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A Remark on the Geometric Jacquet Functor

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Abstract. We give an action of N on the geometric Jacquet functor defined by Emerton-Nadler-Vilonen.

Let $G_{\mathbb{R}}$ be a reductive linear algebraic group over $\mathbb{R}$, $G_{\mathbb{R}} = K_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}$ an Iwasawa decomposition, and $M_{\mathbb{R}}$ the centralizer of $A_{\mathbb{R}}$ in $K_{\mathbb{R}}$. Then $P_{\mathbb{R}} = M_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}$ is a Langlands decomposition of a minimal parabolic subgroup. We use lower-case fraktur letters to denote the corresponding Lie algebras and omit the subscript "$\mathbb{R}$" to denote complexifications. Fix a Cartan involution θ such that $K = \{g \in G \mid \theta(g) = g\}$. For a $(\mathfrak{g}, K)$-module V, the Jacquet module $J(V)$ of V is defined by the space of $\mathfrak{n}$-finite vectors in $\varprojlim_{k\to\infty} V/\theta(\mathfrak{n})^k V$ [Cas80].

For simplicity, assume that V has the same infinitesimal character as the trivial representation. Denote the category of Harish-Chandra modules with the same infinitesimal characters as the trivial representation by HC_ρ. Let X be the flag variety of G, $\mathrm{Perv}_K(X)$ the category of K-equivariant perverse sheaves on X. By the Beilinson-Bernstein correspondence and the Riemann-Hilbert correspondence, we have the localization functor $\Delta\colon \mathrm{HC}_\rho \to \mathrm{Perv}_K(X)$. Emerton-Nadler-Vilonen gave a geometric description of $J(V)$ by the following way [ENV04]. Fix a cocharacter $\nu\colon \mathbb{G}_m \to A$ which is positive on the roots in $\mathfrak{n}$. Define $a\colon \mathbb{G}_m \times X \to X$ by $a(t,x) = \nu(t)x$. Consider the following diagram,

$$X \simeq \{0\} \times X \to \mathbb{A}^1 \times X \leftarrow \mathbb{G}_m \times X \xrightarrow{a} X.$$

Let $R\psi$ be the nearby cycle functor with respect to $\mathbb{A}^1 \times X \to \mathbb{A}^1$. Then the geometric Jacquet functor Ψ is defined by

$$\Psi(\mathscr{F}) = R\psi(a^*(\mathscr{F})).$$

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THEOREM 1 (Emerton-Nadler-Vilonen [ENV04, Theorem 1.1]). *We have* $\Delta \circ J \simeq \Psi \circ \Delta \colon \mathrm{HC}_\rho \to \mathrm{Perv}_K(X)$.

It is easy to see that $J(V)$ is a $(\mathfrak{g}, N)$-module for a $(\mathfrak{g}, K)$-module V. Hence $\Psi(\mathscr{F})$ is N-equivariant for $\mathscr{F} \in \mathrm{Perv}_K(X)$. (See also [ENV04, Remark 1.3].)

In this paper, we give the action of N on $\Psi(\mathscr{F})$ in a geometric way. Roughly speaking, this action is given by the "limit" of the action of K.

We use the following lemma.

LEMMA 2. *Let* $\mathcal{X}$ *be a scheme of finite type over* $\mathbb{A}^1$, $\mathcal{X}^0$ *(resp.* $\mathcal{X}_0$*) the inverse image of* $\mathbb{G}_m$ *(resp.* $\{0\}$*), and* $\mathcal{G}$ *a smooth group scheme over* $\mathbb{A}^1$*. Assume that the action of* $\mathcal{G}$ *on* $\mathcal{X}$ *over* $\mathbb{A}^1$ *is given. If* $\mathscr{F}^0$ *is a* $\mathcal{G}^0$*-equivariant perverse sheaf on* $\mathcal{X}^0$*, then* $R\psi(\mathscr{F}^0)$ *is* $\mathcal{G}_0$*-equivariant.*

PROOF. Define $m \colon \mathcal{G} \times_{\mathbb{A}^1} \mathcal{X} \to \mathcal{X}$ by $m(g, x) = gx$. Then m is a smooth morphism. Let $m^0 \colon \mathcal{G}^0 \times_{\mathbb{G}_m} \mathcal{X}^0 \to \mathcal{X}^0$ and $m_0 \colon \mathcal{G}_0 \times \mathcal{X}_0 \to \mathcal{X}_0$ be its restrictions. Since $\mathscr{F}^0$ is $\mathcal{G}^0$-equivariant, we are given an isomorphism $(m^0)^*(\mathscr{F}^0) \simeq \mathrm{pr}_2^*(\mathscr{F}^0)$. Hence we have an isomorphism $R\psi(m^0)^*(\mathscr{F}^0) \simeq R\psi\mathrm{pr}_2^*(\mathscr{F}^0)$. By the smooth base change theorem, we get an isomorphism $m_0^* R\psi(\mathscr{F}^0) \simeq \mathrm{pr}_2^* R\psi(\mathscr{F}^0)$ [SGA7 II, Exposé XIII]. It is immediate that this isomorphism gives a $\mathcal{G}_0$-equivariant structure on $R\psi(\mathscr{F}^0)$. □

Set

$$\mathcal{K}^0 = \{(t, k) \in \mathbb{G}_m \times G \mid k \in \mathrm{Ad}(\nu(t)^{-1})(K)\} \subset \mathbb{A}^1 \times G.$$

Let $\mathcal{K}$ be the closure of $\mathcal{K}^0$ in $\mathbb{A}^1 \times G$. It is a closed sub-group scheme of $\mathbb{A}^1 \times G$. Then $\mathcal{K}$ is flat over $\mathbb{A}^1$. By [Oor66], a group scheme of finite type over a field of characteristic zero is reduced. Hence each fiber of $\mathcal{K} \to \mathbb{A}^1$ is reduced. Therefore, $\mathcal{K} \to \mathbb{A}^1$ is smooth. Hence $\mathcal{K}$ is smooth over $\mathbb{A}^1$.

Let Σ be the restricted root system of $(\mathfrak{g}, \mathfrak{a})$, Σ^+ the positive system corresponding to $\mathfrak{n}$, and $\mathfrak{g}_\alpha$ the restricted root space for $\alpha \in \Sigma$. Then $\mathfrak{k}$ is spanned by $\mathfrak{m}$ and $\{X + \theta(X) \mid X \in \mathfrak{g}_\alpha, \alpha \in \Sigma^+\}$. Since

$$\mathrm{Ad}(\nu(t)^{-1})(X + \theta(X)) = t^{-\langle \nu, \alpha \rangle}(X + t^{2\langle \nu, \alpha \rangle}\theta(X))$$

for $X \in \mathfrak{g}_\alpha$, the Lie algebra of $\mathrm{Ad}(\nu(t)^{-1})(K)$ is spanned by

$$\mathfrak{m} \text{ and } \{X + t^{2\langle \nu, \alpha \rangle}\theta(X) \mid X \in \mathfrak{g}_\alpha, \alpha \in \Sigma^+\}.$$

Hence the neutral component of $\mathcal{K}_0$ is $M^\circ N$ where M° is the neutral component of M. Since $MK^\circ = K$ and $\mathrm{Ad}(\nu(t)^{-1})(M) = M$, we have $\mathcal{K}_0 = MM^\circ N = MN$.

Define $\widetilde{a}\colon \mathcal{K}^0 \times_{\mathbb{G}_m} (\mathbb{G}_m \times X) \to K \times X$ (resp. $\widetilde{m}\colon \mathcal{K}^0 \times_{\mathbb{G}_m} (\mathbb{G}_m \times X) \to \mathbb{G}_m \times X$, $m\colon K \times X \to X$) by $\widetilde{a}((t,k),(t,x)) = (\mathrm{Ad}(\nu(t))k, \nu(t)x)$ (resp. $\widetilde{m}((t,k),(t,x)) = (t,kx)$, $m(k,x) = kx$). Then we have the following commutative diagrams

$$\begin{array}{ccc} \mathcal{K}^0 \times_{\mathbb{G}_m} (\mathbb{G}_m \times X) & \xrightarrow{\widetilde{m}} & \mathbb{G}_m \times X \\ \downarrow{\scriptstyle \widetilde{a}} & & \downarrow{\scriptstyle a} \\ K \times X & \xrightarrow{m} & X, \end{array} \qquad \begin{array}{ccc} \mathcal{K}^0 \times_{\mathbb{G}_m} (\mathbb{G}_m \times X) & \xrightarrow{\mathrm{pr}_2} & \mathbb{G}_m \times X \\ \downarrow{\scriptstyle \widetilde{a}} & & \downarrow{\scriptstyle a} \\ K \times X & \xrightarrow{\mathrm{pr}_2} & X. \end{array}$$

Let $\mathscr{F} \in \mathrm{Perv}_K(X)$. Then $m^*\mathscr{F} \simeq \mathrm{pr}_2^*\mathscr{F}$. Hence we get $\widetilde{a}^* m^*\mathscr{F} \simeq \widetilde{a}^*\mathrm{pr}_2^*\mathscr{F}$. By the above diagrams, $\widetilde{m}^* a^*\mathscr{F} \simeq \mathrm{pr}_2^* a^*\mathscr{F}$. Therefore $a^*(\mathscr{F})$ is $\mathcal{K}^0$-equivariant. By Lemma 2, $\Psi(\mathscr{F}) = R\psi(a^*(\mathscr{F}))$ is $\mathcal{K}_0 = MN$-equivariant.

Remark 3. Let Γ be the quasi-inverse functor of Δ. Then N acts on $\Gamma(\Psi(\mathscr{F}))$. Moreover, this becomes a $(\mathfrak{g}, N)$-module [Kas89, 9.1.1], namely, the infinitesimal action of N coincides with the action of $\mathfrak{n} \subset \mathfrak{g}$. We also have that $J(\Gamma(\mathscr{F}))$ is a $(\mathfrak{g}, N)$-module. Hence both N-actions have the same infinitesimal actions. Since the action of N is determined by its infinitesimal action, the N-action we defined above coincides with the N-action on $J(\Gamma(\mathscr{F}))$.

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