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A Short Time Asymptotic Behavior of The Brownian Motion on Scale Irregular Sierpinski Gaskets

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Abstract. We study short time asymptotic estimates for the transition probability density of the Brownian motion on scale irregular Sierpinski gaskets which are spatially homogeneous but do not have any exact self-similarity.

1. Introduction

Fractals are ideal examples of disordered media. We cannot define differential calculus on fractals because of the lack of smoothness. This makes difficult to analyze diffusion phenomena in a rigorous way. Several probabilists have tried to solve such problem by constructing diffusion processes on fractals. The first work was the construction of Brownian motion on the Sierpinski gasket done by Goldstein [G] and Kusuoka [Kus]. Then Barlow-Perkins [BP] showed the existence of the transition probability density of Brownian motion and the estimate for them on the Sierpinski gasket. Once a diffusion process is constructed on a metric space, it is of interest to know relations between properties of the diffusion process and that of the metric space. Short time asymptotic behavior of a heat kernel is one of them. The first step in this direction was made by Varadahan [V1]. He showed

$$\lim_{t\to 0} t\log p_t^M(x,y) = -\frac{\rho(x,y)^2}{2} \tag{1.1}$$

for a heat kernel p_t^M on a Riemannian manifold M, where $x, y \in M$ and ρ is the Riemannian metric on M. Let F be the Sierpinski gasket on $\mathbb{R}^2$ with an intrinsic geodesic metric $d_{\mathbf{2}}$, called a shortest path metric. Kumagai

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[Kum] gave Varadhan type short time asymptotic estimates for the transition probability density $p_t^{\mathbf{2}}(x,y)$ of the Brownian motion on the Sierpinski gasket: for $x,y \in F$ and $z \in [2/5,1]$,

$$\begin{aligned} \lim_{n\to\infty} \left(\left(\frac{2}{5}\right)^n z\right)^{1/(d_w^{\mathbf{2}}-1)} \log p^{\mathbf{2}}_{(\frac{2}{5})^n z}(x,y) \\ = -d_{\mathbf{2}}(x,y)^{d_w^{\mathbf{2}}/(d_w^{\mathbf{2}}-1)} G\Big(\frac{z}{d_{\mathbf{2}}(x,y)}\Big), \end{aligned} \tag{1.2}$$

where $d_w^{\mathbf{2}} = \log 5/\log 2 = 2.321928\ldots$ and G is a periodic non-constant positive continuous function with $G(5s/2) = G(s)$. This G is defined as a Legendre transform of some limiting function of a Laplace transform of a certain hitting time of Brownian motion.

Barlow-Hambly [BH] introduced scale irregular Sierpinski gaskets and showed the existence of Brownian motion on them and that of the transition probability density. They also gave the estimate for transition probability density functions as we will state below. In the present paper, we will study their short time asymptotic behaviors.

Let us give some definitions and prepare some notations to state our results. In the present paper we consider the simplest scale irregular Sierpinski gaskets. (See Section 2 for the detail of the scale irregular Sierpinski gaskets and Brownian motion on them.) We call $\eta \in \{2,3\}^{\mathbb{N}}$ an environment. We denote the element in $\{2,3\}^{\mathbb{N}}$ whose all components are 2 (resp. 3) by $\mathbf{2}$ (resp. $\mathbf{3}$).

Define the projection $\pi_j^{\mathbb{K}} : \{2,3\}^{\mathbb{K}} \to \{2,3\}$ by $\pi_j^{\mathbb{K}}\eta = \eta_j, \eta \in \{2,3\}^{\mathbb{K}}$ for each $j \in \mathbb{K}$ and the left shift operator $\theta_{\mathbb{K}} : \{2,3\}^{\mathbb{K}} \to \{2,3\}^{\mathbb{K}}$ by $\pi_j^{\mathbb{K}}\theta_{\mathbb{K}}\eta = \pi_{j+1}^{\mathbb{K}}\eta, \eta \in \{2,3\}^{\mathbb{K}}, j \in \mathbb{K}$, where $\mathbb{K}$ is equal to $\mathbb{N}$ or $\mathbb{Z}$. We will write them simply θ and π_j when there is no possibility of confusion.

We denote by $\bar{\eta}$ an element of $\{2,3\}^{\mathbb{Z}}$ such that $\pi_k\bar{\eta} = \eta_k$ if $k \in \mathbb{N}$ and $\pi_k\bar{\eta} = 2$ otherwise for $\eta \in \{2,3\}^{\mathbb{N}}$. Let $b(i)$, $m(i)$ and $t(i) : \{2,3\} \to \mathbb{R}$ for each $i = 2,3$ be given by

$$(b(2), m(2), t(2)) = (2,3,5) \quad \text{and} \quad (b(3), m(3), t(3)) = (3,6,90/7).$$

Here $b(i)$, $m(i)$ and $t(i)$ are the length, mass and time scaling factors on $\mathrm{SG}(i)$ for $i = 1,2$, see Figure 1. Refer to subsection 2.1 for details of notations. Let $B_n : \{2,3\}^{\mathbb{N}} \to [0,\infty)$, $M_n : \{2,3\}^{\mathbb{N}} \to [0,\infty)$ and T_n :

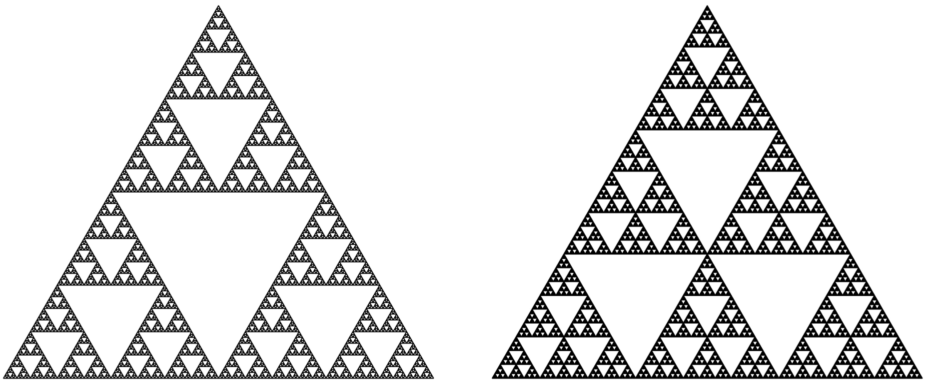


Fig. 1. The standard Sierpinski gasket SG(2) and a variant SG(3).

$\{2,3\}^{\mathbb{N}} \to [0,\infty)$ be given by $B_0(\eta) = M_0(\eta) = T_0(\eta) = 1$ and

$$(1.3) \qquad B_n(\eta) = \prod_{i=1}^{n} b(\pi_i \eta), \quad M_n(\eta) = \prod_{i=1}^{n} m(\pi_i \eta), \quad T_n(\eta) = \prod_{i=1}^{n} t(\pi_i \eta)$$

for each $n \in \mathbb{N}$. Moreover let $d_w^\eta(n)$ and $d_s^\eta(n)$, $\eta \in \{2,3\}^{\mathbb{N}}$, $n \in \mathbb{N}$ be given by

$$d_w^\eta(n) = \frac{\log T_n(\eta)}{\log B_n(\eta)}, \quad d_s^\eta(n) = 2\frac{\log M_n(\eta)}{\log T_n(\eta)}.$$

Since it is clear that $d_w^{\mathbf{2}}(n) = \log t(2)/\log b(2) = 2.321928\ldots$ and $d_w^{\mathbf{3}}(n) = \log t(3)/\log b(3) = 2.324660\ldots$ for each $n \in \mathbb{N}$, we can set $d_w^{\mathbf{2}} = d_w^{\mathbf{2}}(n)$ and $d_w^{\mathbf{3}} = d_w^{\mathbf{3}}(n)$. Also note that $d_w^{\mathbf{2}} \le d_w^\eta(n) \le d_w^{\mathbf{3}}$ for each $\eta \in \{2,3\}^{\mathbb{N}}$ and $n \in \mathbb{N}$. Let $k_\eta(n,m)$ be given by

$$(1.4) \qquad k_\eta(m,n) = \inf\{j \ge 0 : T_{m+j}(\eta)/B_{m+j}(\eta) \ge T_n(\eta)/B_m(\eta)\}$$

for $\eta \in \{2,3\}^{\mathbb{N}}$ and $m,n \in \mathbb{N}$. Let F^η be a scale irregular Sierpinski gasket and d_η a metric on F^η for $\eta \in \{2,3\}^{\mathbb{N}}$. There is a natural 'flat' measure μ^η on F^η which is characterized by the property that it assigns mass $M_n(\eta)^{-1}$ to each triangle in F^η of side length $B_n(\eta)^{-1}$.

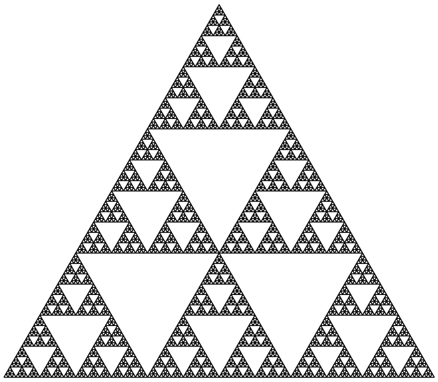


Fig. 2. The scale irregular Sierpinski gasket for $\eta = \{3, 2, 2, 3, 3, \cdots\}$.

On the above setting Barlow and Hambly [BH] have constructed a regular local Dirichlet form $(\mathcal{E}^\eta, \mathcal{F}^\eta)$ on $L^2(F^\eta, \mu^\eta)$. Let $\{P_t^\eta\}_{t\geq 0}$ be the semigroup of Markov operators associated with the Dirichlet form. They proved the following.

Theorem 1.1 (Barlow-Hambly [BH]). *Let $\eta \in \{2, 3\}^{\mathbb{N}}$.*

(a) *P_t^η has a continuous transition probability density $p_t^\eta(x, y)$ with respect to μ^η.*

(b) *There exist positive constants c_1, c_2, c_3, c_4 (independent of $\eta \in \{2, 3\}^{\mathbb{N}}$) such that if $B_m(\eta)^{-1} \leq d_\eta(x, y) < B_{m-1}(\eta)^{-1}$, $T_n(\eta)^{-1} \leq t < T_{n-1}(\eta)^{-1}$, then*

$$c_3 t^{-d_s^\eta(n)/2} \exp\Big(- c_4 \big(\frac{d_\eta(x, y)^{d_w^\eta(m+k)}}{t} \big)^{1/(d_w^\eta(m+k)-1)} \Big)$$
$$\leq p_t^\eta(x, y) \leq c_1 t^{-d_s^\eta(n)/2} \exp\Big(- c_2 \big(\frac{d_\eta(x, y)^{d_w^\eta(m+k)}}{t} \big)^{1/(d_w^\eta(m+k)-1)} \Big),$$

where $k = k_\eta(m, n)$. Also $c_3 t^{-d_s^\eta(n)/2} \leq p_t^\eta(x, x) \leq c_1 t^{-d_s^\eta(n)/2}$ for all $x \in F^\eta$.

In the present paper we consider $\{2, 3\}^{\mathbb{Z}}$ and $\{2, 3\}^{\mathbb{N}}$ as metric spaces in a usual manner. Let G_a (resp. G_b) be a graph as illustrated in Figure 3 (a) (resp. (b)) and $Y^2_\cdot$ (resp. $Y^3_\cdot$) a simple random walk on G_a (resp. G_b), starting at 0. We will denote by $\varphi(\cdot,2)$ (resp. $\varphi(\cdot,3)$) the probability generating function of the first hitting time to $\{z_1,z_2\}$ by $Y^2_\cdot$ (resp. $Y^3_\cdot$). For details see Section 3. Let $D_\delta=\{z\in\mathbb{C}:\mathrm{Re}(z)>-\delta\}$ for $\delta>0$ and fix some $\epsilon>0$. Then we have the following theorem.

THEOREM 1.2. *There exists a unique function $g:D_\epsilon\times\{2,3\}^{\mathbb{N}}\to\mathbb{C}$ satisfying the following properties:*

(a) $g(z,\eta)=\varphi(g(z/T_1(\eta),\theta\eta),\pi_1\eta),\quad g(0,\eta)=1,\quad g'(0,\eta)=-1,$

(b) *$g(z,\eta)$ is holomorphic in D_ϵ for each $\eta\in\{2,3\}^{\mathbb{N}}$,*

(c) *$g:D_\epsilon\times\{2,3\}^{\mathbb{N}}\to\mathbb{C}$ is continuous.*

Let $\Psi:[0,\infty)\times\{2,3\}^{\mathbb{Z}}\to[0,\infty)$ be a function constructed from g given by the theorem above and we will denote by Ψ^* a Legendre transform of Ψ (see Section 4 for details). Our main theorem is the following.

THEOREM 1.3. *For any compact set $K\subset(0,\infty)$,*

$$\lim_{n\to\infty}\sup_{\eta\in\{2,3\}^{\mathbb{N}}}\sup_{\substack{x,y\in F^\eta\\ z\in K}}\Big|\Big(\frac{B_n(\eta)}{T_n(\eta)}\Big)^{1/(d_w^\eta(n)-1)}\log p^\eta_{\frac{B_n(\eta)}{T_n(\eta)}z}(x,y)$$
$$+d_\eta(x,y)\Psi^*\Big(\frac{z}{d_\eta(x,y)},\theta^n\bar{\eta}\Big)\Big|=0,$$

where the second term in the left hand side is 0 when $x=y$.

This theorem corresponds to Varadhan type estimates for usual heat kernel on $\mathbb{R}^d$, (1.1) and is the extension of the result on Sierpinski gaskets, (1.2). The part $\theta^n\bar{\eta}$ expresses the feature of scale irregular Sierpinski gaskets. Also the uniformity with respect to $\eta\in\{2,3\}^{\mathbb{N}}$ of the convergence is remarkable point in this theorem. The speed of convergence is independent of not only $x,y\in F^\eta$ but also $\eta\in\{2,3\}^{\mathbb{N}}$. Theorem 1.3 is one form of asymptotic estimates with respect to the heat kernel p^η_t. Note that the exponent of the distance d_η equals 1, which is different from (1.1) or (1.2). Now, let us rewrite this result into another form. To this end we need some notations.

Similarly to (1.3) let $B_{-n} : \{2,3\}^{\mathbb{Z}} \to [0,\infty)$, and $T_{-n} : \{2,3\}^{\mathbb{Z}} \to [0,\infty)$ be given by

$$B_{-n}(\xi) = \Big\{ \prod_{i=0}^{n} b(\pi_{-i}\xi) \Big\}^{-1}, \quad T_{-n}(\xi) = \Big\{ \prod_{i=0}^{n} t(\pi_{-i}\xi) \Big\}^{-1}$$

for each $n \geq 0$ and

$$B_n(\xi) = B_n(P\xi) \ \text{ and } \ T_n(\xi) = T_n(P\xi) \ \text{ for each } \ n \geq 1,$$

where $P : \{2,3\}^{\mathbb{Z}} \to \{2,3\}^{\mathbb{N}}$ is the projection defined by $\pi_k P(\xi) = \pi_k \xi,\ \xi \in \{2,3\}^{\mathbb{Z}},\ k \in \mathbb{N}$. Let $d_w^{\xi}(n)$, $\xi \in \{2,3\}^{\mathbb{Z}}$, $n \in \mathbb{Z}$ be given by

$$d_w^{\xi}(n) = d_w^{P\xi}(n) \text{ if } n \geq 1 \ \text{ and } \ d_w^{\xi}(-n) = \frac{\log T_{-n}(\xi)}{\log B_{-n}(\xi)} \ \text{ if } \ n \geq 0.$$

Then we obtain two corollaries as another expression of Theorem 1.3. The first has a similar form to (1.2).

Corollary 1.4. *Let $K \subset (0,\infty)$ be a compact set. There exist a non-constant function $G_z : (0,\infty) \times \{2,3\}^{\mathbb{Z}} \to (0,\infty)$ for each $z \in K$ and constants $c = c(K), c' = c'(K) > 0$ such that $c \leq G_z(s,\xi) \leq c'$ for all $s \in (0,\infty)$, $\xi \in \{2,3\}^{\mathbb{Z}}$, $z \in K$ and*

$$\lim_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{x,y\in F^{\eta} \\ z\in K}} \Big| \Big(\frac{B_n(\eta)}{T_n(\eta)}\Big)^{1/(d_w^{\eta}(n)-1)} \log p^{\eta}_{\frac{B_n(\eta)}{T_n(\eta)}z}(x,y) + d_{\eta}(x,y)^{d_w^{\theta^n\bar{\eta}}(m)/(d_w^{\theta^n\bar{\eta}}(m)-1)} G_z\Big(\frac{z}{d_{\eta}(x,y)}, \theta^n\bar{\eta}\Big) \Big| = 0,$$

where $m = m_{\eta,n,z,x,y}$ is an integer with

$$B_m(\theta^n\bar{\eta})/T_m(\theta^n\bar{\eta}) \leq z/d_{\eta}(x,y) < B_{m-1}(\theta^n\bar{\eta})/T_{m-1}(\theta^n\bar{\eta}).$$

The second has a similar form to (1.1).

Corollary 1.5. *There exist a non-constant function $G : (0,\infty) \times \{2,3\}^{\mathbb{Z}} \to (0,\infty)$ and constants $c, c' > 0$ such that $c \leq G(s,\xi) \leq c'$ for all $s \in (0,\infty)$, $\xi \in \{2,3\}^{\mathbb{Z}}$ and*

$$\lim_{t\to 0} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{x,y\in F^{\eta}} \Big| t^{1/(d_w^{\bar{\eta}}(m)-1)} \log p_t^{\eta}(x,y)$$

$$+ d_\eta(x,y)^{d_w^{\bar\eta}(m)/(d_w^{\bar\eta}(m)-1)} G\Big(\frac{t}{d_\eta(x,y)}, \bar\eta\Big)\Big| = 0, \tag{1.5}$$

where $m = m_{\eta,t,x,y}$ is an integer with

$$B_m(\bar\eta)/T_m(\bar\eta) \le t/d_\eta(x,y) < B_{m-1}(\bar\eta)/T_{m-1}(\bar\eta).$$

A short time asymptotic estimate is an interesting topic by itself. Also there are some applications. For example this is a powerful tool to show Schilder type large deviation principle, see for instance [V] and [BK]. In fact our initial motivation of this paper is to study large deviation for Brownian motion on scale irregular Sierpinski gaskets. The result will be presented in a separate paper [N]. The uniformity with respect to $\eta \in \{2,3\}^{\mathbb{N}}$ in Theorem 1.3 plays an important role in that paper.

2. Dirichlet Form and Brownian Motion on Scale Irregular Sierpinski Gaskets

2.1. Scale irregular Sierpinski gaskets

We describe the construction of the simplest scale irregular Sierpinski gaskets. In this paper we restrict our attention to this case for simplicity. However our results in this paper can be extended to the general case after the obvious changes. See [BH] for a more detailed account of the general setting. Our notation is slightly different from that of [BH]. We set $(b(2), m(2)) = (2,3)$ and $(b(3), m(3)) = (3,6)$. Let $F_0 = \{a_1, a_2, a_3\}$ be the set of vertices of a unit equilateral triangle T in $\mathbb{R}^2$. We define $\psi_i^{(2)} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\psi_i^{(2)}(x) = \frac{1}{b(2)}(x - a_i) + a_i \quad \text{for each} \quad 1 \le i \le m(2).$$

Let a_4, a_5, a_6 be the midpoints of the 3 sides of F_0 and define $\psi_i^{(3)} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\psi_i^{(3)}(x) = \frac{1}{b(3)}(x - a_i) + a_i \quad \text{for each} \quad 1 \le i \le m(3).$$

For $j=1,2$, maps $\{\psi_i^{(j)}\}_{1\le i\le m(j)}$ carry the triangle T into each one of the $m(i)$ upward facing smaller triangles obtained by decomposing T into $b(i)^2$ congruent equilateral triangles of side $b(i)^{-1}$. For $B\subset\mathbb{R}^2$ set

$$\Phi^{(a)}(B)=\bigcup_{j=1}^{m(a)}\psi_j^{(a)}(B)\quad\text{for each}\quad a=2,3\quad\text{and}$$
$$\Phi_n^{(\eta)}(B)=\Phi^{(\pi_1\eta)}\circ\cdots\circ\Phi^{(\pi_n\eta)}(B).$$

Then the scale irregular Sierpinski gasket F^η associated with the environment sequence η is defined by the closure of

$$\bigcup_{n=1}^{\infty}\Phi_n^{(\eta)}(F_0).$$

Note that $F^{\mathbf{2}}$ is the standard Sierpinski gasket SG(2) and $F^{\mathbf{3}}$ is a variant SG(3), see Figure 1 . We write $W_n^\eta=\{(w_1,\dots,w_n):1\le w_i\le m(\pi_i\eta),1\le i\le n\}$ for the set of words of length n. For $w\in W_n^\eta$ we define

$$\psi_w=\psi_{w_1}^{(\pi_1\eta)}\circ\cdots\circ\psi_{w_n}^{(\pi_n\eta)}. \tag{2.1}$$

We define $F_n^\eta=\cup_{w\in W_n^\eta}\psi_w(F_0)$, and call sets of the form $\psi_w(F_0)$ n-cells for $w\in W_n^\eta$. We define natural graph structure on F_n^η by letting $\{x,y\}$ be an edge if and only if x,y both belong to the small n-cell. This graph is connected. Write $\rho_n(x,y)$ for the graph distance in F_n^η. In [BH], Barlow-Hambly have defined a metric d_η on F^η which have the following properties:

(2.2) $\quad d_\eta(x,y)=B_n(\eta)^{-1}\rho_n(x,y)$ for all $x,y\in F_n^\eta$ and $n\ge 0$.

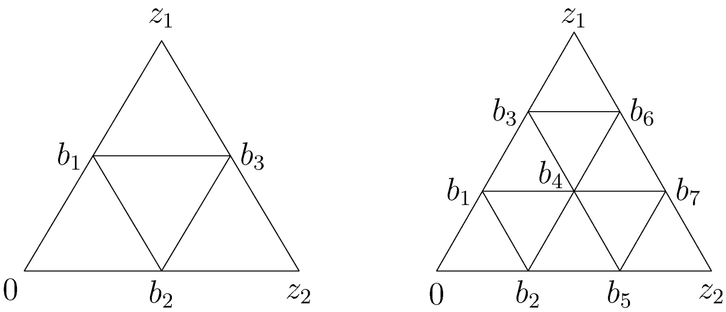
(2.3) $\quad$ There exists a constant $c>0$ such that

$$|x-y|\le d_\eta(x,y)\le c|x-y|\quad\text{for any}\quad x,y\in F^\eta.$$

The time scaling factor $t(2)$ associated SG(2) is defined by the expectation of the first hitting time to $\{z_1,z_2\}$ by $Y_\cdot^2$. The time scaling factors $t(3)$ associated SG(3) is defined similarly. An easy computation shows that $t(2)=5$ and $t(3)=90/7$.

2.2. Dirichlet form and Brownian motion

We can construct a regular local Dirichlet form $\mathcal{E}^\eta$ on $L^2(F^\eta,\mu)$. See [BH] for details. Let $\{P_t^\eta\}_{t\ge0}$ be the semigroup of Markov operators associated with the Dirichlet form $(\mathcal{E}^\eta,\mathcal{F}^\eta)$ on $L^2(F^\eta,\mu^\eta)$. As $(\mathcal{E}^\eta,\mathcal{F}^\eta)$ is regular



$(a)\ \pi_1\eta = 2$ $(b)\ \pi_1\eta = 3$

Fig. 3. F_1^η.

and local, there exists a Feller diffusion $(X_t,\ t \geq 0,\ P_x^\eta,\ x \in F^\eta)$ with semigroup $\{P_t^\eta\}_{t\geq 0}$, which is called *Brownian motion* on F^η in [BH]. Besides they remark that $G_\lambda^\eta = \int e^{-\lambda t} P_t^\eta dt$ has a bounded symmetric density $u_\lambda^\eta(x,y)$ with respect to μ^η and $u_\lambda^\eta(x,\cdot)$ is continuous for each x.

3. Properties of Moment Generating Function of W

For Brownian motion $X_\cdot$ on F^η, define the stopping times S^k and S_i^k by $S^k = S_0^k = \inf\{t \geq 0 : X_t \in F_k^\eta\}$ and

$$S_i^k = \inf\{t > S_{i-1}^k : X_t \in F_k^\eta \setminus \{X_{S_{i-1}^k}\}\} \quad \text{for} \quad i \in \mathbb{N}.$$

These are the times of the successive visits to F_k^η by $X_\cdot$. Also for $X_\cdot$, let $W = \inf\{t \geq 0 : X_t \in F_0^\eta \setminus \{X_0\}\}$. Let $Y_i^m = X_{S_i^m}$, then $Y_\cdot^m$ is a simple random walk on F_m^η. Similarly define stopping times $S^k(Y_\cdot^m)$, $S_i^k(Y_\cdot^m)$, $i \in \mathbb{N}$ and $W(Y_\cdot^m)$ by $S^k(Y_\cdot^m) = S_0^k(Y_\cdot^m) = \inf\{n \in \mathbb{Z}_+ : Y_n^m \in F_k^\eta\}$, $S_i^k(Y_\cdot^m) = \inf\{n > S_{i-1}^k(Y_\cdot^m) : Y_n^m \in F_k^\eta \setminus \{Y_{S_{i-1}^k(Y_\cdot^m)}\}\}$ for $i \in \mathbb{N}$ and $W(Y_\cdot^m) = \inf\{n \in \mathbb{Z}_+ : Y_n^m \in F_0^\eta \setminus \{Y_0^m\}\}$. We have the following theorem in [BH] .

THEOREM 3.1. (i) $E_0^\eta[W] = 1$ *for all* $\eta \in \{2,3\}^{\mathbb{N}}$. (ii) $W(Y_\cdot^n)/T_n(\eta) \to W$ P_0^η*-a.s. for each* $\eta \in \{2,3\}^{\mathbb{N}}$ *and* $\sup_{\eta\in\{2,3\}^{\mathbb{N}}} E_0^\eta[|W(Y_\cdot^n)/T_n(\eta) - W|^2] \to 0$ *as* $n \to \infty$.

Note that a probability generating function of $W(Y_\cdot^1)$ with respect to P_0^η depends only on $\pi_1\eta$. So let us define $\varphi : [0,1] \times \{2,3\} \to \mathbb{R}$ by

$$\varphi(u, \pi_1\eta) = E_0^\eta[u^{W(Y_\cdot^1)}].$$

Also let $f_n(u,\eta)$ be a probability generating function of $W(Y_\cdot^n)$ with respect to P_0^η:

$$f_n(u,\eta) = E_0^\eta[u^{W(Y_\cdot^n)}]. \tag{3.1}$$

Since $E_0^\eta[u^{S_k^1(Y_\cdot^n)-S_{k-1}^1(Y_\cdot^n)}] = E_0^{\theta\eta}[u^{S_k^0(Y_\cdot^{n-1})-S_{k-1}^0(Y_\cdot^{n-1})}] = f_{n-1}(u,\theta\eta)$ and $W(Y_\cdot^n) = \sum_{k=1}^{W(Y_\cdot^1)}(S_k^1(Y_\cdot^n) - S_{k-1}^1(Y_\cdot^n))$ for each $k \in \mathbb{N}$, we have

$$f_n(u,\eta) = \varphi(f_{n-1}(u,\theta\eta), \pi_1\eta). \tag{3.2}$$

In addition we define moment-generating functions $g(z,\eta)$, $g_n(z,\eta) : \{z \in \mathbb{C} : Re(z) \geq 0\} \times \{2,3\}^{\mathbb{N}} \to \mathbb{C}$, $n = 0,1,2,\dots$ to be

$$g_n(z,\eta) = E_0^\eta\Big[\exp\Big(-z\frac{W(Y_\cdot^n)}{T_n(\eta)}\Big)\Big] \quad \text{and} \quad g(z,\eta) = E_0^\eta[\exp(-zW)].$$

We check at once that for all $\eta \in \{2,3\}^{\mathbb{N}}$, $s \geq 0$ and $n \geq 1$

$$g_{n+1}(T_1(\eta)s, \eta) = \varphi(g_n(s,\theta\eta), \pi_1\eta), \tag{3.3}$$

which is clear from (3.2).

Let $G_n(s,\eta) = -\log g_n(s,\eta)$ for $s \geq 0$ and $\eta \in \{2,3\}^{\mathbb{N}}$. Noting that $\varphi(u,\pi_1\eta) = g_1(-T_1(\eta)\log u, \eta)$ for all $u \in (0,1]$ and $\eta \in \{2,3\}^{\mathbb{N}}$, by (3.3) we have

$$G_{n+1}(s,\eta) = G_1\Big(T_1(\eta)G_n\Big(\frac{s}{T_1(\eta)}, \theta\eta\Big), \eta\Big) \tag{3.4}$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $s \geq 0$ and $n \geq 1$. By the way the function $G_1(\cdot,\eta)$ is extensible to a holomorphic function in a neighborhood of the origin. This follows from the exponential decay of the tail probability of $W(Y_\cdot^1)$.

Lemma 3.2. *For any* $\eta \in \{2,3\}^{\mathbb{N}}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log P_0^\eta[W(Y_\cdot^1) \geq n] < 0.$$

PROOF. We only consider the case (b) in Figure 3. Let $v_n^j = P_{b_j}^\eta[W(Y_\cdot^1) = n]$ for $j = 1 \dots 7$ and $v_k^0 = P_0^\eta[W(Y_\cdot^1) = n]$. Note that $v_n^1 = v_n^2$, $v_n^3 = v_n^5$ and $v_n^6 = v_n^7$ by the symmetry of the figure. Set $v_n = (v_n^0, v_n^1, v_n^3, v_n^4, v_n^6)'$. Then the markov property shows that there exists a matrix such that $v_n = Av_{n-1}$. We can solve this difference equation by considering the Jordan canonical form of A. Noting that $v_n^0 \to 0$ as $n \to \infty$, it is easy to complete the proof. □

Let $B(0,\delta) = \{z \in \mathbb{C} : |z| < \delta\}$ for $\delta > 0$. It is easy to check that $G_1(0,\eta)$=0, $G_1'(0,\eta) = 1$. Therefore there are $\epsilon_0 > 0$ and $M > 0$ such that

$$|G_1(z,\eta)| \leq M|z| \quad \text{for any} \quad z \in B(0,\epsilon_0) \quad \text{and} \quad \eta \in \{2,3\}^{\mathbb{N}}.$$

Constants ϵ_0 and M are not depending on $\eta \in \{2,3\}^{\mathbb{N}}$, because G_1 depends only on $\pi_1\eta = \eta_1 \in \{2,3\}$. Let $D_\delta = \{z \in \mathbb{C} : \mathrm{Re}(z) > -\delta\}$ for $\delta > 0$. We can show the following in the same way as Proposition 3.7 in [Kus].

PROPOSITION 3.3. *Let $\epsilon_1 = (\epsilon_0/2) \wedge (1/(2M))$. Then for each $n \in \mathbb{N}$, there are holomorphic functions $H_n(z,\eta)$ defined in $B(0,\epsilon_1)$ such that*

$$G_n(z,\eta) = z(1 + H_n(z,\eta)), \quad |H_n(z,\eta)| \leq 2M|z|$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$. In particular for $\epsilon_2 = \epsilon_1/4$,

$$\sup_{\eta \in \{2,3\}^{\mathbb{N}}, n \in \mathbb{N}} |g_n(-\epsilon_2,\eta)| \vee |g(-\epsilon_2,\eta)| < \infty.$$

PROOF. We only give the proof of the latter part. It follows immediately that $\sup_{n \in \mathbb{N}, \eta \in \{2,3\}^{\mathbb{N}}} g_n(-2\epsilon_2,\eta) < \infty$ from the former part, which implies that the sequence $\{\exp(\epsilon_2 W(Y_\cdot^n)/T_n(\eta))\}_{n=1}^\infty$ is uniformly integrable. Also we have already known that $W(Y_\cdot^n)/T_n(\eta) \to W$ as $n \to \infty$ from Theorem 3.1. These imply $\sup_{\eta \in \{2,3\}^{\mathbb{N}}} E_0^\eta[\exp(\epsilon_2 W)] < \infty$. This completes our assertion. □

Let $\epsilon = \epsilon_2/4$. Then we have the following lemma.

LEMMA 3.4.

(1) *Functions $g_n(\cdot,\eta) : D_\epsilon \to \mathbb{C}$ is holomorphic for each $n \in \mathbb{N}$, $\eta \in \{2,3\}^{\mathbb{N}}$ and $g_n : D_\epsilon \times \{2,3\}^{\mathbb{N}} \to \mathbb{C}$ is continuous for each $n \in \mathbb{N}$.*

(2) *For each compact set* $K \subset D_\epsilon$,

$$\sup_{z\in K,\eta\in\{2,3\}^{\mathbb{N}}} |g(z,\eta) - g_n(z,\eta)| \to 0 \quad \textit{as} \quad n \to \infty.$$

In particular from (1) and (2), $g(\cdot,\eta) : D_\epsilon \to \mathbb{C}$ is holomorphic for each $\eta \in \{2,3\}^{\mathbb{N}}$ and $g : D_\epsilon \times \{2,3\}^{\mathbb{N}} \to \mathbb{C}$ is continuous.

Proof. (1) It is easy to see that $g_n(\cdot,\eta) : D_\epsilon \to \mathbb{C}$ is holomorphic for each $n \in \mathbb{N}$, $\eta \in \{2,3\}^{\mathbb{N}}$ from Proposition 3.3. Also noting that $g_n(z,\eta)$ depends only on first n components of η, the continuity of g_n is obvious.
(2) Since there is a constant $c = c(\epsilon) > 0$ such that

$$\begin{aligned}
&|g(z,\eta) - g_n(z,\eta)| \le E_0^\eta\Big[\Big|\int_{W(Y_\cdot^n)/T(\eta)}^{W} z\exp(-zt)dt\Big|\Big] \\
&\le |z|E_0^\eta\Big[e^{\epsilon(W+W(Y_\cdot^n)/T_n(\eta))}\Big|W - \frac{W(Y_\cdot^n)}{T_n(\eta)}\Big|\Big] \le c|z|E_0^\eta\Big[\Big|W - \frac{W(Y_\cdot^n)}{T_n(\eta)}\Big|^2\Big]^{1/2}
\end{aligned}$$

for all $z \in D_\epsilon$ by Schwarz inequality and Proposition 3.3, Theorem 3.1 (ii) implies our assertion. □

The moment-generating function g is characterized by Theorem 1.2. Now let us prove this theorem.

Proof of Theorem 1.2. It is easy to check that the moment-generating function g satisfies (a), (b) and (c). Let us prove the uniqueness. Though we follow the proof of Theorem 8.2 in [H], we need some improvements. Let $U_1(z,\eta)$ and $U_2(z,\eta)$ be functions satisfying (a), (b) and (c). We choose $\epsilon_1 > 0$ such that $\epsilon_1 < \epsilon$. By assumption for any $\eta \in \{2,3\}^{\mathbb{N}}$ there are $a_k(\eta), b_k(\eta) \in \mathbb{C}, k \ge 2$ such that

$$U_1(z,\eta) = 1 - z + \sum_{k=2}^{\infty} a_k(\eta)z^k \ \text{ and } \ U_2(z,\eta) = 1 - z + \sum_{k=2}^{\infty} b_k(\eta)z^k$$

for any $z \in B(0,\epsilon_1)$. Setting $M_i(r) = \sup\{|U_i(z,\eta)| : |z| = r, \eta \in \{2,3\}^{\mathbb{N}}\}$ for $i = 1,2$ and $r > 0$, we have

$$|a_n(\eta)| \le \frac{M_1(\epsilon_1)}{\epsilon_1^n}, \quad |b_n(\eta)| \le \frac{M_2(\epsilon_1)}{\epsilon_1^n}. \tag{3.5}$$

Note that M_1 and M_2 exist by the continuity of U_1 and U_2. Define the function $\gamma : D_\epsilon \times \{2,3\}^{\mathbb{N}} \to \mathbb{C}$ by

$$\gamma(z,\eta) = z\sum_{k=2}^{\infty}(a_k(\eta) - b_k(\eta))z^{k-2}.$$

Then we have $\lim_{z\to 0}\sup_{\eta\in\{2,3\}^{\mathbb{N}}}|\gamma(z,\eta)| = 0$ from (3.5). By the way there are constants $s_0 = s_0(\epsilon_1) > 0$ and $c = c(\epsilon_1)$ such that

$$\sup_{\eta\in\{2,3\}^{\mathbb{N}}, i=1,2}|U_i(s,\eta)| \leq 1 - s + \frac{c}{\epsilon_1 - s}s^2 < 1$$

for any $s > 0$ with $s < s_0$. Also we have $|\varphi'(z,\pi_1\eta)| \leq T_1(\eta)$ for $z \in B(0,1)$. Therefore since

$$\begin{aligned}&|T_1(\eta)s\gamma(T_1(\eta)s,\eta)|\\ &\qquad = |\varphi\big(U_1(s,\theta\eta),\pi_1(\eta)\big) - \varphi\big(U_2(s,\theta\eta),\pi_1(\eta)\big)| \leq T_1(\eta)|s\gamma(s,\theta\eta)|\end{aligned}$$

for any $s < s_0$ and $\eta \in \{2,3\}^{\mathbb{N}}$ from (a), we get $|\gamma(s,\eta)| \leq |\gamma(s/T_1(\eta),\theta\eta)|$. Hence we obtain

$$|\gamma(s,\eta)| \leq \lim_{n\to\infty}|\gamma(\frac{s}{T_n(\eta)},\theta^n\eta)| = 0 \quad \text{for all} \quad s < s_0 \quad \text{and} \quad \eta \in \{2,3\}^{\mathbb{N}}.$$

Uniqueness theorem implies our assertion. □

At the end of this section, we give another important property of g.

LEMMA 3.5. *There exist constants $c, c' > 0$ such that*

$$\exp(-cs^{1/d_w^\eta(n)}) \leq g(s,\eta) \leq \exp(-c's^{1/d_w^\eta(n)})$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $n \geq 0$, $s \in [T_n(\eta), T_{n+1}(\eta)]$.

PROOF. We follow the argument of Proposition 3.2 in [Kum1].
(i) Proof of the lower bounds: By Jensen inequality $g(s,\eta) = E_0^\eta[e^{-sW}] \geq e^{-s}$ for all $\eta \in \{2,3\}^{\mathbb{N}}$ and $s \in [0,\infty)$. We can choose $c > 0$ such that

$$e^{-s} \geq e^{-t(3)} \geq 4\exp(-cs^{1/d_w^{\mathbf{3}}}) \quad \text{for any} \quad s \in [1, t(3)].$$

Hence we obtain

$$g(s,\hat{\eta}) \geq e^{-s} \geq 4\exp(-cs^{1/d_w^{\mathbf{3}}}) \geq 4\exp(-cs^{1/d_w^{\tilde{\eta}}(n)})$$

for all $n \geq 1$, $\tilde{\eta},\hat{\eta} \in \{2,3\}^{\mathbb{N}}$, $s \in [1,t(3)]$. Therefore by the definition (3.1) of $f_n(s,\eta)$ and Theorem 1.2 (a) we get

$$\begin{aligned} g(T_n(\eta)s,\eta) &= f_n(g(s,\theta^n\eta),\eta) \geq 4\Big(\frac{g(s,\theta^n\eta)}{4}\Big)^{B_n(\eta)} \\ &\geq 4\exp(-cB_n(\eta)s^{1/d_w^{\eta}(n)}) = 4\exp(-c(T_n(\eta)s)^{1/d_w^{\eta}(n)}) \end{aligned}$$

for any $s \in [1,t(3)]$, $\eta \in \{2,3\}^{\mathbb{N}}$. Thus we see that

$$g(s,\eta) \geq 4\exp(-cs^{1/d_w^{\eta}(n)})$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$ and $s \in [T_n(\eta),T_{n+1}(\eta)]$.
(ii) Proof of the upper bounds: It is easy to see that $\sup_{\eta\in\{2,3\}^{\mathbb{N}}} g(1,\eta) < 1$ by Theorem 1.2 (c). So there exists $c' > 0$ such that

$$g(s,\eta) \leq \exp(-c't(3)^{1/d_w^{\mathbf{2}}}) \leq \exp(-c's^{1/d_w^{\mathbf{2}}})$$

for any $s \in [1,t(3)]$ and $\eta \in \{2,3\}^{\mathbb{N}}$. So we have

$$g(s,\hat{\eta}) \leq \exp(-c's^{1/d_w^{\mathbf{2}}}) \leq \exp(-c's^{1/d_w^{\tilde{\eta}}(n)})$$

for any $\tilde{\eta},\hat{\eta} \in \{2,3\}^{\mathbb{N}}, s \in [1,t(3)]$. Therefore by the definition (3.1) of $f_n(s,\eta)$ and Theorem 1.2 (a),

$$\begin{aligned} g(T_n(\eta)s,\eta) &= f_n(g(s,\theta^n\eta),\eta) \leq g(s,\theta^n\eta)^{B_n(\eta)} \\ &\leq \exp(-c'B_n(\eta)s^{1/d_w^{\eta}(n)}) = \exp(-c'(T_n(\eta)s)^{1/d_w^{\eta}(n)}) \end{aligned}$$

for any $s \in [1,t(3)]$. As a result we deduce that

$$g(s,\eta) \leq \exp(-c's^{1/d_w^{\eta}(n)})$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$ and $s \in [T_n(\eta),T_{n+1}(\eta)]$. This completes the proof of the lemma. $\square$

4. Properties of $\Psi(s,\xi)$

From the definition of $\varphi(u,\pi_1\eta)$, we have $\varphi(u,\pi_1\eta) = P_0^\eta[W(Y_\cdot^1) = b(\pi_1\eta)]u^{b(\pi_1\eta)} + \sum_{j=b(\pi_1\eta)+1}^\infty P_0^\eta[W(Y_\cdot^1) = j]u^j$. Let us define $h : [0,1] \times \{2,3\} \to [0,\infty)$ by

$$h(u,k) = -\log \frac{\varphi(u,k)}{u^{b(k)}}.$$

Note that $h(\cdot,k) : [0,1] \to [0,\infty)$ is continuous for each $k \in \{2,3\}$ and there exist constants $c, c' > 0$ such that

$$0 \le h(u,k) \le c \quad \text{and} \quad -c \le h'(u,k) \le -c' \tag{4.1}$$

for any $u \in [0,1]$ and $k \in \{2,3\}$ by the definition. Here we define $L_n : [0,\infty) \times \{2,3\}^{\mathbb{N}} \to [0,\infty)$ by

$$L_n(s,\eta) = -\frac{\log g(T_n(\eta)s,\eta)}{B_n(\eta)}.$$

Note that $L_n(s,\eta)$ is a concave function with respect to s for each $\eta \in \{2,3\}^{\mathbb{N}}$ and $n \in \mathbb{N}$. In the case $\eta = \mathbf{2}$ (i.e. usual Sierpinski gasket) $\lim_{n\to\infty} L_n(s,\mathbf{2})$ exists for all $s \ge 0$ (see Lemma 3.3 in [Kum]). But $\lim_{n\to\infty} L_n(s,\eta)$ does not necessarily exist for all $\eta \in \{2,3\}^{\mathbb{N}}$. By using Theorem 1.2 (a) repeatedly, we have

$$g(T_n(\eta)s,\eta) = g(s,\theta^n\eta)^{B_n(\eta)} \prod_{k=1}^n \exp\big(-B_{k-1}(\eta)h\big(g\big(\frac{T_n(\eta)}{T_k(\eta)}s,\theta^k\eta\big),\pi_k\eta\big)\big)$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $s \in [0,\infty)$ and $n \in \mathbb{N}$. Therefore an easy computation shows that

$$\begin{aligned}
L_n(s,\eta) &= -\log g(s,\theta^n\eta) + \frac{1}{b(\pi_0(\theta^n\bar{\eta}))} h\big(g(s,\theta^n\eta),\pi_0(\theta^n\bar{\eta})\big) \\
&\quad + \sum_{k=1}^{n-1} B_{-(n-k)}(\theta^n\bar{\eta}) h\big(g\big(\frac{s}{T_{-(n-k-1)}(\theta^n\bar{\eta})},\theta^{-(n-k)}\theta^n\eta\big),\pi_{-(n-k)}(\theta^n\bar{\eta})\big)\big) \\
&\le -\log g(s,\theta^n\eta) + \frac{1}{b(\pi_0(\theta^n\bar{\eta}))} h\big(g(s,\theta^n\eta),\pi_0(\theta^n\bar{\eta})\big) \\
&\quad + \sum_{j=1}^{\infty} B_{-j}(\theta^n\bar{\eta}) h\big(g\big(\frac{s}{T_{-(j-1)}(\theta^n\bar{\eta})},P(\theta^{-j}\theta^n\bar{\eta})\big),\pi_{-j}(\theta^n\bar{\eta})\big)\big).
\end{aligned}$$

This motivates us to consider the following functions. Let $F_k : [0,\infty) \times \{2,3\}^{\mathbb{Z}} \to [0,\infty)$ for $k \geq 0$ be given by

$$F_k(s,\xi) = B_{-k}(\xi) h\big(g(s/T_{-(k-1)}(\xi), P(\theta^{-k}\xi)), \pi_{-k}\xi\big), \quad k \geq 1$$

and

$$F_0(s,\xi) = \frac{1}{b(\pi_0\xi)} h\big(g(s, P(\xi)), \pi_0\xi\big).$$

By (4.1) we see that $\sup_{s\in[0,\infty),\xi\in\{2,3\}^{\mathbb{Z}}} \sum_{k=0}^{\infty} F_k(s,\xi) < \infty$. So we can define the function $\Psi : [0,\infty) \times \{2,3\}^{\mathbb{Z}} \to [0,\infty)$ to be

$$\Psi(s,\xi) = -\log g(s, P(\xi)) + \sum_{k=0}^{\infty} F_k(s,\xi). \tag{4.2}$$

Also we have

$$\begin{aligned} F_k'(s,\xi) = &-\frac{B_{-k}(\xi)}{T_{-(k-1)}(\xi)} \\ &\times h'\big(g(\frac{s}{T_{-(k-1)}(\xi)}, P(\theta^{-k}\xi)), \pi_{-k}\xi\big) E_0^{P(\theta^{-k}\xi)}\big[W \exp(-\frac{sW}{T_{-(k-1)}(\xi)})\big] \end{aligned}$$

for each $k \geq 1$ and

$$F_0'(s,\xi) = -\frac{1}{b(\pi_0\xi)} h'\big(g(s, P(\xi)), \pi_0\xi\big) E_0^{P(\xi)}\big[W \exp(-sW)\big].$$

By Lemma 3.5 and Schwarz inequality there are constants $c, c' > 0$ such that

$$E_0^{P(\theta^{-k}\xi)}\big[W \exp(-\frac{sW}{T_{-(k-1)}(\xi)})\big] \leq c \exp(-c'(t(2)^k s)^{1/d_w^{\mathbf{3}}}) \tag{4.3}$$

for any $s > 0$ and $k \geq K$, where $K = K(s)$ is a non-negative integer with $t(2)^K s \geq 1$. For each $s_0 \in (0,\infty)$, let $(a,b) \subset (0,\infty)$ be an open interval containing the s_0. Then from (4.1) and (4.3), we see that

$$\sum_{k=K}^{\infty} |F_k'(s,\xi)| \leq c'' \sum_{k=K}^{\infty} \big(\frac{t(3)}{b(3)}\big)^k \exp(-c'(t(2)^k a)^{1/d_w^{\mathbf{3}}}) < \infty$$

for any $s \in (a,b)$. Hence $\Psi(s,\xi)$ is differentiable with respect to s for each $\xi \in \{2,3\}^{\mathbb{Z}}$ and we obtain

$$\Psi'(s,\xi) = -\frac{g'(s,P(\xi))}{g(s,P(\xi))} + \sum_{k=0}^{\infty} F_k'(s,\xi). \tag{4.4}$$

$\Psi(s,\xi)$ has the following properties and approximates $L_n(s,\eta)$ in the following sense.

Lemma 4.1.

(1) $\Psi \ : \ [0,\infty)\times\{2,3\}^{\mathbb{Z}} \to [0,\infty)$ *is continuous.*

(2) $\sup_{s\in[0,\infty),\xi\in\{2,3\}^{\mathbb{Z}}} |L_n(s,P(\xi)) - \Psi(s,\theta^n\xi)| \to 0$ *as* $n\to\infty$.

(3) *The function* Ψ *satisfies* $\Psi(0,\xi)=0$ *and the following functional equation* $\Psi(T_1(\xi)s,\xi) = B_1(\xi)\Psi(s,\theta\xi)$ *for all* $s\in[0,\infty)$ *and* $\xi\in\{2,3\}^{\mathbb{Z}}$.

Proof. (1) From Theorem 1.2 (c) it suffices to show the continuity the second term of the right hand side of (4.2). It is easy to see that $F_k : [0,\infty)\times\{2,3\}^{\mathbb{N}} \to [0,\infty)$ is continuous by Theorem 1.2 (c) and continuity of $h(\cdot,\pi_{-k}\xi)$ for each $k\geq 0$. Since there is some constant $c>0$ such that

$$\sup_{\substack{s\in[0,\infty)\\ \xi\in\{2,3\}^{\mathbb{Z}}}} \Big|\sum_{k=1}^{\infty} F_k(s,\xi) - \sum_{k=1}^{m} F_k(s,\xi)\Big| \leq c\sum_{k=m+1}^{\infty}\frac{1}{b(2)^{k+1}} = \frac{c}{b(2)^m}$$

for any $m\in\mathbb{N}$ by (4.1), the partial sum $\sum_{k=1}^m F_k(s,\xi)$ converges uniformly with respect to $s\in[0,\infty)$ and $\xi\in\{2,3\}^{\mathbb{Z}}$ as $m\to\infty$. This implies our assertion.

(2) It is easily seen that $B_{-(n-k)}(\theta^n\xi) = B_{k-1}(\xi)/B_n(\xi)$, $T_{-(n-k-1)}(\theta^n\xi) = T_k(\xi)/T_n(\xi)$, $\pi_{-(n-k)}\theta^n\xi = \pi_k P\xi$ and $P(\theta^n\xi) = \theta^n(P\xi)$. These imply that

$$F_{n-k}(s,\theta^n\xi) = \frac{B_{k-1}(\xi)}{B_n(\xi)} h\big(g\big(\frac{T_n(\xi)}{T_k(\xi)}s, \theta^k P\xi\big), \pi_k P\xi\big)$$

for all $k\in\mathbb{N}$ with $1\leq k\leq n-1$ and

$$F_0(s,\theta^n\xi) = \frac{1}{b(\pi_n P\xi)} h\big(g(s,\theta^n P\xi), \pi_n P\xi\big).$$

Therefore it follows that

$$\Big|\frac{\log g(T_n(\xi)s, P\xi)}{B_n(\xi)} - \Psi(s,\theta^n\xi)\Big| = \Big|\sum_{k=n}^{\infty} F_k(s,\theta^n\xi)\Big| \le \frac{c}{b(2)^n}$$

for some constant $c > 0$ by (4.1).
(3) It is easy check that $B_1(\xi)F_k(s,\theta\xi) = F_{k-1}(T_1(\xi)s,\xi)$ for all $k \in \mathbb{N}$ and $B_1(\xi)(-\log g(s, P(\theta\xi)) + F_0(s,\theta\xi)) = -\log g(T_1(\xi)s, P(\xi))$, where we use Theorem 1.2 (a) in the second equation. These imply our assertion. □

Next we state some properties of Ψ'.

LEMMA 4.2.

(1) $\Psi' : (0,\infty) \times \{2,3\}^{\mathbb{Z}} \to (0,\infty)$ *is continuous.*

(2) $\Psi'(s,\xi)$ *is strictly decreasing with respect to* $s \in (0,\infty)$ *for each* $\xi \in \{2,3\}^{\mathbb{Z}}$.

(3) $\min_{\xi\in\{2,3\}^{\mathbb{Z}}} \Psi'(s,\xi) \to \infty$ *as* $s \downarrow 0$ *and* $\max_{\xi\in\{2,3\}^{\mathbb{Z}}} \Psi'(s,\xi) \to 0$ *as* $s \to \infty$.

PROOF. (1) This is the same as that of Lemma 4.1 except for obvious modifications.
(2) Let η be a element of $\{2,3\}^{\mathbb{N}}$ whose the first component $\pi_1\eta$ is $\pi_{-k}\xi$. Then note that by Hölder's inequality

$$\begin{aligned} &h\big(g(s/T_{-(k-1)}(\xi), P(\theta^{-k}\xi)), \pi_{-k}\xi\big) \\ &\quad = -\log E_0^{\eta}\Big[E_0^{P(\theta^{-k}\xi)}\big[\exp\big(-\frac{sW}{T_{-(k-1)}(\xi)}\big)\big]^{W(Y^1_\cdot)-b(\pi_1\eta)}\Big] \end{aligned}$$

is a concave function with respect to $s > 0$ for each $k \ge 1$, $\xi \in \{2,3\}^{\mathbb{Z}}$. Hence the second term in the right hand of (4.4) is monotone decreasing with respect to s for each $\xi \in \{2,3\}^{\mathbb{Z}}$. Since it is easy to check that 2-th derivative of $-\log g(s,\eta)$ is strictly negative for all $s \in (0,\infty)$ and $\eta \in \{2,3\}^{\mathbb{N}}$, the first term in the right hand of (4.4) is strictly monotone decreasing. This implies our assertion.
(3) Fix $\xi \in \{2,3\}^{\mathbb{Z}}$. As we stated above, the first term in the right hand of (4.4) is strictly monotone decreasing. We assume that this term converges to

$c > 0$ as $s \to \infty$. Then an easy computation shows that there is a constant c' such that $-\log g(s, P(\xi)) \geq c's$ for large enough s, which contradicts Lemma 3.5. So the first term in the right hand of (4.4) converges to 0 as $s \to \infty$. Also by (4.1) and the monotone convergence theorem, we see that

$$\sum_{k=1}^{\infty} F_k'(s,\xi) \geq c' \sum_{k=1}^{\infty} \frac{B_{-k}(\xi)}{T_{-(k-1)}(\xi)} E_0^{P(\theta^{-k}\xi)}\Big[W \exp\big(-\frac{sW}{T_{-(k-1)}(\xi)}\big)\Big] \to \infty$$

as $s \downarrow 0$. Similarly we have $\sum_{k=1}^{\infty} F_k'(s,\xi) \to 0$ as $s \to \infty$ by (4.1) and Lebesgue convergence theorem. As a result we have $\Psi'(s,\xi) \to \infty$ as $s \downarrow 0$ and $\Psi'(s,\xi) \to 0$ as $s \to \infty$ for each $\xi \in \{2,3\}^{\mathbb{Z}}$. Since $\Psi'(\cdot,\xi)$ is strictly monotone decreasing from (2), Dini's theorem implies our assertion. □

From Lemma 4.2 (2), $\Psi'(\cdot,\xi)$ is strictly decreasing for each $\xi \in \{2,3\}^{\mathbb{Z}}$. Therefore $\Psi'(\cdot,\xi)$ has the inverse function $(\Psi')^{-1}(\cdot,\xi)$ for each $\xi \in \{2,3\}^{\mathbb{Z}}$.

Lemma 4.3.

(1) *For each compact set $K \subset (0,\infty)$ there exists compact set $K_1 \subset (0,\infty)$ such that*

$$\bigcup_{\xi \in \{2,3\}^{\mathbb{Z}}} (\Psi')^{-1}(K,\xi) \subset K_1. \tag{4.5}$$

(2) $(\Psi')^{-1} : (0,\infty) \times \{2,3\}^{\mathbb{Z}} \to (0,\infty)$ *is continuous.*

Proof. (1) Let us denote by K_2 the left hand side of (4.5). Assume that $\inf K_2 = 0$. Then there is $a_n \in K_2$ such that $0 < a_n < 1/n$ for each $n \in \mathbb{N}$. Further for each a_n there exist $\xi^n \in \{2,3\}^{\mathbb{Z}}$ and $z_n \in K$ such that $\Psi'(a_n, \xi^n) = z_n \in K$. This contradicts Lemma 4.2 (3). Next assume that $\sup K_2 = \infty$. Then there exist $b_n \in K_2$ and $\xi^n \in \{2,3\}^{\mathbb{Z}}$ such that $b_n > n$ and $\Psi'(b_n,\xi^n) \in K$ for each $n \in \mathbb{N}$. This contradicts Lemma 4.2 (3).
(2) It is easily seen that $(\Psi')^{-1}(\cdot,\xi) : (0,\infty) \to (0,\infty)$ is continuous for each $\xi \in \{2,3\}^{\mathbb{Z}}$. By Lemma 4.2 (1), $\Psi'(s,\xi')$ converges to $\Psi'(s,\xi)$ pointwise as $\xi' \to \xi$ for each $s \in (0,\infty)$. So does inverse function. Since $(\Psi')^{-1}(\cdot,\xi)$ is monotone decreasing, this convergence is compact uniform on $(0,\infty)$. This completes the proof. □

Define the Legendre transform $\Psi^*(z,\xi)$ by $\Psi^*(z,\xi)=\sup_{s>0}\{\Psi(s,\xi)-zs\}$ for $z>0$ and $\xi\in\{2,3\}^{\mathbb{Z}}$. Since we have

$$\Psi^*(z,\xi)=\sup_{s>0}\{\Psi(T_1(\xi)s,\xi)-zT_1(\xi)s\}=B_1(\xi)\Psi^*\Big(\frac{T_1(\xi)}{B_1(\xi)}z,\theta\xi\Big)$$

by Lemma 4.1 (3), the function Ψ^* satisfies the following functional equation:

$$\Psi^*\Big(\frac{B_1(\xi)}{T_1(\xi)}z,\xi\Big)=B_1(\xi)\Psi^*(z,\theta\xi).$$

From $T_{-n}(\xi)=T_{n+1}(\theta^{-(n+1)}\xi)^{-1}$ and $B_{-n}(\xi)=B_{n+1}(\theta^{-(n+1)}\xi)^{-1}$, we obtain

$$\Psi^*(z,\xi)=\begin{cases}B_n(\xi)\Psi^*(T_n(\xi)z/B_n(\xi),\theta^n\xi) & \text{if } n\geq 1,\\ B_n(\xi)\Psi^*(T_n(\xi)z/B_n(\xi),\theta^{n-1}\xi) & \text{if } n<0\end{cases}\tag{4.6}$$

for all $\xi\in\{2,3\}^{\mathbb{Z}}$ and $z>0$. At the end of this section, we give some properties of Ψ^*.

Lemma 4.4.

(1) $\Psi^*\ :\ (0,\infty)\times\{2,3\}^{\mathbb{Z}}\to(0,\infty)$ *is continuous.*

(2) *There exist constants* $c,c'>0$ *such that*

$$cz^{-1/(d_w^\xi(n)-1)}\leq\Psi^*(z,\xi)\leq c'z^{-1/(d_w^\xi(n)-1)}$$

for any $z>0$ *and* $\xi\in\{2,3\}^{\mathbb{Z}}$, *where* n *is an integer with* $B_n(\xi)/T_n(\xi)\leq z<B_{n-1}(\xi)/T_{n-1}(\xi)$.

(3) *Let* K *be a compact set on* $(0,\infty)$. *There exists a compact set* $\Gamma=\Gamma(K)\subset(0,\infty)$ *such that* $\Psi^*(z,\xi)=\sup_{s\in\Gamma}\{\Psi(s,\xi)-zs\}$ *for any* $z\in K$ *and* $\xi\in\{2,3\}^{\mathbb{Z}}$.

Proof. (1), (3) Since $\Psi'(\cdot,\xi)$ is strictly decreasing for each $\xi\in\{2,3\}^{\mathbb{Z}}$, $(\Psi')^{-1}(z,\xi)$ is a unique point such that $\Psi^*(z,\xi)=\Psi((\Psi')^{-1}(z,\xi),\xi)-z(\Psi')^{-1}(z,\xi)$ for each $z\in(0,\infty)$, $\xi\in\{2,3\}^{\mathbb{Z}}$. By Lemma 4.1 (1) and Lemma

4.3 (2), $\Psi : [0,\infty) \times \{2,3\}^{\mathbb{Z}} \to [0,\infty)$ and $(\Psi')^{-1} : (0,\infty) \times \{2,3\}^{\mathbb{Z}} \to (0,\infty)$ are continuous. These imply our assertion.

(2) Since $\sup_{\xi\in\{2,3\}^{\mathbb{Z}}} \Psi^*(1,\xi) < \infty$, from (4.6) there is a constant $c' > 0$ such that

$$\Psi^*(z,\xi) \le \Psi^*\Big(\frac{B_n(\xi)}{T_n(\xi)},\xi\Big) = B_n(\xi) \sup_{\xi\in\{2,3\}^{\mathbb{Z}}} \Psi^*(1,\xi) \le c' z^{-1/(d_w^{\xi}(n)-1)}$$

for any $n \in \mathbb{Z}$, $\xi \in \{2,3\}^{\mathbb{Z}}$ and $z \in [B_n(\xi)/T_n(\xi), B_{n-1}(\xi)/T_{n-1}(\xi))$. The lower bound is proved in exactly the same way. □

5. Hitting Time and Distance

In this section, our goal is to prove the next proposition. This shows the relation between hitting times and distances.

Proposition 5.1. *For any $\delta_0 > 0$ and compact set $K \subset (0,\infty)$,*

$$\lim_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{d_\eta(x,y)\ge\delta_0 \\ s\in K}} \left| \frac{\log E_x^\eta[\exp(-T_n(\eta)s\tau_y)]}{\log g(T_n(\eta)s,\eta)} - d_\eta(x,y) \right| = 0,$$

where $\tau_y = \inf\{t \ge 0 : X_t = y\}$.

In the case $\eta = \mathbf{2}$, this proposition corresponds Lemma 3.4 in [Kum]. We make some preparations to show this proposition. First, we consider the estimates of probability generating functions of stopping times $S^0(Y_\cdot^n)$ and $W(Y_\cdot^n)$. Let $c_1 = \max_{\eta\in\{2,3\}^{\mathbb{N}}} \max_{x\in F_1^\eta} E_x^\eta[S^0(Y_\cdot^1)]$.

Lemma 5.2. *For any $\eta \in \{2,3\}^{\mathbb{N}}$, $n \in \mathbb{N}$, $x \in F_n^\eta$ and $u \in [0,1]$,*

$$E_x^\eta[u^{S^0(Y_\cdot^n)}] \ge E_0^\eta[u^{W(Y_\cdot^n)}]^{c_1}. \tag{5.1}$$

Proof. Let $z_1, z_2 \in F_0^\eta$ and $b_1 \in F_1^\eta$ be as in Figure 3. Then by Jensen's inequality we have

$$E_0^\eta[u^{W(Y_\cdot^1)}]^{c_1} \vee E_{b_1}^\eta[u^{S^0(Y_\cdot^1)}]^{c_1} \le u^{c_1} \le E_x^\eta[u^{S^0(Y_\cdot^1)}] \tag{5.2}$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $x \in F_1^\eta$ and $u \in [0,1]$. Thus by strong Markov property we see that

$$\begin{aligned} E_{b_1}^\eta[u^{S^0(Y_\cdot^m)}]^{c_1} &= E_{b_1}^\eta\big[E_0^\eta[u^{S_1^1(Y_\cdot^m)}]^{S^0(Y_\cdot^1)}\big]^{c_1} \\ &\le E_x^\eta\big[E_0^\eta[u^{S_1^1(Y_\cdot^m)}]^{S^0(Y_\cdot^1)}\big] = E_x^\eta[u^{S^0(Y_\cdot^m)}] \end{aligned} \tag{5.3}$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $x \in F_1^\eta$, $m \in \mathbb{N}$ and $u \in [0,1]$. Now we will prove (5.1) by induction on n. Assume that our assertion is true for $n-1$. Then we have

$$E_0^\eta[u^{S_1^1(Y_\cdot^n)}]^{c_1} \le E_x^\eta[u^{S^1(Y_\cdot^n)}] \tag{5.4}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$ and $x \in F_n^\eta$. Note that $S_1^1(Y_\cdot^n)$ is the first hitting time of b_1, b_2 under P_0^η. Hence by (5.3) and (5.4) we obtain

$$\begin{aligned} E_x^\eta[u^{S^0(Y_\cdot^n)}] &= \sum_{y \in C_1(x)} E_x^\eta[1_{\{X_{S^1(Y_\cdot^n)}=y\}} u^{S^1(Y_\cdot^n)}] E_y^\eta[u^{S^0(Y_\cdot^n)}] \\ &\ge E_x^\eta[u^{S^1(Y_\cdot^n)}] E_{b_1}^\eta[u^{S^0(Y_\cdot^n)}]^{c_1} \ge E_0^\eta[u^{S_1^1(Y_\cdot^n)}]^{c_1} E_{b_1}^\eta[u^{S^0(Y_\cdot^n)}]^{c_1} \end{aligned}$$

for all $x \in F_n^\eta \setminus F_1^\eta$, where $C_1(x)$ is the 1-cell which contains x. Note that this is true for all $x \in F_1^\eta$ from (5.3). On the other hand by strong Markov property

$$E_0^\eta[u^{W(Y_\cdot^n)}] = E_0^\eta[u^{S_1^1(Y_\cdot^n)}] E_{b_1}^\eta[u^{\tau_{1,2}(Y_\cdot^n)}] \le E_0^\eta[u^{S_1^1(Y_\cdot^n)}] E_{b_1}^\eta[u^{S^0(Y_\cdot^n)}],$$

where $\tau_{1,2}(Y_\cdot^n) = \inf\{i \in \mathbb{N} : Y_i^n \in \{z_1, z_2\}\}$. This implies (5.1) is true for n. □

Next we prove the continuous version of Lemma 5.2.

Lemma 5.3. *It holds that*

$$E_x^\eta[u^{S^0}] \ge E_0^\eta[u^W]^{c_1} \textit{ for all } \eta \in \{2,3\}^{\mathbb{N}},\ x \in F^\eta \textit{ and } u \in [0,1].$$

Proof. Firstly from strong Markov property and Lemma (5.2) we see that

$$E_0^\eta[u^W]^{c_1} = E_0^\eta\big[E_0^\eta[u^{S_1^n}]^{W(Y_\cdot^n)}\big]^{c_1} \le E_x^\eta\big[E_0^\eta[u^{S_1^n}]^{S^0(Y_\cdot^n)}\big] = E_x^\eta[u^{S^0}] \tag{5.5}$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $n \in \mathbb{N}$, $x \in F_n^\eta$ and $u \in [0,1]$. Next we consider in case $x \in F^\eta \setminus F_\infty^\eta$. We have by strong Markov property

$$E_x^\eta[u^{S^0}] = \sum_{i=0}^{2} E_{w_i^n}^\eta[u^{S^0}] E_x^\eta[u^{S^n} 1_{\{X_{S^n}=w_i^n\}}] \geq E_0^\eta[u^W]^{c_1} E_x^\eta[u^{S^n}] \tag{5.6}$$

for all $n \in \mathbb{N}$, where $\{w_0^n, w_1^n, w_2^n\}$ is the boundary of n-complex which contains x. Since $P_x^\eta[\lim_{n\to\infty} S^n = 0] = 1$, the dominated convergence theorem implies our assertion. □

Let us state some properties of resolvent densities. Let $\eta \in \{2,3\}^{\mathbb{N}}$ and remind the function

$$u_s^\eta(x,y) = \int_0^\infty e^{-st} p_t^\eta(x,y)\, dt, \quad (s,x,y) \in (0,\infty) \times F^\eta \times F^\eta.$$

LEMMA 5.4.

(1) *Let $\eta \in \{2,3\}^{\mathbb{N}}$. It follows that*

$$E_x^\eta[e^{-s\tau_y}] = \frac{u_s^\eta(x,y)}{u_s^\eta(x,x)} \quad \textit{for all} \quad x,y \in F^\eta \quad \textit{and} \quad s > 0,$$

where $\tau_y = \inf\{t \geq 0 : X_t = y\}$.

(2) *There exist constants $c, c' > 0$ such that*

$$c\frac{M_n(\eta)}{T_n(\eta)} \leq u_s(x,x) \leq c'\frac{M_n(\eta)}{T_n(\eta)}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $x \in F^\eta$ and $s \geq 1$, where n is a positive integer with $T_n(\eta) \leq s \leq T_{n+1}(\eta)$.

PROOF. (1) We can deduce this lemma in a similar fashion of Lemma 5.6 in [BP].
(2) Firstly we shall prove the lower bounds. The function $p_t^\eta(x,x)$ is decreasing in t for each $x \in F^\eta$ from a general fact about symmetric processes. So

$$u_s^\eta(x,x) \geq \int_0^{1/s} e^{-st} p_t^\eta(x,x) dt \geq \frac{1-e^{-1}}{s} p_{1/s}^\eta(x,x).$$

By Theorem (1.1) we have the lower bounds.

Next we shall prove the upper bounds. From Theorem (1.1) and the monotonicity of $p_t^\eta(x,x)$ with respect to t again, we have $p_t^\eta(x,x) \le c_1 M_n(\eta)$ if $1/T_{n+1}(\eta) \le t \le 1/T_n(\eta)$ and $p_t^\eta(x,x) \le c_1$ if $1 \le t$. We divide $u_s^\eta(x,x)$ into three parts:

$$u_s^\eta(x,x) = \sum_{j=0}^{n} \int_{1/T_{j+1}(\eta)}^{1/T_j(\eta)} e^{-st} p_t^\eta(x,x)dt + \sum_{j=n+1}^{\infty} \int_{1/T_{j+1}(\eta)}^{1/T_j(\eta)} e^{-st} p_t^\eta(x,x)dt + \int_1^\infty e^{-st} p_t^\eta(x,x)dt.$$

Clearly the third term is smaller than $c_1/T_n(\eta)$. Also it follows easily that there is some constant $c_2 > 0$ such that the first term is bounded $c_2 M_n(\eta)/T_n(\eta)$ for any $\eta \in \{2,3\}^{\mathbb{N}}$, $x \in F^\eta$, $s \ge 1$ and $n \in \mathbb{N}$ with $T_n(\eta) \le s \le T_{n+1}(\eta)$. Finally we estimate the second term. However, noting that $e^{-x/t(3)} - e^{-x} \le (1 - 1/t(3))x$ for all $x > 0$, this is straightforward too. Then we have our assertion. $\square$

From above Lemma 5.4 there is a constant $C > 0$ such that

$$E_x^\eta[\exp(-s\tau_y)] = \frac{u_s^\eta(x,y)}{u_s^\eta(x,x)} \le \frac{c'}{c} \frac{u_s^\eta(y,x)}{u_s^\eta(y,y)} = C E_y^\eta[\exp(-s\tau_x)] \tag{5.7}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $x, y \in F^\eta$ and $s \ge 1$. We can now prove Proposition 5.1.

PROOF OF PROPOSITION 5.1. Let $\eta \in \{2,3\}^{\mathbb{N}}$, $m \in \mathbb{N}$ and a shortest F_m^η-path be $\pi = \{x_0, \cdots, x_l\}$ connecting $x(= x_0)$ and $y(= x_l)$ for each $x, y \in F_m^\eta$. As in the proof of Lemma 3.4 in [Kum], Theorem 1.2 (1) and strong Markov property give

$$\Big(\frac{1}{m(3)}\Big)^l g\Big(\frac{s}{T_m(\eta)}, \theta^m \eta\Big)^l \le E_x^\eta[\exp(-s\tau_y)] \le g\Big(\frac{s}{T_m(\eta)}, \theta^m \eta\Big)^l$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $m \ge 1$ and $s > 0$. Substituting $T_n(\eta)s$ for s, we get

$$\Big| \frac{\log E_x^\eta[\exp(-T_n(\eta)s\tau_y)]}{B_m(\eta) \log g(\frac{T_n(\eta)s}{T_m(\eta)}, \theta^m\eta)} - d_\eta(x,y) \Big| \le \frac{-c \log m(3)}{\log g(\frac{T_n(\eta)s}{T_m(\eta)}, \theta^m\eta)}, \tag{5.8}$$

where c is the constant in (2.3). Note that $d_\eta(x,y) = l/B_m(\eta)$. In particular considering in the case $x, y \in F_0^\eta$, we have

$$\lim_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{s\in K} \left| \frac{\log g(T_n(\eta)s,\eta)}{B_m(\eta)\log g(T_n(\eta)s/T_m(\eta),\theta^m\eta)} - 1\right| = 0 \tag{5.9}$$

for each $m \in \mathbb{N}$. By adding (5.8) and (5.9) we see that

$$\lim_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{x,y\in F_m^\eta \\ s\in K}} \left| \frac{\log E_x^\eta[\exp(-T_n(\eta)s\tau_y)]}{\log g(T_n(\eta)s,\eta)} - d_\eta(x,y)\right| = 0 \tag{5.10}$$

for each $m \in \mathbb{N}$. Next we consider the case $x, y \in F^\eta \setminus F_m^\eta$. Let $A_m(x)$ be a m-complex which contains x. For any $\delta_0 > 0$ there exists $M = M(\delta_0) \in \mathbb{N}$ such that if $m \geq M$ then $A_m(x) \cap A_m(y) = \phi$ for any $x, y \in F^\eta \setminus F_m^\eta$ with $d_\eta(x,y) \geq \delta_0$. By (5.7) and strong Markov property we have

$$\begin{aligned} E_x^\eta[\exp(-s\tau_y)] &= \sum_{i=0}^{2} E_x^\eta[\exp(-sS_0^m)1_{\{X_{S_0^m}=z_i^m\}}]E_{z_i^m}^\eta[\exp(-s\tau_y)] \\ &\geq C^{-1}E_x^\eta[\exp(-sS_0^m)]E_y^\eta[\exp(-sS_0^m)] \min_{i,j} E_{w_j^m}^\eta[\exp(-s\tau_{z_i^m})] \end{aligned}$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $x \in F^\eta$ and $s \geq 1$, where $\partial A_m(x) = \{z_0^m, z_1^m, z_2^m\}$, $\partial A_m(y) = \{w_0^m, w_1^m, w_2^m\}$. In the same way we have

$$E_x^\eta[\exp(-s\tau_y)] \leq C \max_{i,j} E_{w_j^m}^\eta[\exp(-s\tau_{z_i^m})]$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $x \in F^\eta$ and $s \geq 1$. By the way we have

$$\frac{\log E_x^\eta[\exp(-sS_0^m)]}{\log g(s/T_m(\eta),\theta^m\eta)} = \frac{\log E_x^\eta[\exp(-sS_0^m)]}{\log E_0^\eta[\exp(-sS_1^m)]} \leq c_1$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $x \in F^\eta, m \in \mathbb{N}$ and $s > 0$ from Lemma 5.3. Since $\sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{x,y\in F^\eta} \sup_{i,j} |d_\eta(x,y) - d_\eta(z_i^m, w_j^m)| \to 0$ as $m \to \infty$, (5.9) and (5.10) imply the proof. □

Define the function $k_n^{x,y}(s,\eta) : [0,\infty) \times \{2,3\}^{\mathbb{N}} \to \mathbb{R}$ by

$$k_n^{x,y}(s,\eta) = -\frac{\log E_x^\eta[\exp(-T_n(\eta)\tau_y s)]}{B_n(\eta)}$$

for each $x, y \in F^\eta$ and $n \in \mathbb{N}$. Set $c_K = \sup_{s\in K, \eta\in\{2,3\}^{\mathbb{N}}, n\in\mathbb{N}} |L_n(s,\eta)|$ for a compact set $K \subset (0,\infty)$. Then Theorem 1.2 (c) implies $0 < c_K < \infty$. For any $\epsilon > 0$ there exists $N = N(\epsilon, K) \in \mathbb{N}$ such that if $n \geq N$ then

$$\sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{s\in K \\ d_\eta(x,y)\geq\delta_0}} \Big| \frac{k_n^{x,y}(s,\eta)}{L_n(s,\eta)} - d_\eta(x,y) \Big| < \frac{\epsilon}{c_K}$$

from Lemma 5.1. Therefore we deduce that

$$\lim_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{s\in K \\ d_\eta(x,y)\geq\delta_0}} |k_n^{x,y}(s,\xi) - d_\eta(x,y)L_n(s,\xi)| = 0 \tag{5.11}$$

for any compact set $K \subset (0,\infty)$. By using Lemma 4.1 (2) and (5.11), we have

$$\lim_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{s\in K \\ d_\eta(x,y)\geq\delta_0}} |k_n^{x,y}(s,\eta) - d_\eta(x,y)\Psi(s,\theta^n\bar{\eta})| = 0. \tag{5.12}$$

6. Proof of Theorem 1.3

For the uniformity with respect to $\eta \in \{2,3\}^{\mathbb{N}}$ in Theorem 1.3, we divide the proof into two parts. First, we consider the case where the distance between x and y is larger than some constant $\delta_0 > 0$, which is the main part of this section. It is comparatively easy to check the second case where x is near enough to y from Theorem 1.1.

Proposition 6.1. *For any compact set $K \subset (0,\infty)$ and $\delta_0 > 0$,*

$$\lim_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{d_\eta(x,y)\geq\delta_0 \\ z\in K}} \Big| \frac{1}{B_n(\eta)} \log P_x^\eta[\tau_y \leq \frac{B_n(\eta)}{T_n(\eta)} z] \\ + d_\eta(x,y)\Psi^*\Big(\frac{z}{d_\eta(x,y)}, \theta^n\bar{\eta}\Big)\Big| = 0.$$

Proof. We follow the proof of Theorem II.6.1 in [E].
(i) From Chebyshev's inequality we have

$$P_x^\eta\Big[\tau_y \leq \frac{B_n(\eta)}{T_n(\eta)} z\Big] \leq E_x^\eta[e^{s(B_n(\eta)z - T_n(\eta)\tau_y)}] = e^{B_n(\eta)(sz - k_n^{x,y}(s,\eta))}$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $s > 0$, $z > 0$, $x, y \in F^\eta$ and $n \in \mathbb{N}$. We can choose a compact set $\Gamma(K, \delta_0) \subset (0, \infty)$ such that

$$\Psi^*\Big(\frac{z}{d_\eta(x,y)}, \theta^n\bar{\eta}\Big) = \sup_{s\in\Gamma(K,\delta_0)} \Big\{\Psi(s, \theta^n\bar{\eta}) - \frac{z}{d_\eta(x,y)}s\Big\} \tag{6.1}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $x, y \in F^\eta$ with $d_\eta(x, y) \geq \delta_0$, $n \in \mathbb{N}$ and $z \in K$ from Lemma 4.4 (3). By (5.12) and (6.1) for any $\epsilon > 0$ there exists $N = N(\epsilon, K, \delta_0)$ such that if $n \geq N$ then

$$\frac{1}{B_n(\eta)} \log P_x^\eta\Big[\tau_y \leq \frac{B_n(\eta)}{T_n(\eta)} z\Big] \leq -d_\eta(x,y)\Psi^*\Big(\frac{z}{d_\eta(x,y)}, \theta^n\bar{\eta}\Big) + \epsilon$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $z \in K$, $x, y \in F^\eta$ with $d_\eta(x, y) \geq \delta_0$.
(ii) Without loss of generality, we can assume that K is a closed interval $[p, q]$ for $p, q \in (0, \infty)$ with $p < q$. Let $Q_n^{\eta,x,y}$ be the distribution function of $T_n(\eta)\tau_y/B_n(\eta)$ under P_x^η. Define probability measures

$$dQ_{n,t}^{\eta,x,y}(v) = \frac{\exp(-B_n(\eta)tv)}{\exp(-B_n(\eta)k_n^{x,y}(t,\eta))} dQ_n^{\eta,x,y}(v) \quad \text{for each} \quad t \geq 0.$$

To simplify the notation, we will drop the subscript x, y, η and refer to $Q_n^{\eta,x,y}$ (resp. $Q_{n,t}^{\eta,x,y}$) as Q_n (resp. $Q_{n,t}$). By virtue of the continuity of Ψ^*, for any $\epsilon > 0$ we can choose $\gamma_\epsilon > 0$ such that $p > 2\gamma_\epsilon$ and

$$\sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{d_\eta(x,y)\geq\delta_0 \\ z\in K, n\in\mathbb{N}}} \Big|\Psi^*\Big(\frac{z-\gamma_\epsilon}{d_\eta(x,y)}, \theta^n\bar{\eta}\Big) - \Psi^*\Big(\frac{z}{d_\eta(x,y)}, \theta^n\bar{\eta}\Big)\Big| < \epsilon. \tag{6.2}$$

Let us abbreviate $z - \gamma_\epsilon$ by z_{γ_ϵ}. Then we have

$$\begin{aligned}\log Q_n[(0, z]] \geq &- B_n(\eta)k_n^{x,y}(t, \eta) \\ &+ B_n(\eta)(tz_{\gamma_\epsilon} - t\beta) + \log Q_{n,t}[(z_{\gamma_\epsilon} - \beta, z_{\gamma_\epsilon} + \beta)]\end{aligned}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $x, y \in F^\eta$, $z \in K$ and $\beta > 0$ with $\beta < \gamma_\epsilon$ Set $K_\epsilon = [p - \gamma_\epsilon, q - \gamma_\epsilon]$. Note that $z_{\gamma_\epsilon} \in [p - \gamma_\epsilon, q - \gamma_\epsilon]$ and $K_\epsilon \subset [p/2, q]$ for small enough $\epsilon > 0$. Then we can choose a compact set $\Gamma = \Gamma(\epsilon, \delta_0, K) \subset (0, \infty)$ such that

$$\Psi^*\Big(\frac{z_{\gamma_\epsilon}}{d_\eta(x,y)}, \theta^n\bar{\eta}\Big) = \sup_{s\in\Gamma} \Big\{\Psi(s, \theta^n\bar{\eta}) - \frac{z_{\gamma_\epsilon}}{d_\eta(x,y)}s\Big\}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $x, y \in F^{\eta}$ with $d_\eta(x,y) \geq \delta_0$, $n \in \mathbb{N}$ and $z \in K$ from Lemma 4.4 (3). Let $t = t(\eta, x, y, z_{\gamma_\epsilon}, n) = (\Psi')^{-1}(z_{\gamma_\epsilon}/d_\eta(x,y), \theta^n\bar{\eta}) > 0$ for each $\eta \in \{2,3\}^{\mathbb{N}}$, $x, y \in F^{\eta}$ with $d_\eta(x,y) \geq \delta_0$, $n \in \mathbb{N}$ and $z_{\gamma_\epsilon} \in K_\epsilon$. Note that $z_{\gamma_\epsilon} = d_\eta(x,y)\Psi'(t, \theta^n\bar{\eta})$ and $t \in \Gamma$. From (6.2) and (5.12), for any $\epsilon > 0$ there exists $N_1 = N_1(\delta_0, \epsilon, K)$ such that if $n \geq N_1$ then

$$\begin{aligned} \frac{\log Q_n[(0,z]]}{B_n(\eta)} - \frac{\log Q_{n,t}[(z_{\gamma_\epsilon} - \beta, z_{\gamma_\epsilon} + \beta)]}{B_n(\eta)} &\geq -\big(k_n^{x,y}(t,\eta) - z_{\gamma_\epsilon} t\big) - \beta t \\ &\geq -d_\eta(x,y)\Psi^*\Big(\frac{z}{d_\eta(x,y)}, \theta^n\bar{\eta}\Big) - \beta t - (c+1)\epsilon \end{aligned} \tag{6.3}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $x, y \in F^{\eta}$ with $d_\eta(x,y) \geq \delta_0$, $z \in K$ and $\beta > 0$ with $\beta < \gamma_\epsilon$, where c is the constant in (2.3). Now we consider $Q_{n,t}[(z_{\gamma_\epsilon} - \beta, z_{\gamma_\epsilon} + \beta)]$. Let us fix $\epsilon > 0$. First, we have

$$\begin{aligned} &Q_{n,t}[\{v \in [0,\infty) : |v - z_{\gamma_\epsilon}| > \beta\}] \exp(\beta B_n(\eta) s) \\ &\quad\leq \int_0^\infty e^{-sB_n(\eta)(v - z_{\gamma_\epsilon})} dQ_{n,t}(v) + \int_0^\infty e^{sB_n(\eta)(v - z_{\gamma_\epsilon})} dQ_{n,t}(v). \end{aligned}$$

For $s > 0$ we have

$$\begin{aligned} &\frac{1}{B_n(\eta)} \log \int_0^\infty \exp(-B_n(\eta)s(v - z_{\gamma_\epsilon})) dQ_{n,t}(v) \\ &\qquad= -\big(k_n^{x,y}(t+s,\eta) - k_n^{x,y}(t,\eta) - d_\eta(x,y)\Psi'(t,\theta^n\bar{\eta})s\big). \end{aligned} \tag{6.4}$$

Also it is easy to check that there are $s_0 = s_0(\Gamma) > 0$ and a closed interval $C = C(\Gamma) \subset (0,\infty)$ such that

$$\begin{aligned} &|\Psi(t+s,\theta^n\bar{\eta}) - \Psi(t,\theta^n\bar{\eta}) - \Psi'(t,\theta^n\bar{\eta})s| \\ &\qquad\qquad \leq s \sup_{\substack{u,v \in C, |u-v|<s \\ \xi \in \{2,3\}^{\mathbb{Z}}}} |\Psi'(u,\xi) - \Psi'(v,\xi)| \end{aligned}$$

for any $t \in \Gamma$, $s \leq s_0$. Let $r(s), s \leq s_0$ denote the right hand side of above inequality. Then from (5.12) for any $\epsilon' > 0$ there exists $N_2 = N_2(\epsilon, \delta_0, \epsilon', K)$ such that if $n \geq N_2$ then

$$|k_n^{x,y}(t+s,\eta) - k_n^{x,y}(t,\eta) - d_\eta(x,y)\Psi'(t,\theta^n\bar{\eta})s| \leq 2\epsilon' + cr(s)$$

for any $t \in \Gamma$, $s \le s_0$, $\eta \in \{2,3\}^{\mathbb{N}}$ and $x, y \in F^\eta$ with $d_\eta(x,y) \ge \delta_0$. Hence by adding (6.4) we obtain

$$\sup_{\substack{\eta\in\{2,3\}^{\mathbb{N}},t\in\Gamma \\ d_\eta(x,y)\ge\delta_0,z\in K}} \frac{1}{B_n(\eta)} \log \int_0^\infty e^{-B_n(\eta)s(v-z_{\gamma_\epsilon})} dQ_{n,t}(v) \le 2\epsilon' + cr(s)$$

for any $s \le s_0$ and $n \ge N_2$. By replacing s with $-s$ ($s > 0$), we deduce the following in exactly the same way. For any $\epsilon' > 0$ there exists $N_3 = N_3(\epsilon, \delta_0, \epsilon', K) > 0$ such that if $n \ge N_3$ then

$$\sup_{\substack{\eta\in\{2,3\}^{\mathbb{N}},t\in\Gamma \\ d_\eta(x,y)\ge\delta_0,z\in K}} \frac{1}{B_n(\eta)} \log \int_0^\infty e^{B_n(\eta)s(v-z_{\gamma_\epsilon})} dQ_{n,t}(v) \le 2\epsilon' + cr(s)$$

for any $s \le s_0$ and $n \ge N_3$. Therefore it follows that for any $\epsilon' > 0$ there is $N_4 = N_2 \vee N_3$ such that if $n \ge N_4$ then

$$\sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{d_\eta(x,y)\ge\delta_0 \\ z\in K,t\in\Gamma}} \frac{1}{B_n(\eta)} \log Q_{n,t}[\{v \in [0,\infty) : |v - z_{\gamma_\epsilon}| > \beta\}]$$
$$\le -\beta s + 2\epsilon' + cr(s) + \frac{\log 2}{B_n(\eta)}$$

for any $\beta > 0$ with $\beta < \gamma_\epsilon$ and $s < s_0$. From Lemma 4.2 (1) it is easy to check that for any $\epsilon > 0$ there is a $N = N(\epsilon, \delta_0, K)$ such that if $n \ge N$ then

$$\frac{\log Q_n[(0,z]]}{B_n(\eta)} \ge -d(x,y)\Psi^*\Big(\frac{z}{d(x,y)}, \theta^n\bar{\eta}\Big) - \epsilon$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $z \in K$, $x, y \in F^\eta$ with $d_\eta(x,y) \ge \delta_0$. (i) and (ii) imply our assertion. □

By using Theorem 1.1, Lemma 4.4 (1) and Proposition 6.1, we can prove the following in the same way as the proof of Theorem 1.2 in [Kum]. For any compact set $K \subset (0,\infty)$ and $\delta_0 > 0$ we have

$$\text{(6.5)} \qquad \lim_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{d_\eta(x,y)\ge\delta_0 \\ z\in K}} \Big| \frac{1}{B_n(\eta)} \log p^\eta_{\frac{B_n(\eta)}{T_n(\eta)}z}(x,y)$$
$$+ d_\eta(x,y)\Psi^*\Big(\frac{z}{d_\eta(x,y)}, \theta^n\bar{\eta}\Big)\Big| = 0$$

Let us complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Recall that the upper bound of the heat kernel of Theorem (1.1) (b) is written in the following form (4.21) in [BH]. There exist constants $c_1, c_2 > 0$ such that if $1/B_m(\eta) \leq d_\eta(x,y) < 1/B_{m-1}(\eta)$, $1/T_n(\eta) \leq t < 1/T_{n-1}(\eta)$ then

$$p_t(x,y) \leq c_1 M_n(\eta) \exp\Big(-c_2 \frac{B_{m+k_\eta(m,n)}(\eta)}{B_m(\eta)}\Big). \tag{6.6}$$

Therefore we have

$$\frac{1}{B_n(\eta)} \log p^\eta_{\frac{B_n(\eta)}{T_n(\eta)}z}(x,y) \leq \frac{\log(c_1 + M_l)}{B_n(\eta)} - \frac{c_2}{B_n(\eta)} \frac{B_{m+k_\eta(m,l)}(\eta)}{B_m(\eta)},$$

where $m \in \mathbb{N}$ with $1/B_m(\eta) \leq d_\eta(x,y) \leq 1/B_{m-1}(\eta)$, $l = l(n,z,\eta) \in \mathbb{N}$ with $1/T_l(\eta) \leq B_n(\eta)z/T_n(\eta) < 1/T_{l-1}(\eta)$. There exists a constant $c = c(K) \in \mathbb{Z}_+$ such that $B_{n+c}(\eta)/T_{n+c}(\eta) \leq B_n(\eta)z/T_n(\eta)$ for any $z \in K \subset (0,\infty)$. Then we have $T_{l-1}(\eta) \leq T_{n+c}(\eta)/B_{n+c}(\eta)$. Since $t(3) \leq B_m(\eta)$ for $m \geq 4$, we see that

$$\frac{T_l(\eta)}{B_m(\eta)} \leq \frac{T_{l-1}(\eta)t(3)}{B_m(\eta)} \leq \frac{T_{n+c}(\eta)}{B_{n+c}(\eta)}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $n \in \mathbb{N}$, $z \in K$ and $m \geq 4$. Recalling the definition (1.4) of k_η, we deduce $k_\eta(m,l) + m \leq n + c$. Then we obtain

$$\sup_{\eta \in \{2,3\}^{\mathbb{N}}} \frac{1}{B_n(\eta)} \sup_{m \geq M, z \in K} \frac{B_{m+k_\eta(m,l)}(\eta)}{B_m(\eta)} \leq \frac{b(3)^c}{b(2)^M}$$

for any $M \geq 4$, $n \geq 1$. Therefore Theorem 1.3 is valid if we replace $\sup_{x,y \in F^\eta, z \in K}$ by $\sup_{d_\eta(x,y) \leq \delta_0, z \in K}$ from Lemma 4.4 (2). By adding (6.5) this completes the proof. □

Finally let us prove Corollary 1.4 and 1.5. Let $G : (0,\infty) \times \{2,3\}^{\mathbb{Z}} \to (0,\infty)$ be given by

$$G(s,\xi) = s^{1/(d_w^\xi(m)-1)} \Psi^*(s,\xi),$$

where $B_m(\xi)/T_m(\xi) \leq s < B_{m-1}(\xi)/T_{m-1}(\xi)$. Then there exist constants $c, c' > 0$ such that $c \leq G(s,\xi) \leq c'$ for any $s \in (0,\infty)$, $\xi \in \{2,3\}^{\mathbb{Z}}$ by Lemma

4.4 (2). We know that $G(s, \mathbf{2})$ is a non-constant function (See [Kum]), so is G. Letting $G_z(s,\xi) = z^{-1/(d_w^{\xi}(m)-1)} G(s,\xi)$ for each $z > 0$, it is obvious that Corollary 1.4 immediately follows.

PROOF OF COROLLARY 1.5. We see at once that for any $\delta_0 > 0$

$$\sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{z\in K \\ d_\eta(x,y)\geq\delta_0}} \left| \frac{1}{\Psi^*\left(\frac{z}{d_\eta(x,y)}, \theta^n\bar{\eta}\right)} \frac{1}{B_n(\eta)} \log p^{\eta}_{\frac{B_n(\eta)}{T_n(\eta)}z}(x,y) + d_\eta(x,y) \right| \to 0$$

as $n \to \infty$ by (6.5), where $K = [1, t(3)/b(3)]$. Let $n = n(\eta, t)$ be an integer satisfying $B_n(\eta)/T_n(\eta) \leq t < B_{n-1}(\eta)/T_{n-1}(\eta)$ for each $\eta \in \{2,3\}^{\mathbb{N}}$ and $t > 0$. Note that setting $z = z(\eta,t) = tT_{n(\eta,t)}(\eta)/B_{n(\eta,t)}(\eta)$, we have $z \in K$. Since $\inf_{\eta\in\{2,3\}^{\mathbb{N}}} n(\eta,t)$ is large if $t > 0$ is small enough, it follows that

$$\lim_{t\to 0} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{d_\eta(x,y)\geq\delta_0} \left| \frac{1}{\Psi^*\Big(t/d_\eta(x,y), \bar{\eta}\Big)} \log p_t^{\eta}(x,y) + d_\eta(x,y) \right| = 0$$

for any $\delta_0 > 0$ from (4.6). By the definition of G, (1.5) is valid if we replace $\sup_{x,y\in F^\eta}$ by $\sup_{d_\eta(x,y)\geq\delta_0}$. Next we consider the case where x is near enough to y. By (6.6) it is enough to show that for some $\delta_0 > 0$ we have

$$\lim_{t\to 0} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{d_\eta(x,y)<\delta_0} t^{1/(d_w^{\bar{\eta}}(n)-1)} \frac{B_{m+k_\eta(m,l)}(\eta)}{B_m(\eta)} = 0, \tag{6.7}$$

where m, n and l are integers with $1/B_m(\eta) \leq d_\eta(x,y) < 1/B_{m-1}(\eta)$, $B_n(\bar{\eta})/T_n(\bar{\eta}) \leq t/d_\eta(x,y) < B_{n-1}(\bar{\eta})/T_{n-1}(\bar{\eta})$, and $1/T_l(\eta) \leq t < 1/T_{l-1}(\eta)$. Now there are the following three cases : (i) $d_\eta(x,y) > t$ and $m < l$, (ii) $d_\eta(x,y) > t$ and $m \geq l$, (iii) $d_\eta(x,y) \leq t$ and $m \geq l$. Since $k_\eta(m,l) = 0$ in cases (ii) and (iii), it is obvious that (6.7) holds. So we consider the case (i). For any $\epsilon > 0$ there are $c \in \mathbb{N}$ and $\delta_0 = \delta_0(\epsilon, c) > 0$ such that $(b(2)/t(2))^c \leq 1/t(3)$ and

$$b(3)^c \Big(\frac{t(3)}{b(3)}\Big)^{1/(d_w^{\mathbf{2}}-1)} \delta_0^{1/(d_w^{\mathbf{3}}-1)+1} < \epsilon.$$

In this case we have $B_n(\bar{\eta}) = B_n(\eta)$ and $T_n(\bar{\eta}) = T_n(\eta)$ since $n \in \mathbb{N}$. From the definition of c, we see that $m + k_\eta(m,l) \leq n + c$. Since $(B_n(\eta)/T_n(\eta))^{1/(d_w^{\eta}(n)-1)} = 1/B_n(\eta)$ we obtain

$$t^{1/(d_w^{\bar{\eta}}(n)-1)} \frac{B_{m+k_\eta(m,l)}(\eta)}{B_m(\eta)} \leq \Big(\frac{t(3)}{b(3)} \frac{B_n(\eta)}{T_n(\eta)} \delta_0\Big)^{1/(d_w^{\eta}(n)-1)} \frac{B_{m+k_\eta(m,l)}(\eta)}{B_m(\eta)}$$

$$\leq b(3)^c \Big(\frac{t(3)}{b(3)}\Big)^{1/(d_w^{\mathbf{2}}-1)} \delta_0^{1/(d_w^{\mathbf{3}}-1)+1} \frac{B_{m+k_\eta(m,l)}(\eta)}{B_{n+c}(\bar{\eta})} \leq \epsilon$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $t < \delta_0$, and $d_\eta(x,y) < \delta_0$. This completes the proof. $\square$

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