define the order of the zero of a function at $t = 0$.

DEFINITION 2.1. Let $f(t,x) \in C^\infty(I \times V)$. We define *the order of the zero of* $f(t,x)$ *on* V *at* $t = 0$ (which we denote by $ord_t(f,V)$) by

$$ord_t(f,V) = \min\{\, k \in \mathbb{N}\,;\, (\partial_t^k f)(0,x) \not\equiv 0 \text{ on } V\}$$

(if $(\partial_t^k f)(0,x) \equiv 0$ on V for all $k \in \mathbb{N}$, we set $ord_t(f,V) = \infty$).

Under the conditions $a_1) \sim a_4)$ we set

$$k_{j,\alpha} = ord_t \left(\frac{\partial G}{\partial z_{j,\alpha}}(t,x,Du(t,x)), V \right), \quad (j,\alpha) \in \Lambda,$$

that is, $k_{j,\alpha}$ denotes the order of the zero of $(\partial G/\partial z_{j,\alpha})(t,x,Du(t,x))$ on V at $t = 0$. We suppose

$$\begin{cases} k_{j,\alpha} \ge \gamma - m + j, & \text{if } (j,\alpha) \in \Lambda \text{ and } |\alpha| = 0, \\ k_{j,\alpha} \ge \gamma - m + j + 1, & \text{if } (j,\alpha) \in \Lambda \text{ and } |\alpha| > 0 \end{cases} \tag{2.2}$$

and we set

$$s_0 = 1 + \max\left[0, \max_{(j,\alpha)\in\Lambda, |\alpha|>0} \left(\frac{j + \sigma|\alpha| - m}{\min\{k_{j,\alpha} - \gamma + m - j,\ m - j\}} \right) \right]. \tag{2.3}$$

Then, we have the following result:

THEOREM 2.2 (Main Theorem). *Suppose the conditions* $a_1) \sim a_4)$ *and* (2.2)*: then, we have* $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}([0,T] \times V)$ *for any* $s \ge \max\{s_0, s_1, s_2\}$.

REMARK 2.3. (1) In the case $\gamma = 0$, we have

$$s_0 = 1 + \max\left[0, \max_{(j,\alpha)\in\Lambda, |\alpha|>0}\left(\frac{j+\sigma|\alpha|-m}{m-j}\right)\right] : \tag{2.4}$$

therefore, for KdV equation (1.1) we have $s_0 = 3$ under $\sigma = 1$. This coincides with the result in Example 1.2.

(2) In the case $\gamma = m$, we have

$$s_0 = 1 + \max\left[0, \max_{(j,\alpha)\in\Lambda, |\alpha|>0}\left(\frac{j+\sigma|\alpha|-m}{\min\{k_{j,\alpha}-j,\ m-j\}}\right)\right] : \tag{2.5}$$

therefore, for the equation (1.4) we have

$$s_0 = 1 + \frac{2\sigma - 2}{\min\{k, 2\}} = \begin{cases} \sigma, & \text{if } k \geq 2, \\ 2\sigma - 1, & \text{if } k = 1. \end{cases}$$

This coincides with the result in Example 1.3.

The paper is organized as follows. In the next section 3 we will reduce our problem to the one for nonlinear Fuchsian equation

$$(t\partial_t)^m u = F\left(t, x, \{(t\partial_t)^j \partial_x^\alpha u\}_{(j,\alpha)\in\Lambda}\right), \tag{2.6}$$

and in sections $4 \sim 7$ we will solve Problem 1.1 for (2.6). In section 8 we will discuss the necessity of the condition $s \geq \max\{s_0, s_1, s_2\}$ in a particular case. In the last section 9 we will give some applications to KdV type equations, non-singular Kowalewskian equations, and nonlinear Fuchsian equations.

3. Reduction of the Problem

Let $u(t,x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(V))$ be a solution of (2.1) given in a_4). For any $q \in \mathbb{N}^*$ with $q \geq m$ we have the decomposition

$$u(t,x) = \varphi(t,x) + t^q w(t,x) \quad \text{with } \varphi(t,x) = \sum_{i=0}^{q-1} \frac{(\partial_t^i u)(0,x)}{i!} t^i;$$

we have $\varphi(t,x) \in \mathcal{E}^{\{1,\sigma\}}(\mathbb{R}_t \times V)$ and $w(t,x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(V))$. Since

$$\begin{aligned} \partial_t^m u &= \partial_t^m \varphi + t^{q-m}[t\partial_t + q]_m w, \\ Du &= D\varphi + \{t^{q-j}[t\partial_t + q]_j \partial_x^\alpha w\}_{(j,\alpha)\in\Lambda} \end{aligned}$$

(where $[\lambda]_0 = 1$ and $[\lambda]_p = \lambda(\lambda-1)\cdots(\lambda-p+1)$ for $p \geq 1$), and since $u(t,x)$ is a solution of (2.1), we have

$$t^\gamma \left(\partial_t^m \varphi + t^{q-m}[t\partial_t + q]_m w\right) \\ = G\left(t, x, D\varphi + \{t^{q-j}[t\partial_t+q]_j \partial_x^\alpha w\}_{(j,\alpha)\in\Lambda}\right). \tag{3.1}$$

Thus, by setting

$$H(t,x,X) = \frac{1}{t^{\gamma+q-m}}\left[G\left(t,x,D\varphi + \{t^{q-j}X_{j,\alpha}\}_{(j,\alpha)\in\Lambda}\right) - t^\gamma \partial_t^m \varphi\right] \tag{3.2}$$

with $X = \{X_{j,\alpha}\}_{(j,\alpha)\in\Lambda}$ we have the equation

$$[t\partial_t + q]_m w = H\left(t, x, \{[t\partial_t+q]_j \partial_x^\alpha w\}_{(j,\alpha)\in\Lambda}\right) \tag{3.3}$$

which is regarded as an equation with respect to $w(t,x)$. To prove the result $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}(I\times V)$, it is enough to show the condition: $w(t,x) \in \mathcal{E}^{\{s,\sigma\}}(I\times V)$. For simplicity we write

$$\mathfrak{D}_q w = \{[t\partial_t+q]_j \partial_x^\alpha w\}_{(j,\alpha)\in\Lambda}.$$

3.1. Properties of $H(t,x,X)$

In order to discuss the equation (3.3) we need informations on the right-hand side of the equation (3.3).

Proposition 3.1. *If $q \geq \gamma + m$ holds, we have the following results.*

(1) *Set $\Omega_1 = \{(t,x,X) \in \mathbb{R}_t \times V \times \mathbb{R}_X^d\,;\,(t,x,D\varphi(t,x) + \{t^{q-j}X_{j,\alpha}\}_{(j,\alpha)\in\Lambda}) \in \Omega\}$: then Ω_1 is an open subset of $\mathbb{R}_t \times V \times \mathbb{R}_X^d$, and we have $H(t,x,X) \in C^\infty(\Omega_1)$. As to the shape of Ω_1 we have the following: for any compact subset K of V and any $a > 0$ there are $\delta > 0$ such that $[-\delta,\delta]\times K \times \{X\,;\,|X| \leq a\} \subset \Omega_1$.*

(2) *$H(t,x,X) \in \mathcal{E}^{\{s^*,\sigma,s^*\}}(\Omega_1)$ with $s^* = \max\{s_1, s_2\}$.*

(3) *$w(t,x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(V))$ is a solution of (3.3) on $I\times V$; this involves the property: $(t,x) \in I\times V \Longrightarrow (t,x,\mathfrak{D}_q w(t,x)) \in \Omega_1$.*

(4) *For any $(j,\alpha) \in \Lambda$ we have*

$$ord_t\Big(\frac{\partial H}{\partial X_{j,\alpha}}(t,x,\mathfrak{D}_q w(t,x)), V\Big) = k_{j,\alpha} - \gamma + m - j, \tag{3.4}$$

and for any $\nu \in \mathbb{N}^d$ with $|\nu| \geq 2$ we have

$$ord_t\Big(\frac{\partial^{|\nu|}H}{\partial X^\nu}(t,x,\mathfrak{D}_q w(t,x)),V\Big) \geq (q-m+1)(|\nu|-1)-\gamma+1. \tag{3.5}$$

Before the proof of this proposition, let us present some preparatory lemmas. First we note:

LEMMA 3.2. *Set $W = \{(t,x,Y) \in \mathbb{R}_t \times V \times \mathbb{R}_Y^d\,;\,(t,x,D\varphi(t,x)+Y) \in \Omega\}$: then W is an open subset of $\mathbb{R}_t \times V \times \mathbb{R}_Y^d$, and we have $G(t,x,D\varphi(t,x)+Y) \in C^\infty(W)$. As to the shape of W we have the following: for any compact subset K of V there are $\delta > 0$ and $\epsilon > 0$ such that $[-\delta,\delta] \times K \times \{Y\,;\,|Y| \leq \epsilon\} \subset W$.*

PROOF. The former half is clear. Let us show the latter half. Set $f : \mathbb{R}_t \times V \times \mathbb{R}_Y^d \ni (t,x,Y) \longrightarrow (t,x,D\varphi(t,x)+Y) \in \mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}^d$; then we have $W = f^{-1}(\Omega)$ and so W is an open subset of $\mathbb{R}_t \times V \times \mathbb{R}_Y^d$. Take any compact subset K of V. Then, for any $x \in K$ we have $f(0,x,0) = (0,x,D\varphi(0,x)) = (0,x,(Du)(0,x)) \in \Omega$ (by the assumption a_4)); this shows that $\{0\} \times K \times \{0\} \subset W$. Since W is open and $\{0\} \times K \times \{0\}$ is compact, we can take $\delta > 0$ and $\epsilon > 0$ such that $[-\delta,\delta] \times K \times \{Y\,;\,|Y| \leq \epsilon\} \subset W$. This proves the latter half. □

By Lemma 3.2, for any compact subset K of V there are $\delta > 0$ and $\epsilon > 0$ such that $[-\delta,\delta] \times K \times \{Y\,;\,|Y| \leq \epsilon\} \subset W$. Therefore, we see that $G(t,x,D\varphi(t,x))$ is well defined on $W_0 = W \cap \{Y = 0\}$, and the following Taylor expansion of $G(t,x,D\varphi(t,x)+Y)$ in Y makes sense:

$$\begin{aligned} &G(t,x,D\varphi(t,x)+Y) \\ &\quad = G(t,x,D\varphi(t,x)) + \sum_{(j,\alpha)\in\Lambda} \frac{\partial G}{\partial z_{j,\alpha}}(t,x,D\varphi(t,x))Y_{j,\alpha} \\ &\quad + \sum_{(j,\alpha),(i,\beta)\in\Lambda} R_{(j,\alpha),(i,\beta)}(t,x,Y)Y_{j,\alpha}Y_{i,\beta} \end{aligned} \tag{3.6}$$

where

$$R_{(j,\alpha),(i,\beta)}(t,x,Y) = \int_0^1 (1-\theta)\frac{\partial^2 G}{\partial z_{j,\alpha}\partial z_{i,\beta}}\big(t,x,D\varphi+\theta Y\big)d\theta.$$

Similarly, if we replace G by $(\partial G/\partial z_{j,\alpha})$ in the above argument we have the expression

$$\begin{aligned}(3.7)\qquad &\frac{\partial G}{\partial z_{j,\alpha}}(t,x,D\varphi(t,x)+Y)\\ &=\frac{\partial G}{\partial z_{j,\alpha}}(t,x,D\varphi(t,x))+\sum_{(i,\beta)\in\Lambda}R^*_{(j,\alpha),(i,\beta)}(t,x,Y)Y_{i,\beta}\end{aligned}$$

where

$$R^*_{(j,\alpha),(i,\beta)}(t,x,Y)=\int_0^1\frac{\partial^2 G}{\partial z_{j,\alpha}\partial z_{i,\beta}}\big(t,x,D\varphi+\theta Y\big)d\theta.$$

By using these formulas we have

LEMMA 3.3. *If $q\ge\gamma+m$ holds, we have the following results.*
(1) *We set*

$$(3.8)\qquad k_{j,\alpha}(\varphi)=ord_t\Big(\frac{\partial G}{\partial z_{j,\alpha}}(t,x,D\varphi(t,x)),V\Big),\quad (j,\alpha)\in\Lambda.$$

Then, if $k_{j,\alpha}<q-m+1$ we have $k_{j,\alpha}(\varphi)=k_{j,\alpha}$, and if $k_{j,\alpha}\ge q-m+1$ we have $k_{j,\alpha}(\varphi)\ge q-m+1$.

(2) *We have $k_{j,\alpha}(\varphi)\ge\gamma-m+j$ if $|\alpha|=0$, and $k_{j,\alpha}(\varphi)\ge\gamma-m+j+1$ if $|\alpha|>0$.*

(3) *Moreover, $G(t,x,D\varphi(t,x))-t^\gamma\partial_t^m\varphi(t,x)=O(t^{\gamma+q-m})$ uniformly on any compact subset of V (as $t\longrightarrow 0$).*

PROOF. Suppose the condition $q\ge\gamma+m$: we note that $j\le m-1$ holds for any $(j,\alpha)\in\Lambda$. Take any compact subset K of V; then we have $[-\delta,\delta]\times K\times\{Y\,;\,|Y|\le\epsilon\}\subset W$ for some $\delta>0$ and $\epsilon>0$. If $\delta>0$ is small enough, we have $|t^{q-j}[t\partial_t+q]_j\partial_x^\alpha w|\le\varepsilon$ on $[0,\delta]\times K$ (for $(j,\alpha)\in\Lambda$), and so by substituting $Y=\{t^{q-j}[t\partial_t+q]_j\partial_x^\alpha w\}_{(j,\alpha)\in\Lambda}$ into (3.7) we have

$$\frac{\partial G}{\partial z_{j,\alpha}}(t,x,Du(t,x))=\frac{\partial G}{\partial z_{j,\alpha}}(t,x,D\varphi(t,x))+O(t^{q-m+1}).$$

This yields the results in the part (1).

If $k_{j,\alpha}<q-m+1$, by (1) we have $k_{j,\alpha}(\varphi)=k_{j,\alpha}$ and so the result (2) is clear from (2.2). If $k_{j,\alpha}\ge q-m+1$, by (1) and the condition $q\ge\gamma+m$ we have $k_{j,\alpha}(\varphi)\ge q-m+1\ge\gamma+1$ and so the result (2) is also clear.

Let us show (3). By bubstituting $Y = \{t^{q-j}[t\partial_t + q]_j \partial_x^\alpha w\}_{(j,\alpha)\in\Lambda}$ into (3.6) and by using the equality (3.1) we have

$$
\begin{aligned}
(3.9) \quad & t^\gamma \partial_t^m \varphi + t^{\gamma+q-m}[t\partial_t + q]_m w \\
&= G\left(t, x, D\varphi + \{t^{q-j}[t\partial_t + q]_j \partial_x^\alpha w\}_{(j,\alpha)\in\Lambda}\right) \\
&= G(t, x, D\varphi(t,x)) + \sum_{(j,\alpha)\in\Lambda} O(t^{k_{j,\alpha}(\varphi)})O(t^{q-j}) + O(t^{2q-2(m-1)}).
\end{aligned}
$$

Since $k_{j,\alpha}(\varphi) \geq \gamma - m + j$ (for $(j,\alpha) \in \Lambda$) are known by (2), we have $k_{j,\alpha}(\varphi) + q - j \geq q + \gamma - m$; in addition, by the condition $q \geq \gamma + m$ we have $2q - 2(m-1) \geq q + \gamma - m + 2 > q + \gamma - m$. Thus, by aplying these results to (3.9) we have the conclusion: $G(t, x, D\varphi(t,x)) - t^\gamma \partial_t^m \varphi(t,x) = O(t^{\gamma+q-m})$. □

Now, let us give a proof of Proposition 3.1.

Proof of Proposition 3.1. By the same argument as in Lemma 3.2 we can show the conditions on Ω_1 and the fact: $G\left(t, x, D\varphi + \{t^{q-j}X_{j,\alpha}\}_{(j,\alpha)\in\Lambda}\right) \in C^\infty(\Omega_1)$. By substituting $Y = \{t^{q-j}X_{j,\alpha}\}_{(j,\alpha)\in\Lambda}$ into (3.6) and by using Lemma 3.3 we have

$$
\begin{aligned}
(3.10) \quad & G\left(t, x, D\varphi + \{t^{q-j}X_{j,\alpha}\}_{(j,\alpha)\in\Lambda}\right) - t^\gamma \partial_t^m \varphi(t,x) \\
&= (G(t, x, D\varphi) - t^\gamma \partial_t^m \varphi(t,x)) \\
&\quad + \sum_{(j,\alpha)\in\Lambda} O(t^{k_{j,\alpha}(\varphi)})O(t^{q-j}) + O(t^{2q-2(m-1)}) = O(t^{\gamma+q-m})
\end{aligned}
$$

(as $t \longrightarrow 0$) uniformly on $K \times \{X\,;\,|X| \leq a\}$ for any compact subset K of V and any $a > 0$. Hence, by the definition (3.2) we have the result: $H(t, x, X) \in C^\infty(\Omega_1)$. This proves (1).

Since $G(t, x, z) \in \mathcal{E}^{\{s_1,\sigma,s_2\}}(\Omega)$ is assumed, by Proposition 4.2 (in section 4.1) we have $G\left(t, x, D\varphi + \{t^{q-j}X_{j,\alpha}\}_{(j,\alpha)\in\Lambda}\right) \in \mathcal{E}^{\{s^*,\sigma,s^*\}}(\Omega_1)$ for $s^* = \max\{s_1, s_2\}$. Since $t^\gamma \partial_t^m \varphi(t,x) \in \mathcal{E}^{\{1,\sigma,1\}}(\Omega_1)$ is clear, by (3.10) we have the conclusion of (2).

For any $(t,x) \in I \times V$ we have $(t, x, D\varphi + \{t^{q-j}[t\partial_t + q]_j \partial_x^\alpha w\}_{(j,\alpha)\in\Lambda}) = (t, x, Du(t,x)) \in \Omega$ and so we have $(t, x, \{[t\partial_t + q]_j \partial_x^\alpha w\}_{(j,\alpha)\in\Lambda}) \in \Omega_1$ which is equivalent to $(t, x, \mathfrak{D}_q w(t,x)) \in \Omega_1$. This proves (3).

Since

$$
\frac{\partial H}{\partial X_{j,\alpha}}(t, x, \mathfrak{D}_q w(t,x)) = \frac{t^{q-j}}{t^{\gamma+q-m}} \frac{\partial G}{\partial z_{j,\alpha}}(t, x, Du(t,x))
$$

holds, (3.4) is clear. If we write $\mu_t(\nu) = \sum_{(j,\alpha)\in\Lambda} j\nu_{j,\alpha}$ for $\nu = \{\nu_{j,\alpha}\}_{(j,\alpha)\in\Lambda} \in \mathbb{N}^d$, we have

$$ord_t\Big(\frac{\partial^{|\nu|}H}{\partial X^\nu}(t,x,\mathscr{D}_q w(t,x)),V\Big) \geq q|\nu| - \mu_t(\nu) - \gamma - q + m.$$

Since $\mu_t(\nu) \leq (m-1)|\nu|$, we have the result (3.5). □

3.2. Further reduction

Let us do a further reduction of the equation (3.3). By the formula

$$[\lambda+q]_j = (\lambda+q)(\lambda+q-1)\cdots(\lambda+q-j+1) = \sum_{0\leq i\leq j} C^{(q)}_{j,i}\lambda^i$$

we define the constants $C^{(q)}_{j,i}$ $(0 \leq i \leq j < m)$: we see that $C^{(q)}_{j,j} = 1$ holds. Set

$$\Lambda_0 = \bigcup_{(j,\alpha)\in\Lambda} \{(i,\alpha) \in \mathbb{N}\times\mathbb{N}^n\,;\, 0 \leq i \leq j\},$$

$$Z = \{Z_{i,\alpha}\}_{(i,\alpha)\in\Lambda_0} \in \mathbb{R}^{d_0}_Z \quad \text{with } d_0 = \#\Lambda_0 \text{ (the cardinal of } \Lambda_0).$$

It is clear that $\Lambda \subset \Lambda_0$ holds. We define a linear change of variables: $\mathbb{R}^{d_0} \ni Z \longrightarrow X(Z) \in \mathbb{R}^d$ by

$$X(Z) = \{X_{j,\alpha}(Z)\}_{(j,\alpha)\in\Lambda} \quad \text{with } X_{j,\alpha}(Z) = \sum_{0\leq i\leq j} C^{(q)}_{j,i} Z_{i,\alpha}.$$

By setting

$$F(t,x,Z) = H(t,x,X(Z)) - \sum_{0\leq i<m} C^{(q)}_{m,i} Z_{i,0} \tag{3.11}$$

the equation (3.3) is rewritten into the form

$$(t\partial_t)^m w = F\left(t,x,\{(t\partial_t)^i\partial_x^\alpha w\}_{(i,\alpha)\in\Lambda_0}\right). \tag{3.12}$$

For simplicity we write

$$\Theta w = \{(t\partial_t)^i\partial_x^\alpha w\}_{(i,\alpha)\in\Lambda_0}.$$

As to the right-hand side of (3.12), by Proposition 3.1 we have

Proposition 3.4. *If $q \ge \gamma + m$ holds, we have the following results.*

(1) *Set* $\Omega_0 = \{(t,x,Z) \in \mathbb{R}_t \times V \times \mathbb{R}^{d_0}\,;\,(t,x,X(Z)) \in \Omega_1\}$*: then* Ω_0 *is an open subset of* $\mathbb{R}_t \times V \times \mathbb{R}^{d_0}$*, and we have* $F(t,x,Z) \in \mathcal{E}^{\{s^*,\sigma,s^*\}}(\Omega_0)$ *with* $s^* = \max\{s_1, s_2\}$*.*

(2) $w(t,x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(V))$ *is a solution of* (3.12) *on* $I \times V$*; this involves the property:* $(t,x) \in I \times V \Longrightarrow (t,x,\Theta w(t,x)) \in \Omega_0$*.*

(3) *We set*

$$q_{i,\alpha} = ord_t\Big(\frac{\partial F}{\partial Z_{i,\alpha}}(t,x,\Theta w(t,x)), V\Big), \quad (i,\alpha) \in \Lambda_0 : \tag{3.13}$$

if $|\alpha| > 0$ *we have* $q_{i,\alpha} \ge 1$*. Moreover, for any* $\nu \in \mathbb{N}^d$ *with* $|\nu| \ge 2$ *we have*

$$ord_t\Big(\frac{\partial^{|\nu|} F}{\partial Z^\nu}(t,x,\Theta w(t,x)), V\Big) \ge (q-m+1)(|\nu|-1) - \gamma + 1. \tag{3.14}$$

We set

$$s_0(F) = 1 + \max\left[0, \max_{(i,\alpha)\in\Lambda_0, |\alpha|>0}\left(\frac{i+\sigma|\alpha|-m}{\min\{q_{i,\alpha},\, m-i\}}\right)\right]. \tag{3.15}$$

Lemma 3.5. *Let* s_0 *be the one in* (2.3)*, and let* $s_0(F)$ *be the one above; then we have* $s_0(F) = s_0$*.*

Proof. For simplicity we write

$$p_{j,\alpha} = ord_t\Big(\frac{\partial H}{\partial X_{j,\alpha}}(t,x,\mathfrak{D}_q w(t,x)), V\Big), \quad (j,\alpha) \in \Lambda :$$

then, by (3.4) we have $p_{j,\alpha} = k_{j,\alpha} - \gamma + m - j$ and s_0 is given by

$$s_0 = 1 + \max\left[0, \max_{(j,\alpha)\in\Lambda, |\alpha|>0}\left(\frac{j+\sigma|\alpha|-m}{\min\{p_{j,\alpha},\, m-j\}}\right)\right]. \tag{3.16}$$

For any $(i,\alpha) \in \Lambda_0$, by (3.11) we have

$$\begin{aligned}\frac{\partial F}{\partial Z_{i,\alpha}}(t,x,\Theta w) &= \sum_{(j,\alpha)\in\Lambda, j\ge i} \frac{\partial H}{\partial X_{j,\alpha}}(t,x,\mathfrak{D}_q w) C^{(q)}_{j,i} = \sum_{(j,\alpha)\in\Lambda, j\ge i} O(t^{p_{j,\alpha}}) \\ &= O(t^{r_{i,\alpha}}) \quad \text{for } r_{i,\alpha} = \min\{p_{j,\alpha}\,;\,(j,\alpha)\in\Lambda, j \ge i\} :\end{aligned}$$

this implies that $q_{i,\alpha} \geq r_{i,\alpha}$ holds.

Let us show the inequality $s_0(F) \leq s_0$. To do so, it is enough to prove the following: for any $(i, \alpha) \in \Lambda_0$ with $|\alpha| > 0$ we have the inequality

$$\frac{i + \sigma|\alpha| - m}{\min\{q_{i,\alpha}, m - i\}} \leq s_0 - 1. \tag{3.17}$$

If $i + \sigma|\alpha| - m \leq 0$, by the condition $s_0 \geq 1$ we have (3.17).

If $i + \sigma|\alpha| - m > 0$ and $r_{i,\alpha} \leq m - i$, we have

$$\frac{i + \sigma|\alpha| - m}{\min\{q_{i,\alpha}, m - i\}} \leq \frac{i + \sigma|\alpha| - m}{\min\{r_{i,\alpha}, m - i\}} = \frac{i + \sigma|\alpha| - m}{r_{i,\alpha}}. \tag{3.18}$$

By the definition of $r_{i,\alpha}$ we have $r_{i,\alpha} = p_{j,\alpha}$ for some $(j, \alpha) \in \Lambda$ with $j \geq i$ and so

$$\frac{i + \sigma|\alpha| - m}{r_{i,\alpha}} \leq \frac{j + \sigma|\alpha| - m}{p_{j,\alpha}} \leq \frac{j + \sigma|\alpha| - m}{\min\{p_{j,\alpha}, m - j\}}. \tag{3.19}$$

Hence, by (3.18), (3.19) and (3.16) we have the result (3.17).

If $i + \sigma|\alpha| - m > 0$ and $r_{i,\alpha} > m - i$, we have $q_{i,\alpha} > m - i$ and so

$$\frac{i + \sigma|\alpha| - m}{\min\{q_{i,\alpha}, m - i\}} = \frac{i + \sigma|\alpha| - m}{m - i}. \tag{3.20}$$

Since $(i, \alpha) \in \Lambda_0$, by the definition of Λ_0 we have a $(j, \alpha) \in \Lambda$ with $j \geq i$: then we see

$$\frac{i + \sigma|\alpha| - m}{m - i} \leq \frac{j + \sigma|\alpha| - m}{m - j} \leq \frac{j + \sigma|\alpha| - m}{\min\{p_{j,\alpha}, m - j\}}. \tag{3.21}$$

Hence, by (3.20), (3.21) and (3.16) we have the result (3.17).

Thus, we have proved the inequality $s_0(F) \leq s_0$. The converse inequlity $s_0(F) \geq s_0$ can be proved in the same way. □

Thus, to prove Theorem 2.2 it is sufficient to show the following result, using the fact that $w(t, x)$ is a solution of (3.12).

THEOREM 3.6. *Suppose the condition*

$$q \geq \max\{\gamma + m, \gamma + 2(m - 1)\}. \tag{3.22}$$

Then, under the above situation, we have $w(t,x) \in \mathcal{E}^{\{s,\sigma\}}(I \times V)$ *for any* $s \geq \max\{s_0(F), s^*\}$ *with* $s^* = \max\{s_1, s_2\}$.

By (3.14) and (3.22) we see: if $|\nu| \geq 2$ we have $(q-m+1)(|\nu|-1)-\gamma+1 \geq (q-m+1)-\gamma+1 \geq m$ and so

$$ord_t\Big(\frac{\partial^{|\nu|}F}{\partial Z^{\nu}}(t,x,\Theta w(t,x)), V\Big) \geq m, \quad \text{if } |\nu| \geq 2. \tag{3.23}$$

This fact will play an important role in the proof of Theorem 3.6.

The proof of this result (Theorem 3.6) will be given in sections $4 \sim 7$: in section 4 we will summarize basic tools and results which are needed in the proof of Theorem 3.6, in section 5 we will give a proof of Theorem 3.6 in the case $s = 1$, and in sections 6 and 7 we will give a proof of Theorem 3.6 in the case $s > 1$.

For simplicity, we will write $u(t,x)$, Λ, $z = \{z_{j,\alpha}\}_{(j,\alpha)\in\Lambda}$, d and $F(t,x,z)$ instead of $w(t,x)$, Λ_0, $Z = \{Z_{i,\alpha}\}_{(i,\alpha)\in\Lambda_0}$, d_0 and $F(t,x,Z)$; therefore, (3.12) is written as

$$(t\partial_t)^m u = F\left(t, x, \{(t\partial_t)^j \partial_x^\alpha u\}_{(j,\alpha)\in\Lambda}\right). \tag{3.24}$$

4. Basic Tools and Results

In this section we will summarize basic tools and results which are needed in the proof of Theorem 3.6. As to general properties of functions of the Gevrey class, readers can refer to Gevrey [4], Komatsu [12],[13] and Yamanaka [20].

4.1. Addition, product and composition

Let $\sigma \geq 1$, V be an open subset of $\mathbb{R}^n_x$, and W be an open subset of $\mathbb{R}^d_z$. As to addition, product and composition of two functions in the Gevrey class $\mathcal{E}^{\{\sigma\}}$ we have

Proposition 4.1. (1) *If* $f(x)$ *and* $g(x)$ *belong to the class* $\mathcal{E}^{\{\sigma\}}(V)$, *we have* $f(x)g(x) \in \mathcal{E}^{\{\sigma\}}(V)$ *and* $f(x)+g(x) \in \mathcal{E}^{\{\sigma\}}(V)$.

(2) *Let* $f(z) \in \mathcal{E}^{\{\sigma\}}(W)$ *and* $g(x) \in \mathcal{E}^{\{\sigma\}}(V)^d$. *If* $g(V) \subset W$ *holds, we have* $f(g(x)) \in \mathcal{E}^{\{\sigma\}}(V)$.

The proof of (1) is easy, and the proof of (2) is seen in Gevrey [4], Yamanaka [20], etc. The following result is also very useful.

PROPOSITION 4.2. *Let $s \geq 1$, $s_1 \geq 1$, $s_2 \geq 1$, $\sigma \geq 1$, let Ω be an open subset of $\mathbb{R}_t^N \times \mathbb{R}_x^n \times \mathbb{R}_z^d$, and let W be an open subset of $\mathbb{R}_t^N \times \mathbb{R}_x^n$. If the conditions*

1) $F(t,x,z) \in \mathcal{E}^{\{s_1,\sigma,s_2\}}(\Omega)$,
2) $u_i(t,x) \in \mathcal{E}^{\{s,\sigma\}}(W)$ $(i = 1, \ldots, d)$,
3) $(t,x) \in W \Longrightarrow (t,x,u(t,x)) \in \Omega$, *where* $u = (u_1, \ldots, u_d)$,
4) $\sigma \geq s_2$ *and* $s \geq \max\{s_1, s_2\}$

hold, we have $F(t,x,u(t,x)) \in \mathcal{E}^{\{s,\sigma\}}(W)$.

PROOF. Take any compact subset K of W; then the image L of K by the map $(t,x) \longrightarrow (t,x,u(t,x))$ is also a compact subset of Ω. We set

$$\begin{aligned} A_{p,q,\nu} &= \frac{1}{|p|!^{s_1-1}|q|!^{\sigma-1}|\nu|!^{s_2-1}} \max_{(t,x,z)\in L} \left| \frac{1}{p!q!\nu!} F^{(p,q,\nu)}(t,x,z) \right|, \\ B_{i,k,\beta} &= \frac{1}{|k|!^{s-1}(|\beta|-1)!^{\sigma-1}} \max_{(t,x)\in K} \left| \frac{1}{k!\beta!} u_i^{(k,\beta)}(t,x) \right|, \ \text{ if } |\beta| \geq 1, \\ B_{i,k,0} &= \frac{1}{(|k|-1)!^{s-1}} \max_{(t,x)\in K} \left| \frac{1}{k!} u_i^{(k,0)}(t,x) \right|, \ \text{ if } |k| \geq 1, \end{aligned}$$

where $p = (p_1, \ldots, p_N) \in \mathbb{N}^N$, $q = (q_1, \ldots, q_n) \in \mathbb{N}^n$, $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{N}^d$, $F^{(p,q,\nu)}(t,x,z) = (\partial_t^p \partial_x^q \partial_z^\nu F)(t,x,z)$, $k = (k_1, \ldots, k_N) \in \mathbb{N}^N$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, and $u_i^{(k,\beta)}(t,x) = (\partial_t^k \partial_x^\beta u_i)(t,x)$ $(i = 1, \ldots, d)$. We set also

$$\begin{aligned} G(t,x,z) &= \sum_{|p|+|q|+|\nu|\geq 0} A_{p,q,\nu} t^p x^q z^\nu \quad \text{with } z = (z_1, \ldots, z_d), \\ w_i(t,x) &= \sum_{|k|+|\beta|\geq 1} B_{i,k,\beta} t^k x^\beta, \quad i = 1, \ldots, d. \end{aligned}$$

Then, $G(t,x,z) = G(t,x,z_1, \ldots, z_d)$ and $w_i(t,x)$ $(i = 1, \ldots, d)$ are convergent in a neighborhoof of $(t,x,z) = (0,0,0)$ and $(t,x) = (0,0)$, respectively, and so the function

$$\begin{aligned} H(t,x) &= G(t,x,w_1(t,x), \ldots, w_d(t,x)) \\ &= \sum_{|p|+|q|+|\nu|\geq 0} A_{p,q,\nu} t^p x^q \prod_{i=1}^d \left[\sum_{|k_i|+|\beta_i|\geq 1} B_{i,k_i,\beta_i} t^{k_i} x^{\beta_i} \right]^{\nu_i} \end{aligned} \tag{4.1}$$

is also convergent in a neighborhood of $(t,x)=(0,0)$. If we set

$$H(t,x)=\sum_{(m,\alpha)\in\mathbb{N}^N\times\mathbb{N}^n} C_{m,\alpha}t^m x^\alpha$$

we have $C_{0,0}=A_{0,0,0}$ and

$$C_{m,\alpha}=\sum_{1\le|p|+|q|+|\nu|\le|m|+|\alpha|} A_{p,q,\nu}\sum_{\substack{s(k(\nu))=m-p\\ s(\beta(\nu))=\alpha-q\\ |k_i(j)|+|\beta_i(j)|\ge 1}}\prod_{i=1}^{d}\prod_{j=1}^{\nu_i} B_{i,k_i(j),\beta_i(j)} \tag{4.2}$$

where

$$s(k(\nu))=\sum_{i=1}^{d}\sum_{j=1}^{\nu_i}k_i(j)\in\mathbb{N}^N,\quad s(\beta(\nu))=\sum_{i=1}^{d}\sum_{j=1}^{\nu_i}\beta_i(j)\in\mathbb{N}^n.$$

Since $H(t,x)$ is a holomorphic function in a neighborhood of $(t,x)=(0,0)$, by Cauchy's inequality we have $C_{m,\alpha}\le M\eta^{|m|+|\alpha|}$ ($|m|+|\alpha|=0,1,2,\dots$) for some $M>0$ and $\eta>0$.

We set $\varphi(t,x)=F(t,x,u(t,x))$; to prove the condition $\varphi(t,x)\in\mathcal{E}^{\{s,\sigma\}}(K)$ it is sufficient to show the following inequalities:

$$\frac{1}{|m|!^{s-1}|\alpha|!^{\sigma-1}}\max_{(t,x)\in K}\left|\frac{1}{m!\alpha!}\varphi^{(m,\alpha)}(t,x)\right|\le 2^{(|m|+|\alpha|)(s_2-1)}C_{m,\alpha} \tag{4.3}$$
for any $|m|+|\alpha|=1,2,\dots$.

Here, we recall that by Faà di Bruno's formula (see Johnson [7]) or by the same argument we can see:

Lemma 4.3. *Let $F(t,x,z)\in C^\infty(\Omega)$ with $z=(z_1,\dots,z_d)$, and let $u(t,x)=(u_1(t,x),\dots,u_d(t,x))\in C^\infty(W)^d$. If $u(W)\subset\Omega$ holds, we have $\varphi(t,x)=F(t,x,u(t,x))\in C^\infty(W)$. In this case, for any $(m,\alpha)\in\mathbb{N}^N\times\mathbb{N}^n$ with $|m|+|\alpha|\ge 1$ we have*

$$\frac{1}{m!\alpha!}\varphi^{(m,\alpha)}=\sum_{1\le|p|+|q|+|\nu|\le|m|+|\alpha|}\frac{1}{p!q!\nu!}F^{(p,q,\nu)}\times \tag{4.4}$$

$$\times \sum_{\substack{s(k(\nu))=m-p \\ s(\beta(\nu))=\alpha-q \\ |k_i(j)|+|\beta_i(j)|\geq 1}} \prod_{i=1}^{d}\prod_{j=1}^{\nu_i}\Big(\frac{1}{k_i(j)!\beta_i(j)!}u_i^{(k_i(j),\beta_i(j))}\Big),$$

where $\varphi^{(m,\alpha)} = \varphi^{(m,\alpha)}(t,x)$, $F^{(p,q,\nu)} = F^{(p,q,\nu)}(t,x,u(t,x))$ *and* $u_i^{(k,\beta)} = u_i^{(k,\beta)}(t,x)$ *(*$i = 1,\dots,d$*).*

By using this formula, let us show (4.3) from now. By the estimates $(1/p!q!\nu!)|F^{(p,q,\nu)}| \leq |p|!^{s_1-1}|q|!^{\sigma-1}|\nu|!^{s_2-1}A_{p,q,\nu}$ on L and $(1/k!\beta!)|u_i^{(k,\beta)}| \leq (|k|-\epsilon_1)!^{s-1}(|\beta|-\epsilon_2)!^{\sigma-1}B_{i,k,\beta}$ (where $(\epsilon_1,\epsilon_2) = (1,0)$ if $|\beta| = 0$, and $(\epsilon_1,\epsilon_2) = (0,1)$ if $|\beta| \geq 1$) on K, we have

(4.5)

$$\begin{aligned}
&\frac{1}{|m|!^{s-1}|\alpha|!^{\sigma-1}}\Big|\frac{1}{m!\alpha!}\varphi^{(m,\alpha)}\Big| \\
&\leq \frac{1}{|m|!^{s-1}|\alpha|!^{\sigma-1}} \sum_{1\leq|p|+|q|+|\nu|\leq|m|+|\alpha|} |p|!^{s_1-1}|q|!^{\sigma-1}|\nu|!^{s_2-1}A_{p,q,\nu}\times \\
&\qquad \times \sum_{\substack{s(k(\nu))=m-p \\ s(\beta(\nu))=\alpha-q \\ |k_i(j)|+|\beta_i(j)|\geq 1}} \prod_{i=1}^{d}\prod_{j=1}^{\nu_i}\Big((|k_i(j)|-\epsilon_1)!^{s-1}(|\beta_i(j)|-\epsilon_2)!^{\sigma-1}B_{i,k_i(j),\beta_i(j)}\Big) \\
&\leq \sum_{1\leq|p|+|q|+|\nu|\leq|m|+|\alpha|} \frac{|p|!^{s_1-1}|q|!^{\sigma-1}|\nu|!^{s_2-1}}{|m|!^{s-1}|\alpha|!^{\sigma-1}}A_{p,q,\nu}\times \\
&\qquad \times \sum_{\substack{s(k(\nu))=m-p \\ s(\beta(\nu))=\alpha-q \\ |k_i(j)|+|\beta_i(j)|\geq 1}} (|k(\nu)|-n_1)!^{s-1}(|\beta(\nu)|-n_2)!^{\sigma-1}\prod_{i=1}^{d}\prod_{j=1}^{\nu_i}\Big(B_{i,k_i(j),\beta_i(j)}\Big),
\end{aligned}$$

where $|k(\nu)| = \sum_{i=1}^{d}\sum_{j=1}^{\nu_i}|k_i(j)|$, $|\beta(\nu)| = \sum_{i=1}^{d}\sum_{j=1}^{\nu_i}|\beta_i(j)|$, $n_1 = \#\{(i,j)\,;\,|\beta_i(j)|=0\}$, and $n_2 = \#\{(i,j)\,;\,|\beta_i(j)|\geq 1\}$. Since $n_1+n_2 = |\nu|$, $s \geq s_2$ and $\sigma \geq s_2$ hold, we have

$$|\nu|!^{s_2-1} \leq 2^{|\nu|(s_2-1)}n_1!^{s_2-1}n_2!^{s_2-1} \leq 2^{|\nu|(s_2-1)}n_1!^{s-1}n_2!^{\sigma-1},$$

and so

$$\begin{aligned}
&\frac{|p|!^{s_1-1}|q|!^{\sigma-1}|\nu|!^{s_2-1}}{|m|!^{s-1}|\alpha|!^{\sigma-1}}(|k(\nu)|-n_1)!^{s-1}(|\beta(\nu)|-n_2)!^{\sigma-1} \\
&\le \frac{|p|!^{s-1}|q|!^{\sigma-1}2^{|\nu|(s_2-1)}n_1!^{s-1}n_2!^{\sigma-1}}{|m|!^{s-1}|\alpha|!^{\sigma-1}}(|m|-|p|-n_1)!^{s-1}(|\alpha|-|q|-n_2)!^{\sigma-1} \\
&= 2^{|\nu|(s_2-1)}\frac{|p|!^{s-1}n_1!^{s-1}(|m|-|p|-n_1)!^{s-1}}{|m|!^{s-1}}\frac{|q|!^{\sigma-1}n_2!^{\sigma-1}(|\alpha|-|q|-n_2)!^{\sigma-1}}{|\alpha|!^{\sigma-1}} \\
&\le 2^{|\nu|(s_2-1)} \le 2^{(|m|+|\alpha|)(s_2-1)}.
\end{aligned}$$

Thus, by applying this to (4.5) we have

$$\begin{aligned}
&\frac{1}{m!^{s-1}|\alpha|!^{\sigma-1}}\left|\frac{1}{m!\alpha!}\varphi^{(m,\alpha)}\right| \\
&\quad \le 2^{(|m|+|\alpha|)(s_2-1)}\sum_{1\le|p|+|q|+|\nu|\le|m|+|\alpha|}A_{p,q,\nu}\sum_{\substack{s(k(\nu))=m-p\\ s(\beta(\nu))=\alpha-q\\ |k_i(j)|+|\beta_i(j)|\ge 1}}\prod_{i=1}^{d}\prod_{j=1}^{\nu_i}\Big(B_{i,k_i(j),\beta_i(j)}\Big) \\
&\quad = 2^{(|m|+|\alpha|)(s_2-1)}C_{m,\alpha}
\end{aligned}$$

on K. This proves (4.3). □

4.2. On the formal norm $\|\!|f|\!\|_{K,\rho}$

For a function $f(x) \in C^\infty(V)$ and a compact subset K of V we define the formal norm $\|\!|f|\!\|_{K,\rho}$ of $f(x)$ on K by

$$\|\!|f|\!\|_{K,\rho} = \sum_{|\alpha|\ge 0}\frac{\|\partial_x^\alpha f\|_K}{|\alpha|!^\sigma}\rho^{|\alpha|} \quad \text{with } \|\partial_x^\alpha f\|_K = \max_{x\in K}|\partial_x^\alpha f(x)| \tag{4.6}$$

which is a formal power series in ρ. In the case K is fixed, we often write $\|\!|f|\!\|_\rho$ instead of $\|\!|f|\!\|_{K,\rho}$. We write $\sum_{k=0}^\infty a_k\rho^k \gg \sum_{k=0}^\infty b_k\rho^k$ if $a_k \ge |b_k|$ holds for all $k \in \mathbb{N}$. It is easy to see:

PROPOSITION 4.4. (1) $\|\!|fg|\!\|_{K,\rho} \ll \|\!|f|\!\|_{K,\rho}\|\!|g|\!\|_{K,\rho}$.

(2) *If* $\|\partial_x^\alpha f\|_K \le Ch^{|\alpha|}|\alpha|!^\sigma$ *holds for any* $|\alpha| = 0, 1, 2, \dots,$ *we have*

$$\|\!|f|\!\|_{K,\rho} \ll \frac{C}{(1-\rho/R)^n} \quad \text{with } R = 1/h. \tag{4.7}$$

Conversely, if (4.7) *holds, we have* $\|\partial_x^\alpha f\|_K \le (2^{n-1}C)(2h)^{|\alpha|}|\alpha|!^\sigma$ *for any* $|\alpha| = 0, 1, 2, \dots$.

(3) $f(x) \in \mathcal{E}^{\{\sigma\}}(V)$ *holds if and only if for any compact subset* K *of* V *the formal norm* $\|\!|f|\!\|_{K,\rho}$ *is convergent in a neighborhood of* $\rho = 0$.

The following lemma is a variant of Nagumo's lemma:

PROPOSITION 4.5 (Nagumo's type lemma). *Let* $\sigma \ge 1$. *If*

$$\|\!|f|\!\|_{K,\rho} \ll \frac{C}{(1-\rho/R)^a} \tag{4.8}$$

holds for some $C > 0$, $a \ge 1$ *and* $R > 0$, *we have*

$$\|\!|\partial_{x_i} f|\!\|_{K,\rho} \ll \frac{Ce^\sigma(a+\sigma)^\sigma/R}{(1-\rho/R)^{a+\sigma}} \quad \text{for } i = 1, \dots, n. \tag{4.9}$$

PROOF. By the assumption (4.8) we have

$$\sum_{k\ge 0} \frac{1}{k!^\sigma}\Bigl(\sum_{|\alpha|=k} \|\partial_x^\alpha f\|_K\Bigr)\rho^k \ll C\sum_{k\ge 0} \frac{a(a+1)\cdots(a+k-1)}{k!}(\rho/R)^k$$

and so

$$\sum_{|\alpha|=k} \|\partial_x^\alpha f\|_K \le Ck!^\sigma \frac{a(a+1)\cdots(a+k-1)}{k!}(1/R)^k, \quad k = 0, 1, 2, \dots.$$

Therefore, we see:

$$\begin{aligned} \|\!|\partial_{x_i} f|\!\|_{K,\rho} &= \sum_{k\ge 0} \frac{1}{k!^\sigma}\Bigl(\sum_{|\alpha|=k} \|\partial_x^\alpha \partial_{x_i} f\|_K\Bigr)\rho^k \\ &\ll \sum_{k\ge 0} \frac{1}{k!^\sigma}\Bigl(C(k+1)!^\sigma \frac{a(a+1)\cdots(a+k)}{(k+1)!}(1/R)^{k+1}\Bigr)\rho^k \\ &= C(1/R)\sum_{k\ge 0}(k+1)^{\sigma-1}\frac{a(a+1)\cdots(a+k)}{k!}(\rho/R)^k. \end{aligned} \tag{4.10}$$

Hence, if we prove the inequality

$$\begin{aligned} &(k+1)^{\sigma-1}a(a+1)\cdots(a+k) \\ &\qquad \le e^\sigma(a+\sigma)^\sigma(a+\sigma)\cdots(a+\sigma+k-1), \quad k \ge 1, \end{aligned} \tag{4.11}$$

by applying this to (4.10) we have the result (4.9).

The proof of (4.11) is as follows. Since a sharp form of the Stirling's formula (see Whittaker-Watson [19]) for the Γ-function guarantees

$$1 < \frac{\Gamma(x)}{\sqrt{2\pi}\, x^{x-1/2}e^{-x}} < \exp\left(\frac{1}{12x}\right) < \sqrt{e} \quad \text{for } x \geq 1, \tag{4.12}$$

we obtain

$$\begin{aligned}
&(k+1)^{\sigma-1}\frac{a(a+1)\cdots(a+k)}{(a+\sigma)\cdots(a+\sigma+k-1)} = (k+1)^{\sigma-1}\frac{\Gamma(a+k+1)\Gamma(a+\sigma)}{\Gamma(a)\Gamma(a+\sigma+k)} \\
&\leq (k+1)^{\sigma-1}\frac{\sqrt{2\pi}(a+k+1)^{a+k+1/2}e^{-a-k-1}\sqrt{e}\sqrt{2\pi}(a+\sigma)^{a+\sigma-1/2}e^{-a-\sigma}\sqrt{e}}{\sqrt{2\pi}(a)^{a-1/2}e^{-a}\sqrt{2\pi}(a+\sigma+k)^{a+\sigma+k-1/2}e^{-a-\sigma-k}} \\
&= (k+1)^{\sigma-1}\frac{(a+k+1)^{a+k+1/2}(a+\sigma)^{a+\sigma-1/2}}{(a)^{a-1/2}(a+\sigma+k)^{a+\sigma+k-1/2}} \\
&\leq \frac{(a+\sigma)^{a+\sigma-1/2}}{(a)^{a-1/2}} = (a+\sigma)^{\sigma}\left(\frac{a}{a+\sigma}\right)^{1/2}\left(1+\frac{\sigma}{a}\right)^{a} \leq (a+\sigma)^{\sigma}e^{\sigma}. \ \Box
\end{aligned}$$

If $f(t,x)$ is a C^∞ function on $I \times K$ we define the formal norm $\|f\|_{I\times K,\rho}$ of $f(t,x)$ on $I \times K$ by

$$\|f\|_{I\times K,\rho} = \sum_{|\alpha|\geq 0} \frac{\|\partial_x^\alpha f\|_{I\times K}}{|\alpha|!^\sigma}\rho^{|\alpha|} \quad \text{with } \|\partial_x^\alpha f\|_{I\times K} = \max_{(t,x)\in I\times K} |\partial_x^\alpha f(t,x)|$$

which is a formal power series in ρ. It is clear that Propositions 4.5 and 4.6 with $\|f\|_{K,\rho}$ replaced by $\|f\|_{I\times K,\rho}$ are also valid.

The following lemma is also very useful.

PROPOSITION 4.6. *If* $0 < R < R_0$ *and* $a \geq 1$ *we have*

$$\frac{1}{(1-\rho/R_0)}\frac{1}{(1-\rho/R)^a} \ll \frac{1}{(1-R/R_0)}\frac{1}{(1-\rho/R)^a}. \tag{4.13}$$

4.3. Estimate of $F(x, u_1(x), \dots, u_d(x))$

Let $\sigma \geq 1$, let $F(x, z_1, \dots, z_d) \in \mathcal{E}^{\{\sigma\}}(\Omega)$ for some open subset Ω of $\mathbb{R}_x^n \times \mathbb{R}_z^d$, and let $u_i(x) \in \mathcal{E}^{\{\sigma\}}(V)$ $(i = 1, \dots, d)$ for some open subset V of $\mathbb{R}_x^n$ which satisfy the following: $x \in V \Longrightarrow (x, u_1(x), \dots, u_d(x)) \in$

Ω. Then, by Proposition 4.1 we have $F(x, u_1(x), \dots, u_d(x)) \in \mathcal{E}^{\{\sigma\}}(V)$. Let K be a compact subset of V; then the image L of K by the map $x \longrightarrow (x, u_1(x), \dots, u_d(x))$ is also a compact subset of Ω. We write $z = (z_1, \dots, z_d)$. By the assumption we have the estimtes

$$1)\ \frac{1}{q!\nu!} \max_{(x,z)\in L} |F^{(q,\nu)}(x,z)| \le A h^{|q|+|\nu|}(|q|+|\nu|)!^{\sigma-1}, \quad |q| \ge 0, |\nu| \ge 0,$$

$$2)\ \frac{1}{\beta!} \max_{x\in K} |u_i^{(\beta)}(x)| \le B h^{|\beta|}(|\beta|-1)!^{\sigma-1}, \quad i = 1, \dots, d \text{ and } |\beta| \ge 1$$

for some $A > 0$, $B > 0$ and $h > 0$.

Proposition 4.7. *Suppose the above conditions, and we set* $\varphi(x) = F(x, u_1(x), \dots, u_d(x))$. *If* $0 < R_0 < 1/h$ *and*

$$hB\Big[\frac{1}{(1-hR_0)^n} - 1\Big] \le \frac{1}{2} \tag{4.14}$$

hold, we have the estimate

$$|\!|\!|\varphi|\!|\!|_{K,\rho} \ll \frac{2^d A}{(1-\rho/R_0)^n}. \tag{4.15}$$

Proof. We set

$$G(x,z) = \sum_{|q|+|\nu|\ge 0} A h^{|q|+|\nu|} x^q z^\nu, \quad w(x) = \sum_{|\beta|\ge 1} B h^{|\beta|} x^\beta :$$

then under the relation $x_i = \rho$ $(i = 1, \dots, d)$ we have

$$G(x,z) \ll \frac{A}{(1-h\rho)^n} \sum_{|\nu|\ge 0} (hz)^\nu, \quad w(x) \ll B\Big[\frac{1}{(1-h\rho)^n} - 1\Big].$$

By the same argument as in the proof of Proposition 4.2 we have

$$\sum_{|\alpha|\ge 0} \frac{\|\varphi^{(\alpha)}\|_K}{|\alpha|!^\sigma} x^\alpha \ll \sum_{|\alpha|\ge 0} \frac{\|\varphi^{(\alpha)}\|_K}{|\alpha|!^{\sigma-1}\alpha!} x^\alpha \ll G(x, w(x), \dots, w(x))$$

as formal power series in x. Thus, if we set $x_i = \rho$ $(i = 1, \dots, d)$ we have

$$\begin{aligned}
\|\varphi\|_{K,\rho} \ll G(\rho, w(\rho), \dots, w(\rho)) &\ll \frac{A}{(1-h\rho)^n} \sum_{|\nu|\ge 0} (hw(\rho))^{|\nu|} \\
&\ll \frac{A}{(1-\rho/R_0)^n} \sum_{|\nu|\ge 0} \Big[hB\Big(\frac{1}{(1-h\rho)^n} - 1\Big)\Big]^{|\nu|} \\
&\ll \frac{A}{(1-\rho/R_0)^n} \sum_{|\nu|\ge 0} \Big[hB\Big(\frac{1}{(1-hR_0)^n} - 1\Big)\Big]^{|\nu|} \\
&\ll \frac{A}{(1-\rho/R_0)^n} \sum_{|\nu|\ge 0} \Big(\frac{1}{2}\Big)^{|\nu|} = \frac{2^d A}{(1-\rho/R_0)^n}.
\end{aligned}$$

In the above, we have used Proposition 4.6. □

5. Proof of Theorem 3.6 in the Case $s = 1$

In the case $s = 1$, by the condition $s \ge \max\{s_0(F), s^*\}$ with $s^* = \max\{s_1, s_2\}$ we have $s_0(F) = s_1 = s_2 = 1$, and so we have $i + \sigma|\alpha| - m \le 0$ for any $(i, \alpha) \in \Lambda_0$. Therefore, our equation (3.12) (or (3.24)) is written in the form

$$(t\partial_t)^m u = F\big(t, x, \Theta u\big) \quad \text{with } \Theta u = \{(t\partial_t)^j \partial_x^\alpha u\}_{j+\sigma|\alpha|\le m, j<m}. \tag{5.0.1}$$

In this case, Theorem 3.6 is nothing but the following result:

THEOREM 5.0.1. *Suppose:* $m \in \mathbb{N}^*$ *and* $F(t, x, z) \in \mathcal{E}^{\{1,\sigma,1\}}(\Omega)$. *If* $u(t, x)$ *is a solution of* (5.0.1) *on* $I \times V$ *belonging to the class* $C^\infty(I, \mathcal{E}^{\{\sigma\}}(V))$ *and if*

$$\frac{\partial F}{\partial z_{j,\alpha}}(t, x, \Theta u)\Big|_{t=0} \equiv 0 \ \text{ on } V, \ \text{ if } |\alpha| > 0, \tag{5.0.2}$$

then we have $u(t, x) \in \mathcal{E}^{\{1,\sigma\}}(I \times V)$.

We note that the local version of this result can be verified by the results of Koike [11] (uniqueness of the solution) and Pongerard [15] (existence of a solution in the class $\mathcal{E}^{\{1,\sigma\}}$). We will give here a direct proof by estimating the term $\|\partial_t^k u\|_{I\times K,\rho}$ inductively on k. The discussion here gives also a preliminary part of the proof of Theorem 3.6 in the case $s > 1$.

We set

$$\begin{aligned} c_j(x) &= \frac{\partial F}{\partial z_{j,0}}(0, x, \Theta u(0,x)), \quad 0 \le j < m, \\ C(\lambda, x) &= \lambda^m - c_{m-1}(x)\lambda^{m-1} - \cdots - c_1(x)\lambda - c_0(x), \\ F_0(t,x,z) &= F(t,x,z) - \sum_{j<m} c_j(x) z_{j,0}. \end{aligned}$$

Then, our equation (5.0.1) is written in the form

$$C(t\partial_t, x)u = F_0\big(t, x, \Theta u\big) \tag{5.0.3}$$

and we have the condition

$$\frac{\partial F_0}{\partial z_{j,\alpha}}(t,x,\Theta u)\Big|_{t=0} \equiv 0 \ \text{ on } V \ \text{ for any } (j,\alpha) \in \Lambda. \tag{5.0.4}$$

In the proof of Theorem 5.0.1, we will treat this equation (5.0.3) instead of (5.0.1). We denote by $\mathbb{C}\{t,\rho\}$ (resp. by $\mathbb{C}\{t\}$) the ring of convergent power series in (t,ρ) (resp. in t).

5.1. On the recurrent formulas on $\partial_t^k u/k!$

In order to prove Theorem 5.0.1 we must show the following result: for any compact subset K of V we have the condition

$$\sum_{k\ge 0} \frac{1}{k!}\|\partial_t^k u\|_{I\times K,\rho} t^k \in \mathbb{C}\{t,\rho\}. \tag{5.1.1}$$

To do so, let us find recurrent formulas on $w_k = \partial_t^k u/k!$ $(k = 1, 2, \dots)$.

First we note:

Lemma 5.1.1. *For any $p = 1, 2, \dots$ and $k = 1, 2, \dots$ we have $\partial_t^k (t\partial_t)^p u = (t\partial_t + k)^p \partial_t^k u$ and so $\partial_t^k (t\partial_t)^p u = \partial_t (t\partial_t + k - 1)^p \partial_t^{k-1} u$.*

Now, we set $\Lambda = \{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^n\,;\, j + \sigma|\alpha| \le m,\ j < m\}$ and

$$w_k(t,x) = \frac{1}{k!}\partial_t^k u(t,x), \quad k = 1, 2, \dots, \tag{5.1.2}$$

and we apply ∂_t^k to the both sides of the equation (5.0.3): by Lemmas 4.3 and 5.1.1, for $k \geq 1$ we have

$$
\begin{aligned}
(5.1.3)\quad & C(t\partial_t + k, x)w_k \\
&= \frac{1}{k!}F_0^{(k,0,0)}(t,x,\Theta u) \\
&\quad + \sum_{(j,\alpha)\in\Lambda} \frac{\partial F_0}{\partial z_{j,\alpha}}(t,x,\Theta u) \times \frac{1}{k}\partial_t(t\partial_t + k - 1)^j \partial_x^\alpha w_{k-1} \\
&\quad + \sum_{\substack{2\leq p+|\nu|\leq k \\ |\nu|\geq 1}} \frac{1}{p!\nu!}F_0^{(p,0,\nu)}(t,x,\Theta u) \times \\
&\qquad \times \sum_{|k^*|=k-p} \prod_{(j,\alpha)\in\Lambda} \Big[(t\partial_t + k_{j,\alpha}(1))^j \partial_x^\alpha w_{k_{j,\alpha}(1)} \times \\
&\qquad\qquad \times \cdots \times (t\partial_t + k_{j,\alpha}(\nu_{j,\alpha}))^j \partial_x^\alpha w_{k_{j,\alpha}(\nu_{j,\alpha})}\Big],
\end{aligned}
$$

where

$$
\nu = \{\nu_{j,\alpha}\}_{(j,\alpha)\in\Lambda} \in \mathbb{N}^d, \quad |\nu| = \sum_{(j,\alpha)\in\Lambda} \nu_{j,\alpha},
$$
$$
|k^*| = \sum_{(j,\alpha)\in\Lambda} \big(k_{j,\alpha}(1) + \cdots + k_{j,\alpha}(\nu_{j,\alpha})\big)
$$

and $F_0^{(p,0,\nu)} = \partial_t^p \partial_z^\nu F_0$ ($p + |\nu| \geq 2$ and $|\nu| \geq 1$).

By the condition (5.0.4) we have the expression

$$
\frac{\partial F}{\partial z_{j,\alpha}}(t,x,\Theta u) = t a_{j,\alpha}(t,x), \quad (j,\alpha) \in \Lambda
$$

for some $a_{j,\alpha}(t,x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(V))$. Therefore, by setting

$$
f_k(t,x) = \frac{1}{k!}F^{(k,0,0)}(t,x,\Theta u(t,x)), \quad k = 1,2,\dots;
$$
$$
g_{p,\nu}(t,x) = \frac{1}{p!\nu!}F^{(p,0,\nu)}(t,x,\Theta u), \quad p + |\nu| \geq 2, |\nu| \geq 1,
$$

and by using the relation

$$\begin{aligned}
&ta_{j,\alpha}(t,x) \times \frac{1}{k}\partial_t(t\partial_t + k - 1)^j \partial_x^\alpha w_{k-1} \\
&= a_{j,\alpha}(t,x)\Big(\frac{1}{k}(t\partial_t + k - 1)^{j+1}\partial_x^\alpha w_{k-1} - \frac{k-1}{k}(t\partial_t + k - 1)^j \partial_x^\alpha w_{k-1}\Big)
\end{aligned}$$

we have the equation

$$\begin{aligned}
(5.1.4)\quad & C(t\partial_t + k, x)w_k \\
&= f_k(t,x) + \sum_{(j,\alpha)\in\Lambda} a_{j,\alpha}(t,x) \times \frac{1}{k}(t\partial_t + k - 1)^{j+1}\partial_x^\alpha w_{k-1} \\
&\qquad - \sum_{(j,\alpha)\in\Lambda} a_{j,\alpha}(t,x) \times \frac{k-1}{k}(t\partial_t + k - 1)^j \partial_x^\alpha w_{k-1} \\
&\quad + \sum_{2\le p+|\nu|\le k, |\nu|\ge 1} g_{p,\nu}(t,x) \times \\
&\qquad \times \sum_{|k^*|=k-p} \prod_{(j,\alpha)\in\Lambda} \Big[(t\partial_t + k_{j,\alpha}(1))^j \partial_x^\alpha w_{k_{j,\alpha}(1)} \times \\
&\qquad\qquad \times \cdots \times (t\partial_t + k_{j,\alpha}(\nu_{j,\alpha}))^j \partial_x^\alpha w_{k_{j,\alpha}(\nu_{j,\alpha})}\Big].
\end{aligned}$$

In the second term of the right-hand side we have the factor $(t\partial_t + k - 1)^{j+1}$: concerning this, we note that $j + 1 \le m$ holds (by the condition $j < m$). This fact guarantees that we can apply the induction argument.

By the assumption we have:

LEMMA 5.1.2. *For any compact subset K of V there are constants $F_k \ge 0$ ($k \ge 1$), $R_0 > 0$, $A_{j,\alpha} \ge 0$ ($(j,\alpha) \in \Lambda$), $C \ge 0$, and $B_{p,\nu} \ge 0$ ($p + |\nu| \ge 2$ and $|\nu| \ge 1$) which satisfy the following properties:*

1) $\|f_k\|_{I\times K,\rho} \ll \dfrac{F_k}{(1-\rho/R_0)^n}$ *and* $\displaystyle\sum_{k\ge 1} F_k t^k \in \mathbb{C}\{t\}$;

2) $\|a_{j,\alpha}\|_{I\times K,\rho} \ll \dfrac{A_{j,\alpha}}{(1-\rho/R_0)^n}$, $\quad (j,\alpha) \in \Lambda$;

3) $\|c_i\|_{K,\rho} \ll \dfrac{C}{(1-\rho/R_0)^n}$, $\quad i = 0, \dots, m-1$;

4) $\|g_{p,\nu}\|_{I\times K,\rho} \ll \dfrac{B_{p,\nu}}{(1-\rho/R_0)^n}$ *and* $\displaystyle\sum_{p+|\nu|\geq 2, |\nu|\geq 1} B_{p,\nu} t^p z^\nu \in \mathbb{C}\{t,z\}$.

PROOF. Since $F_0(t,x,z) \in \mathcal{E}^{\{1,\sigma,1\}}(\Omega)$ is supposed, by (2) of Proposition 4.1 we have $F_0(t,x,\Theta u(t,x)) \in C^\infty(I,\mathcal{E}^{\{\sigma\}}(V))$ and $(\partial F_0/\partial z_{j,\alpha})(t,x,\Theta u(t,x)) \in C^\infty(I,\mathcal{E}^{\{\sigma\}}(V))$. Hence, by Proposition 4.4 we have the properties 2) and 3).

We write $u_{j,\alpha} = (t\partial_t)^j \partial_x^\alpha u$; we have $\Theta u = \{u_{j,\alpha}\}_{(j,\alpha)\in\Lambda}$. Since $u(t,x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(K))$ is supposed, we have $u_{j,\alpha}(t,x) \in C^\infty(I,\mathcal{E}^{\{\sigma\}}(K))$.

Let K be a compact subset of V; then the image L of K by the map $(t,x) \longrightarrow (t,x,\Theta u(t,x))$ is also a compact subset of Ω. By the assumption we have the estimates

$$\frac{1}{p!\alpha!\nu!}\max_{(t,x,z)\in L}|\partial_t^p\partial_x^\alpha\partial_z^\nu F_0(t,x,z)| \leq A h^{p+|\alpha|+|\nu|}|\alpha|!^{\sigma-1} \quad \text{for any } (p,\alpha,\nu)\in\mathbb{N}\times\mathbb{N}^n\times\mathbb{N}^d, \tag{5.1.5}$$

$$\frac{1}{\beta!}\max_{(t,x)\in I\times K}|\partial_x^\beta u_{j,\alpha}| \leq B h^{|\beta|}(|\beta|-1)!^{\sigma-1} \quad \text{for any } |\beta|>0 \tag{5.1.6}$$

for some $A>0$, $B>0$ and $h>0$. By using these conditions, let us show 1) and 4).

By (5.1.5) we have

$$\frac{1}{\alpha!\nu!}\max_{(t,x,z)\in L}\big|\partial_x^\alpha\partial_z^\nu\big((1/k!)F_0^{(k,0,0)}\big)(t,x,z)\big| \leq A h^{k+|\alpha|+|\nu|}|\alpha|!^{\sigma-1}$$
for any $(\alpha,\nu)\in\mathbb{N}^n\times\mathbb{N}^d$;

therefore, by Proposition 4.7 we have the result: if $R_0>0$ is sufficiently small so that $0<R_0<1/h$ and

$$hB\left[\frac{1}{(1-hR_0)^n}-1\right] \leq \frac{1}{2} \tag{5.1.7}$$

hold, we have the estimate

$$\|f_k\|_{I\times K,\rho} \ll \frac{2^d A h^k}{(1-\rho/R_0)^n}. \tag{5.1.8}$$

This proves 1). Similarly, by (5.1.5) we have

$$\begin{aligned}
&\frac{1}{\alpha!\nu!}\max_{(t,x,z)\in L}\big|\partial_x^\alpha\partial_z^\nu\big((1/(p!\mu!))F_0^{(p,0,\mu)}\big)(t,x,z)\big| \\
&= \frac{(\nu+\mu)!}{\nu!\mu!}\frac{1}{p!\alpha!(\nu+\mu)!}\max_{(t,x,z)\in L}|\partial_t^p\partial_x^\alpha\partial_z^{\nu+\mu}F_0(t,x,z)| \\
&\le 2^{|\nu|+|\mu|}Ah^{p+|\alpha|+|\nu|+|\mu|}|\alpha|!^{\sigma-1} \le Ah^p(2h)^{|\mu|}(2h)^{|\alpha|+|\nu|}|\alpha|!^{\sigma-1} \\
&\qquad\qquad\qquad\qquad\text{for any } (\alpha,\nu)\in\mathbb{N}^n\times\mathbb{N}^d;
\end{aligned}$$

therefore, by Proposition 4.7 we have the result: if $R_0>0$ is sufficiently small so that $0<R_0<1/(2h)$ and

$$2hB\left[\frac{1}{(1-2hR_0)^n}-1\right]\le\frac{1}{2} \tag{5.1.9}$$

hold, we have the estimate

$$|\!|\!| g_{p,\mu}|\!|\!|_{I\times K,\rho}\ll\frac{2^d(Ah^p(2h)^{|\mu|})}{(1-\rho/R_0)^n}. \tag{5.1.10}$$

This proves 4). □

5.2. On the equation $C(t\partial_t+k,x)w=g$

Let K be a compact subset of V, and let us consider the equation

$$C(t\partial_t+k,x)w=g(t,x)\quad\text{on } I\times K, \tag{5.2.1$_k$}$$

where k is a positive integer. We note:

LEMMA 5.2.1. *There are constants $k_0\in\mathbb{N}^*$ and $M_0>0$ which satisfy the following property: if $w(t,x)\in C^\infty(I\times K)$ and $g(t,x)\in C^\infty(I\times K)$ satisfy the equation $(5.2.1)_k$ for some $k\ge k_0$, and if $|g(t,x)|\le A$ holds on $I\times K$, we have*

$$\sum_{i=0}^m\|(t\partial_t+k)^iw\|_{I\times K}\le M_0A. \tag{5.2.2}$$

PROOF. Let $\lambda_i(x)$ $(i=1,\dots,m)$ be the roots of $C(\lambda,x)=0$, and take $k_0\in\mathbb{N}^*$ so that

$$k_0-\mathrm{Re}\lambda_i(x)\ge 1\ \text{ on } K,\ \ i=1,\dots,m.$$

In the case $m = 1$, our equation $(5.2.1)_k$ is written in the form $(t\partial_t + k - \lambda(x))w = g$, and so we have

$$w(t,x) = \int_0^t \left(\frac{\tau}{t}\right)^{k-\lambda(x)} g(\tau,x)\frac{d\tau}{\tau}$$

and

$$|w(t,x)| \le \int_0^t \left(\frac{\tau}{t}\right)^{k-\mathrm{Re}\lambda(x)} A\frac{d\tau}{\tau} = \frac{A}{k-\mathrm{Re}\lambda(x)} \le A:$$

moreover, if we take $B > 0$ so that $B \ge |\lambda(x)|$ on K, we have $|(t\partial_t + k)w| \le |\lambda(x)w| + |g| \le BA + A = A(1+B)$ on $I \times K$. Thus, by taking $M_0 = 1 + (1+B)$ we have the result (5.2.2).

In the case $m = 2$, the proof is as follows. Since our equation $(5.2.1)_k$ is written in the form $(t\partial_t + k - \lambda_1(x))(t\partial_t + k - \lambda_2(x))w = g$, by setting $w_1 = (t\partial_t + k - \lambda_2(x))w$ we have

$$(t\partial_t + k - \lambda_1(x))w_1 = g, \quad (t\partial_t + k - \lambda_2(x))w = w_1.$$

If we take $B > 0$ so that $B \ge |\lambda_i(x)|$ on K for $i = 1, 2$, by the equation $(t\partial_t + k - \lambda_1(x))w_1 = g$ we have $|w_1| \le A$ and $|(t\partial_t + k)w_1| \le A(1+B)$ on $I \times K$. Hence, by the equation $(t\partial_t + k - \lambda_1(x))w = w_1$ we have $|w| \le A$ and $|(t\partial_t + k)w| \le A(1+B)$ on $I \times K$. Then,

$$|(t\partial_t + k)^2 w| \le 2B|(t\partial_t + k)w| + B^2|w| + |g| \le 2BA(1+B) + B^2 A + A.$$

Thus, by setting $M_0 = 1 + (1+B) + (2B(1+B) + B^2 + 1)$ we have the result.

The general case $m \ge 3$ can be proved in the same way. □

LEMMA 5.2.2. *Let $k_0 \in \mathbb{N}^*$ and $M_0 > 0$ be the constants in Lemma 5.2.1. If $w(t,x) \in C^\infty(I \times K)$ and $g(t,x) \in C^\infty(I \times K)$ satisfy the equation $(5.2.1)_k$ for some $k \ge k_0$, and if the conditions*

i) $\|c_i\|_{K,\rho} \ll \dfrac{C}{(1-\rho/R_0)^n}, \quad i = 0, 1, \dots, m-1,$

ii) $\|g\|_{I\times K,\rho} \ll \dfrac{A}{(1-\rho/R)^a}$ *for some $A > 0$ and $a \ge 1$,*

iii) *$R > 0$ satisfies* $M_0 C\left[\dfrac{1}{(1-R/R_0)^n} - 1\right] \le \dfrac{1}{2}$

hold, we have

$$\|\!|(t\partial_t + k)^i w\|\!|_{I\times K,\rho} \ll \frac{2M_0}{k^{m-i}} \frac{A}{(1-\rho/R)^a}, \quad i = 0, 1, \dots, m. \tag{5.2.3}$$

Proof. First, we note:

$$\sum_{i=0}^{m} \|\!|(t\partial_t + k)^i w\|\!|_{I\times K,\rho} \ll \frac{2M_0 A}{(1-\rho/R)^a}. \tag{5.2.4}$$

The proof is as follows. Since $C(t\partial_t + k, x)w = g$ holds on $I \times K$, for any $\alpha \in \mathbb{N}^n$ we have

$$\partial_x^\alpha g = C(t\partial_t + k, x)\partial_x^\alpha w - \sum_{i=0}^{m-1} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ |\alpha_1|\geq 1}} \frac{\alpha!}{\alpha_1!\alpha_2!} \partial_x^{\alpha_1} c_i(x) \times (t\partial_t + k)^i \partial_x^{\alpha_2} w :$$

therefore, by Lemma 5.2.1 we have

$$\begin{aligned}
&\sum_{i=0}^{m} \|(t\partial_t + k)^i \partial_x^\alpha w\|_{I\times K} \leq M_0 \|C(t\partial_t + k)\partial_x^\alpha w\|_{I\times K} \\
&\leq M_0 \|\partial_x^\alpha g\|_{I\times K} \\
&\quad + M_0 \sum_{i=0}^{m-1} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ |\alpha_1|\geq 1}} \frac{|\alpha|!}{|\alpha_1|!|\alpha_2|!} \|\partial_x^{\alpha_1} c_i\|_K \|\partial_x^{\alpha_2} (t\partial_t + k)^i w\|_{I\times K}.
\end{aligned}$$

Hence, we have the inequality

$$\begin{aligned}
&\sum_{i=0}^{m} \|\!|(t\partial_t + k)^i w\|\!|_{I\times ,K\rho} \\
&\leq M_0 \|\!|g\|\!|_{I\times K,\rho} + M_0 \sum_{i=0}^{m-1} \left[\sum_{|\alpha_1|\geq 1} \frac{\|\partial_x^{\alpha_1} c_i\|_K}{|\alpha_1|!^\sigma} \rho^{|\alpha_1|} \right] \|\!|(t\partial_t + k)^i w\|\!|_{I\times K,\rho} \\
&\leq \frac{M_0 A}{(1-\rho/R)^a} + M_0 C \left(\frac{1}{(1-\rho/R_0)} - 1 \right) \sum_{i=0}^{m-1} \|\!|(t\partial_t + k)^i w\|\!|_{I\times K,\rho}.
\end{aligned}$$

Thus, by Proposition 4.6 and the assumption iii) we obtain

$$\begin{aligned}\sum_{i=0}^{m} \|(t\partial_t + k)^i w\|_{I\times,K\rho} &\ll \frac{1}{1 - M_0 C\Big(\dfrac{1}{(1-\rho/R_0)} - 1\Big)} \times \frac{M_0 A}{(1-\rho/R)^a} \\ &\ll \frac{1}{1-1/2} \times \frac{M_0 A}{(1-\rho/R)^a} = \frac{2M_0 A}{(1-\rho/R)^a}.\end{aligned}$$

This proves (5.2.4).

By using (5.2.4) let us show (5.2.3). We note that if $(t\partial_t + k)u = f$ holds, we have $(t\partial_t + k)\partial_x^\alpha u = \partial_x^\alpha f$ for any $\alpha \in \mathbb{N}^n$, and so by using the integral expression

$$\partial_x^\alpha u(t,x) = \int_0^t \Big(\frac{\tau}{t}\Big)^k \partial_x^\alpha f(\tau,x) \frac{d\tau}{\tau}$$

we have the estimate $\|\partial_x^\alpha u\|_{I\times K} \le (1/k)\|\partial_x^\alpha f\|_{I\times K}$; this proves the a-priori estimate $\|u\|_{I\times,K\rho} \le (1/k)\|f\|_{I\times,K\rho}$. Thus, we have

$$\|u\|_{I\times,K\rho} \le \frac{1}{k}\|(t\partial_t + k)u\|_{I\times,K\rho}. \tag{5.2.5}$$

Since (5.2.4) implies

$$\|(t\partial_t + k)^m u\|_{I\times,K\rho} \ll \frac{2M_0 A}{(1-\rho/R)^a},$$

by using (5.2.5) $(m-i)$-times we have the result (5.2.3). □

5.3. Proof of Theorem 5.0.1

Now, let us give a proof of Theorem 5.0.1. As in subsection 5.1 we set

$$w_k(t,x) = \frac{1}{k!}\partial_t^k u(t,x), \quad k = 1, 2, \dots . \tag{5.3.1}$$

Let K be a compact subset of V. Our aim is to show the condition:

$$\sum_{k\ge 1} \|w_k\|_{I\times K,\rho} t^k \in \mathbb{C}\{t,\rho\}. \tag{5.3.2}$$

We know that $w_k(t,x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(K))$ $(k = 1, 2, \dots)$ satisfy the formula (5.1.4); moreover we may suppose that the coefficients of (5.1.4) satisfy the properties in Lemma 5.1.2. We take a $\mu \geq \max\{n, m\}$; let $k_0 \in \mathbb{N}^*$ and $M_0 > 0$ be the constants in Lemma 5.2.1 (we may suppose that $k_0 \geq 2$ holds), and set $\mathscr{S} = \{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^n\, ;\, j = 0, 1, \ldots, m \text{ and } |\alpha| \leq m/\sigma\}$. Take $R > 0$ and $A_k > 0$ ($k = 1, 2, \ldots, k_0 - 1$) so that

$$\text{(5.3.3)} \qquad \begin{aligned} &\|(t\partial_t + k)^j \partial_x^\alpha w_k\|_{I \times K, \rho} \ll \frac{1}{k^{m-j-\sigma|\alpha|}} \frac{A_k}{(1 - \rho/R)^{\mu(2k-1)}} \\ &\text{for any } (j, \alpha) \in \mathscr{S}, \ \ k = 1, 2, \ldots, k_0 - 1. \end{aligned}$$

Since $R > 0$ can be taken as small as possible, we may suppose that

$$M_0 C \left[\frac{1}{(1 - R/R_0)^n} - 1 \right] \leq \frac{1}{2}$$

(where C is the constant in 3) of Lemma 5.1.2); so we can use the result in Lemma 5.2.2. We may assume also $0 < R < 1$.

Let us consider the functional equation with respect to (t, Y):

$$\text{(5.3.4)} \quad \begin{aligned} Y = &\sum_{k=1}^{k_0 - 1} \frac{A_k t^k}{(1 - \rho/R)^{\mu(2k-1)}} + \\ &+ \frac{2M_0}{(1 - \rho/R)^\mu} \left[\sum_{k \geq k_0} \frac{F_k t^k}{(1 - \rho/R)^{\mu(2k-2)}} + 2 \sum_{(j,\alpha) \in \Lambda} \frac{A_{j,\alpha} t}{(1 - \rho/R)^\mu} \beta Y \right. \\ &\left. + \sum_{p + |\nu| \geq 2, |\nu| \geq 1} \frac{B_{p,\nu} t^p}{(1 - R/R_0)^n} \frac{1}{(1 - \rho/R)^{\mu(2p + |\nu| - 2)}} (\beta Y)^{|\nu|} \right], \end{aligned}$$

where ρ is a parameter, and $\beta = (2\mu e/R)^m$. Since this is an analytic functional equation, the implicit function theorem tells us that for any $0 < \rho < R$ this equation (5.3.4) has a unique holomorphic solution $Y = Y(t)$ with $Y(0) = 0$ in a neighborhood of $t = 0$; if we expand this into Taylor series

$$Y = \sum_{k \geq 1} Y_k t^k,$$

we can easily see that

$$\text{(5.3.5)} \qquad Y_k \gg \frac{A_k}{(1 - \rho/R)^{\mu(2k-1)}}, \quad k = 1, 2, \ldots, k_0 - 1$$

and that Y_k (for $k \geq k_0$) is determined by the following recurrent formulas:

$$
\begin{aligned}
(5.3.6)\quad Y_k = &\frac{2M_0}{(1-\rho/R)^{\mu}}\Bigg[\frac{F_k}{(1-\rho/R)^{\mu(2k-2)}} + 2\sum_{(j,\alpha)\in\Lambda}\frac{A_{j,\alpha}}{(1-\rho/R)^{\mu}}\,\beta Y_{k-1} \\
&+ \sum_{2\leq p+|\nu|\leq k, |\nu|\geq 1}\frac{B_{p,\nu}}{(1-R/R_0)^n}\frac{1}{(1-\rho/R)^{\mu(2p+|\nu|-2)}}\times \\
&\times \sum_{|k^*|=k-p}\prod_{(j,\alpha)\in\Lambda}(\beta Y_{k_{j,\alpha}(1)})\times\cdots\times(\beta Y_{k_{j,\alpha}(\nu_{j,\alpha})})\Bigg]
\end{aligned}
$$

(where we used the same notations as in (5.1.4)). Moreover, we can see by induction on k that Y_k has the form

$$
(5.3.7)\qquad Y_k = \frac{C_k}{(1-\rho/R)^{\mu(2k-1)}}, \quad k = 1, 2, \ldots
$$

where C_k $(k \geq 1)$ are constants which are independent of the parameter ρ. We write $Y_k = Y_k(\rho)$ when we hope to emphasize that Y_k depends on ρ.

LEMMA 5.3.1. *For $k = 1, 2, \ldots$ we have*

$$
(5.3.8)_k\quad \|(t\partial_t + k)^j \partial_x^{\alpha} w_k\|_{I\times K,\rho} \ll \frac{1}{k^{m-j-\sigma|\alpha|}}\beta Y_k(\rho) \quad \text{for any } (j,\alpha)\in\mathscr{S}.
$$

PROOF. The cases $k = 1, \ldots, k_0 - 1$ are clear from (5.3.3), (5.3.5) and the condition $\beta > 1$. Let us show the general case by induction on k.

Let $k \geq k_0$; suppose that $(5.3.8)_p$ is already proved for $p = 1, 2, \ldots, k-1$. Then, by (5.1.4), the induction hypothesis and the condition $j + \sigma|\alpha| \leq m$ (for $(j,\alpha)\in\Lambda$) we have

$$
\begin{aligned}
(5.3.9)\quad &\|C(t\partial_t + k)w_k\|_{I\times K,\rho} \\
&\leq \frac{F_k}{(1-\rho/R_0)^n} + \sum_{(j,\alpha)\in\Lambda}\frac{A_{j,\alpha}}{(1-\rho/R_0)^n}\beta Y_{k-1} \\
&\quad + \sum_{(j,\alpha)\in\Lambda}\frac{A_{j,\alpha}}{(1-\rho/R_0)^n}\beta Y_{k-1} \\
&\quad + \sum_{2\leq p+|\nu|\leq k, |\nu|\geq 1}\frac{B_{p,\nu}}{(1-\rho/R_0)^n}\sum_{|k^*|=k-p}\prod_{(j,\alpha)\in\Lambda}\prod_{i=1}^{\nu_{j,\alpha}}\beta Y_{k_{j,\alpha}(i)}.
\end{aligned}
$$

Since $\mu(2|k^*| - |\nu|) = \mu(2k - 2p - |\nu|) = 2\mu(k - p - |\nu|) + \mu|\nu| \geq \mu|\nu| \geq \mu$ and

$$\frac{1}{(1-\rho/R_0)^n} \ll \frac{1}{(1-\rho/R)^{\mu}} \ll \frac{1}{(1-\rho/R)^{\mu(2k-2)}},$$
$$\frac{1}{(1-\rho/R_0)^n} \prod_{(j,\alpha)\in\Lambda} \prod_{i=1}^{\nu_{j,\alpha}} \beta Y_{k_{j,\alpha}(i)} \ll \frac{1}{(1-R/R_0)^n} \prod_{(j,\alpha)\in\Lambda} \prod_{i=1}^{\nu_{j,\alpha}} \beta Y_{k_{j,\alpha}(i)}$$

hold, by comparing (5.3.9) with (5.3.6) we have the estimate

$$\|C(t\partial_t + k)w_k\|_{I\times K,\rho} \ll \frac{(1-\rho/R)^{\mu}}{2M_0} Y_k = \frac{1}{2M_0} \frac{C_k}{(1-\rho/R)^{\mu(2k-2)}},$$

and by applying Lemma 5.2.2 to this estimate we have

$$(5.3.10) \quad \|(t\partial_t + k)^j w_k\|_{I\times K,\rho} \ll \frac{1}{k^{m-j}} \frac{C_k}{(1-\rho/R)^{\mu(2k-2)}}, \quad j = 0, 1, \ldots, m.$$

Then, $(5.3.8)_k$ is verified by the following: by applying Proposition 4.5 to (5.3.10) and by using $|\alpha| \leq \sigma|\alpha| \leq m \leq \mu$ we have

$$\begin{aligned}
(5.3.11) \quad & \|(t\partial_t + k)^j \partial_x^{\alpha} w_k\|_{I\times K,\rho} \\
&\ll \frac{1}{k^{m-j}} \frac{C_k e^{\sigma|\alpha|}/R^{|\alpha|}}{(1-\rho/R)^{\mu(2k-2)+\sigma|\alpha|}} \\
&\qquad \times (\mu(2k-2)+\sigma)^{\sigma} \cdots (\mu(2k-2)+\sigma|\alpha|)^{\sigma} \\
&\ll \frac{1}{k^{m-j}} \frac{C_k e^{\sigma|\alpha|}(2k\mu)^{\sigma|\alpha|}/R^{|\alpha|}}{(1-\rho/R)^{\mu(2k-2)+\sigma|\alpha|}} = \frac{k^{\sigma|\alpha|}}{k^{m-j}} \frac{C_k (2\mu e)^{\sigma|\alpha|}/R^{|\alpha|}}{(1-\rho/R)^{\mu(2k-2)+\sigma|\alpha|}} \\
&\ll \frac{k^{\sigma|\alpha|}}{k^{m-j}} \frac{C_k (2\mu e/R)^m}{(1-\rho/R)^{\mu(2k-2)+\mu}} = \frac{1}{k^{m-j-\sigma|\alpha|}} \beta Y_k. \ \square
\end{aligned}$$

Completion of the Proof of Theorem 5.0.1. By Lemma 5.3.1 we have

$$\sum_{k\geq 1} \|w_k\|_{I\times K,\rho} t^k \ll \sum_{k\geq 1} \frac{1}{k^m} \beta Y_k t^k \ll \beta \sum_{k\geq 1} Y_k t^k \in \mathbb{C}\{t\}$$

for any $0 < \rho < R$. This proves (5.3.2). $\square$

6. Proof of Theorem 3.6 in the Case $s > 1$

In this section we will give a proof of Theorem 3.6 in the case $s > 1$: the proof consists of two results stated below (Theorem 6.1 and Proposition 6.3).

Let us state the first result (Theorem 6.1). Let $m \in \mathbb{N}^*$, $L \in \mathbb{N}^*$, Λ be a subset of $\{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^n\,;\, j < m, |\alpha| \le L\}$, $d = \#\Lambda$ (the cardinal of Λ), and let us consider the following nonlinear partial differential equation

$$C(t\partial_t, x)u = F\big(t, x, \Theta u\big) \quad \text{with } \Theta u = \{(t\partial_t)^j \partial_x^\alpha u\}_{(j,\alpha)\in\Lambda} \tag{6.1}$$

where

$$C(\lambda, x) = \lambda^m - c_{m-1}(x)\lambda^{m-1} - \cdots - c_1(x)\lambda - c_0(x).$$

We denote by $z = \{z_{j,\alpha}\}_{(j,\alpha)\in\Lambda}$ the variable in $\mathbb{R}^d$ (which corresponds to $\Theta u = \{(t\partial_t)^j \partial_x^\alpha u\}_{(j,\alpha)\in\Lambda}$). Let Ω be an open subset of $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d$, and let $F(t,x,z)$ be a C^∞ function on Ω. Let $s_1 \ge 1$, $\sigma \ge 1$ and $s_2 \ge 1$, let $I = [0,T]$ (with $T > 0$), and let V be an open subset of $\mathbb{R}^n$. The main assumptions in this section are as follows.

- b$_1$) $L \ge m \ge 1$, $s_1 \ge 1$ and $\sigma \ge s_2 \ge 1$.
- b$_2$) Λ is a subset of $\{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^n\,;\, j < m, |\alpha| \le L\}$.
- b$_3$) $c_j(x) \in \mathcal{E}^{\{\sigma\}}(V)$ $j = 0, 1, \ldots, m-1$.
- b$_4$) $F(t,x,z) \in \mathcal{E}^{\{s_1,\sigma,s_2\}}(\Omega)$.
- b$_5$) $u(t,x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(V))$ is a solution of (6.1) on $I \times V$; this involves the property: $(t,x) \in I \times V \Longrightarrow (t, x, \Theta u(t,x)) \in \Omega$.

We set

$$q_{p,\nu} = ord_t\Big(\frac{\partial^{p+|\nu|}F}{\partial t^p \partial z^\nu}(t, x, \Theta u(t,x)), V\Big), \quad p + |\nu| \ge 1 \tag{6.2}$$

and set
(6.3)

$$s_0 = 1 + \max\left[0, \sup_{p+|\nu|\ge 1, |\nu|\ge 1}\left(\max_{(j,\alpha)\in\Lambda_\nu}\left(\frac{j + \sigma|\alpha| - m}{p + |\nu| + \min\{q_{p,\nu}, m-j\} - 1}\right)\right)\right],$$

where $\Lambda_\nu = \{(j,\alpha) \in \Lambda\,;\, \nu_{j,\alpha} > 0\}$ for $\nu = \{\nu_{j,\alpha}\}_{(j,\alpha)\in\Lambda} \in \mathbb{N}^d$. Then the first result is stated as follows.

Theorem 6.1. *Suppose the conditions $b_1) \sim b_5)$ and*

$$\frac{\partial F}{\partial z_{j,\alpha}}(t,x,\Theta u)\Big|_{t=0} \equiv 0 \ \ \textit{on } V \ \ \textit{for any } (j,\alpha)\in\Lambda; \tag{6.4}$$

then we have $u(t,x)\in\mathcal{E}^{\{s,\sigma\}}(I\times V)$ for any $s\geq \max\{s_0,s_1,s_2\}$.

By (6.4) we see that if $p=0$ and $|\nu|=1$ we have $q_{p,\nu}\geq 1$: therefore, in the formula (6.3) we have $p+|\nu|+\min\{q_{p,\nu},m-j\}-1\geq 1$. The proof of this result (Theorem 6.1) will be given later (in section 7); let us admit this for a while.

Next, let us state the second result (Proposition 6.3). We set

$$q^*_{j,\alpha} = ord_t\Big(\frac{\partial F}{\partial z_{j,\alpha}}(t,x,\Theta u(t,x)),V\Big), \quad (j,\alpha)\in\Lambda \tag{6.5}$$

(here we used the notation $q^*_{j,\alpha}$ instead of $q_{j,\alpha}$ for avoiding a confusion with $q_{p,\nu}$) and set

$$s_0(F) = 1+\max\left[0,\ \max_{(j,\alpha)\in\Lambda,|\alpha|>0}\left(\frac{j+\sigma|\alpha|-m}{\min\{q^*_{j,\alpha},\ m-j\}}\right)\right]. \tag{6.6}$$

Then we have

Proposition 6.2. *If the condition*

$$q_{p,\nu}\geq \max\{m-p,0\} \quad \textit{for } p+|\nu|\geq 2, |\nu|\geq 1 \tag{6.7}$$

holds, we have $s_0(F)=s_0$.

Proof. We note that if $|\nu|=1$ we have $\Lambda_\nu=\{(j_0,\alpha_0)\}$ for some $(j_0,\alpha_0)\in\Lambda$ and $q_{0,\nu}=q^*_{j_0,\alpha_0}$: this implies that

$$s_0(F) = 1+\max\left[0,\max_{|\nu|=1}\left(\max_{(j,\alpha)\in\Lambda_\nu}\frac{j+\sigma|\alpha|-m}{\min\{q_{0,\nu},m-j\}}\right)\right].$$

Therefore, to show $s_0(F)=s_0$ it is sufficient to prove the following result:

$$\frac{j+\sigma|\alpha|-m}{p+|\nu|+\min\{q_{p,\nu},m-j\}-1}\leq s_0(F)-1 \tag{6.8}$$
$$\text{for any } p+|\nu|\geq 2,\ |\nu|\geq 1 \text{ and } (j,\alpha)\in\Lambda_\nu.$$

Let us show this now. Take any (p,ν) with $p+|\nu| \geq 2$ and $|\nu| \geq 1$; then take any $(j,\alpha) \in \Lambda_\nu$. By (6.7) we have $q_{p,\nu} \geq \max\{m-p,0\}$.

If $p \geq m$ holds, we have $p+|\nu|+\min\{q_{p,\nu}, m-j\}-1 \geq p+|\nu|+0-1 \geq p \geq m \geq m-j$ and so

$$\frac{j+\sigma|\alpha|-m}{p+|\nu|+\min\{q_{p,\nu}, m-j\}-1} \leq \frac{j+\sigma|\alpha|-m}{m-j} \leq s_0(F)-1. \tag{6.9}$$

If $j \leq p < m$ we have $q_{p,\nu} \geq \max\{m-p,0\} = m-p$ and $\min\{q_{p,\nu}, m-j\} \geq \min\{m-p, m-j\} = m-p$: therefore we have $p+|\nu|+\min\{q_{p,\nu}, m-j\}-1 \geq p+|\nu|+(m-p)-1 \geq m \geq m-j$ and so we have (6.8) from (6.9). In the case $p < j$ we can show (6.8) in the same way. □

PROOF OF THEOREM 3.6. In the situation of Theorem 3.6 we have the condition (3.23) which yields the property (6.7). Therefore, by Theorem 6.1 and Proposition 6.2 we have the result: $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}(I \times V)$ for any $s \geq \max\{s_0(F), s_1, s_2\}$. □

Thus, to complete the proof of Theorem 3.6 it is sufficient to show Theorem 6.1.

7. Proof of Theorem 6.1

In this section we will give a proof of Theorem 6.1. Since the case $s=1$ is already proved in Theorem 5.0.1, we may confine ourselves to the case

$$s > 1. \tag{7.0.1}$$

The idea of the proof in this section comes from the proof of Maillet's type theorem in Gérard-Tahara [3].

7.1. Preliminaries

Let $u(t,x)$ be a solution of (6.1) given in b_5) and set

$$w_k(t,x) = \frac{1}{k!}\partial_t^k u(t,x), \quad k = 1,2,\ldots. \tag{7.1.1}$$

Take any $s > 1$ satisfying $s \geq \max\{s_0, s_1, s_2\}$, and fix it. To prove Theorem 6.1 we must show the following assertion: for any compact subset K of V

we have the condition:

$$\sum_{k\geq 1} \frac{1}{(k-1)!^{s-1}} \|w_k\|_{I\times K,\rho}\, t^k \in \mathbb{C}\{t,\rho\}. \tag{7.1.2}$$

Before the proof of this asserion, let us present some preparatory discussions which are needed in the proof of (7.1.2).

First, we note: by the calculation in subsection 5.1 we see that $w_k(t,x)$ (for $k \geq 1$) satisfies the equation

$$\begin{aligned}(7.1.3)\ & C(t\partial_t + k, x)w_k \\ &= f_k(t,x) + \sum_{1\leq p+|\nu|\leq k, |\nu|\geq 1} a_{p,\nu}(t,x) \times \\ &\qquad \times \sum_{|k^*|=k-p} \prod_{(j,\alpha)\in\Lambda} \Bigg[\frac{1}{k_{j,\alpha}(1)!} \partial_t^{k_{j,\alpha}(1)} (t\partial_t)^j \partial_x^\alpha u \times \\ &\qquad\qquad \times \cdots \times \frac{1}{k_{j,\alpha}(\nu_{j,\alpha})!} \partial_t^{k_{j,\alpha}(\nu_{j,\alpha})} (t\partial_t)^j \partial_x^\alpha u \Bigg],\end{aligned}$$

where

$$\begin{aligned} f_k(t,x) &= \frac{1}{k!} F^{(k,0,0)}(t,x,\Theta u(t,x)) \quad (k\geq 1), \\ a_{p,\nu}(t,x) &= \frac{1}{p!\nu!} F^{(p,0,\nu)}(t,x,\Theta u(t,x)) \quad (p+|\nu|\geq 1,\ |\nu|\geq 1). \end{aligned}$$

By the definition, we have $q_{k,0} = ord_t(f_k(t,x),V)$ $(k \geq 1)$ and $q_{p,\nu} = ord_t(a_{p,\nu}(t,x),V)$ $(p+|\nu| \geq 1$ and $|\nu| \geq 1)$. Moreover, by the same argument as in Lemma 5.1.2 we have

Lemma 7.1.1. *For any compact subset K of V, there are constants $F_k \geq 0$ $(k \geq 1)$, $R_0 > 0$, $C \geq 0$ and $A_{p,\nu} \geq 0$ $(p+|\nu| \geq 1,\ |\nu| \geq 1)$ which satisfy the following properties:*

1) $\|f_k\|_{I\times K,\rho} \ll \dfrac{F_k}{(1-\rho/R_0)^n}$ *and* $\displaystyle\sum_{k\geq 1} \frac{F_k}{k!^{s-1}} t^k \in \mathbb{C}\{t\}$;

2) $\|c_i\|_{K,\rho} \ll \dfrac{C}{(1-\rho/R_0)^n}$, $\quad i = 0,\dots,m-1$;

3) $\|a_{p,\nu}\|_{I\times K,\rho} \ll \dfrac{A_{p,\nu}}{(1-\rho/R_0)^n}$ *and* $\displaystyle\sum_{p+|\nu|\geq 1, |\nu|\geq 1} \frac{A_{p,\nu}}{(p+|\nu|)!^{s-1}} t^p z^\nu \in \mathbb{C}\{t,z\}$.

For simplicity we write

$$(7.1.4)\quad g_k(p,\nu) = a_{p,\nu}(t,x) \sum_{|k^*|=k-p} \prod_{(j,\alpha)\in\Lambda} \Bigl[\frac{1}{k_{j,\alpha}(1)!} \partial_t^{k_{j,\alpha}(1)} (t\partial_t)^j \partial_x^\alpha u \times \times \cdots \times \frac{1}{k_{j,\alpha}(\nu_{j,\alpha})!} \partial_t^{k_{j,\alpha}(\nu_{j,\alpha})} (t\partial_t)^j \partial_x^\alpha u \Bigr] :$$

then our equation (7.1.3) is written as

$$(7.1.5)\qquad C(t\partial_t + k, x) w_k = f_k(t,x) + \sum_{1\le p+|\nu|\le k, |\nu|\ge 1} g_k(p,\nu).$$

In the forthcoming subsections, we will give estimates of $g_k(p,\nu)$: the following lemma (Lemma 7.1.2 given below) will play a very important role in the estimation of $g_k(p,\nu)$.

For integers $0 \le r < k$ we define the constants $C^{(0)}_{k,h}$ $(1 \le h \le k)$ and $C^{(+)}_{k,r,h}$ $(0 \le h \le r < k)$ by the coefficients of

$$x(x-1)\cdots(x-k+1) = \sum_{h=1}^{k} C^{(0)}_{k,h} x^h,$$
$$(x-k+1)(x-k+2)\cdots(x-k+r) = \sum_{h=0}^{r} C^{(+)}_{k,r,h} x^h.$$

Lemma 7.1.2. *For* $0 \le r < k$ *we have*

$$(7.1.6)\qquad t^k\partial_t^k = \sum_{h=1}^{k} C^{(0)}_{k,h} (t\partial_t)^h \quad \textit{and} \quad t^r\partial_t^r = \sum_{h=0}^{r} C^{(+)}_{k,r,h} (t\partial_t + k - r)^h.$$

Moreover, we have the following estimates:

$$(7.1.7)\qquad \frac{1}{k!}\sum_{h=1}^{k} |C^{(0)}_{k,h}| = 1 \quad \textit{and} \quad \frac{(k-r)!}{k!}\sum_{h=0}^{r} |C^{(+)}_{k,r,h}|(k-r)^h \le 2^r.$$

Proof. (7.1.6) follows from

$$t^k\partial_t^k = t\partial_t(t\partial_t - 1)\cdots(t\partial_t - k + 1), \quad \text{and}$$
$$t^r\partial_t^r = t\partial_t(t\partial_t - 1)\cdots(t\partial_t - r + 1)$$
$$= (\vartheta - k + r)(\vartheta - k + r - 1)\cdots(\vartheta - k + 1)$$

with $\vartheta = (t\partial_t + k - r)$. Let us show (7.1.7). By the definition of $C^{(0)}_{k,h}$ we have

$$\sum_{h=1}^{k} |C^{(0)}_{k,h}| x^h = x(x+1)\cdots(x+k-1);$$

therefore by setting $x = 1$ we have the former half of (7.1.7). By the definition of $C^{(+)}_{k,r,h}$ we see:

$$C^{(+)}_{k,r,h} = (-1)^{r-h} \sum_{k-r \le p_1 < \cdots < p_{r-h} \le k-1} p_1 p_2 \cdots p_{r-h}$$

and so

$$|C^{(+)}_{k,r,h}| \le \sum_{k-r \le p_1 < \cdots < p_{r-h} \le k-1} \frac{(k-1)!}{(k-r+h-1)!} \le \binom{r}{r-h} \frac{k!}{(k-r+h)!}.$$

Thus we obtain

$$\begin{aligned} &\frac{(k-r)!}{k!} \sum_{h=0}^{r} |C^{(+)}_{k,r,h}| (k-r)^h \\ &\quad \le \frac{(k-r)!}{k!} \sum_{h=0}^{r} \binom{r}{r-h} \frac{k!}{(k-r+h)!} (k-r)^h \\ &\quad = \sum_{h=0}^{r} \binom{r}{r-h} \frac{(k-r)^h}{(k-r+h)\cdots(k-r+1)} \le \sum_{h=0}^{r} \binom{r}{r-h} = 2^r. \ \square \end{aligned}$$

Now, let us take a sufficiently large $\mu \in \mathbb{N}$ such that $\mu > m$ and $j + \sigma|\alpha| \le \mu$ for all $(j, \alpha) \in \Lambda$: take also a sufficiently large $N \in \mathbb{N}$ so that

$$s - 1 \ge \frac{\mu - m}{N - 1} \quad \text{and} \quad N \ge 2. \tag{7.1.8}$$

7.2. Estimate of $g_k(p, \nu)$ in the case $p + |\nu| \ge N$

In this subsection, we will give an estimate of $g_k(p, \nu)$ in the case $p + |\nu| \ge N$. Let $k \ge N$ and suppose the following estimates;

$$|||(t\partial_t + h)^j \partial_x^\alpha w_h|||_{I \times K, \rho} \ll \frac{(h-1)!^{s-1}}{h^{\mu - j - \sigma|\alpha|}} \beta Y_h(\rho) \tag{7.2.1}$$

$$\text{for } j = 0, 1, \dots, m,\ |\alpha| \le L \text{ and } h = 1, 2, \dots, k-1,$$

where $\beta > 0$ and

$$Y_h(\rho) = \frac{C_h}{(1-\rho/R)^{\mu(2h-1)}}, \quad h = 1, 2, \dots, k-1 \tag{7.2.2}$$

for some $C_h \geq 0$ $(h = 1, 2, \dots, k-1)$. Then we have

Proposition 7.2.1. *Suppose the conditions* (7.1.8), $k \geq N$, (7.2.1) *and* (7.2,2). *If* $p + |\nu| \geq N$, *we have the estimate*

$$\begin{aligned} &\|g_k(p,\nu)\|_{I\times K,\rho} \\ &\quad \ll \frac{(k-1)!^{s-1}}{k^{\mu-m}} \frac{A_{p,\nu} e^{N(s-1)}}{(p+|\nu|-N)!^{s-1}} \frac{(p+|\nu|)^{\mu-\mu_\nu}}{(1-\rho/R_0)^n} \times \\ &\qquad \times \sum_{|k^*|=k-p} \prod_{(j,\alpha)\in\Lambda} \Big[\beta Y_{k_{j,\alpha}(1)} \times \cdots \times \beta Y_{k_{j,\alpha}(\nu_{j,\alpha})}\Big], \end{aligned} \tag{7.2.3}$$

where $\mu_\nu = \max\{j + \sigma|\alpha|\,;\, \nu_{j,\alpha} > 0\}$ (*for* $\nu = \{\nu_{j,\alpha}\}_{(j,\alpha)\in\Lambda}$ *with* $|\nu| \geq 1$).

Proof. By (7.1.4), Lemma 7.1.1, (7.2.1) and the definition of μ_ν we have

$$\begin{aligned} &\|g_k(p,\nu)\|_{I\times K,\rho} \\ &\quad \ll \frac{A_{p,\nu}}{(1-\rho/R_0)^n} \sum_{|k^*|=k-p} \prod_{(j,\alpha)\in\Lambda} \Big[\frac{(k_{j,\alpha}(1)-1)!^{s-1}}{k_{j,\alpha}(1)^{\mu-\mu_\nu}} \beta Y_{k_{j,\alpha}(1)} \times \\ &\qquad \times \cdots \times \frac{(k_{j,\alpha}(\nu_{j,\alpha})-1)!^{s-1}}{k_{j,\alpha}(\nu_{j,\alpha})^{\mu-\mu_\nu}} \beta Y_{k_{j,\alpha}(\nu_{j,\alpha})}\Big]. \end{aligned} \tag{7.2.4}$$

Let us note:

Lemma 7.2.2. *In the situation in* (7.2.4) *we have*

$$\prod_{(j,\alpha)\in\Lambda} \prod_{i=1}^{\nu_{j,\alpha}} (k_{j,\alpha}(i)-1)!^{s-1} \leq (k-p-|\nu|)!^{s-1}, \tag{7.2.5}$$

$$\prod_{(j,\alpha)\in\Lambda} \prod_{i=1}^{\nu_{j,\alpha}} \frac{1}{k_{j,\alpha}(i)^{\mu-\mu_\nu}} \leq \frac{(p+|\nu|)^{\mu-\mu_\nu}}{k^{\mu-\mu_\nu}}. \tag{7.2.6}$$

Proof of Lemma 7.2.2. (7.2.5) is clear from the condition $|k^*| = k-p$ (the definition of $|k^*|$ is given under the formula (5.1.3)). Let us show (7.2.6). Since $k_{j,\alpha}(i) \geq 1$ we have

$$k_{i,\beta}(p) \leq \prod_{(j,\alpha)\in\Lambda} \prod_{i=1}^{\nu_{j,\alpha}} k_{j,\alpha}(i) \quad (\text{for } (i,\beta)\in\Lambda \text{ and } p = 1,\dots,\nu_{i,\beta});$$

therefore

$$k = p + |k^*| \leq p + |\nu| \times \prod_{(j,\alpha)\in\Lambda} \prod_{i=1}^{\nu_{j,\alpha}} k_{j,\alpha}(i) \leq (p+|\nu|) \times \prod_{(j,\alpha)\in\Lambda} \prod_{i=1}^{\nu_{j,\alpha}} k_{j,\alpha}(i)$$

which yields (7.2.6). □

Hence, by applying this to (7.2.4) we have

$$\begin{aligned} (7.2.7) \qquad & \| g_k(p,\nu) \|_{I\times K,\rho} \\ & \ll \frac{A_{p,\nu}}{(1-\rho/R_0)^n} \frac{(k-p-|\nu|)!^{s-1}(p+|\nu|)^{\mu-\mu_\nu}}{k^{\mu-\mu_\nu}} \times \\ & \quad \times \sum_{|k^*|=k-p} \prod_{(j,\alpha)\in\Lambda} \Big[\beta Y_{k_{j,\alpha}(1)} \times \cdots \times \beta Y_{k_{j,\alpha}(\nu_{j,\alpha})}\Big]. \end{aligned}$$

Since the Stirling's formula (4.12) gives

$$\begin{aligned} (7.2.8) \qquad \frac{(k-N)!}{(k-1)!} &\leq \frac{\sqrt{2\pi}(k-N+1)^{k-N+1/2}e^{-(k-N+1)+1/2}}{\sqrt{2\pi}k^{k-1/2}e^{-k}} \\ &= \Big(\frac{k-N+1}{k}\Big)^{k-N+1/2} \frac{e^{N-1/2}}{k^{N-1}} \leq \frac{e^N}{k^{N-1}}, \end{aligned}$$

by using this and the condition (7.1.8), that is, $(N-1)(s-1) \geq \mu - m \geq \mu_\nu - m$ we have

$$\begin{aligned} \frac{(k-p-|\nu|)!^{s-1}}{k^{\mu-\mu_\nu}} &\leq \frac{1}{(p+|\nu|-N)!^{s-1}} \frac{(k-N)!^{s-1}}{k^{\mu-\mu_\nu}} \\ &= \frac{(k-1)!^{s-1}}{k^{\mu-m}} \frac{1}{(p+|\nu|-N)!^{s-1}} \times \frac{(k-N)!^{s-1}}{(k-1)!^{s-1}k^{m-\mu_\nu}} \\ &\leq \frac{(k-1)!^{s-1}}{k^{\mu-m}} \frac{1}{(p+|\nu|-N)!^{s-1}} \times \frac{e^{N(s-1)}}{k^{(N-1)(s-1)}k^{m-\mu_\nu}} \\ &\leq \frac{(k-1)!^{s-1}}{k^{\mu-m}} \frac{1}{(p+|\nu|-N)!^{s-1}} \times e^{N(s-1)}. \end{aligned}$$

Thus, by applying this to (7.2.7) we can obtain (7.2.3). □

7.3. In the case $p+|\nu|<N$

Next, we must give an estimate of $g_k(p,\nu)$ in the case $p+|\nu|<N$: we note that $g_k(p,\nu)$ is expressed in the form

$$(7.3.1)\quad g_k(p,\nu)=\frac{a_{p,\nu}(t,x)}{t^{|r^*|}}\times \times\sum_{|k^*|=k-p}\prod_{(j,\alpha)\in\Lambda}\Bigl[t^{r_{j,\alpha}(1)}\frac{1}{k_{j,\alpha}(1)!}\partial_t^{k_{j,\alpha}(1)}(t\partial_t)^j\partial_x^\alpha u\times \times\cdots\times t^{r_{j,\alpha}(\nu_{j,\alpha})}\frac{1}{k_{j,\alpha}(\nu_{j,\alpha})!}\partial_t^{k_{j,\alpha}(\nu_{j,\alpha})}(t\partial_t)^j\partial_x^\alpha u\Bigr]$$

for any $r_{j,\alpha}(i)\in\mathbb{N}$ ($(j,\alpha)\in\Lambda$ and $1\le i\le\nu_{j,\alpha}$), where

$$|r^*|=\sum_{(j,\alpha)\in\Lambda}\bigl(r_{j,\alpha}(1)+\cdots+r_{j,\alpha}(\nu_{j,\alpha})\bigr).$$

In this context, we write $k^*=\{k_{j,\alpha}(i)\,;\,(j,\alpha)\in\Lambda, 1\le i\le\nu_{j,\alpha}\}\in(\mathbb{N}^*)^{|\nu|}$ and $r^*=\{r_{j,\alpha}(i)\,;\,(j,\alpha)\in\Lambda, 1\le i\le\nu_{j,\alpha}\}\in\mathbb{N}^{|\nu|}$.

Let k be sufficiently large. In order to estimate the term $g_k(p,\nu)$ by induction on k, we suppose the conditions (7.2.1) and (7.2.2): since $u(t,x)\in C^\infty(I,\mathcal{E}^{\{\sigma\}}(K))$ is assumed, we may suppose also that

$$(7.3.2)\quad \|(t\partial_t)^j\partial_x^\alpha u\|_{I\times K,\rho}\ll\frac{A^*}{(1-\rho/R_0)^n},\quad j=0,1,\dots,m \text{ and } |\alpha|\le L$$

for some $A^*\ge 0$, where $R_0>0$ can be the same as in Lemma 7.1.1.

LEMMA 7.3.1. *Let $r\in\mathbb{N}$ be such that $0\le r\le\min\{k_{j,\alpha}(i),m-j\}$ (this means that we have three cases: $0\le r<k_{j,\alpha}(i)\le m-j$, $r=k_{j,\alpha}(i)\le m-j$ and $0\le r\le m-j<k_{j,\alpha}(i)$). Suppose that $k_{j,\alpha}(i)-r\le k-1$ holds. Then we have the following results.*

(1) *If $0\le r<k_{j,\alpha}(i)\le m-j$ or $0\le r\le m-j<k_{j,\alpha}(i)$ holds, we have*

$$(7.3.3)\quad \Bigl\|t^r\frac{1}{k_{j,\alpha}(i)!}\partial_t^{k_{j,\alpha}(i)}(t\partial_t)^j\partial_x^\alpha u\Bigr\|_{I\times K,\rho} \ll\frac{(k_{j,\alpha}(i)-r-1)!^{s-1}}{(k_{j,\alpha}(i)-r)^{\mu-j-\sigma|\alpha|}}\times 2^r\beta Y_{k_{j,\alpha}(i)-r}.$$

(2) *If* $r = k_{j,\alpha}(i) \le m - j$, *we have*

$$\Big\|\Big| t^r \frac{1}{k_{j,\alpha}(i)!} \partial_t^{k_{j,\alpha}(i)} (t\partial_t)^j \partial_x^\alpha u \Big\|\Big|_{I\times K,\rho} \ll \frac{A^*}{(1-\rho/R_0)^n}. \tag{7.3.4}$$

Proof. Let us show (1). In this case, we have $0 \le r < k_{j,\alpha}(i)$ and so by (7.1.6) we have

$$\begin{aligned} t^r \frac{1}{k_{j,\alpha}(i)!} \partial_t^{k_{j,\alpha}(i)} (t\partial_t)^j \partial_x^\alpha u &= \frac{1}{k_{j,\alpha}(i)!} t^r \partial_t^r \times \partial_t^{k_{j,\alpha}(i)-r} (t\partial_t)^j \partial_x^\alpha u \\ &= \frac{(k_{j,\alpha}(i)-r)!}{k_{j,\alpha}(i)!} t^r \partial_t^r \times (t\partial_t + k_{j,\alpha}(i) - r)^j \partial_x^\alpha w_{k_{j,\alpha}(i)-r} \\ &= \frac{(k_{j,\alpha}(i)-r)!}{k_{j,\alpha}(i)!} \sum_{h=0}^{r} C^{(+)}_{k_{j,\alpha}(i),r,h} (t\partial_t + k_{j,\alpha}(i) - r)^{j+h} \partial_x^\alpha w_{k_{j,\alpha}(i)-r}. \end{aligned}$$

Since $k_{j,\alpha}(i) - r \le k - 1$ and $j + h \le m$ (for $h = 0, 1, \dots, r$) are assumed, we can apply (7.2.1) to the term $(t\partial_t + k_{j,\alpha}(i) - r)^{j+h} \partial_x^\alpha w_{k_{j,\alpha}(i)-r}$ and we have

$$\begin{aligned} &\Big\|\Big| t^r \frac{1}{k_{j,\alpha}(i)!} \partial_t^{k_{j,\alpha}(i)} (t\partial_t)^j \partial_x^\alpha u \Big\|\Big|_{I\times K,\rho} \\ &\ll \frac{(k_{j,\alpha}(i)-r)!}{k_{j,\alpha}(i)!} \sum_{h=0}^{r} |C^{(+)}_{k_{j,\alpha}(i),r,h}| \frac{(k_{j,\alpha}(i)-r-1)!^{s-1}}{(k_{j,\alpha}(i)-r)^{\mu-j-h-\sigma|\alpha|}} \beta Y_{k_{j,\alpha}(i)-r} \\ &\ll 2^r \times \frac{(k_{j,\alpha}(i)-r-1)!^{s-1}}{(k_{j,\alpha}(i)-r)^{\mu-j-\sigma|\alpha|}} \beta Y_{k_{j,\alpha}(i)-r}; \end{aligned}$$

in the last inequality we have used (7.1.7). This proves (7.3.3).

Next, let us show (2). In this case, by (7.1.6) and $r = k_{j,\alpha}(i)$ we have

$$t^r \frac{1}{k_{j,\alpha}(i)!} \partial_t^{k_{j,\alpha}(i)} (t\partial_t)^j \partial_x^\alpha u = \frac{1}{k_{j,\alpha}(i)!} \sum_{h=0}^{k_{j,\alpha}(i)} C^{(0)}_{k_{j,\alpha}(i),h} (t\partial_t)^{j+h} \partial_x^\alpha u.$$

Since $j + h \le m$ (for $h = 0, 1, \dots, k_{j,\alpha}(i)$) is assumed, we can apply (7.3.2) to the term $(t\partial_t)^{j+h} \partial_x^\alpha u$ and we have

$$\begin{aligned} &\Big\|\Big| t^r \frac{1}{k_{j,\alpha}(i)!} \partial_t^{k_{j,\alpha}(i)} (t\partial_t)^j \partial_x^\alpha u \Big\|\Big|_{I\times K,\rho} \\ &\ll \frac{1}{k_{j,\alpha}(i)!} \sum_{h=0}^{k_{j,\alpha}(i)} |C^{(0)}_{k_{j,\alpha}(i),h}| \frac{A^*}{(1-\rho/R_0)^n} \ll \frac{A^*}{(1-\rho/R_0)^n}; \end{aligned}$$

in the last inequality we have used (7.1.7). This proves (7.3.4). □

By using Lemma 7.3.1 $|\nu|$-times we have

Lemma 7.3.2. *In the notation in* (7.3.1), *we let* $1 \leq p + |\nu| < N$ *with* $|\nu| \geq 1$, *and let* $0 \leq r_{j,\alpha}(i) \leq \min\{k_{j,\alpha}(i), m-j\}$ $((j,\alpha) \in \Lambda$ *and* $1 \leq i \leq \nu_{j,\alpha})$. *Suppose that* $k \geq (m+1)N$ *and* $k = |k^*| + p$ *hold. Suppose also that* $k_{j,\alpha}(i) - r_{j,\alpha}(i) \leq k-1$ *holds for all* (j,α,i). *Then we have the estimate*

$$
(7.3.5) \qquad \Big|\Big|\Big| t^{|r^*|} \prod_{(j,\alpha)\in\Lambda} \prod_{i=1}^{\nu_{j,\alpha}} \frac{1}{k_{j,\alpha}(i)!} \partial_t^{k_{j,\alpha}(i)} (t\partial_t)^j \partial_x^\alpha u \Big|\Big|\Big|_{I\times K,\rho}
$$
$$
\ll \frac{(k-|r^*|-p-|\nu|_+)!^{s-1}(p+(m+1)|\nu|)^{\mu-\mu^*}}{(k-|r^*|)^{\mu-\mu^*}} \times
$$
$$
\times \prod_{(j,\alpha,i)\in J_0} \frac{A^*}{(1-\rho/R_0)^n} \times \prod_{(j,\alpha,i)\in J_+} 2^{r_{j,\alpha}(i)} \beta Y_{k_{j,\alpha}(i)-r_{j,\alpha}(i)},
$$

where

$$
J_0 = \{(j,\alpha,i)\,;\, r_{j,\alpha}(i) = k_{j,\alpha}(i)\}, \quad J_+ = \{(j,\alpha,i)\,;\, r_{j,\alpha}(i) < k_{j,\alpha}(i)\},
$$
$$
|\nu|_0 = \textit{the cardinal of } J_0, \quad |\nu|_+ = \textit{the cardinal of } J_+,
$$
$$
\mu^* = \max_{(j,\alpha,i)\in J^*} \big(j + \sigma|\alpha|\big) \quad \textit{with } J^* = \{(j,\alpha,i)\,;\, m-j < k_{j,\alpha}(i)\}.
$$

(*As is seen below, we have* $|\nu|_0 + |\nu|_+ = |\nu|$, $J_+ \supset J^* \neq \emptyset$, $k - |r^*| - p - |\nu|_+ \geq 1$ *and* $k - |r^*| \geq 1$; *we note also that* μ^* *is independent of* $r^* = \{r_{j,\alpha}(i)\,;\,(j,\alpha)\in\Lambda, 1 \leq i \leq \nu_{j,\alpha}\}$).

Remark 7.3.3. In the above lemma we have supposed that $k_{j,\alpha}(i) - r_{j,\alpha}(i) \leq k-1$ holds for all (j,α,i): but, in the following cases (i) and (ii) this condition is trivially satisfied:

case (i): $p + |\nu| \geq 2$,

case (ii): $p = 0$, $|\nu| = 1$ and $|r^*| \geq 1$.

The reason is as follows. (i) If $p + |\nu| \geq 2$, by the condition $|k^*| + p = k$ and $k_{j,\alpha}(i) \geq 1$ $((j,\alpha) \in \Lambda$ and $1 \leq i \leq \nu_{j,\alpha})$, we have the condition: $k_{j,\alpha}(i) \leq k-1$ holds for all (j,α,i). (ii) If $p = 0$, $|\nu| = 1$ and $|r^*| \geq 1$, then we have $k^* = k \in \mathbb{N} \setminus \{0\}$ and $r^* = r \in \mathbb{N}$; and so by the condition $|r^*| = r \geq 1$ we have $k - r \leq k-1$.

Proof of Lemma 7.3.2. First, let us see the conditions written in the last part of the lemma. Since $r_{j,\alpha}(i) \le k_{j,\alpha}(i)$ is assumed for all (j,α,i), the condition $|\nu|_0 + |\nu|_+ = |\nu|$ is clear. If $(j,\alpha,i) \in J^*$ holds, we have $r_{j,\alpha}(i) \le m - j < k_{j,\alpha}(i)$ and so we have $(j,\alpha,i) \in J_+$; this shows the condition $J^* \subset J_+$.

If $J^* = \emptyset$, we have $k_{j,\alpha}(i) \le m - j \le m$ for all (j,α,i); then by the condition $p + |\nu| < N$ we have $k = p + |k^*| \le p + m|\nu| \le m(p + |\nu|) < mN$. This contradicts the condition $k \ge (m+1)N$. Thus, we have proved the condition $J^* \ne \emptyset$.

Similarly, by the condition $r_{j,\alpha}(i) \le m - j \le m$ we have $|r^*| \le m|\nu| < mN$ and so $|r^*| + p + |\nu|_+ \le |r^*| + (p + |\nu|) < mN + N = (m+1)N \le k$; this proves the condition $k - |r^*| - p - |\nu|_+ \ge 1$. The condition $k - |r^*| \ge 1$ is also verified.

Now, let us show (7.3.5). By Lemma 7.3.1 we have

$$\begin{aligned}(7.3.6)\quad & \Big\|t^{|r^*|} \prod_{(j,\alpha)\in\Lambda} \prod_{i=1}^{\nu_{j,\alpha}} \frac{1}{k_{j,\alpha}(i)!} \partial_t^{k_{j,\alpha}(i)} (t\partial_t)^j \partial_x^\alpha u \Big\|_{I\times K,\rho} \\ & \ll \prod_{(j,\alpha,i)\in J_0} \frac{A^*}{(1-\rho/R_0)^n} \times \\ & \qquad \times \prod_{(j,\alpha,i)\in J_+} \frac{(k_{j,\alpha}(i) - r_{j,\alpha}(i) - 1)!^{s-1}}{(k_{j,\alpha}(i) - r_{j,\alpha}(i))^{\mu-j-\sigma|\alpha|}} 2^{r_{j,\alpha}(i)} \beta Y_{k_{j,\alpha}(i) - r_{j,\alpha}(i)}.\end{aligned}$$

Hence, if we know the inequalities

$$(7.3.7)\qquad \prod_{(j,\alpha,i)\in J_+} (k_{j,\alpha}(i) - r_{j,\alpha}(i) - 1)! \le (k - |r^*| - p - |\nu|_+)!,$$

$$(7.3.8)\qquad \prod_{(j,\alpha,i)\in J_+} \frac{1}{(k_{j,\alpha}(i) - r_{j,\alpha}(i))^{\mu-j-\sigma|\alpha|}} \le \frac{(p + (m+1)|\nu|)^{\mu-\mu^*}}{(k - |r^*|)^{\mu-\mu^*}},$$

by applying these to (7.3.6) we have the result (7.3.5).

Let us show (7.3.7) and (7.3.8). Since $k_{j,\alpha}(i) = r_{j,\alpha}(i)$ holds for all $(j,\alpha,i) \in J_0$, we have

$$\prod_{(j,\alpha,i)\in J_+} (k_{j,\alpha}(i) - r_{j,\alpha}(i) - 1)!$$

$$= \prod_{(j,\alpha,i)\in J_0} (k_{j,\alpha}(i) - r_{j,\alpha}(i))! \times \prod_{(j,\alpha,i)\in J_+} (k_{j,\alpha}(i) - r_{j,\alpha}(i) - 1)!$$
$$\leq (|k^*| - |r^*| - |\nu|_+)! = (k - p - |r^*| - |\nu|_+)! :$$

this proves (7.3.7). In order to rpove (7.3.8) we note that by the condition $J_+ \supset J^*$ and the definition of μ^* we have

$$\prod_{(j,\alpha,i)\in J_+} \frac{1}{(k_{j,\alpha}(i) - r_{j,\alpha}(i))^{\mu-j-\sigma|\alpha|}} \leq \prod_{(j,\alpha,i)\in J^*} \frac{1}{(k_{j,\alpha}(i) - r_{j,\alpha}(i))^{\mu-\mu^*}}. \tag{7.3.9}$$

Set

$$|k^*|_+ = \sum_{(j,\alpha,i)\in J_+} k_{j,\alpha}(i) \quad \text{and} \quad |r^*|_+ = \sum_{(j,\alpha,i)\in J_+} r_{j,\alpha}(i);$$

then we have $|k^*| - |r^*| = |k^*|_+ - |r^*|_+$. By the definition we see that $0 \leq r_{j,\alpha}(i) < k_{j,\alpha}(i) \leq m - j$ holds for all $(j, \alpha, i) \in J_+ \setminus J^*$. We have

$$\begin{aligned} k - |r^*| &= p + (|k^*| - |r^*|) = p + (|k^*|_+ - |r^*|_+) \\ &= p + \sum_{(j,\alpha,i)\in J_+\setminus J^*} (k_{j,\alpha}(i) - r_{j,\alpha}(i)) + \sum_{(j,\alpha,i)\in J^*} (k_{j,\alpha}(i) - r_{j,\alpha}(i)) \\ &\leq p + \#(J_+ \setminus J^*) \times m + \#J^* \times \prod_{(j,\alpha,i)\in J^*} (k_{j,\alpha}(i) - r_{j,\alpha}(i)) \\ &\leq p + |\nu| \times m + |\nu| \times \prod_{(j,\alpha,i)\in J^*} (k_{j,\alpha}(i) - r_{j,\alpha}(i)) \\ &\leq (p + (m+1)|\nu|) \times \prod_{(j,\alpha,i)\in J^*} (k_{j,\alpha}(i) - r_{j,\alpha}(i)) \end{aligned}$$

(where $\#A$ denotes the cardinal of the set A) and so we have

$$\prod_{(j,\alpha,i)\in J^*} \frac{1}{(k_{j,\alpha}(i) - r_{j,\alpha}(i))} \leq \frac{(p + (m+1)|\nu|)}{(k - |r^*|)}.$$

By applying this to (7.3.9) we have the result (7.3.8). □

In Lemma 7.3.2 we see that

$$|r^*| = \sum_{(j,\alpha)\in\Lambda}\sum_{i=1}^{\nu_{j,\alpha}} r_{j,\alpha}(i) \le \sum_{(j,\alpha)\in\Lambda}\sum_{i=1}^{\nu_{j,\alpha}} \min\{m-j, k_{j,\alpha}(i)\} \ (\le m|\nu| < mN).$$

Recall that $q_{p,\nu} = ord_t(a_{j,\alpha}(t,x), V)$ and so we see: if $|r^*| \le q_{p,\nu}$ holds we have the expression $a_{p,\nu}(t,x) = t^{|r^*|} b^*_{p,\nu}(t,x)$ for some $b^*_{p,\nu}(t,x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(V))$ and by Lemma 7.3.2 we have

$$\begin{aligned}(7.3.10)\quad & \|g(p,\nu)\|_{I\times K,\rho} \\ &\ll \|b^*_{p,\nu}\|_{I\times K,\rho} \times \frac{(k-|r^*|-p-|\nu|_+)!^{s-1}(p+(m+1)|\nu|)^{\mu-\mu^*}}{(k-|r^*|)^{\mu-\mu^*}} \times \\ &\quad \times \prod_{(j,\alpha,i)\in J_0} \frac{A^*}{(1-\rho/R_0)^n} \times \prod_{(j,\alpha,i)\in J_+} 2^{r_{j,\alpha}(i)} \beta Y_{k_{j,\alpha}(i)-r_{j,\alpha}(i)}.\end{aligned}$$

7.4. Estimate of $g_k(p,\nu)$ in the case $p+|\nu|<N$

Since $q_{p,\nu} = ord_t(a_{j,\alpha}(t,x), V)$, we have $a_{p,\nu}(t,x) = O(t^{q_{p,\nu}})$ (as $t \longrightarrow 0$): therefore, we can take constants $A^*_{p,\nu} \ge 0$ ($1 \le p+|\nu| < N$ and $|\nu| \ge 1$) and $R_0 > 0$ (as in Lemma 7.1.1) so that

$$(7.4.1)\quad \|t^{-q} a_{p,\nu}\|_{I\times K,\rho} \ll \frac{A^*_{p,\nu}}{(1-\rho/R_0)^n} \quad \text{for any } 0 \le q \le \min\{q_{p,\nu}, mN\}.$$

We take $k \ge (m+1)N$ and fix it: we suppose the conditions (7.2.1), (7.2.2), $\beta \ge 1$ and

$$(7.4.2)\qquad \frac{A^*}{(1-\rho/R_0)^n} \ll Y_1(\rho) \ll \frac{1}{2} Y_2(\rho) \ll \cdots \ll \frac{1}{2^{k-1}} Y_{k-1}(\rho).$$

Take any (p,ν) with $1 \le p+|\nu| < N$ and $|\nu| \ge 1$, and fix it. For these fixed p, $\nu = \{\nu_{j,\alpha}\}_{(j,\alpha)\in\Lambda}$ and k we denote by $\mathcal{K}_{p,\nu,k}$ the set of all $|\nu|$-vectors $k^* = \{k_{j,\alpha}(i)\,;\, (j,\alpha)\in\Lambda,\ 1 \le i \le \nu_{j,\alpha}\} \in (\mathbb{N}^*)^{|\nu|}$ such that $|k^*| = k-p$. For $k^* \in \mathcal{K}_{p,\nu,k}$ we set

$$\begin{aligned} L(k^*) &= \sum_{(j,\alpha)\in\Lambda}\sum_{i=1}^{\nu_{j,\alpha}} \min\{k_{j,\alpha}(i), m-j\} \ (\le m|\nu| < mN), \\ q_{p,\nu}(k^*) &= \min\{q_{p,\nu}, L(k^*)\} \ (< mN),\end{aligned}$$

and we denote by $\mathcal{Q}(p,\nu,k^*)$ the set of all $|\nu|$-vectors $r^* = \{r_{j,\alpha}(i)\,;\,(j,\alpha) \in \Lambda,\ 1 \le i \le \nu_{j,\alpha}\} \in \mathbb{N}^{|\nu|}$ such that

1) $|r^*| = q_{p,\nu}(k^*)\ \ (< mN)$, and
2) $0 \le r_{j,\alpha}(i) \le \min\{k_{j,\alpha}(i), m-j\}$ (for all (j,α,i)).

We denote by $e_{j,\alpha} \in \mathbb{N}^d$ the d-vector $e_{j,\alpha} = \{\nu_{i,\delta}\}_{(i,\delta)\in\Lambda}$ such that $\nu_{i,\delta} = 1$ if $(i,\delta) = (j,\alpha)$ and $\nu_{i,\delta} = 0$ if $(i,\delta) \ne (j,\alpha)$.

PROPOSITION 7.4.1. *Let $1 \le p + |\nu| < N$ with $|\nu| \ge 1$, and $k \ge (m+1)N$. Suppose the conditions* (6.4) *and*

$$s - 1 \ge \max\left[0,\ \max_{k^*\in\mathcal{K}_{p,\nu,k}}\left(\min_{r^*\in\mathcal{Q}(p,\nu,k^*)}\left(\frac{\mu^* - m}{p + |\nu|_+ + |r^*| - 1}\right)\right)\right] \tag{7.4.3}$$

(*under the same notations as in Lemma 7.3.2: as is seen below we have $p + |\nu|_+ + |r^*| - 1 \ge 1$ for all $r^* \in \mathcal{Q}(p,\nu,k^*)$, and so the right-hand side of* (7.4.3) *is well-defined*)*. Suppose also the conditions* (7.2.1), (7.2.2), $\beta \ge 1$, (7.4.1), *and* (7.4.2)*. Then we have the following estimates.*

(1) *In the case $p = 0$ and $\nu = e_{j,\alpha}$ (where $(j,\alpha) \in \Lambda$):*

$$\|g_k(0,e_{j,\alpha})\|_{I\times K,\rho} \ll \frac{(k-1)!^{s-1}}{k^{\mu-m}} \times e^{(s-1)(m+1)N}(m+1)^{\mu} \times \times \frac{A^*_{0,e_{j,\alpha}}(m+1)^{\mu}}{(1-\rho/R_0)^n} 2\beta Y_{k-1}. \tag{7.4.4}$$

(2) *In the case $p + |\nu| \ge 2$:*

$$\begin{aligned}&\|g_k(p,\nu)\|_{I\times K,\rho}\\ &\quad\ll \frac{(k-1)!^{s-1}}{k^{\mu-m}} \times e^{(s-1)(m+1)N}(m+1)^{\mu} \times\\ &\qquad \times \frac{A^*_{p,\nu}(p+(m+1)|\nu|)^{\mu}}{(1-\rho/R_0)^n} \sum_{|k^*|=k-p}\ \prod_{(j,\alpha)\in\Lambda}\ \prod_{i=1}^{\nu_{j,\alpha}} \beta Y_{k_{j,\alpha}(i)}.\end{aligned} \tag{7.4.5}$$

PROOF. First let us show that $p + |\nu|_+ + |r^*| - 1 \ge 1$ holds for all $r^* \in \mathcal{Q}(p,\nu,k^*)$ (under $k^* \in \mathcal{K}_{p,\nu,k}$). Take any $r^* \in \mathcal{Q}(p,\nu,k^*)$; then as is seen in the proof of Lemma 7.3.2 we have $J_+ \supset J^* \ne \emptyset$ and so $|\nu|_+ \ge 1$.

Since $L(k^*) \geq 1$ holds (for $|\nu| \geq 1$), by the condition (6.4) we see: if $p = 0$ and $|\nu| = 1$, we have $|r^*| = q_{p,\nu}(k^*) = \min\{q_{p,\nu}, L(k^*)\} \geq 1$.

By using these facts, the inequality $p+|\nu|_+ + |r^*| - 1 \geq 1$ is verified in the following way: if $p \geq 1$, by the condition $|\nu|_+ \geq 1$ we have $p+|\nu|_+ + |r^*| - 1 \geq 1$; if $p = 0$ and $|\nu| = 1$, by the conditions $|\nu|_+ \geq 1$ and $|r^*| \geq 1$ we have $p + |\nu|_+ + |r^*| - 1 \geq 1$; if $p = 0$, $|\nu| \geq 2$ and $|r^*| \geq 1$, by the condition $|\nu|_+ \geq 1$ we have $p+|\nu|_+ + |r^*| - 1 \geq 1$; if $p = 0$, $|\nu| \geq 2$ and $|r^*| = 0$, by the definition of J_+ we have $|\nu|_+ = |\nu| \geq 2$ and so we have $p+|\nu|_+ + |r^*| - 1 \geq 1$.

Now, let us show (7.4.4) and (7.4.5). Take any $k^* \in \mathscr{K}_{p,\nu,k}$ and fix it. By the assumption (7.4.3) we have an $r^* = \{r_{j,\alpha}(i)\,;\,(j,\alpha) \in \Lambda \text{ and } 1 \leq i \leq \nu_{j,\alpha}\} \in \mathfrak{Q}(p,\nu,k^*)$ such that

$$s - 1 \geq \frac{\mu^* - m}{p + |\nu|_+ + |r^*| - 1} \tag{7.4.6}$$

holds. By Remark 7.3.3 we see that $k_{j,\alpha}(i) - r_{j,\alpha}(i) \leq k - 1$ holds for all (j,α,i); therefore, we can apply Lemma 7.3.2 to this case. Since $|r^*| = q_{p,\nu}(k^*) \leq q_{p,\nu}$ and $|r^*| < mN$ hold, by (7.3.10) and (7.4.1) we have

$$\begin{aligned} &\| g(p,\nu) \|_{I \times K,\rho} \\ &\quad \ll \frac{A^*_{p,\nu}}{(1-\rho/R_0)^n} \times \frac{(k - |r^*| - p - |\nu|_+)!^{s-1}(p + (m+1)|\nu|)^{\mu-\mu^*}}{(k - |r^*|)^{\mu-\mu^*}} \times \\ &\qquad \times \prod_{(j,\alpha,i) \in J_0} \frac{A^*}{(1-\rho/R_0)^n} \times \prod_{(j,\alpha,i) \in J_+} 2^{r_{j,\alpha}(i)} \beta Y_{k_{j,\alpha}(i) - r_{j,\alpha}(i)}. \end{aligned} \tag{7.4.7}$$

Here, we note that

$$\frac{k}{k - |r^*|} \leq \frac{(m+1)N}{(m+1)N - mN} = (m+1);$$

moreover, by (7.2.8) with N replaced by $|r^*| + p + |\nu|_+$ $(\leq mN + N)$ we have

$$\frac{(k - |r^*| - p - |\nu|_+)!}{(k-1)!} \leq \frac{e^{|r^*|+p+|\nu|_+}}{k^{|r^*|+p+|\nu|_+ - 1}} \leq \frac{e^{(m+1)N}}{k^{|r^*|+p+|\nu|_+ - 1}}.$$

Therefore, we have

$$(7.4.8)\qquad \begin{aligned}&\frac{(k-|r^*|-p-|\nu|_+)!^{s-1}}{(k-|r^*|)^{\mu-\mu^*}}\\ &= \frac{(k-1)!^{s-1}}{k^{\mu-\mu^*}} \times \frac{(k-|r^*|-p-|\nu|_+)!^{s-1}}{(k-1)!^{s-1}} \times \frac{k^{\mu-\mu^*}}{(k-|r^*|)^{\mu-\mu^*}}\\ &\le \frac{(k-1)!^{s-1}}{k^{\mu-\mu^*}} \times \frac{e^{(s-1)(m+1)N}}{k^{(s-1)(|r^*|+p+|\nu|_+-1)}} \times (m+1)^{\mu-\mu^*}\\ &= \frac{(k-1)!^{s-1}}{k^{\mu-m}} \times \frac{e^{(s-1)(m+1)N}}{k^{(s-1)(|r^*|+p+|\nu|_+-1)+m-\mu^*}} \times (m+1)^{\mu-\mu^*}\end{aligned}$$

Since (7.4.6) is assumed, we have $(s-1)(|r^*|+p+|\nu|_+-1) \ge \mu^*-m$ and so

$$\frac{1}{k^{(s-1)(|r^*|+p+|\nu|_+-1)+m-\mu^*}} \le 1;$$

this leads us to

$$(7.4.9)\quad \frac{(k-|r^*|-p-|\nu|_+)!^{s-1}}{(k-|r^*|)^{\mu-\mu^*}} \le \frac{(k-1)!^{s-1}}{k^{\mu-m}} \times e^{(s-1)(m+1)N}(m+1)^{\mu-\mu^*}.$$

By (7.4.7) and (7.4.9) we obtain the estimate

$$(7.4.10)\qquad \begin{aligned}&\big\|g(p,\nu)\big\|_{I\times K,\rho}\\ &\ll \frac{(k-1)!^{s-1}}{k^{\mu-m}} \times \frac{A^*_{p,\nu}}{(1-\rho/R_0)^n} \times\\ &\quad \times e^{(s-1)(m+1)N}(m+1)^{\mu-\mu^*}(p+(m+1)|\nu|)^{\mu-\mu^*} \times\\ &\quad \times \prod_{(j,\alpha,i)\in J_0} \frac{A^*}{(1-\rho/R_0)^n} \times \prod_{(j,\alpha,i)\in J_+} 2^{r_{j,\alpha}(i)}\beta Y_{k_{j,\alpha}(i)-r_{j,\alpha}(i)}.\end{aligned}$$

Thus, by applying the conditions $(m+1)^{\mu-\mu^*} \le (m+1)^{\mu}$, $(p+(m+1)|\nu|)^{\mu-\mu^*} \le (p+(m+1)|\nu|)^{\mu}$,

- $\dfrac{A^*}{(1-\rho/R_0)^n} \ll Y_{k_{j,\alpha(i)}} \ll \beta Y_{k_{j,\alpha(i)}}$, and
- $2^{r_{j,\alpha}(i)}\beta Y_{k_{j,\alpha}(i)-r_{j,\alpha}(i)} \ll \beta Y_{k_{j,\alpha}(i)}$

to (7.4.10), we have the estimate (7.4.5) in the case $p+|\nu| \ge 2$. In the above, we have used the fact: if $p+|\nu| \ge 2$ we have $k_{j,\alpha}(i) \le k-1$ for all (j,α,i).

If $p = 0$ and $|\nu| = 1$, we have $J_0 = \emptyset$, $\# J_+ = 1$: for the unique $(j, \alpha, 1) \in J_+$ we have $k_{j,\alpha}(1) = k$ and $r_{j,\alpha}(1) = \min\{q_{p,\nu}, m - j\} \geq 1$. Therefore we have

$$\prod_{(j,\alpha,i) \in J_0} \frac{A^*}{(1-\rho/R_0)^n} \times \prod_{(j,\alpha,i) \in J_+} 2^{r_{j,\alpha}(i)} \beta Y_{k_{j,\alpha}(i) - r_{j,\alpha}(i)}$$
$$= 2^{r_{j,\alpha}(1)} \beta Y_{k - r_{j,\alpha}(1)} \ll 2\beta Y_{k-1} :$$

this proves (7.4.4) in the case $p = 0$ and $|\nu| = 1$. □

7.5. On the condition (7.4.3)

In this subsection, we will compare the condition

(7.5.1) $$s - 1 \geq \max_{k^* \in \mathcal{K}_{p,\nu,k}} \left[\min_{r^* \in \mathfrak{Q}(p,\nu,k^*)} \left(\frac{\mu^* - m}{p + |\nu|_+ + |r^*| - 1} \right) \right]$$

with the other condition

(7.5.2) $$s - 1 \geq \max_{(j,\alpha) \in \Lambda_\nu} \left(\frac{j + \sigma|\alpha| - m}{p + |\nu| + \min\{q_{p,\nu}, m - j\} - 1} \right)$$

(where $\Lambda_\nu = \{(j,\alpha) \in \Lambda \,;\, \nu_{j,\alpha} > 0\}$). We have

Lemma 7.5.1. *Let $1 \leq p + |\nu| < N$ with $|\nu| \geq 1$, $k \geq (m+1)N$ and $s \geq 1$. Then, the two conditions* (7.5.1) *and* (7.5.2) *are equivalent.*

Proof in the Case $|\nu| = 1$. Suppose the condition $|\nu| = 1$; we have $\nu = e_{j_0,\alpha_0}$ for some $(j_0, \alpha_0) \in \Lambda$. In this case, we have $k^* = k - p$, $L(k^*) = \min\{k - p, m - j_0\} = m - j_0$, $|r^*| = q_{p,\nu}(k^*) = \min\{q_{p,\nu}, L(k^*)\} = \min\{q_{p,\nu}, m - j_0\}$, $|\nu|_+ = 1$, $\mu^* = j_0 + \sigma|\alpha_0|$, and $\Lambda_\nu = \{(j_0, \alpha_0)\}$. Therefore, two conditions (7.5.1) and (7.5.2) are written in the same form

(7.5.3) $$s - 1 \geq \frac{j_0 + \sigma|\alpha_0| - m}{p + 1 + \min\{q_{p,\nu}, m - j_0\} - 1} = \frac{j_0 + \sigma|\alpha_0| - m}{p + \min\{q_{p,\nu}, m - j_0\}}.$$

This proves the result. □

Proof in the Case $|\nu| \geq 2$. Suppose the condition $|\nu| \geq 2$. First, let us show that (7.5.1) implies (7.5.2). To do so, we suppose the condition (7.5.1); then, to show (7.5.2) it is sufficient to prove the following:

(7.5.4) $$s - 1 \geq \frac{j_0 + \sigma|\alpha_0| - m}{p + |\nu| + \min\{q_{p,\nu}, m - j_0\} - 1} \quad \text{for any } (j_0, \alpha_0) \in \Lambda_\nu.$$

Take any $(j_0, \alpha_0) \in \Lambda_\nu$, and fix it. If $j_0 + \sigma|\alpha_0| \le m$, by the condition $s \ge 1$ we have (7.5.4). Therefore, from now we may assume: $j_0 + \sigma|\alpha_0| > m$. Since $\nu_{j_0,\alpha_0} \ge 1$ holds, we can write the $|\nu|$-vector k^* as

$$k^* = (k_{j_0,\alpha_0}(1), \dots) = (k_1, k_2, \dots, k_{|\nu|}) :$$

the important condition is that k_1 corresponds to the component $k_{j_0,\alpha_0}(1)$. We set

$$k^* = (k_1, k_2, \dots, k_{|\nu|}) \quad \text{with} \quad \begin{cases} k_1 = k - p - |\nu| + 1, \\ k_i = 1 \ \text{ for } i = 2, \dots, |\nu|. \end{cases}$$

Then we have $k^* \in \mathscr{K}_{p,\nu,k}$, and so by (7.5.1) we have

$$s - 1 \ge \frac{\mu^* - m}{p + |\nu|_+ + |r^*| - 1} \tag{7.5.5}$$

for some $r^* = (r_{j_0,\alpha_0}(1), \dots) = (r_1, r_2, \dots, r_{|\nu|}) \in \mathscr{Q}(p, \nu, k^*)$: by the definition of $\mathscr{Q}(p, \nu, k^*)$ we see: $r_1 \le \min\{k_1, m - j_0\} = m - j_0$, $0 \le r_i \le 1$ (for $i = 2, \dots, |\nu|$), and $|r^*| = r_1 + r_2 + \cdots + r_{|\nu|} = \min\{q_{p,\nu}, L(k^*)\}$. In this case we have $J^* = \{(j_0, \alpha_0, 1)\}$ and so $\mu^* = j_0 + \sigma|\alpha_0|$. If we set

$$a = \#\{i \in \{2, \dots, |\nu|\}\, ;\, r_i = 0\},$$
$$b = \#\{i \in \{2, \dots, |\nu|\}\, ;\, r_i = 1\},$$

we have $a + b = |\nu| - 1$, $|\nu|_+ = 1 + a$, and $|r^*| = r_1 + b$. Therefore, by (7.5.5) we have

$$s - 1 \ge \frac{j_0 + \sigma|\alpha_0| - m}{p + (1 + a) + (r_1 + b) - 1} = \frac{j_0 + \sigma|\alpha_0| - m}{p + |\nu| + r_1 - 1}. \tag{7.5.6}$$

Since $r_1 \le m - j_0$ and $r_1 \le |r^*| = \min\{q_{p,\nu}, L(k^*)\} \le q_{p,\nu}$ hold, we have $r_1 \le \min\{q_{p,\nu}, m - j_0\}$. Thus, by (7.5.6) we obtain

$$s - 1 \ge \frac{j_0 + \sigma|\alpha_0| - m}{p + |\nu| + r_1 - 1} \ge \frac{j_0 + \sigma|\alpha_0| - m}{p + |\nu| + \min\{q_{p,\nu}, m - j_0\} - 1}.$$

This proves (7.5.4). Thus, we have proved that (7.5.1) implies (7.5.2).

Next, let us show that (7.5.2) implies (7.5.1). To do so, we suppose the condition (7.5.2); then, to show (7.5.1) it is sufficient to prove the following: for any $k^* \in \mathscr{K}_{p,\nu,k}$ we can find an $r^* \in \mathscr{Q}(p, \nu, k^*)$ such that

$$s - 1 \ge \frac{\mu^* - m}{p + |\nu|_+ + |r^*| - 1} \tag{7.5.7}$$

holds. Let us show this now.

Take any $k^* = \{k_{j,\alpha}(i)\,;\,(j,\alpha) \in \Lambda,\ 1 \le i \le \nu_{j,\alpha}\} \in \mathscr{K}_{p,\nu,k}$, and fix it. Since $J^* \neq \emptyset$ is known (in Lemma 7.3.2), we have $J^* = \{(j_1,\alpha_1,i_1),\dots,(j_h,\alpha_h,i_h)\}$ for some $h \ge 1$; by the definition of J^* we have

$$\begin{cases} m-j < k_{j,\alpha}(i), & \text{if } (j,\alpha,i) \in J^*, \\ m-j \ge k_{j,\alpha}(i), & \text{if } (j,\alpha,i) \notin J^*. \end{cases}$$

Without loss of generality we may suppose that $\mu^*(= \max\{j_1+\sigma|\alpha_1|,\dots,j_h+\sigma|\alpha_h|\}) = j_1+\sigma|\alpha_1|$: then $\nu_{j_1,\alpha_1} > 0$ is obvious. Since

$$L(k^*) = (m-j_1)+\cdots+(m-j_h) + \sum_{(j,\alpha,i)\notin J^*} k_{j,\alpha}(i),$$

we have $q_{p,\nu}(k^*) = \min\{q_{p,\nu}, L(k^*)\} \ge \min\{q_{p,\nu}, m-j_1\}$.

If $\mu^* \le m$, the condition (7.5.7) is clear for any $r^* \in \mathfrak{Q}(p,\nu,k^*)$. Therefore we may suppose that $\mu^* > m$ (that is, $j_1+\sigma|\alpha_1| > m$) holds. In this case, in order to show (7.5.7), we divide our situation into the following three cases:

1) $0 \le q_{p,\nu}(k^*) \le (m-j_1)+\cdots+(m-j_h)$,
2) $(m-j_1)+\cdots+(m-j_h) < q_{p,\nu}(k^*) \le (m-j_1)+\cdots+(m-j_h)+(|\nu|-h)$,
3) $(m-j_1)+\cdots+(m-j_h)+(|\nu|-h) < q_{p,\nu}(k^*)$.

In the case 1) we take an $r^* = \{r_{j,\alpha}(i)\,;\,(j,\alpha) \in \Lambda,\ 1 \le i \le \nu_{j,\alpha}\} \in \mathfrak{Q}(p,\nu,k^*)$ so that $|r^*| = q_{p,\nu}(k^*)$ and

$$\begin{cases} 0 \le r_{j,\alpha}(i) \le m-j, & \text{if } (j,\alpha,i) \in J^*, \\ r_{j,\alpha}(i) = 0, & \text{if } (j,\alpha,i) \notin J^*: \end{cases}$$

then we have $|\nu|_+ = |\nu|$ and so by (7.5.2)

$$\begin{aligned} \frac{\mu^*-m}{p+|\nu|_+ + |r^*|-1} &= \frac{j_1+\sigma|\alpha_1|-m}{p+|\nu|+q_{p,\nu}(k^*)-1} \\ &\le \frac{j_1+\sigma|\alpha_1|-m}{p+|\nu|+\min\{q_{p,\nu}, m-j_1\}-1} \le s-1. \end{aligned}$$

This proves (7.5.7).

In the case 2) we take an $r^* = \{r_{j,\alpha}(i)\,;\,(j,\alpha) \in \Lambda,\ 1 \le i \le \nu_{j,\alpha}\} \in \mathfrak{Q}(p,\nu,k^*)$ so that $|r^*| = q_{p,\nu}(k^*)$ and

$$\begin{cases} 0 \le r_{j,\alpha}(i) = m-j, & \text{if } (j,\alpha,i) \in J^*, \\ 0 \le r_{j,\alpha}(i) \le k_{j,\alpha}(i), & \text{if } (j,\alpha,i) \notin J^*. \end{cases}$$

In this case, since

$$\begin{aligned}&\#\{(j,\alpha,i)\notin J^*\,;\,r_{j,\alpha}(i)=k_{j,\alpha}(i)\}\\&\quad\le\#\{(j,\alpha,i)\notin J^*\,;\,r_{j,\alpha}(i)\ge 1\}\\&\quad\le q_{p,\nu}(k^*)-((m-j_1)+\cdots+(m-j_h))\le|\nu|-h\end{aligned}$$

holds, we have

$$\begin{aligned}|\nu|_+&=|\nu|-\#\{(j,\alpha,i)\notin J^*\,;\,r_{j,\alpha}(i)=k_{j,\alpha}(i)\}\\&\ge|\nu|-q_{p,\nu}(k^*)+(m-j_1)+\cdots+(m-j_h)\ge|\nu|-|\nu|+h=h\ge 1\end{aligned}$$

and

$$\begin{aligned}p+|\nu|_++|r^*|-1&=p+|\nu|_++q_{p,\nu}(k^*)-1\\&\ge p+(|\nu|-q_{p,\nu}(k^*)+(m-j_1)+\cdots+(m-j_h))+q_{p,\nu}(k^*)-1\\&=p+|\nu|+(m-j_1)+\cdots+(m-j_h))-1\\&\ge p+|\nu|+(m-j_1)-1\\&\ge p+|\nu|+\min\{q_{p,\nu},m-j_1\}-1.\end{aligned}$$

Therefore, (7.5.7) is verifies by the following:

$$\begin{aligned}\frac{\mu^*-m}{p+|\nu|_++|r^*|-1}&=\frac{j_1+\sigma|\alpha_1|-m}{p+|\nu|_++q_{p,\nu}(k^*)-1}\\&\le\frac{j_1+\sigma|\alpha_1|-m}{p+|\nu|+\min\{q_{p,\nu},m-j_1\}-1}\le s-1.\end{aligned}$$

In the case 3) we take the same $r^*=\{r_{j,\alpha}(i)\,;\,(j,\alpha)\in\Lambda,\ 1\le i\le\nu_{j,\alpha}\}\in\mathfrak{Q}(p,\nu,k^*)$ as in the case 2). Then we have $|\nu|_+\ge\#J^*=h$ and so

$$\begin{aligned}p+|\nu|_++|r^*|-1&=p+|\nu|_++q_{p,\nu}(k^*)-1\\&\ge p+h+((m-j_1)+\cdots+(m-j_h)+|\nu|-h)-1\\&=p+|\nu|+(m-j_1)+\cdots+(m-j_h)-1\\&\ge p+|\nu|+(m-j_1)-1\\&\ge p+|\nu|+\min\{q_{p,\nu},m-j_1\}-1.\end{aligned}$$

Therefore, (7.5.7) is verifies in the same way as in the case 2).

Thus, we have proved that (7.5.2) implies (7.5.1). □

By the proof we see:

LEMMA 7.5.2. *Let $1 \le p + |\nu| < N$ with $|\nu| \ge 1$, and $k \ge (m+1)N$. We have the equality*

$$\tag{7.5.8} \begin{aligned} &\max\left[0, \max_{k^* \in \mathscr{K}_{p,\nu,k}} \left(\min_{r^* \in \mathscr{Q}(p,\nu,k^*)} \left(\frac{\mu^* - m}{p + |\nu|_+ + |r^*| - 1} \right)\right)\right] \\ &= \max\left[0, \max_{(j,\alpha)\in\Lambda_\nu} \left(\frac{j + \sigma|\alpha| - m}{p + |\nu| + \min\{q_{p,\nu}, m - j\} - 1}\right)\right]. \end{aligned}$$

Thus, the condition (7.4.3) in Proposition 7.4.1 is trivial under the assumption $s \ge \max\{s_0, s_1, s_2\}$ in Theorem 6.1.

7.6. Completion of the proof of Theorem 6.1

Let us recall our situation again: $u(t,x)$ is a solution of (6.1) given in b_5), $w_k(t,x)$ $(k = 1, 2, \dots)$ are

$$\tag{7.6.1} w_k(t,x) = \frac{1}{k!}\partial_t^k u(t,x), \quad k = 1, 2, \dots,$$

$s > 1$ is a real number satisfying $s \ge \max\{s_0, s_1, s_2\}$, and K is a compact subset of V. Our aim is to prove the following condition:

$$\tag{7.6.2} \sum_{k\ge1} \frac{1}{(k-1)!^{s-1}} \|w_k\|_{I\times K,\rho}\, t^k \in \mathbb{C}\{t, \rho\}.$$

In the proof, we may suppose that Lemmas 7.1.1 and 5.2.1 hold. Let k_0 and M_0 be the constants in Lemma 5.2.1. We take a sufficiently large $\mu \in \mathbb{N}$ such that $\mu > m$, $\mu \ge n$, $\mu \ge \sigma L$ and that $j + \sigma|\alpha| \le \mu$ holds for all $(j,\alpha) \in \Lambda$: take also a sufficiently large $N \in \mathbb{N}$ so that

$$\tag{7.6.3} s - 1 \ge \frac{\mu - m}{N - 1}, \quad (m+1)N \ge k_0 \quad \text{and} \quad N \ge 2.$$

As is seen in (7.1.5), we know that $w_k(t,x)$ (for $k \ge 1$) satisfies

$$\tag{7.6.4} C(t\partial_t + k, x)w_k = f_k(t,x) + \sum_{1\le p+|\nu|\le k, |\nu|\ge1} g_k(p,\nu).$$

Since $w_k(t,x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(K))$ holds for all $k = 1, 2, \ldots$, we can choose $C_k^* \geq 0$ $(k = 1, 2, \ldots, (m+1)N)$ so that

$$\|\!|(t\partial_t + k)^j \partial_x^\alpha w_k\|\!|_{I\times K,\rho} \ll \frac{(k-1)!^{s-1}}{k^{\mu-j-\sigma|\alpha|}} \frac{C_k^*}{(1-\rho/R_0)^n} \tag{7.6.5}$$

for $j = 0, 1, \ldots, m$, $|\alpha| \leq L$ and $k = 1, 2, \ldots, (m+1)N$.

We let A^* be the one in (7.3.2), and let $A_{p,\nu}^* \geq 0$ $(1 \leq p + |\nu| < N$ and $|\nu| \geq 1)$ be the ones in (7.4.1). Set

- $F_k^* = \dfrac{k^{\mu-m} F_k}{(k-1)!^{s-1}} \quad (k \geq 1)$,
- $A_{p,\nu}^* = \dfrac{A_{p,\nu}}{(p+|\nu|-N)!^{s-1}} \quad (p + |\nu| \geq N,\ |\nu| \geq 1)$,

where $F_k \geq 0$ $(k \geq 1)$ and $A_{p,\nu} \geq 0$ $(p + |\nu| \geq N,\ |\nu| \geq 1)$ are the ones in Lemma 7.1.1. We see:

$$\sum_{k\geq 1} F_k^* t^k \in \mathbb{C}\{t\} \quad \text{and} \quad \sum_{p+|\nu|\geq 1, |\nu|\geq 1} A_{p,\nu}^* t^p z^\nu \in \mathbb{C}\{t, z\}. \tag{7.6.6}$$

Now, let $R > 0$ be sufficiently small so that $0 < R < \min\{1, R_0\}$ and Lemma 5.2.2 hold, and let us consider the functional equation with respect to (Y, t):

$$\begin{aligned} Y = {} & \frac{A^*}{(1-\rho/R)^\mu} t + \frac{2}{(1-\rho/R)^{2\mu}} tY \\ & + \sum_{1\leq k\leq (m+1)N} \frac{C_k^*}{(1-\rho/R)^{\mu(2k-1)}} t^k \\ & + \frac{2M_0}{(1-\rho/R)^\mu} \Biggl[\sum_{k>(m+1)N} \frac{F_k^*}{(1-\rho/R)^{\mu(2k-2)}} t^k \\ & \quad + H \sum_{(j,\alpha)\in\Lambda} \frac{A_{0,e_{j,\alpha}}^* (m+1)^\mu}{(1-\rho/R)^\mu} t \times 2\beta Y \\ & \quad + H \sum_{p+|\nu|\geq 2, |\nu|\geq 1} \frac{A_{p,\nu}^* (p + (m+1)|\nu|)^\mu}{(1-R/R_0)^n (1-\rho/R)^{\mu(2p+|\nu|-2)}} t^p (\beta Y)^{|\nu|} \Biggr] \end{aligned} \tag{7.6.7}$$

where $H = e^{(s-1)(m+1)N}(m+1)^\mu$, $\beta = (2\mu e/R)^\mu$ and ρ is regarded as a parameter. Since this is an analytic functional equation (as is quaranteed by the condition (7.6.6)), the implicit function theorem tells us that for any $0 < \rho < R$ this equation (7.6.7) has a unique holomorphic solution $Y(t)$ with $Y(0) = 0$ in a neighborhood of $t = 0$. If we expand this into Taylor series

$$Y = \sum_{k \geq 1} Y_k t^k,$$

we can easily see that

$$Y_1 \gg \frac{A^*}{(1-\rho/R)^\mu} \gg \frac{A^*}{(1-\rho/R_0)^\mu} \gg \frac{A^*}{(1-\rho/R_0)^n}, \tag{7.6.8}$$

$$Y_{k+1} \gg \frac{2}{(1-\rho/R)^{2\mu}} Y_k \gg 2Y_k, \quad k = 1, 2, \dots, \tag{7.6.9}$$

$$\begin{aligned} Y_k &\gg \frac{C_k^*}{(1-\rho/R)^{\mu(2k-1)}} \\ &\gg \frac{C_k^*}{(1-\rho/R_0)^n}, \quad k = 1, 2, \dots, (m+1)N \end{aligned} \tag{7.6.10}$$

(by the conditions $0 < R < R_0$ and $\mu \geq n$) and that Y_k (for $k > (m+1)N$) is determined by the following recurrent formulas:

$$\begin{aligned} Y_k = & \frac{2M_0}{(1-\rho/R)^\mu} \Bigg[\frac{1/M_0}{(1-\rho/R)^\mu} Y_{k-1} + \frac{F_k^*}{(1-\rho/R)^{\mu(2k-2)}} \\ & + H \sum_{(j,\alpha)\in\Lambda} \frac{A^*_{0,e_{j,\alpha}}(m+1)^\mu}{(1-\rho/R)^\mu} \times 2\beta Y_{k-1} \\ & + H \sum_{2 \leq p+|\nu| \leq k, |\nu| \geq 1} \frac{A^*_{p,\nu}(p+(m+1)|\nu|)^\mu}{(1-R/R_0)^n (1-\rho/R)^{\mu(2p+|\nu|-2)}} \times \\ & \times \sum_{|k^*|=k-p} \prod_{(j,\alpha)\in\Lambda} (\beta Y_{k_{j,\alpha}(1)}) \times \cdots \times (\beta Y_{k_{j,\alpha}(\nu_{j,\alpha})}) \Bigg]. \end{aligned} \tag{7.6.11}$$

We note that the condition (7.4.2) follows from (7.6.8) and (7.6.9). In addition, we can see by induction on k that Y_k has the form

$$Y_k = \frac{C_k}{(1-\rho/R)^{\mu(2k-1)}}, \quad k = 1, 2, \dots \tag{7.6.12}$$

where C_k ($k \geq 1$) are constants which are independent of the parameter ρ.

LEMMA 7.6.1. *For $k = 1, 2, \ldots$ we have*

$$(7.6.13)_k \qquad \| (t\partial_t + k)^j \partial_x^\alpha w_k \|_{I\times K,\rho} \ll \frac{(k-1)!^{s-1}}{k^{\mu-j-\sigma|\alpha|}} \beta Y_k(\rho)$$

for $j = 0, 1, \ldots, m$ and $|\alpha| \leq L$.

PROOF. The cases $k = 1, 2, \ldots, (m+1)N$ are clear from the conditions (7.6.5), (7.6.10), and $\beta > 1$. Let us show the general case by induction on k.

Let $k > (m+1)N$; suppose that $(7.6.13)_p$ is already proved for $p = 1, 2, \ldots, k-1$. Then, by (7.6.4), Lemma 7.1.1, Propositions 7.2.1 and 7.4.1 (with Lemma 7.5.1) we have

$$\begin{aligned}
&\| C(t\partial_t + k) w_k \|_{I\times K,\rho} \\
&\ll \frac{F_k}{(1-\rho/R_0)^n} \\
&+ \sum_{N \leq p+|\nu| \leq k, |\nu|\geq 1} \frac{(k-1)!^{s-1}}{k^{\mu-m}} \frac{A_{p,\nu} e^{N(s-1)}}{(p+|\nu|-N)!^{s-1}} \frac{(p+|\nu|)^{\mu-\mu_\nu}}{(1-\rho/R_0)^n} \\
&\qquad\qquad \times \sum_{|k^*|=k-p} \prod_{(j,\alpha)\in\Lambda} \prod_{i=1}^{\nu_{j,\alpha}} \beta Y_{k_{j,\alpha}(i)} \\
&+ \sum_{(j,\alpha)\in\Lambda} \frac{(k-1)!^{s-1}}{k^{\mu-m}} \times e^{(s-1)(m+1)N} (m+1)^\mu \times \frac{A^*_{0,e_{j,\alpha}} (m+1)^\mu}{(1-\rho/R_0)^n} 2\beta Y_{k-1} \\
&+ \sum_{2 \leq p+|\nu| < N, |\nu|\geq 1} \frac{(k-1)!^{s-1}}{k^{\mu-m}} \times e^{(s-1)(m+1)N} (m+1)^\mu \times \\
&\qquad\qquad \times \frac{A^*_{p,\nu} (p+(m+1)|\nu|)^\mu}{(1-\rho/R_0)^n} \sum_{|k^*|=k-p} \prod_{(j,\alpha)\in\Lambda} \prod_{i=1}^{\nu_{j,\alpha}} \beta Y_{k_{j,\alpha}(i)} :
\end{aligned}$$

hence we have

$$\begin{aligned}
(7.6.14) \quad &\| C(t\partial_t + k) w_k \|_{I\times K,\rho} \\
&\ll \frac{(k-1)!^{s-1}}{k^{\mu-m}} \Bigg[\frac{F^*_k}{(1-\rho/R_0)^n} + H \sum_{(j,\alpha)\in\Lambda} \frac{A^*_{0,e_{j,\alpha}} (m+1)^\mu}{(1-\rho/R_0)^n} 2\beta Y_{k-1}
\end{aligned}$$

$$+ H \sum_{2\le p+|\nu|\le k, |\nu|\ge 1} \frac{A^*_{p,\nu}(p+(m+1)|\nu|)^\mu}{(1-\rho/R_0)^n} \times$$

$$\times \sum_{|k^*|=k-p} \prod_{(j,\alpha)\in\Lambda} \prod_{i=1}^{\nu_{j,\alpha}} \beta Y_{k_{j,\alpha}(i)} \Bigg].$$

Since $\mu(2|k^*|-|\nu|) = \mu(2k-2p-|\nu|) = 2\mu(k-p-|\nu|)+\mu|\nu| \ge \mu|\nu| \ge \mu$,

$$\frac{1}{(1-\rho/R_0)^n} \ll \frac{1}{(1-\rho/R)^\mu} \ll \frac{1}{(1-\rho/R)^{\mu(2k-2)}},$$

$$\frac{1}{(1-\rho/R_0)^n} \prod_{(j,\alpha)\in\Lambda} \prod_{i=1}^{\nu_{j,\alpha}} \beta Y_{k_{j,\alpha}(i)} \ll \frac{1}{(1-R/R_0)^n} \prod_{(j,\alpha)\in\Lambda} \prod_{i=1}^{\nu_{j,\alpha}} \beta Y_{k_{j,\alpha}(i)}$$

hold, by comparing (7.6.14) with (7.6.11) we have the estimate

$$\begin{aligned} &\| C(t\partial_t + k) w_k \|_{I\times K,\rho} \\ &\ll \frac{(k-1)!^{s-1}}{k^{\mu-m}} \frac{(1-\rho/R)^\mu}{2M_0} Y_k = \frac{(k-1)!^{s-1}}{2M_0 k^{\mu-m}} \frac{C_k}{(1-\rho/R)^{\mu(2k-2)}}. \end{aligned}$$

Thus, by applying Lemma 5.2.2 we have

$$\begin{aligned} &\| (t\partial_t + k)^j w \|_{I\times K,\rho} \\ &\ll \frac{(k-1)!^{s-1}}{k^{\mu-j}} \frac{C_k}{(1-\rho/R)^{\mu(2k-2)}}, \quad j=0,1,\dots,m. \end{aligned} \tag{7.6.15}$$

This result corresponds to (5.3.10) in section 5, and so by the same argument as in (5.3.11) we have

$$\| (t\partial_t + k)^j \partial_x^\alpha w_k \|_{I\times K,\rho} \ll \frac{(k-1)!^{s-1}}{k^{\mu-j-\sigma|\alpha|}} \beta Y_k.$$

This proves $(7.6.13)_k$. □

Completion of the Proof of (7.6.2). By Lemma 7.6.1 we have

$$\sum_{k\ge 1} \frac{1}{(k-1)!^{s-1}} \| w_k \|_{I\times K,\rho} t^k \ll \sum_{k\ge 1} \frac{1}{k^\mu} \beta Y_k t^k \ll \beta \sum_{k\ge 1} Y_k t^k \in \mathbb{C}\{t\}$$

for any $0<\rho<R$. This proves (7.6.2). □

7.7. A remark on (6.1)

In Theorem 6.1 we have obtained the result $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}(I \times V)$ for any $s \geq \max\{s_0, s_1, s_2\}$ with s_0 defined by (6.3): but, by applying Theorem 2.2 (or, by the same reduction as in section 3) we have the following result which is an improvement of Theorem 6.1.

THEOREM 7.7.1. *Suppose the conditions $b_1) \sim b_5)$ and (6.4): then we have $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}(I \times V)$ for any $s \geq \max\{s_0(F), s_1, s_2\}$ where*

$$s_0(F) = 1 + \max\left[0, \max_{(j,\alpha)\in\Lambda, |\alpha|>0} \left(\frac{j + \sigma|\alpha| - m}{\min\{q^*_{j,\alpha},\, m-j\}}\right)\right]. \tag{7.7.1}$$

PROOF. Our equation is just (2.1) with $\gamma = m$ and $Du(t,x)$ replaced by $\Theta u(t,x)$. Therefore, when we apply Theorem 2,2 to this case, $k_{j,\alpha}$ must be replaced by $q^*_{j,\alpha} + j$: then we have the result. □

8. On the Necessary Condition

In this section, we will derive a necessary condition for a solution $u(t,x)$ to belong to the class $\mathcal{E}^{\{s,\sigma\}}(I \times V)$.

Let $I = [0,T]$ (with $T > 0$), let V be an open neighborhood of $x = 0 \in \mathbb{R}^n$, and let Ω be an open neighborhood of $(t,x,z) = (0,0,0) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^d$. First, we will consider the same equation as (6.1)

$$C(t\partial_t, x)u = F\big(t, x, \Theta u\big) \quad \text{with } \Theta u = \{(t\partial_t)^j \partial_x^\alpha u\}_{(j,\alpha)\in\Lambda} \tag{8.1}$$

on $I \times V$, where

$$C(\lambda, x) = \lambda^m - c_{m-1}(x)\lambda^{m-1} - \cdots - c_1(x)\lambda - c_0(x),$$

and we suppose the same conditions $b_1) \sim b_5)$ as in section 6. In addition we assume:

c_1) $C(k,0) > 0$ for any $k = 1, 2, \ldots$;

c_2) $C(k,0) - C(k,x) \gg 0$ (at $x = 0$) for any $k = 1, 2, \ldots$;

c_3) $F(t,x,z) \gg 0$ (at $(t,x,z) = (0,0,0)$), and

$$\liminf_{|\beta|\to\infty} \left(\frac{F^{(1,\beta,0)}(0,0,0)}{|\beta|!^\sigma}\right)^{1/|\beta|} > 0; \tag{8.2}$$

c_4) $u(0,x)=0$ on V, and

$$\frac{\partial F}{\partial z_{j,\alpha}}(t,x,\Theta u)\Big|_{t=0} \equiv 0 \text{ on } V \text{ for any } (j,\alpha)\in\Lambda. \tag{8.3}$$

In c_2) and c_3) we used the following notations: for $\phi(x)\in C^\infty(V)$ we write $\phi(x)\gg 0$ (at $x=0$) if $\phi^{(\beta)}(0)\geq 0$ holds for all $\beta\in\mathbb{N}^n$, and for $F(t,x,z)\in C^\infty(\Omega)$ we write $F(t,x,z)\gg 0$ (at $(t,x,z)=(0,0,0)$) if $F^{(k,\beta,\nu)}(0,0,0)\geq 0$ holds for all $(k,\beta,\nu)\in\mathbb{N}\times\mathbb{N}^n\times\mathbb{N}^d$.

We note that by c_4) we have $(\Theta u)(0,x)=0$ and so by setting $t=0$ in (8.1) we have $F(0,x,0)=0$ on V. If we set $a(x)=F^{(1,0,0)}(0,x,0)$, the condition (8.2) implies that there is an $h>0$ such that $a^{(\beta)}(0)\geq h^{|\beta|}|\beta|!^\sigma$ holds for any sufficiently large $|\beta|$.

As before, we set $q_{j,\alpha}=ord_t((\partial F/\partial z_{j,\alpha})(t,x,\Theta u(t,x)),V)$ $((j,\alpha)\in\Lambda)$; then we have the expression

$$\frac{\partial F}{\partial z_{j,\alpha}}(t,x,\Theta u(t,x)) = a_{j,\alpha}(x)t^{q_{j,\alpha}} + O(t^{q_{j,\alpha}+1}) \quad (\text{as } t\longrightarrow +0)$$

for some $a_{j,\alpha}(x)\in\mathcal{E}^{\{\sigma\}}(V)$ with $a_{j,\alpha}(x)\gg 0$ (at $x=0$). We set

$$\Lambda(+)=\{(j,\alpha)\in\Lambda\,;\, a_{j,\alpha}(0)>0, |\alpha|>0\}. \tag{8.4}$$

Then we have the following result.

Theorem 8.1. *Suppose the conditions $b_1)\sim b_5)$ and $c_1)\sim c_4)$. Then, if $u(t,x)\in\mathcal{E}^{\{s,\sigma\}}(I\times V)$ holds for some $s\geq 1$, we have*

$$s\geq 1+\max\left[0, \max_{(j,\alpha)\in\Lambda(+)}\left(\frac{j+\sigma|\alpha|-m}{q_{j,\alpha}}\right)\right]. \tag{8.5}$$

Remark 8.2. Compare this with Theorem 7.7.1; there is a gap between $s\geq s_0(F)$ and (8.5). But, at present, the author don't know how to fill this gap.

To prove this theorem, we note:

Lemma 8.3. *We have $u(t,x)\gg 0$ (at $(t,x)=(0,0)$).*

PROOF. We set

$$W_k(x)=\frac{1}{k!}(\partial_t^k u)(0,x), \quad k=0,1,2,\dots . \tag{8.6}$$

By the assumption we have $W_0(x)=u(0,x)=0 \gg 0$ (at $x=0$). By applying ∂_t to (8.1) we have

$$\begin{aligned}C(t\partial_t+1,x)\partial_t u &= F^{(1,0,0)}(t,x,\Theta u)\\ &\quad+\sum_{(i,\beta)\in\Lambda}\frac{\partial F}{\partial z_{i,\beta}}(t,x,\Theta u)\partial_t(t\partial_t)^i\partial_x^\beta u.\end{aligned} \tag{8.7}$$

Since (8.3) is assumed, by setting $t=0$ we have $C(1,x)W_1(x)=F^{(1,0,0)}(0, x,0)\gg 0$ (at $x=0$) and so

$$\begin{aligned}W_1(x)=\frac{F^{(1,0,0)}(0,x,0)}{C(1,x)}&=\frac{F^{(1,0,0)}(0,x,0)}{C(1,0)}\sum_{p=0}^{\infty}\Big(\frac{C(1,0)-C(1,x)}{C(1,0)}\Big)^p\\ &\gg \frac{F^{(1,0,0)}(0,x,0)}{C(1,0)}\gg 0 \quad (\text{at } x=0),\end{aligned} \tag{8.8}$$

where $a(x)\gg b(x)$ (at $x=0$) means that $a^{(\beta)}(0)\ge |b^{(\beta)}(0)|$ holds for all $\beta\in\mathbb{N}^n$.

In the general case, if we use the relation (5.1.3) (or (7.1.3)) we can show by induction on k that $W_k(x)\gg 0$ (at $x=0$) holds for all $k=0,1,2,\dots$. This proves $u(t,x)\gg 0$ (at $(t,x)=(0,0)$). □

PROOF OF THEOREM 8.1. Take any $(j,\alpha)\in\Lambda(+)$; then we have $q_{j,\alpha}\ge 1$ and $a_{j,\alpha}(0)>0$. Our aim is to show the condition

$$s-1\ge\frac{j+\sigma|\alpha|-m}{q_{j,\alpha}}. \tag{8.9}$$

Let us show this now. We set $a(x)=F^{(1,0,0)}(0,x,0)$; by (8.7) we have

$$\begin{aligned}&C(t\partial_t+1,x)\partial_t u\\ &\quad=(a(x)+O(t))+\sum_{(i,\beta)\in\Lambda}(a_{i,\beta}(x)t^{q_{i,\beta}}+O(t^{q_{i,\beta}+1}))\partial_t(t\partial_t)^i\partial_x^\beta u\\ &\quad=a(x)+t^{q_{j,\alpha}}a_{j,\alpha}(x)\partial_t(t\partial_t)^j\partial_x^\alpha u+\cdots\\ &\quad=a(x)+t^{q_{j,\alpha}}a_{j,\alpha}(0)\partial_t(t\partial_t)^j\partial_x^\alpha u+\cdots .\end{aligned}$$

By setting $q = q_{j,\alpha} \geq 1$ and $A = a_{j,\alpha}(0) > 0$ for simplicity and by using the fact $t^q\partial_t(t\partial_t)^j = t^{q-1}(t\partial_t)^{j+1}$ we have

$$C(t\partial_t + 1, x)\partial_t u = a(x) + At^{q-1}(t\partial_t)^{j+1}\partial_x^\alpha u + \cdots . \tag{8.10}$$

Since $u(t,x) \gg 0$ (at $(t,x) = (0,0)$) is known, in the formula (8.10) the part "$\cdots$" satisfies "$\cdots \gg 0$ (at $(t,x) = (0,0)$) ".

Now, let us apply $\partial_t^{q-1+\ell}$ (for $\ell \geq 1$) to (8.10); we have

$$\begin{aligned}
&C(t\partial_t + q + \ell, x)\frac{1}{(q+\ell)!}\partial_t^{q+\ell}u \\
&= \frac{A}{(q+\ell)!}\partial_t^{q-1+\ell}\Big[t^{q-1}(t\partial_t)^{j+1}\partial_x^\alpha u\Big] + \cdots \\
&= \frac{A\ell!}{(q+\ell)!}\Big[(t\partial_t + 1 + \ell)\cdots(t\partial_t + q - 1 + \ell)(t\partial_t + \ell)^{j+1}\partial_x^\alpha \frac{1}{\ell!}\partial_t^\ell u\Big] + \cdots .
\end{aligned}$$

Therefore, by setting $t = 0$ we have

$$C(q+\ell, x)W_{q+\ell} \gg \frac{A\ell!}{(q+\ell)!}(1+\ell)\cdots(q-1+\ell)\ell^{j+1}\partial_x^\alpha W_\ell = \frac{A\ell^{j+1}}{(q+\ell)}\partial_x^\alpha W_\ell$$

and by using the conditions $C(q+\ell, 0) - C(q+\ell, x) \gg 0$ and $W_{q+\ell}(x) \gg 0$ we have

$$C(q+\ell, 0)W_{q+\ell} \gg \frac{A\ell^{j+1}}{(q+\ell)}\partial_x^\alpha W_\ell. \tag{8.11}$$

Thus, by (8.8), (8.11) and $a(x) = F^{(1,0,0)}(0,x,0)$ we can easily see the following estimates:

$$\begin{aligned}
W_{kq+1}(x) \gg \frac{A^k(q+1)^{j+1}\cdots((k-1)q+1)^{j+1}}{C(1,0)C(q+1,0)\cdots C(kq+1,0)} \times \\
\times \frac{1}{(q+1)\cdots(kq+1)}\,\partial_x^{k\alpha}(a(x))
\end{aligned} \tag{8.12}$$

for $k = 1, 2, \ldots$.

Here, we recall: by the assumption $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}(I \times V)$ we have $|W_p(0)| \leq BH^p p!^{s-1}$ $(p = 0, 1, 2, \ldots)$ for some $B > 0$ and $H > 0$, and by the condition (8.2) we have an $h > 0$ such that $a^{(\beta)}(0) \geq h^{|\beta|}|\beta|!^\sigma$ for any sufficiently large $|\beta|$. We note also that $C(k,0) \leq ck^m$(for $k = 1, 2, \ldots$) holds for some $c>0$. By applying these conditions to (8.12), for any sufficiently large k we have

$$\begin{aligned} BH^{kq+1}(kq+1)!^{s-1} &\geq |W_{kq+1}(0)| = W_{kq+1}(0) \\ &\geq \frac{A^k(q+1)^{j+1}\cdots((k-1)q+1)^{j+1}}{c^{k+1}(q+1)^{m+1}\cdots(kq+1)^{m+1}} h^{k|\alpha|}(k|\alpha|)!^{\sigma} \\ &\geq B_1 H_1^k k!^{j+\sigma|\alpha|-m} \end{aligned}$$

for some $B_1>0$ and $H_1>0$. Thus, we have the estimates

$$\frac{k!^{j+\sigma|\alpha|-m}}{(kq+1)!^{s-1}} \leq B_2 H_2^k \quad \text{for any sufficiently large } k$$

for some $B_2>0$ and $H_2>0$: this shows that $j+\sigma|\alpha|-m \leq q(s-1)$ holds. Thus, we have proved (8.9). □

Next, let us apply Theorem 8.1 to the initial value problem for (2.1) with $\gamma=0$

$$(8.13) \qquad \begin{cases} \partial_t^m u = G(t,x,Du) \quad \text{with } Du = \{\partial_t^j \partial_x^\alpha u\}_{(j,\alpha)\in\Lambda}, \\ \partial_t^i u\big|_{t=0} = \varphi_i(x), \quad i=0,1,\dots,m-1 \end{cases}$$

under the same assumptions $\mathrm{a}_1) \sim \mathrm{a}_4)$ as in section 2: in addition, we suppose that Ω is an open neighborhood of a point $(t,x,z)=(0,0,z^0)\in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^d$, and $\varphi_i(x)$ $(i=0,1,\dots,m-1)$ are functions in the class $\mathcal{E}^{\{\sigma\}}(V)$ satisfying $\{\varphi_j^{(\alpha)}(0)\}_{(j,\alpha)\in\Lambda} = z^0 \in \mathbb{R}^d$. We note that $Du(0,x) = \{\varphi_j^{(\alpha)}(x)\}_{(j,\alpha)\in\Lambda}$ holds. We set $\varphi_m(x) = G(0,x,\{\varphi_j^{(\alpha)}(x)\}_{(j,\alpha)\in\Lambda})$ and

$$\begin{aligned} a(x) =& \frac{\partial G}{\partial t}\big(0,x,\{\varphi_j^{(\alpha)}(x)\}_{(j,\alpha)\in\Lambda}\big) \\ &+ \sum_{(j,\alpha)\in\Lambda} \frac{\partial G}{\partial z_{j,\alpha}}\big(0,x,\{\varphi_j^{(\alpha)}(x)\}_{(j,\alpha)\in\Lambda}\big) \times \varphi_{j+1}^{(\alpha)}(x). \end{aligned}$$

Instead of $\mathrm{c}_1) \sim \mathrm{c}_4)$, we assume:

$\mathrm{c}_1)^*$ $G(t,x,z) \gg 0$ (at $(t,x,z)=(0,0,z^0)$),
$\mathrm{c}_2)^*$ $\varphi_i(x) \gg 0$ (at $x=0$) for $i=0,1,\dots,m-1$, and

$$c_3)^* \liminf_{|\beta|\to\infty}\left(\frac{a^{(\beta)}(0)}{|\beta|!^{\sigma}}\right)^{1/|\beta|} > 0.$$

As before, we set $k_{j,\alpha} = ord_t((\partial G/\partial z_{j,\alpha})(t,x,Du(t,x)),V)$ $((j,\alpha) \in \Lambda)$; then we have the expression

$$\frac{\partial G}{\partial z_{j,\alpha}}(t,x,Du(t,x)) = a_{j,\alpha}(x)t^{k_{j,\alpha}} + O(t^{k_{j,\alpha}+1}) \quad (\text{as } t \longrightarrow +0)$$

for some $a_{j,\alpha}(x) \in \mathcal{E}^{\{\sigma\}}(V)$ with $a_{j,\alpha}(x) \gg 0$ (at $x = 0$). We set

$$\Lambda(+) = \{(j,\alpha) \in \Lambda\,;\, a_{j,\alpha}(0) > 0, |\alpha| > 0\}. \tag{8.14}$$

Then we have the following result.

Theorem 8.4. *Suppose the conditions* $b_1) \sim b_5)$, *and* $c_1)^* \sim c_3)^*$. *Then, if* $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}(I \times V)$ *holds for some* $s \geq 1$, *we have*

$$s \geq 1 + \max\left[0, \max_{(j,\alpha)\in\Lambda(+)}\left(\frac{j+\sigma|\alpha|-m}{k_{j,\alpha}+m-j}\right)\right]. \tag{8.15}$$

Proof. By $c_1)^*$ and $c_2)^*$ we have $\varphi_m(x) \gg 0$. We set

$$u(t,x) = \sum_{i=0}^{m} \varphi_i(x)\frac{t^i}{i!} + t^m w(t,x):$$

then, by the same argument as in section 3 we can reduce our equation (8.13) to an equation of type (8.1) with respect to $w(t,x)$, and we can apply Theorem 8.1. We note that the condition (8.2) is verified by $c_3)^*$. □

There is also a gap between (2.4) and (8.15).

9. Some Particular Cases

In this section, we will give some applications to generalized KdV equations, non-singular Kowalwskian equations, and nonlinear Fuchsian equations.

9.1. Generalized KdV type equations

Let $k, \ell \in \{1,2,3,4,5,\dots\}$ and $m \in \{3,4,5,6,\dots\}$, and let us consider the Cauchy problem for the generalized $mk\ell$-KdV equation

$$\partial_t u = \partial_x^m u + u^k \partial_x^\ell u, \quad u(0,x) = \varphi(x), \tag{9.1}$$

where $t \in \mathbb{R}$ is the time variable, $x \in \mathbb{R}$ or $x \in \mathbb{T}$ is the space variable, and $\varphi(x)$ is an appropriate function in the Gevrey class $\mathcal{E}^{\{\sigma\}}$ for some $\sigma \geq 1$. This equation is discussed in Hannah-Himonas-Petronilho [6].

In [6], it is shown that for a suitable $\varphi(x) \in \mathcal{E}^{\{\sigma\}}$ the solution of (9.1) does not belongs to $\mathcal{E}^{\{\sigma\}}$ in the time variable: also, for the KdV equation (that is, (9.1) with $(m,k,\ell) = (3,1,1)$) in the periodic case, it is shown that the solution to the Cauchy problem with analytic data belongs to $\mathcal{E}^{\{3\}}$ in time.

The construction of a solution for (9.1) is a very difficult problem; but, once we have a solution $u(t,x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(\mathbb{R}))$ (or $u(t,x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(\mathbb{T}))$ with $I = (-\delta, \delta)$), the time regularity is obtained by Theorem 2.2, and the non-analyticity of the solution in time is obtained by Theorem 8.4.

THEOREM 9.1. (1) *Let $I = (-\delta, \delta)$, and V be an open subset of $\mathbb{R}$. If $u(t,x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(V))$ is a solution of* (9.1)*, we have $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}(I \times V)$ for any $s \geq \max\{m\sigma, \ell\sigma\}$.*

(2) *In a particular case where the initial data $\varphi(x)$ satisfies $\varphi(0) > 0$, $\varphi(x) \gg 0$ (at $x = 0$) and*

$$\liminf_{\alpha \to \infty} \Big(\frac{\varphi^{(\alpha)}(0)}{\alpha!^\sigma} \Big)^{1/\alpha} > 0, \tag{9.2}$$

we have the necessity of the condition $s \geq \max\{m\sigma, \ell\sigma\}$ in the following sense: the solution $u(t,x)$ of (9.1) *does not belong to the Gevrey class $\mathcal{E}^{\{s\}}$ in time for $1 \leq s < \max\{m\sigma, \ell\sigma\}$.*

Since the author is not familiar with regularity results on the KdV equation, it is not clear whether the assumption $u(t,x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(V))$ in the part (1) is reasonable or not. In the case $\sigma = 1$ for periodic KdV equation, this assumption is verified by Gorsky-Himonas [5]. In the case $\sigma = 3$, see Kato-Ogawa [8].

As to the result (2) we note the following: in the case $\sigma > 1$ there are many functions $\varphi(x) \in \mathcal{E}^{\{\sigma\}}(\mathbb{R})$ with compact support satisfying the condition (9.2). In the case $\sigma = 1$, the necessity of the condition $s \geq \max\{m, \ell\}$ can be verified under the initial data

$$\varphi(x) = \frac{i^{(m-\ell)/k} e^{ix}}{M - e^{ix}} \quad (M > 1)$$

or

$$\varphi(x) = \frac{1}{(i-x)^{(4p+m-\ell)/k}} \quad (p \in \mathbb{N}^*, k < 2m - 2\ell + 8p)$$

by a small modification of the argument in [6] (see also Lysik [14]).

9.2. Non-singular Kowalewskian equations

Let $m \in \mathbb{N}^*$, and let us consider the equation

$$\partial_t^m u = G\left(t, x, \{\partial_t^j \partial_x^\alpha u\}_{j+|\alpha| \leq m, j<m}\right) \tag{9.3}$$

which is a particular case of (2.1) with $\gamma = 0$ and $L = m$ (the "non-singular" means $\gamma = 0$, and the "Kowalewskian" means $L = m$).

We let Ω be an open subset of $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d$, $I = [0, T]$ (with $T > 0$), and V be an open subset of $\mathbb{R}^n$. Then, by Theorem 2.2 we have

THEOREM 9.2. (1) *If $G(t, x, z) \in \mathcal{E}^{\{s_1, \sigma, s_2\}}(\Omega)$ holds for some $s_1 \geq 1$ and $\sigma \geq s_2 \geq 1$, and if $u(t, x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(V))$ is a solution of* (9.3) *on $I \times V$, then we have $u(t, x) \in \mathcal{E}^{\{s, \sigma\}}(I \times V)$ for any $s \geq \max\{\sigma, s_1\}$.*

(2) *In particular, if $G(t, x, z) \in \mathcal{E}^{\{\sigma\}}(\Omega)$ holds for some $\sigma \geq 1$, and if $u(t, x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(V))$ is a solution of* (9.3) *on $I \times V$, then we have $u(t, x) \in \mathcal{E}^{\{\sigma\}}(I \times V)$.*

PROOF. In this case, by (2.4) and the fact that $j + |\alpha| \leq m$ we have $s_0 \leq \sigma$, and so by Theorem 2.2 we have the result. □

Usually, to show the existence of a solution $u(t, x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(V))$ we need some hyperbolicity condition; though, if we have a solution $u(t, x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(V))$, time regularity of this solution is obtained by our result without any hyperbolicity condition.

Kinoshita-Taglialatela [10] considered the time regularity of the solution of the Cauchy problem: $\partial_t^2 u - a(t)\partial_x^2 u = b(t)\partial_t + c(t)\partial_t$, $u(0, x) = u_0(x)$

and $\partial_t u(0,x) = u_1(x)$, and under suitable conditions they proved that the problem is well-posed in $\mathcal{E}^{\{s,\sigma\}}([0,T]\times\mathbb{R})$ with $0 \le \sigma - 1 \le (s-1)/s$. By Theorem 9.2 this condition is improved to $1 \le \sigma \le s$ (that is, $0 \le \sigma - 1 \le s-1$).

9.3. Nonlinear Fuchsian equations

Let $m \in \mathbb{N}^*$, and let us consider

$$(t\partial_t)^m u = G\left(t, x, \{(t\partial_t)^j \partial_x^\alpha u\}_{j+|\alpha|\le m, j<m}\right) \tag{9.3}$$

Let Ω be an open subset of $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d$, and let $G(t,x,z)$ be a C^∞ function on Ω. Let $I = [0,T]$ (with $T > 0$), and V be an open subset of $\mathbb{R}^n$. We suppose:

d$_1$) $m \ge 1, s_1 \ge 1, \sigma \ge s_2 \ge 1$;
d$_2$) $G(t,x,z) \in \mathcal{E}^{\{s_1,\sigma,s_2\}}(\Omega)$;
d$_3$) $u(t,x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(V))$ is a solution of (9.3) on $I \times V$.

Under the notation $\Theta u = \{(t\partial_t)^j \partial_x^\alpha u\}_{j+|\alpha|\le m, j<m}$ we set

$$k_{j,\alpha} = ord_t\Big(\frac{\partial G}{\partial z_{j,\alpha}}(t,x,\Theta u(t,x)), V\Big) \quad (j+|\alpha| \le m, j<m),$$
$$s_0 = 1 + \max\left[0, \max_{j+|\alpha|\le m, |\alpha|>0}\left(\frac{j+\sigma|\alpha|-m}{\min\{k_{j,\alpha}, m-j\}}\right)\right].$$

By Theorem 2.2 (or Theorem 7.7.1) we have

THEOREM 9.3. (1) *Suppose the conditions d$_1$) $\sim$ d$_3$) and the following condition: if $|\alpha| > 0$ we have $k_{j,\alpha} > 0$. Then, we have $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}(I \times V)$ for any $s \ge \max\{s_0, s_1, s_2\}$.*

(2) *In addittion, if $G(t,x,z) \in \mathcal{E}^{\{\sigma\}}(\Omega)$ and if σ satisfies*

$$1 \le \sigma \le 1 + \min\left[\infty, \min_{(j,\alpha)\in\Delta}\left(\frac{m-j-|\alpha|}{|\alpha|-k_{j,\alpha}}\right)\right] \tag{9.5}$$
$$\text{with } \Delta = \{(j,\alpha)\,;\, k_{j,\alpha} < |\alpha|\},$$

we have $u(t,x) \in \mathcal{E}^{\{\sigma\}}(I \times V)$.

PROOF. The result (1) follows from Theorem 7.7.1. Let us show (2). By Theorem 7.7.1 with $s_1 = s_2 = \sigma$ we have the result $u(t,x) \in \mathcal{E}^{\{\sigma\}}(I \times V)$ if the condition $\sigma \geq s_0$ holds, that is, if the following condition holds:

$$\sigma \geq 1 + \max\left[0, \max_{j+|\alpha|\leq m, |\alpha|>0}\left(\frac{j+\sigma|\alpha|-m}{\min\{k_{j,\alpha}, m-j\}}\right)\right] \tag{9.6}$$

which is equivalent to

$$\begin{aligned} &m - j - \min\{k_{j,\alpha}, m-j\} \geq \sigma(|\alpha| - \min\{k_{j,\alpha}, m-j\}) \\ &\text{for any } j + |\alpha| \leq m \text{ with } |\alpha| > 0. \end{aligned} \tag{9.7}$$

If $|\alpha| \leq k_{j,\alpha}$, by the condition $|\alpha| \leq m-j$ we have $|\alpha| \leq \min\{k_{j,\alpha}, m-j\}$ and so the condition (9.7) is clear from the fact that the right-hand side is nonpositive, and the left-hand side is nonnegative. If $|\alpha| > k_{j,\alpha}$, by the condition $|\alpha| \leq m-j$ we have $\min\{k_{j,\alpha}, m-j\} = k_{j,\alpha}$ and the the inequality (9.7) is equivalent to

$$\sigma \leq \frac{m-j-\min\{k_{j,\alpha}, m-j\}}{|\alpha| - \min\{k_{j,\alpha}, m-j\}} = \frac{m-j-k_{j,\alpha}}{|\alpha| - k_{j,\alpha}} = 1 + \frac{m-j-|\alpha|}{|\alpha| - k_{j,\alpha}}.$$

Thus, if we set $\Delta = \{(j,\alpha)\,;\, k_{j,\alpha} < |\alpha|\}$, our condition (9.6) is equivalent to

$$1 \leq \sigma \leq 1 + \min\left[\infty, \min_{(j,\alpha)\in\Delta}\left(\frac{m-j-|\alpha|}{|\alpha| - k_{j,\alpha})}\right)\right]. \tag{9.8}$$

This proves the result (2). □

Let us recall the following example in Tahara [17]:

Example 9.4. Let $(t,x) \in [0,T] \times \mathbb{R}$, $a > 0$, $k \in \mathbb{N}^*(= \{1,2,\ldots\})$ and let us consider

$$(t\partial_t + a)^2 u - t^k \partial_x^2 u = f(t,x). \tag{9.9}$$

The following results are known:

(1) (9.9) is uniquely solvable in $C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(\mathbb{R}))$ for any $\sigma \geq 1$.

(2) If $k \geq 2$, (9.9) is also uniquely solvable in $\mathcal{E}^{\{\sigma\}}([0,T] \times \mathbb{R})$ for any $\sigma \geq 1$.

(3) But, in the case $k=1$, the equation (9.9) is not uniquely solvable in $\mathcal{E}^{\{\sigma\}}([0,T]\times\mathbb{R})$ for any $\sigma>1$.

By (1), for any $\sigma\geq 1$ and any $f(t,x)\in\mathcal{E}^{\{\sigma\}}([0,T]\times\mathbb{R})$ we have a unique solution $u(t,x)\in C^{\infty}([0,T],\mathcal{E}^{\{\sigma\}}(\mathbb{R}))$ of (9.9); therefore the well-posedness problem in $\mathcal{E}^{\{\sigma\}}([0,T]\times\mathbb{R})$ is reduced to the following problem:

Problem 9.5. If $u(t,x)\in C^{\infty}([0,T],\mathcal{E}^{\{\sigma\}}(\mathbb{R}))$ is a solution of (9.9), can we have the result $u(t,x)\in\mathcal{E}^{\{\sigma\}}([0,T]\times\mathbb{R})$?

About this problem, in the linear case the author has given in [17] a sufficient condition for the problem to be affirmative. The result (2) of Theorem 9.3 is a generalization to the nonlinear case.

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