J. Math. Sci. Univ. Tokyo
17 (2010), 267–301.

Remarks on Boundary Values for Temperate Distribution Solutions to Regular-Specializable Systems

By Susumu Yamazaki

Dedicated to Professor Kiyoomi Kataoka on his sixtieth birthday

Abstract. For temperate distribution solutions to regular-specializable systems of analytic linear differential equations, boundary value problems are formulated in the framework of algebraic analysis. Moreover, under a certain hyperbolicity condition, the solvability theorems are discussed for some classes.

Introduction

In this paper, we consider boundary value problems for temperate distribution solutions to regular-specializable systems of analytic linear differential equations in the framework of *Algebraic Analysis*.

Regular-specializable system is first defined by Kashiwara [10], and constitutes a special class of Fuchsian systems in the sense of Laurent-Monteiro Fernandes [19]. In a single equation case, under the assumption that all the characteristic exponents are constants, this corresponds to a Fuchsian operator in the sense of Baouendi-Goulaouic [4] or equivalently, a regular-singular operator with weak sense due to Kashiwara-Oshima [14] (cf. Oshima [28]). For any regular-specializable system, its vanishing cycle and nearby cycle in the $\mathscr{D}$-Module theory are defined (see Dimca [9], Kashiwara [10], Laurent [18], Maisonobe-Mebkhout [22]). After the results by Kashiwara-Oshima [14] and Oshima [28], for any hyperfunction solutions to a regular-specializable system, Monteiro Fernandes ([23], [24]) defined an injective boundary value morphism which takes values in hyperfunction solutions to the nearby cycle of the system. This morphism extends

2010 *Mathematics Subject Classification*. Primary 32C38; Secondary 35A27, 35G15, 58J32.
This research was partially supported by the Grant-in-Aid for Scientific Research (C) No. 20540191, Japan Society for the Promotion of Science.

the non-characteristic boundary value morphism due to Komatsu-Kawai and Schapira. Moreover Laurent-Monteiro Fernandes [20] reformulated this morphism and discussed the solvability under a kind of hyperbolicity condition (see Yamazaki [34] for a microlocal version).

Next, we replace hyperfunctions by distributions. Then, the functor of moderate cohomology due to Kashiwara [11] and its microlocalization due to Andronikof [2] enable us to apply the algebraic and geometrical tools in Kashiwara-Schapira [15] to treat (temperate) distributions. As an application of the theory of these functors, we shall consider boundary value problem for temperate distribution solutions to regular-specializable systems, and prove that the boundary value morphism due to Monteiro Fernandes induces the boundary value morphism for temperate distribution solutions to these systems. Further as examples, we shall show that the boundary value morphism is surjective if the system satisfies regular-singular and hyperbolicity conditions or if the equation under consideration is regular-specializable and Fuchsian strictly hyperbolic operator in the sense of Bove-Lewis-Parenti [6].

We remark that Tahara [32] investigated the structure of hyperfunction solutions to a general Fuchsian hyperbolic operator in depth, and the uniqueness of boundary values was obtained by Oaku [25] without hyperbolicity condition by using F-mild hyperfunctions. Further Oaku-Yamazaki [27] extended the uniqueness results to Fuchsian systems. For the Cauchy problem for hyperfunction and microfunction solutions to a Fuchsian hyperbolic system, we refer to Yamazaki [35].

1. Preliminaries

In this section, we shall recall the notation and several facts needed in later sections.

We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of integers, of real numbers and of complex numbers respectively. Moreover we set $\mathbb{N} := \{n \in \mathbb{Z};\, n \geqslant 1\} \subset \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_{>0} := \{r \in \mathbb{R};\, r > 0\} \subset \mathbb{R}_{\geqslant 0} := \{r \in \mathbb{R};\, r \geqslant 0\}$.

In this paper, we shall mainly follow the notation of Andronikof [2] and Kashiwara-Schapira [15], [16].

Let $\mathcal{A}$ be a Ring on a topological space Z. We denote by $\mathfrak{Mod}(\mathcal{A})$ the category of (left) $\mathcal{A}$-Modules, and by $\mathfrak{Mod}_{\mathrm{coh}}(\mathcal{A})$ the full subcategory of $\mathfrak{Mod}(\mathcal{A})$ consisting of coherent $\mathcal{A}$-Modules. Further we denote by $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ the bounded derived category of complexes of $\mathcal{A}$-Modules, and by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A})$ the full subcategory of $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ consisting of objects with coherent cohomologies. We set $\mathfrak{Mod}(Z) := \mathfrak{Mod}(\mathbb{C}_Z)$, $\mathbf{D}^{\mathrm{b}}(Z) := \mathbf{D}^{\mathrm{b}}(\mathbb{C}_Z)$ etc. for short.

Let $\mathcal{C}$ be any of $\mathfrak{Mod}(\mathcal{A})$, $\mathfrak{Mod}_{\mathrm{coh}}(\mathcal{A})$, $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ or $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A})$. Then by the abuse of the notation, we write simply $\mathcal{F} \in \mathcal{C}$ if $\mathcal{F}$ is an object of $\mathcal{C}$ on an open subset.

In this paper, all the manifolds are assumed to be *paracompact*. For a vector bundle $\tau\colon E \to Z$ over a manifold Z, we set $\dot{\tau}\colon \dot{E} := E \setminus Z \to Z$ (the zero-section removed). Let $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(E) \subset \mathbf{D}^{\mathrm{b}}(E)$ be the subcategory of the bounded derived category of sheaves such that each cohomology is conic.

Throughout this paper, M denotes an $(n+1)$-dimensional real analytic manifold, N a one-codimensional real analytic closed submanifold of M, and $f_N\colon N \hookrightarrow M$ the canonical embedding. Let $\tau_N\colon T_N M \to N$ and $\pi_N\colon T^*_N M \to N$ be the *normal* and the *conormal bundles* to N in M respectively. Let X and Y be complexifications of M and N respectively such that Y is a closed submanifold of X and that $Y \cap M = N$, and $f\colon Y \hookrightarrow X$ the complexification of f_N. We denote by $\mathscr{or}_{N/M}$ the relative orientation sheaf attached to $N \to M$, by $\omega_{N/M}$ the relative dualizing complex, and by $\omega^{\otimes -1}_{N/M}$ its dual. Explicitly, $\omega_{N/M} = \mathscr{or}_{N/M}[-1]$ and $\omega^{\otimes -1}_{N/M} = \mathscr{or}_{N/M}[1]$. Let $i\colon M \hookrightarrow X$ be the canonical embedding.

Let $\mathscr{O}_X$ and $\mathscr{D}_X$ denote the sheaves on X *of holomorphic functions* and *of holomorphic linear differential operators* respectively (see Kashiwara [12] for $\mathscr{D}$-Module theory). Let $\mathscr{D}_{Y\to X} := \mathscr{O}_Y \underset{\mathscr{O}_X}{\otimes} \mathscr{D}_X$ and $\mathscr{D}_{X\leftarrow Y}$ be *transfer bi-Modules*. We set $\mathscr{D}^A_M := i^{-1}\mathscr{D}_X$ etc. (we add the superscript A in order to avoid the confusion with holomorphic cases). Let $\mathscr{B}_M$ and $\mathscr{D}\!\mathscr{b}_M$ be the sheaves on M of *Sato hyperfunctions* and of *Schwartz distributions* respectively. Let $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(M)$ denote the bounded derived category *of* $\mathbb{R}$-*constructible sheaves*. We denote by

$$\mathscr{T}\!\mathit{hom}(*, \mathscr{D}\!\mathscr{b}_M) = TH_M(*)\colon \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(M)^{\circ} \to \mathbf{D}^{\mathrm{b}}(\mathscr{D}^A_M),$$

the *Schwartz functor* due to Kashiwara [11] (see also Kashiwara-Schapira [16]). This functor is characterized by the following properties:

(i) If Z is a closed subanalytic subset of M, then $\mathscr{T}\!\mathit{hom}(\mathbb{C}_Z, \mathscr{D}\!\mathscr{b}_M) = \Gamma_Z(\mathscr{D}\!\mathscr{b}_M)$;

(ii) If U is an open subanalytic subset of M, then $\mathcal{T}hom(\mathbb{C}_U, \mathcal{D}b_M) = \mathcal{D}b_M / \Gamma_{X \setminus U}(\mathcal{D}b_M)$. In particular, we have

$$\begin{array}{ccccccc} \Gamma_{X\setminus U}(\mathcal{D}b_M) & \longrightarrow & \mathcal{D}b_M & \longrightarrow & \mathcal{T}hom(\mathbb{C}_U, \mathcal{D}b_M) & \xrightarrow{+1} & \\ \downarrow & & \| & & \downarrow & & \\ \boldsymbol{R}\Gamma_{X\setminus U}(\mathcal{D}b_M) & \longrightarrow & \mathcal{D}b_M & \longrightarrow & \boldsymbol{R}\Gamma_U(\mathcal{D}b_M) & \xrightarrow{+1} & \end{array} \tag{1.1}$$

Further, the functor of *moderate cohomology* due to Kashiwara [11] (see also [16]) is defined by

$$\mathcal{T}hom(*, \mathcal{O}_X) := \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{T}hom(*, \mathcal{D}b_X)) \colon \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(X)^{\circ} \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X).$$

Note that for any $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(M)$, we have

$$\mathcal{T}hom(F, \mathcal{D}b_M) = i^{-1}\mathcal{T}hom(\boldsymbol{R}i_* F, \mathcal{O}_X) \otimes \omega_{M/X}^{\otimes -1}.$$

Let $\nu_N(*) \colon \mathbf{D}^{\mathrm{b}}(M) \to \mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(T_N M)$ and $\mu_N(*) \colon \mathbf{D}^{\mathrm{b}}(M) \to \mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(T^*_N M)$ be the *specialization* and *microlocalization functors* respectively. Further, we denote by

$$\begin{aligned} T\text{-}\nu_N \mathcal{T}hom(*, \mathcal{D}b_M) &= T\text{-}\nu_N\, TH_M(*) \colon \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(M)^{\circ} \to \mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(T_N M), \\ T\text{-}\mu_N \mathcal{T}hom(*, \mathcal{D}b_M) &= T\text{-}\mu_N\, TH_M(*) \colon \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(M)^{\circ} \to \mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(T^*_N M), \end{aligned}$$

the *temperate Schwartz specialization* and *temperate Schwartz microlocalization functors along* N respectively due to Andronikof [2]. In particular, we set:

$$\begin{aligned} T\text{-}\nu_N(\mathcal{D}b_M) &:= T\text{-}\nu_N \mathcal{T}hom(\mathbb{C}_M, \mathcal{D}b_M), \\ T\text{-}\mu_N(\mathcal{D}b_M) &:= T\text{-}\mu_N \mathcal{T}hom(\mathbb{C}_M, \mathcal{D}b_M). \end{aligned}$$

Then we recall:

1.1. THEOREM ([2]). (1) *T-$\nu_N(\mathcal{D}b_M)$, $\nu_N(\mathcal{D}b_M)$ and $\nu_N(\mathcal{B}_M)$ are concentrated in degree zero, and there exist the following natural monomorphisms of sheaves*:

$$T\text{-}\nu_N(\mathcal{D}b_M) \rightarrowtail \nu_N(\mathcal{D}b_M) \rightarrowtail \nu_N(\mathcal{B}_M).$$

(2) *T-$\mu_N(\mathcal{D}b_M)$ and $\mu_N(\mathcal{B}_M)$ are concentrated in degree zero, and there exist the following natural morphisms of sheaves*:

$$T\text{-}\mu_N(\mathcal{D}b_M) \simeq \mathcal{H}^0\mu_N(\mathcal{D}b_M) \rightarrowtail \mu_N(\mathcal{B}_M).$$

(3) *There exists the following isomorphisms*:

$$\boldsymbol{R}\tau_{N!}\,T\text{-}\nu_N(\mathcal{D}b_M) = \boldsymbol{R}\pi_{N*}\,T\text{-}\mu_N(\mathcal{D}b_M) = \Gamma_N(\mathcal{D}b_M) \simeq \mathcal{D}^A_{M\leftarrow N} \underset{\mathcal{D}^A_N}{\otimes} \mathcal{D}b_N,$$

$$\boldsymbol{R}\tau_{N*}\,T\text{-}\nu_N(\mathcal{D}b_M) = \boldsymbol{R}\pi_{N!}\,T\text{-}\mu_N(\mathcal{D}b_M) \otimes \boldsymbol{or}_{N/M}[1] = f_N^{-1}\mathcal{D}b_M\,.$$

Here $\mathcal{D}^A_{M\leftarrow N} := \mathcal{D}_{X\leftarrow Y}\big|_N \otimes \boldsymbol{or}_{N/M}$.

1.2. Remark. By Sato's fundamental distinguished triangle and Theorem 1.1, there exist the following morphisms of distinguished triangles:

$$\begin{array}{ccccccc}
\Gamma_N(\mathcal{D}b_M) & \longrightarrow & \boldsymbol{R}\dot{\pi}_{N*}\,T\text{-}\mu_N(\mathcal{D}b_M) & \longrightarrow & f_N^{-1}\mathcal{D}b_M \otimes \boldsymbol{or}_{N/M} & \xrightarrow{+1} \\
\downarrow & & \downarrow & & \| & \\
\boldsymbol{R}\Gamma_N(\mathcal{D}b_M) & \longrightarrow & \boldsymbol{R}\dot{\pi}_{N*}\,\mu_N(\mathcal{D}b_M) & \longrightarrow & f_N^{-1}\mathcal{D}b_M \otimes \boldsymbol{or}_{N/M} & \xrightarrow{+1} \\
\downarrow & & \downarrow & & \downarrow & \\
\Gamma_N(\mathcal{B}_M) & \longrightarrow & \boldsymbol{R}\dot{\pi}_{N*}\,\mu_N(\mathcal{B}_M) & \longrightarrow & f_N^{-1}\mathcal{B}_M \otimes \boldsymbol{or}_{N/M} & \xrightarrow{+1}
\end{array} \tag{1.2}$$

Here we remark that $\Gamma_N(\mathcal{B}_M) = \boldsymbol{R}\Gamma_N(\mathcal{B}_M)$ while $\Gamma_N(\mathcal{D}b_M) \neq \boldsymbol{R}\Gamma_N(\mathcal{D}b_M)$.

Next, set

$$P^+ := \{(v,\xi) \in \dot{T}_N M \underset{N}{\times} \dot{T}^*_N M;\ \langle v,\xi\rangle > 0\}$$

and denote by $p_1^+\colon P^+ \to \dot{T}_N M$ and $p_2^+\colon P^+ \to \dot{T}^*_N M$ the canonical projections. Then:

1.3. Proposition ([33, Corollary A.2], cf [30, Chapter I]). *For any $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(T_N M)$, there exists the following distinguished triangle*:

$$F \to \tau_N^{-1}\boldsymbol{R}\tau_{N!}\,F \otimes \omega_{N/M}^{\otimes -1} \to \boldsymbol{R}p^+_{1*}\,p_2^{+\,-1}\,F^\wedge \otimes \omega_{N/M}^{\otimes -1} \xrightarrow{+1}.$$

Here $F^\wedge$ denotes the Fourier-Sato transform of F.

Let $\varphi \in \mathcal{O}_X$ be a local defining function of Y; that is, Y is locally given by $\varphi^{-1}(0)$ and $d\varphi(z) \neq 0$ for any $z \in Y$. Further, we can assume that $\varphi|_M$ is a local defining function of N. Then, mappings $s\colon N \ni x \mapsto (x; d\varphi(x)) \in \dot{T}_N M$ and ${}^t s\colon N \ni x \mapsto (x; {}^t d\varphi(x)) \in \dot{T}^*_N M$ are well defined. Set $T_N M^+ := \mathbb{R}_{>0}\, s(N) \subset \dot{T}_N M$ and $T^*_N M^+ := \mathbb{R}_{>0}\, {}^t s(N) \subset \dot{T}^*_N M$. Then, we have

$$\begin{aligned} s^{-1}\boldsymbol{R}p^+_{1*}\, p_2^{+\,-1} F^\wedge \otimes \omega^{\otimes -1}_{N/M} &= s^{-1}\boldsymbol{R}\Gamma_{T^+_N M}\boldsymbol{R}p^+_{1*}\, p_2^{+\,-1} F^\wedge \otimes \omega^{\otimes -1}_{N/M} \\ &= \boldsymbol{R}\tau_{N*}\boldsymbol{R}p^+_{1*}\, p_2^{+\,-1}\boldsymbol{R}\Gamma_{T^*_N M^+}(F^\wedge) \otimes \omega^{\otimes -1}_{N/M} \\ &\simeq \boldsymbol{R}\pi_{N*}\boldsymbol{R}\Gamma_{T^*_N M^+}(F^\wedge) \otimes \omega^{\otimes -1}_{N/M} \simeq {}^t s^{-1} F^\wedge \otimes \omega^{\otimes -1}_{N/M}. \end{aligned}$$

Taking $F = \nu_N(\mathscr{B}_M) \otimes \mathscr{or}_{N/M}$, $\nu_N(\mathscr{D}\mathscr{b}_M) \otimes \mathscr{or}_{N/M}$ or $T\text{-}\nu_N(\mathscr{D}\mathscr{b}_M) \otimes \mathscr{or}_{N/M}$ in Proposition 1.3, we obtain by Theorem 1.1:

1.4. PROPOSITION. *For any $\mathcal{M} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$, there exist the following morphisms of distinguished triangles*:

$$\begin{array}{ccccccc} \Gamma_N(\mathscr{D}\mathscr{b}_M) & \longrightarrow & {}^t s^{-1}\, T\text{-}\mu_N(\mathscr{D}\mathscr{b}_M) & \longrightarrow & s^{-1}\, T\text{-}\nu_N(\mathscr{D}\mathscr{b}_M) \otimes \mathscr{or}_{N/M} & \xrightarrow{+1} & \\ \downarrow & & \downarrow & & \downarrow & & \\ \boldsymbol{R}\Gamma_N(\mathscr{D}\mathscr{b}_M) & \longrightarrow & {}^t s^{-1}\mu_N(\mathscr{D}\mathscr{b}_M) & \longrightarrow & s^{-1}\nu_N(\mathscr{D}\mathscr{b}_M) \otimes \mathscr{or}_{N/M} & \xrightarrow{+1} & \\ \downarrow & & \downarrow & & \downarrow & & \\ \Gamma_N(\mathscr{B}_M) & \longrightarrow & {}^t s^{-1}\mu_N(\mathscr{B}_M) & \longrightarrow & s^{-1}\nu_N(\mathscr{B}_M) \otimes \mathscr{or}_{N/M} & \xrightarrow{+1} & . \end{array}$$

2. Boundary Value Morphism and Regular-Specializable Systems

In this section, we define boundary values of temperate distribution solutions to a regular-specializable systems as a main result. For the notation and related topics, see Appendix A. In particular, $\mathfrak{Mod}_{\mathcal{R}_Y}(\mathscr{D}_X)$ denotes the category of regular-specializable $\mathscr{D}_X|_Y$-Modules. For a $\mathscr{D}_X$-Module $\mathcal{M}$, let

$$\begin{aligned} \boldsymbol{D}f^*\mathcal{M} &:= \mathscr{D}_{Y\to X} \overset{\boldsymbol{L}}{\underset{\mathscr{D}_X}{\otimes}} \mathcal{M}, \\ \boldsymbol{D}f^!\mathcal{M} &:= \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathcal{M}, \mathscr{D}_{X\leftarrow Y}), \mathscr{D}_Y)[-1], \end{aligned}$$

be the *inverse image* and the *extraordinary inverse image* of $\mathcal{M}$ in the $\mathcal{D}$-Module theory respectively. We take the following local coordinates:

$$
\begin{array}{ccc}
N = \mathbb{R}^n_x \times \{0\} & \xhookrightarrow{f_N} & M = \mathbb{R}^n_x \times \mathbb{R}_t \\
\downarrow & & \downarrow i \\
Y = \mathbb{C}^n_z \times \{0\} & \xhookrightarrow{f} & X = \mathbb{C}^n_z \times \mathbb{C}_\tau
\end{array}
\tag{2.1}
$$

2.1. Proposition. *If $\mathcal{M} \in \mathfrak{Mod}_{\mathcal{R}_Y}(\mathcal{D}_X)$, then the exists the following commutative diagram*:

$$
\begin{array}{ccc}
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_N(\mathcal{D}b_M)) \otimes \omega_{N/M}^{\otimes -1} & = & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^!\mathcal{M}, \mathcal{D}b_N) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_N(\mathcal{B}_M)) \otimes \omega_{N/M}^{\otimes -1} & = & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^!\mathcal{M}, \mathcal{B}_N).
\end{array}
\tag{2.2}
$$

Proof. Let $\mathcal{F}$ be a $\mathcal{D}_Y$-Module. Since $\mathcal{D}_{X\leftarrow Y}$ is flat as a right $\mathcal{D}_Y$-Module, by Theorem A.15 (1) and (A.3) we see

$$
\begin{aligned}
&\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^!\mathcal{M}, \mathcal{F})[-1] \simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^!\mathcal{M}, \mathcal{D}_Y) \overset{\boldsymbol{L}}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{F}[-1] \\
&\simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_{X\leftarrow Y}) \overset{\boldsymbol{L}}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{F} \simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_{X\leftarrow Y} \overset{\boldsymbol{L}}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{F}) \\
&\simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_{X\leftarrow Y} \underset{\mathcal{D}_Y}{\otimes} \mathcal{F}).
\end{aligned}
\tag{2.3}
$$

Then, applying the functor $\boldsymbol{R}\Gamma_N(*) \otimes \mathcal{or}_M$ to (A.4), we obtain

$$
\begin{aligned}
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}^A_{M\leftarrow N} \underset{\mathcal{D}^A_N}{\otimes} \mathcal{B}_N) &= \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^!\mathcal{M}, \mathcal{B}_N) \otimes \omega_{N/M} \\
&\simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_N(\mathcal{B}_M)).
\end{aligned}
\tag{2.4}
$$

We can write a section (or a germ) of $\mathcal{D}^A_{M\leftarrow N}$ as $\sum_j a_j(x, \partial_x)\, \partial_t^j \otimes |dt|^{\otimes -1}$, where $|dt|^{\otimes -1}$ is a generator of $\mathcal{or}_{N/M}$. Then the morphism defined by

$$\mathcal{D}^A_{M\leftarrow N} \ni \partial_t^j \otimes |dt|^{\otimes -1} \mapsto \partial_t^j \delta(t)$$

induces

$$\begin{array}{ccc}\mathscr{D}^A_{M\leftarrow N}\underset{\mathscr{D}^A_N}{\otimes}\mathscr{D}\mathscr{b}_N & = & \Gamma_N(\mathscr{D}\mathscr{b}_M)\\ \downarrow & & \downarrow\\ \mathscr{D}^A_{M\leftarrow N}\underset{\mathscr{D}^A_N}{\otimes}\mathscr{B}_N & \longrightarrow & \Gamma_N(\mathscr{B}_M).\end{array}$$

Thus by (2.3) and (2.4), we have

$$\begin{array}{ccc}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^{!}\mathscr{M},\mathscr{D}\mathscr{b}_N)\otimes\omega_{N/M} & \longrightarrow & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^{!}\mathscr{M},\mathscr{B}_N)\otimes\omega_{N/M}\\ \| & & \|\\ \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}^A_{M\leftarrow N}\underset{\mathscr{D}^A_N}{\otimes}\mathscr{D}\mathscr{b}_N) & \longrightarrow & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}^A_{M\leftarrow N}\underset{\mathscr{D}^A_N}{\otimes}\mathscr{B}_N)\\ \| & & \|\\ \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\Gamma_N(\mathscr{D}\mathscr{b}_M)) & \longrightarrow & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\Gamma_N(\mathscr{B}_M))\end{array}$$

thus we obtain (2.2). □

We recall a result on the boundary value morphism. For any $\mathscr{M}\in\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$, we denote by $\mathrm{Ch}\,\mathscr{M}$ the *characteristic variety.*

2.2. REMARK. $\mathscr{N}\in\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_X|_Y)$ is said to be *near-hyperbolic* at $x_0\in N$ in $\pm dt$-codirection in the sense of Laurent-Monteiro Fernandes [20, Definition 1.3.1] if there exist positive constants C and ε such that the following condition holds:

$$\begin{aligned}&\mathrm{Ch}\,\mathscr{N}\cap\{(z,\tau;\zeta,\eta)\in T^*X;\ |z-x_0|,\ |\tau|<\varepsilon,\ \mathrm{Re}\,\tau\neq 0\}\\ &\qquad\subset\{(z,\tau;\zeta,\eta)\in T^*X;\ |\mathrm{Re}\,\eta|<C\big(|\mathrm{Im}\,\zeta|(|\mathrm{Im}\,z|+|\mathrm{Im}\,\tau|)+|\mathrm{Re}\,\zeta|\big)\}.\end{aligned}$$

Let $\mathscr{B}^A_{N|M}$ denote the subsheaf of $\mathscr{B}_M|_N$ *of hyperfunctions with a real analytic parameter* t (see [30], [35]). If $\mathscr{N}$ is near-hyperbolic and Fuchsian in the sense of [19], then there exists an isomorphism (see [35]):

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{N},\mathscr{B}^A_{N|M})\simeq\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^{*}\mathscr{N},\mathscr{B}_N).$$

2.3. THEOREM ([24], cf. [34]). *For any* $\mathcal{M} \in \mathfrak{Mod}_{\mathcal{R}_Y}(\mathcal{D}_X)$, *there exists the following morphism of distinguished triangles*:

$$\begin{array}{ccc}
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, {}^t s^{-1}\mu_N(\mathcal{B}_M)) \otimes \boldsymbol{or}_{N/M} & \xrightarrow{\alpha} & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Phi_Y(\mathcal{M}), \mathcal{B}_N) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, s^{-1}\nu_N(\mathcal{B}_M)) & \xrightarrow{\beta} & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{B}_N) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_N(\mathcal{B}_M)) \otimes \omega_{N/M}^{\otimes -1} & = & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^!\mathcal{M}, \mathcal{B}_N) \\
\downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1}
\end{array}$$

such that α, β *induce monomorphisms*:

$$H^0(\alpha)\colon \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, {}^t s^{-1}\mu_N(\mathcal{B}_M)) \otimes \boldsymbol{or}_{N/M} \rightarrowtail \mathcal{H}om_{\mathcal{D}_Y}(\Phi_Y(\mathcal{M}), \mathcal{B}_N),$$
$$H^0(\beta)\colon \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, s^{-1}\nu_N(\mathcal{B}_M)) \rightarrowtail \mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{B}_N).$$

In addition, if $\mathcal{M}$ *is near-hyperbolic at any* $x_0 \in N$ *in* $\pm dt$*-codirection, then both* α *and* β *are isomorphisms.*

2.4. REMARK. (1) If Y is non-characteristic for $\mathcal{M}$, then by (A.2) and Theorem 2.3, we have

$$\begin{array}{ccc}
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, s^{-1}\nu_N(\mathcal{B}_M)) & \xrightarrow{\beta} & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{B}_N) \\
\downarrow & & \| \\
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_N(\mathcal{B}_M)) \otimes \omega_{N/M}^{\otimes -1} & = & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mathcal{B}_N),
\end{array}$$

and $H^0(\beta)$ coincides with the boundary value morphism due to Komatsu-Kawai and Schapira.

(2) A microlocal counterpart of β is defined in Yamazaki [34] along the line of [27].

The main result in this paper is the following:

2.5. THEOREM. *For any* $\mathcal{M} \in \mathfrak{Mod}_{\mathcal{R}_Y}(\mathcal{D}_X)$, *there exists the following* *morphism of distinguished triangles*:

$$\begin{array}{ccc}
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, {}^t s^{-1} T\text{-}\mu_N(\mathcal{D}b_M)) \otimes \boldsymbol{or}_{N/M} & \xrightarrow{\alpha^t} & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Phi_Y(\mathcal{M}), \mathcal{D}b_N) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, s^{-1} T\text{-}\nu_N(\mathcal{D}b_M)) & \xrightarrow{\beta^t} & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{D}b_N) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_N(\mathcal{D}b_M)) \otimes \omega_{N/M}^{\otimes -1} & = & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^!\mathcal{M}, \mathcal{D}b_N) \\
\downarrow{+1} & & \downarrow{+1}
\end{array}$$

which is compatible with Proposition 1.4 *and Theorem* 2.3. *In particular*, α^t *and* β^t *induce monomorphisms*:

$$\begin{aligned}
H^0(\alpha^t)\colon \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, {}^t s^{-1} T\text{-}\mu_N(\mathcal{D}b_M)) &\otimes \boldsymbol{or}_{N/M} \\
&\rightarrowtail \mathcal{H}om_{\mathcal{D}_Y}(\Phi_Y(\mathcal{M}), \mathcal{D}b_N), \\
H^0(\beta^t)\colon \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, s^{-1} T\text{-}\nu_N(\mathcal{D}b_M)) &\rightarrowtail \mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{D}b_N).
\end{aligned}$$

PROOF. First we define β^t. The problem is local, so we take coordinates as in (2.1). We follow the method in [23] (for the notation, see Appendix A). Recall the set G in (A.1). For $\alpha \in G$, set for short:

$$\mathcal{S}_p^{(\alpha)} := \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\psi_p^{(\alpha)}(\mathcal{O}_X), \mathcal{O}_X).$$

Then, the inductive system $\{\psi_p^{(\alpha)}(\mathcal{O}_X)\}_{p\in\mathbb{N}_0}$ defines a projective system $\{\mathcal{S}_p^{(\alpha)}\}_{p\in\mathbb{N}_0}$. Set $\Omega_+ := \{(x,t) \in M;\, t > 0\}$, and let $\Omega_+^c \subset X$ be a connected open neighborhood of Ω_+ such that $Y \cap \Omega_+^c = \emptyset$ and that we can take a branch $L(\tau)$ of $\log\tau$ in Ω_+^c. Let $j\colon \Omega_+^c \hookrightarrow X$ be the embedding. Note that $j^{-1}\mathcal{S}_p^{(\alpha)}$ is concentrated in degree zero since $j^{-1}\psi_p^{(\alpha)}(\mathcal{O}_X)$ is a coherent $j^{-1}\mathcal{O}_X$-Module. We define

$$s_p^{(\alpha)\prime} \in j^{-1}\mathcal{S}_p^{(\alpha)} \simeq \mathcal{H}om_{j^{-1}\mathcal{D}_X}(j^{-1}\psi_p^{(\alpha)}(\mathcal{O}_X), j^{-1}\mathcal{O}_X)$$

by the following:

$$\begin{aligned} j^{-1}\psi_p^{(\alpha)}(\mathcal{O}_X) = \sum_{k=0}^{p} \mathcal{O}_{\Omega^+}\, e_k^{(\alpha)} &\ni \sum_{k=0}^{p} m_k\, e_k^{(\alpha)} \\ &\mapsto \sum_{k=0}^{p} m_k \frac{e^{-\alpha L(\tau)} L(\tau)^k}{k!} \in j^{-1}\mathcal{O}_X\,. \end{aligned}$$

Then applying the functor $j_!$, we obtain a $(\mathcal{D}_X)_{\Omega^+}$-linear morphism

$$s_p^{(\alpha)} := j_! s_p^{(\alpha)\prime} \colon \psi_p^{(\alpha)}(\mathcal{O}_X)_{\Omega^+} \to (\mathcal{O}_X)_{\Omega^+}.$$

Since $\psi_p^{(\alpha)}(\mathcal{O}_X)$ is $\mathcal{D}_X$-coherent, (locally) we can take a resolution

$$0 \leftarrow \psi_p^{(\alpha)}(\mathcal{O}_X) \leftarrow \mathcal{D}_X^{r_0} \xleftarrow{\cdot P_0} \cdots \xleftarrow{\cdot P_{k-1}} \mathcal{D}_X^{r_k} \leftarrow 0.$$

Then, both $(\mathcal{S}_p^{(\alpha)})_{\Omega_+^c}$ and $\boldsymbol{R}\mathcal{H}om_{(\mathcal{D}_X)_{\Omega_+^c}}(\psi_p^{(\alpha)}(\mathcal{O}_X)_{\Omega_+^c}, (\mathcal{O}_X)_{\Omega_+^c})$ are represented by $(\mathcal{O}_X^{r_0})_{\Omega_+^c} \xrightarrow{(P_0)_{\Omega_+^c}\cdot} \cdots \xrightarrow{(P_{k-1})_{\Omega_+^c}\cdot} (\mathcal{O}_X^{r_k})_{\Omega_+^c}$, that is,

$$\begin{aligned} s_p^{(\alpha)} &\in \mathcal{H}om_{(\mathcal{D}_X)_{\Omega_+^c}}(\psi_p^{(\alpha)}(\mathcal{O}_X)_{\Omega_+^c}, (\mathcal{O}_X)_{\Omega_+^c}) \\ &\simeq \boldsymbol{R}\mathcal{H}om_{(\mathcal{D}_X)_{\Omega_+^c}}(\psi_p^{(\alpha)}(\mathcal{O}_X)_{\Omega_+^c}, (\mathcal{O}_X)_{\Omega_+^c}) = (\mathcal{S}_p^{(\alpha)})_{\Omega_+^c}\,. \end{aligned}$$

For $p \leqslant q$, the morphism $(\mathcal{S}_q^{(\alpha)})_{\Omega_+^c} \ni s_q^{(\alpha)} \mapsto s_p^{(\alpha)} \in (\mathcal{S}_p^{(\alpha)})_{\Omega_+^c}$ is compatible with the projective system structure of $\{\mathcal{S}_p^{(\alpha)}\}_{p\in\mathbb{N}_0}$. Let $\delta\colon X \hookrightarrow X \times X$ be the diagonal embedding. We denote by $*\boxtimes *$ and $*\underset{\mathcal{D}}{\boxtimes}*$ the external tensor products of sheaves and of $\mathcal{D}$-Modules respectively. We remark that $*\overset{\boldsymbol{L}}{\underset{\mathcal{O}_X}{\otimes}}* = \boldsymbol{D}\delta^*(*\underset{\mathcal{D}}{\boxtimes}*)$. Let $Z \Subset N$ be a compact subset, and set

$$\begin{aligned} A &:= \{(x;t) \in \dot{T}_N M;\, x \in Z,\, t > 0\}, \\ \mathcal{U} &:= \{U \subset M;\ \text{open subanalytic neighborhood of } Z\}. \end{aligned}$$

Since the image of A in $\dot{T}_N M/\mathbb{R}_{>0}$ is compact, by [2, Proposition 2.1.4] we have for any $k \in \mathbb{Z}$:

$$\begin{aligned} H^k(Z; s^{-1}T\text{-}\nu_N(\mathcal{D}\mathcal{b}_M)) &= H^k(A; T\text{-}\nu_N(\mathcal{D}\mathcal{b}_M)) \\ &= \varinjlim_{U \in \mathcal{U}} H^k(M; \mathcal{T}\!hom(\mathbb{C}_{U\cap\Omega_+}, \mathcal{D}\mathcal{b}_M)). \end{aligned}$$

In particular, $H^k(Z; s^{-1}T\text{-}\nu_N(\mathcal{D}\mathcal{b}_M)) = 0$ holds if $k \neq 0$. On the other hand, if $U, V \in \mathcal{U}$ with $V \Subset U$, then the restriction $\Gamma(U; \mathcal{T}\!hom(\mathbb{C}_{\Omega_+}, \mathcal{D}\mathcal{b}_M)) \to \Gamma(V; \mathcal{T}\!hom(\mathbb{C}_{\Omega_+}, \mathcal{D}\mathcal{b}_M))$ factorizes through

$$\begin{array}{ccc} \Gamma(U; \mathcal{T}\!hom(\mathbb{C}_{\Omega_+}, \mathcal{D}\mathcal{b}_M)) & \longrightarrow & \Gamma(V; \mathcal{T}\!hom(\mathbb{C}_{\Omega_+}, \mathcal{D}\mathcal{b}_M)) \\ \searrow & & \nearrow \\ & \Gamma(M; \mathcal{T}\!hom(\mathbb{C}_{V\cap\Omega_+}, \mathcal{D}\mathcal{b}_M)) & \end{array}$$

Hence

$$\begin{aligned} \varinjlim_{U \in \mathcal{U}} \Gamma(M; \mathcal{T}\!hom(\mathbb{C}_{U\cap\Omega_+}, \mathcal{D}\mathcal{b}_M)) &= \varinjlim_{U \in \mathcal{U}} \Gamma(U; \mathcal{T}\!hom(\mathbb{C}_{\Omega_+}, \mathcal{D}\mathcal{b}_M)) \\ &= \Gamma(Z; \mathcal{T}\!hom(\mathbb{C}_{\Omega_+}, \mathcal{D}\mathcal{b}_M)), \end{aligned}$$

that is, $s^{-1}T\text{-}\nu_N(\mathcal{D}\mathcal{b}_M)$ is identified with the soft sheaf $\mathcal{T}\!hom(\mathbb{C}_{\Omega_+}, \mathcal{D}\mathcal{b}_M)|_N$. Thus by Proposition 1.4, there is a triangle

$${}^t s^{-1}T\text{-}\mu_N(\mathcal{D}\mathcal{b}_M) \otimes \mathcal{or}_{N/M} \to \mathcal{T}\!hom(\mathbb{C}_{\Omega_+}, \mathcal{D}\mathcal{b}_M)|_N \to \Gamma_N(\mathcal{D}\mathcal{b}_M) \otimes \omega_{N/M}^{\otimes -1} \xrightarrow{+1}.$$

We define a morphism $\cdot s_p^{(\alpha)} \colon \mathbb{C}_X \ni C \mapsto C s_p^{(\alpha)} \in (\mathcal{S}_p^{(\alpha)})_{\Omega_+^c}$.

Since $\psi_p^{(\alpha)}(\mathcal{M}) \in \mathfrak{Mod}_{\mathcal{R}_Y}(\mathcal{D}_X)$ for any $\alpha \in G$ and $p \in \mathbb{N}_0$, by Proposition 2.1 we obtain

$$\begin{aligned} &\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, s^{-1}T\text{-}\nu_N(\mathcal{D}\mathcal{b}_M)) = \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}\!hom(\mathbb{C}_{\Omega_+}, \mathcal{D}\mathcal{b}_M))|_N \\ &\xrightarrow{1 \otimes \cdot s_p^{(\alpha)}} (\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}\!hom(\mathbb{C}_{\Omega_+}, \mathcal{D}\mathcal{b}_M)) \overset{\boldsymbol{L}}{\otimes} (\mathcal{S}_p^{(\alpha)})_{\Omega_+^c})|_N \\ &\to (\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}\!hom(\mathbb{C}_{\Omega_+}, \mathcal{D}\mathcal{b}_M)) \overset{\boldsymbol{L}}{\otimes} \mathcal{S}_p^{(\alpha)})|_N \\ &\to (\delta^{-1}\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X \boxtimes \mathcal{D}_X}(\mathcal{M} \boxtimes \psi_p^{(\alpha)}(\mathcal{O}_X), \mathcal{T}\!hom(\mathbb{C}_{\Omega_+}, \mathcal{D}\mathcal{b}_M) \boxtimes \mathcal{O}_X))|_N \end{aligned}$$

$$\begin{aligned}
&\to (\delta^{-1}\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_{X\times X}}(\mathcal{M}\underset{\mathcal{D}}{\boxtimes}\psi_p^{(\alpha)}(\mathcal{O}_X), \mathcal{T}hom(\mathbb{C}_{\Omega_+}, \mathcal{D}b_M)\underset{\mathcal{D}}{\boxtimes}\mathcal{O}_X))|_N\\
&\to \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\boldsymbol{D}\delta^*(\mathcal{M}\underset{\mathcal{D}}{\boxtimes}\psi_p^{(\alpha)}(\mathcal{O}_X)), \boldsymbol{D}\delta^*(\Gamma_N(\mathcal{D}b_M)\underset{\mathcal{D}}{\boxtimes}\mathcal{O}_X))\otimes\omega_{N/M}^{\otimes -1}\\
&\simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\psi_p^{(\alpha)}(\mathcal{M}), \Gamma_N(\mathcal{D}b_M))\otimes\omega_{N/M}^{\otimes -1}\\
&\simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^!\psi_p^{(\alpha)}(\mathcal{M}), \mathcal{D}b_N).
\end{aligned}$$

By (A.5), for any $\alpha \in G$ we define a morphism β_α^t by the following way:

$$\begin{aligned}
\beta_\alpha^t\colon \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, s^{-1}T\text{-}\nu_N(\mathcal{D}b_M)) &\to \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^!\psi_p^{(\alpha)}(\mathcal{M}), \mathcal{D}b_N)\\
&\to \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\overline{\mathsf{gr}}_Y^{-\alpha}(\mathcal{M}), \mathcal{D}b_N).
\end{aligned}$$

Thus by Theorem A.18 we can define:

$$\begin{aligned}
\beta^t := \bigoplus_{\alpha\in G}\beta_\alpha^t\colon &\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, s^{-1}T\text{-}\nu_N(\mathcal{D}b_M))\\
&\to \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{D}b_N).
\end{aligned}$$

By (1.1) we have natural morphisms

$$\mathcal{T}hom(\mathbb{C}_{\Omega_+}, \mathcal{D}b_M) \to \Gamma_{\Omega_+}(\mathcal{D}b_M) \to \Gamma_{\Omega_+}(\mathcal{B}_M).$$

Hence by the construction, we have a commutative diagram:

$$\begin{array}{ccc}
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(\mathbb{C}_{\Omega_+}, \mathcal{D}b_M))|_N & \searrow & \\
\downarrow & & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\Omega_+}(\mathcal{B}_M))|_N\\
(\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(\mathbb{C}_{\Omega_+}, \mathcal{D}b_M))\overset{\boldsymbol{L}}{\otimes}(\mathcal{S}_p^{(\alpha)})_{\Omega_+^c})|_N & & \downarrow\\
\downarrow & \searrow & (\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\Omega_+}(\mathcal{B}_M))\overset{\boldsymbol{L}}{\otimes}(\mathcal{S}_p^{(\alpha)})_{\Omega_+^c})|_N\\
(\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(\mathbb{C}_{\Omega_+}, \mathcal{D}b_M))\overset{\boldsymbol{L}}{\otimes}\mathcal{S}_p^{(\alpha)})|_N & & \|\\
\downarrow & \searrow & (\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\Omega_+}(\mathcal{B}_M))\overset{\boldsymbol{L}}{\otimes}\mathcal{S}_p^{(\alpha)})|_N\\
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^!\psi_p^{(\alpha)}(\mathcal{M}), \mathcal{D}b_N) & & \downarrow\\
& \searrow & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^!\psi_p^{(\alpha)}(\mathcal{M}), \mathcal{B}_N)
\end{array}$$

Therefore, comparing this construction of β^t with that of [23], [24], and [20, Proposition 2.3.1], we see that β^t is compatible with β, and we obtain α^t by Proposition 1.4 and Theorem A.15. Next, by Theorem A.15 we have the following morphism of distinguished triangles:

$$\begin{array}{ccc}
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Phi_Y(\mathcal{M}), \mathcal{D}\ell_N) & \longrightarrow & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Phi_Y(\mathcal{M}), \mathcal{B}_N) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{D}\ell_N) & \longrightarrow & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{B}_N) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^!\mathcal{M}, \mathcal{D}\ell_N) & \longrightarrow & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^!\mathcal{M}, \mathcal{B}_N) \\
\downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1}
\end{array}$$

Taking cohomologies, we obtain the following commutative diagram by Proposition 2.1, Theorem 2.3, and the standard construction of distinguished triangles:

$$\begin{array}{ccc}
\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, {}^ts^{-1}T\text{-}\mu_N(\mathcal{D}\ell_M)) \otimes \boldsymbol{or}_{N/M} & \xrightarrow{H^j(\alpha^t)} & \mathcal{E}xt^j_{\mathcal{D}_Y}(\Phi_Y(\mathcal{M}), \mathcal{D}\ell_N) \\
\downarrow & & \downarrow \\
\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, {}^ts^{-1}\mu_N(\mathcal{B}_M)) \otimes \boldsymbol{or}_{N/M} & \xrightarrow{H^j(\alpha)} & \mathcal{E}xt^j_{\mathcal{D}_Y}(\Phi_Y(\mathcal{M}), \mathcal{B}_N).
\end{array}$$

Therefore α^t is compatible with α. The proof is complete. □

2.6. Remark. Under the regular-singular condition and additional assumptions, boundary values are defined by Oshima [28] and Parenti-Tahara [29] for temperate distribution solutions. These boundary values coincide with those obtained in Theorem 2.5 (see Example 2.7 below).

2.7. *Example*. Let $b(\alpha) \in \mathbb{C}[\alpha]$ be a monic polynomial of degree m, and $P \in \mathsf{F}^m\mathsf{V}_Y^{-1}(\mathcal{D}_X)$. Assume that $b(\alpha) = \prod_{j=1}^{\mu}(\alpha - \alpha_j)^{\nu_j}$ ($\alpha_i - \alpha_j \notin \mathbb{Z}$ for $i \neq j$) with $\sum_{j=1}^{\mu}\nu_j = m$. We set $\mathcal{M} := \mathcal{D}_X/\mathcal{D}_X(b(\vartheta) - P)$. By Example A.14, we see that $\Psi_Y(\mathcal{M}) \simeq \mathcal{D}_Y^m$. We take the following local coordinates:

$$\begin{array}{ccccc} N = \mathbb{R}^n_x \times \{0\} & \hookrightarrow & M = \mathbb{R}^n_x \times \mathbb{R}_t & & \\ \downarrow & & \downarrow & \searrow & \\ Y = \mathbb{C}^n_z \times \{0\} & \hookrightarrow & L = \mathbb{C}^n_z \times \mathbb{R}_t & \hookrightarrow & X = \mathbb{C}^n_z \times \mathbb{C}_\tau\,, \end{array}$$

and set (see [26], [27], [34]):

$$\widetilde{\mathcal{B}}_{N|M} := \mathcal{H}^{n+1}_{T_N M}(\nu_Y(\boldsymbol{R}\Gamma_L(\mathcal{O}_X))) \otimes \mathcal{or}_{N/L}\,.$$

Then there exists a natural monomorphism $\nu_N(\mathcal{B}_{N|M}) \rightarrowtail \widetilde{\mathcal{B}}_{N|M}$. Let us take any $v^* = (x_0; 1\,\frac{d}{dt}) \in T_N M^+$ and $u(x,t) \in \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_N(\mathcal{B}_M))_{v^*}$. Then it is known (see [34], cf. [26]) that $u(x,t)$ can be written as

$$u(x,t) = \sum_{j=1}^{\mu} \sum_{k=1}^{\nu_j} \sum_{i=1}^{J_{ij}} F^i_{jk}(x + \sqrt{-1}\,\Gamma^i_{jk}\,0, t)\, t^{\alpha_j}\, (\log t)^{k-1}, \tag{2.5}$$

as a section of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \widetilde{\mathcal{B}}_{N|M})_{v^*}$. Here each $F^i_{jk}(z,\tau)$ is holomorphic on a neighborhood of $\{(z,0) \in X;\, |x_0 - z| < \varepsilon,\, \operatorname{Im} z \in \Gamma^i_{jk}\}$ with a positive constant ε and an open convex cone $\Gamma^i_{jk} \subset \mathbb{R}^n$. Hence

$$u_{jk}(x) := \sum_{i=1}^{J_{ij}} F^i_{jk}(x + \sqrt{-1}\,\Gamma^i_{jk}\,0, 0) \in \mathcal{B}_{N,x_0}$$

are well defined, and $H^0(\beta)(f)$ is equivalent to

$$\{u_{jk}(x,0);\, 1 \leqslant j \leqslant \mu,\, 1 \leqslant k \leqslant \nu_j\} \subset \mathcal{B}^m_{N,x_0}\,.$$

In particular, if

$$u(x,t) \in \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, T\text{-}\nu_N(\mathcal{D}b_M))_{v^*} \text{ or } u(x,t) \in \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_N(\mathcal{D}b_M))_{v^*},$$

then the expression (2.5) can be compared with that of Parenti-Tahara [29, Theorem 2].

3. Non-Characteristic and Regular System Case

In this section, we consider a non-characteristic boundary value problems under a regular singular condition.

Let $\mathscr{E}_X$ be the *Ring of microdifferential operators* on T^*X. We denote by $\{\mathscr{E}_X(m)\}_{m\in\mathbb{Z}}$ the usual order filtration on $\mathscr{E}_X$ (see Sato-Kawai-Kashiwara [30] and Schapira [31]). Let Λ be a $\mathbb{C}^\times$-conic involutory closed subset of $\dot{T}^*X$, and set $\mathcal{I}_\Lambda := \{P \in \mathscr{E}_X(1); \sigma_1(P)\big|_\Lambda \equiv 0\}$. Let $\mathscr{E}_\Lambda \subset \mathscr{E}_X$ be a sheaf of subring generated by $\mathcal{I}_\Lambda$; that is, $\mathscr{E}_\Lambda := \bigcup_{m\in\mathbb{N}_0} \mathcal{I}_\Lambda^m$ with convention $\mathcal{I}_\Lambda^0 := \mathscr{E}_X(0)$. By the definition, we have $\mathscr{E}_X(0) \subset \mathscr{E}_\Lambda$. Further Kashiwara-Oshima [14] proved that $\mathscr{E}_\Lambda$ is a Noetherian Ring in the sense of [15, Definition 11.1.1] (hence coherent), and that every coherent $\mathscr{E}_X$-Module is pseudocoherent as an $\mathscr{E}_\Lambda$-Module.

3.1. DEFINITION (see [14]). Let Λ be a $\mathbb{C}^\times$-conic involutory closed subset of $\dot{T}^*X$ and $\mathfrak{M} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{E}_X)$ defined on an open set of $\dot{T}^*X$. Then we say that $\mathfrak{M}$ *has regular singularities along* Λ if there exists locally an $\mathscr{E}_\Lambda$ sub-Module $\mathfrak{L} \subset \mathfrak{M}$ such that $\mathfrak{L} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{E}_X(0))$ and that $\mathscr{E}_X\mathfrak{L} = \mathfrak{M}$ holds. Note that If $\mathfrak{M}$ has regular singularities along Λ, then $\operatorname{supp}\mathfrak{M} \subset \Lambda$ ([13, Lemma 1.13]).

3.2. DEFINITION. Let Λ be a $\mathbb{C}^\times$-conic involutory closed subset of T^*X and $\mathcal{M} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$. We set $\dot{\Lambda} := \Lambda \setminus T^*_XX$. Then we say that $\mathcal{M}$ *has regular singularities along* $\dot{\Lambda}$ if the corresponding coherent $\mathscr{E}_X$-Module ${}^{\mathscr{E}}\!\mathcal{M} := \mathscr{E}_X \underset{\pi^{-1}\mathscr{D}_X}{\otimes} \pi^{-1}\mathcal{M}$ has regular singularities along $\dot{\Lambda}$.

The notion of regular singularities is closely related to Levi conditions (see for example [8] and references cited therein).

For a subset $A \subset T^*X$, we denote by $C_{T^*_MX}(A)$ the *normal cone of* A *along* T^*_MX. Then $C_{T^*_MX}(A)$ is a closed cone of $T_{T^*_MX}T^*X$. Take a local coordinate system $(z;\zeta) = (x+\sqrt{-1}y;\xi+\sqrt{-1}\eta)$ in T^*X, where T^*_MX is defined by $\{y=0, \xi=0\}$. We may identify $T_{T^*_MX}T^*X$ with T^*X by the coordinates above, hence the corresponding coordinate system of $T_{T^*_MX}T^*X$ is $(x,\sqrt{-1}y;\xi,\sqrt{-1}\eta)$. By [15, Proposition 4.1.2], we see that $(x_0,\sqrt{-1}y_0;\xi_0,\sqrt{-1}\eta_0) \in C_{T^*_MX}(A)$ if and only if there exist sequences $\{(z_n;\zeta_n)\}_{n=1}^\infty \subset A$ and $\{c_n\}_{n=1}^\infty \subset \mathbb{R}_{>0}$ such that

$$(z_n;\zeta_n) \underset{n}{\to} (x_0,\sqrt{-1}\eta_0), \quad c_n(y_n,\xi_n) \underset{n}{\to} (y_0,\xi_0).$$

There exists a canonical inclusion $T^*M \hookrightarrow T_{T^*_MX}T^*X$ which is described by $(x;\xi)\mapsto(x,0;\xi,0)$ in local coordinates above (see [15]). Then $(x_0;\xi_0)\in T^*M\cap C_{T^*_MX}(A)$ if and only if there exists a sequence $\{(z_n;\zeta_n)\}_{n=1}^{\infty}\subset A$ such that

$$(x_n+\sqrt{-1}y_n;\xi_n)\underset{n}{\longrightarrow}(x_0,0;\xi_0),\quad |y_n|\,|\eta_n|\underset{n}{\longrightarrow}0.$$

Let V be a $\mathbb{C}^{\times}$-conic closed subset of T^*X. Recall that $f_N\colon N\hookrightarrow M$ (or simply N) is said to be *hyperbolic for* V if

$$\dot{T}^*_NM\cap C_{T^*_MX}(V)=\emptyset.$$

3.3. Theorem ([36]). *Let $\Lambda\subset T^*X$ be a closed $\mathbb{C}^{\times}$-conic subset, and $\mathcal{M}\in\mathfrak{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$. Suppose that $\dot{\Lambda}\subset\dot{T}^*X$ is a regular involutory complex submanifold, $\mathcal{M}$ has regular singularities along $\dot{\Lambda}$, and $f_N\colon N\hookrightarrow M$ is hyperbolic for Λ. Then*

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},T\text{-}\mu_N(\mathcal{D}b_M))\big|_{\dot{T}^*_NM}=\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mu_N(\mathcal{D}b_M))\big|_{\dot{T}^*_NM}=0.$$

For the proof, see [36]. Using this theorem, we can prove:

3.4. Theorem. *Under the same assumption as Theorem* 3.3, *there exist the following isomorphisms*:

$$\begin{array}{ccccc}
f_N^{-1}\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}b_M) & & \xrightarrow{\ \sim\ } & & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M},\mathcal{D}b_N)\\
 & \diagdown\!\!\diagdown & & \nearrow\sim & \\
\downarrow\wr & & f_N^{-1}\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}b_M) & & \downarrow\wr\\
 & & \downarrow\wr & & \\
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},s^{-1}\nu_N(\mathcal{D}b_M)) & & & & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}),\mathcal{D}b_N)\\
 & \nwarrow\sim & & \nearrow\sim & \\
\downarrow\wr & & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},s^{-1}T\text{-}\nu_N(\mathcal{D}b_M)) & & \downarrow\wr\\
 & & \downarrow\wr & & \\
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\boldsymbol{R}\Gamma_N(\mathcal{D}b_M))\otimes\omega_{N/M}^{\otimes-1} & & & & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^!\mathcal{M},\mathcal{D}b_N)\\
 & \nwarrow\sim & & \nearrow\sim & \\
 & & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\Gamma_N(\mathcal{D}b_M))\otimes\omega_{N/M}^{\otimes-1} & &
\end{array}$$

PROOF. By (A.2), we have

$$\begin{aligned}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(Df^*\mathscr{M},\mathscr{D}\mathscr{b}_N) &\simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}),\mathscr{D}\mathscr{b}_N)\\ &\simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^!\mathscr{M},\mathscr{D}\mathscr{b}_N).\end{aligned}$$

In view of (1.2), Propositions 1.4 and 2.1, we have by Theorem 3.3:

$$\begin{array}{ccc} f_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}\mathscr{b}_M) & \xrightarrow{\sim} & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\Gamma_N(\mathscr{D}\mathscr{b}_M))\otimes\omega_{N/M}^{\otimes -1} \\ \downarrow\wr & & \downarrow\wr \\ \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(Df^*\mathscr{M},\mathscr{D}\mathscr{b}_N) & \xrightarrow{\sim} & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\boldsymbol{R}\Gamma_N(\mathscr{D}\mathscr{b}_M))\otimes\omega_{N/M}^{\otimes -1}, \end{array}$$

$$\begin{array}{ccc} \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},T\text{-}\nu_N(\mathscr{D}\mathscr{b}_M)) & \xrightarrow{\sim} & \tau_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(Df^*\mathscr{M},\mathscr{D}\mathscr{b}_N) \\ \downarrow\wr & & \downarrow\wr \\ \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\nu_N(\mathscr{D}\mathscr{b}_M)) & \xrightarrow{\sim} & \tau_N^{-1}\boldsymbol{R}\Gamma_N\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}\mathscr{b}_M)\otimes\omega_{N/M}^{\otimes -1}. \end{array}$$

Therefore we can obtain the theorem. □

3.5. REMARK. (1) Under the same assumption as Theorem 3.3, there exist the following commutative diagrams:

$$\begin{array}{ccc} f_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}\mathscr{b}_M) & \longrightarrow & f_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}_M) \\ \downarrow\wr & & \downarrow\wr \\ \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(Df^*\mathscr{M},\mathscr{D}\mathscr{b}_N) & \longrightarrow & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(Df^*\mathscr{M},\mathscr{B}_N), \end{array}$$

$$\begin{array}{ccc} \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},T\text{-}\nu_N(\mathscr{D}\mathscr{b}_M)) & \xrightarrow{\sim} & \tau_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(Df^*\mathscr{M},\mathscr{D}\mathscr{b}_N) \\ \downarrow\wr & & \| \\ \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\nu_N(\mathscr{D}\mathscr{b}_M)) & \xrightarrow{\sim} & \tau_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(Df^*\mathscr{M},\mathscr{D}\mathscr{b}_N) \\ \downarrow & & \downarrow \\ \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\nu_N(\mathscr{B}_M)) & \xrightarrow{\sim} & \tau_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(Df^*\mathscr{M},\mathscr{B}_N). \end{array}$$

(2) Each section of

$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},T\text{-}\nu_N(\mathscr{D}\mathscr{b}_M))\big|_{\dot{T}_NM} = \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\nu_N(\mathscr{D}\mathscr{b}_M))\big|_{\dot{T}_NM}$$

is an extensible mild distribution in the sense of Kataoka [17].

(3) Theorems 3.3 and 3.4 can be proved in the higher-codimensional case (see [36]).

4. Example of Regular-Singular Operator

In this section, we consider a hyperbolic regular-singular operator. Since the problem is local, we use the coordinate system in (2.1). Set for short (each double sign in same order):

$$v_\pm := (0; \pm 1 \frac{d}{dt}) \in \dot{T}_N M, \quad p_\pm := (0; \pm 1\, dt) \in \dot{T}^*_N M,$$
$$q_\pm := (0; \pm\sqrt{-1}\, dt) \in \dot{T}^*_M X.$$

Set $\mathscr{D}\mathscr{b}^A_{N|M} := \mathscr{B}^A_{N|M} \cap \mathscr{D}\mathscr{b}_M|_N$ (recall that $\mathscr{B}^A_{N|M}$ is the subsheaf of $\mathscr{B}_M|_N$ of hyperfunctions with a real analytic parameter t). Let $\mathscr{C}_M$ be the sheaf of *Sato's microfunctions* on $T^*_M X$ as usual, and $\mathscr{C}^f_M$ the subsheaf *of temperate microfunctions* on $T^*_M X$ (see [2] for a functorial construction of this sheaf). Then we have the following exact sequence ([35, Proposition 3.2], cf. [30, Chapter I, Theorem 2.2.6]):

$$\begin{array}{ccccccccc} 0 \to & \mathscr{D}\mathscr{b}^A_{N|M,0} & \to & \mathscr{D}\mathscr{b}_{M,0} & \to & \mathscr{C}^f_{M,q_+} \oplus \mathscr{C}^f_{M,q_-} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathscr{B}^A_{N|M,0} & \to & \mathscr{B}_{M,0} & \to & \mathscr{C}_{M,q_+} \oplus \mathscr{C}_{M,q_-} & \to 0 \end{array} \tag{4.1}$$

We consider the following:

4.1. Condition. $\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}\mathscr{b}^A_{N|M}) \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{D}\mathscr{b}_N)$.

Let $b(\alpha) = \alpha^m + \sum_{j=0}^{m-1} a_j \alpha^j$ be a polynomial of degree m with coefficients in $\mathbb{C}$. Assume that P has the following form:

$$P(z, \tau, \partial_z, \vartheta) = b(\vartheta) + \sum_{\nu=0}^{m} \tau b_\nu(z, \tau)\, \vartheta^\nu + \sum_{\substack{1 \leqslant |\alpha| \\ |\alpha|+\nu \leqslant m}} p_{\alpha\nu}(z, \tau)\, (\tau\partial_z)^\alpha \vartheta^\nu \in \mathscr{D}_{X^{(m)}}.$$

Then P has regular singularities along Y in the sense of Kashiwara-Oshima [14] (see also Oshima [28]) and regular-specializable simultaneously. We denote by $\{\alpha_j\}_{j=1}^m := b^{-1}(0) \subset \mathbb{C}$ the set of *characteristic exponents* of P. For any $\alpha \in \mathbb{C}$, we set:

$$P_\alpha(z,\tau,\partial_z,\vartheta) := \tau^\alpha P(z,\tau,\partial_z,\vartheta)\tau^{-\alpha} = P(z,\tau,\partial_z,\vartheta-\alpha),$$

and consider $\mathcal{M} := \mathcal{D}_X/\mathcal{D}_X P$ and $\mathcal{M}_\alpha := \mathcal{D}_X/\mathcal{D}_X P_\alpha$. In particular, $\mathcal{M} = \mathcal{M}_0$. Then, each $\mathcal{M}_\alpha$ is regular-specializable, and the set of the characteristic exponents of P_α is $\{\alpha_j + \alpha\}_{j=1}^m$. Note that for any $\alpha \in \mathbb{C}$ we have (cf. [11, Lemma 3.8])

$$(4.2)\quad \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, s^{-1}T\text{-}\nu_N(\mathcal{D}b_M)) \simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha, s^{-1}T\text{-}\nu_N(\mathcal{D}b_M)),$$

and that $\Psi_Y(\mathcal{M}_\alpha) \simeq \mathcal{D}_Y^{\oplus m}$ and $\Phi_Y(\mathcal{M}_\alpha) \simeq \mathcal{D}_Y^{\oplus m}$ for any α (Example A.14).

We denote by $\mathcal{C}'_M$ the conic sheaf on $T^*_M X$ associated with the presheaf

$$T^*_M X \supset U \mapsto \Gamma(\pi_M(U);\mathcal{D}b_M)/\{u \in \Gamma(\pi_M(U);\mathcal{D}b_M);\ \mathrm{WF}(u) \cap U = \emptyset\}.$$

Here $\pi_M\colon T^*_M X \to M$ is the canonical projection, and $\mathrm{WF}(u) \subset T^*M \simeq T^*_M X$ denotes the *wave-front set* of a distribution u. Let $\mathcal{D}b^\infty_{N|M}$ be the sheaf on M *of distributions with a* C^∞ *parameter* t. Then we have:

$$(4.3)\qquad \begin{array}{ccccccccc} 0 \to & \mathcal{D}b^A_{N|M,0} & \to & \mathcal{D}b_{M,0} & \to & \mathcal{C}^f_{M,q_+} \oplus \mathcal{C}^f_{M,q_-} & \to 0 \\ & \big\downarrow & & \| & & \downarrow & \\ 0 \to & \mathcal{D}b^\infty_{N|M,0} & \to & \mathcal{D}b_{M,0} & \to & \mathcal{C}'_{M,q_+} \oplus \mathcal{C}'_{M,q_-} & \to 0. \end{array}$$

4.2. PROPOSITION (see [6]). *For any α and $\varepsilon = \pm$, there exist the following isomorphisms*:

$$(4.4)\qquad \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha, \mathcal{C}^f_M)_{q_\varepsilon} \xrightarrow{\sim} \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha, \mathcal{C}'_M)_{q_\varepsilon} \simeq \mathcal{D}b^{\oplus m}_{N,0}.$$

In particular,

$$(4.5)\qquad \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha, \mathcal{D}b^A_{N|M})_0 \xrightarrow{\sim} \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha, \mathcal{D}b^\infty_{N|M})_0\,.$$

PROOF. If we prove (4.4), then we obtain (4.5) by (4.3).

We denote by $\mathscr{C}^d_M$ the conic sheaf on $T^*_M X$ associated with the presheaf

$$T^*_M X \supset U \mapsto \Gamma(\pi_M(U); \mathcal{C}^\infty_M) / \{u \in \Gamma(\pi_M(U); \mathcal{C}^\infty_M);\ \mathrm{SS}(u) \cap U = \emptyset\}.$$

Here $\mathcal{C}^\infty_M$ is the sheaf of C^∞ functions on M, and $\mathrm{SS}(u)$ denotes the *singularity spectrum* of u (see Bony [5]). We have an exact sequence (see [1]):

$$0 \to \mathscr{C}^d_M|_{\dot{T}^*_M X} \to \mathscr{C}^f_M|_{\dot{T}^*_M X} \to \mathscr{C}'_M|_{\dot{T}^*_M X} \to 0.$$

Let $\mathscr{E}^{\mathbb{R},f}_X$ be the Ring *of temperate holomorphic microlocal operators* on T^*X due to Andronikof [2]. Since both $\mathscr{C}^f_M$ and $\mathscr{C}^d_M$ are $\mathscr{E}^{\mathbb{R},f}_X$-Modules ([2], [5], [7]) and conically soft (even conically supple), so is $\mathscr{C}'_M$. Let $\mathfrak{C}_M$ be any of $\mathscr{C}^f_M$, $\mathscr{C}^d_M$ or $\mathscr{C}'_M$, and q either q_+ or q_-. By [3, Corollary 3.3] and the fact that ∂_τ is invertible in $\mathscr{E}^{\mathbb{R},f}_{X,q}$, we have

$$\begin{aligned} \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}_\alpha, \mathfrak{C}_M)_q &\simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{E}^{\mathbb{R},f}_X}(\mathscr{E}^{\mathbb{R},f}_X / \mathscr{E}^{\mathbb{R},f}_X \vartheta, \mathfrak{C}_M)^{\oplus m}_q \\ &\simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{E}^{\mathbb{R},f}_X}(\mathscr{E}^{\mathbb{R},f}_X / \mathscr{E}^{\mathbb{R},f}_X \tau, \mathfrak{C}_M)^{\oplus m}_q \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_X / \mathscr{D}_X \tau, \mathfrak{C}_M)^{\oplus m}_q. \end{aligned}$$

We set $\mathcal{C}^{\infty,A}_{N|M} := \mathscr{B}^A_{N|M} \cap \mathcal{C}^\infty_M|_N$. Then we easily obtain

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & 0 & & \mathscr{D}b_{N,0} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathscr{D}b^A_{N|M,0} & \to & \mathscr{D}b_{M,0} & \to & \mathscr{C}^f_{M,q_+} \oplus \mathscr{C}^f_{M,q_-} & \to & 0 \\
 & & t\downarrow & & t\downarrow & & t\downarrow & & \\
0 & \to & \mathscr{D}b^A_{N|M,0} & \to & \mathscr{D}b_{M,0} & \to & \mathscr{C}^f_{M,q_+} \oplus \mathscr{C}^f_{M,q_-} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathscr{D}b_{N,0} & & 0 & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{C}^{\infty,A}_{N|M,0} & \longrightarrow & \mathcal{C}^{\infty}_{M,0} & \longrightarrow & \mathscr{C}^{d}_{M,q_+} \oplus \mathscr{C}^{d}_{M,q_-} & \longrightarrow & 0 \\
 & & t\downarrow & & t\downarrow & & t\downarrow & & \\
0 & \longrightarrow & \mathcal{C}^{\infty,A}_{N|M,0} & \longrightarrow & \mathcal{C}^{\infty}_{M,0} & \longrightarrow & \mathscr{C}^{d}_{M,q_+} \oplus \mathscr{C}^{d}_{M,q_-} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{C}^{\infty}_{N,0} & & \mathcal{C}^{\infty}_{N,0} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Hence by Snake Lemma, we have

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_X/\mathscr{D}_X\tau, \mathscr{C}^f_M)_q = \mathscr{D}\mathscr{b}_{N,0}, \qquad \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_X/\mathscr{D}_X\tau, \mathscr{C}^d_M)_q = 0.$$

By a distinguished triangle

$$\begin{aligned}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_X/\mathscr{D}_X\tau, \mathscr{C}^d_M)_q &\to \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_X/\mathscr{D}_X\tau, \mathscr{C}^f_M)_q \\ &\to \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_X/\mathscr{D}_X\tau, \mathscr{C}'_M)_q \xrightarrow{+1},\end{aligned}$$

we have

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_X/\mathscr{D}_X\tau, \mathscr{C}^f_M)_q = \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_X/\mathscr{D}_X\tau, \mathscr{C}'_M)_q = \mathscr{D}\mathscr{b}_{N,0}.$$

Therefore we obtain (4.4). □

We consider the following condition on α:

4.3. CONDITION. *$\alpha_j + \alpha \notin \mathbb{Z}$ holds for any $1 \leqslant j \leqslant m$.*

4.4. REMARK. By [25] (see also [35]), there is a morphism

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}_\alpha, \mathscr{B}^A_{N|M}) \to \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}_\alpha, \mathscr{B}_N),$$

such that this induces a monomorphism

$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}_\alpha, \mathscr{B}^A_{N|M}) \to \mathscr{H}om_{\mathscr{D}_Y}(Df^*\mathscr{M}_\alpha, \mathscr{B}_N).$$

Next, assume Condition 4.3. Then it is known that $\boldsymbol{D}f^!\mathscr{M}_\alpha = \boldsymbol{D}f^*\mathscr{M}_\alpha = 0$ (see [21]). Since $\mathscr{D}\mathscr{b}^A_M$ is a subsheaf of $\mathscr{B}^A_{N|M}$, it follows that

$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}_\alpha, \mathscr{D}\mathscr{b}^A_{N|M}) = \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}_\alpha, \mathscr{B}^A_{N|M}) = 0,$$

and by Proposition 2.1 we have:

$$(4.6)\quad \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha, \Gamma_N(\mathcal{B}_M)) \otimes \omega_{N/M}^{\otimes -1} = \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^!\mathcal{M}_\alpha, \mathcal{B}_N) = 0,$$
$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha, \Gamma_N(\mathcal{D}b_M)) \otimes \omega_{N/M}^{\otimes -1} = \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^!\mathcal{M}_\alpha, \mathcal{D}b_N) = 0.$$

Further we can write Condition 4.1 as

$$(4.7)\qquad \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha, \mathcal{D}b^A_{N|M}) = 0.$$

4.5. Proposition. *Under Condition* 4.3 *and assumption* (4.7), *there exists the following isomorphisms*:

$$\begin{array}{ccc} \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha, {}^t s^{-1}\, T\text{-}\mu_N(\mathcal{D}b_M)) \otimes \mathcal{o}r_{N/M} & \xrightarrow{\sim} & \mathcal{D}b_N^{\oplus m} \\ \downarrow\wr & & \downarrow\wr \\ \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha, s^{-1}\, T\text{-}\nu_N(\mathcal{D}b_M)) & \xrightarrow{\sim} & \mathcal{D}b_N^{\oplus m}. \end{array}$$

Proof. Since problem is local, we may identify $\mathcal{o}r_{N/M}$ with $\mathbb{Z}_N$, and prove in the stalk at the origin. We have morphisms for $\varepsilon = \pm$:

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha, T\text{-}\mu_N(\mathcal{D}b_M))_{p_\varepsilon} \to \mathcal{D}b_{N,0}^{\oplus m},$$
$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha, T\text{-}\nu_N(\mathcal{D}b_M))_{v_\varepsilon} \to \mathcal{D}b_{N,0}^{\oplus m}.$$

Therefore by Theorem 2.5, we have only to show

$$(4.8)\qquad \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha, T\text{-}\mu_N(\mathcal{D}b_M))_{p_+} \simeq \mathcal{D}b_{N,0}^{\oplus m}.$$

Since $0 \to \mathcal{D}_X \xrightarrow{\cdot P_\alpha} \mathcal{D}_X \to \mathcal{M}_\alpha \to 0$ is exact, for any $j \geqslant 2$ we have

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}_\alpha, T\text{-}\mu_N(\mathcal{D}b_M))_{p_+} = \mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}_\alpha, T\text{-}\mu_N(\mathcal{D}b_M))_{p_-} = 0.$$

By Proposition 4.5, (4.1) and (4.7), we obtain:

$$(4.9)\qquad \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha, \mathcal{D}b_M)_0 \simeq \mathcal{D}b_{N,0}^{\oplus m} \oplus \mathcal{D}b_{N,0}^{\oplus m}.$$

Hence by (1.2), (4.6) and (4.9), we have

$$\mathscr{D}\mathscr{b}_{N,0}^{\oplus m} \oplus \mathscr{D}\mathscr{b}_{N,0}^{\oplus m} \xleftarrow{\sim} \bigoplus_{\varepsilon=\pm} \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}_\alpha, T\text{-}\nu_N(\mathscr{D}\mathscr{b}_M))_{p_\varepsilon} \\ \to \mathscr{D}\mathscr{b}_{N,0}^{\oplus m} \oplus \mathscr{D}\mathscr{b}_{N,0}^{\oplus m}$$

Considering the correspondence of each morphism, we obtain (4.8). □

4.6. THEOREM. *Assume that there exists an* $\alpha \in \mathbb{C}$ *which satisfies Condition* 4.3 *and assumption* (4.7). *Then there exists the following isomorphism of distinguished triangles*:

$$(4.10)\quad \begin{array}{ccc} \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, {}^t s^{-1} T\text{-}\mu_N(\mathscr{D}\mathscr{b}_M)) \otimes \boldsymbol{or}_{N/M} & \xrightarrow{\sim} & \mathscr{D}\mathscr{b}_N^{\oplus m} \\ \downarrow & & \downarrow \\ \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, s^{-1} T\text{-}\nu_N(\mathscr{D}\mathscr{b}_M)) & \xrightarrow{\sim} & \mathscr{D}\mathscr{b}_N^{\oplus m} \\ \downarrow & & \downarrow \\ \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_N(\mathscr{D}\mathscr{b}_M)) \otimes \omega_{N/M}^{\otimes -1} & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^!\mathscr{M}, \mathscr{D}\mathscr{b}_N) \\ \downarrow{+1} & & \downarrow{+1} \end{array}$$

PROOF. By (4.2) and Proposition 4.5, we have

$$\begin{aligned} \mathscr{D}\mathscr{b}_{N,0}^{\oplus m} &\simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}_\alpha, s^{-1} T\text{-}\nu_N(\mathscr{D}\mathscr{b}_M)) \\ &\simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, s^{-1} T\text{-}\nu_N(\mathscr{D}\mathscr{b}_M)) \\ &\to \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{D}\mathscr{b}_N) \simeq \mathscr{D}\mathscr{b}_{N,0}^{\oplus m}. \end{aligned}$$

Hence we can prove

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{D}\mathscr{b}_N) \xrightarrow{\sim} \mathscr{D}\mathscr{b}_{N,0}^{\oplus m}.$$

Then Theorem 2.5 completes the proof. □

By definition we can write

$$\sigma_m(P)(z, \tau; \zeta, \lambda) = \tau^m p_m(z, \tau; \zeta, \lambda),$$

where $p_m(z,\tau;\zeta,\lambda)$ is a polynomial of degree m with respect to (ζ,λ). Here we remark that $p_m(z,\tau;0,\lambda)=\lambda^m$, and for any $\alpha\in\mathbb{C}$,

$$\sigma_m(P_\alpha)(z,\tau;\zeta,\lambda)=\sigma_m(P)(z,\tau;\zeta,\lambda)=\tau^m p_m(z,\tau;\zeta,\lambda).$$

4.7. Condition. *$p_m(z,\tau;\zeta,\lambda)$ is strictly hyperbolic with respect dt; that is, for any $(x,t,\xi)\in M\times(\mathbb{R}^n\setminus\{0\})$, the polynomial (with respect to λ) $\mathbb{C}\ni\lambda\mapsto p_m(x,t;\xi,\lambda)$ has roots $\lambda_j(x,t;\xi)$ $(1\leqslant j\leqslant m)$ such that*

(i) *$\{\lambda_j(x,t;\xi)\}_{j=1}^m$ are real and distinct;*

(ii) *each $\lambda_j(x,t;\xi)$ homogeneous with respect to ξ of degree one;*

(iii) *if $|\xi|=1$, then each $\lambda_j(x,t;\xi)$ is bounded.*

Note that Condition 4.7 entails the near-hyperbolicity of each $\mathcal{M}_\alpha$. Then we recall:

4.8. Theorem ([6]). *Under Conditions* 4.3 *and* 4.7, *it follows that*

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha,\mathcal{D}b^{\infty}_{N|M})_0\simeq 0.$$

In particular by (4.5),

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha,\mathcal{D}b^{A}_{N|M})_0\simeq 0.$$

Hence if we assume Condition 4.7, then there exists an $\alpha\in\mathbb{C}$ which satisfies Conditions 4.1 and 4.3. Therefore, we obtain

4.9. Theorem. *Under Condition* 4.7, *the isomorphism* (4.10) *holds.*

4.10. Remark. Under Conditions 4.3 and 4.7, Parenti-Tahara [29, Theorem 1] proved:

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha,s^{-1}T\text{-}\nu_N(\mathcal{D}b_M))\xrightarrow{\sim}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha,s^{-1}\nu_N(\mathcal{D}b_M)).$$

Appendix A. Specializable Systems and Nearby, Vanishing Cycles

We recall the definitions of (regular-) specializable system and its vanishing and nearby cycle Modules. The contents in this Appendix are known to specialists, but we cannot find suitable references for the purpose of this paper. Therefore, we review the definitions and properties in some detail (cf. [9], [19], [22], [31] for the proof of corresponding contents). Since the problem is local, we fix the coordinates of (2.1). Set $\vartheta := \tau\partial_\tau$. Let $\{\mathscr{D}_{X}(m)\}_{m\in\mathbb{N}_0}$ the usual *order filtration.*

A.1. DEFINITION. Let $\mathcal{I}_Y$ be the defining Ideal of Y in $\mathscr{O}_X$ with a convention that $\mathcal{I}_Y^j = \mathscr{O}_X$ for $j \leqslant 0$. The *V-filtration* $\{\mathsf{V}_Y^k(\mathscr{D}_X)\}_{k\in\mathbb{Z}}$ *(along* Y*)* is a filtration on $\mathscr{D}_X|_Y$ defined by

$$\mathsf{V}_Y^k(\mathscr{D}_X) := \bigcap_{j\in\mathbb{Z}}\{P \in \mathscr{D}_X|_Y;\, P\mathcal{I}_Y^j \subset \mathcal{I}_Y^{j-k}\}.$$

This filtration is given by $\mathsf{V}_Y^k(\mathscr{D}_X) = \{\sum_{j-i\leqslant k} P_{ij}(z,\tau,\partial_z)\,\tau^i\partial_\tau^j \in \mathscr{D}_X\big|_Y\}$.

In what follows we omit the phrase "along Y" since Y is fixed.

A.2. DEFINITION. Let $\mathscr{M} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$.

(1) A V-filtration $\mathsf{V}(\mathscr{M})$ on $\mathscr{M}$ is a family $\{\mathsf{V}^\mu(\mathscr{M})\}_{\mu\in\mathbb{Z}}$ of sub-Groups such that

(i) if $\nu \in \mathbb{N}_0$, then $\mathsf{V}^\mu(\mathscr{M}) \subset \mathsf{V}^{\mu+\nu}(\mathscr{M})$;

(ii) $\bigcup_\mu \mathsf{V}^\mu(\mathscr{M}) = \mathscr{M}$;

(iii) If μ, $\nu \in \mathbb{Z}$, then $\mathsf{V}_Y^\mu(\mathscr{D}_X)\,\mathsf{V}^\nu(\mathscr{M}) \subset \mathsf{V}^{\mu+\nu}(\mathscr{M})$ holds.

(2) A V-filtration $\mathsf{V}(\mathscr{M})$ is said to be *good* if (locally) there exist $m \in \mathbb{N}$, $\{u_j\}_{j=1}^J \subset \mathscr{M}$, $\{\mu_j\}_{j=1}^J \in \mathbb{Z}$ such that for any μ, the following holds:

$$\mathsf{V}^\mu(\mathscr{M}) = \sum_{j=1}^J \mathsf{V}_Y^{\mu-\mu_j}(\mathscr{D}_X)u_j\,.$$

A.3. Proposition. *For any* $\mathcal{M} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_X|_Y)$, *the following conditions are equivalent*:

(1) *there exist* (*locally*) *a good* V*-filtration* $\mathsf{V}_1(\mathcal{M})$ *and a non-zero polynomial* $b(s) \in \mathbb{C}[s]$ *such that for any* $k \in \mathbb{Z}$

$$b(\vartheta + k)\,\mathsf{V}_1^k(\mathcal{M}) \subset \mathsf{V}_1^{k-1}(\mathcal{M}).$$

(2) *for any good* V*-filtration* $\mathsf{V}(\mathcal{M})$, *there exists* (*locally*) *a non-zero polynomial* $b'(s) \in \mathbb{C}[s]$ *such that for any* $k \in \mathbb{Z}$

$$b'(\vartheta + k)\,\mathsf{V}^k(\mathcal{M}) \subset \mathsf{V}^{k-1}(\mathcal{M}).$$

(3) *for any system of* (*local*) *generators* $\{u_j\}_{j=1}^J$ *of* $\mathcal{M}$, *there exists* (*locally*) *a non-zero polynomial* $b(s) \in \mathbb{C}[s]$ *such that for any* $1 \leqslant j \leqslant J$

$$b(\vartheta)u_j \subset \sum_{i=1}^{J} \mathsf{V}_Y^{-1}(\mathscr{D}_X)\,u_i\,.$$

A.4. Definition. $\mathcal{M} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_X|_Y)$ is said to be *specializable* if $\mathcal{M}$ satisfies equivalent conditions of Proposition A.3. We denote by $\mathfrak{Mod}_{\mathcal{B}_Y}(\mathscr{D}_X) \subset \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_X|_Y)$ the subcategory consisting of specializable $\mathscr{D}_X|_Y$-Modules.

Next, we set $\mathsf{F}^p\mathsf{V}_Y^\mu(\mathscr{D}_X) := \mathscr{D}_{X^{(p)}} \cap \mathsf{V}_Y^\mu(\mathscr{D}_X)$, and call

$$\mathsf{FV}_Y(\mathscr{D}_X) := \{\mathsf{F}^p\mathsf{V}_Y^\mu(\mathscr{D}_X)\}_{p\in\mathbb{N}_0,\mu\in\mathbb{Z}}$$

the *bi-filtration.* This enjoys the following properties:

(i) $\bigcup_{p,\mu} \mathsf{F}^p\mathsf{V}_Y^\mu(\mathscr{D}_X) = \mathscr{D}_X$;

(ii) if q, $\nu \in \mathbb{N}_0$, then $\mathsf{F}^p\mathsf{V}_Y^\mu(\mathscr{D}_X) \subset \mathsf{F}^{p+q}\mathsf{V}_Y^{\mu+\nu}(\mathscr{D}_X)$;

(iii) if p, q, μ, $\nu \in \mathbb{Z}$, then $\mathsf{F}^p\mathsf{V}_Y^\mu(\mathscr{D}_X)\,\mathsf{F}^q\mathsf{V}_Y^\nu(\mathscr{D}_X) \subset \mathsf{F}^{p+q}\mathsf{V}_Y^{\mu+\nu}(\mathscr{D}_X)$.

A.5. Definition. Let $\mathcal{M} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_X|_Y)$.

(1) A bi-filtration $\mathsf{FV}(\mathcal{M})$ on $\mathcal{M}$ is a family $\{\mathsf{F}^p\mathsf{V}^\mu(\mathcal{M})\}_{p,\mu\in\mathbb{Z}}$ of sub-Groups such that

(i) if $q,\ \nu \in \mathbb{N}_0$, then $\mathsf{F}^p\mathsf{V}^\mu(\mathcal{M}) \subset \mathsf{F}^{p+q}\mathsf{V}^{\mu+\nu}(\mathcal{M})$;

(ii) $\bigcup\limits_{p,\mu} \mathsf{F}^p\mathsf{V}^\mu(\mathcal{M}) = \mathcal{M}$;

(iii) if $p,\ q,\ \mu,\ \nu \in \mathbb{Z}$, then $\mathsf{F}^p\mathsf{V}^\mu_Y(\mathcal{D}_X)\,\mathsf{F}^q\mathsf{V}^\nu(\mathcal{M}) \subset \mathsf{F}^{p+q}\mathsf{V}^{\mu+\nu}(\mathcal{M})$ holds.

(2) A bi-filtration $\mathsf{FV}(\mathcal{M})$ is said to be *good* if locally there exist $J \in \mathbb{N}$, $\{u_j\}_{j=1}^J \subset \mathcal{M}$, $\{p_j\}_{j=1}^J$, $\{\mu_j\}_{j=1}^J \subset \mathbb{Z}$ such that for any p, μ, the following holds:

$$\mathsf{F}^p\mathsf{V}^\mu(\mathcal{M}) = \sum_{j=1}^J \mathsf{F}^{p-p_j}\mathsf{V}_Y^{\mu-\mu_j}(\mathcal{D}_X)u_j\,.$$

A.6. Proposition. *For any $\mathcal{M} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathcal{D}_X|_Y)$, the following conditions are equivalent*:

(1) *there exist (locally) a good bi-filtration* $\mathsf{FV}(\mathcal{M})$ *and non-zero polynomial $b(s) \in \mathbb{C}[s]$ of degree m such that for any $k,\ l \in \mathbb{Z}$*

$$b(\vartheta + l)\,\mathsf{F}^k\mathsf{V}^l(\mathcal{M}) \subset \mathsf{F}^{k+m}\mathsf{V}^{l-1}(\mathcal{M}).$$

(2) *for any good bi-filtration* $\mathsf{FV}(\mathcal{M})$*, there exist (locally) a non-zero polynomial $b(s) \in \mathbb{C}[s]$ of degree m such that for any $k,\ l \in \mathbb{Z}$*

$$b(\vartheta + l)\,\mathsf{F}^k\mathsf{V}^l(\mathcal{M}) \subset \mathsf{F}^{k+m}\mathsf{V}^{l-1}(\mathcal{M}).$$

(3) *for any system of (local) generators $\{u_j\}_{j=1}^J$ of $\mathcal{M}$, there exists (locally) a non-zero polynomial $b(s) \in \mathbb{C}[s]$ of degree m such that for any $1 \leqslant j \leqslant J$*

$$b(\vartheta)u_j \subset \sum_{i=1}^J \mathsf{F}^m\mathsf{V}_Y^{-1}(\mathcal{D}_X)\,u_i\,.$$

A.7. Remark. The author cannot find literature of the proof of Proposition A.6, and Professor T. Oaku kindly informed the author of the proof.

A.8. Definition. $\mathcal{M} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathcal{D}_X|_Y)$ is said to be *regular-specializable* if $\mathcal{M}$ satisfies the equivalent conditions of Proposition A.6. We denote by $\mathfrak{Mod}_{\mathcal{R}_Y}(\mathscr{D}_X) \subset \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_X|_Y)$ the subcategory consisting of regular-specializable $\mathscr{D}_X|_Y$-Modules. By definition, $\mathfrak{Mod}_{\mathcal{R}_Y}(\mathscr{D}_X) \subset \mathfrak{Mod}_{\mathcal{B}_Y}(\mathscr{D}_X)$.

A.9. Remark. (1) Let $\mathscr{M} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$, and assume that Y is non-characteristic for $\mathscr{M}$. Then $\mathscr{M} \in \mathfrak{Mod}_{\mathcal{R}_Y}(\mathscr{D}_X)$.

(2) It is known that every holonomic $\mathscr{D}_X|_Y$-Module is specializable, and moreover every regular-holonomic $\mathscr{D}_X|_Y$-Module is regular-specializable.

A.10. Proposition. *Let* $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ *be an exact sequence in* $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_X|_Y)$. *Then*

(1) $\mathscr{M} \in \mathfrak{Mod}_{\mathcal{B}_Y}(\mathscr{D}_X)$ *if and only if* $\mathscr{M}', \mathscr{M}'' \in \mathfrak{Mod}_{\mathcal{B}_Y}(\mathscr{D}_X)$.

(2) $\mathscr{M} \in \mathfrak{Mod}_{\mathcal{R}_Y}(\mathscr{D}_X)$ *if and only if* $\mathscr{M}', \mathscr{M}'' \in \mathfrak{Mod}_{\mathcal{R}_Y}(\mathscr{D}_X)$.

Let $\prec$ be the lexicographical order on $\mathbb{C} = \mathbb{R} + \sqrt{-1}\,\mathbb{R}$. We set

$$G := \{\alpha \in \mathbb{C};\, 0 \preccurlyeq \alpha \prec 1\}. \tag{A.1}$$

A.11. Proposition. *For any* $\mathscr{M} \in \mathfrak{Mod}_{\mathcal{B}_Y}(\mathscr{D}_X)$, *there exist a unique good* V*-filtration* $\mathsf{V}_Y(\mathscr{M}) = \{\mathsf{V}_Y^k(\mathscr{M})\}_{k\in\mathbb{Z}}$ *and a non-zero polynomial* $b(s) \in \mathbb{C}[s]$ *such that* $b^{-1}(0) \subset G$ *and for any* $k \in \mathbb{Z}$

$$b(\vartheta + k)\,\mathsf{V}_Y^k(\mathscr{M}) \subset \mathsf{V}_Y^{k-1}(\mathscr{M}).$$

Moreover, for any $k \in \mathbb{N}_0$,

$$\mathsf{V}_Y^{k+1}(\mathscr{M}) = \mathsf{V}^k(\mathscr{D}_X)\mathsf{V}_Y^1(\mathscr{M}), \quad \mathsf{V}_Y^{-k}(\mathscr{M}) = t^k\,\mathsf{V}_Y^0(\mathscr{M}).$$

In particular, $\mathscr{M} = \mathscr{D}_X\mathsf{V}_Y^1(\mathscr{M})$ *holds.*

A.12. Definition. For any $\mathscr{M} \in \mathfrak{Mod}_{\mathcal{B}_Y}(\mathscr{D}_X)$, we denote by $\mathrm{Gr}_Y(\mathscr{M}) = \bigoplus_{j\in\mathbb{Z}} \mathrm{Gr}_Y^j(\mathscr{M})$ the associated graded Module with $\mathsf{V}_Y(\mathscr{M})$ in Proposition A.11 (i.e. $\mathrm{Gr}_Y^j(\mathscr{M}) := \mathsf{V}_Y^j(\mathscr{M})/\mathsf{V}_Y^{j-1}(\mathscr{M})$). Then the *vanishing cycle* $\Phi_Y(\mathscr{M})$ and *nearby cycle* $\Psi_Y(\mathscr{M})$ are defined respectively by

$$\Phi_Y(\mathscr{M}) := \mathrm{Gr}_Y^1(\mathscr{M}), \quad \Psi_Y(\mathscr{M}) := \mathrm{Gr}_Y^0(\mathscr{M}).$$

It is known that $\mathrm{Gr}_Y^k(\mathscr{M}) \in \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_Y)$ for any $k \in \mathbb{Z}$, and for any $k \in \mathbb{N}_0$

$$\mathrm{Gr}_Y^{k+1}(\mathscr{M}) = \partial_t^k \varPhi_Y(\mathscr{M}), \quad \mathrm{Gr}_Y^{-k}(\mathscr{M}) = t^k \varPsi_Y(\mathscr{M}).$$

A.13. REMARK. (1) There are several conventions about the definitions of vanishing and nearby cycles, and this fact causes some confusion. For the relation between our definition (the choice of G in (A.1), Definition A.12 and Theorem A.18 below) and another definition, see [9, § 5.3].

(2) Laurent [18] extended the definitions of nearby and vanishing cycles by using the theory of second microlocalization.

A.14. *Example*. Let $b(s) \in \mathbb{C}[s]$ be a polynomial of degree $m \in \mathbb{N}$, and $P \in \mathrm{Mat}_J(\mathsf{V}_Y^{-1}(\mathscr{D}_X))$. Set $\mathscr{M} := \mathscr{D}_X^J/\mathscr{D}_X^J(b(\vartheta) - P)$. Then $\mathscr{M} \in \mathfrak{Mod}_{\mathcal{B}_Y}(\mathscr{D}_X)$, and as $\mathscr{D}_Y$-Modules

$$\varPhi_Y(\mathscr{M}) \simeq \mathscr{D}_X^{mJ}, \qquad \varPsi_Y(\mathscr{M}) \simeq \mathscr{D}_X^{mJ}.$$

We shall give a sketch of proof.

(1) Set $\mathscr{L} := \mathscr{D}_X/\mathscr{D}_X(\vartheta - \alpha)^\nu$. Then a direct calculation gives

$$\varPhi_Y(\mathscr{L}) \simeq \mathscr{D}_Y^\nu, \qquad \varPsi_Y(\mathscr{L}) \simeq \mathscr{D}_Y^\nu.$$

(2) Set $\mathscr{M}_0 := \mathscr{D}_X/\mathscr{D}_X b(\vartheta)$. We write $b(s) = \prod_{j=1}^{m'} (s - \alpha_j)^{\nu_j}$ ($\alpha_i \neq \alpha_j$ if $i \neq j$), and set $\mathscr{M}^{(j)} := \mathscr{D}_X/\mathscr{D}_X(\vartheta - \alpha_j)^{\nu_j}$. Then by using Chinese remainder theorem, we can show

$$\mathscr{M}_0 \simeq \bigoplus_{j=1}^{m'} \mathscr{M}^{(j)}, \qquad \mathsf{V}_Y^k(\mathscr{M}_0) \simeq \bigoplus_{j=1}^{m'} \mathsf{V}_Y^k(\mathscr{M}^{(j)}),$$

and hence for any $k \in \mathbb{Z}$

$$\mathrm{Gr}_Y^k(\mathscr{M}_0) \simeq \bigoplus_{j=1}^{m'} \mathrm{Gr}_Y^k(\mathscr{M}^{(j)}).$$

(3) By the proof of Proposition A.11, for any $k \in \mathbb{Z}$ we can show

$$\mathrm{Gr}_Y^k(\mathscr{M}) = \mathrm{Gr}_Y^k((\mathscr{M}_0)^J) = \mathrm{Gr}_Y^k(\mathscr{M}_0)^J.$$

A.15. THEOREM. *For any $\mathcal{M} \in \mathfrak{Mod}_{\mathcal{B}_Y}(\mathcal{D}_X)$, the following hold*:
(1) *There exist the following distinguished triangles*:

$$\Phi_Y(\mathcal{M}) \xrightarrow{t} \Psi_Y(\mathcal{M}) \to \boldsymbol{D}f^*\mathcal{M} \xrightarrow{+1},$$
$$\boldsymbol{D}f^!\mathcal{M} \to \Psi_Y(\mathcal{M}) \xrightarrow{\partial_t} \Phi_Y(\mathcal{M}) \xrightarrow{+1},$$

and $\boldsymbol{D}f^*\mathcal{M}$, $\boldsymbol{D}f^!\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_Y)$.

(2) $\Gamma_{[X\smallsetminus Y]}(\mathcal{M})$, $H^j_{[Y]}(\mathcal{M}) \in \mathfrak{Mod}_{\mathcal{B}_Y}(\mathcal{D}_X)$ *for any* $j \in \mathbb{N}_0$. *In addition if* $\mathcal{M} \in \mathfrak{Mod}_{\mathcal{R}_Y}(\mathcal{D}_X)$, *then* $\Gamma_{[X\smallsetminus Y]}(\mathcal{M})$, $H^j_{[Y]}(\mathcal{M}) \in \mathfrak{Mod}_{\mathcal{R}_Y}(\mathcal{D}_X)$ *for any* $j \in \mathbb{N}_0$ (*here note that* $\boldsymbol{R}\Gamma_{[X\smallsetminus Y]}(\mathcal{M}) = \Gamma_{[X\smallsetminus Y]}(\mathcal{M})$).

(3) $\Psi_Y(\mathcal{M}) = \Psi_Y(\Gamma_{[X\smallsetminus Y]}(\mathcal{M}))$ *and* $\Psi_Y(\Gamma_{[Y]}(\mathcal{M})) = 0$.

A.16. PROPOSITION. *Let $\mathcal{M} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathcal{D}_X|_Y)$, and assume that Y is non-characteristic for $\mathcal{M}$. Set $Df^*\mathcal{M} := \mathcal{H}^0\boldsymbol{D}f^*\mathcal{M}$. Then $\Phi_Y(\mathcal{M}) = 0$ and*

$$\boldsymbol{D}f^!\mathcal{M} \simeq \Psi_Y(\mathcal{M}) \simeq \boldsymbol{D}f^*\mathcal{M} \simeq Df^*\mathcal{M}. \tag{A.2}$$

A.17. THEOREM. *For any $\mathcal{M} \in \mathfrak{Mod}_{\mathcal{R}_Y}(\mathcal{D}_X)$, there exist the following isomorphisms*:

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_{X\leftarrow Y})[1] \simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^!\mathcal{M}, \mathcal{D}_Y), \tag{A.3}$$
$$\boldsymbol{R}\Gamma_Y\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)[2] \simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^!\mathcal{M}, \mathcal{O}_Y). \tag{A.4}$$

For any $k \in \mathbb{Z}$ and $\alpha \in G$, we set

$$\overline{\mathsf{gr}}^{k-\alpha}_Y(\mathcal{M}) := \bigcup_{p\in\mathbb{N}} \mathrm{Ker}(\mathrm{Gr}^k_Y(\mathcal{M}) \xrightarrow{(\vartheta+k-\alpha)^p} \mathrm{Gr}^k_Y(\mathcal{M})).$$

Here we remark that each $\beta \in \mathbb{C}$ can be written uniquely as $\beta = k - \alpha$ with $k \in \mathbb{Z}$ and $\alpha \in G$.

A.18. THEOREM. *Let $\mathcal{M} \in \mathfrak{Mod}_{\mathcal{B}_Y}(\mathcal{D}_X)$.*

(1) *There exists a finite subset* $A \subset G$ *such that if* $\alpha \in A$, *then there exists locally* $p \in \mathbb{N}$ *such that for any* $k \in \mathbb{Z}$

$$\overline{\mathrm{gr}}_Y^{k-\alpha}(\mathcal{M}) = \mathrm{Ker}(\mathrm{Gr}_Y^k(\mathcal{M}) \xrightarrow{(\vartheta+k-\alpha)^p} \mathrm{Gr}_Y^k(\mathcal{M})),$$

and if $\alpha \in G \smallsetminus A$, *then* $\overline{\mathrm{gr}}_Y^{k-\alpha}(\mathcal{M}) = 0$ *for any* $k \in \mathbb{Z}$.

(2) *For any* $k \in \mathbb{Z}$, *there exists the following finite direct decomposition*:

$$\mathrm{Gr}_Y^k(\mathcal{M}) = \bigoplus_{\alpha \in G} \overline{\mathrm{gr}}_Y^{k-\alpha}(\mathcal{M}).$$

In particular,

$$\varPhi_Y(\mathcal{M}) = \bigoplus_{\alpha \in G} \overline{\mathrm{gr}}_Y^{1-\alpha}(\mathcal{M}), \qquad \varPsi_Y(\mathcal{M}) = \bigoplus_{\alpha \in G} \overline{\mathrm{gr}}_Y^{-\alpha}(\mathcal{M}).$$

Let $\mathcal{M} \in \mathfrak{Mod}_{\mathcal{B}_Y}(\mathcal{D}_X)$. For any $\alpha \in G$ and p, $j \in \mathbb{N}_0$, we set

$$e_j^{(\alpha)} := \frac{t^{-\alpha}(\log t)^j}{j!}, \quad \psi_p^{(\alpha)}(\mathcal{O}_X) := \sum_{k=0}^{p} \varGamma_{[X \smallsetminus Y]}(\mathcal{O}_X)\, e_j^{(\alpha)},$$
$$\psi_p^{(\alpha)}(\mathcal{M}) := \mathcal{M} \underset{\mathcal{O}_X}{\otimes} \psi_p^{(\alpha)}(\mathcal{O}_X) = \sum_{k=0}^{p} \varGamma_{[X \smallsetminus Y]}(\mathcal{M})\, e_j^{(\alpha)}.$$

It is known that each $\psi_p^{(\alpha)}(\mathcal{M}) \in \mathfrak{Mod}_{\mathcal{B}_Y}(\mathcal{D}_X)$ for any α and p, and if $\mathcal{M} \in \mathfrak{Mod}_{\mathcal{R}_Y}(\mathcal{D}_X)$, so is each $\psi_p^{(\alpha)}(\mathcal{M})$. In particular, $\psi_p^{(\alpha)}(\mathcal{O}_X) \in \mathfrak{Mod}_{\mathcal{R}_Y}(\mathcal{D}_X)$. By a natural inclusion $\psi_p^{(\alpha)}(\mathcal{O}_X) \hookrightarrow \psi_q^{(\alpha)}(\mathcal{O}_X)$ for any $p \leqslant q$, $\{\psi_p^{(\alpha)}(\mathcal{O}_X)\}_{p \in \mathbb{N}_0}$ has a structure of an inductive system. Moreover for any $\alpha \in G$, the system $\{\psi_p^{(\alpha)}(\mathcal{O}_X)\}_{p \in \mathbb{N}_0}$ induces an inductive system $\{\boldsymbol{D}f^!\psi_p^{(\alpha)}(\mathcal{M})\}_{p \in \mathbb{N}_0}$. We set $D^j f^! \psi_p^{(\alpha)}(\mathcal{M}) := \mathcal{H}^j \boldsymbol{D} f^! \psi_p^{(\alpha)}(\mathcal{M})$.

A.19. PROPOSITION. *Let* $\mathcal{M} \in \mathfrak{Mod}_{\mathcal{B}_Y}(\mathcal{D}_X)$, *and take any* $Z \Subset Y$. *Then there exists a* $p_0 \in \mathbb{N}$ *such that the following hold*:

(1) $\varGamma(Z; \overline{\mathrm{gr}}_Y^{-\alpha}(\mathcal{M})) \simeq \varGamma(Z; D^0 f^! \psi_p^{(\alpha)}(\mathcal{M}))$ *holds for any* $p \geqslant p_0$,

(2) $\varGamma(Z; D^1 f^! \psi_p^{(\alpha)}(\mathcal{M})) \to \varGamma(Z; D^1 f^! \psi_{p+p_0}^{(\alpha)}(\mathcal{M}))$ *is the zero morphism for any* $p \in \mathbb{N}_0$.

By Proposition A.19, for any $Z \Subset Y$, there exists the following natural morphism on Z if p is large enough:

$$\overline{\mathsf{gr}}_Y^{-\alpha}(\mathscr{M}) \to \boldsymbol{D} f^! \psi_p^{(\alpha)}(\mathscr{M}), \tag{A.5}$$

which is compatible with the inductive system structure.

References

[1] Andronikof, E., On the $\mathscr{C}^\infty$-singularities of regular holonomic distributions, Ann. Inst. Fourier (Grenoble) **42** (1992), 695–705.

[2] Andronikof, E., *Microlocalisation Tempérée*, Mém. Soc. Math. Fr. (N.S.) **57**, 1994.

[3] Andronikof, E. and T. Monteiro Fernandes, On the tempered solutions of regular systems, *D-Modules and Microlocal Geometry*, Proceedings Lisbon 1990, Walter de Gruyter, Berlin–New York, 1992, pp. 107–124.

[4] Baouendi, M. S. and C. Goulaouic, Cauchy problems with characteristic initial hypersurface, Comm. Pure Appl. Math. **26** (1973), 455–475.

[5] Bony, J.-M., Propagation des singularités différentiables pour une classe d'opérateurs différentiels à coefficients analytiques, Équ. Dériv. Partielles de Rennes (1975), Astérisque **34–35**, 1976, pp. 43–91.

[6] Bove, A., Lewis, J. E. and C. Parenti, Structure properties of solutions of some Fuchsian hyperbolic equations, Math. Ann. **273** (1986), 553–571.

[7] Colin, V., Formal microlocalization, C. R. Acad. Sci. Paris Sér. I Math. **327** (1998), 289–293.

[8] D'Agnolo, A. and G. Taglialatela, Sato-Kashiwara determinant and Levi conditions for systems, J. Math. Sci. Univ. Tokyo **7** (2000), 401–422.

[9] Dimca, A., *Sheaves in Topology*, Universitext, Springer, 2004.

[10] Kashiwara, M., Vanishing cycle sheaves and holonomic systems of differential equations, Algebraic Geometry, Proceedings Japan-France, Tokyo/Kyoto 1982 (M. Raynaud and T. Shioda, eds.), Lecture Notes in Math. **1016**, Springer, Berlin–Heidelberg–New York, 1983, pp. 134–142.

[11] Kashiwara, M., The Riemann-Hilbert problem for holonomic systems, Publ. Res. Inst. Math. Sci. **20** (1984), 319–365.

[12] Kashiwara, M., *An Introduction to Algebraic Analysis*, Iwanami Shoten, Tokyo, 2000 (Japanese); English transl., *D-Modules and Microlocal Calculus*, Transl. of Math. Monogr. **217**, Amer. Math. Soc., 2003.

[13] Kashiwara, M. and T. Kawai, On holonomic systems of microdifferential equations. III. Systems with regular singularities, Publ. Res. Inst. Math. Sci. **17** (1981), 813–979.

[14] Kashiwara, M. and T. Oshima, Systems of differential equations with regular singularities and their boundary value problems, Ann. of Math. **106** (1977), 145–200.
[15] Kashiwara, M. and P. Schapira, *Sheaves on Manifolds*, Grundlehren Math. Wiss. **292**, Springer, Berlin–Heidelberg–New York, 1990.
[16] Kashiwara, M. and P. Schapira, Moderate and Formal Cohomology Associated with Constructible Sheaves, Mém. Soc. Math. Fr. (N.S.) **94**, 1996.
[17] Kataoka, K., Some properties of mild hyperfunctions and an application of micro-analyticity in boundary value problems, J. Fac. Sci. Univ. Tokyo Sect. IA **30** (1983), 279–296.
[18] Laurent, Y., Vanishing cycles of D-modules, Invent. Math. **112** (1993), 491–539.
[19] Laurent, Y. and T. Monteiro Fernandes, Systèmes différentiels fuchsiens le long d'une sous-variété, Publ. Res. Inst. Math. Sci. **24** (1988), 397–431.
[20] Laurent, Y. and T. Monteiro Fernandes, Topological boundary values and regular $\mathscr{D}$-modules, Duke Math. J. **93** (1998), 207–230.
[21] Laurent, Y. and P. Schapira, Images inverse des modules différentiels, Compositio Math. **61** (1987), 229–251.
[22] Maisonobe, Ph. and Z. Mebkhout, Le théorèmes de comparaison pour les cycles évanescents, Sémin. Congr. **8** (2004), 311–389.
[23] Monteiro Fernandes, T., Formulation des valeurs au bord pour les systèmes réguliers, Compositio Math. **81** (1992), 121–142.
[24] Monteiro Fernandes, T., Holmgren theorem and boundary values for regular systems, C. R. Acad. Sci. Paris Sér. I Math. **318** (1994), 913–918.
[25] Oaku, T., F-mild hyperfunctions and Fuchsian partial differential equations, *Group Representation and Systems of Differential Equations*, Proceedings Tokyo 1982 (K. Okamoto, ed.), Advanced Studies in Pure Math. **4**, Kinokuniya, Tokyo; North-Holland, Amsterdam–New York–Oxford, 1984, pp. 223–242.
[26] Oaku, T., Boundary value problems for a system of linear partial differential equations and propagation of micro-analyticity, J. Fac. Sci. Univ. Tokyo, Sect. IA Math. **33** (1986), 175–232.
[27] Oaku, T. and S. Yamazaki, Higher-codimensional boundary value problems and F-mild microfunctions, Publ. Res. Inst. Math. Sci. **34** (1998), 383–437.
[28] Oshima, T., A definition of boundary values of solutions of partial differential equations with regular singularities, Publ. Res. Inst. Math. Sci. **19** (1983), 1203–1230.
[29] Parenti C. and H. Tahara, Asymptotic expansions of distribution solutions of some Fuchsian hyperbolic equations, Publ. Res. Inst. Math. Sci. **20** (1987), 909–922.
[30] Sato, M., Kawai, T. and M. Kashiwara, Microfunctions and pseudo-differential equations, Hyperfunctions and Pseudo-Differential Equations, Proceedings Katata 1971 (H. Komatsu, ed.), Lecture Notes in Math. **287**, Springer, Berlin–Heidelberg–New York, 1973, pp. 265–529.

[31] Schapira, P., *Microdifferential Systems in the Complex Domain*, Grundlehren Math. Wiss. **269**, Springer, Berlin–Heidelberg–New York, 1985.
[32] Tahara, H., Fuchsian type equations and Fuchsian hyperbolic equations, Japan J. Math. (N.S.) **5** (1979), 245–347.
[33] Uchida, M., A generalization of Bochner's tube theorem in elliptic boundary value problems, Publ. Res. Inst. Math. Sci. **31** (1995), 1065–1077.
[34] Yamazaki, S., Microlocal boundary value problem for regular-specializable systems, J. Math. Soc. Japan **56** (2004), 1109–1129.
[35] Yamazaki, S., Hyperfunction solutions to Fuchsian hyperbolic systems, J. Math. Sci. Univ. Tokyo **12** (2005), 191–209.
[36] Yamazaki, S., Microsupports of Gevrey class or ultradistribution solutions to systems with irregularities and their applications, in preparation, see: Gevrey class or ultradistribution solutions to Cauchy and boundary value problems for systems with irregularities, to appear in RIMS Kôkyûroku Bessatsu.

(Received May 7, 2010)
(Revised November 5, 2010)

Department of General Education
College of Science and Technology
Nihon University
24–1 Narashinodai 7-chome
Funabashi-shi, Chiba 274–8501
Japan