

Analysis and Estimation of Error Constants for P_0 and P_1 Interpolations over Triangular Finite Elements

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Abstract. We give some fundamental results on the error constants for the piecewise constant interpolation function and the piecewise linear one over triangles. For the piecewise linear one, we mainly analyze the conforming case, but the present results also appear to be available for the non-conforming case. We obtain explicit relations for the upper bounds of the constants, and analyze dependence of such constants on the geometric parameters of triangles. In particular, we explicitly determine some special constants including the Babuška-Aziz constant, which plays an essential role in the interpolation error estimation of the linear triangular finite element. The obtained results are expected to be widely used for a priori and a posteriori error estimations in adaptive computation and numerical verification of numerical solutions based on the triangular finite elements. We also give some numerical results for the error constants and for a posteriori estimates of some eigenvalues related to the error constants.

1. Introduction

The finite element method (FEM) is now recognized as a powerful numerical method for wide classes of partial differential equations. Furthermore, it also has sound mathematical bases such as highly refined a priori and a posteriori error estimations. In the classical a priori error analysis of FEM, the interpolation error analysis is essential to derive final error estimates in various norms and/or semi-norms [10, 11, 21]. In this process, there appear a number of positive constants besides the standard discretization (or mesh) parameter h and norms (or seminorms), but it has been very difficult to evaluate such constants explicitly. For quantitative purposes, however,

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it is indispensable to evaluate or bound them as accurately as possible, because sharper estimates enable more efficient finite element computations. Thus such evaluation has become progressively more important and has been attempted especially for adaptive finite element calculations based on a posteriori error estimation as well as for numerical verification by FEM [3, 6, 8, 10, 25]. In this paper, we will give some fundamental results on various interpolation error constants of the most popular triangular finite elements.

More specifically, we derive some fundamental estimates for the interpolation error constants appearing in the popular P_0 (piecewise constant) and P_1 (piecewise linear) triangular finite elements. Inspired by the monumental paper of Babuška-Aziz [5], we analyze the dependence of several constants on the geometric parameters such as the maximum interior angle and the minimum edge length of a triangle more quantitatively than works precedent to ours. Among them, the optimal constant (C_4 in the present paper) appearing in the H^1 error estimate of the P_1 interpolation of H^2 functions over the unit isosceles right triangle is essential and frequently used, and it was explicitly evaluated firstly by Natterer [27]. On the other hand, this constant was shown to be closely related to the one (C_1 in this paper) presented and effectively used by Babuška and Aziz in conjunction with the maximum angle condition [5]. More precisely, C_1 gives an upper bound quite close to the optimal constant C_4 , and the relation between C_4 and C_1 was further discussed in [25, 30]. Thus a precise estimation of these two constants is very important, and a number of researchers have given bounds for these using various approximation methods including numerical verification, see e. g. [4, 7, 22, 25, 26, 27, 30].

For the above Babuška-Aziz constant, we already obtained a value which is in a sense optimal [18]. That is, by analytically solving an eigenvalue problem for the 2D Laplacian over the above triangular domain with the aid of the reflection method [28], we showed that the constant can be easily determined from a solution of $\mu^{-1} + \tan(\mu^{-1}) = 0$. In this paper, we will also give some additional results for exact values or bounds of various error constants. Moreover, we will present some explicit relations for the dependence of such constants on the geometry of triangles. In particular, emphasis is put on the maximum angle condition presented in [5]. We also derive some analytical results on asymptotic behaviors when a right triangle becomes very thin or slender as may be seen in anisotropic triangulations, cf. [8].

Thus our results can be effectively used in the quantitative a priori and a posteriori error estimations of the finite element solutions by the P_1 triangular element and also those based on the P_0 triangle. The former is the most classical and fundamental one but still in frequent use, while the latter appears in some mixed finite element methods and implicitly on various occasions. Moreover, we also give some concrete a posteriori error estimates to eigenvalues related to several error constants. Numerical results are also obtained for the error constants and a posteriori estimates of some eigenvalues.

The plan of this paper is as follows. Section 1 is the present one on some historical remarks and overview of our analysis. Section 2 gives necessary notations and definitions, and also introduces various error constants to be analyzed. Section 3 deals with estimation of various interpolation error constants, and Section 4 analyzes asymptotic behaviors of such constants when the triangle is a thin right one. Section 5 gives application of our results to a posteriori estimation of some error constants by using the P_1 FEM. Section 6 is the one for numerical results, while Section 7 is for concluding remarks and acknowledgements. Appendix is also attached to give some additional theoretical and numerical results related to Section 4.

2. Preliminaries: Error Constants

Let h , α and θ be positive constants such that

$$(1) \quad h > 0, \quad 0 < \alpha \leq 1, \quad (\pi/3 \leq) \cos^{-1}(\alpha/2) \leq \theta < \pi.$$

Then we can define the triangle $T_{\alpha,\theta,h}$ by $\triangle OAB$ with three vertices $O(0,0)$, $A(h,0)$ and $B(\alpha h \cos \theta, \alpha h \sin \theta)$. From (1), AB is the edge of maximum length, i. e. $\overline{AB} \geq h \geq \alpha h$ with $h = \overline{OA}$ and $\alpha h = \overline{OB}$ being the medium and the minimum edge lengths, respectively. Notice here that the notation h is mostly used as the largest edge length as in [11], but our usage of h as the medium one may be convenient for the present purposes. A point on the closure of $T_{\alpha,\theta,h}$ is denoted by $x = \{x_1, x_2\}$, and the three edges e_1 , e_2 and e_3 of $T_{\alpha,\theta,h}$ are defined as

$$(2) \quad e_1 = OA, \quad e_2 = OB, \quad e_3 = AB.$$

We can configure any triangle as $T_{\alpha,\theta,h}$ by a congruent transformation with suitable α , θ and h . As in [5], we will use abbreviated notations $T_{\alpha,\theta} = T_{\alpha,\theta,1}$,

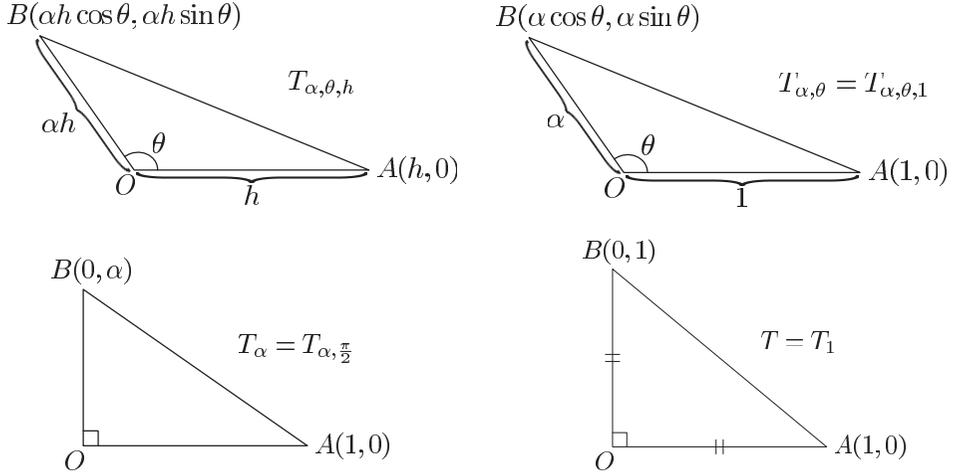


Fig. 1. Notations for triangles : $T_{\alpha, \theta} = T_{\alpha, \theta, 1}$, $T_{\alpha} = T_{\alpha, \pi/2}$, $T = T_1$

$T_{\alpha} = T_{\alpha, \pi/2}$ and $T = T_1$ (Fig. 1).

We will use the popular Hilbert space $L_2(T_{\alpha, \theta, h})$ with the norm $\|\cdot\|_{T_{\alpha, \theta, h}}$, where the subscript $T_{\alpha, \theta, h}$ will be often omitted. If we need the L_2 space and its norm for other domains like Ω , we will use $L_2(\Omega)$ and $\|\cdot\|_{\Omega}$. Let us define closed linear spaces for functions on $T_{\alpha, \theta, h}$ by

$$(3) \quad V_{\alpha, \theta, h}^0 = \{v \in H^1(T_{\alpha, \theta, h}) \mid \int_{T_{\alpha, \theta, h}} v(x) dx = 0\},$$

$$(4) \quad V_{\alpha, \theta, h}^i = \{v \in H^1(T_{\alpha, \theta, h}) \mid \int_{e_i} v ds = 0\} \quad (i = 1, 2, 3),$$

$$(5) \quad V_{\alpha, \theta, h}^4 = \{v \in H^2(T_{\alpha, \theta, h}) \mid v(O) = v(A) = v(B) = 0\},$$

where $H^1(T_{\alpha, \theta, h})$ and $H^2(T_{\alpha, \theta, h})$ are respectively the first- and second-order Sobolev spaces for real square integrable functions over $T_{\alpha, \theta, h}$ [2], and ds is the line element. For other domains like Ω , we will also use spaces such as $H^1(\Omega)$ and $H^2(\Omega)$ later. For the above spaces, we will again use abbreviated notations $V_{\alpha, \theta}^i = V_{\alpha, \theta, 1}^i$, $V_{\alpha}^i = V_{\alpha, \pi/2}^i$ and $V^i = V_1^i$ ($0 \leq i \leq 4$).

Let us consider the usual P_0 interpolation operator $\Pi_{\alpha, \theta, h}^0$ and P_1 one $\Pi_{\alpha, \theta, h}^1$ for functions on $T_{\alpha, \theta, h}$ [10, 11, 21]: $\Pi_{\alpha, \theta, h}^0 v$ for any $v \in H^1(T_{\alpha, \theta, h})$ is

a constant function well-defined by

$$(6) \quad (\Pi_{\alpha,\theta,h}^0 v)(x) = \int_{T_{\alpha,\theta,h}} v(y) dy / \int_{T_{\alpha,\theta,h}} dy \quad (\forall x \in T_{\alpha,\theta,h}),$$

while $\Pi_{\alpha,\theta,h}^1 v$ for any $v \in H^2(T_{\alpha,\theta,h})$ is an at most linear polynomial function such that

$$(7) \quad (\Pi_{\alpha,\theta,h}^1 v)(x) = v(x) \quad \text{for } x = O, A, B.$$

To analyze these interpolation operators, let us introduce the positive constants by

$$(8) \quad C_i(\alpha, \theta, h) = \sup_{v \in V_{\alpha,\theta,h}^i \setminus \{0\}} \|v\| / |v|_1 \quad (i = 0, 1, 2, 3),$$

$$C_4(\alpha, \theta, h) = \sup_{v \in V_{\alpha,\theta,h}^4 \setminus \{0\}} |v|_1 / |v|_2,$$

$$(9) \quad C_5(\alpha, \theta, h) = \sup_{v \in V_{\alpha,\theta,h}^4 \setminus \{0\}} \|v\| / |v|_2,$$

where $|v|_1 = (\sum_{i=1}^2 \|\partial v / \partial x_i\|^2)^{1/2}$, and $|v|_2 = (\sum_{i,j=1}^2 \|\partial^2 v / \partial x_i \partial x_j\|^2)^{1/2}$. When we need to specify a domain like Ω for the above semi-norms $|\cdot|_1$ and $|\cdot|_2$, we will use $|\cdot|_{1,\Omega}$ and $|\cdot|_{2,\Omega}$, respectively. The existence of these positive constants follows from the Rellich compactness theorem. Due to the properties to become clear soon, such constants together with some related ones are often called *interpolation error constants*. We will again use abbreviated notations $C_i(\alpha, \theta) = C_i(\alpha, \theta, 1)$, $C_i(\alpha) = C_i(\alpha, \pi/2)$ and $C_i = C_i(1)$ for $0 \leq i \leq 5$ as in [5].

By a simple scale change, we find that $C_i(\alpha, \theta, h) = hC_i(\alpha, \theta)$ ($i = 0, 1, 2, 3, 4$) and $C_5(\alpha, \theta, h) = h^2C_5(\alpha, \theta)$. These relations and constants are used to derive popular interpolation error estimates for $\Pi_{\alpha,\theta,h}^i$ ($i = 0, 1$) applied to functions on $T_{\alpha,\theta,h}$ [10, 11, 21]:

$$(10) \quad \|v - \Pi_{\alpha,\theta,h}^0 v\| \leq C_0(\alpha, \theta) h |v|_1 \quad (\forall v \in H^1(T_{\alpha,\theta,h})),$$

$$(11) \quad |v - \Pi_{\alpha,\theta,h}^1 v|_1 \leq C_4(\alpha, \theta) h |v|_2 \quad (\forall v \in H^2(T_{\alpha,\theta,h})),$$

$$(12) \quad \|v - \Pi_{\alpha,\theta,h}^1 v\| \leq C_5(\alpha, \theta) h^2 |v|_2 \quad (\forall v \in H^2(T_{\alpha,\theta,h})),$$

by noting $v - \Pi_{\alpha,\theta,h}^0 v \in V_{\alpha,\theta,h}^0$ for $v \in H^1(T_{\alpha,\theta,h})$ and $v - \Pi_{\alpha,\theta,h}^1 v \in V_{\alpha,\theta,h}^4$ for $v \in H^2(T_{\alpha,\theta,h})$.

Moreover, for the partial derivative $\partial v/\partial x_1$ of $v \in H^2(T_{\alpha,\theta,h})$, we have

$$(13) \quad \left\| \partial(v - \Pi_{\alpha,\theta,h}^1 v)/\partial x_1 \right\| \leq C_1(\alpha, \theta)h |\partial v/\partial x_1|_1 ,$$

since $\partial(v - \Pi_{\alpha,\pi/2,h}^1 v)/\partial x_1 \in V_{\alpha,\theta,h}^1$. On the other hand, to obtain an estimate in terms of $C_2(\alpha, \theta)$, let us rotate the x_1 - x_2 plane around the origin O by angle $\theta - \pi/2$ so that the edge OB becomes the ordinate. Then the coordinate transformation $\hat{x} = \Phi_\theta(x)$ between the original variable $x = \{x_1, x_2\}$ and the new one $\hat{x} = \{\hat{x}_1, \hat{x}_2\}$ is given by, along with the associated transformation $\hat{v} = v \circ \Phi_\theta^{-1}$ for $v \in H^2(T_{\alpha,\theta,h})$,

$$(14) \quad \hat{x}_1 = x_1 \sin \theta - x_2 \cos \theta, \quad \hat{x}_2 = x_1 \cos \theta + x_2 \sin \theta ,$$

$$(15) \quad \hat{v}(\hat{x}) = v(x) = v(\hat{x}_1 \sin \theta + \hat{x}_2 \cos \theta, -\hat{x}_1 \cos \theta + \hat{x}_2 \sin \theta) .$$

Based on essentially the same arguments as for $\partial v/\partial x_1$, we can show for $\partial \hat{v}/\partial \hat{x}_2$ that

$$(16) \quad \left\| \partial(\hat{v} - \hat{\Pi}_{\alpha,\theta,h}^1 \hat{v})/\partial \hat{x}_2 \right\| \leq C_2(\alpha, \theta)h |\partial \hat{v}/\partial \hat{x}_2|_1 ,$$

where $\hat{\Pi}_{\alpha,\theta,h}^1$ is $\Pi_{\alpha,\theta,h}^1$ for the rotated $T_{\alpha,\theta,h}$. The above two estimates (13) and (16) are in a sense sharper than (11) as noted in [21]. Similar relation also holds for $C_3(\alpha, \theta)$.

Thus we can give quantitative interpolation estimates, if we succeed in bounding the constants $C_i(\alpha, \theta)$'s explicitly by fairly simple functions of α and θ . Notice that each constant can be characterized by minimization of a kind of Rayleigh quotient. Then it is equivalent to finding the minimum eigenvalue of an eigenvalue problem expressed by a weak formulation, which is further expressed by a partial differential equation with some auxiliary conditions.

That is, such constants are characterized by minimization of Rayleigh's quotients $R_{\alpha,\theta}^{(i)}$'s:

$$(17) \quad C_i^{-2}(\alpha, \theta) = \inf_{v \in V_{\alpha,\theta}^i \setminus \{0\}} R_{\alpha,\theta}^{(i)}(v) ;$$

$$R_{\alpha,\theta}^{(i)}(v) = |v|_1^2 / \|v\|^2 \quad (i = 0, 1, 2, 3) ,$$

$$(18) \quad C_4^{-2}(\alpha, \theta) = \inf_{v \in V_{\alpha,\theta}^4 \setminus \{0\}} R_{\alpha,\theta}^{(4)}(v) ; \quad R_{\alpha,\theta}^{(4)}(v) = |v|_2^2 / |v|_1^2 ,$$

$$(19) \quad C_5^{-2}(\alpha, \theta) = \inf_{v \in V_{\alpha, \theta}^4 \setminus \{0\}} R_{\alpha, \theta}^{(5)}(v); \quad R_{\alpha, \theta}^{(5)}(v) = |v|_2^2 / \|v\|^2,$$

where all notations and functions are for $T_{\alpha, \theta}$.

By the standard compactness arguments, each infimum above is actually a minimum, and is the smallest eigenvalue of a certain eigenvalue problem. For example, the eigenvalue problem associated with $C_0(\alpha, \theta)$ is to find $\lambda \in \mathbf{R}$ and $u \in V_{\alpha, \theta}^0 \setminus \{0\}$ that satisfy

$$(20) \quad (\nabla u, \nabla v) = \lambda(u, v) \quad (\forall v \in V_{\alpha, \theta}^0),$$

where ∇ denotes the gradient operator, and (\cdot, \cdot) the inner products of $L_2(T_{\alpha, \theta})$ and $L_2(T_{\alpha, \theta})^2$. Notations such as $(\cdot, \cdot)_\Omega$ will be also used to specify the domains like Ω . The above is also expressed by a differential equation, a linear constraint and a boundary condition [25, 26]:

$$(21) \quad -\Delta u = \lambda u \text{ in } T_{\alpha, \theta}, \quad \int_{T_{\alpha, \theta}} u(x) dx = 0, \quad \partial u / \partial n = 0 \text{ on } \partial T_{\alpha, \theta},$$

where $\partial / \partial n$ denotes the outward normal derivative on edges, and $\partial T_{\alpha, \theta}$ is the boundary of $T_{\alpha, \theta}$. The above boundary condition is the homogeneous Neumann one, and the desired minimum eigenvalue is the second (and positive) one for the same problem without the linear constraint. Since $T_{\alpha, \theta}$ is a triangle, it is difficult to solve the above explicitly except in special cases.

As for $C_1(\alpha, \theta)$, it is characterized in the same fashion as (20), just by replacing $V_{\alpha, \theta}^0$ with $V_{\alpha, \theta}^1$. However, the equations corresponding to (21) become more complicated [25, 26]:

$$(22) \quad \begin{aligned} -\Delta u &= \lambda u \text{ in } T_{\alpha, \theta}, \quad \int_0^1 u(x_1, 0) dx_1 = 0, \\ \partial u / \partial n &= \begin{cases} 0 & \text{on edges } OB \text{ and } AB, \\ c & \text{on edge } OA, \end{cases} \end{aligned}$$

where c denotes an unknown constant to be decided simultaneously with u and λ , cf. Sec. 5.3.

The other constants are characterized similarly. For example, the eigenvalue problem associated to $C_4(\alpha, \theta)$ is to find $\lambda \in \mathbf{R}$ and $u \in V_{\alpha, \theta}^4 \setminus \{0\}$ that satisfy

$$(23) \quad \sum_{i, j=1}^2 (\partial^2 u / \partial x_i \partial x_j, \partial^2 v / \partial x_i \partial x_j) = \lambda(\nabla u, \nabla v) \quad (\forall v \in V_{\alpha, \theta}^4).$$

But the partial differential equation related to the above and also that to $C_5(\alpha, \theta)$ are of fourth order with special linear constraints and boundary conditions [4, 7], and are more difficult to deal with than the second order equations as in (21) and (22).

3. Estimation of Interpolation Error Constants

It is generally difficult to obtain exact values of the error constants $C_i(\alpha, \theta)$'s. So we will first give some formulas to bound them in terms of their special values such as $C_i(= C_i(1, \pi/2))$'s. Such formula can be useful for various purposes if some selected values are evaluated with sufficient accuracy, and we will also perform exact evaluation of some special constants.

3.1. Reconsideration of Natterer's results

Natterer [27] derived an upper bound formula for $C_4(\alpha, \theta)$ in terms of $C_4 = C_4(1, \pi/2)$, α and θ . He also gave an upper bound for C_4 , so that his formula has been effectively used in quantitative error estimates of finite element solutions including numerical verifications [25, 26]. Here we begin by applying his techniques to bound the error constants in Section 2.

To this end, let us introduce the following affine transformation $\xi = \Psi_{\alpha, \theta}(x)$ between $x = \{x_1, x_2\} \in T_{\alpha, \theta}$ and $\xi = \{\xi_1, \xi_2\} \in T = T_{1, \pi/2}$:

$$(24) \quad \begin{aligned} \xi_1 &= x_1 - x_2 / \tan \theta, \quad \xi_2 = x_2 / (\alpha \sin \theta) \quad ; \\ x_1 &= \xi_1 + \alpha \xi_2 \cos \theta, \quad x_2 = \alpha \xi_2 \sin \theta. \end{aligned}$$

By utilizing the above, we obtain the following results including Natterer's one for $C_4(\alpha, \theta)$.

THEOREM 1. *For $\alpha \in]0, +\infty[$ and $\theta \in]0, \pi[$, $C_i(\alpha, \theta)$'s are bounded as*

$$(25) \quad \psi_i(\alpha, \theta) C_i \leq C_i(\alpha, \theta) \leq \phi_i(\alpha, \theta) C_i \quad (0 \leq i \leq 5),$$

where $C_i = C_i(1, \pi/2)$ ($0 \leq i \leq 5$),

$$(26) \quad \begin{aligned} \psi_i(\alpha, \theta) &= \sqrt{\frac{\nu_-(\alpha, \theta)}{2}} \quad (0 \leq i \leq 3), \\ \psi_4(\alpha, \theta) &= \frac{\nu_-(\alpha, \theta)}{\sqrt{2\nu_+(\alpha, \theta)}}, \quad \psi_5(\alpha, \theta) = \frac{\nu_-(\alpha, \theta)}{2}, \end{aligned}$$

$$(27) \quad \begin{aligned} \phi_i(\alpha, \theta) &= \sqrt{\frac{\nu_+(\alpha, \theta)}{2}} \quad (0 \leq i \leq 3), \\ \phi_4(\alpha, \theta) &= \frac{\nu_+(\alpha, \theta)}{\sqrt{2\nu_-(\alpha, \theta)}}, \quad \phi_5(\alpha, \theta) = \frac{\nu_+(\alpha, \theta)}{2} \end{aligned}$$

with

$$(28) \quad \begin{aligned} \nu_-(\alpha, \theta) &= 1 + \alpha^2 - \sqrt{1 + 2\alpha^2 \cos 2\theta + \alpha^4}, \\ \nu_+(\alpha, \theta) &= 1 + \alpha^2 + \sqrt{1 + 2\alpha^2 \cos 2\theta + \alpha^4}. \end{aligned}$$

REMARK 1. In general, the upper bounds are more important than the lower ones, but the latter may be effective to evaluate the accuracy of the boundings. The above estimates should be used essentially in the range $0 < \alpha \leq 1$ and $\frac{\pi}{3} \leq \cos^{-1} \frac{\alpha}{2} \leq \theta < \pi$, where the upper bounds except for $i = 4$ are uniformly bounded. Thus these constants are robust to deformation of $T_{\alpha, \theta}$. On the other hand, the upper bound for $C_4(\alpha, \theta)$ is not so, and, to assure such uniform boundedness, we use the so-called minimum angle condition [11]: the minimum interior angle is bounded from below by a certain positive constant. This may be seen by using the identity $\nu_-(\alpha, \theta)\nu_+(\alpha, \theta) = 4\alpha^2 \sin^2 \theta$ to rewrite the upper bound inequality for $C_4(\alpha, \theta)$ as

$$(29) \quad C_4(\alpha, \theta) \leq C_4[2^{-1}\nu_+(\alpha, \theta)]^{\frac{3}{2}}/(\alpha \sin \theta).$$

The right-hand side above diverges to $+\infty$ as $\alpha \rightarrow +0$ for each $\theta \in]0, \pi[$, but α cannot approach 0 under the minimum angle condition. Actually, $C_4(\alpha, \theta)$ is uniformly bounded under a weaker condition called the maximum angle one of Babuška-Aziz [5], see Remark 2.

PROOF. We will use the coordinate transformation (24) between $T_{\alpha, \theta}$ and T . By simple calculations, we have for $\tilde{v}(\xi_1, \xi_2) = v(x_1, x_2)$, i. e., $\tilde{v} = v \circ \Psi_{\alpha, \theta}^{-1}$ under the present transformation:

$$\begin{aligned} &\sum_{i=1}^2 (\partial v / \partial x_i)^2 \\ &= [(\partial \tilde{v} / \partial \xi_1)^2 - 2\alpha^{-1} \cos \theta \partial \tilde{v} / \partial \xi_1 \cdot \partial \tilde{v} / \partial \xi_2 + \alpha^{-2} (\partial \tilde{v} / \partial \xi_2)^2] / \sin^2 \theta, \end{aligned}$$

where v and \tilde{v} are assumed to be smooth. The two eigenvalues associated to the quadratic form for $\partial \tilde{v} / \partial \xi_i$ ($i = 1, 2$) in $[\cdot]$ above are two solutions

of the characteristic equation $\mu^2 - \alpha^{-2}(1 + \alpha^2)\mu + \alpha^{-2} \sin^2 \theta = 0$, and given by $\nu_-(\alpha, \theta)/(2\alpha^2)$ and $\nu_+(\alpha, \theta)/(2\alpha^2)$. Thus

$$\begin{aligned} \nu_-(\alpha, \theta) \sum_{i=1}^2 (\partial \tilde{v} / \partial \xi_i)^2 &\leq 2\alpha^2 \sin^2 \theta \sum_{i=1}^2 (\partial v / \partial x_i)^2 \\ &\leq \nu_+(\alpha, \theta) \sum_{i=1}^2 (\partial \tilde{v} / \partial \xi_i)^2. \end{aligned}$$

Moreover, the Jacobian of the present transformation is evaluated as $\partial(x_1, x_2) / \partial(\xi_1, \xi_2) = \alpha \sin \theta$. From these estimates and the identity $\nu_-(\alpha, \theta) \cdot \nu_+(\alpha, \theta) = 4\alpha^2 \sin^2 \theta$, we have

$$(a.1) \quad \begin{aligned} \|v\|_{T_{\alpha, \theta}}^2 &= \alpha \sin \theta \|\tilde{v}\|_T^2, \\ 2\alpha \sin \theta |\tilde{v}|_{1, T}^2 / \nu_+(\alpha, \theta) &\leq |v|_{1, T_{\alpha, \theta}}^2 \leq 2\alpha \sin \theta |\tilde{v}|_{1, T}^2 / \nu_-(\alpha, \theta), \end{aligned}$$

where $|\cdot|_{1, T_{\alpha, \theta}}$, for example, denotes $|\cdot|_1$ for $T_{\alpha, \theta}$. The results for $i = 0, 1, 2, 3$ are now easy to obtain by using the above and the definitions of the constants $C_i(\alpha, \theta)$'s.

Similarly, by using notations $v_{x_i x_j} = \partial^2 v / \partial x_i \partial x_j$ and $\tilde{v}_{\xi_i \xi_j} = \partial^2 \tilde{v} / \partial \xi_i \partial \xi_j$, we obtain

$$\begin{aligned} \sum_{i, j=1}^2 (v_{x_i x_j})^2 &= [(\tilde{v}_{\xi_1 \xi_1})^2 + \alpha^{-4} (\tilde{v}_{\xi_2 \xi_2})^2 + 2\alpha^{-2} (1 + \cos^2 \theta) (\tilde{v}_{\xi_1 \xi_2})^2 \\ &\quad + 2\alpha^{-2} \cos^2 \theta \tilde{v}_{\xi_1 \xi_1} \tilde{v}_{\xi_2 \xi_2} - 4\alpha^{-1} \cos \theta \tilde{v}_{\xi_1 \xi_1} \tilde{v}_{\xi_1 \xi_2} \\ &\quad - 4\alpha^{-3} \cos \theta \tilde{v}_{\xi_2 \xi_2} \tilde{v}_{\xi_1 \xi_2}] / \sin^4 \theta. \end{aligned}$$

Let us consider the following real symmetric matrix related to the quadratic form for $\tilde{v}_{\xi_1 \xi_1}$, $\tilde{v}_{\xi_2 \xi_2}$ and $\sqrt{2} \tilde{v}_{\xi_1 \xi_2}$ in $[\cdot]$ of the right-hand side above :

$$\begin{pmatrix} 1 & \alpha^{-2} \cos^2 \theta & -\sqrt{2} \alpha^{-1} \cos \theta \\ \alpha^{-2} \cos^2 \theta & \alpha^{-4} & -\sqrt{2} \alpha^{-3} \cos \theta \\ -\sqrt{2} \alpha^{-1} \cos \theta & -\sqrt{2} \alpha^{-3} \cos \theta & \alpha^{-2} (1 + \cos^2 \theta) \end{pmatrix}.$$

The associated characteristic equation is

$$\begin{aligned} &\mu^3 - \alpha^{-4} \{1 + (1 + \cos^2 \theta) \alpha^2 + \alpha^4\} \mu^2 \\ &\quad + \alpha^{-6} \sin^2 \theta \{1 + (1 + \cos^2 \theta) \alpha^2 + \alpha^4\} \mu - \alpha^{-6} \sin^6 \theta \\ &= (\mu - \alpha^{-2} \sin^2 \theta) \{\mu^2 - \alpha^{-4} (1 + 2\alpha^2 \cos^2 \theta + \alpha^4) \mu + \alpha^{-4} \sin^4 \theta\} = 0, \end{aligned}$$

which has three eigenvalues $\nu_-^2(\alpha, \theta)/(4\alpha^4) \leq \alpha^{-2} \sin^2 \theta \leq \nu_+^2(\alpha, \theta)/(4\alpha^4)$ with $\nu_-(\alpha, \theta)$ and $\nu_+(\alpha, \theta)$ defined by (28). Now we have the estimates

$$\nu_-^2(\alpha, \theta) \sum_{i, j=1}^2 (\tilde{v}_{\xi_i \xi_j})^2 \leq 4\alpha^4 \sin^4 \theta \sum_{i, j=1}^2 (v_{x_i x_j})^2$$

$$\leq \nu_+^2(\alpha, \theta) \sum_{i,j=1}^2 (\tilde{v}_{\xi_i \xi_j})^2,$$

which gives, as (a.1),

$$(a.2) \quad 4\alpha \sin \theta |\tilde{v}|_{2,T}^2 / \nu_+^2(\alpha, \theta) \leq |v|_{2,T_{\alpha,\theta}}^2 \leq 4\alpha \sin \theta |\tilde{v}|_{2,T}^2 / \nu_-^2(\alpha, \theta).$$

From (a.1) and (a.2), we obtain the results for $i = 4, 5$. \square

As a corollary of the preceding theorem, we can bound each $C_i(\alpha, \theta)$ in terms of $C_i(\alpha)$ and θ . Such estimates can be effective when the dependence of $C_i(\alpha)$ on α is known as we will see later. The bounding can be achieved by utilizing the affine transformation between $x = \{x_1, x_2\} \in T_{\alpha,\theta}$ and $\xi = \{\xi_1, \xi_2\} \in T_\alpha$: $\xi_1 = x_1 - x_2 \cos \theta / \sin \theta$, $\xi_2 = x_2 / \sin \theta$. But the same can be attained by comparing this transformation with (24) and using (25) for $\alpha = 1$.

COROLLARY 1. For $\alpha \in]0, +\infty[$ and $\theta \in]0, \pi[$, $C_i(\alpha, \theta)$'s are bounded as

$$(30) \quad \psi_i(\theta) C_i(\alpha) \leq C_i(\alpha, \theta) \leq \phi_i(\theta) C_i(\alpha) \quad (0 \leq i \leq 5),$$

where $\psi_i(\theta) = \psi_i(1, \theta)$ and $\phi_i(\theta) = \phi_i(1, \theta)$ for $0 \leq i \leq 5$. More specifically,

$$(31) \quad \begin{aligned} \psi_i(\theta) &= \sqrt{1 - |\cos \theta|} \quad (0 \leq i \leq 3), \\ \psi_4(\theta) &= \frac{1 - |\cos \theta|}{\sqrt{1 + |\cos \theta|}}, \quad \psi_5(\theta) = 1 - |\cos \theta|, \end{aligned}$$

$$(32) \quad \begin{aligned} \phi_i(\theta) &= \sqrt{1 + |\cos \theta|} \quad (0 \leq i \leq 3), \\ \phi_4(\theta) &= \frac{1 + |\cos \theta|}{\sqrt{1 - |\cos \theta|}}, \quad \phi_5(\theta) = 1 + |\cos \theta|. \end{aligned}$$

REMARK 2. The function $\phi_4(\theta)$ is consistent with the maximum angle condition [5]: it is bounded on $[\pi/3, \pi - \delta]$ for each sufficiently small $\delta > 0$. Thus, $C_4(\alpha, \theta)$ is uniformly bounded for $0 < \alpha \leq 1$ and $\frac{\pi}{3} \leq \theta < \pi - \delta$, if $C_4(\alpha)$ is uniformly bounded for such α , cf. Theorem 4.

3.2. Estimation of $C_4(\alpha, \theta)$ by $C_1(\alpha, \theta)$ and $C_2(\alpha, \theta)$

We can also give an upper bound for $C_4(\alpha, \theta)$ in terms of $C_1(\alpha, \theta)$ and $C_2(\alpha, \theta)$.

THEOREM 2. *For any $\alpha \in]0, +\infty[$ and $\theta \in]0, \pi[$, $C_4(\alpha, \theta)$ is bounded as*

$$(33) \quad C_4(\alpha, \theta) \leq \frac{1}{\sqrt{2} \sin \theta} \nu(\alpha, \theta) \leq C_1 \frac{1 + |\cos \theta|}{\sin \theta} \sqrt{\nu_+(\alpha, \theta)/2},$$

where $\nu_+(\alpha, \theta)$ is defined by (28), and $\nu(\alpha, \theta)$ by

$$(34) \quad \begin{aligned} \nu(\alpha, \theta) = & \left[C_1^2(\alpha, \theta) + C_2^2(\alpha, \theta) + 2C_1(\alpha, \theta)C_2(\alpha, \theta) \cos^2 \theta \right. \\ & \left. + (C_1(\alpha, \theta) + C_2(\alpha, \theta)) \right. \\ & \left. \times \sqrt{C_1^2(\alpha, \theta) + C_2^2(\alpha, \theta) + 2C_1(\alpha, \theta)C_2(\alpha, \theta) \cos 2\theta} \right]^{1/2}. \end{aligned}$$

REMARK 3. We can easily see that the maximum angle condition applies to the present estimate (33), cf. [5, 21]. It is also possible to bound $C_4(\alpha, \theta)$ in terms of $C_1(\alpha, \theta)$ and $C_3(\alpha, \theta)$. Moreover, it may be meaningful to compare two estimates (29) and (33) for $C_4(\alpha, \theta)$:

$$\begin{aligned} C_4(\alpha, \theta) &\leq C_4 \frac{1}{\alpha \sin \theta} [\nu_+(\alpha, \theta)/2]^{\frac{3}{2}} =: \gamma_1(\alpha, \theta), \\ C_4(\alpha, \theta) &\leq C_1 \frac{1 + |\cos \theta|}{\sin \theta} \sqrt{\nu_+(\alpha, \theta)/2} =: \gamma_2(\alpha, \theta). \end{aligned}$$

Noting the relations $2\alpha |\cos \theta| \leq \sqrt{1 + 2\alpha^2 \cos 2\theta + \alpha^4} \leq 1 + \alpha^2$ and $2\alpha \leq 1 + \alpha^2$, we find

$$\frac{\alpha(1 + |\cos \theta|)}{1 + \alpha^2} \frac{C_1}{C_4} \leq \frac{\gamma_2(\alpha, \theta)}{\gamma_1(\alpha, \theta)} = \frac{2\alpha(1 + |\cos \theta|)}{1 + \alpha^2 + \sqrt{1 + 2\alpha^2 \cos 2\theta + \alpha^4}} \frac{C_1}{C_4} \leq \frac{C_1}{C_4}.$$

It is known that $C_4 \approx 0.489$ by numerical computations without verification [4, 22, 30]. On the other hand, C_1 is theoretically shown to be an upper bound of C_4 , but is quite close to C_4 thanks to the numerically verified bounding $0.492 < C_1 < 0.493$ [25, 26], see also Theorem 3 later. Thus (33) is practically better than (29) for almost all values of α and θ .

PROOF. We will use temporary notations $C_{i,\alpha,\theta}$'s as $C_i(\alpha,\theta)$'s. From the definition, we have

$$(b.1) \quad C_{4,\alpha,\theta}^2 = \sup_{v \in V_{\alpha,\theta}^4 \setminus \{0\}} |v|_1^2 / |v|_2^2 \\ = \sup_{v \in V_{\alpha,\theta}^4 \setminus \{0\}} (\|\partial v / \partial x_1\|^2 + \|\partial v / \partial x_2\|^2) / (|\partial v / \partial x_1|_1^2 + |\partial v / \partial x_2|_1^2).$$

Recall here the transformation rules (14) and (15). Then, for the present $v \in V_{\alpha,\theta}^4 \setminus \{0\}$ and the associated $\hat{v} = v \circ \Phi_\theta^{-1}$, we can show as (13) and (16) that

$$(b.2) \quad \|\partial v / \partial x_1\| \leq C_{1,\alpha,\theta} |\partial v / \partial x_1|_1, \quad \|\partial \hat{v} / \partial \hat{x}_2\| \leq C_{2,\alpha,\theta} |\partial \hat{v} / \partial \hat{x}_2|_1,$$

where $\partial \hat{v} / \partial \hat{x}_2 = \cos \theta \partial v / \partial x_1 + \sin \theta \partial v / \partial x_2$ at $x = \{x_1, x_2\} \in T_{\alpha,\theta}$ and $\hat{x} = \{\hat{x}_1, \hat{x}_2\} = \Phi_\theta(x)$. Then $\partial v / \partial x_2 = (\partial \hat{v} / \partial \hat{x}_2 - \cos \theta \partial v / \partial x_1) / \sin \theta$ can be evaluated as

$$\sin^2 \theta \|\partial v / \partial x_2\|^2 \leq \|\partial \hat{v} / \partial \hat{x}_2\|^2 + 2|\cos \theta| \cdot \|\partial \hat{v} / \partial \hat{x}_2\| \cdot \|\partial v / \partial x_1\| \\ + \cos^2 \theta \|\partial v / \partial x_1\|^2.$$

By (b.2) and the present inequality, we can bound $\|\partial v / \partial x_1\|^2 + \|\partial v / \partial x_2\|^2$ from above as

$$(b.3) \quad \sin^2 \theta [\|\partial v / \partial x_1\|^2 + \|\partial v / \partial x_2\|^2] \\ \leq \|\partial v / \partial x_1\|^2 + 2|\cos \theta| \cdot \|\partial v / \partial x_1\| \cdot \|\partial \hat{v} / \partial \hat{x}_2\| + \|\partial \hat{v} / \partial \hat{x}_2\|^2 \\ \leq C_{1,\alpha,\theta}^2 |\partial v / \partial x_1|_1^2 + 2 C_{1,\alpha,\theta} C_{2,\alpha,\theta} |\cos \theta| \cdot |\partial v / \partial x_1|_1 \cdot |\partial \hat{v} / \partial \hat{x}_2|_1 \\ + C_{2,\alpha,\theta}^2 |\partial \hat{v} / \partial \hat{x}_2|_1^2.$$

To evaluate $|\partial \hat{v} / \partial \hat{x}_2|_1$ above, we again use $\partial \hat{v} / \partial \hat{x}_2 = \cos \theta \partial v / \partial x_1 + \sin \theta \partial v / \partial x_2$. Then

$$|\partial \hat{v} / \partial \hat{x}_2|_1 \leq |\cos \theta| \cdot |\partial v / \partial x_1|_1 + \sin \theta |\partial v / \partial x_2|_1.$$

Substituting the above into the right-hand side of (b.3), we obtain

$$\sin^2 \theta [\|\partial v / \partial x_1\|^2 + \|\partial v / \partial x_2\|^2] \\ \leq \{C_{1,\alpha,\theta}^2 + 2 C_{1,\alpha,\theta} C_{2,\alpha,\theta} \cos^2 \theta + C_{2,\alpha,\theta}^2 \cos^2 \theta\} |\partial v / \partial x_1|_1^2$$

$$\begin{aligned}
& + 2 C_{2,\alpha,\theta} \{C_{1,\alpha,\theta} + C_{2,\alpha,\theta}\} \sin \theta |\cos \theta| \cdot |\partial v / \partial x_1|_1 \cdot |\partial v / \partial x_2|_1 \\
& + C_{2,\alpha,\theta}^2 \sin^2 \theta |\partial v / \partial x_2|_1^2 .
\end{aligned}$$

By eigenvalue analysis of the quadratic form above for $|\partial v / \partial x_1|_1$ and $|\partial v / \partial x_2|_1$, we have

$$\|\partial v / \partial x_1\|^2 + \|\partial v / \partial x_2\|^2 \leq \nu^2(\alpha, \theta) [|\partial v / \partial x_1|_1^2 + |\partial v / \partial x_2|_1^2] / (2 \sin^2 \theta) .$$

by using $\nu(\alpha, \theta)$ in (34). This gives the former part of (33) by (b.1).

To derive the latter part of (33), we should evaluate $\nu(\alpha, \theta)$ by using $C_i(\alpha, \theta) \leq \phi_i(\alpha, \theta) C_i$ ($i = 1, 2$) and $\phi_1(\alpha, \theta) = \phi_2(\alpha, \theta) = \sqrt{\nu_+(\alpha, \theta)/2}$ in (25) along with equality $C_1 = C_2$. \square

3.3. Determination of some constants

Theorem 1 tells us that we can obtain upper bounds of the constants $C_i(\alpha, \theta)$ ($0 \leq i \leq 5$), if correct values of $C_i = C_i(1, \pi/2)$ are known. The upper bounds thus evaluated may be rough but anyway correct, so that they can be used for various theoretical purposes. According to some preceding works [18, 19, 25, 26], such exact evaluation is possible at least for C_0 and $C_1 = C_2$. We quote the results below, along with an additional new result for C_3 .

THEOREM 3. *It holds for $C_i = C_i(1, \pi/2)$ ($0 \leq i \leq 3$) that*

- 1) $i = 0$: $C_0 = 1/\pi$,
- 2) $i = 1, 2$: $C_1 = C_2 =$ *the maximum positive solution of the following equation for μ ;*

$$(35) \quad 1/\mu + \tan(1/\mu) = 0 .$$

The concrete value of C_1 can be obtained numerically with verification. For example,

$$(36) \quad 0.49282 < C_1 < 0.49293 .$$

- 3) $i = 3$: $C_3 = C_1/\sqrt{2}$, $0.34847 < C_3 < 0.34856$.

REMARK 4. i) Numerical computation without verification gives $C_1 = 0.49291245\dots$ and $C_3 = 0.34854173\dots$. Eq. (35) is popular in vibration analysis of strings [28], and the constant C_1 , called the Babuška-Aziz one [18, 19], plays an important role in various situations.

ii) Exact values of C_4 and C_5 are unavailable to the best of the authors' knowledge. Fortunately, $C_1 (= C_2)$ is a nice upper bound of C_4 , see Sections 4.2 and 6.2. Numerically, $C_4 \approx 0.489$ as reported in [4, 22, 30]. As for C_5 , estimate $C_5 < 0.361$ in [14] is correct but probably rough, while an exact lower bound estimation $C_5 \geq [(15 + \sqrt{193})/1440]^{1/2} = 0.1416\dots$ can be derived by the Ritz-Galerkin method using the basis functions $x_1 + x_2 - x_1^2 - x_2^2$ and x_1x_2 proposed in [25]. Our own numerical computations suggest that $C_5 \approx 0.168$.

PROOF. As 1) and 2), we can prove 3) by using a kind of symmetry method [18, 19, 25, 26].

1] Similarly to (20), the eigenvalue problem for C_3 is to find $\{\lambda, u\} \in \mathbf{R} \times V^3 \setminus \{0\}$ such that

$$(c.1) \quad (\nabla u, \nabla v)_T = \lambda(u, v)_T \quad (\forall v \in V^3).$$

Here, T is the unit right isosceles triangle $T_{1, \pi/2, 1}$, and $V^3 = V_{1, \pi/2, 1}^3$ is defined by (4). Notice that we are interested only in the minimum eigenvalue and the associated eigenfunctions.

Let us divide T into two congruent parts using the line $x_2 = x_1$, which is also the line of symmetry for T . Moreover, one of the congruent parts is denoted by \tilde{T} :

$$\tilde{T} = \{x = \{x_1, x_2\} \in T; x_1 > x_2\}.$$

The eigenfunction $u \neq 0$ can be uniquely decomposed into the symmetric part u_s and the antisymmetric one u_a with respect to the line $x_2 = x_1$:

$$u = u_s + u_a.$$

Since u_s and u_a are orthogonal to each other both for $(\cdot, \cdot)_T$ and $(\nabla \cdot, \nabla \cdot)_T$, they can be dealt with separately: u_s and u_a both belong to V^3 and satisfy (c.1) for the minimum eigenvalue λ .

2] We first consider the case where $u_s \neq 0$. We can see that the restriction \tilde{u} of u_s to \tilde{T} is not zero and satisfies the following eigenvalue problem related

to \tilde{T} :

$$(c.2) \quad \tilde{u} \in \tilde{V}^3 \setminus \{0\}; \quad (\nabla \tilde{u}, \nabla \tilde{v})_{\tilde{T}} = \lambda(\tilde{u}, \tilde{v})_{\tilde{T}} \quad (\forall \tilde{v} \in \tilde{V}^3),$$

where λ is identical to the former one, the inner products are for \tilde{T} , and \tilde{V}^3 is defined by

$$\tilde{V}^3 = \{\tilde{v} \in H^1(\tilde{T}); \int_0^{\frac{1}{2}} \tilde{v}(1-s, s) ds = 0\}.$$

Clearly, (c.2) is essentially the same as the eigenvalue problem for $C_1(1, \pi/2, 1/\sqrt{2})$, since \tilde{T} is congruent to $T_{1, \pi/2, 1/\sqrt{2}}$. It is also easy to see that the eigenpair for the minimum eigenvalue of (c.2) satisfies (c.1), if the eigenfunction is extended to whole T symmetrically with respect to $x_2 = x_1$. Thus \tilde{u} is an eigenfunction for the minimum eigenvalue of (c.2) in the present case. Then we find that $C_3 = C_1/\sqrt{2}$, since $C_1(\alpha, \theta, 1/\sqrt{2}) = C_1(\alpha, \theta)/\sqrt{2}$, cf. Section 2.

3] Secondly, we consider the case where $u_a \neq 0$. Due to the antisymmetry, the trace of u_a to the line of symmetry $x_2 = x_1$ inside T is shown to be 0. Moreover, any antisymmetric function in $H^1(T)$ automatically satisfies the line integration condition imposed on V^3 . Thus the restriction u^\dagger of u_a to \tilde{T} is not zero and is an eigenfunction of the eigenvalue problem:

$$(c.3) \quad u^\dagger \in V^\dagger \setminus \{0\}; \quad (\nabla u^\dagger, \nabla v^\dagger)_{\tilde{T}} = \lambda(u^\dagger, v^\dagger)_{\tilde{T}} \quad (\forall v^\dagger \in V^\dagger),$$

where λ is identical to the former one, and V^\dagger is defined by

$$V^\dagger = \{v^\dagger \in H^1(\tilde{T}); v^\dagger(s, s) = 0 \quad (0 < s < 1/2)\}.$$

If we consider the reflection with respect to the line $x_1 = 1/2$, (c.3) becomes the problem of the same form with V^\dagger replaced by

$$V^* = \{v^* \in H^1(\tilde{T}); v^*(1-s, s) = 0 \quad (0 < s < 1/2)\}.$$

Clearly, the eigenvalues remain the same. Since $V^* \subset \tilde{V}^3$, the minimum eigenvalue of (c.3) cannot be smaller than that of (c.2), due to the characterization of the minimum eigenvalue by the Rayleigh quotient. Thus it is actually sufficient to consider the case where $u_s \neq 0$. \square

3.4. Application to interpolation and a priori error estimates

In this subsection, we show how to apply the obtained results to interpolation error estimates and some a priori error estimates for FEM.

From the preceding considerations, especially equations (10) through (12) and Theorems 1 and 2, we have for example the following P_0 and P_1 interpolation error estimates:

$$(37) \quad \|v - \Pi_{\alpha,\theta,h}^0 v\| \leq C_0 \phi_0(\alpha, \theta) h |v|_1 \quad (\forall v \in H^1(T_{\alpha,\theta,h})),$$

$$(38) \quad |v - \Pi_{\alpha,\theta,h}^1 v|_1 \leq C_1 \frac{1 + |\cos \theta|}{\sin \theta} [\nu_+(\alpha, \theta)/2]^{\frac{1}{2}} h |v|_2$$

$$(\forall v \in H^2(T_{\alpha,\theta,h})),$$

$$(39) \quad \|v - \Pi_{\alpha,\theta,h}^1 v\| \leq C_5 \phi_5(\alpha, \theta) h^2 |v|_2 \quad (\forall v \in H^2(T_{\alpha,\theta,h})).$$

These may be rough but are correct quantitative upper bounds, if the values of C_0 , C_1 and C_5 or at least their upper bounds are known. For C_0 and C_1 , we have obtained exact values in Theorem 3, while, presumably, C_5 has been evaluated only approximately, cf. Remark 4.

As was already noted, such error bounds are available for triangles of general configuration by applying appropriate congruent transformations [5, 10, 11, 21]. Then such interpolation error estimates can be directly used in a priori error estimates of finite element solutions. In what follows, we will briefly explain an example of such process. See e. g. [11] for the details.

As a model problem, let us consider the Dirichlet problem of the Poisson equation over an bounded polygonal domain $\Omega \subset \mathbf{R}^2$: given $f \in L_2(\Omega)$, find $u \in H_0^1(\Omega)$ such that $-\Delta u = f$ in Ω . Here, $H_0^1(\Omega)$ is the popular subspace of $H^1(\Omega)$ with the homogeneous Dirichlet condition imposed. In the standard weak formulation, the condition for $u \in H_0^1(\Omega)$ is stated as

$$(40) \quad (\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad (\forall v \in H_0^1(\Omega)).$$

Since this is a well-posed problem, we can define an operator $G : f \in L_2(\Omega) \mapsto u \in H_0^1(\Omega)$.

To apply the FEM based on the P_1 triangle to this problem, we consider a regular family of triangulations $\{\mathcal{T}^\eta\}_{\eta>0}$ of Ω and construct a P_1 finite element space $V^\eta \subset H_0^1(\Omega)$ for each \mathcal{T}^η , where $\eta > 0$ is the discretization

parameter. For the terminology *regular*, cf. [11]. The finite element solution $u_\eta \in V^\eta$ for $u = Gf$ is uniquely determined by imitating (40) in V^η :

$$(41) \quad (\nabla u_\eta, \nabla v_\eta)_\Omega = (f, v_\eta)_\Omega \quad (\forall v_\eta \in V^\eta).$$

An important fact in the Ritz-Galerkin FEM is the best approximation property [11]:

$$(42) \quad |u - u_\eta|_{1,\Omega} = \min_{v_\eta \in V^\eta} |u - v_\eta|_{1,\Omega} \quad (|\cdot|_{1,\Omega} = |\cdot|_1 \text{ for } \Omega).$$

Another important fact is the L_2 error estimation based on the Aubin-Nitsche trick [11]:

$$(43) \quad \|u - u_\eta\|_\Omega \leq |u - u_\eta|_{1,\Omega} \sup_{g \in L_2(\Omega) \setminus \{0\}} \inf_{v_\eta \in V^\eta} |Gg - v_\eta|_{1,\Omega} / \|g\|_\Omega.$$

From (42), an error estimation based on the standard nodal interpolation function $\Pi^{\eta,1}u \in V^\eta$ using the vertex values of u is given by

$$(44) \quad |u - u_\eta|_{1,\Omega} \leq |u - \Pi^{\eta,1}u|_{1,\Omega}.$$

Here, the global interpolation operator $\Pi^{\eta,1}$ is closely related to the local one $\Pi_{\alpha,\theta,h}^1$. That is, for each triangle $K \in \mathcal{T}^\eta$, we can find a $T_{\alpha,\theta,h}$ congruent to K under a congruent transformation $\Phi_K : K \rightarrow T_{\alpha,\theta,h}$, and it then holds that $(\Pi^{\eta,1}u)|_K = [\Pi_{\alpha,\theta,h}^1\{(u|_K) \circ \Phi_K^{-1}\}] \circ \Phi_K$. If $u \in H^2(\Omega)$, we have by (11), using notations $\{\alpha_K, \theta_K, h_K\}$ as $\{\alpha, \theta, h\}$ of $T_{\alpha,\theta,h}$ associated to K ,

$$(45) \quad |u - \Pi^{\eta,1}u|_{1,\Omega}^2 = \sum_{K \in \mathcal{T}^\eta} |u - \Pi^{\eta,1}u|_{1,K}^2 \\ \leq \sum_{K \in \mathcal{T}^\eta} h_K^2 C_4^2(\alpha_K, \theta_K) |u|_{2,K}^2.$$

Thus we obtain from (44) an a priori error estimate

$$(46) \quad |u - u_\eta|_{1,\Omega} \leq |u - \Pi^{\eta,1}u|_{1,\Omega} \leq C_{4,\eta} \eta |u|_{2,\Omega} \quad ; \\ C_{4,\eta} = \max_{K \in \mathcal{T}^\eta} C_4(\alpha_K, \theta_K), \quad \eta = \max_{K \in \mathcal{T}^\eta} h_K.$$

To evaluate $C_{4,\eta}$ from above, we can utilize various upper bounds already derived for $C_4(\alpha, \theta)$, an example of which can be also found in (38). In problems more general than (40), we may also need upper bounds for $C_5(\alpha, \theta)$ to obtain global L_2 error bounds, although we can avoid the use of such

bounds to a certain extent by adopting the Aubin-Nitsche trick [11]. The constants $C_i(\alpha, \theta)$ for $0 \leq i \leq 3$ may appear to be subsidiary here, but they actually play essential roles in the analysis of the non-conforming P_1 FEM as is noted in [19].

To apply the above to verification of various differential equations by FEM, it is often required to evaluate norms or semi-norms of the solutions by various data. A typical example is to give upper bounds of $|u|_{2,\Omega}$ in (46) by a norm of f . Here we can use the relation $|u|_{2,\Omega} \leq \|f\|_\Omega$, if Ω is convex in addition to the assumptions already stated [15]. Then we have

$$(47) \quad |u - u_\eta|_{1,\Omega} \leq C_{4,\eta}\eta \|f\|_\Omega,$$

and moreover, by applying (43) with v_η taken as $\Pi^{\eta,1}(Gg)$,

$$(48) \quad \|u - u_\eta\|_\Omega \leq C_{4,\eta}\eta |u - u_\eta|_{1,\Omega} \leq C_{4,\eta}^2\eta^2 \|f\|_\Omega,$$

where we have used the estimate $|Gg - \Pi^{\eta,1}(Gg)|_{1,\Omega} \leq C_{4,\eta}\eta |Gg|_{2,\Omega} \leq C_{4,\eta}\eta \|g\|_\Omega$. The present estimation can be compared with the L_2 interpolation estimate

$$(49) \quad \|u - \Pi^{\eta,1}u\|_\Omega \leq C_{5,\eta}\eta^2 |u|_{2,\Omega} \leq C_{5,\eta}\eta^2 \|f\|_\Omega$$

with $C_{5,\eta} = \max_{K \in \mathcal{T}^\eta} C_5(\alpha_K, \theta_K)$.

3.5. Application to a posteriori error estimates

A posteriori error estimation is also feasible and effective in various situations by using the interpolation error constants considered in the preceding subsections. So, before closing the present section, we also show how to apply the obtained results to a posteriori error estimates for FEM. Here we only explain a special and rather classical approach [12, 20, 24] briefly.

Let q be an element of $H(\operatorname{div}; \Omega) := \{q \in L_2(\Omega)^2 \mid \operatorname{div} q \in L_2(\Omega)\}$ [12, 20]. Then, some simple calculations give, with the same notations as in Section 3.4,

$$\begin{aligned} |u - u_\eta|_{1,\Omega}^2 &= (\nabla(u - u_\eta), \nabla(u - u_\eta))_\Omega \\ &= (u - u_\eta, -\Delta u)_\Omega + (\nabla(u - u_\eta), q - \nabla u_\eta - q)_\Omega \\ &= (u - u_\eta, f + \operatorname{div} q)_\Omega + (\nabla(u - u_\eta), q - \nabla u_\eta)_\Omega \\ &\leq \|u - u_\eta\|_\Omega \cdot \|f + \operatorname{div} q\|_\Omega + |u - u_\eta|_{1,\Omega} \cdot \|q - \nabla u_\eta\|_\Omega. \end{aligned}$$

By (48), we have $|u - u_\eta|_{1,\Omega}^2 \leq (C_{4,\eta}\eta \|f + \operatorname{div} q\|_\Omega + \|q - \nabla u_\eta\|_\Omega)|u - u_\eta|_{1,\Omega}$, and hence

$$(50) \quad |u - u_\eta|_{1,\Omega} \leq C_{4,\eta}\eta \|f + \operatorname{div} q\|_\Omega + \|q - \nabla u_\eta\|_\Omega .$$

Here $C_{4,\eta}$ appears again, and this becomes an a posteriori estimate by specifying q . The most elegant but a restrictive choice is based on the hypercircle method [12, 20], which employs q with $f + \operatorname{div} q = 0$ so that $C_{4,\eta}$ is unnecessary. More practical choice is to obtain q by post-processing of u_η , for example, by smoothing ∇u_η so as to belong to $H(\operatorname{div}; \Omega)$. For this to be effective, we need that $\|q - \nabla u_\eta\|_\Omega = O(\eta)$ and, preferably, $\|f + \operatorname{div} q\|_\Omega = o(1)$. Combining (50) with (43) as in (48), we also obtain an a posteriori L_2 -error estimate using $C_{4,\eta}$:

$$(51) \quad \|u - u_\eta\|_\Omega \leq C_{4,\eta}^2 \eta^2 \|f + \operatorname{div} q\|_\Omega + C_{4,\eta}\eta \|q - \nabla u_\eta\|_\Omega .$$

4. Dependence of $C_i(\alpha)$ on α

Up to now, we have given some basic results for dependence of error constants on h , α and θ . In this section, we will consider the dependence of such constants on $\alpha > 0$ in the special case when $\theta = \pi/2$ and $h = 1$. Actually, we need their behaviors in the range $0 < \alpha \leq 1$, and, in view of (30), we want to find their maxima or nice upper bounds there. Furthermore, the limiting case $\alpha \rightarrow +0$ is of some practical interests in the anisotropic mesh refinements [1, 13].

4.1. Definitions and notations

Since each $C_i(\alpha) = C_i(\alpha, \pi/2, 1)$ is defined through minimization of a Rayleigh quotient in terms of norms and/or seminorms over T_α (see (17) through (19)), it is natural to introduce the following transformation $\xi = \Psi_\alpha(x)$ between $x = \{x_1, x_2\} \in T_\alpha$ and $\xi = \{\xi_1, \xi_2\} \in T$:

$$(52) \quad \xi_1 = x_1, \quad \xi_2 = x_2/\alpha,$$

together with the associated transformation $\tilde{v} = v \circ \Psi_\alpha^{-1}$ between functions v over T_α and \tilde{v} over T : $\tilde{v}(\xi) = v(x) = v(\xi_1, \alpha\xi_2)$. Notice that $\Psi_\alpha = \Psi_{\alpha, \pi/2}$ for $\Psi_{\alpha, \theta}$ in (24).

Then we have the following expressions to (semi-)norms for T_α in terms of those for T :

$$(53) \quad \|v\|_{T_\alpha}^2 = \alpha \|\tilde{v}\|_T^2,$$

$$(54) \quad |v|_{1,T_\alpha}^2 = \alpha a_\alpha^{(1)}(\tilde{v}); \quad a_\alpha^{(1)}(\tilde{v}) := \|\partial\tilde{v}/\partial\xi_1\|_T^2 + \alpha^{-2} \|\partial\tilde{v}/\partial\xi_2\|_T^2,$$

$$(55) \quad |v|_{2,T_\alpha}^2 = \alpha a_\alpha^{(2)}(\tilde{v}); \quad a_\alpha^{(2)}(\tilde{v}) \\ := \|\partial^2\tilde{v}/\partial\xi_1^2\|_T^2 + 2\alpha^{-2} \|\partial^2\tilde{v}/\partial\xi_1\partial\xi_2\|_T^2 + \alpha^{-4} \|\partial^2\tilde{v}/\partial\xi_2^2\|_T^2,$$

where, for example in (53), $v \in L_2(T_\alpha)$ and $\tilde{v} \in L_2(T)$ with $v = \tilde{v} \circ \Psi_\alpha$. By using these forms, $R_\alpha^{(i)}(v) = R_{\alpha,\pi/2}^{(i)}(v)$ ($0 \leq i \leq 5$) for $R_{\alpha,\theta}^{(i)}$'s in (17) through (19) are expressed as

$$(56) \quad R_\alpha^{(i)}(v) = \tilde{R}_\alpha^{(i)}(\tilde{v}) := a_\alpha^{(1)}(\tilde{v})/\|\tilde{v}\|_T^2; \\ v \in V_\alpha^i \setminus \{0\}, \quad \tilde{v} = v \circ \Psi_\alpha^{-1} \in V^i \setminus \{0\} \quad (0 \leq i \leq 3),$$

$$(57) \quad R_\alpha^{(4)}(v) = \tilde{R}_\alpha^{(4)}(\tilde{v}) := a_\alpha^{(2)}(\tilde{v})/a_\alpha^{(1)}(\tilde{v}); \\ v \in V_\alpha^4 \setminus \{0\}, \quad \tilde{v} = v \circ \Psi_\alpha^{-1} \in V^4 \setminus \{0\},$$

$$(58) \quad R_\alpha^{(5)}(v) = \tilde{R}_\alpha^{(5)}(\tilde{v}) := a_\alpha^{(2)}(\tilde{v})/\|\tilde{v}\|_T^2; \\ v \in V_\alpha^4 \setminus \{0\}, \quad \tilde{v} = v \circ \Psi_\alpha^{-1} \in V^4 \setminus \{0\}.$$

We can now analyze the constants $C_i(\alpha)$'s over the common triangle T .

We also present the bilinear forms associated to the quadratic forms $a_\alpha^{(i)}(\cdot)$'s for $i = 1, 2$:

$$(59) \quad a_\alpha^{(1)}(u, v) := (\partial u/\partial x_1, \partial v/\partial x_1)_T \\ + \alpha^{-2} (\partial u/\partial x_2, \partial v/\partial x_2)_T \quad (\forall u, v \in H^1(T)),$$

$$(60) \quad a_\alpha^{(2)}(u, v) := (\partial^2 u/\partial x_1^2, \partial^2 v/\partial x_1^2)_T \\ + 2\alpha^{-2} (\partial^2 u/\partial x_1\partial x_2, \partial^2 v/\partial x_1\partial x_2)_T \\ + \alpha^{-4} (\partial^2 u/\partial x_2^2, \partial^2 v/\partial x_2^2)_T \quad (\forall u, v \in H^2(T)).$$

Here we have used u, v and $x = \{x_1, x_2\}$ instead of \tilde{u}, \tilde{v} and $\xi = \{\xi_1, \xi_2\}$, respectively.

The following function spaces will play important roles later:

$$(61) \quad H^{k,Z}(T) = \{v \in H^k(T); \partial v/\partial x_2 = 0\} \quad (k = 1, 2),$$

$$(62) \quad V^{i,Z} = \{v \in V^i; \partial v / \partial x_2 = 0\} \quad (0 \leq i \leq 4),$$

which are actually identified with the spaces of functions dependent only on the variable x_1 as we will see later. By considering bilinear forms $a^{(i)}(\cdot, \cdot)$ for $i = 1, 2$ over the above type of function spaces, we are naturally led to the following bilinear forms:

$$(63) \quad a_Z^{(1)}(u, v) := (\partial u / \partial x_1, \partial v / \partial x_1)_T \quad (\forall u, v \in H^1(T)),$$

$$(64) \quad a_Z^{(2)}(u, v) := (\partial^2 u / \partial x_1^2, \partial^2 v / \partial x_1^2)_T \quad (\forall u, v \in H^2(T)).$$

As a characterization of $H^{1,Z}(T)$ above, let us state a lemma to be used later. Its proof is omitted since it can be performed by slightly modifying that for Theorem 3.1.4' of [16].

LEMMA 1. *Any $v \in H^{1,Z}(T)$ can be identified with a function v^* of single variable x_1 :*

$$(65) \quad v(x_1, x_2) = v^*(x_1) \quad \text{for a.e. } x = \{x_1, x_2\} \in T.$$

4.2. Monotonicity and upper bounds of $C_i(\alpha)$

We first derive some fundamental results for $C_i(\alpha)$'s for $0 < \alpha \leq 1$, especially for their upper bounds. With this regard, we owe much the following analysis to Babuška and Aziz [5]. In particular, the estimation $C_4(\alpha) \leq C_1$, an important part of the following theorem, was derived in [5] and also referred to in [25, 30], so that we here call C_1 the Babuška-Aziz constant.

THEOREM 4. *Constants $C_i(\alpha) = C_i(\alpha, \pi/2, 1)$ ($0 \leq i \leq 5$) are continuous positive-valued functions of $\alpha \in]0, +\infty[$ (here we consider also for $\alpha > 1$). In addition, except for $i = 4$, they are monotonically increasing in α . Thus, in particular,*

$$(66) \quad C_i(\alpha) \leq C_i (= C_i(1)) \quad ; \quad \forall \alpha \in]0, 1] \quad (i = 0, 1, 2, 3, 5).$$

On the other hand, it holds for $i = 4$ that

$$(67) \quad C_4(\alpha) \leq \max\{C_1(\alpha), C_2(\alpha)\} \leq C_1 (= C_2); \quad \forall \alpha \in]0, 1].$$

REMARK 5. From (66) and (67), $C_4(\alpha)$ is bounded from above by a monotonically increasing function of α . Moreover, numerical results in Section 6 suggest that it is also monotonically increasing.

PROOF. We just give sketches since the arguments below are popular. As in [5], we utilize the Rayleigh quotients $\tilde{R}_\alpha^{(i)}$'s defined in Section 4.1 for functions over the common domain T .

For the continuity, we first note that each Rayleigh quotient for a fixed $\tilde{v} \neq 0$ is a continuous positive function of α , so that its infimum over all \tilde{v} is uniformly bounded over any compact interval for α of the form $[\alpha_1, \alpha_2]$; $0 < \alpha_1 < \alpha_2 < +\infty$. It is also clear that the infimum for each $\alpha > 0$ is actually the minimum and cannot be zero (i. e., it is positive), as is shown by the Rellich compactness theorem and the reduction to absurdity. Then we can assure the existence of both $\overline{\lim}_{\beta \rightarrow \alpha} C_i^{-2}(\beta) (\leq C_i^{-2}(\alpha))$ and $\underline{\lim}_{\beta \rightarrow \alpha} C_i^{-2}(\beta)$ for each $\alpha > 0$ and i ; $0 \leq i \leq 5$. Choosing an appropriate bounded sequence in V^i associated to the above lower limit, we can prove $C_i^{-2}(\alpha) \leq \underline{\lim}_{\beta \rightarrow \alpha} C_i^{-2}(\beta)$, i. e., the continuity at α , by adopting the weakly lower semi-continuity of the numerator and the continuity of the denominator appearing in the definition of $\tilde{R}_\alpha^{(i)}$ with respect to the metric of V^i . Here, the Rellich type compactness theorem is again needed, and arguments similar to those in the subsequent subsection are used as well.

The monotonicity and (67) can be concluded in completely the same fashion as in [5]. \square

4.3. Asymptotic behaviors of constants as $\alpha \rightarrow +0$

We will now analyze asymptotic behaviors of the constants $C_i(\alpha)$'s ($0 \leq i \leq 5$) as $\alpha \rightarrow +0$ by adopting techniques developed e. g. in [23]. In particular, the right limit values $C_i(+0)$'s are determined from certain transcendental equations (derived from eigenvalue problems of ordinary differential equations) in terms of the hypergeometric functions [32]. For example, $C_2(+0)^{-1}$ is equal to the first positive zero of the Bessel function $J_0(\cdot)$. Moreover, these right limits give lower bounds for respective $C_i(\alpha)$'s, including the non-trivial case $i = 4$. Such results can be of use for analyzing the ‘‘anisotropic triangulations’’ discussed e. g. in [1, 8, 13].

4.3.1 Main results

We first present the main results as a theorem below.

THEOREM 5. *For each i ($0 \leq i \leq 5$), $C_i(+0) = \lim_{\alpha \rightarrow +0} C_i(\alpha)$ exists and is positive. Moreover, they are the lower limits of the respective constants, i. e., $C_i(+0) = \inf_{\alpha > 0} C_i(\alpha)$ for $0 \leq i \leq 5$. They are characterized by the relations $C_i(+0) = 1/\sqrt{\lambda^{(i)}}$ for $0 \leq i \leq 5$, where $\lambda^{(i)}$'s are the minimum eigenvalues of the following eigenvalue problems:*

$0 \leq i \leq 3$: Find $\lambda(= \lambda^{(i)}) \in \mathbf{R}$ and $u \in V^{i,Z} \setminus \{0\}$ such that

$$(68) \quad a_Z^{(1)}(u, v) = \lambda(u, v)_T \quad (\forall v \in V^{i,Z}),$$

$i = 4$: Find $\lambda(= \lambda^{(4)}) \in \mathbf{R}$ and $u \in V^{4,Z} \setminus \{0\}$ such that

$$(69) \quad a_Z^{(2)}(u, v) = \lambda a_Z^{(1)}(u, v) \quad (\forall v \in V^{4,Z}),$$

$i = 5$: Find $\lambda(= \lambda^{(5)}) \in \mathbf{R}$ and $u \in V^{4,Z} \setminus \{0\}$ such that

$$(70) \quad a_Z^{(2)}(u, v) = \lambda(u, v)_T \quad (\forall v \in V^{4,Z}).$$

These eigenvalue problems are also expressed by those for the following 2nd- or 4th-order ordinary differential equations for $u = u(s)$ over the interval $[0, 1]$.

$$(71) \quad i = 0 : -[(1-s)u'(s)]' = \lambda^{(0)}(1-s)u(s) \quad (0 < s < 1),$$

$$\int_0^1 (1-s)u(s) ds = u'(0) = 0,$$

$$(72) \quad i = 1 : -[(1-s)u'(s)]' = \lambda^{(1)}(1-s)u(s) + C \quad (0 < s < 1),$$

$$\int_0^1 u(s) ds = u'(0) = 0,$$

$$(73) \quad i = 2 : -[(1-s)u'(s)]' = \lambda^{(2)}(1-s)u(s) \quad (0 < s < 1), \quad u(0) = 0,$$

$i = 3$: essentially the same as for $i = 1$;

$$(74) \quad -[(1-s)u'(s)]' = \lambda^{(3)}(1-s)u(s) + C \quad (0 < s < 1),$$

$$\int_0^1 u(s) ds = u'(0) = 0,$$

$i = 4$: actually reduces to the case $i = 1$;

$$(75) \quad \begin{aligned} [(1-s)u''(s)]'' &= -\lambda^{(4)}[(1-s)u'(s)]' \quad (0 < s < 1), \\ u(0) = u(1) = u''(0) &= 0, \end{aligned}$$

$$(76) \quad \begin{aligned} i = 5 : [(1-s)u''(s)]'' &= \lambda^{(5)}(1-s)u(s) \quad (0 < s < 1), \\ u(0) = u(1) = u''(0) &= 0. \end{aligned}$$

Here, C is an unknown constant to be determined simultaneously with u and $\lambda^{(i)}$ ($i = 1, 3$).

REMARK 6. Two conditions $\int_0^1 (1-s)u(s) ds = 0$ and $u'(0) = 0$ in (71) are actually identical as shown by integrating the differential equation in (71) from $s = 0$ to $s = 1$. In the above, the numbers of boundary conditions are smaller than the orders of differential equations. This is mainly attributed to the singularities of coefficients at $s = 1$ in the differential equations, so that the usual full numbers of boundary conditions are excessive to decide eigenfunctions in respective spaces V^j 's ($0 \leq j \leq 4$). The eigenpairs expressed by the hypergeometric functions [32], together with numerical values of $C_i(+0)$'s ($0 \leq i \leq 5$), are given in Appendix.

4.3.2 Proof of main results

Let us prove Theorem 5. The proofs for the statements from $i = 0$ to $i = 5$ are more or less alike, and we will give descriptions almost exclusively for $i = 4$, the most complicated case.

1] To analyze $C_4(\alpha)$, let us define $\lambda_4(\alpha)$ for $\alpha > 0$ by $\lambda_4(\alpha) := C_4^{-2}(\alpha) > 0$, that is,

$$(77) \quad \lambda_4(\alpha) = \inf_{v \in V^4 \setminus \{0\}} \tilde{R}_\alpha^{(4)}(v); \quad \tilde{R}_\alpha^{(4)}(v) = a_\alpha^{(2)}(v)/a_\alpha^{(1)}(v).$$

By the standard arguments, the infimum is shown to be actually the minimum and attained by a certain $u \in V^4 \setminus \{0\}$. Moreover, $\{\lambda_4(\alpha), u\}$ is an eigenpair of the eigenvalue problem:

$$(78) \quad a_\alpha^{(2)}(u, v) = \lambda_4(\alpha)a_\alpha^{(1)}(u, v) \quad (\forall v \in V^4),$$

where $a_\alpha^{(i)}(u, v)$ for $i = 1, 2$ are the bilinear forms (59) and (60) associated to $a_\alpha^{(i)}(\cdot)$'s. The present $\lambda_4(\alpha) > 0$ is also shown to be the minimum eigenvalue of (78). Since $\tilde{R}_\alpha^{(4)}(v)$ is a homogeneous form of order 0, we can normalize the eigenfunction u as

$$(79) \quad a_\alpha^{(1)}(u) = 1.$$

2] Let us show that $\lambda_4(+0) = \lim_{\alpha \rightarrow +0} \lambda_4(\alpha)$ exists and is positive. Taking $v \in V^4 \setminus \{0\}$ in (77) as $v(x_1, x_2) = x_1(1 - x_1)$, we can see that $\lambda_4(\alpha)$ is uniformly bounded for $\alpha \in]0, \infty[$, and hence $\alpha = 0$ is an accumulation point. In particular, both $\lambda_4^* := \underline{\lim}_{\alpha \rightarrow +0} \lambda_4(\alpha) \geq 0$ and $\lambda_4^\dagger := \overline{\lim}_{\alpha \rightarrow +0} \lambda_4(\alpha)$ exist. Then we can find a sequence $\{\alpha_n\}_{n=1}^\infty$ in $]0, 1]$ such that

$$(80) \quad \lim_{n \rightarrow \infty} \alpha_n = 0, \quad \lim_{n \rightarrow \infty} \lambda_4(\alpha_n) = \lambda_4^*.$$

We must show that λ_4^* coincides with λ_4^\dagger to conclude the existence of the right limit $\lambda_4(+0)$.

Associated to the above sequence $\{\alpha_n\}$, there exists a sequence $\{u_n\}$ in $V^4 \setminus \{0\}$ such that each member satisfies (78) and (79), i. e., $a_{\alpha_n}^{(1)}(u_n) = 1$, $a_{\alpha_n}^{(2)}(u_n) = \lambda_4(\alpha_n)$, and

$$(81) \quad a_{\alpha_n}^{(2)}(u_n, v) = \lambda_4(\alpha_n) a_{\alpha_n}^{(1)}(u_n, v) \quad (\forall v \in V^4).$$

Since $|u|_{1,T}^2 = \sum_{i=1}^2 \|\partial u / \partial x_i\|_T^2 \leq a_\alpha^{(1)}(u)$ and $|u|_{2,T}^2 = \sum_{i,j=1}^2 \|\partial^2 u / \partial x_i \partial x_j\|_T^2 \leq a_\alpha^{(2)}(u)$ for $\alpha \in]0, 1]$, the sequence $\{u_n\}$ is bounded with respect to the semi-norms of $H^1(T)$ and $H^2(T)$:

$$(82) \quad |u_n|_{1,T}^2 + |u_n|_{2,T}^2 \leq 1 + \lambda_4(\alpha_n) \quad (n = 1, 2, \dots).$$

Moreover, $\{\|u_n\|_T\}$ is also shown to be bounded by noting that $\{u_n\}$ is a sequence in V^4 and utilizing the Rellich theorem. Thus $\{u_n\}$ is bounded in $H^2(T)$, so that there exist a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, and $u_0 \in V^4$ such that, for $n \rightarrow \infty$,

$$(83) \quad u_n \rightarrow u_0 \text{ weakly in } V^4 \subset H^2(T) \text{ and strongly in } H^1(T),$$

where the strong convergence is concluded by the Rellich theorem. Substituting $v_Z \in V^{4,Z}$ as $v \in V^4$ in (81) and then taking the limit for $n \rightarrow \infty$, we find

$$(84) \quad a_Z^{(2)}(u_0, v_Z) = \lambda_4^* a_Z^{(1)}(u_0, v_Z) \quad (\forall v_Z \in V^{4,Z}).$$

Furthermore, since $\|\partial u_n / \partial x_2\|_T^2 = \alpha_n^2 \left[a_{\alpha_n}^{(1)}(u_n) - \|\partial u_n / \partial x_1\|_T^2 \right]$ from (59), we can show that

$$(85) \quad \partial u_0 / \partial x_2 = 0, \text{ i. e., } u_0 \in V^{4,Z}.$$

Thus we have obtained (69), provided that $u_0 \neq 0$. For the moment, we cannot exclude the possibility that $u_0 = 0$, so that we will now consider the two cases below.

3] (Case: $u_0 \neq 0$) In this case, $\{\lambda_4^*, u_0\} \in \mathbf{R} \times V^{4,Z}$ is an eigenpair of (84), and is also associated with the following minimization problem:

$$(86) \quad \lambda = \inf_{v \in V^{4,Z} \setminus \{0\}} a_Z^{(2)}(v) / a_Z^{(1)}(v).$$

It is not difficult to show that this minimization problem has a minimum $\mu_4 > 0$, which is at the same time the minimum eigenvalue of (84) and whose arbitrary minimizer $v^Z \in V^{4,Z} \setminus \{0\}$ is an associated eigenfunction. Noting that λ_4^* is an eigenvalue of (84) and, for all $\alpha \in]0, \infty[$,

$$(87) \quad \begin{aligned} \mu_4 &= \inf_{v \in V^{4,Z} \setminus \{0\}} a_Z^{(2)}(v) / a_Z^{(1)}(v) = a_Z^{(2)}(v^Z) / a_Z^{(1)}(v^Z) \\ &= a_\alpha^{(2)}(v^Z) / a_\alpha^{(1)}(v^Z) = \tilde{R}_\alpha^{\{4\}}(v^Z) \geq \lambda_4(\alpha), \end{aligned}$$

we have $\mu_4 \leq \lambda_4^* = \underline{\lim}_{\alpha \rightarrow +0} \lambda_4(\alpha) \leq \lambda_4^\dagger = \overline{\lim}_{\alpha \rightarrow +0} \lambda_4(\alpha) \leq \mu_4$, that is, λ_4^* coincides with both λ_4^\dagger and μ_4 , so that it is the minimum eigenvalue of (84) and u_0 is an associated eigenfunction. Thus, if $u_0 \neq 0$ for all possible subsequences, λ_4^* is uniquely determined independently of the sequences like original $\{u_n\}$, so that the present λ_4^* is the true right limit $\lambda_4(+0)$. Furthermore, from the above consideration, λ_4^* is also the upper limit of $\lambda_4(\alpha)$ for $\alpha \in]0, \infty[$, that is, $1/\sqrt{\lambda_4^*}$ is the lower limit of $C_4(\alpha)$. By using $v(x_1, x_2) = \sin \pi x_1 \in V^{4,Z} \setminus \{0\}$ in the Rayleigh quotient in (86), we can also show that

$$(88) \quad 0 < \mu_4 = \lambda_4^* \leq \pi^2 < 10.$$

4] (Case: $u_0 = 0$) Let us define w_n by $w_n = \alpha_n^{-1} \partial u_n / \partial x_2$ ($n = 1, 2, \dots$). Then we can see that $w_n \in V^2 \subset H^1(T)$. Since $u_n \rightarrow u_0 = 0$ strongly in $H^1(T)$ and $a_{\alpha_n}^{(1)}(u_n) = 1$, it holds that $\|w_n\|_T^2 = 1 - \|\partial u_n / \partial x_1\|_T^2 \rightarrow 1$. Moreover, $a_{\alpha_n}^{(2)}(u_n) = \lambda_4(\alpha_n)$, i. e.,

$$(89) \quad \begin{aligned} & \|\partial^2 u_n / \partial x_1^2\|_T^2 + 2\|\partial w_n / \partial x_1\|_T^2 \\ & + \alpha_n^{-2} \|\partial w_n / \partial x_2\|_T^2 = \lambda_4(\alpha_n) \quad (n = 1, 2, \dots), \end{aligned}$$

is uniformly bounded, so that $\{w_n\}$ is bounded in $H^1(T)$ and $\|\partial w_n / \partial x_2\|_T \rightarrow 0$ ($n \rightarrow \infty$). Thus, further choosing a subsequence of $\{w_n\}$ and denoting it by the same notation, we can show the existence of $w_0 \in V^{2,Z} \setminus \{0\}$ with $\|w_0\|_T = 1$ such that, for $n \rightarrow \infty$,

$$(90) \quad w_n \rightarrow w_0 \text{ weakly in } V^2 \subset H^1(T), \text{ and strongly in } L^2(T).$$

Let v^* be an arbitrary function of x_1 such that $v^* \in C^2([0, 1])$ with $v^*(0) = 0$, and take $v \in V^4$ in (81) as $v(x_1, x_2) = v^*(x_1)x_2$. For simplicity, we will identify v^* with $v^* \otimes 1_{x_2}$, where 1_{x_2} is the unit constant function of x_2 . Then (81) becomes

$$(91) \quad \alpha_n a_Z^{(2)}(u_n, v) + 2a_Z^{(1)}(w_n, v^*) = \lambda_4(\alpha_n)[\alpha_n a_Z^{(1)}(u_n, v) + (w_n, v^*)_T].$$

Letting $n \rightarrow \infty$ above, we find that $w_0 \in V^{2,Z} \setminus \{0\}$ satisfies

$$(92) \quad a_Z^{(1)}(w_0, v^*) = \frac{1}{2} \lambda_4^*(w_0, v^*)_T.$$

Moreover, the above holds even for any v^* taken from $V^{2,Z}$, since any functions in $V^{2,Z}$ can be approximated by C^2 functions of x_1 vanishing at $x_1 = 0$. Thus the present relation can be viewed as an eigenvalue problem in $V^{2,Z}$ with an eigenpair $\{\lambda_4^*/2, w_0\}$. As usual, all the eigenvalues are positive, so that $\lambda_4^* > 0$.

By Lemma 1, w_0 can be identified with a function $w^*(x_1)$, so that (92) is rewritten as

$$(93) \quad \int_0^1 (1-x_1) \frac{dw^*}{dx_1}(x_1) \frac{dw^*}{dx_1}(x_1) dx_1 = \frac{1}{2} \lambda_4^* \int_0^1 (1-x_1) w^*(x_1) v^*(x_1) dx_1.$$

Taking v^* from $C_0^\infty(]0, 1[)$, we find, in the distributional (actually classical) sense on $]0, 1[$,

$$(94) \quad -\frac{d}{dx_1}[(1-x_1) \frac{dw^*}{dx_1}(x_1)] = \frac{1}{2} \lambda_4^* (1-x_1) w^*(x_1).$$

Moreover, it follows from the condition $w_0 \in V^{2,Z}$ that $w^*(0) = 0$. Since $\lambda_4^* > 0$, the general solution of the above is of the form, for arbitrary constants c_1 and c_2 ,

$$(95) \quad w^*(x_1) = c_1 J_0\left(\sqrt{\frac{\lambda_4^*}{2}}(1-x_1)\right) + c_2 Y_0\left(\sqrt{\frac{\lambda_4^*}{2}}(1-x_1)\right),$$

where $J_0(\cdot)$ and $Y_0(\cdot)$ are the 0-th order Bessel functions of the first and second kinds, respectively. As is well known, $J_0(\cdot)$ is sufficiently smooth, while $Y_0(\cdot)$ is of the form $Y_0(s) = c_3 \log s + r(s)$ for $s > 0$, where $c_3 \neq 0$ is a constant and $r(s)$ is a sufficiently smooth remainder term [32]. Consequently, c_2 must be zero for w_0 to belong to $V^{2,Z} \subset H^1(T)$. Then by considering the conditions $w^*(0) = 0$ and $c_1 \neq 0$, $J_0(\sqrt{\lambda_4^*/2})$ must be zero, that is, $\sqrt{\lambda_4^*/2}$ is equal to a positive zero of $J_0(\cdot)$. In fact $J_0(\cdot)$ has countably infinite positive zeros without any accumulation points except $+\infty$ [32]. Denoting the smallest positive zero by $\gamma_0 > 0$, we have

$$(96) \quad \lambda_4^* \geq 2\gamma_0^2.$$

We can show that $\gamma_0 > 2.25 = 9/4$, so that $\lambda_4^* > 10$. Comparing this with (88), i. e., $10 > \mu_4 \geq \sup_{\alpha>0} \lambda_4(\alpha) \geq \lambda_4^*$, we have a contradiction, and can exclude the possibility that $u_0 = 0$.

REMARK 7. Although it is well known that $\gamma_0 = 2.4048\dots$ numerically, we must verify that $\gamma_0 > 2.25$ for strict analysis. This can be done for example by using the well-known power series expansion $J_0(s) = \sum_{m=0}^{\infty} (-s^2/4)^m / (m!)^2$ and numerical verification techniques, cf. [18, 33].

5] We have now proved that λ_4^* and $u_0 \neq 0$ are actually the minimum eigenvalue and the associated eigenfunction of (69), respectively, and that $C_4(+0) = 1/\sqrt{\lambda_4^*} = \inf_{\alpha>0} C_4(\alpha)$.

It is not difficult to prove (75). That is, the differential equation can be obtained just as we derived (94) from (93), while $u(0) = u(1) = 0$ follow from the condition $u_0 \in V^{4,Z}$. Finally, $u''(0) = 0$ is obtained as a natural boundary condition associated to (69).

Let us also show that (75) reduces to (72). Denoting u' by v and integrating the differential equation in (75) with respect to the variable s , we have, for an arbitrary constant C ,

$$(97) \quad -[(1-s)v'(s)]' = \lambda^{(4)}[(1-s)v(s)] + C,$$

which coincides with the differential equation in (72) after rewriting v as u . The boundary condition $v'(0) = 0$ follows from $u''(0) = 0$, and the condition $\int_0^1 v(s) ds = 0$ is derived from the relation $\int_0^1 u'(s) ds = u(1) - u(0) = 0$. Once v is determined, u can be reconstructed by integration: $u(s) = \int_0^s v(t) dt$. Consequently, the present case $i = 4$ reduces to the case $i = 1$.

6] In the cases other than $i = 4$, the analyses are a bit easier since the denominators of $\tilde{R}_\alpha^{(i)}$'s do not depend on α . We just show how to derive (72) from (68) for $i = 1$.

For $i = 1$, $u = u(x_1, x_2)$ and $v = v(x_1, x_2)$ in $V^{1,Z}$ can be identified with functions $u^* = u^*(x_1)$ and $v^* = v^*(x_1)$, respectively, so that (68) for $i = 1$ can be expressed by

$$(98) \quad \int_0^1 (1-x_1) \frac{du^*}{dx_1}(x_1) \frac{dv^*}{dx_1}(x_1) dx_1 = \lambda \int_0^1 (1-x_1) u^*(x_1) v^*(x_1) dx_1.$$

Let us consider dw^*/dx_1 for an arbitrary $w^* \in C_0^\infty(]0, 1[)$. Then it can be identified with a function in $V^{1,Z}$, so that, by substituting it into (98) as v^* , we find that

$$(99) \quad \frac{d^2}{dx_1^2} [(1-x_1) \frac{du^*}{dx_1}(x_1)] = -\lambda \frac{d}{dx_1} [(1-x_1) u^*(x_1)],$$

from which follows the differential equation in (72). The definition of $V^{1,Z}$ implicates that $\int_0^1 u(s) ds = 0$, while $u'(0) = 0$ is a natural boundary condition associated to (68) for $i = 1$.

5. A Posteriori Estimation of Some Constants

It is in general difficult to determine exact values of various constants defined in Section 2 for $T_{\alpha,\theta}$ of general shape. Numerically, we can adopt the FEM to obtain approximate values to them as in [4, 7, 22, 30], but their quantitative error estimates are often unavailable. So, as an application of our results, let us give a kind of a posteriori estimation of $C_i(\alpha, \theta)$'s ($0 \leq i \leq 3$) by adopting the P_1 (piecewise linear) FEM. At present, our approach gives only approximate boundings of constants, which we expect will be turned into mathematically correct ones by appropriate numerical verification methods in future. Our approach is based on the classical a priori error estimates for the finite element approximations to the smallest

non-zero eigenvalue of the minus Laplacian with the Neumann or Dirichlet conditions, cf. e. g. Schultz [29].

5.1. Preliminaries

First let us make some preparations. Let Ω be a bounded convex polygonal domain, which will often be the triangle $T_{\alpha,\theta}$ later. Let us also consider a closed linear subspace $H_s^1(\Omega)$ of $H^1(\Omega)$, such as $H_0^1(\Omega)$ typically, which can be infinite-dimensional and satisfies

$$(100) \quad H_s^1(\Omega) \neq \{0\}, \quad 1 \notin H_s^1(\Omega) \quad (1 = \text{constant function of unit value}).$$

As a generalization of (40), let us consider the problem of finding $u \in H_s^1(\Omega)$, for a given $f \in L_2(\Omega)$, such that

$$(101) \quad (\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad (\forall v \in H_s^1(\Omega)).$$

Since the uniqueness and existence of u in $H_s^1(\Omega)$ are trivial, we can define an operator G_s by

$$(102) \quad G_s : f \in L_2(\Omega) \mapsto u \in H_s^1(\Omega) \text{ determined by (101)} .$$

As a generalization of the problem related to (17), let us also consider a minimization problem for the Rayleigh quotient

$$(103) \quad R^s(v) := |v|_{1,\Omega}^2 / \|v\|_\Omega^2 ; \quad v \in H_s^1(\Omega) \setminus \{0\} .$$

The minimum actually exists and is positive under (100) as may be shown by the compactness arguments. Moreover, denoting the minimum and an associated minimizer by $\lambda > 0$ and $u \in H_s^1(\Omega) \setminus \{0\}$, respectively, they satisfy

$$(104) \quad (\nabla u, \nabla v)_\Omega = \lambda(u, v)_\Omega \quad (\forall v \in H_s^1(\Omega)).$$

By using G_s in (102), the present $u \in H_s^1(\Omega)$ is shown to satisfy $u = \lambda G_s u$.

To apply the P_1 FEM to the above two problems, we first introduce a regular family of triangulations $\{\mathcal{T}^\eta\}_{\eta>0}$ of Ω mentioned in Section 3.4, and then construct the piecewise linear finite element space $S^\eta \subset H^1(\Omega)$ for each \mathcal{T}^η as

$$(105) \quad S^\eta := \{v_\eta \in C(\overline{\Omega}) \mid v_\eta|_K \text{ is a linear function for each } K \in \mathcal{T}^\eta\} .$$

For $u \in H^2(\Omega) \subset C(\overline{\Omega})$, we can define the piecewise linear interpolant $\Pi^{\eta,1}u \in S^\eta$ by

$$(106) \quad (\Pi^{\eta,1}u)(x^*) = u(x^*) \quad \text{for any vertex } x^* \text{ of } \mathcal{T}^\eta .$$

We will also use $\eta = \max_{K \in \mathcal{T}^\eta} h_K$, $C_{4,\eta} = \max_{K \in \mathcal{T}^\eta} C_4(\alpha_K, \theta_K)$ and $C_{5,\eta} = \max_{K \in \mathcal{T}^\eta} C_5(\alpha_K, \theta_K)$ defined in Section 3.4. Then we have the interpolation estimates ((46), (49)):

$$(107) \quad |u - \Pi^{\eta,1}u|_{1,\Omega} \leq C_{4,\eta}\eta|u|_{2,\Omega} , \quad \|u - \Pi^{\eta,1}u\|_\Omega \leq C_{5,\eta}\eta^2|u|_{2,\Omega} .$$

To construct approximate problems to (101) and the minimization of (103), let us consider the subspace S_s^η of S^η defined by

$$(108) \quad S_s^\eta := S^\eta \cap H_s^1(\Omega) ,$$

which we assume to be different from $\{0\}$. Although various other finite-dimensional subspaces of $H_s^1(\Omega)$ may be available in place of S_s^η , the above is anyway one possible choice.

Then an approximation to (101) is to find $u_\eta \in S_s^\eta$, for a given $f \in L_2(\Omega)$, such that

$$(109) \quad (\nabla u_\eta, \nabla v_\eta)_\Omega = (f, v_\eta)_\Omega \quad (\forall v_\eta \in S_s^\eta) .$$

The uniqueness and existence of u_η in S_s^η are trivial, so that we can define an operator G_s^η :

$$(110) \quad G_s^\eta : f \in L_2(\Omega) \mapsto u_\eta \in S_s^\eta \text{ determined by (109) ,}$$

which is approximating G_s . As generalizations of (42) and (43), we have

$$(111) \quad |G_s f - G_s^\eta f|_{1,\Omega} = \min_{v_\eta \in S_s^\eta} |G_s f - v_\eta|_{1,\Omega} ,$$

$$(112) \quad \begin{aligned} & \|G_s f - G_s^\eta f\|_\Omega \\ & \leq |G_s f - G_s^\eta f|_{1,\Omega} \sup_{g \in L_2(\Omega) \setminus \{0\}} \inf_{v_\eta \in S_s^\eta} |G_s g - v_\eta|_{1,\Omega} / \|g\|_\Omega . \end{aligned}$$

On the other hand, an approximation problem related to $R^s(\cdot)$ is to find its minimum in $S_s^\eta \setminus \{0\}$. In this case, the existence of the minimum is again

trivial, and the minimum λ^η and an associated minimizer $u_\eta \in S_s^\eta \setminus \{0\}$ satisfies the relation analogous to (104):

$$(113) \quad (\nabla u_\eta, \nabla v_\eta)_\Omega = \lambda^\eta (u_\eta, v_\eta)_\Omega \quad (\forall v_\eta \in S_s^\eta).$$

The following results (see e. g. Theorem 8.3 of [29]) play an essential role in our approach.

LEMMA 2. *Let λ and λ^η be respectively defined by $\lambda = \min_{v \in H_s^1(\Omega) \setminus \{0\}} R^s(v)$ and $\lambda^\eta = \min_{v_\eta \in S_s^\eta \setminus \{0\}} R^s(v_\eta)$, and $u \in H_s^1(\Omega)$ be an minimizer associated to λ such that $\|u\|_\Omega = 1$. Then it holds that, for any $v_\eta \in S_s^\eta \setminus \{0\}$ with $\|u - v_\eta\|_\Omega < 1$,*

$$(114) \quad \lambda \leq \lambda^\eta \leq \lambda + [|u - v_\eta|_{1,\Omega} / (1 - \|u - v_\eta\|_\Omega)]^2.$$

The following results are also well known and will be used later, cf. [15].

LEMMA 3. *For the present Ω and a given $f \in L_2(\Omega)$, let us consider $u \in H^1(\Omega)$ such that*

$$(115) \quad (\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad (\forall v \in H^1(\Omega)).$$

Then such u exists if and only if

$$(116) \quad (f, 1)_\Omega = 0,$$

and is unique up to an additive arbitrary constant function. Moreover, $u \in H^2(\Omega)$ with

$$(117) \quad |u|_{2,\Omega} \leq \|\Delta u\|_\Omega = \|f\|_\Omega.$$

REMARK 8. One possible condition to assure the uniqueness of u is $(u, 1)_\Omega = 0$. The present problem corresponds to the Poisson equation with the homogeneous Neumann condition:

$$(118) \quad -\Delta u = f \quad \text{in } \Omega, \quad \partial u / \partial n = 0 \quad \text{on } \partial \Omega.$$

5.2. A posteriori estimation of $C_0(\alpha, \theta)$

We first give a posteriori estimates to $C_0(\alpha, \theta)$, for which $\Omega = T_{\alpha, \theta}$ and $H_s^1(\Omega) = V_{\alpha, \theta}^0$. Let \mathcal{P}^0 be the orthogonal projection operator from $L_2(T_{\alpha, \theta})$ to $L_2^0(T_{\alpha, \theta}) := \{g \in L_2(\alpha, \theta) \mid (g, 1)_{T_{\alpha, \theta}} = 0\}$:

$$(119) \quad \mathcal{P}^0 g := g - |T_{\alpha, \theta}|^{-1}(g, 1)_{T_{\alpha, \theta}} \quad (\forall g \in L_2(T_{\alpha, \theta})) ,$$

where $|T_{\alpha, \theta}|$ is the measure of $T_{\alpha, \theta}$. Moreover, $\mathcal{P}^0|H^1(T_{\alpha, \theta})$ is also an orthogonal projection operator from $H^1(T_{\alpha, \theta})$ to $V_{\alpha, \theta}^0$ with respect to the inner product of $H^1(T_{\alpha, \theta})$: $(u, v)_{1, T_{\alpha, \theta}} := (u, v)_{T_{\alpha, \theta}} + (\nabla u, \nabla v)_{T_{\alpha, \theta}}$ ($\forall u, v \in H^1(T_{\alpha, \theta})$). We also denote the present G_s , G_s^η and S_s^η respectively by G_0 , G_0^η and S_0^η . Since S^η contains the constant functions, we have

$$(120) \quad S_0^\eta = \mathcal{P}^0 S^\eta .$$

We will again omit the subscript $T_{\alpha, \theta}$ for the (semi-)norms and inner products. Since $\nabla \mathcal{P}^0 v = \nabla v$ and $(f, \mathcal{P}^0 v) = (\mathcal{P}^0 f, v)$ for any $v \in H^1(T_{\alpha, \theta})$, eq. (101) for $u \in V_{\alpha, \theta}^0$ in the present case becomes

$$(121) \quad (\nabla u, \nabla v) = (\mathcal{P}^0 f, v) \quad (\forall v \in H^1(T_{\alpha, \theta})) ,$$

where $\mathcal{P}^0 f = f$ under (116). Likewise, eq. (104) for $\{\lambda, u\} \in \mathbf{R} \times V_{\alpha, \theta}^0 \setminus \{0\}$ becomes

$$(122) \quad (\nabla u, \nabla v) = \lambda(u, v) \quad (\forall v \in H^1(T_{\alpha, \theta})) ,$$

since $\mathcal{P}^0 u = u$. By Lemma 3, the above u belongs to $H^2(T_{\alpha, \theta}) \cap V_{\alpha, \theta}^0$ with

$$(123) \quad |u|_2 \leq \lambda \|u\| .$$

Under the preceding preparations, let us apply Lemma 2 to estimate the minimum eigenvalue λ_0^η of (113) for $S_s^\eta = S_0^\eta$ in terms of the minimum eigenvalue λ_0 of (104) or (122). The minimizer associated to λ_0 is denoted by u_0 with the normalization condition $\|u_0\| = 1$. As v_η in (112), we can take various candidates from S_0^η . One possibility is to utilize the interpolant $\Pi^{\eta, 1} u_0 \in S^\eta$ of u_0 . Unfortunately, it may be outside S_0^η , but its projection $\mathcal{P}^0 \Pi^{\eta, 1} u_0$ can be used thanks to (120). By using the properties of the orthogonal projection (119), we find that

$$(124) \quad |u_0 - \mathcal{P}^0 \Pi^{\eta, 1} u_0|_1 = |u_0 - \Pi^{\eta, 1} u_0|_1 ,$$

$$(125) \quad \|u_0 - \mathcal{P}^0 \Pi^{\eta,1} u_0\| = \|\mathcal{P}^0(u_0 - \Pi^{\eta,1} u_0)\| \leq \|u_0 - \Pi^{\eta,1} u_0\| .$$

Using (107) and (123), we can evaluate the above in terms of η , λ_0 , $C_{4,\eta}$ and $C_{5,\eta}$. Unfortunately, we have not necessarily obtained accurate theoretical upper bounds for $C_{5,\eta}$ as was noted in Remark 4. So we should also try to avoid the use of such a constant.

Another possibility is to use $\tilde{u}_{\eta,0} := \lambda_0 G_0^\eta u_0$, which is surely in S_0^η and is suggested by the identity $u_0 = \lambda_0 G^0 u_0$. For this choice, we have

$$(126) \quad |u_0 - \tilde{u}_{\eta,0}|_1 \leq |u_0 - \mathcal{P}^0 \Pi^{\eta,1} u_0|_1 = |u_0 - \Pi^{\eta,1} u_0|_1 ,$$

$$(127) \quad \|u_0 - \tilde{u}_{\eta,0}\| \leq |u_0 - \tilde{u}_{\eta,0}|_1 \sup_{g \in L_2(T_{\alpha,\theta}) \setminus \{0\}} \inf_{v_\eta \in S_0^\eta} |G_0 g - v_\eta|_1 / \|g\| .$$

Here, only former part of (107) is needed: η , λ_0 and $C_{4,\eta}$ are necessary but $C_{5,\eta}$ is not so.

Based on the above considerations, we have now the following two a priori error estimates.

LEMMA 4 (A priori estimates for λ_0^η). *If $C_{5,\eta} \eta^2 \lambda_0 < 1$, it holds for the above λ_0 and λ_0^η :*

$$(128) \quad \lambda_0 \leq \lambda_0^\eta \leq \lambda_0 + [C_{4,\eta} \eta \lambda_0 / (1 - C_{5,\eta} \eta^2 \lambda_0)]^2 .$$

Similarly, if $C_{4,\eta}^2 \eta^2 \lambda_0 < 1$, then

$$(129) \quad \lambda_0 \leq \lambda_0^\eta \leq \lambda_0 + [C_{4,\eta} \eta \lambda_0 / (1 - C_{4,\eta}^2 \eta^2 \lambda_0)]^2 .$$

REMARK 9. In actual application of the above estimates, where the exact value of $C_{4,\eta}$ ($C_{5,\eta}$, resp.) may not be available, we can use an appropriate upper bound $\tilde{C}_{4,\eta}$ ($\tilde{C}_{5,\eta}$, resp.). From the considerations in Remark 4, (128) would give a better bounding than (129).

Let us define two functions of variable t related to (128) and (129):

$$(130) \quad \varphi_{0,1}(t) := t + [C_{4,\eta} \eta t / (1 - C_{5,\eta} \eta^2 t)]^2 \quad (0 < t < 1 / (C_{5,\eta} \eta^2)) ,$$

$$(131) \quad \varphi_{0,2}(t) := t + [C_{4,\eta} \eta t / (1 - C_{4,\eta}^2 \eta^2 t)]^2 \quad (0 < t < 1 / (C_{4,\eta} \eta^2)) ,$$

where quantities other than t are considered just parameters. Since these functions are continuous and monotonically increasing, they have their inverse functions, which are defined in $]0, +\infty[$ and will be denoted by $\varphi_{0,1}^{-1}$ and $\varphi_{0,2}^{-1}$. Then we can easily obtain the following a posteriori estimates or boundings of λ_0 by numerically obtained λ_0^η .

THEOREM 6 (A posteriori estimates for λ_0). *The above λ_0 , λ_0^η , $\varphi_{0,1}^{-1}$ and $\varphi_{0,2}^{-1}$ satisfy*

$$(132) \quad \varphi_{0,1}^{-1}(\lambda_0^\eta) \leq \lambda_0 \leq \lambda_0^\eta \quad \text{if } \lambda_0^\eta < 1/(C_{5,\eta}\eta^2),$$

$$(133) \quad \varphi_{0,2}^{-1}(\lambda_0^\eta) \leq \lambda_0 \leq \lambda_0^\eta \quad \text{if } \lambda_0^\eta < 1/(C_{4,\eta}\eta^2).$$

PROOF. Lemma 4 gives $(0 <) \lambda_0^\eta \leq \varphi_{0,i}(\lambda_0) \leq \varphi_{0,i}(\lambda_0^\eta)$ if $\lambda_0^\eta < 1/(C_{5,\eta}\eta^2)$ ($i = 1$) or $\lambda_0^\eta < 1/(C_{4,\eta}\eta^2)$ ($i = 2$), from which the conclusion follows by operating $\varphi_{0,i}^{-1}$. \square

Now we can easily obtain boundings to the constant $C_0(\alpha, \theta)$. For example, from (132),

$$(134) \quad 1/\sqrt{\lambda_0^\eta} \leq C_0(\alpha, \theta) \leq 1/\sqrt{\varphi_{0,1}^{-1}(\lambda_0^\eta)} \quad \text{if } \lambda_0^\eta < 1/(C_{5,\eta}\eta^2).$$

The results (132) and (133) can be also viewed as a posteriori error estimates for λ_0^η , since (132), for example, can be rewritten as $0 \leq \lambda_0^\eta - \lambda_0 \leq \lambda_0^\eta - \varphi_{0,1}^{-1}(\lambda_0^\eta)$.

REMARK 10. Estimates (128), (129), (132) and (133) also hold for λ and λ^η of Lemma 2 with $H_s^1(\Omega) = H_0^1(\Omega)$ and $S_s^\eta = S^\eta \cap H_0^1(\Omega)$. Here, Lemma 3 cannot be used, but the corresponding G_s has the property $G_s : L_2(\Omega) \rightarrow H_0^1(\Omega) \cap H^2(\Omega)$ with $|G_s f|_2 \leq \|f\|$ ($\forall f \in L_2(\Omega)$), cf. Section 3.4. Moreover, we cannot utilize projection operators like \mathcal{P}^0 above, but can take advantage of the property $\Pi^{\eta,1} G_s f \in S_s^\eta$ ($\forall f \in L_2(\Omega)$) for the present $\Pi^{\eta,1}$, G_s and S_s^η . Such a case is related to approximating Poincaré's constant [2], i.e., numerical evaluation of the smallest eigenvalue of $-\Delta$ with the homogeneous Dirichlet condition.

5.3. A posteriori estimation of $C_i(\alpha, \theta)$'s ($i = 1, 2, 3$)

Secondly, let us give a posteriori estimates to $C_i(\alpha, \theta)$'s ($1 \leq i \leq 3$), where $\Omega = T_{\alpha, \theta}$ and $H_s^1(\Omega) = V_{\alpha, \theta}^i$. As notations G_s , G_s^η and S_s^η for each $i \in \{1, 2, 3\}$, we will use G_i , G_i^η and S_i^η , respectively. Let us define an operator $\mathcal{P}^i : H^1(T_{\alpha, \theta}) \rightarrow V_{\alpha, \theta}^i$ ($i \in \{1, 2, 3\}$) by

$$(135) \quad \mathcal{P}^i v := v - |e_i|^{-1} \int_{e_i} v \, ds \quad (\forall v \in V_{\alpha, \theta}^i),$$

where $|e_i|$ denotes the length of edge e_i . Unlike \mathcal{P}^0 , the above operators are not well-defined over $L_2(T_{\alpha, \theta})$, but the following relations similar to (120) still hold :

$$(136) \quad S_i^\eta = \mathcal{P}^i S^\eta \quad (1 \leq i \leq 3).$$

Suggested by [26], let us introduce quadratic functions f_i 's ($1 \leq i \leq 3$) of $x = \{x_1, x_2\}$ by

$$(137) \quad f_i(x_1, x_2) := |e_i| [(x_1 - x_1^i)^2 + (x_2 - x_2^i)^2] / (4|T_{\alpha, \theta}|),$$

where $x^1 = \{x_1^1, x_2^1\} = B(\alpha \cos \theta, \alpha \sin \theta)$, $x^2 = \{x_1^2, x_2^2\} = A(1, 0)$ and $x^3 = \{x_1^3, x_2^3\} = O(0, 0)$. These functions are sufficiently smooth and satisfy

$$(138) \quad \partial f_i / \partial n = \delta_{ij} \quad \text{on } e_j \quad (\forall i, j \in \{1, 2, 3\}).$$

Then $\int_{e_i} v \, ds = (\nabla f_i, \nabla v) + (\Delta f_i, v)$ ($\forall v \in H^1(T_{\alpha, \theta})$), so that (135) can be rewritten by

$$(139) \quad \mathcal{P}^i v := v - |e_i|^{-1} [(\nabla f_i, \nabla v) + (\Delta f_i, v)] \quad (\forall v \in H^1(T_{\alpha, \theta})).$$

Similarly to (121), eq. (101) for $u \in V_{\alpha, \theta}^i$ in the present case becomes $(\nabla u, \nabla v) = (f, \mathcal{P}^i v)$ ($\forall v \in H^1(T_{\alpha, \theta})$), which can be rewritten by

$$(140) \quad \begin{aligned} & (\nabla [u + |e_i|^{-1} (f, 1) f_i], \nabla v) \\ & = (f - |e_i|^{-1} (f, 1) \Delta f_i, v) \quad (\forall v \in H^1(T_{\alpha, \theta})). \end{aligned}$$

By Lemma 3, we find that $u + |e_i|^{-1} (f, 1) f_i \in H^2(T_{\alpha, \theta})$ with $|u + |e_i|^{-1} (f, 1) f_i|_2 \leq \|f - |e_i|^{-1} (f, 1) \Delta f_i\|$, and hence, by using the triangle and Schwarz inequalities,

$$(141) \quad |u|_2 \leq \|f\| + |e_i|^{-1} |(f, 1)| (|f_i|_2 + \|\Delta f_i\|)$$

$$\leq \|f\| \left[1 + |e_i|^{-1} \sqrt{|T_{\alpha,\theta}|} (|f_i|_2 + \|\Delta f_i\|) \right].$$

Clearly, it holds that $2|T_{\alpha,\theta}| = \alpha \sin \theta$, $|e_1| = 1$, $|e_2| = \alpha$, $|e_3| = \sqrt{1 + \alpha^2 - 2\alpha \cos \theta}$, $|f_i|_2 = \|\Delta f_i\|/\sqrt{2}$ and $\Delta f_i(x_1, x_2) = |e_i|/|T_{\alpha,\theta}|$, so that we have, for all $i \in \{1, 2, 3\}$,

$$(142) \quad |u|_2 \leq (2 + 1/\sqrt{2})\|f\| .$$

Similarly, eq. (104) for $\{\lambda, u\} \in \mathbf{R} \times V_{\alpha,\theta}^i$ ($1 \leq i \leq 3$) in the present case becomes

$$(143) \quad (\nabla u, \nabla v) = \lambda(u, \mathcal{P}^i v) \quad (\forall v \in H^1(T_{\alpha,\theta})) .$$

Thus, we can utilize the results for (140) by taking f in (140) as λu in (143). The approximation problems corresponding to (109) and (113) are also given by using S_i^η 's ($1 \leq i \leq 3$). Then, just as in the case of $C_0(\alpha, \theta)$, we have the following results for $C_i(\alpha, \theta)$'s ($1 \leq i \leq 3$).

THEOREM 7 (A priori and a posteriori estimates for eigenvalues ($i = 1, 2, 3$)). *For each $i \in \{1, 2, 3\}$, let λ_i and λ_i^η be respectively the smallest eigenvalues of (104) and (113) for $H_s^1(\Omega) = V_{\alpha,\theta}^i$ and $S_s^\eta = S_i^\eta$. Then, if $(MC_{4,\eta})^2 \lambda_i < 1$ with $M := 2 + 1/\sqrt{2}$, it holds that*

$$(144) \quad \lambda_i \leq \lambda_i^\eta \leq \lambda_i + [MC_{4,\eta} \lambda_i / (1 - M^2 C_{4,\eta}^2 \lambda_i)]^2 .$$

and, if $\lambda_i^\eta < 1/(MC_{4,\eta})^2$,

$$(145) \quad \varphi_i^{-1}(\lambda_i^\eta) \leq \lambda_i \leq \lambda_i^\eta ,$$

where φ_i^{-1} is the inverse function of the following monotonically increasing continuous function:

$$(146) \quad \varphi_i(t) := t + [MC_{4,\eta} \eta t / (1 - M^2 C_{4,\eta}^2 \eta^2 t)]^2 \\ (0 < t < 1/(MC_{4,\eta})^2 ; \quad 1 \leq i \leq 3) .$$

REMARK 11. Because of the factor $M \approx 2.7071\dots$, efficiency of (144) is worse than that of (129). In the present case, estimates corresponding

to (128) using $C_{5,\eta}$ do not appear to be fully effective unlike (128). This is attributed to the fact that we cannot at present obtain desirable estimates for $\|u - \mathcal{P}^i \Pi^{\eta,1} u\|$ ($\forall u \in V_{\alpha,\theta}^i \cap H^2(T_{\alpha,\theta}); 1 \leq i \leq 3$), since \mathcal{P}^i is not definable over $L_2(T_{\alpha,\theta})$ and hence we cannot use the best approximation property there.

6. Numerical Results

We performed numerical computations to see the actual dependence of various constants on α and θ . Furthermore, we also utilized the obtained exact values or upper bounds of such constants to give quantitative a posteriori error estimates for some eigenvalue problems.

6.1. Computational methods

To obtain approximate values of error constants, we can utilize the FEM quite effectively. In particular, we used the most popular P_1 triangular finite element for numerical computations of $C_i(\alpha, \theta)$'s for $0 \leq i \leq 3$ by preparing appropriate triangulations of $T_{\alpha,\theta}$. For $C_4(\alpha, \theta)$ and $C_5(\alpha, \theta)$, it is natural to use various triangular finite elements for Kirchhoff plate bending problems, since the associated partial differential equations are of 4th order as is noted in Section 2. In our actual computations, we used the discrete Kirchhoff triangular element presented in [17]. On the other hand, we can also use the Siganevich approach for computation of $C_4(\alpha, \theta)$, which adopts the P_1 triangle together with a penalty method for a system of 2nd order partial differential equations similar to the incompressible Stokes system [30].

In every case, we have a matrix eigenvalue problem as the discretization of the original eigenvalue problem described by a weak form. More specifically, it is a generalized matrix eigenvalue problem with respect to unknown eigenvectors of nodal values of approximate eigenfunctions, and it can be solved for example by the inverse iteration method [9]. A difficulty in deriving such matrix eigenvalue problems is how to deal with linear constraint conditions imposed on the spaces $V_{\alpha,\theta}^i$ for $i = 0, 1, 2, 3$. Similar constraint conditions also appear if we use the Siganevich method to compute $C_4(\alpha, \theta)$. One possible method is to eliminate some unknown nodal values by using the linear constraints, but then we have non-sparse coefficient matrices in general. Another method is to employ the Lagrange multiplier method, which does not destroy the global sparseness of the matrices. We tested

both approaches with reasonable results. On the other hand, we do not have such a difficulty in computing $C_4(\alpha, \theta)$ and $C_5(\alpha, \theta)$ by Kirchhoff type triangular elements, where the linear constraints $v(O) = v(A) = v(B) = 0$ for $V_{\alpha, \theta}^4$ can be handled as homogeneous “point” conditions.

The numerical results below are obtained by FEM in the double or quadruple precision arithmetics, without strict numerical verification. But their accuracy appears to be reasonable at least in graphical level, since finer mesh computations give essentially the same graphs.

6.2. Numerical results for error constants

We first show some results for $C_i(\alpha)$'s ($0 \leq i \leq 5$) by the P_1 finite element and the Kirchhoff triangular element in [17] with the uniform triangulation of the domain T_α . In such calculations, T_α is subdivided into a number of small triangles congruent to $T_{\alpha, \pi/2, h}$ with e.g. $h = 1/20$. The Siganevich method [30] is also tested to calculate $C_4(\alpha)$ approximately.

Figure 2 consists of two parts and illustrates the graphs of approximate $C_i(\alpha)$'s ($0 \leq i \leq 5$) versus $\alpha \in]0, 1]$. Exact values of C_0 and $C_1 = C_2$ together with an approximate value of C_5 are also included as horizontal lines. At $\alpha = 1$, the approximate values coincide well with the available exact ones in Theorem 3, and we can numerically see that $C_1 (= C_2)$ is a nice upper bound of C_4 . For general α , the monotonically increasing behaviors theoretically predicted for $C_i(\alpha)$'s ($i = 0, 1, 2, 3, 5$) as well as the relation $C_4(\alpha) \leq \min\{C_1(\alpha), C_2(\alpha)\}$ are also well observable in the graphs. The present numerical results suggest that $C_4(\alpha)$ is also monotonically increasing, but we have not succeeded in proving such a conjecture. Moreover, when $\alpha \approx 0$, the numerical results agree well with the exact right limits given in Appendix. For $C_4(\alpha)$, we tested two methods, that is, the P_1 triangle with the Siganevich method and the Kirchhoff triangle. The graph for $C_4(\alpha)$ in Fig. 2 is actually obtained by the latter approach, but is indistinguishable in graphical level from the one by the former method.

We also check numerically the validity of the upper bounds for $C_4(\alpha, \theta)$ in Corollary 1 and Theorem 2. We here show one example with $\theta = 2\pi/3$: $C_4(\alpha, 2\pi/3)$, $C_4^{(1)}(\alpha, 2\pi/3) := C_4(\alpha)\phi_4(2\pi/3)$ and $C_4^{(2)}(\alpha, 2\pi/3) := \nu(\alpha, 2\pi/3)/(\sqrt{2}\sin\frac{2\pi}{3})$, where two functions ϕ_4 and ν come from (32) and (34). We also obtained $C_4(\alpha)$, $C_1(\alpha, 2\pi/3)$ and $C_2(\alpha, 2\pi/3)$ numerically to use in the above two formulas. The results are shown in Fig. 3, and we can see that both $C_4^{(1)}(\alpha, 2\pi/3)$ and $C_4^{(2)}(\alpha, 2\pi/3)$ give upper bounds to

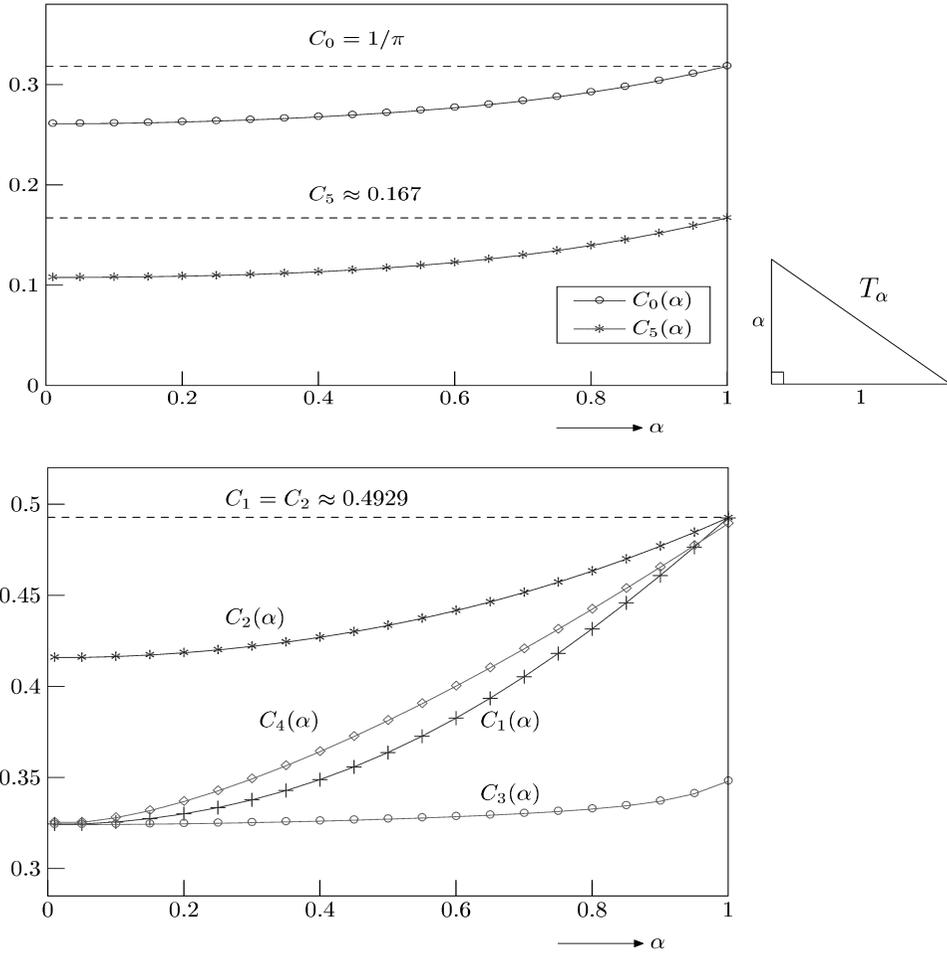


Fig. 2. Numerically obtained graphs for $C_i(\alpha)$'s ($0 \leq i \leq 5$; $0 < \alpha \leq 1$)

$C_4^{(1)}(\alpha, 2\pi/3)$ numerically. In the present case, $C_4^{(2)}(\alpha, 2\pi/3)$ appears to be superior to $C_4^{(1)}(\alpha, 2\pi/3)$ as upper bounds.

Figures 4 and 5 illustrate numerically obtained contour lines for $C_i(\alpha, \theta)$'s in the $\alpha - \theta$ polar coordinates, where the abscissa denotes $\alpha \cos \theta$, and the ordinate does $\alpha \sin \theta$. The unit circle $\alpha = 1$ is also shown by a dotted curve. The minimum required range for α and θ is specified by (1),

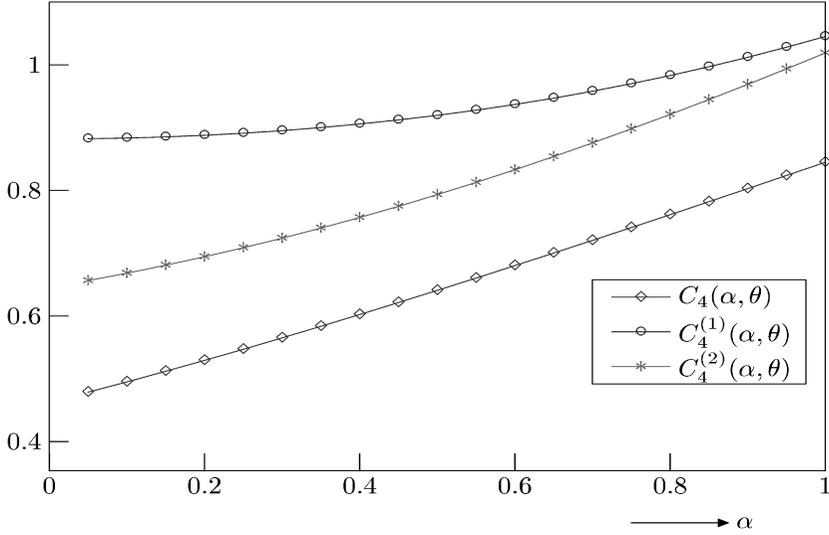


Fig. 3. Two upper bounds of $C_4(\alpha, \theta)$ for $\theta = 2\pi/3$

but the contour lines are shown in wider ranges to see their global behaviors. The contour lines are sometimes cut off in the portions $\alpha \approx 0$ and $|\theta - \pi/2| \approx \pi/2$, where the expected accuracy may be insufficient. The behavior of $C_4(\alpha, \theta)$ appears to be the most complicated among all, and the necessity of the maximum angle condition is visually seen. As is expected, the other constants seem to be uniformly bounded for $\alpha \leq 1$.

6.3. A posteriori estimates of eigenvalues

To apply the results in Section 5, let us consider a posteriori estimates for $C_0 = C_0(1, \pi/2)$ and $C_1 = C_1(1, \pi/2)$ based on the P_1 FEM, or rather for the eigenvalues $\lambda_0 = C_0^{-2}$ and $\lambda_1 = C_1^{-2}$.

Table 1 gives boundings for λ_0 by (132) and (133) of Theorem 6 and those for λ_1 by (145) of Theorem 7. We tested several meshes for T , which are uniform ones as is shown in the table. The parameters $\tilde{C}_{4,\eta}$, $\tilde{C}_{5,\eta}$ and η in (132), (133) and (145) are specified here as

$$(147) \quad \tilde{C}_{4,\eta} = 0.5, \quad \tilde{C}_{5,\eta} = 0.17, \quad \eta = 1/N,$$

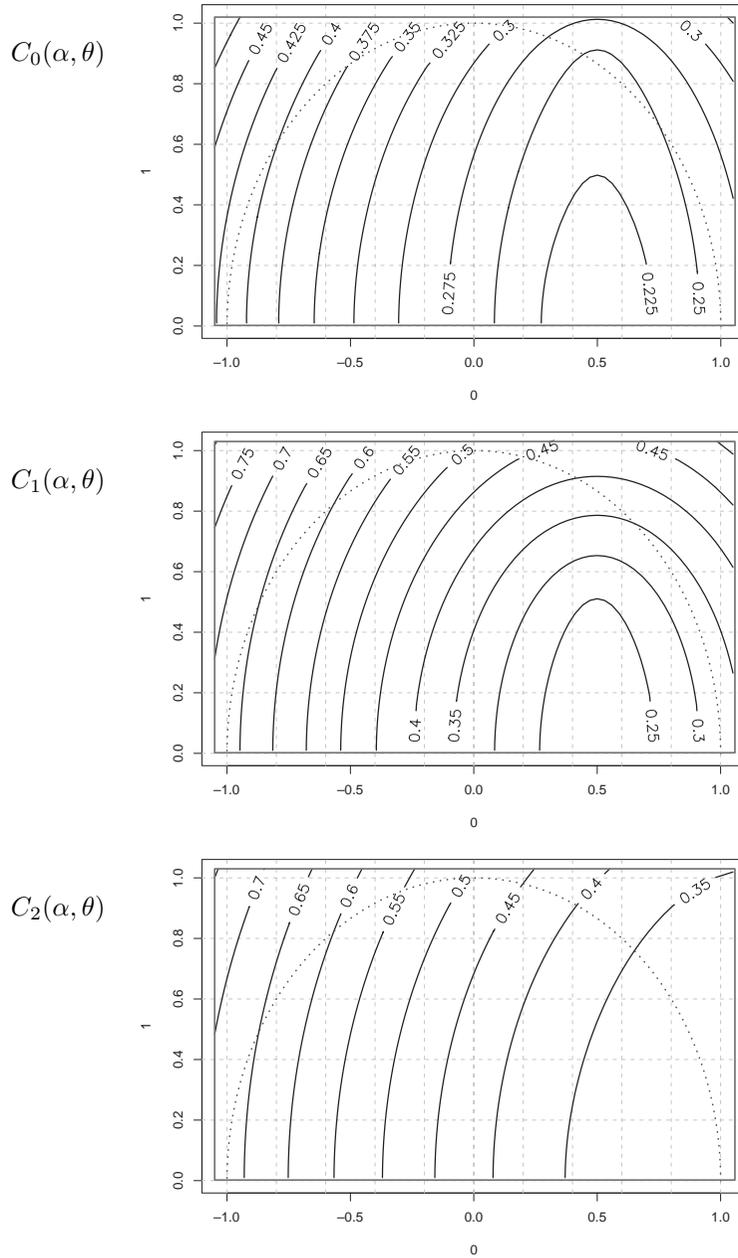


Fig. 4. Contour lines of $C_i(\alpha, \theta)$ for $i = 0, 1, 2$

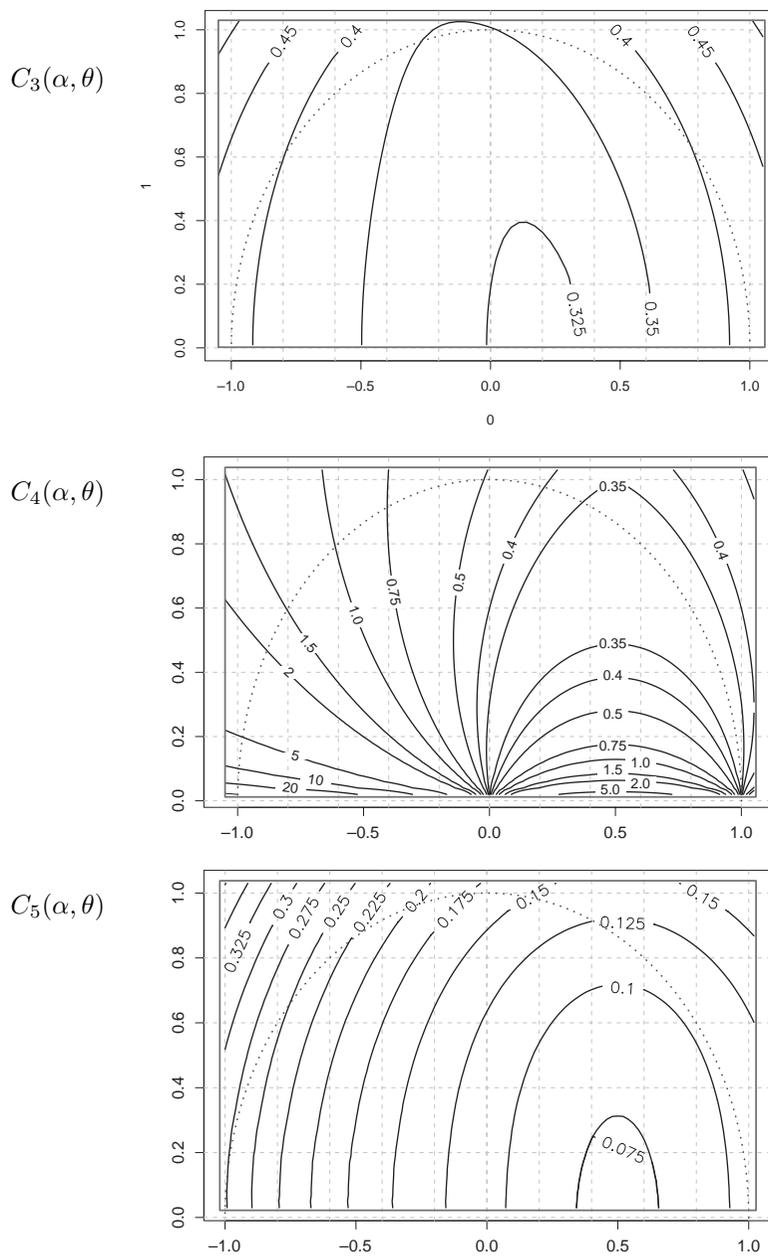
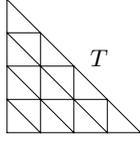


Fig. 5. Contour lines of $C_i(\alpha, \theta)$ for $i = 3, 4, 5$

Table 1. A posteriori estimates for λ_0 and λ_1



$\eta = 1/N$, $N = 4$ in the left figure

† Approximate eigenvalue is outside the domain of definition for φ_1^{-1} .

N	bounds for λ_0 by $\varphi_{0,1}^{-1}$	bounds for λ_0 by $\varphi_{0,2}^{-1}$	bounds for λ_1 by φ_1^{-1}
2	$6.5550 < \lambda_0 < 11.7155$	$5.9890 < \lambda_0 < 11.7155$	$\lambda_1 < 4.3071^\dagger$
3	$8.1463 < \lambda_0 < 10.7213$	$7.8874 < \lambda_0 < 10.7213$	$1.9780 < \lambda_1 < 4.2102$
4	$8.8616 < \lambda_0 < 10.3570$	$8.7512 < \lambda_0 < 10.3570$	$2.6006 < \lambda_1 < 4.1713$
8	$9.6143 < \lambda_0 < 9.9946$	$9.6055 < \lambda_0 < 9.9946$	$3.6537 < \lambda_1 < 4.1304$
16	$9.8060 < \lambda_0 < 9.9012$	$9.8054 < \lambda_0 < 9.9012$	$3.9982 < \lambda_1 < 4.1196$
32	$9.8537 < \lambda_0 < 9.8776$	$9.8537 < \lambda_0 < 9.8776$	$4.0864 < \lambda_1 < 4.1168$
64	$9.8656 < \lambda_0 < 9.8716$	$9.8656 < \lambda_0 < 9.8716$	$4.1085 < \lambda_1 < 4.1161$
(∞)	$\lambda_0 = \pi^2 = 9.869604\dots$		$\lambda_1 \approx 4.115858$

where N is the number of elements along each edge of T ($N = 4$ in the figure of Table 1). Here $\tilde{C}_{4,\eta}$ is a correct upper bound of $C_{4,\eta}$ (cf. Theorems 3 and 4), while $\tilde{C}_{5,\eta}$ is only an approximate one of $C_{5,\eta}$ at present. We tested (132) to see its effectiveness experimentally.

We can observe that the present simple methods can actually bound C_0 and C_1 from both above and below. As is expected, (132) gives better lower bounds than (133) for coarser meshes. Table 1 also shows that the lower bounds obtained for C_1 are in general rougher than those for C_0 , which is probably attributed to the factor $M = 2 + 1/\sqrt{2}$.

As another application, let us bound the first eigenvalue of $-\Delta$ with the Dirichlet condition for the regular n -polygonal domain Ω_n ($n \geq 3$), circumscribing the unit disk Ω_∞ centered at the origin O . In this case, the formulas in Lemma 4 and Theorem 6 can be used without modifications since each Ω_n is convex, cf. Remark 10. It is well known that the first eigenvalue for Ω_n is monotonically increasing in n and is bounded from above by that for Ω_∞ . The eigenvalues for $n = 4$ and $n = \infty$ are known as $\pi^2/2$

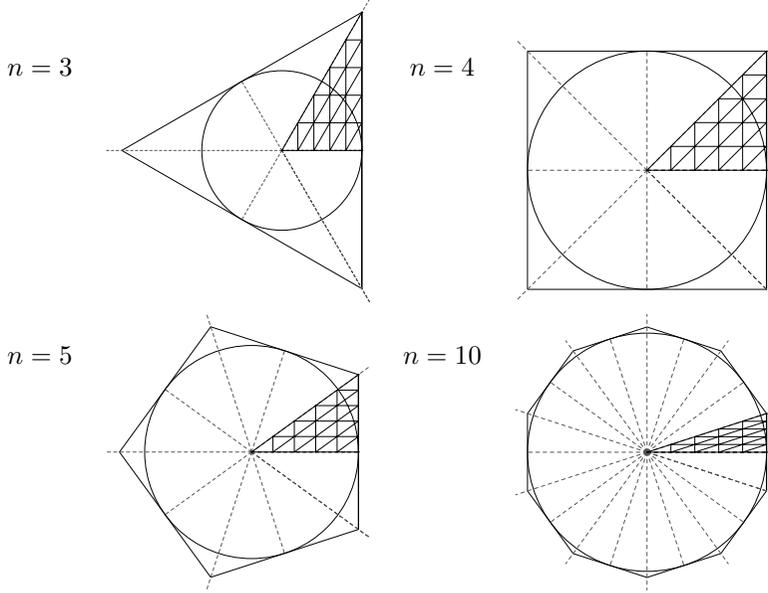


Fig. 6. Meshes for n -polygonal domains Ω_n with $N = 5$: $n = 3, 4, 5, 10$

and the square of the first zero of the Bessel function J_0 , respectively, but it is difficult to determine the exact values for general n . So we will numerically evaluate such eigenvalues for several n with a posteriori estimates.

As meshes, we first triangulate the right triangle $\triangle OAB$ with $OA = 1$, $AB = \tan(\pi/n)$ and $\angle OAB = \pi/2$ by dividing each edges uniformly into N segments. Then by a reflection and rotations, we can obtain whole meshes for Ω_n , see Fig. 6. Then we can use (133) with

$$(148) \quad \tilde{C}_{4,\eta} = 0.5, \quad \eta = \sqrt{3}/N \text{ if } n = 3, \quad \eta = 1/N \text{ if } n \geq 4,$$

where $\alpha \leq 1$ in all the cases. The obtained results are summarized in Table 2, from which we can experimentally see the effectiveness of our bounding (133). Such results will be strictly mathematical one when appropriate verification methods become available.

Table 2. A posteriori estimates for the first eigenvalue λ associated to Ω_n

n	N	bounds for λ	N	bounds for λ	N	bounds for λ
3	5	$3.9082 < \lambda < 4.4963$	10	$4.2688 < \lambda < 4.4147$	100	$4.3853 < \lambda < 4.3868$
4	5	$4.7700 < \lambda < 5.0211$	10	$4.8954 < \lambda < 4.9569$	100	$4.9344 < \lambda < 4.9351$
5	5	$5.0049 < \lambda < 5.2826$	10	$5.1590 < \lambda < 5.2273$	100	$5.2075 < \lambda < 5.2082$
6	5	$5.1387 < \lambda < 5.4323$	10	$5.3114 < \lambda < 5.3839$	100	$5.3659 < \lambda < 5.3667$
7	5	$5.2220 < \lambda < 5.5257$	10	$5.4078 < \lambda < 5.4831$	100	$5.4666 < \lambda < 5.4674$
8	5	$5.2774 < \lambda < 5.5879$	10	$5.4727 < \lambda < 5.5498$	100	$5.5346 < \lambda < 5.5354$
9	5	$5.3160 < \lambda < 5.6313$	10	$5.5185 < \lambda < 5.5969$	100	$5.5827 < \lambda < 5.5836$
10	5	$5.3440 < \lambda < 5.6628$	10	$5.5520 < \lambda < 5.6313$	100	$5.6181 < \lambda < 5.6190$

7. Concluding Remarks

We have obtained some explicit relations for the dependence of several interpolation error constants on geometric parameters of triangular finite elements. We can effectively utilize these results to give upper bounds of various a priori and a posteriori error estimates of finite element solutions based on the P_1 and/or P_0 approximate functions. Some numerical results were also given to see the effectiveness of our analysis and the actual behaviors of the error constants. To obtain clearer picture of the interpolation error constants, we should also perform various analyses including numerical analysis with verifications, asymptotic analysis etc.

We have mainly considered the conforming P_1 triangle, which can naturally construct subspaces of H^1 space over the entire domain. But there also exists a non-conforming counterpart, which is also based on the piecewise linear polynomials but uses the midpoint nodes of edges [11, 31]. Analysis of such an element is more complicated, since we must evaluate the errors induced by the interelement discontinuity of the approximate functions. Still we can obtain some interpolation error estimates as given in [19] by using the constants for the P_0 and the conforming P_1 triangles. We will report more refined results in subsequent papers.

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Appendix A. Determination of $\lambda^{(i)} = C_i^{-2}(+0)$ ($0 \leq i \leq 5$)

Recall Theorem 5 for the determination of $C_i(+0) := \lim_{\alpha \rightarrow +0} C_i(\alpha)$ or $\lambda^{(i)} = C_i^{-2}(+0)$ ($0 \leq i \leq 5$). Fortunately, all the ordinary differential equations (ODE) there can be solved by means of the hypergeometric functions [32], so that we can obtain the determination equations in terms of such functions. All the numerical results below are obtained by using Mathematica[®].

A.1 $\lambda^{(0)}$

From Theorem 5, the ODE and the boundary condition in this case are given by

$$(149) \quad -((1-s)u'(s))' = \lambda^{(0)}(1-s)u(s) \quad \text{for } s \in]0, 1[, \quad u'(0) = 0.$$

The general solution of the above that can be identified with an element of $H^1(T) \supset V^{0,Z}$ is

$$(150) \quad u(s) = c^{(1)} J_0(\sqrt{\lambda^{(0)}}(1-s)),$$

where $c^{(1)}$ is an arbitrary constant and J_0 is the 0-th order Bessel function of the first kind. Actually $Y_0(\sqrt{\lambda^{(0)}}(1-s))$ (Y_0 = the 0-th order Bessel function of the second kind) also satisfies the ODE but cannot be identified with an element of $H^1(T)$. Thus applying the boundary condition above and the relation $J_1 = -J_0'$, we have the equation for $\lambda^{(0)}$ as $J_1(\sqrt{\lambda^{(0)}}) = 0$, which means that $\sqrt{\lambda^{(0)}}$ is the smallest positive zero of J_1 . Approximately, we get

$$(151) \quad \lambda^{(0)} \approx 3.83171^2, \quad C_0(+0) \approx 0.260980.$$

A.2 $\lambda^{(1)} = \lambda^{(3)} = \lambda^{(4)}$

In this case, the ODE, the linear constraint and the boundary condition are given by (72) as

$$(152) \quad \begin{aligned} & -((1-s)u'(s))' = \lambda^{(1)}(1-s)u(s) + C \quad \text{for } s \in]0, 1[, \\ & \int_0^1 u(s)ds = 0, \quad u'(0) = 0, \end{aligned}$$

where C is an arbitrary constant. Then the general solution of ODE that can be identified with an element of $H^1(T) \supset V^{1,Z}$ is expressed by

$$(153) \quad \begin{aligned} u(s) = c^{(1)} & J_0(\sqrt{\lambda^{(1)}}(1-s)) \\ & - C(1-s) {}_1F_2(1; 3/2, 3/2; -\lambda^{(1)}(1-s)^2/4), \end{aligned}$$

where $c^{(1)}$ is an arbitrary constant, and ${}_1F_2(\cdot; \cdot, \cdot; \cdot)$ is a kind of hypergeometric function. Using the linear constraint and the boundary condition, we have the following for $\lambda = \lambda^{(1)}$:

$$(154) \quad \begin{aligned} & \frac{\lambda}{4} {}_0F_1(; 2; -\frac{\lambda}{4}) {}_2F_3(1, 1; \frac{3}{2}, \frac{3}{2}, 2; -\frac{\lambda}{4}) \\ & + {}_1F_2(\frac{1}{2}; 1, \frac{3}{2}; -\frac{\lambda}{4}) {}_1F_2(1; \frac{1}{2}, \frac{3}{2}; -\frac{\lambda}{4}) = 0, \end{aligned}$$

where ${}_0F_1$ and ${}_2F_3$ are also hypergeometric functions. Approximately, we have

$$(155) \quad \lambda^{(1)} \approx 3.08126^2, \quad C_1(+0) \approx 0.324542.$$

A.3 $\lambda^{(2)}$

By Theorem 5, the ODE and the boundary condition associated with $\lambda^{(2)}$ are given as

$$(156) \quad -((1-s)u'(s))' = \lambda^{(2)}(1-s)u(s) \quad \text{for } s \in]0, 1[, \quad u(0) = 0.$$

Then the general solution of the above ODE belonging to $H^1(T) \supset V^{2,Z}$ is the same as (150): $u(x) = c^{(1)}J_0(\sqrt{\lambda^{(2)}}(1-s))$, so that the determination equation for $\lambda^{(2)}$ is obtained as $J_0(\sqrt{\lambda^{(2)}}) = 0$, that is, $\sqrt{\lambda^{(2)}}$ is the minimum positive zero of J_0 . Approximately, we have

$$(157) \quad \lambda^{(2)} \approx 2.40483^2, \quad C_2(+0) \approx 0.415831.$$

A.4 $\lambda^{(5)}$

By Theorem 5, the ODE and the boundary conditions associated to $\lambda^{(5)}$ are given as

$$(158) \quad \begin{aligned} ((1-s)u''(s))'' &= \lambda^{(5)}(1-s)u(s) \\ \text{for } s \in]0, 1[, \quad u(0) &= u(1) = u''(0) = 0. \end{aligned}$$

Then the general solution of the ODE belonging to $H^2(T) \supset V^{4,Z}$ is

$$(159) \quad \begin{aligned} u(s) &= c^{(1)} {}_0F_3\left(\frac{1}{2}, \frac{3}{4}, \frac{3}{4}; \frac{\lambda^{(5)}(1-s)^4}{256}\right) \\ &\quad + c^{(2)}(1-s) {}_0F_3\left(\frac{3}{4}, 1, \frac{5}{4}; \frac{\lambda^{(5)}(1-s)^4}{256}\right) \\ &\quad + c^{(3)}(1-s)^2 {}_0F_3\left(\frac{5}{4}, \frac{5}{4}, \frac{3}{2}; \frac{\lambda^{(5)}(1-s)^4}{256}\right), \end{aligned}$$

where $c^{(1)}$, $c^{(2)}$ and $c^{(3)}$ are arbitrary constants, and ${}_0F_3$ is a hypergeometric function. Then, introducing two functions $f(\lambda, t) = t {}_0F_3\left(\frac{3}{4}, 1, \frac{5}{4}; \frac{\lambda}{256}t^4\right)$ and $g(\lambda, t) = t^2 {}_0F_3\left(\frac{5}{4}, \frac{5}{4}, \frac{3}{2}; \frac{\lambda}{256}t^4\right)$, the determination equation for $\lambda = \lambda^{(5)}$ is given by

$$(160) \quad f''(\lambda, 1)g(\lambda, 1) - g''(\lambda, 1)f(\lambda, 1) = 0,$$

where $'' = \partial^2/\partial t^2$. Approximately, we find that

$$(161) \quad \lambda^{(5)} \approx 9.26775^2, \quad C_5(+0) \approx 0.107901.$$

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