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Stabilizing Effect of Diffusion and Dirichlet Boundary Conditions

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Abstract. It is known that diffusion together with Dirichlet boundary conditions can inhibit the occurrence of blow-up. We examine the question how strong is this stabilizing effect for reaction-diffusion equations in one space-dimension. We show that if all positive solutions of an ODE blow up in finite time then for the corresponding parabolic PDE (obtained by adding diffusion and the Dirichlet boundary condition) there is either an unbounded sequence of stationary solutions or an unbounded time-dependent solution.

1. Introduction

Consider the problem

$$\begin{cases} u_t = u_{xx} + f(u), & |x| < L,\ t > 0, \\ u(\pm L, t) = 0, & t > 0, \\ u(x, 0) = u_0(x) \geq 0, & |x| \leq L, \end{cases} \tag{1}$$

where $L > 0$, $u_0 \in C([-L, L])$ and $f \in C^1([0, \infty))$ satisfies

$$f(u) > 0 \qquad \text{for } u > 0, \tag{2}$$

$$\int_1^\infty \frac{du}{f(u)} < \infty. \tag{3}$$

Under these assumptions, the solution of the initial value problem

$$\begin{cases} U_t = f(U), \\ U(0) = U_0 \end{cases} \tag{4}$$

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blows up for every $U_0 > 0$. However, it was shown in [2] that there is a smooth function f satisfying (2), (3) such that the solution of (1) is global and bounded for every bounded u_0, see also [1] and Section 19.3 in [3].

In this paper, we are interested in the question how strong is the stabilizing effect of the diffusion together with the Dirichlet boundary condition. More precisely, we address the question whether or not there is a function f satisfying (2), (3) such that the solution u of (1) is global for every bounded u_0 and there is a constant $C > 0$ which depends on L and f but not on u_0 such that

$$\lim_{t\to\infty} \|u(\cdot,t)\|_{C([-L,L])} \le C.$$

For the nonlinearities from the examples in [2] or [1] there exists an unbounded sequence of solutions of

$$\begin{cases} v_{xx} + f(v) = 0, \quad |x| < L,\ v \ge 0, \\ v(\pm L) = 0, \end{cases} \tag{5}$$

which means that a constant C with the required property does not exist, see the remark after Proposition 3.1.

If $f(u) = ku$, $k > 0$, then the solution of (4) grows exponentially for every $U_0 > 0$ but the solution of (1) decays to zero exponentially if $L < \pi/\sqrt{k}$. We wonder if a similar change of behavior may occur for some superlinear function f.

In [2], one can find an example of a pair of functions f, g such that some solutions of the system

$$\begin{aligned} U_t &= f(U,V), \\ V_t &= g(U,V), \end{aligned}$$

blow up in finite time while all solutions of the corresponding parabolic system

$$\begin{aligned} u_t &= d_1\Delta u + f(u,v), \qquad x \in \Omega,\ t > 0, \\ v_t &= d_2\Delta v + g(u,v), \qquad x \in \Omega,\ t > 0, \end{aligned}$$

$d_1, d_2 > 0$, with the Dirichlet boundary condition

$$u(x,t) = v(x,t) = 0, \qquad x \in \partial\Omega,\ t > 0,$$

are global and converge to zero as $t \to \infty$.

We denote the set of all solutions of (5) by S. If f satisfies (2) then it is very easy to see that S is ordered. This means that if $v_1, v_2 \in S$, $v_1 \not\equiv v_2$ then either $v_1(x) < v_2(x)$ or $v_1(x) > v_2(x)$ for $|x| < L$.

Our main result is the following:

THEOREM 1.1. *Let $L > 0$ be arbitrary. Assume that $f \in C^1([0, \infty))$ satisfies (2) and*

$$\limsup_{u \to \infty} \frac{F(u)}{u^2} = \infty, \qquad F(u) := \int_0^u f(v)\,dv. \tag{6}$$

Then one of the following holds:
(i) *$S = \emptyset$ and for every $u_0 \in C([-L, L])$, $u_0 \geq 0$, there is $0 < T_{max} \leq \infty$ such that*

$$\lim_{t \uparrow T_{max}} \max_{|x| \leq L} u(x, t) = \infty; \tag{7}$$

(ii) *$S \neq \emptyset$, S is bounded, and if $v^* \in S$ is the maximal stationary solution then for every $u_0 \in C([-L, L])$, $u_0 \geq v^*$, $u_0 \not\equiv v^*$, there is $0 < T_{max} \leq \infty$ such that (7) holds;*
(iii) *$S \neq \emptyset$, and there is a sequence $\{v_n\} \subset S$ such that*

$$\lim_{n \to \infty} v_n(0) = \infty.$$

We show in Proposition 2.1 that (3) implies (6). It follows then from Theorem 1.1 that the diffusion and the Dirichlet boundary condition are not able to create a bounded set which would attract all solutions of (1) even if the growth of f is weaker than in (3). In particular, if f satisfies (2), (6) and is such that the solution u of (1) is global for every bounded initial function u_0 then the set S of all stationary solutions is unbounded (as in the examples from [1], [2]).

As an example of a function f for which (i) occurs when L is large enough, we can take $f(u) = u \log(u + a)$, $a > 1$ (then $T_{max} = \infty$ for every $u_0 \geq 0$) or $f(u) = e^u$ (then $T_{max} < \infty$ for every u_0).

The second case (ii) occurs for instance if $f(u) = u^p$, $p > 1$, and $L > 0$ is arbitrary. Then $S = \{0, v^*\}$ and $T_{max} < \infty$ for every $u_0 \geq v^*$, $u_0 \not\equiv v^*$.

Some sufficient conditions under which $S = \emptyset$, $S \neq \emptyset$ is bounded or S is unbounded will be given in Section 3.

Section 2 is devoted to the proof of the fact that (3) implies (6). We study the set S in Section 3 and prove Theorem 1.1 in Section 4.

2. Growth of the Nonlinearity f

Proposition 2.1. *Let $f \in C([0,\infty))$ satisfy* (2) *and* (3). *Then*

$$\lim_{u\to\infty} \frac{F(u)}{u^2} = \infty.$$

Proof. Suppose that

$$\liminf_{u\to\infty} \frac{F(u)}{u^2} < \infty.$$

Then there are $c > 0$ and an increasing sequence $\{u_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} u_n = \infty, \qquad \frac{F(2u_n)}{u_n^2} \le c.$$

Thus we have

$$\begin{aligned} u_n^2 &= \left(\int_{u_n}^{2u_n} ds\right)^2 \le \int_{u_n}^{2u_n} f(s)\,ds \int_{u_n}^{2u_n} \frac{ds}{f(s)} \le F(2u_n) \int_{u_n}^{2u_n} \frac{ds}{f(s)} \\ &\le c u_n^2 \int_{u_n}^{2u_n} \frac{ds}{f(s)}, \end{aligned}$$

hence

$$\int_{u_n}^{2u_n} \frac{ds}{f(s)} \ge \frac{1}{c},$$

which contradicts (3). □

3. Stationary Solutions

In this section we study the set S of solutions of (5). It is well known that v is a nontrivial solution of (5) if and only if the equation

$$T(m) := \frac{1}{\sqrt{2}} \int_0^m \frac{ds}{\sqrt{F(m) - F(s)}} = L \tag{8}$$

has a solution $m > 0$. Then

$$m = v(0) = \max_{|x| \le L} v(x), \qquad v(x) = v(-x),$$

and v is given by the formula

$$\frac{1}{\sqrt{2}} \int_{v(x)}^{m} \frac{ds}{\sqrt{F(m) - F(s)}} = |x|.$$

PROPOSITION 3.1. *Let $f \in C([0,\infty))$ satisfy (2). Then*

$$\liminf_{m \to \infty} T(m) = 0 \tag{9}$$

if and only if

$$\limsup_{u \to \infty} \frac{F(u)}{u^2} = \infty. \tag{10}$$

PROOF. Assume that (10) holds. Choose $a > 0$, an integer $n > 2F(a)/a^2$, and the smallest number $u_n > a$ such that

$$\frac{F(u_n)}{u_n^2} = n.$$

Then take $y_n \in (a, u_n)$ such that

$$\frac{F(y_n)}{y_n^2} = \frac{n}{2} \qquad \text{and} \qquad \frac{n}{2} \le \frac{F(z)}{z^2} \le n \quad \text{for } z \in [y_n, u_n].$$

Now we find an upper bound for $T(u_n)$:

$$\begin{aligned} T(u_n) &= \frac{1}{\sqrt{2}} \int_0^{u_n} \frac{ds}{\sqrt{F(u_n) - F(s)}} \\ &= \frac{1}{\sqrt{2}} \left(\int_0^{y_n} \frac{ds}{\sqrt{F(u_n) - F(s)}} + \int_{y_n}^{u_n} \frac{ds}{\sqrt{F(u_n) - F(s)}} \right) \\ &=: \frac{1}{\sqrt{2}} (I_1 + I_2), \end{aligned}$$

where

$$I_2 \le \int_{y_n}^{u_n} \frac{ds}{\sqrt{nu_n^2 - ns^2}} \le \frac{1}{\sqrt{n}} \int_0^{u_n} \frac{ds}{\sqrt{u_n^2 - s^2}} = \frac{\pi}{2\sqrt{n}},$$

and

$$I_1 \le \int_0^{y_n} \frac{ds}{\sqrt{F(u_n) - F(y_n)}} = \frac{y_n}{\sqrt{nu_n^2 - ny_n^2/2}} \le \frac{u_n}{\sqrt{nu_n^2 - nu_n^2/2}} = \sqrt{\frac{2}{n}}.$$

It follows that

$$T(u_n) \le \frac{1}{\sqrt{2}} \left(\frac{\pi}{2\sqrt{n}} + \sqrt{\frac{2}{n}} \right) \to 0 \quad \text{as } n \to \infty.$$

Since $u_n \to \infty$, we obtain (9).

Next we prove that (9) implies (10). Suppose (10) does not hold. Then there is $M > 0$ such that $F(m) < Mm^2$ for $m > 1$, and

$$T(m) = \frac{1}{\sqrt{2}} \int_0^m \frac{ds}{\sqrt{F(m) - F(s)}} \ge \frac{1}{\sqrt{2}} \int_0^m \frac{ds}{\sqrt{F(m)}} = \frac{m}{\sqrt{2F(m)}}.$$

Therefore,

$$T(m) > \frac{1}{\sqrt{2M}} \quad \text{for } m > 1. \ \square$$

REMARK. From Propositions 2.1, 3.1 it follows that if f is as in [1] or [2] then (9) holds. On the other hand, the construction of f in [1] or [2] immediately yields that

$$\limsup_{m \to \infty} T(m) = \infty.$$

This means that for every $L > 0$ there is a sequence $\{m_n\}_{n=1}^{\infty}$, $m_n \to \infty$, of solutions of (8).

Next we continue our study of the behavior of $T(m)$ as $m \to \infty$.

PROPOSITION 3.2. *Let $f \in C([0,\infty))$ satisfy* (2)*. Then*

$$\text{(i)} \qquad \lim_{u\to\infty} \frac{f(u)}{u} = \infty \quad \textit{implies that} \quad \lim_{m\to\infty} T(m) = 0,$$

$$\text{(ii)} \qquad \liminf_{u\to\infty} \frac{f(u)}{u} > 0 \quad \textit{implies that} \quad \limsup_{m\to\infty} T(m) < \infty,$$

$$\text{(iii)} \qquad \liminf_{u\to\infty} \frac{F(u)}{u^2} = 0 \quad \textit{implies that} \quad \limsup_{m\to\infty} T(m) = \infty.$$

PROOF. (i) For every $n > 0$ there is $a > 0$ such that

$$f(u) > nu \quad \text{for} \quad u > a. \tag{11}$$

Choose $m > 2a$. We find an upper bound for $T(m)$ as follows:

$$\begin{aligned} T(m) &= \frac{1}{\sqrt{2}} \int_0^m \frac{ds}{\sqrt{F(m) - F(s)}} \\ &= \frac{1}{\sqrt{2}} \left(\int_0^a \frac{ds}{\sqrt{F(m) - F(s)}} + \int_a^m \frac{ds}{\sqrt{F(m) - F(s)}} \right) \\ &=: \frac{1}{\sqrt{2}} (I_1 + I_2). \end{aligned}$$

To estimate I_1 we use (11):

$$F(m) - F(s) > F(m) - F(a) = \int_a^m f(u)\,du > \frac{n}{2}(m^2 - a^2) > \frac{3}{2} n a^2.$$

Hence

$$I_1 < \sqrt{\frac{2}{3n}}\,.$$

From the mean value theorem and (11) we have for some $y \in (s, m)$ that

$$F(m) - F(s) = f(y)(m - s) > ny(m - s) > ns(m - s), \quad s > a.$$

Therefore

$$I_2 < \int_a^m \frac{ds}{\sqrt{ns(m - s)}} < \frac{1}{\sqrt{n}} \int_0^m \frac{ds}{\sqrt{s(m - s)}} = \frac{\pi}{\sqrt{n}}\,,$$

and

$$T(m) < \frac{1}{\sqrt{3n}} + \frac{\pi}{\sqrt{2n}}.$$

To prove (ii) we proceed similarly. Under our assumption, there exist $c, a > 0$ such that

$$f(u) > cu \quad \text{for} \quad u > a.$$

We can now repeat the proof of (i) with n replaced by c to obtain that

$$T(m) < \frac{1}{\sqrt{3c}} + \frac{\pi}{\sqrt{2c}} \quad \text{for} \quad m > 2a.$$

To show (iii) we choose a sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0$, and $\{m_n\}$, $m_n \to \infty$, such that $F(m_n) < \varepsilon_n m_n^2$. Then

$$T(m_n) = \frac{1}{\sqrt{2}} \int_0^{m_n} \frac{ds}{\sqrt{F(m_n) - F(s)}} > \frac{1}{\sqrt{2}} \int_0^{m_n} \frac{ds}{\sqrt{F(m_n)}} > \frac{1}{\sqrt{\varepsilon_n}}. \ \square$$

From Proposition 3.2 one can draw some conclusions about the solvability of (8). For example, if f is as in Proposition 3.2 (iii) and satisfies (10) then S is unbounded for every $L > 0$.

If f is as in Proposition 3.2 (i) then $S = \emptyset$ or $S \neq \emptyset$ is bounded.

If $f(0) = 0$ and $f'(0) > 0$ or $f(0) > 0$ (which guarantees that $\lim_{m \to 0} T(m) < \infty$, see [4]) and f is as in Proposition 3.2 (ii) then $S = \emptyset$ if L is large enough.

If $f(0) = f'(0) = 0$ (which implies that $\lim_{m \to 0} T(m) = \infty$, see [4]) and (10) holds then $S \neq \emptyset$ for every $L > 0$.

4. Proof of Theorem 1.1

Proof. Assume that $S \neq \emptyset$, S is bounded and $v^* \not\equiv 0$. Let $m^* = v^*(0)$ be the largest root of (8). Proposition 3.1 guarantees that (9) holds, thus $T(m^* + \varepsilon) < L$ for $\varepsilon > 0$. Let w_ε denote the solution of the initial value problem

$$\begin{aligned} &w_{xx} + f(w) = 0, \\ &w(0) = m^* + \varepsilon, \quad w_x(0) = 0. \end{aligned}$$

Then there is a unique $x_\varepsilon \in (0, L)$ such that $w_\varepsilon(x_\varepsilon) = v^*(x_\varepsilon)$. Set

$$\varphi_\varepsilon(x) = \begin{cases} w_\varepsilon(x), & |x| \le x_\varepsilon, \\ v^*(x), & x_\varepsilon < |x| \le L. \end{cases}$$

For the solution u^ε of (1) with $u_0(x) = \varphi_\varepsilon(x)$ we have that $u_t^\varepsilon(x,t) \ge 0$ for $(x,t) \in [-L, L] \times (0, T^\varepsilon_{max})$ where $T^\varepsilon_{max} \le \infty$ is the maximal existence time of u^ε, see Section 52.6 in [3]. We can now easily see that u^ε cannot stay bounded because then $T^\varepsilon_{max} = \infty$ (cf. Section 16 in [3]) and $u^\varepsilon(x,t) \to \psi^\varepsilon(x)$ as $t \to \infty$ where $\psi^\varepsilon \in S$, $\psi^\varepsilon > v^*$ in $(-L, L)$, see Lemma 53.10 in [3].

If u_0 is as in Theorem 1.1 (ii) then for every $\tau \in (0, T_{max})$ there is $\varepsilon > 0$ such that $u(\cdot, \tau) \ge \varphi_\varepsilon$ in $[-L, L]$ and it follows by comparison that u is unbounded.

If $f(0) = 0$ and $S = \{0\}$ then we modify the argument slightly. In this case, there is no solution of (8) but we can repeat the proof with $v^* \equiv 0$ and w_ε which is the solution of

$$\begin{aligned} & w_{xx} + f(w) = 0, \\ & w(0) = \varepsilon, \quad w_x(0) = 0. \end{aligned}$$

If $S = \emptyset$ then necessarily $f(0) > 0$ and the solution u of (1) with $u_0 \equiv 0$ cannot stay bounded. □

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