J. Math. Sci. Univ. Tokyo
17 (2010), 179–199.

Abel-Jacobi Equivalence and a Variant of the Beilinson-Hodge Conjecture

By James D. Lewis

Abstract. Let $X/\mathbb{C}$ be a smooth projective variety and $\mathrm{CH}^r(X)$ the Chow group of codimension r algebraic cycles modulo rational equivalence. Let us assume the (conjectured) existence of the Bloch-Beilinson filtration $\{F^\nu\mathrm{CH}^r(X)\otimes\mathbb{Q}\}_{\nu=0}^r$ for all such X (and r). If $\mathrm{CH}^r_{AJ}(X)\subset\mathrm{CH}^r(X)$ is the subgroup of cycles Abel-Jacobi equivalent to zero, then there is an inclusion $F^2\mathrm{CH}^r(X)\otimes\mathbb{Q}\subset\mathrm{CH}^r_{AJ}(X)\otimes\mathbb{Q}$. Roughly speaking we show that this inclusion is an equality for all X (and r) if and only if a certain variant of Beilinson-Hodge conjecture holds for K_1.

1. Introduction

Let $X/\mathbb{C}$ be a smooth projective variety and $\mathrm{CH}^r(X;\mathbb{Q})$ the Chow group of codimension r algebraic cycles modulo rational equivalence, tensored with $\mathbb{Q}$. The existence of a descending filtration $\{F^\nu\mathrm{CH}^r(X;\mathbb{Q})\}_{\nu=0}^r$ on $\mathrm{CH}^r(X;\mathbb{Q})$ whose graded pieces factor through the Grothendieck motive, is a consequence of the classical Hodge conjecture (HC), together with a conjecture of Bloch and (independently) Beilinson (BBC) on the injectivity of the Abel-Jacobi map for Chow groups of smooth projective varieties over number fields. Assuming such a filtration, then one has $F^1\mathrm{CH}^r(X;\mathbb{Q}) = \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q})$ (= the nullhomologous cycles) and an inclusion $F^2\mathrm{CH}^r(X;\mathbb{Q})\subset\mathrm{CH}^r_{AJ}(X;\mathbb{Q})$, where the latter term are the cycles that are Abel-Jacobi equivalent to zero. The question as to whether this inclusion is (conjecturally) an equality, has generated some debate.

For a mixed $\mathbb{Q}$-Hodge structure H, we put $\Gamma(H) := \hom_{\mathrm{MHS}}(\mathbb{Q}(0),H)$. Evidently, by a mixed Hodge theory argument one can show that $\Gamma\big(H^{2r}(\mathbb{C}(X),\mathbb{Q}(r))\big) = 0$ for all smooth projective $X/\mathbb{C}$ and all $r>0$ is

2010 *Mathematics Subject Classification.* Primary 14C25; Secondary 14C30, 14C35.
Key words: Bloch-Beilinson filtration, normal function, Chow group.
Partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

equivalent to the statement of the classical Hodge conjecture. In this paper we consider the candidate Bloch-Beilinson filtration $\{F^\nu \mathrm{CH}^r(X;\mathbb{Q})\}_{\nu\geq 0}$ introduced in [Lew1], and put $D^r(X) := \bigcap_{\nu\geq 0} F^\nu \mathrm{CH}^r(X;\mathbb{Q})$. Evidently HC + BBC $\Rightarrow D^r(X) = 0$ (Theorem 4.1(vi)). Our main result is the following.

Theorem 1.1. *Consider these two statements:*

(i) $\Gamma\big(H^{2r-1}(\mathbb{C}(X),\mathbb{Q}(r))\big) = 0$ *for all all smooth projective* $X/\mathbb{C}$ *and all* $r > 1$.

(ii) $F^2\mathrm{CH}^r(X;\mathbb{Q}) = \mathrm{CH}^r_{AJ}(X;\mathbb{Q})$ *for all smooth projective* $X/\mathbb{C}$, *and all* r.

If we assume the HC, then (i) $\Rightarrow$ *(ii). If we further assume that* $D^r(X) \subset N^1\mathrm{CH}^r(X;\mathbb{Q})$, *then (ii)* $\Rightarrow$ *(i). (Here* $N^1\mathrm{CH}^r(X;\mathbb{Q})$ *is the subspace of cycles homologous to zero on codimension* ≥ 1 *algebraic subsets of* X.*)*

In section 5 we provide some evidence in support of the statement in Theorem 1.1(i). In particular we arrive at:

Theorem 1.2. *Let* $X/\mathbb{C}$ *be a smooth projective variety of dimension* d, *and let* $r > 1$.

(i) *Suppose that* $\mathrm{CH}^r_{AJ}(X;\mathbb{Q}) \subset N^1\mathrm{CH}^r(X;\mathbb{Q})$ *and either (i)* $d \leq 4$, *or (ii)* $r \in \{2, d-1\}$, *or (iii)* r, d *arbitrary and the HC holds. Then* $\Gamma\big(H^{2r-1}(\mathbb{C}(X),\mathbb{Q}(r))\big) = 0$. *(The statement* $\Gamma\big(H^{2d-1}(\mathbb{C}(X),\mathbb{Q}(d))\big) = 0$ *for* $d > 1$ *holds unconditionally.)*

(ii) *Let us further assume that* X *is a complete intersection with* $H^0(X,\Omega^d_X) = 0$. *Assume that either (i)* $d \leq 4$, *or (ii)* $r \in \{2, d-1\}$, *or (iii)* r, d *arbitrary and the HC holds. Then* $\Gamma\big(H^{2r-1}(\mathbb{C}(X),\mathbb{Q}(r))\big) = 0$.

(iii) *Again let* X *be a complete intersection and assume the HC. Then for all* r *with* $d \neq 2r-1$ *and* $D^r(X) \subset N^1\mathrm{CH}^r(X;\mathbb{Q})$, *we have* $\Gamma\big(H^{2r-1}(\mathbb{C}(X),\mathbb{Q}(r))\big) = 0$.

For the convenience to the reader, we also relate the statement in Theorem 1.1(ii) to the field of definition of the torsion locus of a cycle induced normal function, a result which seems known only among experts ([K-P]).

Let $\mathrm{CH}^r_{\mathrm{alg}}(X;\mathbb{Q})$ be the subspace of cycles that are algebraically equivalent to zero. As a result of Corollaries 4.9 & 5.1 below, we deduce the following.

COROLLARY 1.3. *Let $X/\mathbb{C}$ be a smooth projective variety. Then:*

(i)

$$\Gamma\big(H^3(\mathbb{C}(X),\mathbb{Q}(2))\big)=0\Rightarrow\begin{cases}F^2\mathrm{CH}^2(X;\mathbb{Q})=\mathrm{CH}^2_{AJ}(X;\mathbb{Q})\\ F^2\mathrm{CH}^2(X;\mathbb{Q})\subset\mathrm{CH}^2_{\mathrm{alg}}(X;\mathbb{Q})\end{cases}.$$

(ii) Conversely, if we further assume that X is either a complete intersection or an Abelian variety, and if $D^2(X)\subset\mathrm{CH}^2_{\mathrm{alg}}(X;\mathbb{Q})$, then:

$$F^2\mathrm{CH}^2(X;\mathbb{Q})=\mathrm{CH}^2_{AJ}(X;\mathbb{Q})\Rightarrow\Gamma\big(H^3(\mathbb{C}(X),\mathbb{Q}(2))\big)=0.$$

(iii) For any smooth projective $X/\mathbb{C}$ satisfying $\mathrm{CH}^2_{AJ}(X;\mathbb{Q})\subset\mathrm{CH}^2_{\mathrm{alg}}(X;\mathbb{Q})$, we have $\Gamma\big(H^3(\mathbb{C}(X),\mathbb{Q}(2))\big)=0$. (This also follows from Theorem 1.2(i), using the fact that $N^1\mathrm{CH}^2(X;\mathbb{Q})=\mathrm{CH}^2_{\mathrm{alg}}(X;\mathbb{Q})$.)

We are grateful to the referee for doing a splendid job, and raising some interesting points.

2. Notation

(i) Throughout this paper we will assume that $k\subset\mathbb{C}$ is an algebraically closed subfield. Let $\mathcal{V}_k$ be the category of smooth projective varieties over k.

(ii) $\mathbb{Q}(n)=(2\pi\mathrm{i})^n\mathbb{Q}=$ Tate twist (a pure HS on $\mathbb{Q}$ of pure weight $-2n$ (and Hodge type $(-n,-n)$).

(iii) For a mixed Hodge structure (MHS) H, we put

$$\Gamma(H):=\hom_{\mathrm{MHS}}(\mathbb{Q}(0),H),$$

$$J(H):=\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0),H).$$

(iv) For $X\in\mathcal{V}_k$, $H^i(X,\mathbb{Q}):=H^i(X(\mathbb{C}),\mathbb{Q})$ (singular cohomology). For $X\in\mathcal{V}_{\mathbb{C}}$,

$$H^i(\mathbb{C}(X),\mathbb{Q}):=\varinjlim_{U}H^i(U,\mathbb{Q}),$$

where the limit is taken over all non-empty Zariski open subsets $U\subset X$.

(v) For $X \in \mathcal{V}_k$, the coniveau filtration is given by

$$N_k^{\nu} H^i(X,\mathbb{Q}) := \ker\left(H^i(X,\mathbb{Q}) \to \varinjlim_{Y \subset X/k, \mathrm{cd}_X Y \geq \nu} H^i(X \backslash Y, \mathbb{Q}) \right).$$

(vi) The statement of the classical Hodge conjecture *for all* $X \in \mathcal{V}_{\mathbb{C}}$, will be abbreviated by HC. For $X \in \mathcal{V}_{\mathbb{C}}$ of dimension d, the hard Lefschetz conjecture $B(X)$ states that the inverse to the hard Lefschetz ismorphism

$$L_X^{d-i} : H^i(X,\mathbb{Q}) \xrightarrow{\sim} H^{2d-i}(X,\mathbb{Q}),$$

is algebraic cycle induced for all $i \leq d$, where L_X is the operation of cupping with a hyperplane section of X.

(vii) Let $\mathrm{CH}^r(X,m)$ be the higher Chow group introduced in [B]. We put $\mathrm{CH}^r(X,m;\mathbb{Q}) := \mathrm{CH}^r(X,m) \otimes \mathbb{Q}$. The classical Chow group is given by $\mathrm{CH}^r(X) = \mathrm{CH}^r(X,0)$. The subgroup of cycles algebraically equivalent to zero is denoted by $\mathrm{CH}^r_{\mathrm{alg}}(X) \subset \mathrm{CH}^r(X)$.

(viii) Let $Y \subset X$ be a Zariski closed subset, where $X \in \mathcal{V}_k$. If $d = \dim X$, we put $\mathrm{CH}_{d-r}(X) = \mathrm{CH}^r(X)$. Likewise let $\mathrm{CH}^r_Y(X) := \mathrm{CH}_{d-r}(Y)$, and $\mathrm{CH}^r_{Y,\mathrm{hom}}(X;\mathbb{Q}) := \ker\big(\mathrm{CH}^r_Y(X;\mathbb{Q}) \to H^{2r}_Y(X,\mathbb{Q})\big)$.

(ix) For $X \in \mathcal{V}_k$, consider the Abel-Jacobi map

$$AJ_X : \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) \to J\big(H^{2r-1}(X,\mathbb{Q}(r))\big).$$

We put $\mathrm{CH}^r_{AJ}(X;\mathbb{Q}) := \ker AJ_X$.

3. A Variant of the Hodge Conjecture for K_1

We review some of the ideas in [K-L], some of which goes back to the work of Jannsen ([Ja2]). Let $X \in \mathcal{V}_k$ be given with algebraic subset $Y \subset X/k$. The localization sequence yields a s.e.s. of MHS:

$$0 \to \frac{H^{2r-1}(X,\mathbb{Q}(r))}{H^{2r-1}_Y(X,\mathbb{Q}(r))} \to H^{2r-1}(X\backslash Y,\mathbb{Q}(r)) \to H^{2r}_Y(X,\mathbb{Q}(r))^{\circ} \to 0,$$

where

$$H^{2r}_Y(X,\mathbb{Q}(r))^{\circ} := \ker\big(H^{2r}_Y(X,\mathbb{Q}(r)) \to H^{2r}(X,\mathbb{Q}(r))\big).$$

Note that $H^{2r-1}(X,\mathbb{Q}(r))/H_Y^{2r-1}(X,\mathbb{Q}(r))$ is a pure Hodge structure of weight -1. Corresponding to this is a commutative diagram:

$$\begin{array}{ccccccc} & \mathrm{CH}^r(X\backslash Y,1;\mathbb{Q}) & \to & \mathrm{CH}^r_Y(X;\mathbb{Q})^\circ & \xrightarrow{\beta} & \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) \\ (1) & \mathrm{cl}_{r,1}\Big\downarrow & & \lambda\Big\downarrow & & \underline{AJ}_X\Big\downarrow \\ 0 \to & \Gamma\big(H^{2r-1}(X\backslash Y,\mathbb{Q}(r))\big) & \xrightarrow{\alpha} & \Gamma\big(H^{2r}_Y(X,\mathbb{Q}(r))^\circ\big) & \to & J\left(\frac{H^{2r-1}(X,\mathbb{Q}(r))}{H_Y^{2r-1}(X,\mathbb{Q}(r))}\right), \end{array}$$

where $\mathrm{CH}^r_Y(X;\mathbb{Q})^\circ$ are the cycles in $\mathrm{CH}^r_Y(X;\mathbb{Q})$ that are homologous to zero on X, and where $\underline{AJ}_X$ is the composite Abel-Jacobi map

$$\mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) \xrightarrow{AJ_X} J\big(H^{2r-1}(X,\mathbb{Q}(r))\big) \to J\left(\frac{H^{2r-1}(X,\mathbb{Q}(r))}{H_Y^{2r-1}(X,\mathbb{Q}(r))}\right).$$

Let us assume that λ is surjective. Such is the case if the HC[1] holds for Y. Then the serpent lemma gives:

$$(2)\qquad \frac{\ker\big(\underline{AJ}_X\big|_{\mathrm{Im}(\beta)}\big)}{\beta\big(\ker\lambda\big)} \simeq \frac{\Gamma\big(H^{2r-1}(X\backslash Y,\mathbb{Q}(r))\big)}{\mathrm{cl}_{r,1}\big(\mathrm{CH}^r(X\backslash Y,1;\mathbb{Q})\big)}.$$

Let $\widetilde{Y} \xrightarrow{\approx} Y$ be a desingularization. If we assume for the moment that the Gysin map $H^{2r-2\mathrm{cd}_X Y-1}(\widetilde{Y},\mathbb{Q}) \to H^{2r-1}(X,\mathbb{Q})$ has a cycle induced right inverse (as implied by the HC), then as argued in [K-L],

$$(3)\qquad \frac{\ker\big(\underline{AJ}_X\big|_{\mathrm{Im}(\beta)}\big)}{\beta\big(\ker\lambda\big)} = \frac{\beta\big(\ker\lambda\big)+\ker\big(AJ_X\big|_{\mathrm{Im}(\beta)}\big)}{\beta\big(\ker\lambda\big)}.$$

We recall that Bloch and Beilinson ([Be] 5.6) independently conjectured the following:

Conjecture 3.1 (BBC = Bloch-Beilinson Conjecture). *If $k=\overline{\mathbb{Q}}$, then*

$$AJ_X : \mathrm{CH}^r_{\mathrm{hom}}(X/\overline{\mathbb{Q}};\mathbb{Q}) \hookrightarrow J\big(H^{2r-1}(X,\mathbb{Q}(r))\big),$$

[1] Homological version, see [Ja2](§7); or if the reader prefers, assume the HC holds for a desingularization $\widetilde{Y}$.

is injective.

Two extreme cases comes to mind:

• If $k = \overline{\mathbb{Q}}$, then the HC + BBC $\Rightarrow \mathrm{cl}_{r,1}\big(\mathrm{CH}^r(X\backslash Y, 1; \mathbb{Q})\big) = \Gamma\big(H^{2r-1}(X\backslash Y, \mathbb{Q}(r))\big)$[2].

• (Jannsen [Ja2]) If $k = \mathbb{C}$ and $\mathrm{codim}_X Y = r$, then λ in (1) is an isomorphism, $H^{2r-1}_Y(X, \mathbb{Q}(r)) = 0$; moreover $\mathrm{cl}_{r,1}$ is surjective $\Leftrightarrow AJ_X$ is injective on $\mathrm{Im}(\beta)$. This implies surjectivity in the case $r = 1$, by the theory of the Picard variety; however for $r > 1$, AJ_X need not be injective (Mumford), hence $\mathrm{cl}_{r,1}$ need not be surjective.

A natural question is whether one can tweak the second scenario situation so that surjectivity is a possibility. As the higher Chow groups involve numerator conditions in the definition, this appears to be the case if one passes to the generic point. Namely:

Conjecture 3.2 ([K-L]).

$$\mathrm{cl}_{r,1} : \mathrm{CH}^r(\mathbb{C}(X), 1; \mathbb{Q}) \twoheadrightarrow \Gamma\big(H^{2r-1}(\mathbb{C}(X), \mathbb{Q}(r))\big),$$

is surjective.

Here we wish to make it clear that $\mathrm{CH}^r(\mathbb{C}(X), 1; \mathbb{Q}) := \mathrm{CH}^r(\mathrm{Spec}(\mathbb{C}(X)), 1; \mathbb{Q})$ and

$$H^i(\mathbb{C}(X), \mathbb{Q}) := \varinjlim_{\mathrm{cd}_X Y = 1} H^i(X\backslash Y, \mathbb{Q}).$$

Proposition 3.3. *The following statements are equivalent:*

(i) $\Gamma\big(H^{2r-1}(\mathbb{C}(X), \mathbb{Q}(r))\big) = 0$ *for all* $X \in \mathcal{V}_{\mathbb{C}}$ *and all* $r > 1$.

(ii) *Conjecture 3.2 holds for all* $X \in \mathcal{V}_{\mathbb{C}}$ *and all* r.

Proof. First, we may assume that $r > 1$, as $\mathrm{cl}_{1,1}$ is surjective. Secondly, for dimension reasons $\mathrm{CH}^r(\mathbb{C}(X), 1) = 0$ for $r > 1$. Thirdly

[2] As originally shown by M. Saito ([MSa]), this statement generalizes to $\mathrm{CH}^r(X\backslash Y, m; \mathbb{Q})$. A different proof of that generalization appears in [Ke-L].

$\Gamma\big(H^{2r-1}(\mathbb{C}(X),\mathbb{Q}(r))\big)=0$ implies $\mathrm{cl}_{r,1}$ is obviously surjective. The proposition follows from this. □

To see why Conjecture 3.2 is plausible[3], observe that by passing to a limit over all codimension 1 subvarieties of X, (2) becomes

$$\frac{\ker\bigg(\underline{AJ}_X : \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) \to J\bigg(\frac{H^{2r-1}(X,\mathbb{Q}(r))}{N^1H^{2r-1}(X,\mathbb{Q}(r))}\bigg)\bigg)}{N^1\mathrm{CH}^r(X;\mathbb{Q})} \tag{4}$$
$$\simeq \frac{\Gamma\big(H^{2r-1}(\mathbb{C}(X),\mathbb{Q}(r))\big)}{\mathrm{cl}_{r,1}\big(\mathrm{CH}^r(\mathbb{C}(X),1;\mathbb{Q})\big)},$$

where $N^p\mathrm{CH}^r(X;\mathbb{Q}) \subset \mathrm{CH}^r(X;\mathbb{Q})$ is the subspace of cycles that are homologous to zero on algebraic subsets of codimension $\geq p$ in X, and $N^pH^i(X,\mathbb{Q}) := N^p_{\mathbb{C}}H^i(X,\mathbb{Q})$ is the coniveau filtration. Then (3) translates to

$$\frac{\ker(AJ_X)+N^1\mathrm{CH}^r(X;\mathbb{Q})}{N^1\mathrm{CH}^r(X;\mathbb{Q})} \simeq \frac{\Gamma\big(H^{2r-1}(\mathbb{C}(X),\mathbb{Q}(r))\big)}{\mathrm{cl}_{r,1}\big(\mathrm{CH}^r(\mathbb{C}(X),1;\mathbb{Q})\big)}. \tag{5}$$

Remark 3.4. Note that the isomorphisms in (4) and (5) hinge on HC assumptions. For instance, (4) requires λ in (1) to be surjective.

In the case $r = d := \dim X$, the reader can easily check that the map $\mathrm{cl}_{d,1}$ in Conjecture 3.2 is unconditionally surjective. Further, according to [Ja1], there is some evidence to suggest that $\ker(AJ_X) \subset N^1\mathrm{CH}^r(X;\mathbb{Q})$. Next, observe that $\mathrm{CH}^r_{\mathrm{alg}}(X;\mathbb{Q}) = N^{r-1}\mathrm{CH}^r(X;\mathbb{Q})$, and that the restricted Abel-Jacobi map,

$$\mathrm{CH}^r_{\mathrm{alg}}(X;\mathbb{Q}) \twoheadrightarrow J\big(N^{r-1}H^{2r-1}(X,\mathbb{Q}(r))\big),$$

is surjective. When $r = 2$ one can easily check that

$$\frac{\mathrm{CH}^2_{\mathrm{alg}}(X;\mathbb{Q})+\ker(AJ_X)}{\mathrm{CH}^2_{\mathrm{alg}}(X;\mathbb{Q})} \simeq \frac{\Gamma\big(H^3(\mathbb{C}(X),\mathbb{Q}(2))\big)}{\mathrm{cl}_{2,1}\big(\mathrm{CH}^2(\mathbb{C}(X),1;\mathbb{Q})\big)} \tag{6}$$
$$= \Gamma\big(H^3(\mathbb{C}(X),\mathbb{Q}(2))\big),$$

[3] Quite generally ([dJ-L]), we also conjecture that $\Gamma\big(H^{2r-m}(\mathbb{C}(X),\mathbb{Q}(r))\big)=0$ for all $X \in \mathcal{V}_{\mathbb{C}}$ and $r \neq m$, and $\mathrm{cl}_{m,m} : \mathrm{CH}^m(\mathbb{C}(X),m) \twoheadrightarrow \Gamma\big(H^m(X,\mathbb{Z}(r))\big)$ is surjective. (Note: The vanishing $\Gamma\big(H^{2r-m}(\mathbb{C}(X),\mathbb{Q}(r))\big)=0$ for $r<m$ is a simple consequence of mixed Hodge theory.)

holds unconditionally.

4. A Descending Filtration

We recall the candidate Bloch-Beilinson (B-B) filtration constructed in [Lew1].

THEOREM 4.1. *Let $X \in \mathcal{V}_{\mathbb{C}}$ be of dimension d. Then for all r, there is a filtration*

$$\mathrm{CH}^r(X;\mathbb{Q}) = F^0 \supset F^1 \supset \cdots \supset F^\nu \supset F^{\nu+1} \supset \cdots \supset F^r \supset F^{r+1} = F^{r+2} = \cdots ,$$

which satisfies the following

(i) $F^1 = \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q})$.

(ii) $F^2 \subset \mathrm{CH}^r_{AJ}(X;\mathbb{Q})$.

(iii) $F^{\nu_1}\mathrm{CH}^{r_1}(X;\mathbb{Q}) \bullet F^{\nu_2}\mathrm{CH}^{r_2}(X;\mathbb{Q}) \subset F^{\nu_1+\nu_2}\mathrm{CH}^{r_1+r_2}(X;\mathbb{Q})$, *where $\bullet$ is the intersection product.*

(iv) *F^ν is preserved under the action of correspondences between smooth projective varieties in $\mathcal{V}_{\mathbb{C}}$*

(v) *Let $\mathrm{Gr}_F^\nu := F^\nu/F^{\nu+1}$ and assume that the Künneth components of the diagonal class $[\Delta_X] = \oplus_{p+q=2d}[\Delta_X(p,q)] \in H^{2d}(X \times X, \mathbb{Q}(d)))$ are algebraic and defined over K. Then*

$$\Delta_X(2d-2r+\ell, 2r-\ell)_*\big|_{\mathrm{Gr}_F^\nu \mathrm{CH}^r(X;\mathbb{Q})} = \delta_{\ell,\nu} \cdot \mathrm{Identity}.$$

[If we assume the conjecture that homological and numerical equivalence coincide, then (v) says that Gr_F^ν factors through the Grothendieck motive.]

(vi) *Let $D^r(X) := \bigcap_\nu F^\nu$, and $k = \overline{\mathbb{Q}}$. If the BBC together with the HC holds, $D^r(X) = 0$.*[4]

REMARK 4.2. The way this filtration is constructed is as follows. Consider a $\overline{\mathbb{Q}}$-spread $\rho : \mathfrak{X} \to \mathcal{S}$, where ρ is smooth and proper morphism of

[4]The formulation in [Lew1] states that if the analog of the BBC holds for smooth quasiprojective varieties defined over a number field, then $D^r(X) = 0$. That analog however, is implied by the BBC + HC.

quasiprojective varieties, and $K = \overline{\mathbb{Q}}(\mathcal{S})$. Let η be the generic point of $\mathcal{S}/\overline{\mathbb{Q}}$, and hence $K := \overline{\mathbb{Q}}(\eta)$, with $X_K := \mathfrak{X}_\eta$. Using the cycle class map into absolute Hodge cohomology, $\mathrm{CH}^r(\mathfrak{X}/\overline{\mathbb{Q}};\mathbb{Q}) \to H^{2r}_{\mathcal{H}}(\mathfrak{X},\mathbb{Q}(r))$, there is a decreasing filtration $\mathcal{F}^\nu\mathrm{CH}^r(\mathfrak{X}/\overline{\mathbb{Q}};\mathbb{Q})$, with the property that $\mathrm{Gr}^\nu_{\mathcal{F}}\mathrm{CH}^r(\mathfrak{X}/\overline{\mathbb{Q}};\mathbb{Q}) \hookrightarrow E_\infty^{\nu,2r-\nu}(\rho)$, where $E_\infty^{\nu,2r-\nu}(\rho)$ is the ν-th graded piece of a Leray filtration associated to ρ. The term $E_\infty^{\nu,2r-\nu}(\rho)$ fits in a short exact sequence:

$$0 \to \underline{E}_\infty^{\nu,2r-\nu}(\rho) \to E_\infty^{\nu,2r-\nu}(\rho) \to \underline{\underline{E}}_\infty^{\nu,2r-\nu}(\rho) \to 0,$$

where

$$\underline{\underline{E}}_\infty^{\nu,2r-\nu}(\rho) = \Gamma\big(H^\nu(\mathcal{S}(\mathbb{C}), R^{2r-\nu}\rho_*\mathbb{Q}(r))\big),$$

$$\begin{aligned}\underline{E}_\infty^{\nu,2r-\nu}(\rho) &= \frac{J\big(W_{-1}H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu}\rho_*\mathbb{Q}(r))\big)}{\Gamma\big(\mathrm{Gr}^0_W H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu}\rho_*\mathbb{Q}(r))\big)} \\ &\subset J\big(H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu}\rho_*\mathbb{Q}(r))\big).\end{aligned}$$

[Here the latter inclusion is a result of the s.e.s.:

$$\begin{aligned}W_{-1}H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu}\rho_*\mathbb{Q}(r)) &\hookrightarrow W_0H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu}\rho_*\mathbb{Q}(r)) \\ &\twoheadrightarrow \mathrm{Gr}^0_W H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu}\rho_*\mathbb{Q}(r)).]\end{aligned}$$

One then has (by definition)

$$F^\nu\mathrm{CH}^r(X_K;\mathbb{Q}) = \lim_{\substack{\to \\ U\subset\mathcal{S}/\overline{\mathbb{Q}}}} \mathcal{F}^\nu\mathrm{CH}^r(\mathfrak{X}_U/\overline{\mathbb{Q}};\mathbb{Q}), \quad \mathfrak{X}_U := \rho^{-1}(U).$$

Now put,

$$E_\infty^{\nu,2r-\nu}(\eta_{\mathcal{S}}) = \lim_{\substack{\longrightarrow \\ U\subset\mathcal{S}/\overline{\mathbb{Q}}}} E_\infty^{\nu,2r-\nu}(\rho)$$

and the same definition for $\underline{E}_\infty^{\nu,2r-\nu-m}(\eta_{\mathcal{S}})$ and $\underline{\underline{E}}_\infty^{\nu,2r-\nu}(\eta_{\mathcal{S}})$. Specifically,

$$\underline{\underline{E}}_\infty^{\nu,2r-\nu}(\eta_{\mathcal{S}}) = \Gamma\big(H^\nu(\eta_{\mathcal{S}}, R^{2r-\nu}\rho_*\mathbb{Q}(r))\big),$$

$$\underline{E}_\infty^{\nu,2r-\nu}(\eta_{\mathcal{S}}) = J\big(W_{-1}H^{\nu-1}(\eta_{\mathcal{S}}, R^{2r-\nu}\rho_*\mathbb{Q}(r))\big)/\Gamma(\mathrm{Gr}^0_W).$$

We have a s.e.s.:

$$0 \to \underline{E}_{\infty}^{\nu,2r-\nu}(\eta_{\mathcal{S}}) \to E_{\infty}^{\nu,2r-\nu}(\eta_{\mathcal{S}}) \to \underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\eta_{\mathcal{S}}) \to 0,$$

and an injection:

$$\mathrm{Gr}_F^{\nu}\mathrm{CH}^r(X_K;\mathbb{Q}) \hookrightarrow E_{\infty}^{\nu,2r-\nu}(\eta_{\mathcal{S}}).$$

We then define

$$F^{\nu}\mathrm{CH}^r(X/\mathbb{C};\mathbb{Q}) = \lim_{\substack{\to \\ K\subset\mathbb{C}}} F^{\nu}\mathrm{CH}^r(X_K;\mathbb{Q}),$$

over all finitely generated subfields $K \subset \mathbb{C}$ over $\overline{\mathbb{Q}}$, which becomes a candidate B-B filtration on $\mathrm{CH}^r(X_{\mathbb{C}};\mathbb{Q})$.

Now let $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$. Then the action of σ on $\mathrm{CH}^r(\mathcal{X}/\overline{\mathbb{Q}};\mathbb{Q})$ is the identity; however in the limit, and after identifying K with its embedding in $\mathbb{C}$, we arrive at $\sigma\big(F^{\nu}\mathrm{CH}^r(X_K;\mathbb{Q}) = F^{\nu}\mathrm{CH}^r(X_{\sigma K};\mathbb{Q})$. In particular, we deduce the following:

Proposition 4.3. *Let $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$, and $X = X/\mathbb{C}$ be a smooth projective variety. Then*

$$\sigma : F^{\nu}\mathrm{CH}^r(X;\mathbb{Q}) \xrightarrow{\sim} F^{\nu}\mathrm{CH}^r(X_{\sigma};\mathbb{Q}),$$

is an isomorphism.

Now let us further assume that $\mathcal{S}$ is affine. Let $V \subset \mathcal{S}(\mathbb{C})$ be smooth, irreducible, closed subvariety of dimension $\nu - 1$ (note that $\mathcal{S}$ affine $\Rightarrow V$ affine). One has a commutative square

$$\begin{array}{ccc} \mathcal{X}_V & \hookrightarrow & \mathcal{X}(\mathbb{C}) \\ \rho_V \downarrow & & \downarrow \rho \\ V & \hookrightarrow & \mathcal{S}(\mathbb{C}), \end{array}$$

and a commutative diagram

$$\begin{array}{ccccccccc}
 & & & \xi \in & \mathrm{Gr}^{\nu}_{\mathcal{F}}\mathrm{CH}^r(\mathfrak{X};\mathbb{Q}) & \mapsto & \mathrm{Gr}^{\nu}_{F}\mathrm{CH}^r(X_K;\mathbb{Q}) & & \\
 & & & & \downarrow & & & & \\
0 & \to & \underline{E}^{\nu,2r-\nu}_{\infty}(\rho) & \to & E^{\nu,2r-\nu}_{\infty}(\rho) & \to & \underline{\underline{E}}^{\nu,2r-\nu}_{\infty}(\rho) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \underline{E}^{\nu,2r-\nu}_{\infty}(\rho_V) & \to & E^{\nu,2r-\nu}_{\infty}(\rho_V) & \to & \underline{\underline{E}}^{\nu,2r-\nu}_{\infty}(\rho_V) & \to & 0 \\
 & & & & & & || & & \\
 & & & & & & 0 & &
\end{array}$$

where $\underline{\underline{E}}^{\nu,2r-\nu}_{\infty}(\rho_V) = 0$ follows from the weak Lefschetz theorem for locally constant systems over affine varieties (see for example [Ar], and the references cited there). Thus for any $\xi \in \mathrm{Gr}^{\nu}_{\mathcal{F}}\mathrm{CH}^r(\mathfrak{X}/\overline{\mathbb{Q}};\mathbb{Q})$, we have a "normal function" ν_ξ with the property that for any such smooth irreducible closed $V \subset S(\mathbb{C})$ of dimension $\nu - 1$, we have a value $\nu_\xi(V) \in \underline{E}^{\nu,2r-\nu}_{\infty}(\rho_V)$. Here we think of V as a point on a suitable open subset of the Chow variety of dimension $\nu - 1$ subvarieties of $\mathcal{S}(\mathbb{C})$ and ν_ξ defined on that subset. Note that it is rather clear from this that $F^2\mathrm{CH}^r(X;\mathbb{Q}) \subset \mathrm{CH}^r_{AJ}(X;\mathbb{Q})$.

Definition 4.4 ([Ke-L]). ν_ξ is called an arithmetic normal function.

An important observation which seems to be acknowledged only among experts (see [K-P], Prop. 86 for their version of all of this), is the following:

Proposition 4.5. *The following statements are equivalent:*

(i) $F^2\mathrm{CH}^r(X;\mathbb{Q}) = \mathrm{CH}^r_{AJ}(X;\mathbb{Q})$ *for all* $X \in \mathcal{V}_{\mathbb{C}}$.

(ii) *For any smooth and proper morphism* $\rho : \mathfrak{X} \to \mathcal{S}$ *of smooth quasiprojective varieties over* $\overline{\mathbb{Q}}$, *and cycle induced normal function*

$$\nu_\xi : \mathcal{S}(\mathbb{C}) \to \coprod_{t \in \mathcal{S}(\mathbb{C})} J\big(H^{2r-1}(X_t, \mathbb{Q}(r))\big),$$

$\xi \in \mathcal{F}^1\mathrm{CH}^r(\mathcal{X}/\overline{\mathbb{Q}};\mathbb{Q})$, *the zero locus (equiv. torsion locus of a corresponding integrally defined normal function)* $\mathcal{Z}(\nu_\xi)$ *of* ν_ξ *is a countable union of algebraic subvarieties over* $\overline{\mathbb{Q}}$.

(iii) *For any smooth and proper morphism* $\rho_V : \mathcal{X}_V \to V$ *of smooth quasiprojective varieties over a subfield* $L \subset \mathbb{C}$ *finitely generated over* $\overline{\mathbb{Q}}$, *and cycle induced normal function*

$$\nu_\xi : V(\mathbb{C}) \to \coprod_{t \in V(\mathbb{C})} J\big(H^{2r-1}(X_t, \mathbb{Q}(r))\big),$$

$\xi \in \mathcal{F}^1\mathrm{CH}^r(\mathcal{X}_V/L;\mathbb{Q})$ *(= relatively homologous to zero with respect to* ρ_V*), the zero locus* $\mathcal{Z}(\nu_\xi)$ *of* ν_ξ *is a countable union of algebraic subvarieties over* $\overline{L}$.

Proof. The implication (ii) $\Rightarrow$ (i) is easy and left to the reader. Going the other way, we know that $\mathcal{Z}(\nu_\xi)$ is a countable union of analytic varieties. For any $p \in \mathcal{Z}(\nu_\xi)$, the $\overline{\mathbb{Q}}$ closure $\overline{\{p\}} \subset \mathcal{S}/\overline{\mathbb{Q}}$ defines a subfamily $\mathcal{X}_{\overline{\{p\}}} \to \overline{\{p\}}$, whose generic fiber satisfies $F^2\mathrm{CH}^r(\mathcal{X}_{\overline{\{p\}},\eta};\mathbb{Q}) = \mathrm{CH}^r_{AJ}(\mathcal{X}_{\overline{\{p\}},\eta};\mathbb{Q})$. Thus ν_ξ vanishes on $\overline{\{p\}}$. Thus $\overline{\{p\}} \subset \mathcal{Z}(\nu_\xi)$. Since the set of all $\overline{\mathbb{Q}}$ subvarieties of $\mathcal{S}/\overline{\mathbb{Q}}$ is countable, likewise $\mathcal{Z}(\nu_\xi)$ is a countable union of varieties over $\overline{\mathbb{Q}}$. To show (ii) $\Rightarrow$ (iii), consider $\rho_V : \mathcal{X}_V \to V$ defined over L. Let $\mathcal{S} \to \mathcal{T}$ be a $\overline{\mathbb{Q}}$-spread of V, with generic points $\eta \in \mathcal{S}/\overline{\mathbb{Q}}$ and $\eta_{\mathcal{T}} \in \mathcal{T}/\overline{\mathbb{Q}}$, and where we have $L = \overline{\mathbb{Q}}(\eta_{\mathcal{T}})$, $V/L = \mathcal{S}_{\eta_{\mathcal{T}}}$, $\mathcal{X}_V = \mathcal{X}_{\eta_{\mathcal{T}}}$. Correspondingly we have a $\overline{\mathbb{Q}}$-spread $\mathcal{X} \to \mathcal{S}$ with $\mathcal{X}_\eta = \mathcal{X}_{\eta_V}$. Note that $\xi \in \mathcal{F}^1\mathrm{CH}^r(\mathcal{X}_V/L;\mathbb{Q})$ is the restriction of a spread cycle $\tilde{\xi} \in \mathcal{F}^1\mathrm{CH}^r(\mathcal{X}/\overline{\mathbb{Q}};\mathbb{Q})$, and if $\Sigma \subset \mathcal{S}/\overline{\mathbb{Q}}$ is an irreducible component of the torsion locus of $\nu_{\tilde{\xi}}$, then $\Sigma_{\eta_{\mathcal{T}}}$ corresponds to a component of the locus of ν_ξ over $\overline{L}$ in $V/\overline{L}$. Finally, the converse (iii) $\Rightarrow$ (ii) is obvious. $\square$

It is instructive to give a direct proof of the following result, which can be deduced from [Ja1] (Thm 6.1). We will need this result in the sections to follow. Recall $\dim X = d$, and the statement $B(X)$ of the hard Lefschetz conjecture for X.

Proposition 4.6. *Let us assume* $B(X)$ *and that* $D^r(X) \subset N^{\nu-1}\mathrm{CH}^r(X;\mathbb{Q})$. *Then*

$$F^\nu\mathrm{CH}^r(X;\mathbb{Q}) \subset N^{\nu-1}\mathrm{CH}^r(X;\mathbb{Q}),$$

for $\nu \geq 1$.

Proof. For simplicity, we will assume that $D^r(X) = 0$, keeping in mind that the situation $D^r(X) \subset N^{\nu-1}\mathrm{CH}^r(X;\mathbb{Q})$ is similar. According to Theorem 4.1, and under the above assumptions,

$$\mathrm{Gr}_F^{\nu}\mathrm{CH}^r(X/\mathbb{C};\mathbb{Q}) \simeq \Delta_X(2d-2r+\nu, 2r-\nu)_*\mathrm{CH}^r(X/\mathbb{C};\mathbb{Q}),$$

and $F^{r+1}\mathrm{CH}^r(X/\mathbb{C};\mathbb{Q}) = 0$. Let $\xi \in F^{\nu}\mathrm{CH}^r(X/\mathbb{C};\mathbb{Q})$ be given. By writing $\xi = \Delta_X(2d-2r+\nu, 2r-\nu)_*\xi + \big(\xi - \Delta_X(2d-2r+\nu, 2r-\nu)_*\xi\big)$, observing that $\big(\xi - \Delta_X(2d-2r+\nu, 2r-\nu)_*\xi\big) \in F^{\nu+1}\mathrm{CH}^r(X;\mathbb{Q})$, and applying downward induction on ν, we can replace ξ by $\Delta_X(2d-2r+\nu, 2r-\nu)_*\xi$. If $2r-\nu < d$, then $H^{2r-\nu}(X,\mathbb{Q}(r)) \hookrightarrow H^{2r-\nu}(Y,\mathbb{Q}(r))$ for any smooth hypersurface $Y \subset X$. Then $B(X)$ implies a cycle induced right inverse $[w]_* : H^{2r-\nu}(Y,\mathbb{Q}(r)) \twoheadrightarrow H^{2r-\nu}(X,\mathbb{Q}(r))$. Hence $w_* : \mathrm{Gr}_F^{\nu}\mathrm{CH}^r(Y;\mathbb{Q}) \twoheadrightarrow \mathrm{Gr}_F^{\nu}\mathrm{CH}^r(X;\mathbb{Q})$ is surjective and $w_*\big(N^{\nu-1}\mathrm{CH}^r(Y;\mathbb{Q})\big) \subset N^{\nu-1}\mathrm{CH}^r(X;\mathbb{Q})$. So by induction on dimension, we are done in this case. So let us assume that $2r-\nu \geq d$, and put $\underline{r} = d-r$. Then $d \geq 2\underline{r}+\nu = 2\underline{r}+m+1$, where $m = \nu-1$. According to [Ja1] (Prop. 4.8(b)), based on a corresponding result of Nori, there exists a smooth complete intersection $Y \subset X$ of codimension $m = \nu-1$ such that ξ is in the image of $\mathrm{CH}_{\underline{r},\mathrm{hom}}(Y;\mathbb{Q}) \to \mathrm{CH}_{\underline{r}}(X;\mathbb{Q}) = \mathrm{CH}^r(X;\mathbb{Q})$. Thus $\xi \in N^{\nu-1}\mathrm{CH}^r(X;\mathbb{Q})$ and we are done. $\square$

Recall that $N^{r-1}\mathrm{CH}^r(X;\mathbb{Q}) = \mathrm{CH}^r_{\mathrm{alg}}(X;\mathbb{Q})$, and hence (and as also pointed out in [Ja1]), under the assumptions in Proposition 4.6, $F^r\mathrm{CH}^r(X;\mathbb{Q}) \subset \mathrm{CH}^r_{\mathrm{alg}}(X;\mathbb{Q})$. However it is worthwhile noting that:

Proposition 4.7. *Suppose that $X/\mathbb{C} = X_0 \times \mathbb{C}$, where $X_0 = X_0/\overline{\mathbb{Q}}$. Assume that the BBC holds. Then $F^2\mathrm{CH}^r(X/\mathbb{C};\mathbb{Q}) \subset \mathrm{CH}^r_{\mathrm{alg}}(X/\mathbb{C};\mathbb{Q})$.*

Proof. Let $\xi \in \mathrm{CH}^r(X/\mathbb{C};\mathbb{Q})$. Then there exists a smooth quasiprojective variety $S/\overline{\mathbb{Q}}$ and cycle $\tilde{\xi} \in \mathrm{CH}^r(S \times_{\overline{\mathbb{Q}}} X_0;\mathbb{Q})$ such that $\xi = \tilde{\xi}_{\eta}$ in $\mathrm{CH}^r(X/\mathbb{C};\mathbb{Q})$, where η is the generic point of $S/\overline{\mathbb{Q}}$, with appropriate embedding $\overline{\mathbb{Q}}(\eta) \hookrightarrow \mathbb{C}$. Since $S(\overline{\mathbb{Q}}) \neq \emptyset$ (Nullstellensatz), we can choose $p \in S(\overline{\mathbb{Q}})$, and set $\xi_0 = \tilde{\xi}_p \in \mathrm{CH}^r(X_0;\mathbb{Q})$. Note that $\xi - \xi_0 \in \mathrm{CH}^r_{\mathrm{alg}}(X/\mathbb{C};\mathbb{Q})$. Now assume that $\xi \in F^2\mathrm{CH}^r(X;\mathbb{Q})$. Then $\xi_0 \in \mathrm{CH}^r_{\mathrm{hom}}(X_0;\mathbb{Q})$ and $AJ_X(\xi_0) = AJ_X(\xi_0 - \xi) \in J^r_{\mathrm{alg}}(X(\mathbb{C}))_{\mathbb{Q}}$, where $J^r_{\mathrm{alg}}(X(\mathbb{C})) := AJ_X(\mathrm{CH}^r_{\mathrm{alg}}(X/\mathbb{C}))$. Note that $J^r_{\mathrm{alg}}(X)$ has an underlying $\overline{\mathbb{Q}}$-structure given by $AJ_X(\mathrm{CH}^r(X_0/\overline{\mathbb{Q}}))$; moreover the action of $\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ is compatible with

$$AJ_X\big|_{\mathrm{CH}^r_{\mathrm{alg}}(X/\mathbb{C};\mathbb{Q})} : \mathrm{CH}^r_{\mathrm{alg}}(X/\mathbb{C};\mathbb{Q}) \twoheadrightarrow J^r_{\mathrm{alg}}(X(\mathbb{C}))_{\mathbb{Q}}.$$

For any $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$, we have

$$\begin{aligned}\sigma\big(AJ_X(\xi_0)\big) &= \sigma\big(AJ_X(\xi_0 - \xi)\big) = AJ_X((\xi_0-\xi)^\sigma)\\ &= AJ_X(\xi_0^\sigma - \xi^\sigma) = AJ_X(\xi_0^\sigma) = AJ_X(\xi_0),\end{aligned}$$

using $\xi^\sigma \in F^2\mathrm{CH}^r(X;\mathbb{Q}) \subset \mathrm{CH}^r_{AJ}(X;\mathbb{Q})$. Hence $AJ_X(\xi_0) \in J^r_{\mathrm{alg}}(X_0(\overline{\mathbb{Q}}))_{\mathbb{Q}}$, and so there exists $\xi_0' \in \mathrm{CH}^r_{\mathrm{alg}}(X_0/\overline{\mathbb{Q}};\mathbb{Q})$ such that $AJ(\xi_0) = AJ_X(\xi_0')$. By the BBC, $\xi_0 = \xi_0' \in \mathrm{CH}^r_{\mathrm{alg}}(X_0/\overline{\mathbb{Q}};\mathbb{Q})$. Thus $\xi \in \mathrm{CH}^r_{\mathrm{alg}}(X/\mathbb{C};\mathbb{Q})$. $\square$

Remark 4.8. Recall $X \in \mathcal{V}_{\mathbb{C}}$. As pointed out in [K-P] (Theorem 88), and based on a similar argument and result in [S], we have

$$\begin{aligned}&F^2\bigcap \mathrm{CH}^r_{\mathrm{alg}}(X;\mathbb{Q})\\ &\quad= \ker\big(AJ_X\big|_{\mathrm{CH}^r_{\mathrm{alg}}(X;\mathbb{Q})} : \mathrm{CH}^r_{\mathrm{alg}}(X;\mathbb{Q}) \to J\big(H^{2r-1}(X,\mathbb{Q}(r))\big)\big).\end{aligned}$$

[This really stems from the fact that $AJ_X(\mathrm{CH}^r_{\mathrm{alg}}(X))$ is an Abelian variety defined over the same field of definition as X.] Then with regard to the expression in (6), we have:

Corollary 4.9. (i)

$$\Gamma\big(H^3(\mathbb{C}(X),\mathbb{Q}(2))\big) = 0 \Rightarrow F^2\mathrm{CH}^2(X;\mathbb{Q}) = \mathrm{CH}^2_{AJ}(X;\mathbb{Q}).$$

(ii) *Conversely, if $B(X)$ holds and $D^2(X) \subset \mathrm{CH}^2_{\mathrm{alg}}(X;\mathbb{Q})$, then*

$$F^2\mathrm{CH}^2(X;\mathbb{Q}) = \mathrm{CH}^2_{AJ}(X;\mathbb{Q}) \Rightarrow \Gamma\big(H^3(\mathbb{C}(X),\mathbb{Q}(2))\big) = 0.$$

5. Some Evidence for Conjecture 3.2

Let $X \in \mathcal{V}_{\mathbb{C}}$ be given with $\dim X = d$. Recall that $\Gamma\big(H^{2d-1}(\mathbb{C}(X),\mathbb{Q}(d))\big) = 0$ for $d > 1$. Our next piece of evidence is an immediate consequence of (6) above.

Corollary 5.1. *Let $X \in \mathcal{V}_{\mathbb{C}}$ be given such that $\mathrm{CH}^2_{AJ}(X;\mathbb{Q}) \subset \mathrm{CH}^2_{\mathrm{alg}}(X;\mathbb{Q})$. Then $\Gamma\big(H^3(\mathbb{C}(X),\mathbb{Q}(2))\big) = 0$.*

Quite generally, if one considers (4) and Remark 3.4 above, then we deduce[5]:

Corollary 5.2. *Let $X \in \mathcal{V}_{\mathbb{C}}$, $\dim X = d$, $r > 1$ be given such that $\mathrm{CH}^r_{AJ}(X;\mathbb{Q}) \subset N^1\mathrm{CH}^r(X;\mathbb{Q})$. Let us further assume either (i) $d \leq 4$, or (ii) $r \in \{2, d-1\}$, or (iii) r, d arbitrary and the HC holds. Then $\Gamma\big(H^{2r-1}(\mathbb{C}(X),\mathbb{Q}(r))\big) = 0$.*

We also have:

Theorem 5.3. (i) *Let X be a smooth complete intersection of dimension d with $H^0(X,\Omega^d_X) = 0$. Assume $r > 1$ and that either (i) $d \leq 4$, or (ii) $r \in \{2, d-1\}$, or (iii) r, d arbitrary and the HC holds. Then $\Gamma\big(H^{2r-1}(\mathbb{C}(X),\mathbb{Q}(r))\big) = 0$.*

(ii) *Let X be a smooth complete intersection of dimension d. Let us assume the HC. Then for all $r > 1$ with $d \neq 2r-1$ and $D^r(X) \subset N^1\mathrm{CH}^r(X;\mathbb{Q})$, we have $\Gamma\big(H^{2r-1}(\mathbb{C}(X),\mathbb{Q}(r))\big) = 0$.*

Proof. Both parts rely on showing that $\mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) = N^1\mathrm{CH}^r(X;\mathbb{Q})$, using (4), and whatever is required to ensure that λ in (1) is surjective.

Part (i). According to [Ro], $\mathrm{CH}_0(X) \simeq \mathbb{Z}$. Thus by a standard diagonal argument due to J.-L. Colliot-Thélène/S. Bloch, we have

$$N \cdot \Delta_X \sim_{\mathrm{rat}} \Gamma_1 + \Gamma_2,$$

where $|\Gamma_1| \subset X \times D$, $|\Gamma_2| \subset p \times X$, $\mathrm{codim}_X D = 1$, $p \in X$ a point, for some $N \in \mathbb{N}$. Thus

$$\begin{aligned}\mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) &= N \cdot \Delta_{X,*}\mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) \\ &= \Gamma_{1,*}\mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) + \Gamma_{2,*}\mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) \subset N^1\mathrm{CH}^r(X;\mathbb{Q}).\end{aligned}$$

Then (i) follows from (4) and Remark 3.4.

[5]This can also be deduced from [K-L].

Part (ii). By the Lefschetz theorems, one can choose a decomposition of the diagonal class

$$\Delta_X = \bigoplus_{p+q=2d} \Delta_X(p,q), \quad \bigl[\Delta_X(p,q)\bigr] \in H^p(X,\mathbb{Q}) \otimes H^q(X,\mathbb{Q}),$$

such that $\Delta_X(p,q)_*\mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) \subset N^1\mathrm{CH}^r(X;\mathbb{Q})$ for $(p,q) \neq (d,d)$. So it suffices to show that $\Delta_X(d,d)_*\mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) \subset N^1\mathrm{CH}^r(X;\mathbb{Q})$ as well. But $d = 2r-\nu$ for some $\nu \in \mathbb{Z}$, and $\mathrm{Gr}_F^\nu \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) \simeq \Delta_X(d,d)_*\mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q})$. This is zero modulo $D^r(X)$ if $\nu \leq 0$. For $\nu \geq 2$, we apply Proposition 4.6. Finally the case $\nu = 1$ is excluded. □

6. Main Theorem

Theorem 6.1. *Consider these two statements:*

(i) *Conjecture 3.2 holds for all $X \in \mathcal{V}_{\mathbb{C}}$, (and all r).*

(ii) $F^2\mathrm{CH}^r(X;\mathbb{Q}) = \mathrm{CH}^r_{AJ}(X;\mathbb{Q})$ *for all $X \in \mathcal{V}_{\mathbb{C}}$, (and all r).*

If we assume the HC, then (i) $\Rightarrow$ (ii). If we further assume that $D^r(X) \subset N^1\mathrm{CH}^r(X;\mathbb{Q})$, then (ii) $\Rightarrow$ (i).

Remark 6.2. (1) Although statement (i) is no more accessible than (ii), the evidence in support of (i) is more apparent, in light of the results and remarks in [K-L], [Ja1], and the previous section.

(ii) The proof of this theorem relies only on the *properties* of the filtration in Theorem 4.1.

Proof. (of theorem) (ii) $\Rightarrow$ (i): Under the given assumptions and according to Proposition 4.6, $F^2\mathrm{CH}^r(X;\mathbb{Q}) \subset N^1\mathrm{CH}^r(X;\mathbb{Q})$. Thus (ii) $\Rightarrow$ (i) is immediate from (5). Thus we need only show that (i) $\Rightarrow$ (ii). Since $F^2\mathrm{CH}^r(X;\mathbb{Q}) \subset \ker(AJ_X)$, it suffices to prove the reverse inclusion $\ker(AJ_X) \subset F^2\mathrm{CH}^r(X;\mathbb{Q})$. Let $\xi \in \ker(AJ_X)$. Since we are assuming Conjecture 3.2, it follows from (5) that $\xi \in N^1\mathrm{CH}^r(X;\mathbb{Q})$. Thus ξ is homologous to zero on some pure codimension one algebraic subset $Y \subset X$. We need the following ingredient.

Lemma 6.3. *Let us assume the HC and let Y be a pure codimension one subvariety of a smooth projective variety X. Then there is a smooth* *variety* $\widetilde{Y}$ *of [pure]* $\dim \widetilde{Y} = \dim Y$*, and a morphism* $\widetilde{Y} \to Y$ *such that*

$$\mathrm{CH}^{\bullet}_{\mathrm{hom}}(\widetilde{Y};\mathbb{Q}) \twoheadrightarrow \mathrm{CH}^{\bullet}_{\mathrm{hom}}(Y;\mathbb{Q}),$$

is surjective.

Remark 6.4. (i) This lemma seems to be related to a statement in Remark 5.13 in [Ja1]. More precisely, and in our notation, is the following statement:

If $f : \tilde{Z} \to Z$ is a surjective, generically finite morphism of irreducible projective varieties, with $\tilde{Z}$ smooth, then $f_* : \mathrm{CH}^{\bullet}_{\mathrm{hom}}(\tilde{Z};\mathbb{Q}) \twoheadrightarrow \mathrm{CH}^{\bullet}_{\mathrm{hom}}(Z;\mathbb{Q})$ is surjective.

From a conjectural standpoint, we expect that this statement is true.

(ii) The assumption that Y has codimension one in the lemma is only used to simplify the proof. We leave it to the reader to generalize the statement of the lemma for arbitrary codimension Y; one possibility being aforementioned statement in (i) above, under the assumption of the HC.

Proof. (of the lemma.) Let $\rho_X : X' \xrightarrow{\approx} X$ be a proper modification of X for which $Y' := \rho_X^{-1}(Y)$ is a NCD, with inclusions $j : Y \hookrightarrow X$, $j' : Y' \hookrightarrow X'$, and morphism $\rho_Y := \rho_X\big|_{Y'}$, and where $X'\backslash Y' \simeq X\backslash Y$. This observation, together with the localization sequences associated to j and j' and the cohomology of blow-ups, leads to the commutative diagram:

$$\begin{tikzcd}
 & \mathrm{CH}^r_{Y'}(X';\mathbb{Q}) \arrow[r, two heads, "\rho_Y"] \arrow[d] & \mathrm{CH}^r_{Y}(X,;\mathbb{Q}) \arrow[d] \\
\ker \rho_{Y,*} \arrow[r, hook] \arrow[d, equal, "\wr"] & H^{2r}_{Y'}(X',\mathbb{Q}) \arrow[r, "\rho_{Y,*}"] \arrow[d] & H^{2r}_{Y}(X,\mathbb{Q}) \arrow[d] \\
\ker \rho_{X,*} \arrow[r, hook] & H^{2r}(X',\mathbb{Q}) \arrow[r, "\rho_{X,*}"] & H^{2r}(X,\mathbb{Q})
\end{tikzcd} \tag{7}$$

Now let $\xi_0 \in \mathrm{CH}^{r-1}_{\mathrm{hom}}(Y;\mathbb{Q})$. Then using $X\backslash Y \simeq X'\backslash Y'$ together with the localization sequence for Chow groups associated to the pairs (X',Y') and (X,Y), there exists $\xi_1 \in \mathrm{CH}^{r-1}(Y';\mathbb{Q})$ for which $j'_*(\xi_1) = \rho_X^*(j_*(\xi_0))$ and $\rho_{Y,*}(\xi_1) = \xi_0$. This is accomplished with the aid of the diagram below.

$$
\begin{array}{cccccc}
 & \mathrm{CH}^r(X'\backslash Y',1;\mathbb{Q}) & === & \mathrm{CH}^r(X\backslash Y,1;\mathbb{Q}) & \\
 & \Big\downarrow & & \Big\downarrow & \\
\xi_1 \in & \mathrm{CH}^{r-1}(Y';\mathbb{Q}) & \xrightarrow{\rho_{Y,*}} & \mathrm{CH}^{r-1}(Y;\mathbb{Q}) & \ni \xi_0 \\
\downarrow & j'_*\Big\downarrow & & \Big\downarrow j_* & \downarrow \\
\rho_X^*(j_*(\xi_0)) \in & \mathrm{CH}^r(X';\mathbb{Q}) & \begin{array}{c}\xrightarrow{\rho_{X,*}} \\ \xleftarrow{\rho_X^*}\end{array} & \mathrm{CH}^r(X;\mathbb{Q}) & \ni j_*(\xi_0) \\
\downarrow & \Big\downarrow & & \Big\downarrow & \downarrow \\
0 \in & \mathrm{CH}^r(X'\backslash Y';\mathbb{Q}) & === & \mathrm{CH}^r(X\backslash Y;\mathbb{Q}) & \ni 0
\end{array}
$$

Note that

$$\mathrm{CH}^r(X';\mathbb{Q}) = \rho_X^*\mathrm{CH}^r(X;\mathbb{Q}) \bigoplus \ker\big\{\rho_{X,*} : \mathrm{CH}^r(X';\mathbb{Q}) \to \mathrm{CH}^r(X;\mathbb{Q})\big\}.$$

$$H^{2r}(X',\mathbb{Q}) = \rho_X^* H^{2r}(X,\mathbb{Q}) \bigoplus \ker \rho_{X,*}.$$

Then on cohomology $[\xi_1] \in \ker \rho_{Y,*}$ in (7), and yet by construction $[\xi_1] \mapsto 0 \in \ker \rho_{X,*}$. Thus by diagram (7), $\xi_1 \in \mathrm{CH}^{r-1}_{\mathrm{hom}}(Y';\mathbb{Q})$ and hence $\rho_{Y,*} : \mathrm{CH}^{r-1}_{\mathrm{hom}}(Y';\mathbb{Q}) \twoheadrightarrow \mathrm{CH}^{r-1}_{\mathrm{hom}}Y;\mathbb{Q})$ is surjective[6] for all r. Write $Y' = \bigcup_{i=1}^N Y'_j$,

[6]Quite generally, this result can be deduced from the s.e.s. $0 \to \mathrm{CH}^r_Y(X,m;\mathbb{Q}) \to \mathrm{CH}^r_{Y'}(X',m;\mathbb{Q}) \bigoplus \mathrm{CH}^r(X,m;\mathbb{Q}) \to \mathrm{CH}^r(X',m;\mathbb{Q}) \to 0$, together with a corresponding s.e.s. on cohomology, given in [Lew2]. In this generalization, Y is any proper closed subset of X, with Y' a NCD in X'.

$Y'_{[1]} = \coprod_{i=1}^{N} Y_j$. For $I = \{i_1 < \cdots < i_\ell\}$, put $Y'_I = \bigcap_{j=1}^{\ell} Y'_{i_j}$, $Y'_{[\ell]} = \coprod_{|I|=\ell} Y'_I$. From the simplicial complex $Y'_{[\bullet]} \to Y'$, we arrive at:

$$\mathrm{CH}^r(Y') \simeq \frac{z^r(Y'_{[1]})}{\mathrm{Gy}\big(z^{r-1}(Y'_{[2]})\big) + z^r_{\mathrm{rat}}(Y'_{[1]})} \simeq \frac{\mathrm{CH}^r(Y'_{[1]})}{\mathrm{Gy}\big(\mathrm{CH}^{r-1}(Y'_{[2]})\big)},$$

where Gy is the (signed) Gysin map. Further, relating this to a corresponding cohomological complex, together with the HC, we arrive at:

$$\frac{\mathrm{CH}^r(Y';\mathbb{Q})}{\mathrm{CH}^r_{\mathrm{hom}}(Y';\mathbb{Q})} \simeq \frac{H^{2r}_{\mathrm{alg}}(Y'_{[1]};\mathbb{Q})}{\mathrm{Gy}\big(H^{2r-2}_{\mathrm{alg}}(Y'_{[2]},\mathbb{Q})\big)},$$

where $H^{2p}_{\mathrm{alg}}(W,\mathbb{Q}) \subset H^{2p}(W,\mathbb{Q})$ is the subspace of of algebraic cocycles, for $W \in \mathcal{V}_{\mathbb{C}}$. Now put $\widetilde{Y} = Y'_{[1]}$. With the aid of the diagram,

$$\begin{array}{ccc}
\mathrm{CH}^{r-1}(Y'_{[2]};\mathbb{Q}) & \twoheadrightarrow & H^{2r-2}_{\mathrm{alg}}(Y'_{[2]},\mathbb{Q}) \\
\mathrm{Gy}\downarrow & & \downarrow\mathrm{Gy} \\
\mathrm{CH}^r(Y'_{[1]};\mathbb{Q}) & \twoheadrightarrow & H^{2r}_{\mathrm{alg}}(Y'_{[1]},\mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathrm{CH}^r(Y';\mathbb{Q}) & \twoheadrightarrow & H^{2r}_{\mathrm{alg}}(Y'_{[1]},\mathbb{Q})/\mathrm{Gy}\big(H^{2r-2}_{\mathrm{alg}}(Y'_{[2]},\mathbb{Q})\big),
\end{array}$$

it follows that the induced proper pushforward $\mathrm{CH}^\bullet_{\mathrm{hom}}(\widetilde{Y};\mathbb{Q}) \twoheadrightarrow \mathrm{CH}^\bullet_{\mathrm{hom}}(Y;\mathbb{Q})$ is surjective. $\square$

Now returning to the proof of Theorem 6.1, consider the composite map $\sigma : \widetilde{Y} \to Y \hookrightarrow X$. By the lemma there exists $\xi_0 \in F^1\mathrm{CH}^{r-1}(\widetilde{Y};\mathbb{Q})$ for which $\sigma_*(\xi_0) = \xi \in \ker(AJ_X)$. The graph of σ determines a fundamental class $[\sigma] \in H^{2d}(\widetilde{Y} \times X, \mathbb{Q})$, where $d = \dim X$. Let $[\sigma]_0 \in H^{2d-2r+1}(\widetilde{Y},\mathbb{Q}) \otimes H^{2r-1}(X,\mathbb{Q})$ be the corresponding Künneth component. Let σ_0 be any algebraic cycle with $[\sigma_0] = [\sigma]_0$. Then as a class $[\ ]_1 \in \mathrm{Gr}^1_F\mathrm{CH}^\bullet$, $[\xi]_1 = [\sigma_{0,*}(\xi_0)]_1$. In particular

$$\sigma_{0,*}(\xi_0) - \xi \in F^2\mathrm{CH}^r(X;\mathbb{Q}).$$

The key issue is the *choice* of representative σ_0 of $[\sigma]_0$. Choose a subHS $V \subset H^{2r-3}(\widetilde{Y},\mathbb{Q})$ such that

$$[\sigma]_{0,*}\big|_V : V \xrightarrow{\sim} [\sigma]_{0,*}\big(H^{2r-3}(\widetilde{Y},\mathbb{Q})\big) \subset H^{2r-1}(X,\mathbb{Q}),$$

is an isomorphism. By the HC, there exists $w \in \mathrm{CH}^{d-1}(X \times \widetilde{Y};\mathbb{Q})$, with $[w] \in \big\{[\sigma]_{0,*}\big(H^{2r-3}(\widetilde{Y},\mathbb{Q})\big)\big\}^\vee \otimes V$, where $\big\{[\sigma]_{0,*}\big(H^{2r-3}(\widetilde{Y},\mathbb{Q})\big)\big\}^\vee \otimes V$ is a subquotient of $H^{2d-2r+1}(X,\mathbb{Q}) \otimes H^{2r-3}(\widetilde{Y},\mathbb{Q})$, (which we can regard as an inclusion by semi-simplicity of polarized Hodge structures over $\mathbb{Q}$), such that

$$[\sigma]_{0,*} \circ [w]_*\big|_{\mathrm{Im}([\sigma]_{0,*})} = \mathrm{Id}_{\mathrm{Im}([\sigma]_{0,*})}.$$

Now by construction, $\sigma_{0,*} \circ w_* \circ \sigma_*(\xi_0) = \sigma_{0,*} \circ w_*(\xi)$; moreover $w_*(\xi) \in \ker(AJ_{\widetilde{Y}})$ by functoriality of the Abel-Jacobi map. By induction on dimension, $\ker(AJ_{\widetilde{Y}}) = F^2\mathrm{CH}^{r-1}(\widetilde{Y};\mathbb{Q})$. Since $\xi - \sigma_{0,*} \circ w_*(\xi) \in F^2\mathrm{CH}^r(X;\mathbb{Q})$, and $\sigma_{0,*} \circ w_*(\xi) \in F^2\mathrm{CH}^r(X;\mathbb{Q})$, it follows that $\xi \in F^2\mathrm{CH}^r(X;\mathbb{Q})$. $\square$

References

[Ar] Arapura, D., The Leray spectral sequence is motivic, Invent. Math. **160** (2005), 567–589.

[Be] Beilinson, A., Height pairings between algebraic cycles, Springer Lecture Notes **1289**, (1987), 1–26.

[B] Bloch, S., Algebraic cycles and higher K-theory, Advances in Math. **61** (1986), 267–304.

[dJ-L] de Jeu, R. and J. D. Lewis, Work in progress.

[Ja1] Jannsen, U., Equivalence relations on algebraic cycles, in Proceedings of the NATO Advanced Study Institute on the Arithmetic and Geometry of Algebraic Cycles Vol. **548**, (Lewis, Yui, Gordon, Müller-Stach, S. Saito, eds.), Kluwer Academic Publishers, Dordrecht, The Netherlands, (2000), 225–260.

[Ja2] Jannsen, U., Mixed Motives and Algebraic K-Theory, Lecture Notes in Mathematics **1400**, (1988).

[K-L] Kang, S.-J. and J. D. Lewis, Beilinson's Hodge conjecture for K_1 revisited, To appear in the Proceedings of the International Colloquium on Cycles, Motives and Shimura Varieties (Mumbai, 2008).

[Ke-L] Kerr, M. and J. D. Lewis, The Abel-Jacobi map for higher Chow groups II, Invent. Math. **170** (2), (2007), 355–420.

[K-P] Kerr, M. and G. Pearlstein, An exponential history of functions with logarithmic growth, Proceedings of the MSRI workshop on the topology of stratified spaces, September 2008.

[Lew1] Lewis, J. D., A filtration on the Chow group of a complex projective variety, Compositio Math. **128** (2001), 299–322.

[Lew2] Lewis, J. D., (Co-)Homological models for higher Chow groups, Preprint.

[Ro] Roitman, A. A., The torsion group of 0-cycles modulo rational equivalence, Ann. of Math. **111** (1980), 553–569.

[S] Saito, S., Motives and filtrations on Chow groups, Invent. Math. **125** (1996), 149–196.

[MSa] Saito, M., Hodge-type conjecture for higher Chow groups, arXiv:math. AG/0201113 v4.

(Received December 10, 2009)
(Revised July 7, 2010)

632 Central Academic Building
University of Alberta
Edmonton, Alberta T6G 2G1, CANADA
E-mail: lewisjd@ualberta.ca