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Torus Fibrations and Localization of Index I – Polarization and Acyclic Fibrations

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Abstract. We define a local Riemann-Roch number for an open symplectic manifold when a completely integrable system without Bohr-Sommerfeld fiber is provided on its end. In particular when a structure of a singular Lagrangian fibration is given on a closed symplectic manifold, its Riemann-Roch number is described as the sum of the number of nonsingular Bohr-Sommerfeld fibers and a contribution of the singular fibers. A key step of the proof is formally explained as a version of Witten's deformation applied to a Hilbert bundle.

1. Introduction

The purpose of this paper is to give a localization technique for the index of spin^c Dirac operator. Our localization makes use of no group action, but a family of acyclic flat connections on tori. A typical example is given by a symplectic manifold with an integrable system.

For a completely integrable system on a closed symplectic manifold, the Riemann-Roch number is sometimes equal to the number of Bohr-Sommerfeld fibers. For Kähler case D. Borthwick, T. Paul and A. Uribe gave a relationship between Bohr-Sommerfeld Lagrangians and the kernels of twisted spin^c Dirac operators using Fourier integral operators [2]. A problem here would be how to count singular Bohr-Sommerfeld fibers, and how to count the contribution from other singular fibers if there are. M. D. Hamilton and E. Miranda dealt with the singular Bohr-Sommerfeld fibers using J. Śniatycki's framework and found a discrepancy between quantizations via real and complex polarizations under this framework [10, 11].

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When an integrable system is associated to a Hamiltonian group action, the geometric quantization conjecture of V. Guillemin and S. Sternberg is a localization property of the Riemann-Roch character. H. Duistermaat, V. Guillemin, E. Meinrenken and S. Wu gave a proof of the conjecture for torus actions using the geometric localization provided by E. Lerman's symplectic cuts [3] [12]. Y. Tian and W. Zhang gave a proof using a direct analytic localization with perturbation of the Dirac operator [14].

In this paper we give a localization property of the Riemann-Roch numbers for a completely integrable system possibly with singular fibers. We define multiplicities for Bohr-Sommerfeld fibers and singular fibers so that the total sum of the multiplicities for all the fibers is exactly equal to the Riemann-Roch number. Our method is flexible and allows various generalizations. In this paper, however, we explain only the simplest case. In our subsequent papers we will explain some of the generalizations and, as an application, an approach to the conjecture of the Guillemin and Sternberg for Hamiltonian torus actions [4, 5].

Our idea is simple. Let M be a $2n$-dimensional closed symplectic manifold with a prequantizing line bundle L. Suppose X is an n-dimensional affine space, and $\pi : M \to X$ a completely integrable system. The symplectic structure ω gives rise to an element $[D \otimes L]$ of the K-homology group $K_0(M)$, where D is the Dolbeault operator for an almost complex structure compatible with ω. The projection π gives an element of $K_0(X)$ defined to be the pushforward $\pi_*[D \otimes L]$. The Riemann-Roch number is calculated as the further pushforward of $\pi_*[D \otimes L]$ with respect to the map from X to a point.

If we have a formulation of K-homology group in terms of some geometric data, and if a geometric representative of $\pi_*[D \otimes L]$ has a *localized support*, then the Riemann-Roch number can be calculated by the data on the support.

In fact it is possible to realize this idea rigorously if we use a formulation of K-homology in terms of some notion of generalized vector bundle with Clifford module bundle action, which is developed in [7], [9][1]. In this short paper, however, we explain only the localization property without appealing

[1]Strictly speaking what we actually need here is the notion of a *K-cohomology cocycle with local coefficient* which is defined using some action of a Clifford algebra bundle. For a manifold if the Clifford algebra bundle is the one generated by its tangent space, the twisted K-cohomology group is identified with its K-homology through a duality.

to the framework of [7] at the expense of giving up identifying $\pi_*[D \otimes L]$ in terms of geometric data.

Some key points of our arguments are:

1. The restriction of the Dolbeault operator, or the spin^c Dirac operator, to a Lagrangian submanifold is equal to its de Rham operator at least on the level of principal symbol.

2. A flat line bundle over a torus is trivial if and only if the corresponding twisted de Rham complex is not acyclic.

3. The localization we use is closely related to adiabatic limit.

4. The Laplacian corresponding to the de Rham operator along Lagrangian fibers plays the role of the potential term of the Dirac-type operator on the base manifold.

5. Since the Laplacian is a second-order elliptic differential operator, when it is strictly positive, it can absorb the effect of first-order term.

The last property makes our construction rather flexible.

The organization of this paper is as follows. In Sections 2, 3 and 4, we give localization properties for symplectic setting, topological setting, and analytical setting respectively. The latter is a generalization of the former for each stage. In Section 5 we prove the localization. In Section 6 we give examples for 2-dimensional topological case. Finally in Section 7 we give comments for possible generalizations.

2. Symplectic Formulation

Let M be a $2n$-dimensional closed symplectic manifold with a symplectic form ω. Suppose L is a complex Hermitian line bundle over M with Hermitian connection ∇ satisfying that the curvature of ∇ is equal to $-2\pi\sqrt{-1}\omega$. M has an almost complex structure compatible with ω, and we can define the Riemann-Roch number $RR(M, L)$ as the index of the Dolbeault operator with coefficients in L.

Let X be an n-dimensional affine space, and $\pi : M \to X$ a completely integrable system. Then generic fibers of π are empty or disjoint unions of finitely many n-dimensional tori with canonical affine structures.

DEFINITION 2.1. $x \in X$ is *L-acyclic* if x is a regular value of π and L does not have any non-vanishing parallel section over the fiber at x.

The main purpose of this paper is to show that $RR(M, L)$ is localized at non L-acyclic points. More precisely

THEOREM 2.2. *Suppose $X = U_\infty \cup (\cup_{i=1}^m U_i)$ is an open covering satisfying the following properties.*

1. *$\{U_i\}_{i=1}^m$ are mutually disjoint.*

2. *U_∞ consists of L-acyclic points.*

Let $V_i = \pi^{-1}(U_i)$. Then for each $i = 1, \dots, m$ there exists an integer $RR(V_i, L)$, which depends only on the data restricted on V_i, such that

$$RR(M, L) = \sum_{i=1}^{m} RR(V_i, L).$$

Here the integer $RR(V_i, L)$ is invariant under continuous deformations of the data.

REMARK 2.3. The theorem asserts that the Riemann-Roch number $RR(M, L)$ is localized at non-singular Bohr-Sommerfeld fibers and singular fibers.

REMARK 2.4. While the Riemann-Roch number $RR(M, L)$ for the closed symplectic manifold M depends only on the symplectic structure ω, the localized Riemann-Roch number $RR(V_i, L)$ depends on the restrictions of ω, π, ∇ as well, though we omit this dependence in the notation if there is no confusion.

REMARK 2.5. In fact Theorem 2.2 holds for a Lagrangian fibration possibly with singular fibers.

In the next section we reduce the above localization theorem to a slightly more general localization (Theorem 3.3) formulated purely topologically.

3. Topological Formulation

In this section let M be a $2n$-dimensional closed spin^c manifold,[2] and E a complex Hermitian vector bundle over M. We define the Riemann-Roch number $RR(M,E)$ as the index of the spin^c Dirac operator with coefficients in E.[3] Let V be an open subset of M.

Definition 3.1. *A real polarization* on V is the data (U, π, ϕ, J) satisfying the following properties.

1. U is an n-dimensional smooth manifold.
2. $\pi : V \to U$ is a fiber bundle whose fibers are disjoint unions of finitely many n-dimensional tori with affine structures.
3. $\phi : \pi_* T_{\text{fiber}} V \to TU$ is an isomorphism between two real vector bundles, where $\pi_* T_{\text{fiber}} V$ is the vector bundle on U consisting of parallel sections of the tangent bundle of the fiber $\pi^{-1}(x)$ for each $x \in U$.
4. J is an almost complex structure on V which is a reduction of the given spin^c-structure.
5. The composition of $J : T_{\text{fiber}} V \to TV$ and $\pi_* : TV \to TU$ is equal to the map induced from ϕ.

Suppose that $V \subset M$ has a real polarization (U, π, ϕ, J) such that the restriction $E|_V$ has a unitary connection ∇ along fibers for the bundle structure $\pi : V \to U$.

Definition 3.2. (E, ∇) is *acyclic* if the restriction $(E, \nabla)|_{\pi^{-1}(x)}$ is a flat vector bundle and the twisted de Rham cohomology group $H^*(\pi^{-1}(x), (E, \nabla)|_{\pi^{-1}(x)})$ is zero for every $x \in U$.

Theorem 3.3. *Let M be a closed spin^c manifold and E a complex Hermitian vector bundle over M. Suppose $M = V_\infty \cup (\cup_{i=1}^m V_i)$ is an open covering satisfying the following properties.*

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[2]In this paper we take a convention of spin^c structures which do not need any Riemannian metrics. See Appendix A for the convention.

[3]Precisely, in order to define the spin^c Dirac operator with coefficients in E we need a unitary connection on E. But it is well-known that $RR(M,E)$ itself does not depend on the choice of connections. So we do not mention it here.

1. *$\{V_i\}_{i=1}^m$ are mutually disjoint.*

2. *V_∞ has a real polarization (U, π, ϕ, J).*

3. *$E|_{V_\infty}$ has a unitary connection ∇ along fibers for the bundle structure $\pi : V_\infty \to U$.*

4. *$(E|_{V_\infty}, \nabla)$ is acyclic.*

Then for each $i = 1, \dots, m$ there exists an integer $RR(V_i, E)$, which depends only on the data restricted on V_i, such that

$$RR(M, E) = \sum_{i=1}^{m} RR(V_i, E).$$

Here the integer $RR(V_i, E)$ is invariant under continuous deformations of the data.

PROOF OF THEOREM 2.2 ASSUMING THEOREM 3.3. Note that the symplectic structure gives an isomorphism $T^*U_\infty \cong \pi_* T_{\mathrm{fiber}} V_\infty$. Fix a Riemannian metric on TU_∞ so that we have an isomorphism $TU_\infty \cong T^*U_\infty$. Define ϕ by using these two. By fixing a Lagrangian splitting $TV_\infty \cong T_{\mathrm{fiber}} V_\infty \oplus \pi^* TU_\infty$, ϕ induces an almost complex structure, which determines the spin^c structure. We take E to be L. Then the rest would be obvious. □

In the next section we further reduce Theorem 3.3 to more general localization for some Dirac-type operator (Theorem 4.5).

4. Analytical Formulation

In this section let M be a closed Riemannian manifold. We denote by $Cl(TM)$ the Clifford algebra bundle over M generated by TM. Let $W = W^0 \oplus W^1$ be a $\mathbb{Z}/2\mathbb{Z}$-graded complex Hermitian vector bundle over M with a structure of $Cl(TM)$-module such that for each vector v in TM, the action of v is skew-Hermitian and of degree-one.

We use the following definition of *Dirac-type operator*.

DEFINITION 4.1. A first order differential operator $D : \Gamma(W) \to \Gamma(W)$ is *a Dirac-type operator on W* if D is a degree-one formally self-adjoint operator with smooth coefficient whose symbol is given by the Clifford action on W.

Dirac-type operators on W are not unique, but their indices are equal.

DEFINITION 4.2. We denote by $\mathrm{Ind}\,(M, W)$ the index of a Dirac-type operator on W.

Let V be an open subset of M.

DEFINITION 4.3. *A generalized real polarization* on V is the data $(U, \pi, D_{\mathrm{fiber}})$ satisfying the following properties.

1. U is a Riemannian manifold.

2. $\pi : V \to U$ is a fiber bundle with fiber a closed manifold.

3. Let $TV = T_{\mathrm{fiber}}V \oplus T_{\mathrm{fiber}}^{\perp}V$ be the orthogonal decomposition with respect to the Riemannian metric on V. Then the projection gives an isometric isomorphism $T_{\mathrm{fiber}}^{\perp}V \cong \pi^*TU$.

4. $D_{\mathrm{fiber}} : \Gamma(W|_V) \to \Gamma(W|_V)$ is a family of Dirac-type operators along fibers anti-commuting with the Clifford action of TU in the following sense.

   (a) D_{fiber} is an order-one, formally self-adjoint differential operator of degree-one.

   (b) D_{fiber} contains only the derivatives along fibers, i.e, D_{fiber} commutes with multiplication of the pullback of smooth functions on U.

   (c) The principal symbol of D_{fiber} is given by the Clifford action of $T_{\mathrm{fiber}}V$.

   (d) The Clifford action of TU on $W|_V$ anti-commutes with D_{fiber}. Here the Clifford action of TU on $W|_V$ is defined through the horizontal lift $\pi^*TU \cong T_{\mathrm{fiber}}^{\perp}V \subset TV$, where the first isomorphism is the one given in 3 above.

DEFINITION 4.4. A generalized real polarization $(U, \pi, D_{\text{fiber}})$ on V is *acyclic* if for each $x \in U$, the restriction of D_{fiber} to $\Gamma(W|_{\pi^{-1}(x)})$ has zero kernel.

THEOREM 4.5. *Let M be a closed Riemannian manifold and $W = W^0 \oplus W^1$ a $Cl(TM)$-module bundle as above. Suppose $M = V_\infty \cup (\cup_{i=1}^m V_i)$ is an open covering satisfying the following properties.*

1. *$\{V_i\}_{i=1}^m$ are mutually disjoint.*
2. *V_∞ has an acyclic generalized real polarization $(U, \pi, D_{\text{fiber}})$.*

Then for each $i = 1, \dots, m$ there exists an integer $\operatorname{Ind}(V_i, W)$, which depends only on the data restricted on V_i, such that

$$\operatorname{Ind}(M, W) = \sum_{i=1}^m \operatorname{Ind}(V_i, W).$$

Here the integer $\operatorname{Ind}(V_i, W)$ is invariant under continuous deformations of the data.

The proof of Theorem 3.3 follows from the next obvious lemma.

LEMMA 4.6. *Let T be an n-dimensional torus with an affine structure. Let $\mathfrak{X}$ be the n-dimensional vector space of parallel vector fields. For any Euclidean metric on $\mathfrak{X}$, T has an induced flat Riemannian metric. Then each element of the dual space $\mathfrak{X}^*$ gives a harmonic 1-form.*

PROOF OF THEOREM 3.3 ASSUMING THEOREM 4.5. Fix a Riemannian metric on U. Combining it with the flat metric associated with the affine structures on fibers of V_∞ via the almost complex structure J, we define a Riemannian metric on V_∞ and extend it to M. Take W to be the tensor product of the spinor bundle of the spin^c manifold M and E. Then we define the family of Dirac-type operators along fibers acting on $\Gamma(W|_{V_\infty})$ by $D_{\text{fiber}} := d_{E,\text{fiber}} + d^*_{E,\text{fiber}}$, where $d_{E,\text{fiber}}$ is the exterior derivative on fibers twisted by the unitary connection ∇ on E and $d^*_{E,\text{fiber}}$ is its formal adjoint. The above lemma implies the anti-commutativity between D_{fiber} and the Clifford action of TU. The acyclic condition for (E, ∇) implies the acyclicity for $(U, \pi, D_{\text{fiber}})$. $\square$

5. Local Index

Let M be a Riemannian manifold and $W = W^0 \oplus W^1$ a $Cl(TM)$-module bundle. Suppose V is an open subset of M with an acyclic generalized real polarization $(U, \pi, D_{\text{fiber}})$ such that $M \smallsetminus V$ is compact. We will define the *local index* $\operatorname{Ind}(M, V, W)$ (or $\operatorname{Ind}(V, W)$ for short) and show deformation invariance and an excision property. The local index depends on the acyclic generalized real polarization on V though we omit it in the notation for simplicity.

REMARK 5.1. When $\pi : V \to U$ is a diffeomorphism, D_{fiber} is a degree-one self-adjoint homomorphism on $W|_V$ anti-commuting with the Clifford multiplication of TV. In this special case the definition of $\operatorname{Ind}(V, W)$ is already given in [6, Chapter 3]. (See Definition 3.14 for the setting, Definition 3.21 for the definition in the case of cylindrical end, Theorem 3.20 for deformation invariance, Theorem 3.29 for the excision property, and Section 3.3 for the definition for general case.) We will generalize the argument there.

5.1. Vanishing lemmas

We will show the following lemma later.

LEMMA 5.2. *Suppose M is closed and $M = V$ has an acyclic generalized real polarization. Take any Dirac-type operator D on W and write $D = \widetilde{D}_U + D_{\text{fiber}}$ for some operator $\widetilde{D}_U$. For a real number t, put $D_t = \widetilde{D}_U + tD_{\text{fiber}}$. Then for any large $t > 0$, $\operatorname{Ker} D_t = 0$.*

We also need a slightly generalized version, which is shown later.

LEMMA 5.3. *Suppose $M = V$ and M has a cylindrical end with translationally invariant acyclic generalized real polarization on it. Take any Dirac-type operator D on W with translationally invariance on the end, and write $D = \widetilde{D}_U + D_{\text{fiber}}$ for some operator $\widetilde{D}_U$. For a real number t, put $D_t = \widetilde{D}_U + tD_{\text{fiber}}$. Then for any large $t > 0$, $\operatorname{Ker} D_t \cap \{L^2\text{-sections}\} = 0$.*

Admitting these lemmas we first give the definition and properties of the local index.

5.2. Cylindrical end

We first give the definition for the special case that M has a cylindrical end and every data is translationally invariant on the end.

LEMMA 5.4. *Suppose M has a cylindrical end $V = N \times (0, \infty)$ with translationally invariant acyclic generalized real polarization on it. Let ρ be a non-negative smooth cut-off function on M satisfying $\rho = 1$ on $N \times [1, \infty)$ and $\rho = 0$ on $M \setminus V$. For $t > 1$, put $\rho_t := 1 + t\rho$. Take any Dirac-type operator D on W with translationally invariance on the end, and write $D = \widetilde{D}_U + D_{\mathrm{fiber}}$ for some operator $\widetilde{D}_U$ on the end. Put $D_t = \widetilde{D}_U + \rho_t D_{\mathrm{fiber}}$ on the end and $D_t = D$ on $M \setminus V$. Then for any large $t > 0$, $\mathrm{Ker}\, D_t \cap \{L^2\text{-sections}\}$ is finite dimensional. Moreover its super-dimension is independent of large t and any other continuous deformations of data.*

PROOF. The restriction of D_t to $N \times [1, \infty)$ is of the form $\alpha(\partial_r + D_{N,t})$ where α is the Clifford multiplication of ∂_r and $D_{N,t}$ is a formally self-adjoint operator on N depending on the parameter t. We show that $\mathrm{Ker}\, D_{N,t} = 0$ for large t. It is technically convenient to introduce a Dirac-type operator $D_{N \times S^1, t}$ on $N \times S^1$ as follows: $D_{N \times S^1, t}$ is written as the same expression $\alpha(\partial_r + D_{N,t})$ where we use the identification $S^1 = \mathbb{R}/\mathbb{Z}$. Since $N \times S^1$ is a closed manifold, we can apply Lemma 5.2 to obtain $\mathrm{Ker}\, D_{N \times S^1, t} = 0$ for large t, which implies our claim $\mathrm{Ker}\, D_{N,t} = 0$ for large t.

When $D_{N,t}$ does not have zero as an eigenvalue, any L^2-solution f for the equation $D_t f = 0$ on M is exponentially decreasing on the end. Then it is well-known that the space of L^2-solutions is finite dimensional, and its super-dimension is deformation invariant as far as $\mathrm{Ker}\, D_{N,t} = 0$. $\square$

The super-dimension of the space of L^2-solutions is equal to the index of $D_t|_{M \setminus V}$ for the Atiyah-Patodi-Singer boundary condition. We use this index as the definition of our local index for the case of cylindrical end.

DEFINITION 5.5. Under the assumption of Lemma 5.4, $\mathrm{Ind}\,(M, V, W)$ is defined to be the super-dimension of $\mathrm{Ker}\, D_t \cap \{L^2\text{-sections}\}$.

The following sum formula follows from a standard argument.

LEMMA 5.6. *Suppose $(M, V = N \times (0, \infty), W)$ and $(M', V' = N' \times (0, \infty), W')$ satisfy the assumption of Lemma 5.4. Let N_0 and N'_0 be a* *connected component of N and N' respectively. Suppose N_0 is isometric to N_0' via $\phi : N_0 \to N_0'$, and for some $R > 0$ the map $\phi : N_0 \times (0, R) \to N_0' \times (0, R)$ given by $(x, r) \mapsto (\phi(x), R - r)$ can be lifted to the isomorphism between the acyclic generalized real polarizations on them. Then we can glue $M \smallsetminus (N_0 \times [R, \infty))$ and $M' \smallsetminus (N_0' \times [R, \infty))$ to obtain a new manifold $\hat{M}$ with cylindrical end $\hat{V} = \hat{N} \times (0, \infty)$ for $\hat{N} = (N \smallsetminus N_0) \cup (N' \smallsetminus N_0')$, and we also have a Clifford module bundle $\hat{W}$ obtained by gluing W and W' on $N_0 \times (0, R) \cong N_0' \times (0, R)$. Then we have*

$$\operatorname{Ind}(\hat{M}, \hat{V}, \hat{W}) = \operatorname{Ind}(M, V, W) + \operatorname{Ind}(M', V', W').$$

PROOF. A proof is given by the APS formula of the indices. An alternative direct argument is to apply the excision property [6, Theorem 5.40]. □

5.3. General case

Now we would like to define the local index for general case.

Let V be an open subset of M with an acyclic generalized real polarization $(U, W_U, \pi, D_{\text{fiber}})$ such that $M \smallsetminus V$ is compact. We can take a codimension-one closed submanifold N_U of U such that $N = \pi^{-1}(N_U)$ divides M into compact part and non-compact part: For instance, let $f : M \to [0, \infty)$ be the distance from $M \smallsetminus V$ and define $g : U \to [0, \infty)$ to be the maximal value of f on the fiber of π. Take a small real number $r > 0$ such that $f^{-1}([0, r])$ is a compact subset of M. Let $h : U \to [0, \infty)$ be a smooth functions on U satisfying $|h(x) - g(x)| < r/2$ for $x \in U$. Take a regular value r_0 of h satisfying $0 < r_0 < r/2$. Then $N_U = h^{-1}(r_0)$ satisfies the required property.

Let K be the compact part of $M \smallsetminus N$. Note that a neighborhood V_N of N in V is diffeomorphic to $N \times (-\epsilon, \epsilon)$. Then we can construct the Riemannian metric and the translationally invariant acyclic generalized real polarization on (M', V'), where $M' := K \cup (N \times [0, \infty))$ and $V' := (K \cap V) \cup (N \times [0, \infty))$. For instance, let $\phi : M' \to M$ be a smooth map which is the identity on the complement of $N \times (-\epsilon, \infty)$ and is given by $(x, r) \mapsto (x, \beta(r))$ on $N \times (-\epsilon, \infty)$ for a smooth function β satisfying $\beta(r) = r$ for $-\epsilon < r < -(2/3)\epsilon$, and $\beta(r) = 0$ for $r \geq (1/3)\epsilon$. Define a bundle endmorphism $\widetilde{\phi} : TM' \to TM$ covering ϕ as follows. On the complement of $N \times (-\epsilon, \infty)$, $\widetilde{\phi}$ is the identity. On $T(N \times (-\epsilon, \infty)) = TN \times T(-\epsilon, \infty)$, define $\widetilde{\phi}$ by

$((x,r),(u,v)) \mapsto (\phi(x,r),(u,v))$, where $x \in N$, $r \in (-\epsilon,\infty)$, $u \in T_xN$ and $v \in T_r(-\epsilon,\infty) = \mathbb{R}$. The required deformed Riemannian metric is defined to be the pullback of the original Riemannian metric by $\widetilde{\phi}$ as a section of the symmetric tensor product of T^*M. The required deformed Clifford module bundle W' is defined to be the pullback ϕ^*W. Note that we can also construct a Riemannian manifold U' with a cylindrical end $N_U \times (0,\infty)$ and a fiber bundle $\pi' : V' \to U'$. We define a Dirac-type operator D'_{fiber} acting on $\Gamma(W'|_{V'})$ by $\phi^* D_{\mathrm{fiber}}$. Then $(U', \pi', D'_{\mathrm{fiber}})$ is a translationally invariant acyclic generalized real polarization on (M', V', W'). For this structure the local index is defined by Definition 5.5.

DEFINITION 5.7. We define $\mathrm{Ind}\,(M,V,W)$ to be the local index for (M',V',W') with the translationally invariant acyclic generalized real polarization $(U', \pi', D'_{\mathrm{fiber}})$.

We have to show the local index is well-defined for the various choice of our construction.

LEMMA 5.8. *Suppose we take two codimension-one closed submanifolds N_U and N'_U in U so that M is divided in two ways. Then the local indices defined by these data coincide.*

PROOF. Let K and K' be the compact parts of M divided by $N = \pi^{-1}(N_U)$ and $N' = \pi^{-1}(N'_U)$ respectively. Then we can take another N''_U so that the corresponding compact part K'' is contained in the intersection of the interiors of K and K'. It suffices to show that the local indices coincide for K and K''. Deform the structures on neighborhoods of K and K'' simultaneously to make the structures translationally invariant near K and K'' respectively. Let M_0, M'_0 and $\hat{M}_0$ be the following manifolds with cylindrical ends.

$$\begin{aligned} M_0 &= K'' \cup (N'' \times [0,\infty)) \\ M'_0 &= (N'' \times (-\infty, 0]) \cup (K \smallsetminus K'') \cup (N \times [0,\infty)) \\ \hat{M}_0 &= K \cup (N \times [0,\infty)) \end{aligned}$$

On the cylindrical ends we have translationally invariant Clifford modules W_0, W'_0 and $\hat{W}_0$, and translationally invariant acyclic generalized real polarizations. On M'_0 the acyclic generalized real polarization is given globally.

The sum formula of Lemma 5.6 implies $\mathrm{Ind}\,(M_0, W_0) + \mathrm{Ind}\,(M_0', W_0') = \mathrm{Ind}\,(\hat{M}_0, \hat{W}_0)$. The vanishing of Lemma 5.3 implies $\mathrm{Ind}\,(M_0', W_0') = 0$. Therefore we have $\mathrm{Ind}\,(M_0, W_0) = \mathrm{Ind}\,(\hat{M}_0, \hat{W}_0)$. This is the required equality. □

5.4. Excision

The well-definedness shown in Lemma 5.8 is the key point for the following formulation of excision property.

THEOREM 5.9 (Excision property). *Let W be a $\mathbb{Z}_2$-graded Clifford module bundle over $Cl(TM)$. Let V be an open subset of M with an acyclic generalized real polarization $(U, W_U, \pi, D_{\mathrm{fiber}})$ such that $M \smallsetminus V$ is compact. Suppose U' is an open subset of U such that $M' := V' \cup (M \smallsetminus V)$ is an open neighborhood of $M \smallsetminus V$, where we put $V' := \pi^{-1}(U')$. Note that V' has the restricted acyclic generalized real polarization. Then we have*

$$\mathrm{Ind}\,(M, V, W) = \mathrm{Ind}\,(M', V', W|_{M'}).$$

PROOF. Take a codimension-one submanifold $N_{U'}$ in U' to define $\mathrm{Ind}\,(M', V', W|_{M'})$. Then $N_{U'}$ can be used to define $\mathrm{Ind}\,(M, V, W)$. □

PROOF OF THEOREM 4.5. We first note that when M is a closed manifold the local index $\mathrm{Ind}\,(M, V, W)$ defined by Definition 5.7 is equal to the usual index $\mathrm{Ind}\,(M, W)$ of a Dirac-type operator. Under the assumption of Theorem 4.5 from the excision property we have $\mathrm{Ind}\,(M, W) = \mathrm{Ind}\,(\cup_{i=1}^m V_i, W|_{\cup_{i=1}^m V_i})$, which implies the theorem. □

5.5. Proof of vanishing lemmas

In this subsection we show the vanishing lemmas Lemma 5.2 and Lemma 5.3. Suppose V is an open subset of M with an acyclic generalized real polarization $(U, \pi, D_{\mathrm{fiber}})$. Take any order-one formally self-adjoint differential operator $\widetilde{D}$ over $W|_V$ with degree one whose principal symbol is given by the composition of the projection $\pi_* : TV \to TU$ and the Clifford action of TU on $W|_V$. Then $\widetilde{D} + D_{\mathrm{fiber}}$ is a Dirac-type operator on $W|_V$.

LEMMA 5.10. *The anticommutator $D_{\mathrm{fiber}}\widetilde{D} + \widetilde{D}D_{\mathrm{fiber}}$ is an order-one differential operator on $W|_V$ which contains only the derivatives along fibers,* *i.e, it commutes with the multiplication of the pullback of smooth functions on U.*

PROOF. Recall that, the principal symbol of $\widetilde{D}$ anti-commutes not only with the symbol of D_{fiber}, but also with the whole operator D_{fiber}. The claim follows from this property. It is straightforward to check it using local description. Instead of giving the detail of the local calculation, however, we here give an alternative formal explanation for the above lemma.

For $x \in U$ let $\mathcal{W}_x$ be the space of sections of the restriction of W to the fiber $\pi^{-1}(x)$. Then $\mathcal{W} = \coprod \mathcal{W}_x$ is formally an infinite dimensional vector bundle over U. We can regard D_{fiber} as an endmorphism on $\mathcal{W}$. Then D_{fiber} is a order-zero differential operator on $\mathcal{W}$ whose principal symbol is equal to D_{fiber} itself. Then, as a differential operator on $\mathcal{W}$, $D_{\mathrm{fiber}}\widetilde{D} + \widetilde{D}D_{\mathrm{fiber}}$ is an (at most) order-one operator whose principal symbol is given by the anticommutator between the Clifford action by TU and D_{fiber}. This principal symbol vanishes, which implies that the anticommutator is order-zero as a differential operator on $\mathcal{W}$, i.e., it does not contain derivatives of U-direction. $\square$

PROOF OF LEMMA 5.2. Let f be a section of W. On each fiber of π at $x \in U$, the second order elliptic operator D_{fiber}^2 is strictly positive. Since $D_{\mathrm{fiber}}\widetilde{D} + \widetilde{D}D_{\mathrm{fiber}}$ gives a first order operator on the fiber, a priori estimate implies the estimate

$$\left| \int_{\pi^{-1}(x)} ((D_{\mathrm{fiber}}\widetilde{D} + \widetilde{D}D_{\mathrm{fiber}})f, f) \right| \leq C \int_{\pi^{-1}(x)} (D_{\mathrm{fiber}}^2 f, f)$$

for some positive constant C. Since M is compact we can take C uniformly. Therefore we have

$$\begin{aligned} \int_M ((\widetilde{D} + tD_{\mathrm{fiber}})^2 f, f) &= \int (\widetilde{D}^2 f, f) + t^2 \int (D_{\mathrm{fiber}}^2 f, f) \\ &\quad + t \int ((D_{\mathrm{fiber}}\widetilde{D} + \widetilde{D}D_{\mathrm{fiber}})f, f) \\ &\geq \int (\widetilde{D}^2 f, f) + (t^2 - Ct) \int (D_{\mathrm{fiber}}^2 f, f) \\ &= \int_M |\widetilde{D}f|^2 + (t^2 - Ct) \int_M |D_{\mathrm{fiber}} f|^2. \end{aligned}$$

In particular if $t > C$ and $(\widetilde{D} + tD_{\text{fiber}})f = 0$, then $D_{\text{fiber}}f$ is zero, which implies f itself is zero. □

Proof of Lemma 5.3. The proof is almost identical to the above one for the compact case. What we still need is to guarantee the validity of partial integration. This validity follows from the fact that when $(\widetilde{D} + tD_{\text{fiber}})f = 0$ and f is in L^2, then f and any derivative of f decay exponentially on the cylindrical end of V. □

Note that the exponential decay in the above proof relies on the vanishing lemma for compact case, so we had to separate the proof.

6. 2-Dimensional Case

Let Σ be a compact oriented surface with non-empty boundary $\partial\Sigma$. Let L be a complex line bundle over Σ. Suppose a $U(1)$-connection is given on the restriction of L to $\partial\Sigma$. When the connection is non-trivial on every boundary component, we can define the local Riemann-Roch number for the data as follows. Fix a product structure $(-\epsilon, 0] \times \partial\Sigma$ on an open neighborhood of the boundary. Then on the collar neighborhood of each connected component of $\partial\Sigma$ the projection onto $(-\epsilon, 0]$ is a circle bundle. Extend the connection smoothly on the neighborhood of the boundary so that we have a flat non-trivial connection on each fiber of the circle bundle. Let $V = V_1$ be the interior of Σ, and V_∞ be the intersection of the open collar neighborhood and V. Then we have the local Riemann-Roch number $RR(V, L)$ (see Theorem 3.3). The deformation invariance of the local Riemann-Roch number implies that it depends only on the initially given data. We often write

$$[\Sigma] = RR(V, L).$$

In this section we calculate $[\Sigma]$ explicitly for several examples (Theorem 6.7). We also show that the local Riemann-Roch number for a non-singular Bohr-Sommerfeld fiber in symplectic case is equal to one (Theorem 6.11). Let us first recall our convention of orientation for boundary. We use the convention for which Stokes theorem holds with positive sign. In other words: Suppose $\hat{X}$ is an oriented manifold, and f is a smooth real function on $\hat{X}$ with 0 a regular value. The orientations of $X = f^{-1}((-\infty, 0])$ and $Y = f^{-1}(0)$

are related to each other as follows. If ω_X and ω_Y are non-vanishing top-degree forms on X and Y compatible with their orientations, then we have $\omega_X|_Y = \rho\, df \wedge \omega_Y$ for some positive smooth function ρ on Y.

6.1. Type of singularities

6.1.1 BS type singularities

For a positive number ϵ let A_ϵ be the annulus $[-\epsilon, \epsilon] \times S^1$ with the orientation given by $dx \wedge d\theta/2\pi$,where x is the coordinate of $[-\epsilon, \epsilon]$. The projection map $(x, \theta) \mapsto x$ gives a circle bundle structure of A_ϵ. Let L be a complex line bundle over A_ϵ, and ∇ a $U(1)$-connection on L. Let $e^{\sqrt{-1}h(x)}$ be the holonomy along the circle of $\{x\} \times S^1$ for the orientation given by $d\theta$. Suppose $h(x)$ is continuous and $h(0) = 0$.

DEFINITION 6.1 (positive/negative BS). When $h(x) > 0$ for $x > 0$ and $h(x) < 0$ for $x < 0$, we call the fiber at 0 a *positive BS type*. When $h(x) < 0$ for $x > 0$ and $h(x) > 0$ for $x < 0$, we call the fiber at 0 a *negative BS type*. See Figure 1.

6.1.2 Disk type singularities

Let D^2 be an oriented disk. Choose a polar coordinate r and θ so that D^2 becomes a unit disk and the orientation of D^2 is compatible with $dr \wedge d\theta$ outside the origin. In particular the orientation of the boundary ∂D^2 is compatible with $d\theta$. The projection map $(r, \theta) \mapsto r$ gives a circle bundle structure on neighborhood of the boundary of D^2.

Let L be a complex line bundle over D^2, and ∇ a $U(1)$-connection on L. Let $e^{\sqrt{-1}h(r)}$ be the holonomy along the circle of radius r centered in the origin. The orientation of the circle is defined as the boundary of the disk of radius r centered in the origin. We can take $h(r)$ continuous with limit value $\lim_{r\to 0} h(r) = 0$.

DEFINITION 6.2 (positive/negative disk). When $h(r) > 0$ for small $r > 0$ we call the singularity positive-disk type. When $h(r) < 0$, then we call it negative-disk type. See Figure 2.

6.1.3 Pants type singularities

Let Σ be a genus 0 oriented surface with three holes, i.e., Σ is a pair of pants. For each boundary component its collar is diffeomorphic to the

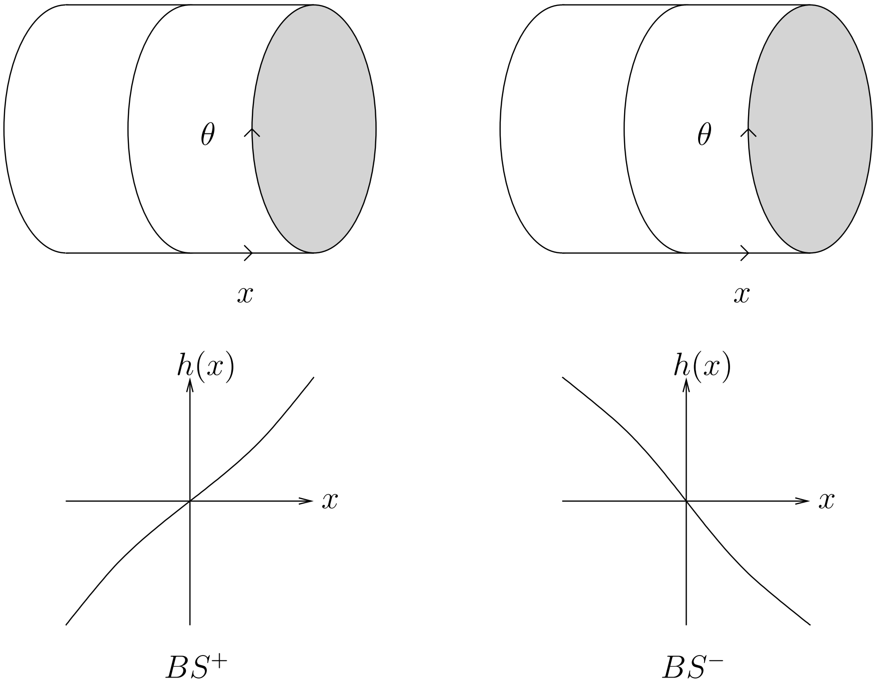

Fig. 1. positive/negative BS

product of a circle and an interval. The projection onto the interval gives a circle bundle structure near the boundary.

Let L be a complex line bundle over Σ, and ∇ a $U(1)$-connection on L. Suppose ∇ is flat and its holonomy along each component of the boundary of Σ is non-trivial. Let $e^{\sqrt{-1}h_k}$ $(k = 1, 2, 3)$ be the three holonomies along the three components of the boundary of Σ. The orientations of the boundary components are defined as the boundary of the oriented manifold Σ. Then the product of these three holonomies is equal to 1. We can take h_k satisfying $0 < h_k < 2\pi$.

From our assumption, there is no Bohr-Sommerfeld fiber in the collar neighborhood of the boundary.

Definition 6.3 (small/large pants). When $h_1 + h_2 + h_3 = 2\pi$, we call

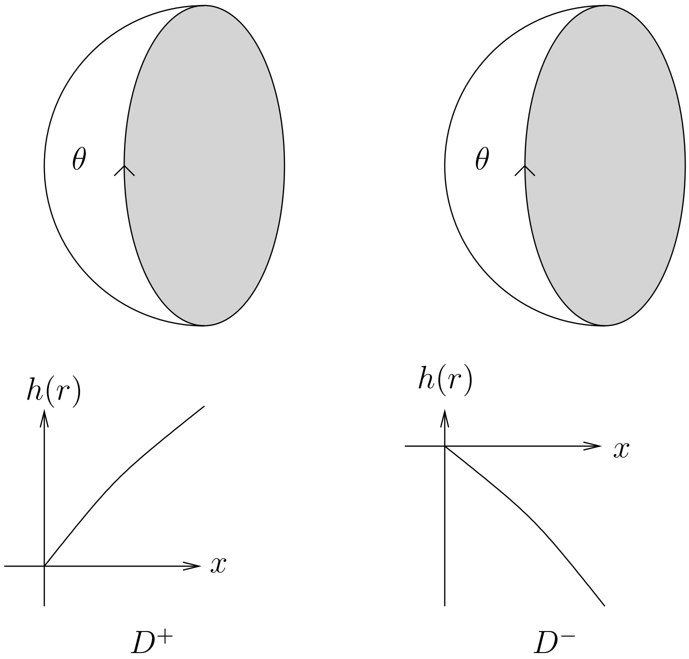

Fig. 2. positive/negative disk

Σ a small pants. When $h_1 + h_2 + h_3 = 4\pi$, we call Σ a large pants.

6.2. Examples

Example 6.4 (a torus over a circle with degree n line bundle). Let x and y the coordinate of $\mathbb{R}^2$. Let $\hat{\nabla}$ be the $U(1)$-connection on the trivial complex line bundle over $\mathbb{R}^2$ with connection form $-\sqrt{-1}xdy$.

1. The curvature $F_{\hat{\nabla}}$ of $\hat{\nabla}$ is $-\sqrt{-1}dx \wedge dy$. In particular

$$\frac{\sqrt{-1}}{2\pi}F_{\hat{\nabla}} = \frac{1}{2\pi}dx \wedge dy$$

2. The holonomy along the straight line from the point $(x, 0)$ to (x, y) is equal to $\exp(\sqrt{-1}xy)$.

3. The connection is invariant under the action of $(m, 2\pi n) \in \mathbb{Z} \oplus 2\pi\mathbb{Z}$ given by
$$(x, y, u) \mapsto (x + m, y + 2\pi n, e^{\sqrt{-1}my}u).$$

For a positive integer N let T^2_N be the quotient of $\mathbb{R}^2$ divided by the subgroup $N\mathbb{Z} \oplus 2\pi\mathbb{Z}$, on which we have the quotient complex line bundle L_N with the quotient connection ∇_N. Then $(\sqrt{-1}/2\pi)F_{\nabla_N}$ gives a symplectic structure on T^2_N. The projection onto the first factor $T^2_N \to \mathbb{R}/N\mathbb{Z}$ is a Lagrangian fibration. The holonomy along the fiber at x mod N is $e^{2\pi\sqrt{-1}x}$. The Bohr-Sommerfeld fibers are the fibers at $x \in \mathbb{Z}/N\mathbb{Z}$, and all of them are positive BS singularities. The degree of L_N is equal to N and the Riemann-Roch number is N from the Riemann-Roch theorem and it is equal to the number of positive BS singularities.

Example 6.5 (a sphere with degree-zero line bundle). Let D^+ and D^- be disks, and L^+ and L^- complex line bundles over D^+ and D^- with $U(1)$-connections such that the connections are flat near the boundaries, respectively. Suppose D^+ and D^- are a positive disk and a negative disk respectively, and the holonomies along the boundaries are $e^{\sqrt{-1}\epsilon}$ and $e^{-\sqrt{-1}\epsilon}$ for a small positive number ϵ. Patch (D^+, L^+) and (D^-, L^-) together to obtain a complex line bundle over an oriented sphere with a $U(1)$-connection. Since the degree of the $U(1)$-bundle is zero, the Riemann-Roch number is 1.

Example 6.6 (a surface with a pants decomposition with a flat line bundle). Let ϵ be a small positive number. Let P^S be a pair of pants with a flat $U(1)$-bundle whose holonomies along the boundary components are $e^{\sqrt{-1}(\pi-\epsilon)}$, $e^{\sqrt{-1}(\pi-\epsilon)}$, and $e^{\sqrt{-1}2\epsilon}$. Let P^L be a pair of pants with a flat $U(1)$-connection whose holonomies along the boundary components are $e^{\sqrt{-1}(\pi+\epsilon)}$, $e^{\sqrt{-1}(\pi+\epsilon)}$, and $e^{-\sqrt{-1}2\epsilon}$. Then P^S is a small pants and P^L is a large pants. For an integer $g \geq 2$, take $(g-1)$ copies of P^S and $(g-1)$-copies of P^L, and patch them together to obtain a flat connection on a closed oriented surface with genus g. The Riemann-Roch number is $1 - g$.

6.3. Local Riemann-Roch numbers

Let $[BS^+]$ and $[BS^-]$ be the contribution of a positive and negative BS respectively. Let $[D^+]$ and $[D^-]$be the contribution of a positive and negative disk respectively. Let $[P^S]$ and $[P^L]$ be the contribution of a small and large pants respectively.

THEOREM 6.7.

$$[BS^+] = 1, \quad [BS^-] = -1, \quad [D^+] = 1, \quad [D^-] = 0,$$
$$[P^S] = 0, \quad [P^L] = -1$$

PROOF. These are consequences of Lemma 6.8 and Lemma 6.9 below. □

LEMMA 6.8. $[P^L] + [BS^+] = [P^S]$, $[BS^-] + [BS^+] = 0$, $[D^-] + [BS^+] = [D^+]$.

PROOF. The three relations are shown in a similar way. We just show the first relation. Let ϵ be a small positive number. Let P^S and P^L be the small pants and the large pants in Example 6.6. Let A be an oriented annulus with $U(1)$-connection such that the connection is flat near the boundary. Suppose A is positive-BS type and both the holonomies of the two boundary components are $e^{\sqrt{-1}2\epsilon}$ for the orientation as boundary of A. Patch P^L and A together along the boundary components with holonomies $e^{-\sqrt{-1}2\epsilon}$ and $e^{\sqrt{-1}2\epsilon}$ to obtain an another pair of pants with a $U(1)$-connection. The glued $U(1)$-connection can be deformed to a flat $U(1)$-connection isomorphic to the one on P^S without changing the connection near boundary components. □

LEMMA 6.9. $[BS^+] = 1$, $[P^S] + [P^L] = -1$, $[D^+] + [D^-] = 1$.

PROOF. The three relations are consequences of Example 6.4, Example 6.5, and Example 6.6 respectively. □

REMARK 6.10. It is possible to show $[BS^+] = 1$ directly without appealing the Riemann-Roch theorem in the following way. We put $M := \mathbb{R} \times S^1$ and consider the Hermitian structure (g, J) on it, which is defined by

$$g(a_1\partial_x + b_1\partial_\theta, a_2\partial_x + b_2\partial_\theta) = a_1a_2 + b_1b_2,$$
$$J(\partial_x) = \partial_\theta, J(\partial_\theta) = -\partial_x.$$

Let $T_{\text{fiber}}M$ be the tangent bundle along fibers of the first projection. Then, as complex vector bundles, $T_{\text{fiber}}M \otimes_{\mathbb{R}} \mathbb{C}$ is identified with (TM, J)

by

$$(1) \qquad T_{\text{fiber}}M \otimes_{\mathbb{R}} \mathbb{C} \to (TM, J),\ \partial_\theta \otimes_{\mathbb{R}} (x + \sqrt{-1}y) \mapsto\ x\partial_\theta + yJ\partial_\theta.$$

Let $L = M \times \mathbb{C}$ be the trivial complex line bundle on M. For $0 < \epsilon < 1$ let $\rho(x)$ be a smooth increasing function on $\mathbb{R}$ with $\rho(0) = 0$, $\rho(x) \equiv \epsilon$ for sufficiently large x and $\rho(x) \equiv -\epsilon$ for sufficiently small x. Consider the $U(1)$-connection on L of the form $\nabla = d - \sqrt{-1}\rho(x)d\theta$.

Let $W := \wedge^\bullet(TM, J) \otimes_{\mathbb{C}} L$ be the $\mathbb{Z}/2$-graded $Cl(TM)$-module bundle over M whose $Cl(TM)$-module structure is defined by [13, pp.38, (5.25)]. We take a Dirac-type operator D acting on $\Gamma(W)$ to be the Dolbeault operator. Under the identification (1) we also take D_{fiber} to be the family of de Rham operators along fibers. It is easy to check that D_{fiber} satisfies the fourth property in Definition 4.3. Then the deformed operator $D_t = D + tD_{\text{fiber}}$ is written in the following way

$$\begin{aligned} D_t s =& \partial_x \otimes \big(\partial_x s_0 + \sqrt{-1}(1+t)\partial_\theta s_0 + (1+t)\rho(x)s_0\big) \\ &- \big(\partial_x s_1 - \sqrt{-1}(1+t)\partial_\theta s_1 - (1+t)\rho(x)s_1\big) \end{aligned}$$

for $s = s_0 + \partial_x \otimes s_1 \in \Gamma(W)$, where $s_0 \in \Gamma(L)$ and $\partial_x \otimes s_1 \in \Gamma((TM, J) \otimes_{\mathbb{C}} L)$ are even and odd parts of s, respectively.

For an L^2-section s_0 of L, we first solve the equation

$$(2) \qquad 0 = \partial_x s_0 + \sqrt{-1}(1+t)\partial_\theta s_0 + (1+t)\rho(x)s_0.$$

By taking the Fourier expansion of s_0 with respect to θ, s_0 is written as

$$s_0 = \sum_{n \in \mathbb{Z}} a_n(x) e^{\sqrt{-1}n\theta}.$$

Then, s_0 satisfies (2) if and only if each a_n is of the form

$$a_n(x) = c_n \exp\left((1+t)\int_0^x n - \rho(x)dx\right)$$

for some constant c_n. Since $\rho(x) \equiv \pm\epsilon$ for sufficiently large, or small x and since s is a L^2-section, it is easy to see that $c_n = 0$ except for $n = 0$. This implies that the kernel of the even part of D_t is one-dimensional.

Next we solve the equation

$$0 = \partial_x s_1 - \sqrt{-1}(1+t)\partial_\theta s_1 - (1+t)\rho(x)s_1.$$

By the similar argument we can show that $c_n = 0$ for all $n \in \mathbb{Z}$. This implies that the kernel of the odd part of D_t is zero-dimensional. Thus, we have $[BS^+] = 1$.

By similar arguments we can also show that $[BS^-] = -1$, $[D^+] = 1$, and $[D^-] = 0$.

6.4. Higher dimensional Bohr-Sommerfeld fibers

We show the following.

THEOREM 6.11. *In symplectic formulation, the local Riemann-Roch number of a non-singular connected Bohr-Sommerfeld fiber is one.*

PROOF. It is known that the neighborhoods of two Bohr-Sommerfeld fibers are isomorphic to each other together with prequantizing line bundle with connection: Recall that the fibers in a neighborhood of a Bohr-Sommerfeld fiber are parameterized by their periods. If we fix a local Lagrangian section, and a trivialization of the first homology group of the local fibers, then we can write down a canonical coordinate. Therefore it suffices to give one example for which the claim is satisfied. An example of a Lagrangian fibration with exactly one n-dimensional Bohr-Sommerfeld fiber is given by the product of n-copies of the fiber bundle structure of the torus T^2_N for $N = 1$ in Example 6.4. In this case our convention of the orientation for the symplectic manifold coincides with the product orientation. The Riemann-Roch number is equal to one because it is equal to the n-th power of $RR(T^2_1) = 1$. □

As a corollary we have the following, which is already shown by J. E. Andersen by using index theorem.

COROLLARY 6.12 ([1]). *For a Lagrangian fibration without singular fibers over a closed symplectic manifold with a prequantizing line bundle, the number of Bohr-Sommerfeld fibers is equal to the Riemann-Roch number.*

REMARK 6.13. It would be expected that, if we use appropriate boundary condition, then it would be possible to define a local Riemann-Roch number for the product $D^+ \times P^L$, and moreover it would be equal to the product $[D^+][P^L]$, i.e., -1. A crucial problem here is that there is no Lagrangian fibration structure on the whole neighborhood of the boundary of $D^+ \times P^L$. In fact it is possible to extend our formulation to such cases. We will discuss this elsewhere [4].

7. Comments

It is possible to extend our construction for various situations.

1. Isotropic fibrations: When we have a integrable system which is not necessary completely integrable, if all the orbits are "periodic" and form tori, we can extend our argument. It would be an interesting problem to investigate the case when the orbits are not periodic.

2. Manifolds with boundaries and corners: Our definition of the local index is related to manifolds with boundaries. For manifolds with coners, it is possible to extend our construction.

3. Equivariant and family version: Our constructin is natural, so if a compact Lie group acts and preserves the data, then everything is formulated equivariantly. Similarly we have a family version of our construction.

4. Equivariant mod-2 indices: A modification of the localization property explained in this paper can be applied to define *G-equivariant mod-2 indices* valued in $R(G)/RO(G)$ or $R(G)/RSp(G)$ for even dimensional G-spin^c-manifolds with G-spin structures on its end [8].

We will discuss them elsewhere [4, 5].

A. Spin^c-Structures

In this appendix we recall our convention for spin^c-strctures on oriented manifolds. (See [6, pp.54] and [13, pp.131-132].) A spin^c structure is usually defined for an oriented Riemannian manifold. In this paper we take a convention of spin^c structures which do not need any Riemannian metrics. In fact a spin^c structure itself is defined at the principal bundle level as follows. Let $GL_m^+(\mathbb{R})$ be the group of orientation preserving linear automorphisms of $\mathbb{R}^m$. Since $GL_m^+(\mathbb{R})$ has the same homotopy type as that of $SO_m(\mathbb{R})$, there is a unique non-trivial double covering of $GL_m^+(\mathbb{R})$ when $m > 1$. We denote it by $p : \widetilde{GL_m^+}(\mathbb{R}) \to GL_m^+(\mathbb{R})$. When $m = 1$, we define $\widetilde{GL_1^+}(\mathbb{R}) := GL_1^+(\mathbb{R}) \times (\mathbb{Z}/2\mathbb{Z})$ and p to be the projection onto the first factor.

Consider the diagonal action of $\mathbb{Z}/2\mathbb{Z}$ on $\widetilde{GL_m^+}(\mathbb{R}) \times \mathbb{C}^\times$, where the action on the first factor is the deck transformation and the action on the second factor is defined by $z \mapsto -z$. Let $\widetilde{GL_m^+}(\mathbb{R}) \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{C}^\times$ be the quotient group by this diagonal action. Note that there is a canonical homomorphism $\hat{p} : \widetilde{GL_m^+}(\mathbb{R}) \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{C}^\times \to GL_m^+(\mathbb{R})$ defined by

$$\hat{p} : [\widetilde{g}, z] \mapsto p(\widetilde{g}).$$

DEFINITION A.1. Let M be an oriented manifold and P_M the associated frame bundle over M, which is a principal $GL_m^+(\mathbb{R})$-bundle. A *spin^c-structure* on M is a pair $(\widetilde{P}_M, q_M)$ satisfying the following two conditions.

1. $\widetilde{P}_M$ is a principal $\widetilde{GL_m^+}(\mathbb{R}) \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{C}^\times$-bundle over M.

2. q_M is a bundle map from $\widetilde{P}_M$ to P_M which is equivariant with respect to the canonical homomorphism $\hat{p}$.

REMARK A.2. Though a spin^c structure can be defined without any Riemannian metric, we need to fix a Riemannian metric to define a Clifford module bundle over an oriented manifold.

REMARK A.3. The natural embedding $GL_n(\mathbb{C}) \hookrightarrow GL_{2n}^+(\mathbb{R})$ induces a homomorphism $GL_n(\mathbb{C}) \to \widetilde{GL_{2n}^+}(\mathbb{R}) \times_{\mathbb{Z}/2} \mathbb{C}^\times$. This means that an almost complex manifold has a canonical spin^c structure in our convention.

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