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Uniform Estimate for Distributions of the Sum of i.i.d. Random Variables with Fat Tail

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Abstract. The research on asymptotic behavior of distributions of the sum of i.i.d random variables has a long history and a lot of facts are known. The authors consider the case where the distribution of a random variable has the second moment but has a fat tail, and they show a new limit theorem for large deviations.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and X_n, $n = 1, 2, \dots$, be independent identically distributed random variables with the same probability law μ.

In the present paper we assume that
(A-1) $E[X_1^2] = 1$ and $E[X_1] = 0$.
Let $F : \mathbf{R} \to [0, 1]$ and $\bar{F} : \mathbf{R} \to [0, 1]$ be given by

$$F(x) = \mu((-\infty, x]) = P(X_1 \leqq x) \text{ and}$$
$$\bar{F}(x) = \mu((x, \infty)) = P(X_1 > x), \qquad x \in \mathbf{R}.$$

We also assume the following.
(A-2) $\bar{F}(x)$ is a regularly varying function of index $-\alpha$ for some $\alpha > 2$, as $x \to \infty$, i.e., if we let

$$L(x) = x^\alpha \bar{F}(x), \qquad x \geqq 1,$$

then $L(x) > 0$ for any $x \geqq 1$, and for any $a > 0$

$$\frac{L(ax)}{L(x)} \to 1, \qquad x \to \infty.$$

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(A-3) $|x|^{\alpha+2}F(x) \to 0, \quad x \to -\infty.$

Recently people in finance are interested in computing the quantile of the distribution of $\sum_{k=1}^n X_k$ for the purpose of measuring market risk.

There are many works on this topic. In particular, there are many results on large deviation results (e.g. Borovkov-Borovkov [1], also see books, Borovkov-Borovkov [2] and Petrov [7]). However, there are not so many results on uniform estimates. Nagaev [5] and [6] proved the following theorem (also see Linnik [4]), and this is the best result so far to our best knowledge.

THEOREM 1 (Nagaev). *Assume* (A-1)-(A-3). *Then we have*

$$\sup_{s\in[1,\infty)} |\frac{P(\sum_{k=1}^n X_k > sn^{1/2})}{\Phi_0(s) + n\bar{F}(sn^{1/2})} - 1| \to 0, \qquad n \to \infty.$$

Here $\Phi_0 : \mathbf{R} \to \mathbf{R}$ is given by

$$\Phi_0(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp(-\frac{y^2}{2})dy, \qquad x \in \mathbf{R}.$$

In this paper, we show two theorems. Combining them, we can improve Nagaev's result a little bit.

Let us explain our results. We assume the following assumption furthermore.

(A-4) The probability law μ is absolutely continuous and has a density function $\rho : \mathbf{R} \to [0, \infty)$ which is right continuous and has a finite total variation.

To state our theorem (Theorem 2), we need some preparations.

Let K be an integer such that $K - 1 < \alpha \leqq K$. Then $K \geqq 3$. From the assumptions (A-2) and (A-3), we see that the probability law μ has $(K-1)$-th moment. So let η_k, $k = 1, \ldots, K-1$, be given by

$$\eta_k = \int_{\mathbf{R}} x^k \mu(dx).$$

Then we see that $\eta_1 = 0$ and $\eta_2 = 1$. Also, let us define $\Phi_k : \mathbf{R} \to \mathbf{R}$, $k = 1, 2, \ldots$, by

$$\Phi_1(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) = -\frac{d}{dx}\Phi_0(x),$$

and

$$\Phi_k(x) = (-1)^{k-1} \frac{d^{k-1}}{dx^{k-1}} \Phi_1(x), \qquad k = 2, 3, \dots .$$

THEOREM 2. *Assume* (A-1)-(A-4). *Then there are* $\delta > 0$ *and* $C > 0$ *such that*

$$\sup_{s \in [1,\infty)} |P(\sum_{k=1}^{n} X_k > sn^{1/2}) - G(n,s)| \leqq Cn^{-(\alpha-2)/2-\delta}, \qquad n = 3, 4, \dots .$$

Here

$$G(n,s)$$

$$= \Phi_0(s) + n \int_{-\infty}^{s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx - \sum_{k=1}^{K-1} \frac{n^{-(k-2)/2}}{k!} \Phi_k(s) \int_0^{\infty} x^k \mu(dx)$$

$$+ \frac{n^{-(K-2)/2}}{K!} \Phi_K(s) \int_{-\infty}^{0} x^K \mu(dx) + \sum_{k=3}^{K-1} n^{-k/6} q_k(n^{-1/3}, \eta_2, \dots, \eta_k) \Phi_k(s)$$

$$+ \sum_{k=K}^{3(K-1)} n^{-k/6} q_k(n^{-1/3}, \eta_2, \dots, \eta_{K-1}, 0, \dots, 0) \Phi_k(s),$$

and q_k*'s are polynomials defined in the next section.*

For the next theorem we assume the following also.
(A-5) There is an $x_0 > 0$ such that $\bar{F}$ is twice continuously differentiable on (x_0, ∞) and that

$$x^2 \frac{d^2}{dx^2} \log \bar{F}(x) \to \alpha, \qquad x \to \infty.$$

Then we have the following.

THEOREM 3. *Assume the assumptions* (A-1)- (A-5) *and let* $\beta : \mathbf{N} \to (0, \infty)$ *be such that*

$$\frac{\beta(n)}{(\log n)^{1/2}} \to \infty, \qquad n \to \infty.$$

Then we have

$$\sup_{s \geqq n^{1/2}\beta(n)} \frac{s^2}{n}\left|\frac{P(\sum_{k=1}^n X_k > s)}{n\bar{F}(s)} - (1 + \frac{\alpha(\alpha+1)n}{2s^2})\right| \to 0, \qquad n \to \infty.$$

Let

$$H(n,s)$$

$$= \Phi_0(s) + n\int_{-\infty}^{s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx - \sum_{k=1}^{2} \frac{n^{-(k-2)/2}}{k!}\Phi_k(s)\int_0^\infty x^k \mu(dx)$$

for $s \geqq 1$, and $n \geqq 1$.

Then we also show the following.

THEOREM 4. *Assume* (A-1)-(A-5). *Then there exist a $C > 0$, $\delta > 0$ and $n_0 \geqq 1$ such that*

$$\sup_{s\in[1,\infty)} \left|\frac{P(\sum_{k=1}^n X_k > sn^{1/2})}{H(n,s)} - 1\right| \leqq Cn^{-\delta}, \qquad n \geqq n_0.$$

Note that by Theorem 3, we see that

$$2(\log n)^2 (\frac{P(\sum_{k=1}^n X_k > (\log n)n^{1/2})}{\Phi_0(\log n) + n\bar{F}((\log n)n^{1/2})} - 1) \to \alpha(\alpha+1) \qquad n \to \infty.$$

Therefore we see that $H(n,s)$ is a better approximation for $P(\sum_{k=1}^n X_k > sn^{1/2})$ than $\Phi_0(s) + n\bar{F}(sn^{1/2})$.

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2. Algebraic Preparation

In this section, we think of formal power series in z. First, we think of the following formal power series in z.

$$(1) \quad \log(1 + \sum_{k=2}^\infty \frac{a_k}{k!}z^k) = \sum_{\ell=1}^\infty \frac{(-1)^{\ell-1}}{\ell}(\sum_{k=2}^\infty \frac{a_k}{k!}z^k)^\ell = \sum_{\ell=2}^\infty c_\ell(a_2,\dots,a_\ell)\frac{z^\ell}{\ell!}$$

Then we see that $c_\ell(a_2,\dots,a_\ell)$, $\ell \geqq 2$, are polynomials in a_2, ..., a_ℓ, and

$$c_\ell(t^2 a_2,\dots,t^\ell a_\ell) = t^\ell c_\ell(a_1,\dots,a_\ell)$$

for any $t, a_1, \dots, a_\ell \in \mathbf{R}$. Moreover, we see that

$$c_2(a_2) = a_2 \quad \text{and} \quad c_\ell(a_2, \dots, a_{\ell-1}, a_\ell) = c_\ell(a_2, \dots, a_{\ell-1}, 0) + a_\ell, \qquad \ell \geqq 2.$$

We also think of the following formal power series in z.

$$\exp(y^{-3} \sum_{\ell=3}^{\infty} c_\ell(a_2, \dots, a_\ell) \frac{(yz)^\ell}{\ell!})$$

$$(2) \quad = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} (\sum_{\ell=3}^{\infty} c_\ell(a_2, \dots, a_\ell) \frac{y^{\ell-3} z^\ell}{\ell!})^k = 1 + \sum_{k=3}^{\infty} q_k(y, a_2, \dots, a_k) z^k.$$

Then we see that $q_k(y, a_2, \dots, a_k)$, $k \geqq 3$, are polynomials in y, $a_2, \dots, a_\ell$. Note that

$$q_k(y, t^2 a_2 \dots, t^k a_k) = t^k q_k(y, a_2, \dots, a_k)$$

and that

$$q_k(y, a_2, \dots, a_k) = q_k(y, a_2, \dots, a_{k-1}, 0) + \frac{y^{k-3}}{k!} a_k, \qquad k \geqq 3.$$

Also we have

$$\exp(y^{-6} \sum_{\ell=3}^{\infty} c_\ell(a_2, \dots, a_\ell) \frac{(y^3 z)^\ell}{\ell!})$$

$$= \exp((y^2)^{-3} \sum_{\ell=3}^{\infty} c_\ell(y^2 a_2, \dots, y^\ell a_\ell) \frac{(y^2 z)^\ell}{\ell!})$$

$$(3) \qquad = 1 + \sum_{k=3}^{\infty} q_k(y^2, y^2 a_2, \dots, y^k a_k) z^k = 1 + \sum_{k=3}^{\infty} y^k q_k(y^2, a_2, \dots, a_k) z^k$$

as a formal power series in z.

3. Property of the Function L

PROPOSITION 5. *We have*

$$\sup_{1/2 \leqq a \leqq 2} \frac{L(ax)}{L(x)} \to 1, \qquad x \to \infty,$$

and

$$\inf_{1/2 \leqq a \leqq 2} \frac{L(ax)}{L(x)} \to 1, \qquad x \to \infty.$$

PROOF. Since the proof is similar, we prove the first equation only. If not, there are $\varepsilon > 0$, $\{a_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ such that $1/2 \leqq a_n \leqq 2$, $x_n \geqq 1$, $n = 1, 2, \dots$, $x_n \to \infty$, $n \to \infty$, and that

$$\frac{L(a_n x_n)}{L(x_n)} > 1 + \varepsilon, \qquad n = 1, 2, \dots .$$

Then taking a subsequence if necessary, we may assume that there is an $a \in [1/2, 2]$ such that $a_n \to a$, $n \to \infty$. Then we see that for any $m \geqq 3$ there is a $n(m) \geqq 1$ such that

$$(a - \frac{1}{m})^{-\alpha} L((a - \frac{1}{m})x_n) = \bar{F}((a - \frac{1}{m})x_n) \geqq \bar{F}(a_n x_n)$$
$$= a_n^{-\alpha} L(a_n x_n), \qquad n \geqq n(m).$$

So we have

$$(1 - \frac{1}{ma})^{-\alpha} \geqq \lim_{n \to \infty} \frac{L(a_n x_n)}{L((a - 1/m)x_n)} \geqq 1 + \varepsilon, \qquad m \geqq 3.$$

Since m is arbitrary, this implies a contradiction. □

PROPOSITION 6. *For any $\varepsilon \in (0, 1)$, there is an $M \geqq 1$ such that*

$$M^{-1} y^{-\varepsilon} \leqq \frac{L(yx)}{L(x)} \leqq M y^{\varepsilon} \qquad x, y \geqq 1.$$

PROOF. For any $\varepsilon \in (0, 1)$ there is an $m \geqq 1$ such that

$$|\frac{L(ex)}{L(x)} - 1| \leqq \varepsilon \qquad x \geqq e^m.$$

Let

$$C = \sup_{x \in [1, e^m]} (\frac{L(ex)}{L(x)} + \frac{L(x)}{L(ex)}) < \infty.$$

Then we have

$$C^{-m}(1-\varepsilon)^n \leqq \frac{L(e^n x)}{L(x)} \leqq C^m (1+\varepsilon)^n, \qquad x \geqq 1,\ n \geqq 0.$$

For any $y \geqq 1$, there is an $n \geqq 1$ such that $e^{n-1} \leqq y \leqq e^n$. Then we have

$$\bar{F}(e^{n-1}x) \geqq \bar{F}(yx) \geqq \bar{F}(e^n x).$$

So we have for any $x, y \geqq 1$

$$(e^{-1}yx)^{-\alpha} L(e^{n-1}x) \geqq (e^{n-1}x)^{-\alpha} L(e^{n-1}x) \geqq (yx)^{-\alpha} L(yx)$$

$$\geqq (e^n x)^{-\alpha} L(e^n x) \geqq (eyx)^{-\alpha} L(e^n x),$$

which implies

$$C^{-m} e^{-\alpha} (1-\varepsilon)^n \leqq \frac{L(yx)}{L(x)} \leqq C^m e^{\alpha} (1+\varepsilon)^{n-1}.$$

Therefore we have

$$C^{-m} e^{-\alpha} (1-\varepsilon) y^{\log(1-\varepsilon)} \leqq \frac{L(yx)}{L(x)} \leqq C^m e^{\alpha} y^{\log(1+\varepsilon)}, \qquad x \geqq 1,\ y \geqq 1.$$

This implies our assertion. $\square$

The following is known as Karamata's theorem (c.f.[3] Appendix), but we give a proof.

Proposition 7. (1) *For any* $\beta < -1$,

$$\frac{1}{t^{\beta+1} L(t)} \int_t^{\infty} x^{\beta} L(x) dx \to -\frac{1}{\beta+1}, \qquad t \to \infty.$$

(2) *For any* $\beta > -1$,

$$\frac{1}{t^{\beta+1} L(t)} \int_1^{t} x^{\beta} L(x) dx \to \frac{1}{\beta+1}, \qquad t \to \infty.$$

(3) *Let* $f : [1, \infty) \to (0, \infty)$ *be given by*

$$f(t) = \int_1^t x^{-1} L(x) dx \qquad t \geqq 1.$$

Then f *is slowly varying.*

PROOF. Note that for $t > 1$

$$\frac{1}{t^{\beta+1} L(t)} \int_t^\infty x^\beta L(x) dx = \int_1^\infty x^\beta \frac{L(tx)}{L(t)} dx, \text{ if } \beta < -1$$

and

$$\frac{1}{t^{\beta+1} L(t)} \int_1^t x^\beta L(x) dx = \int_{1/t}^1 x^\beta (\frac{L(t)}{L(tx)})^{-1} dx \text{ if } \beta > -1$$

Then the assertions (1) and (2) follow from this equation and Proposition 5.

Let us prove (3). If $\lim_{t\to\infty} f(t) < \infty$, the assertion is obvious. So we assume that $\lim_{t\to\infty} f(t) = \infty$. Then for any $a > 0$ and $t_0 > 1$

$$f(at) = \int_{1/a}^t x^{-1} L(ax) dx = \int_{1/a}^{t_0} x^{-1} L(ax) dx + \int_{t_0}^t x^{-1} L(x) \frac{L(ax)}{L(x)} dx.$$

So we have

$$\inf_{x \geqq t_0} \frac{L(ax)}{L(x)} \leqq \underline{\lim}_{t\to\infty} \frac{f(at)}{f(t)} \leqq \overline{\lim}_{t\to\infty} \frac{f(at)}{f(t)} \leqq \sup_{x \geqq t_0} \frac{L(ax)}{L(x)}.$$

Therefore by Proposition 6 and Lebesgue's convergence theorem, we have our assertion. □

4. Estimate for Moments and Characteristic Functions

Remind that K is an integer such that $K - 1 < \alpha \leqq K$ and

$$\eta_k = \int_{-\infty}^\infty x^k \mu(dx), \qquad k = 1, 2, \ldots, K - 1.$$

Then by the assumption (A4) we have $\eta_1 = 0$ and $\eta_2 = 1$. Note that

$$1 - \bar{F}(t) \geqq 1 - \int_2^\infty \frac{x^2}{4} \mu(dx) \geqq \frac{3}{4}$$

for any $t \geqq 2$. Let

$$\eta_k(t) = \int_{(-\infty,t]} x^k \mu(dx), \qquad t > 0,\ k = 1, 2, \dots, K+1,$$

and

$$\bar{\eta}_k(t) = \int_{(t,\infty)} x^k \mu(dx), \qquad t > 0,\ k = 0, 1, 2, \dots, K-1.$$

Then we have

$$\eta_k(t) = \int_{(-\infty,0)} x^k \mu(dx) + k \int_0^t x^{k-1} \bar{F}(x) dx - t^k \bar{F}(t),$$
$$t > 0,\ k = 1, 2, \dots, K+1,$$

and

$$\bar{\eta}_k(t) = k \int_t^\infty x^{k-1} \bar{F}(x) dx + t^k \bar{F}(t) \qquad t > 0,\ k = 0, 1, 2, \dots, K-1.$$

Then by Propositions 6 and 7 we have the following.

PROPOSITION 8. *For any $\varepsilon > 0$, there is a $C(\varepsilon) > 0$ such that*

$$L(t) \leqq C(\varepsilon) t^{\varepsilon},$$

$$|\eta_K(t)| \leqq C(\varepsilon) t^{-\alpha+K+\varepsilon},$$

$$|\bar{\eta}_k(t)| \leqq C(\varepsilon) t^{-\alpha+k+\varepsilon}, \qquad k = 0, 1, 2, \dots K-1,$$

and

$$\int_{(-\infty,t]} |x|^{K+1} \mu(dx) \leqq C(\varepsilon) t^{-\alpha+K+1+\varepsilon}$$

for any $t \geqq 1$.

The following is well known.

PROPOSITION 9. (1) *For any $m \geqq 0$, let $r_{e,m} : \mathbf{R} \to \mathbf{C}$ be given by*

$$r_{e,m}(t) = \exp(it) - (1 + \sum_{k=1}^m \frac{(it)^k}{k!}), \qquad t \in \mathbf{R}.$$

Then we have

$$|r_{e,m}(t)| \leqq \frac{|t|^{m+1}}{(m+1)!} \qquad t \in \mathbf{R}.$$

(2) *For any $m \geqq 1$, let $r_{l,m} : \{z \in \mathbf{C};\ |z| \leqq 1/2\} \to \mathbf{C}$ be given by*

$$r_{l,m}(z) = \log(1+z) - \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} z^k, \qquad z \in \mathbf{C},\ |z| \leqq 1/2.$$

Then we have

$$|r_{l,m}(z)| \leqq 2|z|^{m+1}, \qquad z \in \mathbf{C},\ |z| \leqq 1/2.$$

Let $\mu(t)$, $t > 0$, be a probability measure on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ given by

$$\mu(t)(A) = (1 - \bar{F}(t))^{-1} \mu(A \cap (-\infty, t]),$$

for any $A \in \mathcal{B}(\mathbf{R})$.

Let $\varphi(\cdot; \mu(t))$, $t > 0$, be the characteristic function of the probability measure $\mu(t)$, i.e.,

$$\varphi(\xi; \mu(t)) = \int_{\mathbf{R}} \exp(ix\xi)\mu(t)(dx), \qquad \xi \in \mathbf{R}.$$

By the assumption (A3), we see that the density function $\rho(x) \to 0$ as $|x| \to \infty$. Also we see that the probability measure $\mu(t)$, $t \geqq 2$, is absolutely continuous and its density function is $(1 - \bar{F}(t))^{-1}\rho(x)1_{(-\infty,t]}(x)$, whose total variation is dominated by twice of that of ρ.

Therefore we have the following.

Proposition 10. (1) *For any $t \geqq 2$ and $\xi \in \mathbf{R}$,*

$$\begin{aligned} i\xi\varphi(\xi; \mu(t)) &= (1 - \bar{F}(t))^{-1} \int_{\mathbf{R}} i\xi e^{i\xi x} \rho(x) 1_{(-\infty,t]}(x) dx \\ &= -(1 - \bar{F}(t))^{-1} \int_{\mathbf{R}} e^{i\xi x} d(\rho(x) 1_{(-\infty,t]}(x)). \end{aligned}$$

(2) *There is a $C > 0$ such that*

$$|\varphi(\xi, \mu(t))| \leqq C(1 + |\xi|)^{-1} \textit{ for any } t \geqq 2 \textit{ and } \xi \in \mathbf{R}.$$

Then we have the following.

PROPOSITION 11. (1) *There is a* $c_0 > 0$ *such that*

$$|\varphi(\xi, \mu(t))| \leqq (1 + c_0|\xi|^2)^{-1/4} \text{ for any } t \geqq 2 \text{ and } \xi \in \mathbf{R}.$$

(2) *For any* $t \geqq 2$, $\xi \in \mathbf{R}$, *and integers* n, m *with* $n \geqq m$,

$$|\varphi(n^{-1/2}\xi, \mu(t))|^n \leqq (1 + \frac{c_0}{m}|\xi|^2)^{-m/4}.$$

PROOF. Let $g(x) = \rho(x)1_{(-2,2)}(x)$, $x \in \mathbf{R}$. Then we have

$$p = \int_{\mathbf{R}} g(x)dx \geqq 1 - \int_{\mathbf{R}} \frac{x^2}{4}\rho(x)dx \geqq 3/4.$$

Note that

$$\begin{aligned} &|\varphi(\xi, \mu(t))|^2 \\ &= (1 - \bar{F}(t))^{-2} \int_{\mathbf{R}} \int_{\mathbf{R}} \exp(i\xi(x - y))\rho(x)1_{(-\infty,t]}(x)\rho(y)1_{(-\infty,t]}(y)dxdy \\ &\leqq (1 - p^2) + \int_{\mathbf{R}} \int_{\mathbf{R}} \exp(i\xi(x - y))g(x)g(y)dxdy = 1 - f(\xi), \end{aligned}$$

where

$$f(\xi) = \int_{\mathbf{R}} \int_{\mathbf{R}} (1 - \cos(\xi(x - y)))g(x)g(y)dxdy.$$

So we see that

$$\lim_{\xi \to 0} |\xi|^{-2} f(\xi) = \frac{1}{2} \int_{\mathbf{R}} \int_{\mathbf{R}} (x - y)^2 g(x)g(y)dxdy > 0.$$

Also, it is easy to see that $f(\xi) > 0$, for all $\xi \in \mathbf{R} \setminus \{0\}$, and so we see that

$$a(r) = \inf_{|\xi| \leqq r} |\xi|^{-2} f(\xi) > 0 \qquad \text{for all } \ r > 0.$$

Therefore we see that

$$|\varphi(\xi, \mu(t))| \leqq (1 - a(r)|\xi|^2)^{1/2} \leqq (1 + a(r)|\xi|^2)^{-1/4}, \qquad |\xi| \leqq r.$$

Also by Proposition 10(2), we see that there is an $r_0 > 0$ such that

$$|\varphi(\xi, \mu(t))| \leqq (1 + |\xi|^2)^{-1/4}, \qquad |\xi| \geqq r_0$$

So we have the assertion (1).

It is easy to chack that $(1 + x/\beta)^\beta \geqq 1 + x$ for any $\beta \geqq 1$ and $x \geqq 0$. Therefore if $n \geqq m$, we have

$$(1 + c_0|n^{-1/2}\xi|^2)^{n/m} \geqq 1 + \frac{c_0}{m}|\xi|^2.$$

This implies the assertion (2). □

5. Asymptotic Expansion of Characteristic Functions

Let

$$\varphi_1(\xi, t) = -\sum_{k=1}^{K-1} \frac{(i\xi)^k}{k!}\bar{\eta}_k(t) + \frac{(i\xi)^K}{K!}\eta_K(t)$$

and

$$\psi_0(n, \xi) = \sum_{k=3}^{K-1} n^{-k/6} q_k(n^{-1/3}, \eta_2, \dots, \eta_k)(i\xi)^k$$

$$+ \sum_{k=K}^{3(K-1)} n^{-k/6} q_k(n^{-1/3}, \eta_2, \dots, \eta_{K-1}, 0, \dots, 0)(i\xi)^k$$

for $t \geqq 2$, $n \geqq 1$ and $\xi \in \mathbf{R}$. Let $\delta = ((\alpha-2)\wedge 1)/(4(K+2))$, $\delta' = \delta/(4(K+2))$, and $t_n = n^{1/2-\delta}$, $n = 1, 2, 3, \dots$. Then $t_n \geqq 2$ for any $n \geqq 8$.

In this section, we prove the following.

LEMMA 12. *Let*

$$R_{n,0}(\xi) = \exp(\frac{1}{2}\xi^2)\varphi(n^{-1/2}\xi, \mu(t_n))^n - (1 + \psi_0(n, \xi) + n\varphi_1(n^{-1/2}\xi, t_n))$$

$$R_{n,1}(\xi) = \exp(\frac{1}{2}\xi^2)\varphi(n^{-1/2}\xi, \mu(t_n))^n - 1$$

$$R_{n,2}(\xi) = \exp(\frac{1}{2}\xi^2)\varphi(n^{-1/2}\xi, \mu(t_n))^{n-1} - 1$$

Then there is a $C > 0$ *such that*

$$|R_{n,0}(\xi)| \leqq Cn^{-(\alpha-2)/2-\delta/4}|\xi|$$

and

$$|R_{n,1}(\xi)| + |R_{n,2}(\xi)| \leqq Cn^{-2K\delta}|\xi|$$

for any $n \geqq 8$ *and* $\xi \in \mathbf{R}$ *with* $|\xi| \leqq n^{\delta'}$.

We make some preparations to prove this lemma. First we prove the following.

PROPOSITION 13. *Let*

$$\varphi_0(\xi) = \sum_{k=2}^{K-1} \frac{(i\xi)^k}{k!}\eta_k,$$

and

$$R_0(\xi, t) = \varphi(\xi; \mu(t)) - (1 + \varphi_0(\xi) + \varphi_1(\xi, t)).$$

Then we have for any $n \geqq 8$, *and* $\xi \in \mathbf{R}$ *with* $|\xi| \leqq n^{\delta'}$,

$$|\varphi(n^{-1/2}\xi; \mu(t_n)) - 1| \leqq \frac{2\sqrt{3}}{3}n^{-1/2}|\xi|,$$

$$|\varphi_1(n^{-1/2}\xi, t_n)| \leqq KC(\delta)n^{-\alpha/2+(K+1)\delta}|\xi|$$

and

$$|R_0(n^{-1/2}\xi, t_n)| \leqq 3C(\delta)n^{-\alpha/2-\delta/4}|\xi|.$$

Here $C(\delta)$ *is as in Proposition 8.*

PROOF. We can easily see that

$$\varphi(\xi; \mu(t)) = \int_{\mathbf{R}} \exp(ix\xi)\mu(t)(dx)$$

$$= 1 + \sum_{k=1}^{K} \frac{(i\xi)^k}{k!}\eta_k(t) + \int_{(-\infty,t]} r_{e,K}(x\xi)\mu(dx)$$

$$+\bar{F}(t)(1 - \bar{F}(t))^{-1} \int_{(-\infty,t]} r_{e,0}(x\xi)\mu(dx)$$

So we see that

$$R_0(\xi,t)=\bar{F}(t)(1-\bar{F}(t))^{-1}\int_{(-\infty,t]} r_{e,0}(x\xi)\mu(dx)+\int_{(-\infty,t]} r_{e,K}(x\xi)\mu(dx).$$

By Propositions 8 and 9 we have

$$|\varphi_1(\xi,t)|\leqq C(\delta)\sum_{k=1}^{K}\frac{|\xi|^k}{k!}t^{-\alpha+k+\delta},\qquad \xi\in\mathbf{R},\ t\geqq 2,$$

and

$$|R_0(\xi,t)|\leqq C(\delta)|\xi|t^{-\alpha+\delta}\int_{\mathbf{R}}|x|\mu(t)(dx)+C(\delta)|\xi|^{K+1}t^{-\alpha+K+1+\delta},$$
$$\xi\in\mathbf{R},\ t\geqq 2.$$

Also, we have

$$|\varphi(\xi;\mu(t))-1|\leqq|\xi|\int_{\mathbf{R}}|x|\mu(t)(dx)\leqq(1-\bar{F}(t))^{-1/2}|\xi|\leqq\frac{2\sqrt{3}}{3}|\xi|,$$
$$\xi\in\mathbf{R},\ t\geqq 2.$$

Note that

$$(n^{-1/2+\delta'})^k(n^{1/2-\delta})^{-\alpha+k+\delta}=n^{-\alpha/2+(\alpha+1/2)\delta-k(\delta-\delta')-\delta^2}.$$

So we have our assertion. □

PROPOSITION 14. *Let*

$$\psi_1(\xi)=\sum_{k=3}^{K-1}\frac{(i\xi)^k}{k!}c_k(\eta_2,\dots,\eta_k)+\frac{(i\xi)^K}{K!}c_K(\eta_2,\dots,\eta_{K-1},0),\qquad \xi\in\mathbf{R}.$$

Also, for any $n\geqq 8$, and $\xi\in\mathbf{R}$ with $|\xi|\leqq n^{\delta'}$, let

$$R_1(n,\xi)=\log(\varphi(n^{-1/2}\xi,\mu(t_n)))-\{-\frac{1}{2n}\xi^2+\psi_1(n^{-1/2}\xi)+\varphi_1(n^{-1/2}\xi,t_n)\}.$$

Then there is a constant $C>0$ such that

$$|R_1(n,\xi)|\leqq Cn^{-\alpha/2-\delta/4}|\xi|$$

for any $n \geqq 8$, *and* $\xi \in \mathbf{R}$ *with* $|\xi| \leqq n^{\delta'}$.

PROOF. Let

$$R_{1,1}(\xi) = \sum_{k=1}^{K} \frac{(-1)^{k-1}}{k} (\varphi_0(\xi))^k + \frac{1}{2}\xi^2 - \psi_1(\xi).$$

Note that

$$\log(1 + \sum_{k=2}^{K-1} \eta_k \frac{z^k}{k!})$$

$$= \sum_{k=2}^{K-1} c_k(\eta_2, \dots, \eta_k) \frac{z^k}{k!} + \sum_{k=K}^{\infty} c_k(\eta_2, \dots, \eta_{K-1}, 0, \dots, 0) \frac{z^k}{k!}$$

as a formal power series of z. So we see that there is a constant $C > 0$ such that

$$|R_{1,1}(\xi)| \leqq C|\xi|^{K+1} \tag{4}$$

for any $\xi \in \mathbf{R}$ with $|\xi| \leqq 1$.

We can easily see that

$$R_1(n, \xi)$$

$$= \log(1 + \varphi_0(n^{-1/2}\xi) + \varphi_1(n^{-1/2}\xi, t_n) + R_0(n^{-1/2}\xi, t_n))$$
$$- \{-\frac{1}{2n}\xi^2 + \psi_1(n^{-1/2}\xi) + \varphi_1(n^{-1/2}\xi, t_n)\}$$
$$= R_{1,1}(n^{-1/2}\xi) + r_{l,K}(\varphi(n^{-1/2}\xi, \mu(t_n)) - 1) + R_0(n^{-1/2}\xi, t_n)$$
$$+ \sum_{k=2}^{K} (-1)^{k-1} (\varphi_0(n^{-1/2}\xi))^{k-1} (\varphi_1(n^{-1/2}\xi, t_n) + R_0(n^{-1/2}\xi, t_n))$$
$$+ \sum_{k=1}^{K} \frac{(-1)^{k-1}}{k} \sum_{j=2}^{k} \binom{k}{j} (\varphi_0(n^{-1/2}\xi))^{k-j} (\varphi_1(n^{-1/2}\xi, t_n) + R_0(n^{-1/2}\xi, t_n))^j.$$

Then we have our assertion from Equation (4) and Proposition 13.

PROPOSITION 15. *Let*

$$R_2(n, \xi) = \exp(n\psi_1(n^{-1/2}\xi)) - (1 + \psi_0(n, \xi)).$$

Then there is a constant $C > 0$ such that

$$|R_2(n,\xi)| \leqq Cn^{-(\alpha-2)/2-1/4}|\xi|$$

for any $n \geqq 8$, and $\xi \in \mathbf{R}$ with $|\xi| \leqq n^{\delta'}$.

PROOF. Note that

$$\exp(y^{-6}(\sum_{k=3}^{K-1} \frac{(y^3 z)^k}{k!} c_k(\eta_2, \dots, \eta_k) + \sum_{k=K}^{\infty} \frac{(y^3 z)^k}{k!} c_k(\eta_2, \dots, \eta_{K-1}, 0, \dots, 0)))$$

$$= 1 + \sum_{k=3}^{K-1} y^k q_k(y^2, \eta_2, \dots, \eta_k) z^k + \sum_{k=K}^{\infty} y^k q_k(\eta_2, \dots, \eta_{K-1}, 0, \dots, 0))) z^k$$

as a formal power series in z. This implies our assertion. □

Now let us prove Lemma 12.

Note that for any $n \geqq 8$, and $\xi \in \mathbf{R}$ with $|\xi| \leqq n^{\delta'}$,

$$\begin{aligned}
&\exp(\frac{1}{2}\xi^2)\varphi(n^{-1/2}\xi; \mu(t_n))^n \\
&= \exp(n\varphi_1(n^{-1/2}\xi, t_n) + n\psi_1(n^{-1/2}\xi) + nR_1(n,\xi))) \\
&= (1 + n\varphi_1(n^{-1/2}\xi, t_n) + r_{e,1}(n\varphi_1(n^{-1/2}\xi, t_n))) \\
&\times (1 + \psi_0(n,\xi) + R_2(n,\xi))(1 + r_{e,0}(nR_1(n,\xi))).
\end{aligned}$$

So we see that

$$\begin{aligned}
R_{n,0}(n,\xi) &= r_{e,0}(nR_1(n,\xi)) \exp(n\varphi_1(n^{-1/2}\xi, t_n) + n\psi_1(n^{-1/2}\xi)) \\
&+ R_2(n,\xi) \exp(n\varphi_1(n^{-1/2}\xi, t_n)) \\
&+ r_{e,1}(n\varphi_1(n^{-1/2}\xi, t_n)) + \psi_0(n,\xi)(n\varphi_1(n^{-1/2}\xi, t_n) + r_{e,1}(n\varphi_1(n^{-1/2}\xi, t_n))).
\end{aligned}$$

Thus we have the first equation from Propositions 13, 14, 15.

Also, we have

$$R_{n,1}(n,\xi) = \exp(n\varphi_1(n^{-1/2}\xi, t_n) + n\psi_1(n^{-1/2}\xi) + nR_1(n,\xi))) - 1,$$

and

$$\begin{aligned}
R_{n,2}(n,\xi) &= \exp((n-1)\varphi_1(n^{-1/2}\xi, t_n) + (n-1)\psi_1(n^{-1/2}\xi) \\
&+ (n-1)R_1(n,\xi)) - \frac{\xi^2}{n}) - 1.
\end{aligned}$$

So, again from Propositions 13, 14, 15 we have the second equation.

6. Proof of Theorem 2

First, we prove the following.

LEMMA 16. *Let ν be a probability measure on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ such that $\int_{\mathbf{R}} x^2 \nu(dx) < \infty$. Also, assume that there is a constant $C > 0$ such that the characteristic function $\varphi(\cdot, \nu) : \mathbf{R} \to \mathbf{C}$ satisfies*

$$|\varphi(\xi; \nu)| \leqq C(1 + |\xi|)^{-2}, \qquad \xi \in \mathbf{R}.$$

Then for any $x \in \mathbf{R}$

$$\nu((x, \infty)) = \Phi_0(x) + \frac{1}{2\pi} \int_{\mathbf{R}} \frac{e^{-ix\xi}}{i\xi} (\varphi(\xi, \nu) - \exp(-\frac{\xi^2}{2})) d\xi.$$

PROOF. From the assumption, ν has a continuous density function β and

$$\beta(x) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{-ix\xi} \varphi(\xi, \nu) d\xi.$$

So we have

$$\nu((x, x+n])) = \Phi_0(x) - \Phi_0(x+n)$$

$$+ \frac{1}{2\pi} \int_{\mathbf{R}} (\int_x^{x+n} e^{-iz\xi} dz)(\varphi(\xi, \nu) - \exp(-\frac{\xi^2}{2})) d\xi.$$

$$= \Phi_0(x) - \Phi_0(x+n) + \frac{1}{2\pi} \int_{\mathbf{R}} \frac{e^{-ix\xi} - e^{-i(x+n)\xi}}{i\xi} (\varphi(\xi, \nu) - \exp(-\frac{\xi^2}{2})) d\xi.$$

Since

$$\int_{\mathbf{R}} \frac{1}{|\xi|} |\varphi(\xi, \nu) - \exp(-\frac{\xi^2}{2})| d\xi < \infty,$$

letting $n \to \infty$, we have the assertion. $\square$

Note that

$$P(\sum_{k=1}^{n} X_k > sn^{1/2}) = \sum_{m=0}^{n} I_m(n, s),$$

where

$$I_m(n, s) = P(\sum_{k=1}^{n} X_k > sn^{1/2}, \sum_{k=1}^{n} 1_{\{X_k > t_n\}} = m), \qquad m = 0, 1, \ldots, n.$$

Then we have

$$I_m(n,s) = \binom{n}{m} P(\sum_{k=1}^{n} X_k > sn^{1/2},\ X_i > t_n,\ i=1,\dots,m,$$
$$X_j \leqq t_n,\ j=m+1,\dots,n),$$

for $m = 0, 1, \dots, n$.

PROPOSITION 17. *There is a $C > 0$ such that*

$$\sum_{m=2}^{n} I_m(n,s) \leqq Cn^{-(\alpha-2)/2-\delta}$$

for any $s \geqq 1$ and $n \geqq 8$.

PROOF. We see that by Proposition 8

$$\sum_{m=2}^{n} I_m(n,s) \leqq \sum_{m=2}^{n} \frac{n(n-1)}{m(m-1)} \binom{n-2}{m-2} \bar{F}(t_n)^m (1-\bar{F}(t_n))^{n-m}$$
$$\leqq \frac{n(n-1)}{2} \bar{F}(t_n)^2 \leqq C(\delta)^2 n^{-(\alpha-2)/2-\delta}.$$

This implies our assertion. □

PROPOSITION 18. *There is a $C > 0$ such that*

$$\sup_{s\in[1,\log n]} |I_0(n,s) - \{(1-n\bar{F}(t_n))\Phi_0(s) - \sum_{k=1}^{K-1} \frac{n^{-(k-2)/2}}{k!} \bar{\eta}_k(t_n)\Phi_k(s)$$
$$+ \frac{(n^{1/2})^{K-2}}{K!} \eta_K(t_n)\Phi_K(s) + g(n,s)\}| \leqq Cn^{-(\alpha-2)/2-\delta/4}$$

for any $n \geqq 8$. Here

$$g(n,s) = \sum_{k=3}^{K-1} n^{-k/6} q_k(n^{-1/3}, \eta_2, \dots, \eta_k)\Phi_k(s)$$
$$+ \sum_{k=K}^{3(K-1)} n^{-k/6} q_k(n^{-1/3}, \eta_2, \dots, \eta_{K-1}, 0, \dots, 0)\Phi_k(s).$$

PROOF. Note that

$$I_0(n,s) = (1-\bar{F}(t_n))^n \mu(t_n)^{*n}((sn^{1/2},\infty))$$
$$= I_{0,0}(n,s) + I_{0,1}(n,s) + I_{0,2}(n,s),$$

where

$$I_{0,0}(n,s) = \mu(t_n)^{*n}((sn^{1/2},\infty)),$$
$$I_{0,1}(n,s) = -n\bar{F}(t_n)\mu(t_n)^{*n}((sn^{1/2},\infty)),$$
$$I_{0,2}(n,s) = ((1-\bar{F}(t_n))^n - 1 + n\bar{F}(t_n))\mu(t_n)^{*n}((sn^{1/2},\infty)).$$

We remark that

$$\Phi_k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi)^{k-1}\exp(-i\xi x - \frac{\xi^2}{2})d\xi, \qquad k=1,2,\dots .$$

By Proposition 11 and Lemma 16, we have

$$I_{0,0}(n,s)$$
$$= \Phi_0(s) + \frac{1}{2\pi}\int_{\mathbf{R}}\frac{e^{-is\xi}}{i\xi}(\varphi(n^{-1/2}\xi,\mu(t_n))^n - \exp(-\frac{\xi^2}{2}))d\xi.$$

Let

$$\tilde{R}_{0,0}(n,s) = I_{0,0}(n,s)$$
$$-\{\Phi_0(s) + \frac{1}{2\pi}\int_{\mathbf{R}}\frac{e^{-is\xi}}{i\xi}(\psi_0(n,\xi) + n\varphi_1(n^{-1/2}\xi,t_n))e^{-\xi^2/2}d\xi\}$$

Then by Lemma 12 we have

$$|\tilde{R}_{0,0}(n,s)|$$
$$\leqq \int_{|\xi|\leqq n^{\delta'}}\frac{|R_{n,0}(\xi)|}{|\xi|}\exp(-\frac{\xi^2}{2})d\xi$$
$$+\int_{|\xi|>n^{\delta'}}\frac{1}{|\xi|}(|\varphi(n^{-1/2}\xi,\mu(t_n))|^n + \exp(-\frac{\xi^2}{2}))d\xi$$
$$+\int_{|\xi|>n^{\delta'}}(|\psi_0(n,\xi)| + n|\varphi_1(n^{-1/2}\xi,t_n)|)e^{-\xi^2/2}d\xi$$

So by Proposition 11 and Lemma 12, we see that there is a $C_0 > 0$ such that

$$|\tilde{R}_{0,0}(n,s)| \leqq C_0 n^{-(\alpha-2)/2-\delta/4}, \qquad n \geqq 8,\ s \geqq 1. \tag{5}$$

Also, we see that

$$\frac{1}{2\pi}\int_{\mathbf{R}} \frac{e^{-is\xi}}{i\xi} n\varphi_1(n^{-1/2}\xi, t_n)\exp(-\frac{1}{2}\xi^2)d\xi$$

$$= -\sum_{k=1}^{K-1} \frac{(n^{-1/2})^{k-2}}{k!}\bar{\eta}_k(t_n)\Phi_k(s) + \frac{(n^{-1/2})^{K-2}}{K!}\eta_K(t_n)\Phi_K(s),$$

and

$$\frac{1}{2\pi}\int_{\mathbf{R}} \frac{e^{-is\xi}}{i\xi}\psi_0(n,\xi)\exp(-\frac{1}{2}\xi^2)d\xi = g(n,s)$$

Similarly by Lemma 12, we see that there is a $C_1 > 0$ such that

$$\sup_{s\in[1,\log n]} |I_{0,1}(n,s) - n\bar{F}(t_n)\Phi_0(s)| \leqq C_1 n^{-(\alpha-2)/2-\delta}, \qquad n \geqq 8. \tag{6}$$

Note that $|(1-x)^n - (1-nx)| \leqq n^2x^2$ for any $x \in [0,1]$, $n \geqq 1$. So we have

$$|I_{0,2}(n,s)| \leqq n^2\bar{F}(t_n)^2 \leqq C(\delta)^2 n^{-(\alpha-2)/2-\delta}.$$

This and Equations 5, 6 imply our assertion. □

PROPOSITION 19. *There is a $C > 0$ such that*

$$\sup_{s\in[1,\log n]} |I_1(n,s) - \{n\int_{-\infty}^{s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx + n\bar{F}(t_n)\Phi_0(s)$$

$$-\sum_{k=1}^{K} \frac{n^{-(k-2)/2}}{k!}\Phi_k(s)\int_0^{t_n} x^k\mu(dx)\}| \leqq Cn^{-(\alpha-2)/2-\delta/4}.$$

PROOF. We see that

$$I_1(n,s) = n(1-\bar{F}(t_n))^{n-1}\int_{\mathbf{R}} P(X_1 + x > sn^{1/2},\ X_1 > t_n)\,\mu(t_n)^{*(n-1)}(dx)$$

$$= n(1 - \bar{F}(t_n))^{n-1} \int_{-\infty}^{\infty} \bar{F}((sn^{1/2} - x) \vee t_n) \mu(t_n)^{*(n-1)}(dx)$$
$$= nJ_0(n,s) + nJ_1(n,s) + nJ_2(n,s),$$

where

$$J_0(n,s) = \int_{-\infty}^{\infty} \bar{F}((s-x)n^{1/2} \vee t_n) \Phi_1(x) dx, \tag{7}$$

$$J_1(n,s) = \int_{-\infty}^{\infty} \bar{F}((sn^{1/2} - x) \vee t_n)(\mu(t_n)^{*(n-1)}(dx) - n^{-1/2}\Phi_1(xn^{1/2})dx), \tag{8}$$

and

$$J_2(n,s) = -(1 - (1 - \bar{F}(t_n))^{n-1}) I_1(n,s). \tag{9}$$

Note that

$$J_0(n,s) = J_{0,0}(n,s) + J_{0,1}(n,s) + J_{0,2}(n,s),$$

where

$$J_{0,0}(n,s) = \int_{-\infty}^{s} \bar{F}((s-x)n^{1/2}) \Phi_1(x) dx,$$
$$J_{0,1}(n,s) = -\int_{s-n^{-\delta}}^{s} \bar{F}((s-x)n^{1/2}) \Phi_1(x) dx$$
$$= -\int_0^{n^{-\delta}} \bar{F}(xn^{1/2}) \Phi_1(s-x) dx,$$

and

$$J_{0,2}(n,s) = \bar{F}(t_n) \int_{s-n^{-\delta}}^{\infty} \Phi_1(x) dx = \bar{F}(t_n) \Phi_0(s - n^{-\delta}).$$

We see that

$$J_{0,1}(n,s) = -\sum_{k=1}^{K} \frac{1}{(k-1)!} \Phi_k(s) \int_0^{n^{-\delta}} \bar{F}(xn^{1/2}) x^{k-1} dx + R_{J,1}(n,s),$$

where

$$R_{J,1}(n,s) = -\int_0^{n^{-\delta}} \bar{F}(xn^{1/2})(\Phi_1(s-x) - \sum_{k=1}^{K} \frac{x^{k-1}}{(k-1)!} \Phi_k(s)) dx.$$

Then

$$|R_{J,1}(n,s)|$$

$$\leqq \sup_{x\in[0,n^{-\delta}]} |\Phi_{K+1}(s-x)|(\int_{n^{-1/2}}^{n^{-\delta}} x^K (xn^{1/2})^{-\alpha} L(xn^{1/2})dx + \int_0^{n^{-1/2}} x^K dx).$$

$$\leqq \sup_{x\in\mathbf{R}} |\Phi_{K+1}(x)|(C(\delta)n^{-\alpha/2+\delta/2}\int_0^{n^{-\delta}} x^{\delta+(K-\alpha)}dx + n^{-(K+1)/2})$$

$$\leqq \sup_{x\in\mathbf{R}} |\Phi_{K+1}(x)|(C(\delta)+1)n^{-\alpha/2-\delta/2}. \tag{10}$$

Also, we see that

$$J_{0,2}(n,s)$$

$$= \bar{F}(t_n)\Phi_0(s) + \sum_{k=1}^K \bar{F}(t_n)\frac{(n^{-\delta})^k}{k!}\Phi_k(s) + R_{J,2}(n,s),$$

where

$$R_{J,2}(n,s) = \bar{F}(t_n)(\Phi_0(s-n^{-\delta}) - \sum_{k=0}^K \frac{(-n^{-\delta})^k}{k!}\frac{d^k\Phi_0}{dx^k}(s)).$$

We see that

$$|R_{J,2}(n,s)| \leqq \bar{F}(t_n)n^{-(K+1)\delta}\sup_{x\in\mathbf{R}}|\Phi_{K+1}(x)|$$

$$\leqq C(\delta)\sup_{x\in\mathbf{R}}|\Phi_{K+1}(x)|n^{-\alpha/2-\delta/4}. \tag{11}$$

It is easy to see that

$$\int_0^{n^{-\delta}} \bar{F}(xn^{1/2})x^{k-1}dx = n^{-k/2}\int_0^{t_n}\bar{F}(x)x^{k-1}dx$$

$$= n^{-k/2}(-\frac{1}{k}\int_0^{t_n} x^k\mu(dx) + \frac{n^{\delta k}}{k}\bar{F}(t_n)), \qquad k=1,\dots,K.$$

So we have

$$J_{0,1}(n,s) + J_{0,2}(n,s) = \bar{F}(t_n)\Phi_0(s)$$

$$(12) \qquad -\sum_{k=1}^{K} \frac{n^{-k/2}}{k!}\Phi_k(s)\int_0^{t_n} x^k \mu(dx) + R_{J,1}(n,s) + R_{J,2}(n,s)$$

Also, we have

$$J_1(n,s) = J_{1,1}(n,s) + J_{1,2}(n,s)$$

where

$$J_{1,1}(n,s) = \bar{F}(t_n)(\mu(t_n)^{*(n-1)}(((s-n^{-\delta})n^{1/2},\infty)) - \Phi_0(s-n^{-\delta}))$$

and

$$J_{1,2}(n,s) = \int_{-\infty}^{s-n^{-\delta}} dx \bar{F}((s-x)n^{1/2})$$

$$\times \frac{1}{2\pi}\int_{\mathbf{R}} e^{-ix\xi}(\varphi(n^{-1/2}\xi;\mu(t_n))^{n-1} - \exp(-\frac{\xi^2}{2}))d\xi$$

By Proposition 11 and Lemma 16, we see that there is a $C_1 > 0$ such that

$$|\mu(t_n)^{*(n-1)}((xn^{1/2},\infty)) - \Phi_0(x)|$$

$$\leqq |\int_{-\infty}^{\infty} \frac{e^{-ix\xi}}{\xi}(\varphi(n^{-1/2}\xi;\mu(t_n))^{n-1} - \exp(-\frac{\xi^2}{2}))d\xi|$$

$$\leqq \int_{|\xi|>n^{\delta'}} \frac{1}{|\xi|}(|\varphi(\xi;\mu(t_n))|^{n-1} + \exp(-\frac{\xi^2}{2}))d\xi$$

$$+ \int_{|\xi|<n^{\delta'}} \frac{1}{|\xi|}|R_{n,2}(\xi)|\exp(-\frac{\xi^2}{2}))d\xi$$

$$\leqq C_1 n^{-2K\delta}, \text{ for any } x \in \mathbf{R} \text{ and } n \geqq 8.$$

Therefore we have

$$|J_{1,1}(n,s)| \leqq C_1\bar{F}(t_n)n^{-2K\delta} \leqq C(\delta)C_1 n^{-\alpha/2-\delta}.$$

Similarly by Lemma 16, we see that there is a $C_2 > 0$ such that

$$|\int_{\mathbf{R}} e^{-ix\xi}(\varphi(n^{-1/2}\xi;\mu(t_n))^{n-1} - \exp(-\frac{\xi^2}{2}))d\xi|$$

$$\leqq C_2 n^{-2K\delta}, \text{ for any } x \in \mathbf{R} \text{ and } n \geqq 8.$$

Then we have

$$|J_{1,2}(n,s)| \leqq C_2 n^{-2K\delta} C(\delta) \int_{n^{-\delta}}^{\infty} (xn^{1/2})^{-\alpha+\delta} dx \leqq C_2 C(\delta) n^{-\alpha/2-\delta}$$

So we see that there is a $C > 0$ such that

$$\sup_{s \in [1, \log n]} |J_1(n,s)| \leqq C n^{-(\alpha-2)/2-\delta} \tag{13}$$

Note that

$$|J_2(n,s)| \leqq n^2 \bar{F}(t_n)^2 \tag{14}$$

So Equations (7) - (14) imply our assertion. □

PROPOSITION 20. *Then there is a $C > 0$ such that*

$$\sup_{s \in [1, \log n)} |P(\sum_{k=1}^{n} X_k > sn^{1/2}) - G(n,s)| \leqq C n^{-(\alpha-2)/2-\delta/4}, \qquad n = 3, 4, \dots .$$

PROOF. Note that

$$\bar{\eta}_k(t_n) + \int_0^{t_n} x^k \mu(dx) = \int_0^{\infty} x^k \mu(dx), \qquad k = 1, 2, \dots, K-1,$$

and

$$\eta_K(t_n) - \int_0^{t_n} x^K \mu(dx) = \int_{-\infty}^{0} x^K \mu(dx).$$

So our assetion is an easy consequence of Propositions 17, 18, 19. □

PROPOSITION 21. *There is a $C > 0$ such that*

$$\sup_{s \in [\log n, \infty)} I_0(n,s) \leqq C n^{-(\alpha-2)/2-\delta/4}, \qquad n = 3, 4, \dots .$$

PROOF. We have

$$I_0(n,s) \leqq \mu(t_n)^{*n}((sn^{1/2}, \infty))$$

$$= \Phi_0(s) + \frac{1}{2\pi} \int_{\mathbf{R}} \frac{e^{-is\xi}}{i\xi} (\varphi(n^{-1/2}\xi, \mu(t_n))^n - \exp(-\frac{\xi^2}{2}))d\xi.$$

$$= \Phi_0(s) + \frac{1}{2\pi} \int_{\mathbf{R}} \frac{e^{-is\xi}}{i\xi} (\varphi(n^{-1/2}\xi, \mu(t_n))^n - \exp(-\frac{\xi^2}{2}))d\xi.$$

$$\leqq \Phi_0(s) + \frac{1}{2\pi} \int_{\mathbf{R}} \frac{e^{-is\xi}}{i\xi} (\psi_0(n,\xi) + n\varphi_1(n^{-1/2}\xi, t_n) \exp(-\frac{\xi^2}{2}))d\xi.$$

$$+\frac{1}{2\pi} \int_{\mathbf{R}} \frac{e^{-is\xi}}{i\xi} R_{n,0}(\xi) \exp(-\frac{\xi^2}{2}))d\xi.$$

Since $\sup_{s \geqq \log n} |\Phi_k(s)|$ is of $O(n^{-M})$ for any $M \geqq 1$, we have our assetion similar to the proof of Proposition 18.

PROPOSITION 22. *There is a $C > 0$ such that*

$$\sup_{s \in [\log n, \infty)} |I_1(n,s) - n \int_{-\infty}^{s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx| \leqq Cn^{-(\alpha-2)/2-\delta/4}.$$

PROOF. Remind that

$$I_1(n,s) = n(1-\bar{F}(t_n))^{n-1} \int_{-\infty}^{\infty} \bar{F}((sn^{1/2}-x) \vee t_n)\mu(t_n)^{*(n-1)}(dx)$$

$$= n(1-\bar{F}(t_n))^{n-1} \{ \int_{-\infty}^{\infty} \bar{F}(((s-x)n^{1/2}) \vee t_n)\Phi_1(x)dx$$

$$+\frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{F}(((s-x)n^{1/2}) \vee t_n)$$

$$\times (\int_{\mathbf{R}} \frac{e^{-ix\xi}}{i\xi} (\varphi(n^{-1/2}\xi, \mu(t_n))^{n-1} - \exp(-\frac{\xi^2}{2}))d\xi \}.$$

Then similarly to the proof of Propositions 19 and 21, we have our assertion. □

Now Theorem 2 is a consequence of Propositions 20 and 22.

7. Preliminary for Theorem 3

PROPOSITION 23. *Let Y be a random variable , and assume that*

$$E[|Y|^2] < \infty \quad \textit{and} \qquad E[Y] = 0.$$

Then for any $s \in \mathbf{R} \setminus \{0\}$ and $b > 0$

$$E[\exp(sY1_{\{|Y| \leqq b\}})] \leqq 1 + |s|^2(1 + \frac{1}{|s|b}) \exp(|s|b)E[|Y|^2].$$

PROOF. First, note that

$$|\exp(x) - 1| \leqq 1 \vee \exp(x),$$

and

$$|\exp(x) - 1| = |\int_0^x e^y dy| \leqq |x|(1 \vee \exp(x)), \qquad x \in \mathbf{R}.$$

So we have

$$|\exp(x) - (1 + x)| = |\int_0^x (e^y - 1)dy| \leqq (|x| \wedge |x|^2)(1 \vee \exp(x))$$

for any $x \in \mathbf{R}$. Therefore we see that

$$|\exp(x) - (1 + x)| \leqq |x|^2 \exp(|x|), \qquad x \in \mathbf{R}.$$

This implies that

$$|E[\exp(sY1_{\{|Y| \leqq b\}})] - (1 + E[sY1_{\{|Y| \leqq b\}}])|$$
$$\leqq |s|^2 \exp(|s|b)E[|Y|^2].$$

Since

$$|E[sY1_{\{|Y| \leqq b\}}]| = |sE[Y, |Y| > b]| \leqq |s|b^{-1}E[|Y|^2],$$

we have our assertion. □

PROPOSITION 24. *Let X be a random variable and assume that*

$$E[|X|^2] < \infty \textit{ and } E[X] = 0.$$

Then for any $t > 0$ *and* $n \geqq 1$

$$n \log E[\exp(\pm \frac{1}{tn^{1/2}} X 1_{\{|X| \leqq tn^{1/2}\}})] \leqq \frac{6}{t^2} E[|X|^2].$$

PROOF. Let $Y = (1/t)X$, $s = \pm n^{-1/2}$, $b = n^{1/2}$, and apply Proposition 23. Since $\log(1+x) \leqq x$, $x \geqq 0$, we have our assertion. $\square$

Now let X_n, $n = 1, 2, \ldots$, be independent identically distributed random variables. Throughout this section we assume that

$$E[|X_1|^2] < \infty \text{ and } E[X_1] = 0.$$

PROPOSITION 25. *For any* $s, t > 0$ *and* $\varepsilon > 0$

$$P(|\sum_{k=1}^n X_k 1_{\{|X_k| \leqq tn^{1/2}\}}| \geqq sn^{1/2}) \leqq 2 \exp(\frac{6}{t^2} E[|X_1|^2]) \exp(-\frac{s}{t}).$$

PROOF. We see that

$$P(\pm \sum_{k=1}^n X_k 1_{\{|X_k| \leqq tn^{1/2}\}} \geqq sn^{1/2})$$

$$\leqq \exp(-\frac{s}{t}) E[\exp(\frac{\pm 1}{tn^{1/2}} \sum_{k=1}^n X_k 1_{\{|X_k| \leqq tn^{1/2}\}})]$$

$$\leqq \exp(-\frac{s}{t}) E[\exp(\frac{\pm 1}{tn^{1/2}} X_1 1_{\{|X_1| \leqq tn^{1/2}\}})]^n.$$

Then by Proposition 24 we have our assertion.

Let $F : \mathbf{R} \to [0, 1]$ and $\bar{F} : \mathbf{R} \to [0, 1]$ be given by

$$F(x) = P(X_1 \leqq x), \qquad x \in \mathbf{R}$$

and

$$\bar{F}(x) = P(X_1 > x), \qquad x \in \mathbf{R}.$$

Then we have the following.

PROPOSITION 26. (1) *For any* $t, s > 0$, *and* $n \geqq 2$,

$$P(|\sum_{k=2}^{n} X_k 1_{\{|X_k| \leqq tn^{1/2}\}}| > sn^{1/2}) \leqq 2\exp(\frac{6}{t^2}E[|X_1|^2])\exp(-\frac{s}{t}).$$

(2) *For any* $s, t > 0$, $\varepsilon \in (0,1)$ *with* $t < (1-\varepsilon)s$,

$$\begin{aligned}|P(\sum_{k=1}^{n} X_k > sn^{1/2}) - nP(X_1 + \sum_{k=2}^{n} X_k 1_{\{|X_k| \leqq tn^{1/2}\}} > sn^{1/2},\\ |\sum_{k=2}^{n} X_k 1_{\{|X_k| \leqq tn^{1/2}\}}| \leqq \varepsilon sn^{1/2})|\\ \leqq 2n(n-1)(F(-tn^{1/2}) + \bar{F}(tn^{1/2}))^2 + 2\exp(\frac{6}{t^2}E[|X_1|^2])\exp(-\frac{s}{t})\\ +2n(F(-tn^{1/2}) + \bar{F}(tn^{1/2}))\exp(\frac{6}{t^2}E[|X_1|^2])\exp(-\frac{\varepsilon s}{2t})\end{aligned}$$

PROOF. Note that

$$\begin{aligned}&P(|\sum_{k=2}^{n} X_k 1_{\{|X_k| \leqq tn^{1/2}\}}| > sn^{1/2})\\ &= P(|\sum_{k=1}^{n-1} X_k 1_{\{|X_k| \leqq \tilde{t}(n-1)^{1/2}\}}| > \tilde{s}(n-1)^{1/2}),\end{aligned}$$

where

$$\tilde{t} = t(\frac{n}{n-1})^{1/2}, \qquad \tilde{s} = s(\frac{n}{n-1})^{1/2}.$$

So we have the assertion (1) from Proposition 25.

Let us denote

$$\tilde{F}(x) = P(|X_1| > x) \leqq F(-x) + \bar{F}(x), \qquad x > 0.$$

Note that

$$P(\sum_{k=1}^{n} X_k > sn^{1/2}) = \sum_{m=0}^{n} I_m,$$

where

$$I_m = P(\sum_{k=1}^n X_k > sn^{1/2}, \sum_{k=1}^n 1_{\{|X_k|>tn^{1/2}\}} = m), \qquad m = 0, 1, \dots, n.$$

Then we have

$$I_m = \binom{n}{m} P(\sum_{k=1}^n X_k > sn^{1/2},\ |X_i| > tn^{1/2},\ i = 1, \dots, m,$$
$$|X_j| \leqq tn^{1/2},\ j = m+1, \dots, n),$$

for $m = 0, 1, \dots, n$. So we see that

$$\sum_{m=2}^n I_m \leqq \sum_{m=2}^n \frac{n(n-1)}{m(m-1)} \binom{n-2}{m-2} \tilde{F}(tn^{1/2})^m (1 - \tilde{F}(tn^{1/2}))^{n-m}$$
$$\leqq \frac{n(n-1)}{2} \tilde{F}(tn^{1/2})^2. \tag{15}$$

Also, by Proposition 25, we have

$$I_0 \leqq 2\exp(-\frac{s}{t})\exp(\frac{6}{t^2}E[|X_1|^2]). \tag{16}$$

Let

$$A_1 = \{|X_1| > tn^{1/2}\}, \qquad A_2 = \{|X_k| \leqq tn^{1/2}, k = 2, 3, \dots, n\},$$
$$B_1 = \{X_1 + \sum_{k=2}^n X_k 1_{\{|X_k| \leqq tn^{1/2}\}} > sn^{1/2}\},$$

and

$$B_2 = \{|\sum_{k=2}^n X_k 1_{\{|X_k| \leqq tn^{1/2}\}}| \leqq \varepsilon sn^{1/2}\}.$$

Note that $B_1 \cap B_2 \subset A_1$, since $t < (1-\varepsilon)s$. So we see that

$$|P(B_1 \cap A_1 \cap A_2) - P(B_1 \cap B_2)|$$
$$\leqq P(B_1 \cap B_2^c \cap A_1 \cap A_2) + P(B_1 \cap B_2 \cap A_1 \cap A_2^c)$$
$$\leqq P(A_1)P(B_2^c) + P(A_1)P(A_2^c). \tag{17}$$

Note that

$$P(A_2^c) \leqq \sum_{k=2}^{n} P(|X_k| > tn^{1/2}) = (n-1)\tilde{F}(tn^{1/2}).$$

Also, by the assertion (1) we have

$$P(B_2^c) \leqq 2\exp(\frac{6}{t^2}E[|X_1|^2])\exp(-\frac{\varepsilon s}{2t}).$$

Since $I_1 = nP(B_1 \cap A_1 \cap A_2)$, we have the assertion from Equations (15), (16) and (17).

This completes the proof. $\square$

8. Some Estimates

In this section, we assume that (A-1) and (A-5).

Let $g : (x_0, \infty) \to \mathbf{R}$, $H : [-1/2, 1/2] \times (2x_0, \infty) \to (0, \infty)$ and $R : [-1/2, 1/2] \times (2x_0, \infty) \to (0, \infty)$ be given by

$$g(x) = x^2 \frac{d^2}{dx^2}(\log \bar{F})(x) - \alpha, \qquad x > x_0,$$

$$H(y; x) = \frac{\bar{F}(x(1+y))}{\bar{F}(x)}, \qquad y \in [-1/2, 1/2],\ x > 2x_0,$$

and

$$R(y; x) = H(y; x) - \{1 - \alpha y + \frac{\alpha(\alpha+1)y^2}{2}\}, \qquad y \in [-1/2, 1/2],\ x > 2x_0,$$

We prove the following in this section.

Proposition 27. *There are functions $a : (2x_0, \infty) \to \mathbf{R}$, $c : (2x_0, \infty) \to [0, \infty)$ and a constant $C > 0$ such that $a(x) \to 0$ and $c(x) \to 0$, as $x \to \infty$, and that*

$$|R(y; x) - a(x)y| \leqq C(c(x)y^2 + |y|^3), \qquad y \in [-1/2, 1/2],\ x > 2x_0.$$

First we prove the following.

Proposition 28. (1) *For any $x > x_0$,*

$$\frac{d}{dx}\log(x^\alpha \bar{F}(x)) = -\int_x^\infty \frac{g(z)}{z^2}.$$

(2) *For any $y \in [-1/2, 1/2]$ and $x > 2x_0$,*

$$H(y;x) = (1+y)^{-\alpha}\exp(-\int_0^y dy' \int_{1+y'}^\infty \frac{g(xz)}{z^2}dz).$$

Proof. Note that

$$g(x) = x^2 \frac{d^2}{dx^2}(\log(x^\alpha \bar{F}(x)))$$

and $g(x) \to 0$ as $x \to \infty$. Then we see that

$$\frac{d}{dy}(\log(y^\alpha \bar{F}(y))) - \frac{d}{dx}(\log(x^\alpha \bar{F}(x))) = \int_x^y \frac{g(z)}{z^2}dz, \tag{18}$$

and so we see that

$$c_0 = \lim_{y\to\infty}\frac{d}{dy}(\log(y^\alpha \bar{F}(y)))$$

exists. Note that

$$\exp(\int_x^{2x}\frac{d}{dy}(\log(y^\alpha \bar{F}(y)))dy) = \frac{L(2x)}{L(x)} \to 1, \qquad x \to \infty.$$

So we see that $c_0 = 0$. Therefore letting $y \to \infty$ in Equation (18) we have the assertion (1).

By the assertion (1), we have

$$\frac{d}{dy}\log((1+y)^\alpha H(y;x)) = -x\int_{x(1+y)}^\infty \frac{g(z)}{z^2}dz = -\int_{1+y}^\infty \frac{g(xz)}{z^2}dz.$$

Since $H(0;x) = 1$, we have the assertion (2). $\square$

Proposition 29. *Let $\tilde{a} : (2x_0, \infty) \to \mathbf{R}$ and $\tilde{c} : (2x_0, \infty) \to \mathbf{R}$ be given by*

$$\tilde{a}(x) = \frac{d}{dy}((1+y)^\alpha H(y,x))|_{y=0},$$

and

$$\tilde{c}(x) = \sup_{y\in[-1/2,1/2]} |\frac{d^2}{dy^2}((1+y)^\alpha H(y,x))|.$$

Then $\tilde{a}(x) \to 0$ *and* $\tilde{c}(x) \to 0,$ *as* $x \to \infty,$ *and that*

$$|H(y;x) - (1+y)^{-\alpha} - \tilde{a}(x)y(1+y)^{-\alpha}| \leqq 2^\alpha \tilde{c}(x)y^2, \qquad y \in [-1/2, 1/2], \ x > 2x_0.$$

PROOF. By Proposition 11 We have

$$\frac{d}{dy}((1+y)^\alpha H(y;x)) = -(1+y)^\alpha H(y;x)\int_{1+y}^\infty \frac{g(xz)}{z^2}dz$$

and so

$$\tilde{a}(x) = -\int_1^\infty \frac{g(xz)}{z^2}dz$$

Similarly, we have

$$\frac{d^2}{dy^2}((1+y)^\alpha H(y;x))$$

$$= (1+y)^\alpha H(y;x)\{(\int_{1+y}^\infty \frac{g(xz)}{z^2}dz)^2 - (1+y)^{-2}g(x(1+y))\}$$

Therefore we have

$$\tilde{c}(x) \leqq 2^\alpha\{(\int_{1/2}^\infty \frac{|g(xz)|}{z^2}dz)^2 + 4\sup\{|g(x(1+y))|;\ y\in[-1/2,1/2]\}\}$$

$$\times \exp(\int_{1/2}^\infty \frac{|g(xz)|}{z^2}dz)$$

These imply that $\tilde{a}(x) \to 0$, $\tilde{c}(x) \to 0$, as $x \to \infty$. Also we have

$$|(1+y)^\alpha H(y;x) - (1+\tilde{a}(x))| \leqq \tilde{c}(x)y^2, \qquad x \geqq 2x_0,\ y \in [-1/2.1/2].$$

This implies our assetion. $\square$

Now Proposition 27 is an easy corollary to Proposition 29.

9. Proof of Thoerem 3.

In this section, we assume that X_n, $n = 1, 2, \dots$, are i.i.d. random variables, $\alpha > 2$ and (A-1) - (A-5) are satisfied. Let $p = (\alpha + 2)/2$ and $\beta = (\alpha + p)/2$. Then we see that $E[|X_1|^p] < \infty$ and there is a $C_0 > 1$ such that

$$F(-x) + \bar{F}(x) \leqq C_0 x^{-\beta}, \qquad x \geqq 1.$$

PROPOSITION 30. *Let $b(x) = E[X_1, |X_1| \leqq x] = -E[X_1, |X_1| > x]$, $x > 0$. Then we have the following.*
(1) $|b(x)| \leqq E[|X_1|^p]^{1/p}(F(-x) + \bar{F}(x))^{1-1/p} \leqq C_0 x^{-\beta(p-1)/p} E[|X_1|^p]^{1/p}$, $x \geqq 1$.
(2) *There is a constant $C_1 > 1$ only dependent on p such that*

$$E[|\sum_{k=1}^n X_k 1_{\{|X_k| \leqq x\}}|^p]^{1/p} \leqq C_1 n^{1/2}(E[|X_1|^p]^{1/p} + |b(x)|) + n|b(x)|$$

$$\leqq C_1 E[|X_1|^p]^{1/p}(1 + C_0)(n^{1/2} + n x^{-\beta(p-1)/p})$$

for any $n = 1, 2, \dots$, and $x \geqq 1$.

PROOF. The assertion (1) is an easy consequence of Hölder's inequality. So we prove the assertion (2). Since $E[X_k 1_{\{|X_k| \leqq x\}} - b(x)] = 0$, $k = 1, 2, \dots$, we see by Burkholder-Davis-Gundy's theorem that there is a constant $C_1 > 0$ depending on p only such that

$$E[|\sum_{k=1}^n (X_k 1_{\{|X_k| \leqq x\}} - b(x))|^p]^{1/p} \leqq C_1 E[|\sum_{k=1}^n (X_k 1_{\{|X_k| \leqq x\}} - b(x))^2|^{p/2}]^{1/p}$$

Then by Hölder's inequality, we have

$$E[|\sum_{k=1}^n (X_k 1_{\{|X_k| \leqq x\}} - b(x))|^p]^{1/p} \leqq C_1 E[n^{p/2-1} \sum_{k=1}^n |X_k 1_{\{|X_k| \leqq x\}} - b(x)|^p]^{1/p}$$

$$= C_1 n^{1/2} E[|X_1 1_{\{|X_1| \leqq x\}} - b(x)|^p]^{1/p} \leqq C_1 n^{1/2}(E[|X_1 1_{\{|X_1| \leqq x\}}|^p]^{1/p} + |b(x)|)$$

This implies our assertion. □

Let $a : (2x_0, \infty) \to \mathbf{R}$ and $c : (2x_0, \infty) \to [0, \infty)$ be as in Proposition 27. Also, let

$$Y_n(t) = \sum_{k=1}^{n} X_k 1_{\{|X_k| \leqq tn^{1/2}\}}, \qquad n \geqq 2,\ t > 0.$$

Then we have the following.

PROPOSITION 31. *Let $r \in ((\alpha+2)/(2\alpha), 1)$. Then for any $\varepsilon \in (0, 1/2)$*

$$\begin{aligned} \overline{\lim}_{n\to\infty} \sup\{s^2 E[|H(-\frac{1}{sn^{1/2}} Y_n(t), sn^{1/2}),& \\ |Y_n(t)| \leqq \varepsilon sn^{1/2}] - (1 + \frac{\alpha(\alpha+1)}{2s^2})|;& \\ s \geqq (\log n)^{1/2},\ t \geqq (\log n)^{-1} s^{(1+r)/2}\} = 0.& \end{aligned}$$

PROOF. Let $s \geqq (\log n)^{1/2}$, $t \geqq (\log n)^{-1} s^{(1+r)/2}$, and $n \geqq 3$. Then $tn^{1/2} \geqq 1$. Note that

$$r\beta(p-1)/p > 1 + \frac{3(\alpha-2)}{8} > 1.$$

We see that

$$\begin{aligned} &E[H(-\frac{1}{sn^{1/2}} Y_n(t), sn^{1/2}),\ |Y_n(t)| \leqq \varepsilon sn^{1/2}] - (1 + \frac{\alpha(\alpha+1)}{2s^2}) \\ &= \frac{a(sn^{1/2}) - \alpha}{sn^{1/2}} E[Y_n(t)] + \frac{\alpha(\alpha+1)}{2s^2 n}(E[Y_n(t)^2] - n) \\ &-E[1 + \frac{a(sn^{1/2}) - \alpha}{sn^{1/2}} Y_n(t) + \frac{\alpha(\alpha+1)}{2s^2 n} Y_n(t)^2, |(sn^{1/2})^{-1} Y_n(t)| > \varepsilon] \\ &\quad + E[R(-\frac{1}{sn^{1/2}} Y_n(t), sn^{1/2}), |(sn^{1/2})^{-1} Y_n(t)| \leqq \varepsilon]. \end{aligned}$$

Note that

$$\begin{aligned} s|E[Y_n(t)]| = ns|b(tn^{1/2})| &\leqq C_0 E[|X_1|^p]^{1/p} s (tn^{1/2})^{-\beta(p-1)/p} \\ &\leqq C_0 E[|X_1|^p]^{1/p} (n^{1/2}(\log n)^{-1})^{-\beta(p-1)/p} s^{1-(1+r)\beta(p-1)/2p}, \\ E[Y_k(t)^2] - n = n(E[(X_1 1_{|X_1| \leqq tn^{1/2}})^2] &- b(tn^{1/2})^2) + E[Y_n(t)]^2 - n \end{aligned}$$

$$= -nE[X_1^2, |X_1| > tn^{1/2}] + n(n-1)b(tn^{1/2})^2,$$

and

$$n^{-p/2}E[|Y_n(t)|^p] \leqq C_1^p(1+C_0)^p E[|X_1|^p](1+t^{-\beta(p-1)/p)}n^{1/2(1-\beta(p-1)/p)})^p.$$

So we see that

$$\begin{aligned}\frac{1}{n}|E[Y_k(t)^2] - n| \leqq E[X_1^2, |X_1| > tn^{1/2}]&\\ + C_0(n-1)n^{-\beta(p-1)/p}(\log n)^{2\beta(p-1)/p}E[|X_1|^p]^{2/p}&,\\ s^2(sn^{1/2})^{-k}E[|Y_n(t)|^k, |(sn^{1/2})^{-1}Y_n(t)| > \varepsilon]&\\ \leqq s^{2-p}\varepsilon^{-p+k}n^{-p/2}E[|Y_n(t)|^p],& \qquad k = 0,1,2,\end{aligned}$$

and

$$\begin{aligned}&s^2E[|R(-\frac{1}{sn^{1/2}}Y_n(t), sn^{1/2})|, |(sn^{1/2})^{-1}Y_n(t)| \leqq \varepsilon],\\ &\leqq C_2(|c(sn^{1/2})|n^{-1}E[Y_n(t)^2] + \varepsilon^{-p}s^{2-p}n^{-p/2}E[|Y_n(t)|^p]).\end{aligned}$$

Combining them, we have our assertion. □

Now we prove Theorem 3. Let $\beta : \mathbf{N} \to (0,\infty)$ be such that

$$\frac{\beta(n)}{(\log n)^{1/2}} \to \infty, \qquad n \to \infty.$$

Assume that Theorem 3 is not valid. Then there is a sequence of positive numbers $\{s_n'\}_{n=1}^\infty$ such that $s_n' \geqq n^{1/2}\beta(n)$, $n = 1,2,\ldots$, and

$$\overline{\lim}_{n\to\infty}\frac{(s_n')^2}{n}|\frac{P(\sum_{k=1}^n X_k > s_n')}{n\bar{F}(s_n')} - (1+\frac{\alpha(\alpha+1)vn}{(s_n')^2})| > 0.$$

Let $s_n = n^{-1/2}s_n' \geqq \beta(n)$. Let us take an $r \in ((\alpha+2)/(2\alpha), 1)$ and fix it. Let t_n, $n = 1,2,\ldots$, be a sequence of positive numbers given by

$$t_n = (\log n)^{-1/2} + (\log n)^{-1}s_n^{(1+r)/2}, \qquad n \geqq 2.$$

Then we have the following.

$$t_n s_n \geqq \frac{\beta(n)}{(\log n)^{1/2}} \to \infty, \quad n \to \infty, \tag{19}$$

$$\frac{2s_n}{t_n} \geqq ((\log n)^{1/2} s_n) \wedge ((\log n) s_n^{(1-r)/2}), \quad n \geqq 2, \tag{20}$$

$$\frac{2s_n}{(\log n)t_n} \geqq (\frac{\beta(n)}{(\log n)^{1/2}}) \wedge (s_n^{(1-r)/2}) \to \infty, \quad n \to \infty, \tag{21}$$

and

$$\frac{(t_n n^{1/2})^2}{(s'_n)^{1+r}} \geqq \frac{(\log n)^{-2} s_n^{1+r} n}{s_n^{1+r} n^{(1+r)/2}} \to \infty, \quad n \to \infty. \tag{22}$$

Therefore by Equation (22), we have

$$\frac{(F(-t_n n^{1/2}) + \bar{F}(t_n n^{1/2}))^2}{\bar{F}(s'_n)^{2r}} \to 0, \quad n \to \infty.$$

Since $2r - 1 > 2/\alpha$, we have

$$(s'_n)^2 \frac{(F(-t_n n^{1/2}) + \bar{F}(t_n n^{1/2}))^2}{\bar{F}(s'_n)} \to 0, \quad n \to \infty. \tag{23}$$

Also, by Equations (19), (20) and (21) we see that for any $m \geqq 1$

$$(ns'_n)^m \exp(\frac{m}{t_n^2} - \frac{1}{m}\frac{s_n}{t_n})$$

$$= \exp(\frac{m}{t_n^2}(1 - \frac{1}{3m^2} t_n s_n)) n^{3m/2} \exp(-(\log n)\frac{1}{3m}\frac{s_n}{(\log n)t_n})$$

$$\times (s_n)^m \exp(-\frac{1}{3m}\frac{s_n}{t_n}) \to 0, \qquad n \to \infty.$$

Note that

$$\frac{(n+1)P(X_{n+1} + Y_n(t_{n+1}) > s_{n+1}(n+1)^{1/2},\ |Y_n(t_{n+1})| \leqq \varepsilon s_{n+1}(n+1)^{1/2})}{(n+1)\bar{F}(s_{n+1}(n+1)^{1/2})}$$

$$= E[H(-\frac{1}{s_{n+1}(n+1)^{1/2}} Y_n(t_{n+1}), s_{n+1}(n+1)^{1/2}),\ |Y_n(t_{n+1})| \leqq \varepsilon s_{n+1}(n+1)^{1/2}].$$

From Proposition 31, we see that

$$s_{n+1}^2 |E[H(-\frac{1}{s_{n+1}(n+1)^{1/2}} Y_n(t_{n+1}), s_{n+1}(n+1)^{1/2}),$$

$$|Y_n(t_{n+1})| \leqq \varepsilon s_{n+1}(n+1)^{1/2}]$$

$$- (1 + \frac{\alpha(\alpha+1)(n+1)}{s_{n+1}^2(n+1)})| \to 0$$

as $n \to \infty$, by letting $s = s_{n+1}(1+1/n)^{1/2}$, $t = t_{n+1}$. Then from this, Proposition 26(2), Equations (19), (20), (21), (22) and (23), we have

$$\frac{(s'_n)^2}{n}\left|\frac{P(\sum_{k=1}^n X_k > s'_n)}{n\bar{F}(s'_n)} - (1+\frac{\alpha(\alpha+1)n}{(s'_n)^2})\right| \to 0$$

as $n \to \infty$. This is a contradiction.

This proves Theorem 3.

10. Proof of Theorem 4

Let $\hat{F}_n : [1,\infty) \to [0,1]$, $n \geqq 1$, be given by

$$\hat{F}_n(s) = \int_{-\infty}^{s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx.$$

Then we have the following.

Proposition 32. *Let $\beta : \mathbf{N} \to (0,\infty)$ be such that*

$$\frac{\beta(n)}{(\log n)^{1/2}} \to \infty, \qquad n \to \infty.$$

Then

$$\sup_{s \geqq \beta(n)} s^2\left|\frac{\hat{F}_n(s)}{\bar{F}(n^{1/2}s)} - (1+\frac{\alpha(\alpha+1)n}{s^2})\right| \to 0, \qquad n \to \infty.$$

Proof. By Proposition 27, we see that

$$\left|\frac{\hat{F}_n(s)}{\bar{F}(sn^{1/2})} - (1+\frac{\alpha(\alpha+1)}{s^2})\right|$$

$$\leqq \int_{[-s/2,s/2]} |R(y/s; sn^{1/2}) - a(sn^{1/2})(y/s)|\Phi_1(y)dy$$

$$+\int_{[-s/2,s/2]^c} 4(1+(|a(sn^{1/2})|+\alpha(\alpha+1))(\frac{y^2}{s^2})\Phi_1(y)dy.$$

This and Proposition 27 imply our assetion. □

It is well known (e.g. Williams [8]) that there is a $C_0 > 0$ such that

$$|\Phi_k(x)| \leqq C_0(1+x)^{k-1}\Phi_1(x), \qquad x \geqq 0,\ k = 1, \dots, 3K, \tag{24}$$

and

$$C_0^{-1}\Phi_1(x) \leqq x\Phi_0(x) \leqq C_0\Phi_1(x), \qquad x \geqq 1. \tag{25}$$

Let

$$H_0(n,s) = \Phi_0(s) + n\hat{F}_n(s),$$

$$A_1(n,s) = \sum_{k=1}^{2} \frac{n^{-(k-2)/2}}{k!}\Phi_k(s)\int_0^\infty x^k \mu(dx),$$

and

$$A(n,s) = n\hat{F}_n(s) - A_1(n,s).$$

First we prove the following.

Proposition 33.

$$\sup_{s\in[1,\log n]} \frac{|A(n,s) - n\bar{F}(n^{1/2}s)|}{H_0(n,s)} \to 0, \qquad n \to \infty.$$

Proof. Let us take a $\gamma \in (0, (\alpha-2)/(4\alpha))$ and fix it. Let $s \geqq 0$ and $n \geqq 3$. Note that

$$\hat{F}_n(s) = \sum_{k=1}^{4} I_k(n,s),$$

where

$$I_1(n,s) = \int_{s-n^{-\gamma}}^{s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx,$$

$$I_2(n,s) = \int_{-s}^{7s/8} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx,$$

$$I_3(n,s) = \int_{7s/8}^{s-n^{-\gamma}} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx,$$

and

$$I_4(n,s) = \int_{-\infty}^{-s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx.$$

It is easy to see that

$$I_1(n,s) = n^{-1/2}\int_0^{n^{(1/2-\gamma)}} \bar{F}(y)\Phi_1(s-n^{-1/2}y)dy.$$

Let

$$R(n,s,y) = \Phi_1(s-n^{-1/2}y) - (\Phi_1(s) + n^{-1/2}y\Phi_2(s))$$

Then for $y \in [0, sn^{1/2-\gamma}]$

$$|R(n,s,y)| \leqq n^{-1}y^2 \sup_{z\in[s-n^{-\gamma},s]} |\Phi_3(z)|$$

$$\leqq C_0 n^{-1}y^2(1+s)^2\Phi_1(s-n^{-\gamma}) = C_0 n^{-1}y^2(1+s)^2\Phi_1(s)\exp(sn^{-\gamma} - n^{-2\gamma}/2)$$
$$\leqq C_0^2 n^{-1}y^2(1+s)^3\exp(n^{-\gamma}s)\Phi_0(s).$$

So we see that

$$n|I_1(n,s) - \sum_{k=1}^2 \frac{n^{-k/2}}{k!}\Phi_k(s)\int_0^\infty x^k\mu(dx)|$$

$$\leqq C_0^2(1+s)^3 n^{-1/2}\exp(n^{-\gamma}s)(\int_0^{n^{1/2-\gamma}} y^2\bar{F}(y)dy)\Phi_0(s)$$
$$+C_0^2(1+s)n^{1/2}(\int_{n^{1/2-\gamma}}^\infty \bar{F}(y)dy)\Phi_0(s) + C_0^2(1+s)^2(\int_{n^{1/2-\gamma}}^\infty y\bar{F}(y)dy)\Phi_0(s)$$

This implies that

$$\sup_{s\in[1,\log n]} \Phi_0(s)^{-1}|nI_1(n,s) - \sum_{k=1}^2 \frac{n^{-(k-2)/2}}{k!}\Phi_k(s)\int_0^\infty x^k\mu(dx)| \to 0, \quad n\to\infty. \tag{26}$$

Note that

$$I_2(n,s) = \bar{F}(sn^{1/2})\int_{-s}^{7s/8}(1-\frac{x}{s})^{-\alpha}\frac{L((s-x)n^{1/2})}{L(sn^{1/2})}\Phi_1(x)dx$$

It is easy to see that

$$\sup_{s\in[(\log n)^{1/4},\log n]} \left|\int_{-s}^{7s/8}(1-\frac{x}{s})^{-\alpha}\frac{L((s-x)n^{1/2})}{L(sn^{1/2})}\Phi_1(x)dx-1\right| \to 0,$$

$$n\to\infty$$

Also we see that

$$n|I_2(n,s)| \leqq n\bar{F}(sn^{1/2})8^{\alpha}\int_{-s}^{7s/8}\frac{L((s-x)n^{1/2})}{L(sn^{1/2})}\Phi_1(x)dx$$

Therefore we have

$$\sup_{s\in[1,(\log n)^{1/4}]} \Phi_0(s)^{-1}(n|I_2(n,s)|+n\bar{F}(sn^{1/2}))\to 0, \qquad n\to\infty.$$

Thus we have

$$\sup_{s\in[1,\log n]} H_0(n,s)^{-1}|nI_2(n,s)-n\bar{F}(sn^{1/2})|\to 0, \qquad n\to\infty. \tag{27}$$

Note that $\sqrt{3}/2\leqq 7/8$. Then we have

$$\Phi_1(7s/8)\leqq(\Phi_1(s))^{3/4},$$

and so we have

$$\begin{aligned} nI_3(n,s) &\leqq ns\bar{F}(n^{1/2-\gamma})\Phi_1(7s/8) \\ &\leqq (n\bar{F}(n^{1/2}\log n))^{1/2}(s\Phi_1(s))^{3/4}\frac{ns^{1/4}\bar{F}(n^{1/2-\gamma})}{(n\bar{F}(n^{1/2}\log n))^{1/2}}. \end{aligned}$$

Since

$$\sup_{n\geqq 3}\sup_{s\in[1,\log n]}\frac{ns^{1/4}\bar{F}(n^{1/2-\gamma})}{(n\bar{F}(n^{1/2}\log n))^{1/2}}<\infty,$$

we see that there is a constant $C>0$ such that

$$nI_3(n,s)\leqq C(n\bar{F}(sn^{1/2}))^{1/2}\Phi_0(s)^{3/4}\leqq C(n\bar{F}(sn^{1/2}))^{1/4}H_0(n,s),$$

$$n\geqq 3,\ s\in[1,\log n].$$

So we have

$$\sup_{s\in[1,\log n]} H_0(n,s)^{-1}|nI_3(n,s)|\to 0, \qquad n\to\infty. \tag{28}$$

Also we have

$$n|I_4(n,s))| \leqq n\bar{F}(2sn^{1/2})\Phi_0(s).$$

So this equation, Equations (26) (27) and (28) imply our assertion. □

PROPOSITION 34. (1) *There is a $C > 0$ such that*

$$\Phi_0(s) + |A_1(n,s)| \leqq Cn^{-2}\bar{F}(n^{1/2}s), \qquad n \geqq 2,\ s \geqq \log n.$$

(2)

$$\sup_{s\in[1,\infty)} \frac{|A(n,s) - n\bar{F}(n^{1/2}s)|}{H_0(n,s)} \to 0, \qquad n \to \infty.$$

PROOF. The assertion (1) is obvious from Equations (24) and (25). To prove the assertion (2), because of Proposition 33, it is sufficient to prove

$$\sup_{s\in[\log n,\infty)} \frac{|A(n,s) - n\bar{F}(n^{1/2}s)|}{H_0(n,s)} \to 0, \qquad n \to \infty.$$

However, by Theorem 3 and Propsosition 32, we see that

$$\sup_{s\in[\log n,\infty)} |\frac{n\hat{F}(n^{1/2}s)}{n\bar{F}(n^{1/2}s)} - 1| \to 0, \qquad n \to \infty.$$

Therefore, combining this with the assertion (1), we have the assertion (2). □

Now let us prove Theorem 4.

By Proposition 34(2), we see that

$$\sup_{s\in[1,\infty)} |\frac{H(n,s)}{H_0(n,s)} - 1| \to 0, \qquad n \to \infty.$$

Therefore there is an n_0 such that

$$H(n,s) \geqq \frac{1}{2}H_0(n,s), \qquad n \geqq n_0,\ s \geqq 1.$$

By Equation (24), we see that there is a $C > 0$ such that

$$\sup_{s\in[1,n^{1/12}]} |G(n,s) - H(n,s)| \leqq Cn^{-1/12}, \qquad n \geqq 1$$

Then combining this with Theorem 2, we see that there are $C > 0$ and $\delta_0 \in (0, 1/12)$ such that

$$\sup_{s\in[1,n^{\delta_0}]} \left|\frac{P(\sum_{k=1}^n X_k > n^{1/2}s)}{H(n,s)} - 1\right| \leqq Cn^{-\delta_0}, \qquad n \geqq n_0.$$

On the other hand, by Theorem 3 and Propsosition 32 we see that there is a $C > 0$ such that

$$\sup_{s\in[n^{\delta_0},\infty)} \left|\frac{P(\sum_{k=1}^n X_k > n^{1/2}s)}{n\hat{F}_n(s)} - 1\right| \leqq Cn^{-2\delta_0}, \qquad n \geqq n_0.$$

So we see by Proposition 34 that we see that there is a $C > 0$ such that

$$\sup_{s\in[n^{\delta_0},\infty)} \left|\frac{P(\sum_{k=1}^n X_k > n^{1/2}s)}{H(n,s)} - 1\right| \leqq Cn^{-2\delta_0}, \qquad n \geqq n_0.$$

These imply Theorem 4.

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