J. Math. Sci. Univ. Tokyo
17 (2010), 123–133.

The Generalized Gluškov-Iwasawa Local Splitting Theorem

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Abstract. In this paper we obtain an extension of Gluškov-Iwasawa local splitting theorem for locally compact groups as the product of a compact group and a local Lie group, to inverse limits of finite dimensional Lie groups.

For any topological group G let G_0 be its identity component and call G almost connected if G/G_0 is compact. If G is a locally compact group which is either connected in [8, Theorem 11] or almost connected in [4, Theorem A] it is proved there that for every $1 \in U$ open $\subseteq G$, there exists a compact normal $N \leq G$, $N \subseteq U$ such that G/N Lie group and $N \times G/N$ is locally isomorphic to G. We call this result the Gluškov-Iwasawa local splitting theorem for locally compact groups . We wish to generalize this result to almost connected projective Lie groups, i.e. inverse limits of finite dimensional Lie groups or equivalently complete Hausdorff topological groups such that every neighborhood of the identity element of which contains a closed normal subgroup that gives rise to a finite dimensional Lie quotient group [9]. An almost connected locally compact group is a projective Lie group [10, p.175], so our result generalizes the Gluškov-Iwasawa local splitting theorem for locally compact groups and we call this generalization the generalized Gluškov-Iwasawa local splitting theorem. We obtain a sufficient criterion for the validity of the generalized Gluškov-Iwasawa local splitting theorem which extends the condition that is automatically satisfied in the locally compact case. This generalization has also been considered in [7]. We show that our sufficient condition is equivalent to the one obtained in [7, Theorem 3.2] but the proofs we give are different and provide a vast simplification of the ones in [7]. We remark that the proof of [7, Theorem 3.2] gives the stated assertion only for connected groups so our Theorem 2.2 actually

2000 *Mathematics Subject Classification*. 22A05.
Key words: Gluškov-Iwasawa local splitting theorem, inverse limit of finite dimensional Lie groups, nilpotent radical of a Lie algebra.

extends the proof of [7, Theorem 3.2] to the almost connected case. Our approach gives a new proof of the Gluškov-Iwasawa local splitting theorem for locally compact groups. Note that, as observed in [6, p.600, Example 14.23], the Gluškov-Iwasawa local splitting theorem does not apply even to connected projective Lie groups in general.

The paper is divided into three sections. In section 1, we state our criterion and show that it is equivalent to the one in [7, Theorem 3.2]. In section 2, we prove that our criterion yields the generalized Gluškov-Iwasawa local splitting theorem in Theorem 2.2. In section 3, we derive the classical Gluškov-Iwasawa local splitting theorem for locally compact groups from our generalized version in Theorem 3.2.

1. The Sufficiency Criterion

We shall need the following Lemma.

Lemma 1.1.

Let L_1, L_2 be two finite dimensional Lie algebras over K, a field of characteristic 0, and let s_1, s_2 (resp. d_1, d_2) be the nilpotent radicals (resp. ideals) of L_1, L_2 respectively such that $s_i \subseteq d_i$ for $i = 1, 2$. Then for any surjective Lie homomorphism $\varphi : L_1 \to L_2$ such that $\varphi(d_1) = d_2$ the following statements hold:

1) *There exists a Lie homomorphism $a : L_2 \to L_1/(d_1 \cap \ker\varphi)$ such that $a(L_2)$ is an ideal in $L_1/(d_1 \cap \ker\varphi)$ and $\varphi' \circ a = 1$ where $\varphi' : L_1/(d_1 \cap \ker\varphi) \to L_2$ is the Lie homomorphism induced by φ.*

2) *Suppose further that there exists a group $G \leq \mathrm{Aut}(L_1, \varphi) = \{m \in \mathrm{Aut}(L_1) : \varphi \circ m = \varphi\}$ and H normal $\leq G, G/H$ finite such that*

i. *$\theta(d_1) = d_1$ for all $\theta \in G$*

ii. *the homomorphism a of part 1) satisfies*

$a(L_2) \subseteq (L_1/(d_1 \cap \ker\varphi))^H = \{\bar{x} \in L_1/(d_1 \cap \ker\varphi) : h(\bar{x}) = \bar{x}$ for all $h \in H\}$, then the homomorphism a of part 1) can further be chosen such that $a(L_2) \subseteq (L_1/(d_1 \cap \ker\varphi))^G$.

Proof.
We have the following commutative diagram of Lie homomorphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & d_1 & \longrightarrow & L_1 & \xrightarrow{c_1} & L_1/d_1 & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow \varphi & & \big\downarrow \bar{\varphi} & & \\ 0 & \longrightarrow & d_2 & \longrightarrow & L_2 & \xrightarrow{c_2} & L_2/d_2 & \longrightarrow & 0 \end{array}$$

where the vertical maps are surjections, c_1, c_2 are the canonical maps and $\bar{\varphi}$ is induced by φ.

Note that since L_i/d_i reductive for $i = 1, 2$, the homomorphism $\bar{\varphi}$ is a projection onto a direct factor [3, p.57, Corollary], hence there exists $\alpha : L_2/d_2 \to L_1/d_1$ Lie homomorphism such that $\bar{\varphi} \circ \alpha = 1$, $L_1/d_1 = \alpha(L_2/d_2) \oplus \ker \bar{\varphi}$ and $\alpha(L_2/d_2)$ ideal in L_1/d_1.

1) The above commutative diagram gives $c_2^{-1} \circ \bar{\varphi}(\bar{x}) = \varphi \circ c_1^{-1}(\bar{x})$ for all $\bar{x} \in L_1/d_1$. It follows that for all $m \in L_2$, we have $c_2^{-1} \circ \bar{\varphi}(\alpha \circ c_2(m)) = \varphi \circ c_1^{-1}(\alpha \circ c_2(m))$ and $m + d_2 = \varphi \circ c_1^{-1}(\alpha \circ c_2(m))$, hence $\varphi^{-1}(m) \cap c_1^{-1}(\alpha \circ c_2(m)) = x_m + (\ker \varphi \cap d_1)$ for some $x_m \in L_1$. We get a function $a : L_2 \to L_1/(\ker \varphi \cap d_1)$ defined by $a(m) = x_m + (\ker \varphi \cap d_1)$ for all $m \in L_2$. It is immediate that a is a Lie homomorphism and that $\varphi' \circ a = 1$. Finally, note that for $m \in L_2$, $\bar{k} \in \ker \varphi/(\ker \varphi \cap d_1)$ and if $c_1' : L_1/(\ker \varphi \cap d_1) \to L_1/d_1$ is the canonical map, then $c_1'([a(m), \bar{k}]) = [\alpha \circ c_2(m), c_1(k)] = 0$ since $c_1(k) \in \ker \bar{\varphi}$ so that $[a(m), \bar{k}] \in (d_1/d_1 \cap \ker \varphi) \cap (\ker \varphi/d_1 \cap \ker \varphi) = 0$ and $L_1/(d_1 \cap \ker \varphi) = a(L_2) \oplus \ker \varphi/(d_1 \cap \ker \varphi)$, $a(L_2)$ ideal in $L_1/(d_1 \cap \ker \varphi)$.

2) Assumption i. and the fact that $m(\ker \varphi) = \ker \varphi$ all $m \in \mathrm{Aut}(L_1, \varphi)$ shows that G acts by automorphisms on $L_1/(\ker \varphi \cap d_1)$ and on L_1/d_1. Therefore $\bar{\varphi} \circ g = \bar{\varphi}$ for all $g \in G$. Note that if $g \in G$, $x \in L_2$, then

$$\begin{aligned} 0 = \bar{\varphi}(g(c_1' \circ a(x)) \quad &\Leftrightarrow \quad 0 = \bar{\varphi}(c_1' \circ a(x)) \\ &\Rightarrow \quad c_1' \circ a(x) = \alpha \circ c_2(x) \in \alpha(L_2/d_2) \cap \ker \bar{\varphi} = 0 \end{aligned}$$

hence $g(c_1' \circ a(L_2)) \cap \ker \bar{\varphi} = g(\alpha(L_2/d_2)) \cap \ker \bar{\varphi} = 0$ for all $g \in G$. Observe that $(L_1/d_1, L_1/d_1) = \alpha(L_2/d_2) \cap (L_1/d_1, L_1/d_1) \oplus \ker \bar{\varphi} \cap (L_1/d_1, L_1/d_1)$ and since $g(\alpha(L_2/d_2)) \cap \ker \bar{\varphi} = 0$ for all $g \in G$, the semi simplicity of $(L_1/d_1, L_1/d_1)$ shows that $g(\alpha(L_2/d_2) \cap (L_1/d_1, L_1/d_1)) = \alpha(L_1/d_2) \cap (L_1/d_1, L_1/d_1)$ for all $g \in G$ [3, p.53, Corollary 1].

Observe also that since $a(L_2) \subseteq (L_1/(d_1 \cap \ker \varphi))^H$ we have $\alpha(L_2/d_2) = c_1' \circ a(L_2) \subseteq (L_1/d_1)^h$, hence $(Z(L_1/d_1))^H \alpha(L_2/d_2) \cap Z(L_1/d_1) \oplus \ker \bar{\varphi} \cap$

$(Z(L_1/d_1))^H$ where $Z(L_1/d_1) =$ Center of L_1/d_1. Therefore the finite group G/H acts linearly on $(Z(L_1/d_1))^H$ and $\ker\bar{\varphi} \cap (Z(L_1/d_1))^H$ is a G/H-stable subspace, hence $(Z(L_1/d_1))^H = (M_Z/d_1)) \oplus \ker\bar{\varphi} \cap (Z(L_1/d_1))^H$ where M_Z/d_1 is a G/H-stable subspace [5, p.22, Theorem 2.2]. We have $\dim M_Z/d_1 = \dim \alpha(L_2/d_2) \cap Z(L_1/d_1)$ and $I = (M_Z/d_1) \oplus \alpha(L_2/d_2) \cap (L_1/d_1, L_1/d_1)$ is an ideal of L_1/d_1. Note that $\dim I = \dim \alpha(L_2/d_2)$ and $I \cap \ker\bar{\varphi} = 0$, hence $\bar{\varphi}$ restricts to an isomorphism $I \to L_2/d_2$. Let α' be the inverse of this isomorphism, then $\alpha'(L_2/d_2)$ is an ideal of L_1/d_1 such that $g(\alpha'(L_2/d_2)) = \alpha'(L_2/d_2)$ for all $g \in G$. Replacing the homomorphism α by α', the construction of part 1) shows that we can get the homomorphism a to satisfy the further condition that $a(L_2) \subseteq (L_1/(d_1 \cap \ker\varphi))^G$ as desired. □

In the following proposition we propose a criterion on a projective Lie group and show that it is equivalent to the one in [7, Theorem 3.2]. For any finite dimensional Lie group G, we denote by $L(G)$ its Lie algebra.

Proposition 1.2.

Let G be a projective Lie group, then the following two statements are equivalent:

i. *there exists a filter base $\{\ker p_i : i \in I\}$ of normal subgroups of G such that $G/\ker p_i$ Lie group for all $i \in I$ that converge to $1 \in G$ and $\sup_i \dim s_i < \infty$, where $s_i =$ nilpotent radical of $L(G/\ker p_i)$.*

ii. *there exists a filter base $\{kerp_i : i \in I\}$ of normal subgroups of G such that $G/\ker p_i$ Lie group for all $i \in I$ that converge to $1 \in G$ and $\sup_i \dim n_i/z_i < \infty$, where $n_i =$ largest nilpotent ideal of $L(G/\ker p_i)$ and $z_i =$ center of $L(G/\ker p_i)$.*

Proof.

Claim. Let L_i, $i = 1, 2$ be finite dimensional Lie algebras over a field K of characteristic 0 and let s_i (resp. n_i, resp. z_i) for $i = 1, 2$ be their respective nilpotent radicals (resp. largest nilpotent ideals, resp. centers) such that $\dim(s_1/(s_1 \cap z_1)) = \dim(s_2/(s_2 \cap z_2))$ and let $\varphi : L_1 \to L_2$ be a surjective Lie homomorphism, then $\varphi(n_1) = n_2$.

Proof of Claim.

Let r_i, $i = 1, 2$ be the radicals of L_i, $i = 1, 2$ respectively. We have

$$n_2 = \{\varphi(x) \in \varphi(r_1) : (\mathrm{ad}_{\varphi(r_1)} \varphi(x))^n = 0 \text{ for some } n \geq 1\}$$

$$\begin{aligned} & \quad [3, \text{p.47, Corollary 7}] + [3, \text{p.53, Corollary 2}] \\ = \; & \varphi(\{x \in r_1 : (\operatorname{ad} x)^n(r_1) \subseteq \ker \varphi \text{ for some } n \geq 1\}) \\ = \; & \varphi(\{x \in r_1 : (\operatorname{ad} x)^n(r_1) \subseteq s_1 \cap \ker \varphi \subseteq s_1 \cap z_1 \text{ for some } n \geq 1\}) \\ \subseteq \; & \varphi(\{x \in r_1 : \operatorname{ad} x)^{n+1}(r_1) = 0 \text{ for some } n \geq 1\}) \\ \subseteq \; & \varphi(n_1), \end{aligned}$$

hence we have equality. □

For $i, j \in I$, $j > i$ let $\varphi_{ij} : G/\ker p_j \to G/\ker p_i$ be the canonical homomorphism, then φ_{ij} induces a surjective Lie homomorphism $(\varphi_{ij})_\# : L(G/\ker p_j) \to L(G/\ker p_i)$.

i. ⇒ ii. Since $\dim s_i/s_i \cap z_i \leq \dim s_i$, we may assume that $\dim s_i = K$, $\dim s_i/s_i \cap z_i = L$ for all $i \in I$. For all $i, j \in I$, $i < j$, $\ker(\varphi_{ij})_\# \cap s_j = 0$.

Therefore,

$$\begin{aligned} x \quad \in \quad & n_j \cap ((\varphi_{ij})_\#)^{-1}(z_i) \\ \Rightarrow \quad & [x, L(G/\ker p_j)] \subseteq \ker(\varphi_{ij})_\# \cap s_j = 0 \\ \Rightarrow \quad & x \in z_j, \end{aligned}$$

hence $z_j = n_j \cap ((\varphi_{ij})_\#)^{-1})(z_i)$. By the above claim $(\varphi_{ij})_\#(n_j) = n_i$, hence $(\varphi_{ij})_\#$ induces an isomorphism $n_j/z_j \to n_i/z_i$.

ii. ⇒ i. Since $\dim s_j/s_j \cap z_j \leq \dim n_j/z_j$, we may assume that $\dim s_j/s_j \cap z_j = L$ for all $j \in I$. Choose $i \in I$ such that $\dim n_i/z_i = \sup_{j \in I} \dim n_j/z_j < \infty$.

For all $i < j \in I$ the above claim gives $(\varphi_{ij})_\#(n_j) = n_j$, hence $z_j = n_j \in ((\varphi_{ij})_\#)^{-1}(z_i)$. Apply Lemma 1.1, part 1) with the substitution $d_1 \to n_j/z_j$, $d_2 \to n_i/z_i$, $L_1 \to L(G/\ker p_j)/z_j$, $L_2 \to L(G/\ker p_i)/z_i$, we get an injective Lie homomorphism

$$\begin{aligned} a_j : L(G/\ker p_i)/z_i \quad \to \quad & L(G/\ker p_j)/n_j \cap (\varphi_{ij})_\#)^{-1}(z_i) \\ = \quad & L(G/\ker p_j)/z_j \end{aligned}$$

such that $L(G/\ker p_j)/z_j = M_j/z_j \oplus ((\varphi_{ij})_\#)^{-1}(z_i)/z_j$ hence $((\varphi_{ij})_\#)^{-1}(z_j)/z_j$ has no nilpotent ideals and therefore is semi simple and the radical of $L(G/\ker p_j) = r_j \subseteq M_j$. We have the radical of $M_j = r(M_j) = r_j$ and therefore the nilpotent radical of $M_j = s(M_j) = [r_j, M_j] \subseteq s_j$.

Observe that $L(G/\ker p_j) = M_j + ((\varphi_{ij})_\#)^{-1}(z_i)$ and if $x \in r_j$, then $y \in M_j \Rightarrow [x,y] \in s(M_j)$ and since $((\varphi_{ij})_\#)^{-1}(z_i)/z_j$ is semi simple $v, w \in ((\varphi_{ij})_\#)^{-1}(z_i) \Rightarrow [x,[v,w]] = 0$ since $[r_j, ((\varphi_{ij})_\#)^{-1}(z_i)] \subseteq z_j$ so that radical of

$$\begin{aligned} L(G/\ker p_j)/s(M_j) &= r_j/s(M_j) \\ &= Z(L(G/\ker p_j)/s(M_j)) \\ &= \text{center of } L(G/\ker p_j)/s(M_j) \end{aligned}$$

and $L(G/\ker p_j)/s(M_j)$ is reductive, hence $s_j = s(M_j)$.

Let $M_j = z_j \oplus P$ where P is a real vector space of dimension $= \dim(M_j/z_j) = \dim(L(G/\ker p_i)/z_i)$ and let $\{x_1, x_2, \ldots, x_{\dim L(G/\ker p_i)/z_i}\}$ be a basis of P. We get

$$\begin{aligned} \dim s_j &= \dim s(M_j) \\ &\leq \dim[M_j, M_j] \\ &\leq \dim_{1\leq k<l\leq \dim(L(G/\ker p_i)/z_i} R[x_k, x_l] \\ &\leq \binom{\dim L(G/\ker p_i)/z_i}{2}. \ \square \end{aligned}$$

2. The Generalized Gluškov-Iwasawa Local Splitting Theorem

In this section we show that the criterion given in Proposition 1.2 is sufficient to establish the generalized Gluškov-Iwasawa local splitting theorem in Theorem 2.2. We remark that the proof of [7, Theorem 3.2] gives the stated assertion only for connected groups so our Theorem 2.2 actually extends the proof of [7, Theorem 3.2] to the almost connected case

Theorem 2.1.

Let G be an almost connected projective Lie group and let $\{\ker p_i : i \in I\}$ be a filter base of normal subgroups of G such that $G/\ker p_i$ Lie group for all $i \in I$ that converges to $1 \in G$ and $\dim s_i = K < \infty$ for all $i \in I$, where $s_i =$ nilpotent radical of $L(G/\ker p_i)$. Fix $i \in I$ and let $\tilde{G}_i$ be the universal cover of $(G/\ker p_i)_0$, then there exists a continuous homomorphism $a_i : \tilde{G}_i \to G$ such that the composition $\tilde{G}_i \xrightarrow{a_i} G \xrightarrow{p_i} G/\ker p_i$ is the canonical map $c_i : \tilde{G}_i \to (G/\ker p_i)_0 \to G/\ker p_i$ and $(a_i(\tilde{G}_i), \ker p_i) = 1$.

Proof.

Fix $i \in I$, for $i < j \in I$ let $\varphi_{ij} : G/\ker p_j \to G/\ker p_i$ be the canonical homomorphism, then φ_{ij} induces a surjective Lie homomorphism $(\varphi_{ij})_\# : L(G/\ker p_j) \to L(G/\ker p_i)$ and define $A_j = \{\sigma \in \mathrm{Hom}(L(G/\ker p_i), L(G/\ker p_j)) : (\varphi_{ij})_\# \circ \sigma = 1,\ \sigma(L(G/\ker p_i))$ ideal in $L(G/\ker p_j)$, $\sigma(L(G/\ker p_i)) \leq (L(G/\ker p_j))^{Ad(\ker p_i/\ker p_j)}\}$. Note that $(A_j, (\varphi_{jk})_\#)$ is an inverse system of sets.

Claim. $A_j \neq \emptyset$ for all $i < j \in I$.

Proof of Claim. By lemma 1.1 part 1), there exists $\sigma \in \mathrm{Hom}(L(G/\ker p_i), L(G/\ker p_j)), (\varphi_{ij})_\# \circ \sigma = 1$, $\sigma(L(G/\ker p_i))$ ideal in $L(G/\ker p_j)$. Let $m_j : \tilde{G} \to G/\ker p_j$ be the continuous homomorphism such that $(m_j)_\# = \sigma$, $\varphi_{ij} \circ m_j = c_i$ (where $(c_i)_\# = 1$) [3, p.305, Theorem 1], then $m_j(\tilde{G}_i)$ normal $\leq (G/\ker p_j)_0$ [3, p.307, Proposition 1] + [3, p.316, Proposition 14]. Therefore

$$\begin{aligned} 1 &\in (m_j(\tilde{G}_i), (\ker p_i/\ker p_j) \cap (G/\ker p_j)_0) \\ &\subseteq m_j(\tilde{G}_i) \cap (\ker p_i/\ker p_j) \\ &\subseteq m_j(\ker c_i) \end{aligned}$$

and since $\ker c_i$ discrete $\leq \tilde{G}_i$, we get $\ker c_i \leq Z(\tilde{G}_i)$ and $\ker c_i$ is a finitely generated abelian group [11, Theorem 3.1]. Hence $m_j(\tilde{G}_i) \cap (\ker p_i/\ker p_j)$ is a countable Hausdorff topological group and therefore totally disconnected and $(m_j(\tilde{G}_i), (\ker p_i/\ker p_j) \cap (G/\ker p_j)_0) = 1$. It follows that $\sigma(L(G/\ker p_i)) \leq (L(G/\ker p_j))^{Ad((\ker p_i/\ker p_j)\cap(G/\ker p_j)_0)}$. Since $(\ker p_i/\ker p_j)/(\ker p_i/\ker p_j) \cap (G/\ker p_j)_0 \leq (G/\ker p_j)/(G/\ker p_j)_0$ finite, Lemma 1.1 part 2) shows that $A_j \neq \emptyset$ as desired. □

We show that $\varprojlim A_j \neq \emptyset$.

Let $S_j = \{\emptyset, (\sigma + M) \cap \{a \in \mathrm{Hom}(L(G/\ker p_i), L(G/\ker p_j)) : a(L(G/\ker p_i))$ ideal in $L(G/\ker p_j), a(L(G/\ker p_i)) \leq (L(G/\ker p_j))^{Ad(\ker p_i/\ker p_j)}\} : \sigma \in A_j,\ M \leq \mathrm{Hom}(G/\ker p_i), L(G/\ker p_j))\}$ for all $i < j \in I$, then these sets satisfy all the conditions of [1, p.198, Theorem 1]. Condition ii) is the only condition that needs to be verified, so suppose that $\{(y_a + M_a) \cap \{a \in \mathrm{Hom}(L(G/\ker p_i), L(G/\ker p_j)) : a(L(G/\ker p_i))$ ideal in $L(G/\ker p_j), a(L(G/\ker p_i)) \leq (L(G/\ker p_j))^{Ad(\ker p_i/\ker p_j)}\} : \alpha \in \Gamma\}$ is a family of sets with the non-trivial finite intersection property. We may assume that the M_α's are closed under finite intersections so if M_0 is one of them of minimum dimension, then $(y_0 + M_0) \cap \{a \in \mathrm{Hom}(L(G/\ker p_i), L(G/\ker p_j)) : a(L(G/\ker p_i))$ ideal in $L(G/\ker p_j), a(L(G/\ker p_i)) \leq (L(G/\ker p_j))^{Ad(\ker p_i/\ker p_j)}\} \subseteq (y_\alpha + M_\alpha) \cap \{a \in \mathrm{Hom}(L(G/\ker p_i), L(G/\ker p_j)) : a(L(G/\ker p_i))$ ideal in $L(G/\ker p_j), a(L(G/\ker p_i)) \leq (L(G/\ker p_j))^{Ad(\ker p_i/\ker p_j)}\}$.

Finally observe that since $(c_i)_\# = 1$ and $(L(G/\ker p_j))^{Ad(\ker p_i/\ker p_j)} = L(C_{G/\ker p_i}(\ker p_i/\ker p_j))$ [3, p.346, Proposition 8], then by virtue of [3, p.305, Theorem 1] + [3, p.316, Proposition 14] there exists a continuous homomorphism $a_i : \tilde{G}_i \to \varprojlim_{j>i} C_{G/\ker p_i}(\ker p_i/\ker p_j) \leq \varprojlim_{j>i} G/\ker p_j = G$ such that $p_i \circ a_i = c_i$. Now observe that the isomorphism $G \to \varprojlim_{j>i} G/\ker p_j$ restricts to an isomorphism $C_G(\ker p_i) \to \varprojlim_{j>i} C_{G/\ker p_i}(\ker p_i/\ker p_j)$ (the surjectivity of the last arrow follows since if $z \in G$ and $(p_j(z), p_j(\ker p_i)) = 1$ for all $i < j \in I$, then $(z, \ker p_i) \subseteq \bigcap_{j>i} \ker p_j = 1$). Hence $(a_i(\tilde{G}_i), \ker p_i) = 1$ as desired. □

Theorem 2.2. *Generalized Gluškov-Iwasawa local splitting theorem.*

Let G be an almost connected projective Lie group such that there exists a filter base $\{\ker p_i : i \in I\}$ of normal subgroups of G such that $G/\ker p_i$ Lie group for all $i \in I$ that converges to $1 \in G$ and $\dim s_i = K < \infty$ for all $i \in I$, where $s_i =$ nilpotent radical of $L(G/\ker p_i)$, then for every $1 \in U$ open $\subseteq G$, there exists N almost connected normal $\leq G$, $N \subseteq U$, G/N Lie group and $1 \in W$ open $\subseteq G/N$, a map $\varphi : W \to G$ such that the map $N \times W \to G$ defined by $(n, w) \to n\varphi(w)$ is a homeomorphism onto an open neighborhood of $1 \in G$ that defines a local isomorphism of $N \times G/N$ to G [2, TG III. 6].

Proof.

Let $1 \in U$ open $\subseteq G$. By hypothesis , there exists $i \in I$ such that $\ker p_i \subseteq U$. By [6, Theorem 4.28 (i)] $G/(\ker p_i)_0$ is a projective Lie group and since all Lie quotients of it are of dimension $\leq \dim G/\ker p_i$, it follows from [6, Theorem 9.44] that $G_0/(\ker p_i)_0$ is locally compact, hence $G/(\ker p_i)_0$ is locally compact [2, TGIII.75, EX 10e]. There exists $(\ker p_i)_0 \leq$

$\ker p_i \leq \ker p_i$ such that $\ker p_i/(\ker p_i)_0$ compact, $G/\ker p_i$ Lie group. For all $j \geq i$, $\ker p_j/\ker p_j \cap \ker p'_i \leq \ker p_i/\ker p'_i$ discrete and $L(G/\ker p_j) \cong L(G/\ker p_j \cap \ker p'_i)$. Therefore, considering the filter base $\{\ker p_j \cap \ker p'_i : j \geq i\}$ we may assume that $\ker p_i$ is almost connected.

Let $\tilde{G}_i$ be the universal covering group of $G/(\ker p_i)_0$, $c_i : \tilde{G}_i \to (G/\ker p_i)_0 \to G/\ker p_i$ the canonical projection and let $a_i : \tilde{G}_i \to G$ be the homomorphism provided by Theorem 2.1. Let $1 \in V$ open $\subseteq \tilde{G}_i$ such that c_i restricts to a homeomorphism $\bar{c}_i : V \to c_i(V)$ open $\subseteq G/\ker p_i$. Let $1 \in W$ open $\subseteq W^2 \subseteq c_i(V)$, then the map ker $p_i \times W \to G$ defined by $(n, w) \to na_i \circ \bar{c}_i^{-1}(w)$ defines a local isomorphism of $\ker p_i \times G/\ker p_i$ to G as desired [2, TGIII.6, Proposition 3]. □

Finally we remark that the sufficiency criterion in Theorem 2.2 for the validity of local splitting of almost connected projective Lie groups is not necessary. In fact, a product of infinitely many copies of a connected finite dimensional Lie group with non-reductive Lie algebra is an example of a connected projective Lie group that admits local splitting for every neighborhood of the identity but for which the sufficiency condition of the theorem obviously fails.

3. The Classical Gluškov-Iwasawa Local Splitting Theorem

In this section we show that our criterion stated in Proposition 1.2 for projective Lie groups is automatically satisfied for almost connected locally compact groups in Lemma 3.1 and hence we obtain a new proof of the classical Gluškov-Iwasawa Local Splitting Theorem for locally compact groups in Theorem 3.2.

LEMMA 3.1.
Let G be an almost connected locally compact group and let K be the maximal compact normal G [5, p.180, Theorem 3.1], then if $\ker p$ *is a compact normal $\leq G$ such that $G/\ker p$ Lie group [10, p.175] we have* $\dim n_p/z_p \leq \dim G/K < \infty$ *where $n_p =$ largest nilpotent ideal in* $L(G/\ker p)$ *and $z_p =$ center of* $L(G/\ker p)$.

PROOF.
Note that $L(K_0 \ker p/\ker p)$ is an ideal of $L(G/\ker p)$ and ad x is semi simple linear transformation of $(L(G/\ker p))$ for all $x \in L(K_0 \ker p/\ker p)$,

hence $n_p \cap L(K_0 \ker p/\ker p) \leq z_p$. It follows that

$$\begin{aligned} \dim n_p/z_p &\leq \dim n_p/n_p \cap L(K_0 \ker p/\ker p) \\ &\leq \dim L(G/\ker p)/L(K_0 \ker p/\ker p) \\ &= \dim G/K_0 \ker p \\ &= \dim G/K < \infty. \ \square \end{aligned}$$

Theorem 3.2. *Classical Gluškov-Iwasawa Local Splitting Theorem.*

Let G be locally compact group, then for any $1 \in U$ open $\subseteq G$, there exists N compact $\leq G$, $N \subseteq U$ and a local Lie group L such that G is locally isomorphic to $N \times L$.

Proof.

Let G^* open $\leq G$, G^*/G_0 compact $\leq G$ [2, TGIII, p.36, Corollary 1], hence we may assume that G is an almost connected projective Lie group [10, p.175]. Our sufficiency criterion is satisfied by virtue of Lemma 3.1 + Proposition 1.2 hence our assertion follows from Theorem 2.2. $\square$

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(Received October 23, 2009)

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