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Local Existence for Nonlinear Cauchy Problems with Small Analytic Data

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Abstract. We study the lifespan of solutions to fully nonlinear second-order Cauchy problems with small real- or complex-analytic data. In each case, the nonlinear term is analytic in (the complex conjugates of) the derivatives of the unknown function. This is an improvement of our previous result.

1. Introduction

Cauchy problems with small initial data have been studied by many authors. Most results are about nonlinear wave equations or nonlinear Schrödinger equations in the $\mathcal{C}^\infty$-category. On the other hand, some results about the Kirchhoff equation were derived in [1] and [2] in the real-analytic category, and m-th order equations have been solved in the Gevrey class in [2]. In our previous article [10], we studied second-order fully nonlinear Cauchy problems with small data in the real- and complex-analytic categories without hyperbolicity assumption, namely in the spirit of the Cauchy-Kowalevsky theorem.

We generalize these results in the present paper: now the nonlinear term is an analytic function not only in ∇u and $\nabla^2 u$ but also in u, $\partial_t u$ and $\nabla\partial_t u$. Moreover, we can deal with equations involving the modulus of the unknown function like $(\partial_t^2 - \partial_x^2)u = |\partial_x u|^2 = \partial_x u\, \overline{\partial_x u}$.

Now we state our result.

Let Ω be an open set of $\mathbb{R}^n_x$, $x = (x_1, \dots, x_n)$. A $\mathcal{C}^\infty$-function $\varphi(x)$ on Ω is said to be *uniformly analytic* on Ω if it satisfies

$$\exists C > 0, \forall \alpha \in \mathbb{N}^n, \sup_{x\in\Omega} |\partial^\alpha \varphi(x)| \le C^{|\alpha|+1}|\alpha|!,$$

where $\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$. We define the function space $A(\Omega)$ to be the totality of uniformly analytic functions on Ω.

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Let t be a point of $\mathbb{R}$. For $T > 0$, the open interval $]-T, T[$ is denoted by I_T. We set $\Omega_T = I_T \times \Omega \subset \mathbb{R}_t \times \mathbb{R}^n_x$.

For $k \in \mathbb{N}$, a continuous function $u(t,x)$ on $\Omega_T = I_T \times \Omega$ is said to belong to $\mathcal{C}^k(T; A(\Omega))$ if it satisfies the following two conditions:

(i) $\forall j \in \{0, \dots, k\}, \forall \alpha \in \mathbb{N}^n, \partial_t^j \partial^\alpha u \in \mathcal{C}(\Omega_T)$,

(ii) $\forall T' \in]0, T[$, $\exists C = C_{T'} > 0$, $\forall j \in \{0, \dots, k\}$, $\forall \alpha \in \mathbb{N}^n$,

$$\sup_{|t| \le T', x \in \Omega} |\partial_t^j \partial^\alpha u(t,x)| \le C^{|\alpha|+1} |\alpha|!.$$

Let $P(\partial_t, \partial_x) = \sum_{j=1}^n p_j \partial_t \partial_j + \sum_{k=1}^n \sum_{j=1}^k p_{jk} \partial_j \partial_k$ be a second-order linear partial differential operator with constant coefficients, where $\partial_j = \partial/\partial x_j$ and $p_j, p_{jk} \in \mathbb{C}$. We consider the following Cauchy problem for a fully nonlinear equation:

$$\text{(CP1)} \begin{cases} \left(\partial_t^2 - P(\partial_t, \partial_x)\right) u = f_1(t; u;\ \partial_t u, \nabla u;\ \nabla \partial_t u, \nabla^2 u), \\ u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x), \end{cases}$$

where $\partial_t = \partial/\partial t, \nabla u = (\partial_j u)_{1 \le j \le n}$ and $\nabla^2 u = (\partial_j \partial_k u)_{1 \le j \le k \le n}$. Here $\varphi(x)$ and $\psi(x)$ are uniformly analytic in an open subset Ω of $\mathbb{R}^n$. We assume that $f_1(t; X; Y; Z)$ is continuous and bounded on $\mathbb{R}_t \times \mathcal{U}$, where $\mathcal{U}$ is an open neighborhood of $(X, Y, Z) = 0 \in \mathbb{C} \times \mathbb{C}^{n+1} \times \mathbb{C}^N$, $N = n(n+3)/2$. Moreover we assume that it is complex-analytic in $\mathcal{U}$ for each fixed $t \in \mathbb{R}$ and has an expansion of the form

$$(1) \qquad f_1(t; X; Y; Z) = \sum_{L \ge 4} a_{\alpha\beta\gamma}(t) X^\alpha Y^\beta Z^\gamma, \quad L = \alpha + 2|\beta| + 3|\gamma|.$$

We shall study the lifespan of a solution when the data are small in some sense.

THEOREM 1.1. *There exist $\delta > 0$ and $\varepsilon_0 > 0$ such that the following holds for all ε with $0 < \varepsilon \le \varepsilon_0$:*

If $\sup_{x \in \Omega} |\partial^\alpha \varphi| \le \varepsilon^{|\alpha|+1} |\alpha|!$ and $\sup_{x \in \Omega} |\partial^\alpha \psi| \le \varepsilon^{|\alpha|+2} |\alpha|!$ for all $\alpha \in \mathbb{N}^n$, then (CP1) *has a solution $u(t,x) \in \mathcal{C}^2(T; A(\Omega))$ for $T = \delta/\varepsilon$.*

We formulate (CP1c), the complex version of (CP1), in the following way. Let $\varphi(x)$ and $\psi(x)$ be complex-analytic functions on an open set U of

$\mathbb{C}^n_x$. We assume[1] that f_1 is independent of t . For $T > 0$, set $B_T = \{t \in \mathbb{C}; |t| < T\}$.

THEOREM 1.2. *There exist $\delta > 0$ and $\varepsilon_0 > 0$ such that the following holds for all ε with $0 < \varepsilon \le \varepsilon_0$:*

If $\sup_{x\in U} |\partial^\alpha \varphi| \le \varepsilon^{|\alpha|+1}|\alpha|!$ and $\sup_{x\in U} |\partial^\alpha \psi| \le \varepsilon^{|\alpha|+2}|\alpha|!$ for all $\alpha \in \mathbb{N}^n$, then (CP1c) *has a unique solution $u(t,x)$ which is complex-analytic on $B_T \times U$ for $T = \delta/\varepsilon$ and satisfies the following estimate: for all T' with $0 < T' < T = \delta/\varepsilon$, there exists $C = C_{T'} > 0$ such that*

$$\sup_{|t|\le T', x\in U} |\partial^\alpha u(t,x)| \le C^{|\alpha|+1}|\alpha|!$$

holds for any $\alpha \in \mathbb{N}^n$.

REMARK 1.3. The functions φ and ψ in Theorem 1.1 extends to the $1/(4\varepsilon)$-neighborhood of Ω in $\mathbb{C}^n$ and satisfies $|\varphi^{(\alpha)}(x)| \le 2^n\varepsilon^{|\alpha|+1}|\alpha|!$, $|\psi^{(\alpha)}(x)| \le 2^n\varepsilon^{|\alpha|+2}|\alpha|!$ there. If f_1 in (CP1) is independent of t, we can apply Theorem 1.2 for a larger value of ε (hence a more modest estimate of lifespan). We get a unique real-analytic solution u to (CP1) for $|t| < \delta/(2^n\varepsilon)$, $x \in \Omega$ and it is uniformly analytic in x. The same can be said about the other theorems.

We can relax the condition on ψ when the nonlinear term belongs to a smaller class and $P = P(\partial_x) = \sum_{k=1}^n \sum_{j=1}^k p_{jk}\partial_j\partial_k$ is free from ∂_t. The second Cauchy problem is:

$$\text{(CP2)} \begin{cases} \big(\partial_t^2 - P(\partial_x)\big)u = f_2(t, u, \partial_t u, \nabla u, \nabla\partial_t u, \nabla^2 u), \\ u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x), \end{cases}$$

where $f_2(t, X, Y, Z, \Theta, \Xi)$ is continuous and bounded on $\mathbb{R}_t \times \mathcal{V}$, where $\mathcal{V}$ is an open neighborhood of $(X, Y, Z, \Theta, \Xi) = 0 \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^{n(n+1)/2}$. Moreover we assume that it is complex-analytic in $\mathcal{V}$ for each fixed $t \in \mathbb{R}$ and has an expansion of the form

$$(2) \qquad f_2(t, X, Y, Z, \Theta, \Xi) = \sum_{L_1\ge 2, L_2\ge 2} a_{\alpha\beta\gamma\lambda\mu}(t) X^\alpha Y^\beta Z^\gamma \Theta^\lambda \Xi^\mu,$$

$$L_1 = \alpha + |\gamma| + |\mu|,\ L_2 = \beta + |\gamma| + 2|\lambda| + 2|\mu|.$$

[1] We assume that f_1 is a bounded entire function in t. It is equivalent to saying that f_1 is independent of t in view of Liouville's theorem.

THEOREM 1.4. *There exist $\delta > 0$ and $\varepsilon_0 > 0$ such that the following holds for all ε with $0 < \varepsilon \le \varepsilon_0$:*

If $\sup_{x\in\Omega} |\partial^\alpha \varphi| \le \varepsilon^{|\alpha|+1}|\alpha|!$ and $\sup_{x\in\Omega} |\partial^\alpha \psi| \le \varepsilon^{|\alpha|+1}|\alpha|!$ for all $\alpha \in \mathbb{N}^n$, then (CP2) *has a solution $u(t,x) \in \mathcal{C}^2(T; A(\Omega))$ for $T = \delta/\varepsilon$.*

(This is a generalization of Theorem 1.1 of [10].*)*

REMARK 1.5. In the definitions of L, L_1 and L_2, the unknown function u and its derivatives have weights as in the following table:

	u	$\partial_t u$	∇u	$\nabla\partial_t u$	$\nabla^2 u$
$L\,(\ge 4)$	1	2	2	3	3
$L_1(\ge 2)$	1	0	1	0	1
$L_2(\ge 2)$	0	1	1	2	2

For example, $(\partial_1 u)^2$ satisfies $L \ge 4$, $L_1 \ge 2, L_2 \ge 2$. On the other hand, $(\partial_t u)^2$ does not satisfy $L_1 \ge 2$, although it satisfies $L \ge 4, L_2 \ge 2$.

REMARK 1.6. The complex-analytic version of Theorem 1.4 can be formulated in an obvious way.

Our method extends to nonlinearities involving the complex conjugates of the derivatives of the unknown function. We can deal with

$$\text{(CP3)}\begin{cases} \left(\partial_t^2 - P(\partial_x)\right)u \\ \qquad = f_3(t; u, \bar{u};\ \partial_t u, \partial_t \bar{u};\ \nabla u, \nabla \bar{u};\ \nabla\partial_t u, \nabla\partial_t \bar{u};\ \nabla^2 u, \nabla^2 \bar{u}), \\ u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x), \end{cases}$$

where $f_3(t; \tilde{X}; \tilde{Y}; \tilde{Z}; \tilde{\Theta}; \tilde{\Xi})$ is continuous and bounded on $\mathbb{R}_t \times \widetilde{\mathcal{V}}$, where $\widetilde{\mathcal{V}}$ is an open neighborhood of $(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{\Theta}, \tilde{\Xi}) = 0 \in \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^{2n} \times \mathbb{C}^{2n} \times \mathbb{C}^{n(n+1)}$. Moreover we assume that it is complex-analytic in $\widetilde{\mathcal{V}}$ for each fixed $t \in \mathbb{R}$ and has an expansion of the form

$$f_3(t; \tilde{X}; \tilde{Y}; \tilde{Z}; \tilde{\Theta}; \tilde{\Xi}) = \sum_{\tilde{L}_1 \ge 2, \tilde{L}_2 \ge 2} a_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\lambda}\tilde{\mu}}(t) \tilde{X}^{\tilde{\alpha}} \tilde{Y}^{\tilde{\beta}} \tilde{Z}^{\tilde{\gamma}} \tilde{\Theta}^{\tilde{\lambda}} \tilde{\Xi}^{\tilde{\mu}},$$

$$\tilde{L}_1 = |\tilde{\alpha}| + |\tilde{\gamma}| + |\tilde{\mu}|,\ \tilde{L}_2 = |\tilde{\beta}| + |\tilde{\gamma}| + 2|\tilde{\lambda}| + 2|\tilde{\mu}|.$$

THEOREM 1.7. *There exist $\delta > 0$ and $\varepsilon_0 > 0$ such that the following holds for all ε with $0 < \varepsilon \le \varepsilon_0$:*

If $\sup_{x\in\Omega} |\partial^\alpha \varphi| \le \varepsilon^{|\alpha|+1}|\alpha|!$ and $\sup_{x\in\Omega} |\partial^\alpha \psi| \le \varepsilon^{|\alpha|+1}|\alpha|!$ for all $\alpha \in \mathbb{N}^n$, then (CP3) *has a solution $u(t,x) \in \mathcal{C}^2(T; A(\Omega))$ for $T = \delta/\varepsilon$.*

(This is a variant of Theorem 1.4. There is a variant of Theorem 1.1, too.)

Example 1.8. The theorem above can be applied to a nonlinear wave equation

$$(\partial_t^2 - \Delta)u = |\nabla u|^2 = \sum_{j=1}^n \partial_j u\, \partial_j \bar{u}.$$

We can deal with operators with first-order terms. Let $P'(\partial_x) = \sum_{j=1}^n p'_j \partial_j$ $(p'_j \in \mathbb{C})$ be a vector field. We consider, with P involving ∂_t,

$$\text{(CP4)} \begin{cases} \big(\partial_t^2 - P(\partial_t, \partial_x) - P'(\partial_x)\big)u = f_4(t, u, \partial_t u, \nabla u, \nabla\partial_t u, \nabla^2 u), \\ u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x), \end{cases}$$

where $f_4(t, X, Y, Z, \Theta, \Xi)$ is continuous and bounded on $\mathbb{R}_t \times \mathcal{V}$, where $\mathcal{V}$ is as in (CP2). Moreover we assume that it is complex-analytic in $\mathcal{V}$ for each fixed $t \in \mathbb{R}$ and has an expansion of the form

$$\begin{aligned} &f_4(t, X, Y, Z, \Theta, \Xi) = \sum_{\ell \ge 5/2} a_{\alpha\beta\gamma\lambda\mu}(t) X^\alpha Y^\beta Z^\gamma \Theta^\lambda \Xi^\mu, \\ &\ell = \alpha + \frac{3}{2}\beta + 2|\gamma| + \frac{5}{2}|\lambda| + \frac{5}{2}|\mu|. \end{aligned}$$

THEOREM 1.9. *There exist $\delta > 0$ and $\varepsilon_0 > 0$ such that the following holds for all ε with $0 < \varepsilon \le \varepsilon_0$:*

If $\sup_{x\in\Omega} |\partial^\alpha \varphi| \le \varepsilon^{|\alpha|+1}|\alpha|!$ and $\sup_{x\in\Omega} |\partial^\alpha \psi| \le \varepsilon^{|\alpha|+3/2}|\alpha|!$ for all $\alpha \in \mathbb{N}^n$, then (CP4) *has a solution $u(t,x) \in \mathcal{C}^2(T; A(\Omega))$ for $T = \delta/\sqrt{\varepsilon}$.*

REMARK 1.10. One can easily formulate and prove variants of Theorem 1.9 like Theorems 1.2 (in the complex domain) and 1.7 (involving complex conjugates).

REMARK 1.11. The Nagumo type argument (using scales of Banach spaces consisting of bounded holomorphic functions) as in [8], [9] etc. can be useful for the study of small initial data in the complex domain. The key would be to choose several constants independently of ε. If one wants to get results in the real domain by using this method, one has to escape to the complex domain as in Remark 1.3 with some loss of lifespan.

2. The Banach Algebra $\mathcal{G}_{T,\zeta}(\Omega)$

We recall some results about a Banach algebra which will be useful in the proofs of the theorems. Set $\theta(X) = K^{-1}\sum_{k=0}^{\infty} X^k/(k+1)^2$, $K = 4\pi^2/3$ and let $D^j\theta(X)$ be its j-th derivative. If $\zeta > 0$, then a continuous function $u(t,x)$ on Ω_T is said to be an element of $\mathcal{G}_{T,\zeta}(\Omega)$ if it is infinitely differentiable in x and there exists an constant $C > 0$ such that

$$\forall \alpha \in \mathbb{N}^n, \forall t \in I_T, \quad \sup_{x\in\Omega} |\partial^\alpha u(t,x)| \le C\zeta^{|\alpha|} D^{|\alpha|}\theta(|t|/T). \tag{3}$$

We define the norm $\|u\|$ to be the infimum of such C's. Then $\mathcal{G}_{T,\zeta}(\Omega)$ becomes a Banach algebra. Moreover it is a subspace of $\mathcal{C}^0(T; A(\Omega))$ for any $T, \zeta > 0$. See [2] or [10] for proof.

For a positive integer m, we equip the direct sum $\overset{m}{\oplus}\mathcal{G}_{T,\zeta}(\Omega)$ with the norm $\|\cdot\|_m$ defined by

$$\|\tau(t,x)\|_m = \left[\sum_{j=1,\dots,m} \|\tau_j(t,x)\|^2\right]^{1/2},$$

$$\tau(t,x) = \big(\tau_1(t,x), \dots, \tau_m(t,x)\big) \in \overset{m}{\oplus}\mathcal{G}_{T,\zeta}(\Omega).$$

Set $\partial_t^{-1}u(t,x) = \int_0^t u(s,x)ds$.

PROPOSITION 2.1. *For all $(k,\alpha) \in (-\mathbb{N}) \times \mathbb{N}^n$ with $k + |\alpha| \le 0$, there exists a constant $C_{k,|\alpha|} > 0$ such that $\partial_t^k\partial^\alpha$ is an endomorphism of the Banach space $\mathcal{G}_{T,\zeta}(\Omega)$ and its norm is not larger than $C_{k,|\alpha|}T^{-k}\zeta^{|\alpha|}$.*

3. Proof of Theorems 1.1 and 1.2

First we shall show Theorem 1.1. Notice that ψ is “smaller” than φ.

Proposition 3.1. *For any $T > 0$ and any $\zeta \geq 2e^2\varepsilon$, we have $\varphi, \psi \in \mathcal{G}_{T,\zeta}(\Omega)$ and $\|\varphi\| \leq K\varepsilon$, $\|\psi\| \leq K\varepsilon^2$, $\|\partial_j\varphi\| \leq K\varepsilon^2$, $\|\partial_j\psi\| \leq K\varepsilon^3$, $\|\partial_j\partial_k\varphi\| \leq 3K\varepsilon^3$, $\|\partial_j\partial_k\psi\| \leq 3K\varepsilon^4$ for $j, k \in \{1, 2, \ldots, n\}$.*

Proof. See [2] or Propositions 3.3 and 3.4 in [10]. □

Proposition 3.2. *Set $\zeta = 2e^2\varepsilon, T = \delta/\varepsilon$ for $\varepsilon > 0, 0 < \delta < 1$. Then there exist positive constants C_1 and C_2 independent of ε and δ such that*

$$
\begin{aligned}
&\|P\partial_t^{-2}w\| \leq C_1\delta\|w\|, \quad \|P(\varphi + t\psi)\| \leq C_2\varepsilon^3,\\
&\|\partial_t^{-2}w + \varphi + t\psi\| \leq C_{-2,0}\varepsilon^{-2}\|w\| + 2K\varepsilon,\\
&\|\partial_t(\partial_t^{-2}w + \varphi + t\psi)\| \leq C_{-1,0}\varepsilon^{-1}\|w\| + K\varepsilon^2,\\
&\|\partial_j(\partial_t^{-2}w + \varphi + t\psi)\| \leq 2e^2C_{-2,1}\varepsilon^{-1}\|w\| + 2K\varepsilon^2,\\
&\|\partial_t\partial_j(\partial_t^{-2}w + \varphi + t\psi)\| \leq 2e^2C_{-1,1}\|w\| + K\varepsilon^3,\\
&\|\partial_j\partial_k(\partial_t^{-2}w + \varphi + t\psi)\| \leq (2e^2)^2C_{-2,2}\|w\| + 6K\varepsilon^3
\end{aligned}
$$

hold for any $w \in \mathcal{G}_{T,\zeta}(\Omega)$.

Proof. Apply Propositions 2.1 and 3.1. We neglect $\delta(< 1)$ in most places with the only exception of the first estimate, in which we use $\delta^2 < \delta$. For example, we have

$$
\begin{aligned}
&\|\partial_j\partial_k(t\psi)\| = \|t\partial_j\partial_k\psi\| \leq T \cdot 3K\varepsilon^4 = 3K\delta\varepsilon^3 < 3K\varepsilon^3,\\
&\|\partial_t^{-2}w\| \leq C_{-2,0}T^2\|w\| = C_{-2,0}\delta^2\varepsilon^{-2}\|w\| \leq C_{-2,0}\varepsilon^{-2}\|w\|.
\end{aligned}
$$

The inequality $\|P(\varphi + t\psi)\| \leq C_2\varepsilon^3$ follows from the estimates on $\partial_j\partial_k\varphi$, $\partial_j\partial_k(t\psi)$ and $\partial_j\psi$. □

Set $w(t, x) = \partial_t^2 u(t, x)$, then $u = \partial_t^{-2}w + \varphi + t\psi$. We define the mappings Q and $\mathcal{L}_1$ by

$$
\begin{aligned}
&Qu = (u; \partial_t u, \nabla u; \nabla\partial_t u, \nabla^2 u),\\
&\mathcal{L}_1(w) = P(\partial_t^{-2}w + \varphi + t\psi) + f_1\big(t; Q(\partial_t^{-2}w + \varphi + t\psi)\big).
\end{aligned}
$$

Then (CP1) is reduced to $w = \mathcal{L}_1(w)$. We shall find a fixed point w of $\mathcal{L}_1$ by showing that $\mathcal{L}_1$ is a contraction from a closed ball of $\mathcal{G}_{T,\zeta}(\Omega)$ to itself, where

$$T = \delta/\varepsilon, \quad \zeta = 2e^2\varepsilon \quad (\varepsilon > 0, 0 < \delta < 1). \tag{4}$$

Set $r = 2C_2\varepsilon^3/(1-2C_1\delta)$, where C_1 and C_2 are as in Proposition 3.2. If δ is sufficiently small, then there exists a positive constant C_3 independent of δ and ε such that $2C_2\varepsilon^3 \leq r \leq C_3\varepsilon^3$ holds. Let $B(r,T,\zeta) = B(r, \delta/\varepsilon, 2e^2\varepsilon) \subset \mathcal{G}_{T,\zeta}(\Omega)$ be the closed ball of radius r centered at 0. For $w \in B(r,T,\zeta)$, set $u = \partial_t^{-2}w + \varphi + t\psi$. A combination of Proposition 3.2 and (4) implies that there exists a positive constant C_4 for which we have

$$\|u\| \leq C_4\varepsilon, \|\partial_t u\| \leq C_4^2\varepsilon^2, \|\partial_j u\| \leq C_4^2\varepsilon^2, \tag{5}$$

$$\|\partial_t\partial_j u\| \leq C_4^3\varepsilon^3, \|\partial_j\partial_k u\| \leq C_4^3\varepsilon^3 \tag{6}$$

for $j, k \in \{1, 2, \ldots, n\}$. Therefore there exists a positive constant C_5 independent of ε and δ such that

$$\|f_1(t; Qu)\| \leq \sum_{L \geq 4} |a_{\alpha\beta\gamma}(t)|(C_4\varepsilon)^L \leq C_5\varepsilon^4$$

holds if $\varepsilon > 0$ is sufficiently small. The estimate $r \geq 2C_2\varepsilon^3$ implies $\|f_1(t; Qu)\| \leq r/2$ for a sufficiently small ε. Thus we find that

$$\begin{aligned}\|\mathcal{L}_1(w)\| &\leq (C_1\delta\|w\| + C_2\varepsilon^3) + \|f_1(t; Qu)\| \\ &\leq (C_1\delta r + C_2\varepsilon^3) + \|f_1(t; Qu)\| \leq r/2 + r/2 = r\end{aligned}$$

holds for $w \in B(r,T,\zeta)$. It means that $\mathcal{L}_1$ is a mapping from $B(r,T,\zeta)$ to itself if ε and δ are sufficiently small.

Next we shall show that $\mathcal{L}_1$ is a contraction mapping. We have

$$\begin{aligned}&f_1(t; X'; Y'; Z') - f_1(t; X; Y; Z) \\ &= (X' - X,\ Y' - Y,\ Z' - Z) \cdot g_1 \\ &= (X' - X) \cdot g_1^X + (Y' - Y) \cdot g_1^Y + (Z' - Z) \cdot g_1^Z,\end{aligned} \tag{7}$$

where

$$g_1 = (g_1^X, g_1^Y, g_1^Z) = \int_0^1 \nabla_{X,Y,Z} f_1\big(t; (1-s)(X,Y,Z) + s(X',Y',Z')\big) ds.$$

For $w, w' \in B(r, T, \zeta)$, set $u = \partial_t^{-2} w + \varphi + t\psi$, $u' = \partial_t^{-2} w' + \varphi + t\psi$. Then for $(X, Y, Z) = Qu$ and $(X', Y', Z') = Qu'$, we have

$$\begin{aligned}
\|X' - X\| &= \|u' - u\| = \|\partial_t^{-2}(w' - w)\| \le C_6 \varepsilon^{-2} \|w' - w\| \\
\|Y' - Y\|_{n+1} &\le \|\partial_t \partial_t^{-2}(w' - w)\| + \|\nabla \partial_t^{-2}(w' - w)\|_n \\
&\le C_6 \varepsilon^{-1} \|w' - w\|, \\
\|Z' - Z\|_N &\le \|\nabla \partial_t \partial_t^{-2}(w' - w)\|_n + \|\nabla^2 \partial_t^{-2}(w' - w)\|_{\frac{n(n+1)}{2}} \\
&\le C_6 \|w' - w\|,
\end{aligned}$$

where C_6 is a positive constant independent of ε and δ.

On the other hand, $\nabla_X f_1, \nabla_Y f_1$ and $\nabla_Z f_1$ consist of terms of the form "(a function in t)$X^\alpha Y^\beta Z^\gamma$" with $L = \alpha + 2|\beta| + 3|\gamma| \ge 3, 2, 1$ respectively. This fact, together with (5) and (6), implies that there exists a positive constant C_7 independent of ε and δ such that

$$\begin{aligned}
&\|g_1^X(t, Qu, Qu')\| \le C_7 \varepsilon^3, \quad \|g_1^Y(t, Qu, Qu')\|_{n+1} \le C_7 \varepsilon^2, \\
&\|g_1^Z(t, Qu, Qu')\|_N \le C_7 \varepsilon.
\end{aligned}$$

A combination of (7) and the inequalities above yields

$$\begin{aligned}
\|f_1(t; Qu') - f_1(t; Qu)\| &\le C_6 C_7 (\varepsilon^{-2} \cdot \varepsilon^3 + \varepsilon^{-1} \cdot \varepsilon^2 + 1 \cdot \varepsilon) \|w' - w\| \\
&= 3 C_6 C_7 \varepsilon \|w' - w\|.
\end{aligned}$$

Hence

$$\|\mathcal{L}_1(w') - \mathcal{L}_1(w)\| \le (C_1 \delta + 3 C_6 C_7 \varepsilon) \|w' - w\|,$$

which implies that $\mathcal{L}_1 \colon B(r, T, \zeta) \to B(r, T, \zeta)$ is a contraction mapping if δ and ε are sufficiently small. Its fixed point $w \in \mathcal{G}_{T,\zeta}(\Omega) \subset \mathcal{C}^0(T; A(\Omega))$ gives us a solution $u = \partial_t^{-2} w + \varphi + t\psi \in \mathcal{C}^2(T; A(\Omega))$.

Proof of Theorem 1.2. Uniqueness follows from the Cauchy-Kowalevsky theorem. We sketch the proof of existence. A complex-analytic function on $B_T \times U$ is said to be an element of $\mathcal{G}_{T,\zeta}^{\mathbb{C}}(U)$ if there exists an constant $C > 0$ such that

$$\forall \alpha \in \mathbb{N}^n, \forall t \in B_T, \quad \sup_{x \in U} |\partial^\alpha u(t, x)| \le C \zeta^{|\alpha|} D^{|\alpha|} \theta(|t|/T). \tag{8}$$

The theorem can be proved in the same way as in the real case, because $\mathcal{G}^{\mathbb{C}}_{T,\zeta}(U)$ is a Banach algebra.

4. Proof of Theorem 1.4

Proposition 3.1 must be revised in an obvious way: now ψ has the same bound as φ and $P(\partial_x)$ is free from ∂_t. The estimates in Proposition 3.2 must be replaced by the following:

$$\begin{aligned}
&\|P\partial_t^{-2}w\| \leq C_1\delta\|w\| \,(\text{unchanged}), \quad \|P(\varphi+t\psi)\| \leq C_2\varepsilon^2(\varepsilon+\delta),\\
&\|\partial_t^{-2}w+\varphi+t\psi\| \leq C_{-2,0}\varepsilon^{-2}\|w\| + K(\varepsilon+\delta),\\
&\|\partial_t(\partial_t^{-2}w+\varphi+t\psi)\| \leq C_{-1,0}\varepsilon^{-1}\|w\| + K\varepsilon \,(\text{much worse}),\\
&\|\partial_j(\partial_t^{-2}w+\varphi+t\psi)\| \leq 2e^2C_{-2,1}\varepsilon^{-1}\|w\| + K\varepsilon(\varepsilon+\delta),\\
&\|\partial_t\partial_j(\partial_t^{-2}w+\varphi+t\psi)\| \leq 2e^2C_{-1,1}\|w\| + K\varepsilon^2 \,(\text{much worse}),\\
&\|\partial_j\partial_k(\partial_t^{-2}w+\varphi+t\psi)\| \leq (2e^2)^2C_{-2,2}\|w\| + 3K\varepsilon^2(\varepsilon+\delta).
\end{aligned}$$

Only one remains unchanged and all the others have worsened more or less. Especially, two have become much worse because smaller powers of ε have appeared. Set $r = 2C_2\varepsilon^2(\varepsilon+\delta)/(1-2C_1\delta)$. Then $2C_2\varepsilon^2(\varepsilon+\delta) \leq r \leq C_3\varepsilon^2(\varepsilon+\delta)$ for some constant C_3 if δ is sufficiently small. If $\|w\| \leq r$, the estimates (5) and (6) for $u = \partial_t^{-2}w+\varphi+t\psi$ should be replaced by

$$\begin{aligned}
&\|u\| \leq C_4(\varepsilon+\delta),\ \|\partial_t u\| \leq C_4\varepsilon,\ \|\partial_j u\| \leq C_4^2\varepsilon(\varepsilon+\delta),\\
&\|\partial_t\partial_j u\| \leq C_4^2\varepsilon^2,\ \|\partial_j\partial_k u\| \leq C_4^3\varepsilon^2(\varepsilon+\delta).
\end{aligned}$$

We have, for some positive constant C_5 independent of ε and δ,

$$\begin{aligned}
&\|f_2(t,u,\partial_t u,\nabla u,\nabla\partial_t u,\nabla^2 u)\|\\
&\leq \sum_{L_1\geq 2, L_2\geq 2} |a_{\alpha\beta\gamma\lambda\mu}(t)| C_4^{L_1+L_2}(\varepsilon+\delta)^{L_1}\varepsilon^{L_2} \leq C_5(\varepsilon+\delta)^2\varepsilon^2,
\end{aligned}$$

and it is smaller than $r/2$ if ε and δ are sufficiently small. It follows that the mapping $\mathcal{L}_2$, the counterpart of $\mathcal{L}_1$, is a mapping from $B(r,T,\zeta)$ if δ and ε are sufficiently small.

Next, we have

$$
\begin{aligned}
(9) \qquad & f_2(t, X', Y', Z', \Theta', \Xi') - f_2(t, X, Y, Z, \Theta, \Xi) \\
=&(X' - X, \dots, \Xi' - \Xi) \cdot g_2(t, X, \dots, \Xi, X', \dots, \Xi') \\
=&(X' - X) g_2^X + \dots + (\Xi' - \Xi) \cdot g_2^\Xi,
\end{aligned}
$$

where $g_2 = (g_2^X, g_2^Y, g_2^Z, g_2^\Theta, g_2^\Xi)$ is the counterpart of g_1.

For $w, w' \in B(r, T, \zeta)$, set $u = \partial_t^{-2} w + \varphi + t\psi,\ u' = \partial_t^{-2} w' + \varphi + t\psi$. Then for $(X, Y, Z, \Theta, \Xi) = Qu, (X', Y', Z', \Theta', \Xi') = Qu'$, we have

$$
\begin{aligned}
&\|X' - X\| = \|u' - u\| = \|\partial_t^{-2}(w' - w)\| \le C_6 \varepsilon^{-2} \|w' - w\|, \\
&\|Y' - Y\| \le \|\partial_t \partial_t^{-2}(w' - w)\| \le C_6 \varepsilon^{-1} \|w' - w\|, \\
&\|Z' - Z\|_n = \|\nabla \partial_t^{-2}(w' - w)\|_n \le C_6 \varepsilon^{-1} \|w' - w\|, \\
&\|\Theta' - \Theta\|_n = \|\nabla \partial_t \partial_t^{-2}(w' - w)\|_n \le C_6 \|w' - w\|, \\
&\|\Xi' - \Xi\|_{n(n+1)/2} = \|\nabla^2 \partial_t^{-2}(w' - w)\|_{n(n+1)/2} \le C_6 \|w' - w\|,
\end{aligned}
$$

where C_6 is a positive constant independent of ε and δ.

On the other hand, the gradients $\nabla_X f_2, \dots, \nabla_\Xi f_2$ are sums of terms like "(a function in t)$X^\alpha Y^\beta Z^\gamma \Theta^\lambda \Xi^\mu$" with L_1 and L_2 as in the following table:

	$\nabla_X f_2$	$\nabla_Y f_2$	$\nabla_Z f_2$	$\nabla_\Theta f_2$	$\nabla_\Xi f_2$
$L_1 = \alpha + \lvert\gamma\rvert + \lvert\mu\rvert$	≥ 1	≥ 2	≥ 1	≥ 2	≥ 1
$L_2 = \beta + \lvert\gamma\rvert + 2\lvert\lambda\rvert + 2\lvert\mu\rvert$	≥ 2	≥ 1	≥ 1	≥ 0	≥ 0

There exists a positive constant C_7 independent of ε and δ such that

$$
\begin{aligned}
&\|g_2^X(t, Qu, Qu')\| \le C_7(\varepsilon + \delta)\varepsilon^2, \quad \|g_2^Y(t, Qu, Qu')\|_{n+1} \le C_7(\varepsilon + \delta)^2 \varepsilon, \\
&\|g_2^Z(t, Qu, Qu')\|_N \le C_7(\varepsilon + \delta)\varepsilon, \quad \|g_2^\Theta(t, Qu, Qu')\|_{n+1} \le C_7(\varepsilon + \delta)^2, \\
&\|g_2^\Xi(t, Qu, Qu')\|_{n+1} \le C_7(\varepsilon + \delta).
\end{aligned}
$$

A combination of (9) and the inequalities above yields, if $\varepsilon + \delta < 1$,

$$
\|f_2(t, Qu) - f_2(t, Qu')\| \le 5 C_6 C_7 (\varepsilon + \delta) \|w' - w\|.
$$

Hence we have

$$
\|\mathcal{L}_2(w') - \mathcal{L}_2(w)\| \le \big[C_1 \delta + 5 C_6 C_7 (\varepsilon + \delta) \big] \|w' - w\|,
$$

which implies that $\mathcal{L}_1\colon B(r,T,\zeta) \to B(r,T,\zeta)$ is a contraction mapping if δ and ε are sufficiently small.

5. Proof of Theorem 1.7

We shall solve

$$(\partial_t^2 - P(\partial_x))u_1 = f_3(t, Qu_1, Qu_2), \tag{10}$$

$$(\partial_t^2 - \overline{P}(\partial_x))u_2 = \overline{f_3}(t, Qu_2, Qu_1), \tag{11}$$

$$u_1(0,x) = \varphi(x),\ \partial_t u_1(0,x) = \psi(x), \tag{12}$$

$$u_2(0,x) = \overline{\varphi(x)},\ \partial_t u_2(0,x) = \overline{\psi(x)}, \tag{13}$$

where $f_3(t, Qu_1, Qu_2) = f_3(t; u_1, u_2; \partial_t u_1, \partial_t u_2; \nabla u_1, \nabla u_2; \dots)$ by abuse of notation: the order of the arguments should be changed. Moreover $\overline{P}$ and $\overline{f_3}$ are defined by

$$\overline{P}(\partial_x) = \sum_{k=1}^{n}\sum_{j=1}^{k} \overline{p_{jk}}\partial_j\partial_k,$$

$$\overline{f_3}(t; \tilde{X}; \tilde{Y}; \tilde{Z}; \tilde{\Theta}; \tilde{\Xi}) = \sum_{\tilde{L}_1 \geq 2, \tilde{L}_2 \geq 2} \overline{a_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\lambda}\tilde{\mu}}(t)}\tilde{X}^{\tilde{\alpha}}\tilde{Y}^{\tilde{\beta}}\tilde{Z}^{\tilde{\gamma}}\tilde{\Theta}^{\tilde{\lambda}}\tilde{\Xi}^{\tilde{\mu}}.$$

Set $\overrightarrow{u} = {}^t(u_1, u_2)$, $\overrightarrow{\varphi} = {}^t(\varphi, \bar{\varphi})$, $\overrightarrow{\psi} = {}^t(\psi, \bar{\psi})$, $\overrightarrow{w} = \partial_t^2\overrightarrow{u} - \overrightarrow{\varphi} - t\overrightarrow{\psi}$. Here ∂_t^2 acts componentwise on a column vector. It is also the case with $P(\partial_t, \partial_x), \overline{P}(\partial_t, \partial_x)$ and Q below. We have only to solve $\overrightarrow{w} = \mathcal{L}_3(\overrightarrow{w})$, where

$$\mathcal{L}_3(\overrightarrow{w}) = P(\partial_t^{-2}\overrightarrow{w} + \overrightarrow{\varphi} + t\overrightarrow{\psi}) + F\big(Q(\partial_t^{-2}\overrightarrow{w} + \overrightarrow{\varphi} + t\overrightarrow{\psi})\big),$$

$$F(V_1, V_2) = {}^t(f_3(t, V_1, V_2), \overline{f_3}(t, V_2, V_1)).$$

By using almost the same estimates as in the proof of Theorem 1.4, we can show that it has a unique fixed point in the closed ball $B_2(r,T,\zeta) \subset \oplus^2\mathcal{G}_{T,\zeta}(\Omega)$ for some r if ε and δ are sufficiently small. It gives a unique solution (u_1, u_2) to (10), … , (13) in $B_2(r,T,\zeta)$. By taking complex conjugates and rearranging the order, we obtain

(14) $$(\partial_t^2 - P)\overline{u_2} = f_3(t, Q\overline{u_2}, Q\overline{u_1}),$$

(15) $$(\partial_t^2 - \overline{P})\overline{u_1} = \overline{f_3}(t, Q\overline{u_1}, Q\overline{u_2}),$$

(16) $$\overline{u_2}(0,x) = \varphi(x),\ \partial_t\overline{u_2}(0,x) = \psi(x),$$

(17) $$\overline{u_1}(0,x) = \overline{\varphi(x)},\ \partial_t\overline{u_1}(0,x) = \overline{\psi(x)}.$$

The pair $(\overline{u_2}, \overline{u_1})$ satisfies the same condition as (u_1, u_2). The uniqueness of the fixed point implies that these pairs are identical. In particular, we have $u_2 = \overline{u_1}$. The solution (u_1, u_2) of (10), ... , (13) gives a solution $u = u_1$ to (CP3).

6. Proof of Theorem 1.9

For $\zeta = 2e^2\varepsilon, T = \delta/\sqrt{\varepsilon}$ $(0 < \delta < 1, 0 < \varepsilon < 1)$, there exist positive constants C_1, C_1', C_2, C_2' such that

$$\|P\partial_t^{-2}w\| \le C_1(\delta\varepsilon^{1/2} + \delta^2\varepsilon)\|w\|, \qquad \|P'\partial_t^{-2}w\| \le C_1'\delta^2\|w\|,$$
$$\|P(\varphi + t\psi)\| \le C_2\varepsilon^{5/2}, \qquad \|P'(\varphi + t\psi)\| \le C_2'\varepsilon^2.$$

Set $r = 2(C_2\varepsilon^{5/2} + C_2'\varepsilon^2)/\{1 - 2[C_1(\delta\varepsilon^{1/2} + \delta^2\varepsilon) + C_1'\delta^2]\}$. There exists a positive constant C_3 such that $C_3\varepsilon^2 \le r \le 2C_3\varepsilon^2$ for sufficiently small δ and ε. For $w \in B(r, T, \zeta)$, set $u = \partial_t^{-2}w + \varphi + t\psi$ as usual. Then there exists a positive constant C_4 for which we have

$$\|u\| \le C_4\varepsilon,\ \|\partial_t u\| \le C_4^{3/2}\varepsilon^{3/2},\ \|\partial_j u\| \le C_4^2\varepsilon^2,$$
$$\|\partial_t\partial_j u\| \le C_4^{5/2}\varepsilon^{5/2},\ \|\partial_j\partial_k u\| \le C_4^3\varepsilon^3 < C_4^3\varepsilon^{5/2}.$$

The remaining part of the proof is now routine.

7. Systems of Nonlinear Wave Equations

Some authors (e.g. [6] and [7]) have studied systems of nonlinear wave equations with different speeds of propagation in the $\mathcal{C}^\infty$-category. They obtained some results about existence or blow-up of solutions. We can consider similar systems in the real-analytic category. Let the space dimension

n be 1 for simplicity (Ω is an open interval) and assume $0 < c_1 < c_2$. We study the following simple example of a system of nonlinear wave equations:

$$\text{(CP4)} \begin{cases} (\partial_t^2 - c_1^2 \partial_x^2) u_1 = \partial_x u_1 \partial_x u_2, \\ (\partial_t^2 - c_2^2 \partial_x^2) u_2 = \partial_x u_2 \partial_x u_1, \\ u_j(0,x) = \varphi_j(x) \ (j = 1,2), \\ \partial_t u_j(0,x) = \psi_j(x) \ (j = 1,2). \end{cases}$$

There exist $\delta > 0$ and $\varepsilon_0 > 0$, *dependent on* c_2 *but independent of* c_1, such that the following holds for all ε with $0 < \varepsilon \le \varepsilon_0$:

If $\sup_{x\in\Omega} |\partial^\ell \varphi_j(x)| \le \varepsilon^{\ell+1} \ell!$ and $\sup_{x\in\Omega} |\partial^\ell \psi_j(x)| \le \varepsilon^{\ell+1} \ell!$ for $j = 1,2$ and $\ell \in \mathbb{N}$, then (CP4) has a solution $u(t,x) \in \mathcal{C}^2(T; A(\Omega))$ for $T = \delta/\varepsilon$.

This fact can be proved by introducing a mapping on a closed ball of $\oplus^2 \mathcal{G}_{T,\zeta}(\Omega)$ as in §5.

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