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Smooth Right Quasigroup Structures on 1-Manifolds

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Abstract. Smooth loop structures on one- manifolds for which the groups topologically generated by right translations are locally compact, are known. In this article we study smooth right loop structures on one-manifolds.

1. Introduction

The smooth loop (quasigroup) structures on one dimensional manifolds are completely classified. One dimensional manifolds are (i) $\mathbb{R}$, (ii) $\mathbb{S}^1$ and (iii) Alexandroff half line [1, 2], [[3], page 235]. But on the Alexandroff half line, no smooth quasigroup structure can be defined. However on $\mathbb{R}$ and $\mathbb{S}^1$ we have smooth quasigroup structures [2, 3] which are obtained by looking at copies of $\mathbb{S}^1$ appearing as transversals to 2-dimensional subgroups of $PSL(2, \mathbb{R})$ and that on $\mathbb{R}$ appear as transversals to special type of subgroups in the universal cover of $PSL(2, \mathbb{R})$. This paper is devoted mainly to study smooth right quasigroup structures on $\mathbb{R}$ and $\mathbb{S}^1$ obtained by deformation of their group structures.

2. Preliminaries

Let S be a nonempty set and $\circ$ be a binary operation on S. Then the groupoid $(S, \circ)$ is called a right quasigroup if for all $x, y \in S$, the equation $X \circ x = y$, where X is unknown in the equation, has a unique solution in S. If there exists $e \in S$ such that $e \circ x = x = x \circ e$ for all $x \in S$, then $(S, \circ)$ is called a right quasigroup with identity also called a right loop. A right quasigroup (right loop) $(S, \circ)$ is called a quasigroup (loop) if the equation $x \circ X = y$, where X is unknown in the equation and $x, y \in S$, has a unique solution in S. Throughout the paper a right quasigroup will always be assumed to contain the identity.

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Let S be a fixed right transversal to a subgroup H in a group G. Then every right transversal to H in G determines and is determined uniquely by a map $g : S \to H$ such that $g(e) = e$, the identity of G (see, [5]). The right transversal S_g determined by a map $g : S \to H$ is given by $S_g = \{g(x)x | x \in S\}$. S and S_g are right quasigroups with identities with respect to the operation o on S and o' on S_g given by

$$\{xoy\} = Hxy \cap S$$

and

$$\{g(x)xo'g(y)y\} = S_g \cap Hg(x)xg(y)y$$

respectively. Further, H acts on S from right through an action θ given by $\{x\theta h\} = Hxh \cap S, \forall\, x \in S, h \in H$

PROPOSITION 2.1. [5] *The right quasigroup (S_g, o') is isomorphic to the right quasigroup (S, o_g) where o_g is the operation given by $xo_gy = x\theta g(y)oy$.*

The operation o_g will be termed as the deformation of o through the map $g : S \to H$.

DEFINITION 2.2. Let H be a subgroup of a group G. Then a map $s : G/H \to G$ is called a *section* if $s(H) = e$, the identity of group G and $\nu s = \mathbf{1}$ where ν denotes the quotient map given by $\nu(x) = Hx$.

If s is a section then the image $s(G/H)$ is a right transversal to H in G. Conversely every right transversal S determines a section s given by $\{s(Hx)\} = S \cap Hx$.

PROPOSITION 2.3. [5] *Let H be a closed subgroup of a topological group G and S a right transversal to H in G. Suppose that the section s from the quotient space $G/^rH$ (the set of right cosets of H in G) to G given by $\{s(Hg)\} = S \cap Hg$ is continuous. Then the binary operation o on S given by $\{xoy\} = S \cap Hxy$ and the map $\chi : S \times S \to S$ given by $\chi(x, y)\ ox = y$ are continuous (Here S is given the subspace topology).*

PROPOSITION 2.4. [5] *Let S be a right transversal to a closed subgroup H of a topological group G which is the image of a continuous section $s : G/^r H \to G$. Then the right transversal $S_g = \{g(x)x \mid x \in S\}$ corresponding to the map $g : S \to H$ with $g(e) = e$ is the image of a continuous section if and only if g is continuous from S (with subspace topology) to H.*

3. Main Results

The main results of the paper are:

THEOREM 3.1. *The circle group $\mathbb{S}^1 = \left\{\overline{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}} \mid 0 \le t < \pi\right\}$ is a right transversal to the subgroup $H = \left\{\overline{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}} | a > 0, b \in \mathbb{R}\right\}$ in the projective special linear group $PSL(2,\mathbb{R})$. Let g be a map from $\mathbb{S}^1$ to H given by*

$$(1) \qquad g\left(\overline{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}}\right) = \overline{\begin{pmatrix} u(t) & v(t) \\ 0 & (u(t))^{-1} \end{pmatrix}}$$

where $u : [0,\pi) \to \mathbb{R} \setminus \{0\}$ and $v : [0,\pi) \to \mathbb{R}$ are smooth maps with $u(0) = 1,\ v(0) = 0$. Then $(\mathbb{S}^1, o_g)$ is a smooth right quasigroup with identity where o_g is given by

$$(2) \qquad \overline{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}} o_g \overline{\begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}} = \overline{\begin{pmatrix} \cos(t_1+s) & \sin(t_1+s) \\ -\sin(t_1+s) & \cos(t_1+s) \end{pmatrix}}$$

and

$$(3) \qquad t_1 = \tan^{-1}\left(\frac{u(s)\sin t}{u(s)^{-1}\cos t - v(s)\sin t}\right).$$

PROOF. A two dimensional subgroup of $PSL(2,\mathbb{R})$ is a conjugate of the subgroup

$$H = \left\{ \overline{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}} | a > 0, b \in \mathbb{R} \right\}.$$

The circle group $\mathbb{S}^1$ is given by $\mathbb{S}^1 = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \mid 0 \leq t < 2\pi \right\}$. Clearly $\mathbb{S}^1 \subseteq SL(2,\mathbb{R})$. The subgroup $\mathbb{S}^1/\{I,-I\}$ of the group $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{I,-I\}$ is again a circle group which is given by

$$\mathbb{S}^1 = \left\{ \overline{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}} \mid 0 \leq t < \pi \right\}.$$

We claim that $\mathbb{S}^1$ is a right transversal to the subgroup H in the group $PSL(2,\mathbb{R})$. To prove this it is sufficient to prove that $PSL(2,\mathbb{R}) = H\mathbb{S}^1$ and $H \cap \mathbb{S}^1 = \{\bar{I}\}$.

Let $\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \in PSL(2,\mathbb{R})$. Then $ad - bc = 1$. Thus c and d will not be zero simultaneously. If $c \neq 0$. Without loss we may assume that $c > 0$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} u(t) & v(t) \\ 0 & (u(t))^{-1} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

where $u(t) = a\cos t + b\sin t$, $v(t) = -a\sin t + b\cos t$ and $t = \tan^{-1}\left(\frac{-c}{d}\right)$.

If $c = 0$ then $d = a^{-1}$. In this case,

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{cases} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } a > 0 \\ \begin{pmatrix} -a & -b \\ 0 & -a^{-1} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } a < 0 \end{cases}$$

Thus $PSL(2,\mathbb{R}) = H\mathbb{S}^1$. Clearly $H \cap \mathbb{S}^1 = \{\bar{I}\}$. This shows that $\mathbb{S}^1$

is a right transversal to H in $PSL(2,\mathbb{R})$. The induced right quasigroup structure o on $\mathbb{S}^1$ is given by

$$(4)\qquad \overline{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}}\, o\, \overline{\begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}} = \overline{\begin{pmatrix} \cos(t+s) & \sin(t+s) \\ -\sin(t+s) & \cos(t+s) \end{pmatrix}}$$

and the action θ of H on $\mathbb{S}^1$ is given by

$$(5)\qquad \overline{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}}\theta\overline{\begin{pmatrix} u(s) & v(s) \\ 0 & (u(s))^{-1} \end{pmatrix}} = \overline{\begin{pmatrix} \cos t_1 & \sin t_1 \\ -\sin t_1 & \cos t_1 \end{pmatrix}}$$

where t_1 is given by Eq.(3).

Let $g : \mathbb{S}^1 \to H$ be a map given by Eq.(1). Since u, v are smooth, therefore g is smooth. By Eq.(4), Eq.(5) and Proposition 2.1, the right quasigroup structure o_g on $\mathbb{S}^1$ is given by

$$\overline{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}}\, o_g\, \overline{\begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}} = \overline{\begin{pmatrix} \cos(t_1+s) & \sin(t_1+s) \\ -\sin(t_1+s) & \cos(t_1+s) \end{pmatrix}}$$

where t_1 is given by Eq.(3). By Proposition 2.4, the operation o_g is smooth. □

REMARK 3.2. In general $(\mathbb{S}^1, o_g)$ need not be a loop and it may not have right inverse property. For example, if we take u and v given by $u(t) = \sin\frac{t}{2} + \cos\frac{t}{2}$ and $v(t) = \sin t\cos t$ then $\overline{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \in \mathbb{S}^1$ has no right inverse. However for certain g, for example

$$g\left(\overline{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}}\right) = \overline{\begin{pmatrix} 1 & \frac{1}{4}t\sin t \\ 0 & 1 \end{pmatrix}},$$

$(\mathbb{S}^1, o_g)$ is indeed a loop.

Consider the additive group of real numbers. Since a differentiable function can be approximated by a polynomial function. In this light, we establish the following

PROPOSITION 3.3. *Let $n \geq 3$. Then a polynomial $P_n(x,y) = \sum_{i=0}^{n} a_i y^i + x\sum_{j=0}^{n-1} b_j y^j$ of degree n in two variables x and y over the field $\mathbb{R}$ determines a smooth right quasigroup structure o with identity 0 given by $xoy = P_n(x,y)$ if and only if $P_n(x,y) = x + y + x\phi(y)$ where $\phi(y)$ is a polynomial in y of degree $n-1$ such that $\phi(a) \neq -1$ for every $a \in \mathbb{R}$ and $\phi(0) = 0$. In particular $P_{2n+1}(x,y) = x + y + xy^{2n}$ where $n \in \mathbb{N}$ determines a smooth right quasigroup structure on $\mathbb{R}$.*

PROOF. Suppose that $n \geq 3$ and $P_n(x,y) = \sum_{i=0}^{n} a_i y^i + x\sum_{j=0}^{n-1} b_j y^j$. Suppose that the structure o given by $xoy = P_n(x,y)$ is a right quasigroup structure with identity 0. Then $0o0 = 0$ and $xo0 = x = 0ox$ implies that

$$a_0 = 0, \quad a_1 = b_0 = 1, \quad a_i = 0, \ \forall\ i \geq 2 \tag{6}$$

Thus $P_n(x,y) = x + y + x\phi(y)$, where $\phi(y) = \sum_{j=1}^{n-1} b_j y^j$ is a polynomial of degree $n-1$ such that $\phi(0) = 0$. Consider the equation $Xoa = 0$ where $a \in \mathbb{R}$ and X is unknown in the equation. Then the existence of solution to the equation implies that $\phi(a) \neq -1$.

Conversely, suppose that $P_n(x,y) = x + y + x\phi(y)$ and $xoy = P_n(x,y)$, where $\phi(y)$ is a polynomial of degree $n-1$ such that $\phi(a) \neq -1$ for every $a \in \mathbb{R}$ and $\phi(0) = 0$. Let $a, b \in \mathbb{R}$. Then $U = \dfrac{b-a}{1+\phi(a)}$ is the solution to the equation $Xoa = b$, where X is unknown in the equation. Also $xo0 = x = 0ox$. The smoothness of polynomial function implies the smoothness of o. Thus $(\mathbb{R}, o)$ is a smooth right quasigroup with identity. In particular if $\phi(y) = y^{2n}, n \in \mathbb{N}$ then $\phi(a) = a^{2n} \geq 0$ for every $a \in \mathbb{R}$. Thus the result follows. □

More generally we have the following proposition:

PROPOSITION 3.4. *Let $\phi(y)$ be a differentiable function on $\mathbb{R}$ such that $\phi(y) \neq -1$, for all $y \in \mathbb{R}$ and $\phi(0) = 0$. Then $(\mathbb{R}, o)$ is a smooth right quasigroup with identity 0, where $xoy = x + y + x\phi(y)$.*

REMARK 3.5. The right quasigroup $(\mathbb{R}, o)$ where $xoy = x + y + xy^{2n}$ does not have right inverse property for 1 has no right inverse.

Smooth right quasigroup structures on $\mathbb{R}$ are obtained also in the following manner :

THEOREM 3.6. *Let $\phi : \mathbb{R} \longrightarrow \mathbb{R} \setminus \{0\}$ be a differentiable map such that $\phi(0) = 1$. Then $(\mathbb{R}, \odot)$ is a smooth right quasigroup with identity 0, where $\odot$ is given by $x \odot y = y + x[\phi(y)]^{-2}$.*

PROOF. Consider the Borel subgroup $\mathcal{B} = \left\{ \overline{\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}} \mid \alpha, \beta \in \mathbb{R}, \alpha > 0 \right\}$ of the projective special linear group $PSL(2, \mathbb{R})$. Then $\mathcal{B} = HS$, $H \cap S = \{\overline{I_2}\}$ where

$$H = \left\{ \overline{\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}} \mid \alpha \in \mathbb{R}, \alpha > 0 \right\} \quad \text{and} \quad S = \left\{ \overline{\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}} \mid \beta \in \mathbb{R} \right\}.$$

Let $A_\alpha = \overline{\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}}$, $\forall \alpha \in \mathbb{R}$. Then (S, o) is a right quasigroup with identity $\overline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$ where o is given by $A_\alpha o A_\beta = A_{\alpha+\beta}$. Let $g : S \to H$ be a deformation map given by $g(A_\alpha) = \overline{\begin{pmatrix} \phi(\alpha) & 0 \\ 0 & \phi(\alpha)^{-1} \end{pmatrix}}$ where $\phi : \mathbb{R} \longrightarrow$

$\mathbb{R} \setminus \{0\}$ is a differentiable map such that $\phi(0) = 1$. The induced right quasigroup structure o_g on S is given by

$$A_\alpha o_g A_\beta = S \cap H[g(A_\alpha)A_\alpha g(A_\beta)A_\beta].$$

Since,

$$g(A_\alpha)A_\alpha g(A_\beta)A_\beta = g(A_\alpha)g(A_\beta)\begin{pmatrix} 1 & \beta+\alpha\phi(\beta)^{-2} \\ 0 & 1 \end{pmatrix}.$$

Therefore $A_\alpha o_g A_\beta = A_{\beta+\alpha\phi(\beta)^{-2}}$. Clearly the map $\psi : \mathbb{R} \longrightarrow S$ defined by $\psi(\alpha) = A_\alpha$ is bijective. This in turn induces the operation $\odot$ on $\mathbb{R}$ given by $\alpha \odot \beta = \beta + \alpha\phi(\beta)^{-2}$ so that ψ is an isomorphism. Clearly the operation $\odot$ on $\mathbb{R}$ is smooth. $\square$

Remark 3.7. The right quasigroup structure o on $\mathbb{R}$ given by $xoy = x + y + x\phi(y)$, is determined by the structure $\odot$ on $\mathbb{R}$ given by $x \odot y = y + x[\psi(y)]^{-2}$, where $\psi(y) = \frac{1}{+\sqrt{1+\phi(y)}}$ and ϕ is a differentiable map from $\mathbb{R}$ into $(-1, \infty)$ such that $\phi(0) = 0$. Further, none of the right loops on $\mathbb{R}$ are loops except the trivial one which is the additive group of real numbers.

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References

[1] Kneser, H. and M. Kneser, Reell analytische Strukturen der Alexandroff-Halbgeraden und der Alexandroff-Geraden, Archiv Math. **11** (1960), 104–106.

[2] Strambach, K., Rechtsdistributive Quasigruppen auf 1− mannigfaltigkeiten, Math. Z. **145** (1975), 63–68.

[3] Nagy, P. T. and K. Strambach, Loops in group theory and Lie theory, De Grueter Expos. Maths. **35** (2002).

[4] Lal, R., Transversals in groups, Journal of algebra **181** (1996), 70-81.

[5] Lal, R. and A. C. Yadav, Topological gyrogroups, gyrotransversals and their deformations, preprint 2008.

[6] Lang, S., Algebra, Rev. Third Edit., second Indian reprint, (2006).

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