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The Growth of the Nevanlinna Proximity Function

By Atsushi Nitanda

Abstract. Let f be a meromorphic mapping from $\mathbf{C}^n$ into a compact complex manifold M. In this paper we give some estimates of the growth of the proximity function $m_f(r, D)$ of f with respect to a divisor D. J.E. Littlewood [2] (cf. Hayman [1]) proved that every non-constant meromorphic function g on the complex plane $\mathbf{C}$ satisfies $\limsup_{r\to\infty} \frac{m_g(r,a)}{\log T(r,g)} \leq \frac{1}{2}$ for almost all point a of the Riemann sphere. We extend this result to the case of a meromorphic mapping $f : \mathbf{C}^n \to M$ and a linear system $P(E)$ on M. The main result is an estimate of the following type: For almost all divisor $D \in P(E)$, $\limsup_{r\to\infty} \frac{m_f(r,D) - m_f(r,\mathcal{I}_{B(E)})}{\log T_{f_E}(r,H_E)} \leq \frac{1}{2}$.

1. Introduction

J.E. Littlewood [2] (cf. [1]) proved that every non-constant meromorphic function g on $\mathbf{C}$ satisfies

$$\limsup_{r\to\infty} \frac{m_g(r,a)}{\log T(r,g)} \leq \frac{1}{2}$$

for almost all $a \in \mathbf{C}$, where $T(r, g)$ denotes the Nevanlinna characteristic function of g. Our main aim is to generalize this result to the case of several complex variables. Cf. A. Sadullaev [8], A. Sadullaev and P.V. Degtjar' [9], and S. Mori [2] for related results (see *Remark* at the end of §6).

Let $L \to M$ be a holomorphic line bundle over a compact complex manifold M. Let $\Gamma(M, L)$ be the vector space of all holomorphic sections of L over M, and $E \subset \Gamma(M, L)$ a vector subspace of dimension at least 2. Then we have a natural meromorphic mapping

$$\rho_E : M \to P(E^*),$$

where $P(E^*)$ is the projective space of the dual E^* of E. Let H_E be the hyperplane bundle over $P(E^*)$ and $B(E) \subset M$ the base of E. Let $f : \mathbf{C}^n \to$

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M be a meromorphic mapping such that $f(\mathbf{C}^n) \not\subset B(E)$. Then we have the composite meromorphic mapping $f_E = \rho_E \circ f : \mathbf{C}^n \to P(E^*)$.

Our main result is as follows (cf. section 2 for more notation):

MAIN THEOREM. *Let $f_E = \rho_E \circ f : \mathbf{C}^n \to P(E^*)$ be as above. If $T_{f_E}(r, H_E) \to \infty \quad (r \to \infty)$, then*

$$\limsup_{r\to\infty} \frac{m_f(r, D) - m_f(r, \mathcal{I}_{B(E)})}{\log T_{f_E}(r, H_E)} \leq \frac{1}{2}$$

for almost all divisor $D \in P(E)$.

In section 4 we first prove the Main Theorem in the case where $E = \Gamma(M, L)$ and $B(E) = \phi$. In section 5 we show an estimate of different type. In section 6 we deal with the general case.

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2. Notation

Let $z = (z^1, \ldots, z^n)$ be the natural coordinate system of $\mathbf{C}^n$. We set

$$\|z\|^2 = \sum_{j=1}^{n} |z^j|^2, \quad d^c = \frac{i}{4\pi}\left(\overline{\partial} - \partial\right),$$

$$\alpha = dd^c\|z\|^2, \quad \eta = d^c \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{n-1},$$

$$B(r) = \{z \in \mathbf{C}^n;\ \|z\| < r\}, \quad \Gamma(r) = \{z \in \mathbf{C}^n;\ \|z\| = r\}.$$

Let M be a compact complex manifold and (L, h) a Hermitian holomorphic line bundle over M. For a meromorphic mapping $f : \mathbf{C}^n \to M$ we define the order function of f with respect to the Chern form ω of (L, h) by

$$T_f(r, \omega) = \int_1^r \frac{dt}{t^{2n-1}} \int_{B(t)} f^*\omega \wedge \alpha^{n-1}$$

and we define the order function of f with respect to L by

$$T_f(r, L) = T_f(r, \omega).$$

$T_f(r, L)$ is well-defined up to a bounded term. We denote the space of holomorphic sections of L by $\Gamma(M, L)$. We have the natural identification

$$P(\Gamma(M, L)) = \{(\sigma);\ \sigma \in \Gamma(M, L) \setminus \{0\}\},$$

where the notation (σ) stands for the effective divisor of σ. Let $D \in P(\Gamma(M, L))$. Then we may take an element $\sigma \in \Gamma(M, L)$ which satisfies

$$D = (\sigma), \quad \|\sigma(x)\| = \sqrt{h(\sigma(x), \sigma(x))} \le 1.$$

When $f(\mathbf{C}^n) \not\subset \operatorname{supp} D$ (the support of D), the proximity function of f with respect to D is defined by

$$m_f(r, D) = \int_{z \in \Gamma(r)} \log \frac{1}{\|\sigma \circ f(z)\|} \eta(z)$$

and we define the counting function of f^*D by

$$N(r, f^*D) = \int_1^r \frac{dt}{t^{2n-1}} \int_{B(t) \cap f^*D} \alpha^{n-1},$$

where f^*D is the pullback of D by f. If L is non-negative, then we have the First Main Theorem

$$T_f(r, L) = N(r, f^*D) + m_f(r, D) + O(1). \tag{1}$$

3. Lemma

Let M be a compact complex manifold and $L \to M$ a holomorphic line bundle. Set

$$V = \Gamma(M, L), \quad N + 1 = \dim\ V.$$

Here we assume that the set $B(V)$ of base points of V is empty, i.e.,

$$B(V) = \{x \in M;\ \sigma(x) = 0, \forall \sigma \in V\} = \phi.$$

We fix a Hermitian inner product $(\ ,\ )$ in V. Let $(\{U_\lambda\}, \{s_\lambda\})$ be a local trivialization covering of L and $\{\sigma_0, \ldots, \sigma_N\}$ an orthonormal base of V. We identify $V^* = \mathbf{C}^{N+1}$ by the dual base of $\{\sigma_0, \ldots, \sigma_N\}$. We define a holomorphic mapping Φ_L from M into $P(V^*) = \mathbf{P}^N(\mathbf{C})$ by

$$\Phi_L(x) = [\sigma_{0\lambda}(x) : \ldots : \sigma_{N\lambda}(x)],\ x \in U_\lambda,$$

where $\sigma_{j\lambda}$ are holomorphic functions on U_λ with $\sigma_j | U_\lambda = \sigma_{j\lambda} s_\lambda$. If $U_\lambda \cap U_\mu \neq \phi$, there exists a holomorphic function $T_{\lambda\mu} : U_\lambda \cap U_\mu \to \mathbf{C} \setminus \{0\}$ such that $s_\lambda(x) T_{\lambda\mu}(x) = s_\mu(x)$ for $x \in U_\lambda \cap U_\mu$. Therefore, Φ_L is well-defined. Then it follows that $L = \Phi_L^* H_{V^*}$, where H_{V^*} is the hyperplane bundle over $P(V^*)$. Hence Fubini-Study metric in H_{V^*} induces a Hermitian metric h in L satisfying

$$h(s_\lambda(x), s_\lambda(x)) = \frac{1}{\sum_{j=0}^N |\sigma_{j\lambda}(x)|^2}. \tag{2}$$

We denote the Chern form of (L, h) by ω. Clearly, ω is non-negative. Hence L is non-negative. Let ω_V denote the Fubini-Study metric form on $P(V)$ induced by the Hermitian inner product $(\ ,\ )$. Since $\omega_V^N = \wedge^N \omega_V$ is a volume element on $P(V)$, it is considered as positive measure μ. We define a C^∞-function S_x on $P(V)$ by

$$S_x(D) = \frac{\sqrt{h(\sigma(x), \sigma(x))}}{\sqrt{(\sigma, \sigma)}}, \quad D = (\sigma) \in P(V).$$

We now prove the following key lemma.

Lemma 1. *Let the notation be as above and $X \subset P(V)$ a Lebesgue measurable subset with $\mu(X) > 0$. Then,*

$$\int_{D \in X} \log \frac{1}{S_x(D)} d\mu(D) \leq \frac{\mu(X)}{2} \left(N + \log \frac{N}{\mu(X)} \right)$$

for all $x \in M$.

Proof. We identify $P(V) = \mathbf{P}^N(\mathbf{C})$ by the base $\{\sigma_0, \ldots, \sigma_N\}$. Then we equate $[z^0 : \ldots : z^N] \in \mathbf{P}^N(\mathbf{C})$ with a divisor $\left(\sum_{j=0}^N z^j \sigma_j\right)$. For $x \in U_\lambda$ and $[z^0 : \ldots : z^N] \in \mathbf{P}^N(\mathbf{C})$ it follows from (2) that

$$S_x([z^0 : \ldots : z^N]) = \frac{\left|\sum_{j=0}^N z^j \sigma_{j\lambda}(x)\right|}{\left(\sum_{j=0}^N |\sigma_{j\lambda}(x)|^2\right)^{1/2} \left(\sum_{j=0}^N |z^j|^2\right)^{1/2}}. \tag{3}$$

Since $B(V) = \phi$, there exists a unitary matrix $G = (g_{ij})$ and a non-zero constant $a \in \mathbf{C}$ such that

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = a\ {}^tG \begin{pmatrix} \sigma_{0\lambda}(x) \\ \vdots \\ \sigma_{N\lambda}(x) \end{pmatrix}.$$

Let $\rho : \mathbf{C}^{N+1} \setminus \{0\} \to \mathbf{P}^N(\mathbf{C})$ be the Hopf fibering. We define a biholomorphic mapping G by $G(\rho(z)) = \rho(Gz),\ z = {}^t(z^0, \dots, z^N) \in \mathbf{C}^{N+1}$. Since G is unitary, we easily see by (3) that

$$S_x(G([z^0 : \dots : z^N])) = \frac{|z^0|}{\left(\sum_{k=0}^N |z^k|^2\right)^{1/2}}. \tag{4}$$

We denote the characteristic function of a subset $S \subset P(V)$ by χ_S. Since ω_V is unitary invariant, it follows from (4) that

$$\begin{aligned} &\int_{\rho(w)\in X} \log \frac{1}{S_x(\rho(w))} \omega_V^N \\ &= \int_{\rho(w)\in \mathbf{P}^N(\mathbf{C})} \chi_X(\rho(w)) \log \frac{1}{S_x(\rho(w))} \omega_V^N \\ &= \int_{\rho(z)\in \mathbf{P}^N(\mathbf{C})} G^* \left(\chi_X(\rho(w)) \log \frac{1}{S_x(\rho(w))} \omega_V^N\right) \\ &= \int_{\rho(z)\in \mathbf{P}^N(\mathbf{C})} \chi_{G^{-1}(X)}(\rho(z)) \log \frac{1}{S_x(G(\rho(z)))} \omega_V^N \\ &= \int_{\rho(z)\in G^{-1}(X)} \log \frac{\left(\sum_{k=0}^N |z^k|^2\right)^{1/2}}{|z^0|} \omega_V^N. \end{aligned} \tag{5}$$

We put

$$V_0 = \{[z^0 : \dots : z^N] \in \mathbf{P}^N(\mathbf{C});\ z^0 \neq 0\}$$

and we set an affine coordinate system on V_0 by

$$\zeta = (\zeta^1, \dots, \zeta^N) = \left(\frac{z^1}{z^0}, \dots, \frac{z^N}{z^0}\right).$$

Then by (5) we have

$$\begin{aligned}
&\int_{\rho(w)\in X} \log \frac{1}{S_x(\rho(w))}\omega_V^N \\
&= \int_{\zeta\in\mathbf{C}^N} \frac{\chi_{G^{-1}(X)} N! \log\left(1+\|\zeta\|^2\right)^{1/2}}{\left(1+\|\zeta\|^2\right)^{N+1}} \bigwedge_{k=1}^{N}\left(\frac{i}{2\pi} d\zeta^k \wedge d\overline{\zeta^k}\right) \\
&= \int_{\zeta\in\mathbf{C}^N} \frac{\chi_{G^{-1}(X)} \log\left(1+\|\zeta\|^2\right)^{1/2}}{\left(1+\|\zeta\|^2\right)^{N+1}}\alpha^N.
\end{aligned}$$

Furthermore, $\mu(X) = \mu(G^{-1}(X))$, so that it suffices to prove that

$$\int_{\zeta\in\mathbf{C}^N} \frac{\chi_X \log\left(1+\|\zeta\|^2\right)^{1/2}}{\left(1+\|\zeta\|^2\right)^{N+1}}\alpha^N \le \frac{\mu(X)}{2}\left(N + \log\frac{N}{\mu(X)}\right) \tag{6}$$

for a Lebesgue measurable set $X \subset \mathbf{C}^N$. Set

$$\Phi(r) = \int_{X\cap\{\zeta\in\mathbf{C}^N;\ \|\zeta\|>r\}} \omega_V^N.$$

Then, $\Phi(r)$ is a continuous decreasing function on $[0,\infty)$ and $0 \le \Phi(r) \le \mu(X) \le 1$. Moreover,

$$\begin{aligned}
\Phi(r) &= \int_{\{\zeta\in\mathbf{C}^N;\ \|\zeta\|>r\}} \frac{\chi_X}{\left(1+\|\zeta\|^2\right)^{N+1}}\alpha^N \qquad (7)\\
&= \int_r^\infty \left\{\int_{\Gamma(t)} \frac{\chi_X 2N t^{2N-1}}{(1+t^2)^{N+1}}\eta\right\} dt,
\end{aligned}$$

so that $\Phi(r)$ is an absolutely continuous function on $[0,s]$ $(s\in[0,\infty))$.

Therefore it follows that

$$\begin{aligned}
&\int_0^s \log(1+r^2)^{1/2} d(-\Phi(r)) \qquad (8)\\
&= \int_0^s \log(1+r^2)^{1/2}\left\{\int_{\Gamma(r)} \frac{\chi_X 2N r^{2N-1}}{(1+r^2)^{N+1}}\eta\right\} dr \\
&= \int_{\zeta\in B(s)} \frac{\chi_X \log(1+\|\zeta\|^2)^{1/2}}{(1+\|\zeta\|^2)^{N+1}}\alpha^N.
\end{aligned}$$

On the other hand, we have

$$\int_0^s \log(1+r^2)^{1/2} d(-\Phi(r)) = \int_0^s \frac{r\Phi(r)}{1+r^2}dr - \Phi(s)\log(1+s^2)^{1/2}. \tag{9}$$

The following convergence will be proved later:

$$\Phi(s)\log(1+s^2)^{1/2} \to 0 \quad (s \to \infty). \tag{10}$$

Hence by (8), (9), (10) the left side of (6) is

$$\int_{\zeta\in\mathbf{C}^N} \frac{\chi_X \log(1+\|\zeta\|^2)^{1/2}}{(1+\|\zeta\|^2)^{N+1}}\alpha^N = \int_0^\infty \frac{r\Phi(r)}{1+r^2}dr. \tag{11}$$

To estimate (11), we put

$$\Psi(r) = \int_{\{\zeta\in\mathbf{C}^N;\ \|\zeta\|>r\}} \omega_V^N.$$

Then, $\Psi(r)$ is a strictly decreasing and continuous function on $[0,\infty)$ such that $0 \le \Phi(r) \le \Psi(r) \le 1$, $\Psi(0) = 1$, and $\lim_{r\to\infty}\Psi(r) = 0$.

We compute $\Psi(r)$ as follows.

$$\begin{aligned}\Psi(r) &= \int_{\{\zeta\in\mathbf{C}^N;\ \|\zeta\|>r\}} \frac{1}{(1+\|\zeta\|^2)^{N+1}}\alpha^N \\ &= \int_r^\infty \left\{\int_{\Gamma(t)} \frac{2Nt^{2N-1}}{(1+t^2)^{N+1}}\eta\right\} dt \\ &= \int_r^\infty \frac{2Nt^{2N-1}}{(1+t^2)^{N+1}}dt \\ &= \sum_{j=1}^N \frac{r^{2(j-1)}}{(1+r^2)^j}.\end{aligned}$$

Therefore we have

$$\frac{1}{1+r^2} \le \Psi(r) \le \frac{N}{1+r^2}. \tag{12}$$

We show (10) as follows.

$$0 \le \Phi(s)\log(1+s^2)^{1/2} \le \Psi(s)\log(1+s^2)^{1/2}$$

$$\leq \frac{N}{1+s^2}\log(1+s^2)^{1/2} \to 0 \quad (s\to\infty).$$

Because of $\mu(X)>0$ we can take a real number $r_1 \geq 0$ such that $\Psi(r_1) = \mu(X)$. By (12)

$$\frac{1}{\mu(X)} \leq 1 + r_1^2 \leq \frac{N}{\mu(X)}. \tag{13}$$

Note that $\Phi(0) = \mu(X)$, $\Phi(r)$ is decreasing, and that $\Phi(r) \leq \min\{\Psi(r), \mu(X)\}$. Therefore, we get

$$\int_0^\infty \frac{r\Phi(r)}{1+r^2}dr \leq \int_0^{r_1} \frac{r\mu(X)}{1+r^2}dr + \int_{r_1}^\infty \frac{r\Psi(r)}{1+r^2}dr$$

$$= \frac{\mu(X)}{2}\log(1+r_1^2) + \int_{r_1}^\infty \frac{r\Psi(r)}{1+r^2}dr.$$

Furthermore by (12) and (13) we see that

$$\int_0^\infty \frac{r\Phi(r)}{1+r^2}dr \leq \frac{\mu(X)}{2}\log\frac{N}{\mu(X)} + \int_{r_1}^\infty \frac{rN}{(1+r^2)^2}dr$$

$$= \frac{\mu(X)}{2}\log\frac{N}{\mu(X)} + \frac{N}{2(1+r_1^2)} \leq \frac{\mu(X)}{2}\left(N + \log\frac{N}{\mu(X)}\right).$$

Therefore, (6) follows from (11). □

4. Growth of the Nevanlinna Proximity Function 1

We show the following theorem.

THEOREM 2. *Let M be a compact complex manifold and $L \to M$ a holomorphic line bundle satisfying $B(\Gamma(M,L)) = \phi$. Let $f : \mathbf{C}^n \to M$ be a meromorphic mapping such that $T_f(r,L) \to \infty$ $(r\to\infty)$. Then we have that for almost all divisor $D \in P(\Gamma(M,L))$*

$$\limsup_{r\to\infty} \frac{m_f(r,D)}{\log T_f(r,L)} \leq \frac{1}{2}.$$

PROOF. Set $V = \Gamma(M, L)$. Let ω, ω_V and S_x be as in the section 3. Then

$$T_f(r, \omega) = T_f(r, L) + O(1).$$

Since $T_f(r, L) \to \infty \ \ (r \to \infty)$, for all positive integer $m \in \mathbf{N}$ we can choose real number $r_m \in (1, \infty)$ such that

$$T_f(r_m, \omega) = m.$$

Let $\beta > 1/2$ be an arbitrary real number and set

$$G(m, \beta) = \{D \in P(V);\ m_f(r_m, D) > \beta \log m\}.$$

We denote by $I(f)$ the indeterminacy locus of f. Because the codimension of $I(f)$ is greater than or equal to 2, it follows from lemma 1 that if $\mu(G(m, \beta)) > 0$, then

$$\mu(G(m, \beta))\beta \log m < \int_{D \in G(m,\beta)} m_f(r_m, D)\omega_V^N$$

$$= \int_{D \in G(m,\beta)} \left\{ \int_{z \in \Gamma(r_m) \setminus I(f)} \log \frac{1}{S_{f(z)}(D)} \eta(z) \right\} \omega_V^N$$

$$= \int_{z \in \Gamma(r_m) \setminus I(f)} \left\{ \int_{D \in G(m,\beta)} \log \frac{1}{S_{f(z)}(D)} \omega_V^N \right\} \eta(z)$$

$$\leq \int_{z \in \Gamma(r_m) \setminus I(f)} \frac{\mu(G(m, \beta))}{2} \left(N + \log \frac{N}{\mu(G(m, \beta))} \right) \eta(z)$$

$$= \frac{\mu(G(m, \beta))}{2} \left(N + \log \frac{N}{\mu(G(m, \beta))} \right).$$

Hence we deduce that

$$\mu(G(m, \beta)) < \frac{Ne^N}{m^{2\beta}}.$$

We set

$$G(\beta) = \bigcap_{m_0=1}^{\infty} \bigcup_{m=m_0}^{\infty} G(m, \beta).$$

Because of $\beta > 1/2$ it follows that

$$\mu(G(\beta)) \le \lim_{m_0\to\infty} \sum_{m=m_0}^{\infty} \mu(G(m,\beta)) < \lim_{m_0\to\infty} \sum_{m=m_0}^{\infty} \frac{Ne^N}{m^{2\beta}} = 0. \tag{14}$$

Note that the set $X(f)$ defined by

$$X(f) = \{D \in P(V);\ \mathrm{supp}\ D \supset f(\mathbf{C}^n)\}$$

has zero measure. Let $D \notin G(\beta) \cup X(f)$. Then there exists an integer $m_D \in \mathbf{N}$ such that for all $m > m_D$

$$m_f(r_m, D) \le \beta \log m. \tag{15}$$

We choose an arbitrary number $s \ge r_{m_D}$ and we take an integer $m_s \in \mathbf{N}$ satisfying $r_{m_s} \le s < r_{m_s+1}$. Then $m_s \ge m_D$. Since $\omega \ge 0$ and $D \notin X(f)$, we have by the First Main Theorem (1) and (15)

$$\begin{aligned} m_f(s,D) &= T_f(s,\omega) - N(s, f^*D) + O(1) \\ &\le T_f(r_{m_s+1},\omega) - N(r_{m_s}, f^*D) + O(1) \\ &= T_f(r_{m_s},\omega) - N(r_{m_s}, f^*D) + O(1) \\ &= m_f(r_{m_s}, D) + O(1) \le \beta \log m_s + O(1) \\ &\le \beta \log T_f(s,\omega) + O(1). \end{aligned}$$

Therefore it follows that for an arbitrary $D \notin G(\beta) \cup X(f)$

$$\limsup_{r\to\infty} \frac{m_f(r,D)}{\log T_f(r,\omega)} \le \beta. \tag{16}$$

We set

$$G = \bigcup_{k=1}^{\infty} G\left(\frac{1}{2} + \frac{1}{k}\right) \cup X(f).$$

Then by (14), (16) we see that

$$\mu(G) \le \sum_{k=1}^{\infty} \mu\left(G\left(\frac{1}{2} + \frac{1}{k}\right)\right) + \mu(X(f)) = 0$$

and that for $D \notin G$

$$\limsup_{r \to +\infty} \frac{m_f(r, D)}{\log T_f(r, \omega)} \leq \frac{1}{2}. \square$$

In general, let M be a compact complex manifold with a Hermitian metric form ω. Let $f : \mathbf{C}^n \to M$ be a meromorphic mapping. Then the order function of f with respect to ω is defined by

$$T_f(r, \omega) = \int_1^r \frac{dt}{t^{2n-1}} \int_{B(t)} f^* \omega \wedge \alpha^{n-1}.$$

We define the order of f by

$$\rho_f = \limsup_{r \to \infty} \frac{\log T_f(r, \omega)}{\log r},$$

which is independent of the choice of the Hermitian metric form ω.

We easily deduce the following corollary from Theorem 2.

COROLLARY 3. *Let M be a compact complex manifold and L a very ample holomorphic line bundle over M. Let $f : \mathbf{C}^n \to M$ be a meromorphic mapping. Assume that the order of f is finite and $T_f(r, L) \to \infty$ $(r \to \infty)$. Then,*

$$\limsup_{r \to \infty} \frac{m_f(r, D)}{\log r} \leq \frac{\rho_f}{2}$$

for almost all effective divisor $D \in P(\Gamma(M, L))$.

5. Growth of the Nevanlinna Proximity Function 2

We now define the projective logarithmic capacity of a subset in the $\mathbf{P}^N(\mathbf{C})$ (See Molzon-Shiffman-Sibony [3]). Let K be a compact subset of $\mathbf{P}^N(\mathbf{C})$. We denote by $\mathcal{M}(K)$ the space of positive Borel measures on K with total mass 1. For $x = [x^0 : \ldots : x^N] \in \mathbf{P}^N(\mathbf{C})$ and $\nu \in \mathcal{M}(K)$ we set

$$u_\nu(x) = \int_{[w^0:\ldots:w^N] \in K} \log \frac{\left(\sum_{j=0}^N |x^j|^2\right)^{1/2} \left(\sum_{j=0}^N |w^j|^2\right)^{1/2}}{\left|\sum_{j=0}^N x^j w^j\right|} d\nu,$$

and

$$V(K) = \inf_{\nu \in \mathcal{M}(K)} \sup_{x \in \mathbf{P}^N(\mathbf{C})} u_\nu(x).$$

Define the projective logarithmic capacity of K by

$$C(K) = \frac{1}{V(K)}.$$

When $V(K) = \infty$, we set $C(K) = 0$. For an arbitrary subset E of $\mathbf{P}^N(\mathbf{C})$ we define the projective logarithmic capacity of E by

$$C(E) = \sup_{K \subset E} C(K),$$

where the supremum is taken over compact subsets K of E.

For real valued functions $A(r)$ and $B(r)$ on $[1, \infty)$ we write

$$A(r) \leq B(r)||$$

if there is a Borel subset $J \subset [1, \infty)$ with finite measure such that $A(r) \leq B(r)$ for $r \in [1, \infty) \setminus J$.

Let the notation be as in the previous section. We now show the following theorem.

THEOREM 4. *Let M be a compact complex manifold, and $L \to M$ a holomorphic line bundle with $B(\Gamma(M, L)) = \phi$. Let $f : \mathbf{C}^n \to M$ be a meromorphic mapping. Let $\varphi(r) > 0$ be a Borel measurable function on $[1, \infty)$ which satisfies*

$$\int_1^\infty \frac{dr}{\varphi(r)} < \infty.$$

Then there exists a subset F of $P(\Gamma(M, L))$ such that $C(F) = 0$ and that

$$m_f(r, D) \leq \varphi(r) + O(1)||$$

for an arbitrary divisor $D \in P(\Gamma(M, L)) \setminus F$.

PROOF. We identify $P(\Gamma(M, L)) = \mathbf{P}^N(\mathbf{C})$ by the base $\{\sigma_0, \ldots, \sigma_N\}$. Then we equate $[\zeta^0 : \ldots : \zeta^N] \in \mathbf{P}^N(\mathbf{C})$ with a divisor $\left(\sum_{j=0}^N \zeta^j \sigma_j\right)$. We set

$$F = \left\{ D \in P(\Gamma(M, L));\ \int_1^\infty \frac{m_f(r, D)}{\varphi(r)} dr = \infty \right\}.$$

Assume that $C(F) > 0$. Then there is a compact subset K of F with $C(K) > 0$. Therefore there exists a $\nu \in \mathcal{M}(K)$ such that

$$\sup_{x \in \mathbf{P}^N(\mathbf{C})} u_\nu(x) < \infty. \tag{17}$$

It follows from (3) and (17) that

$$\int_{[\zeta^0:\ldots:\zeta^N] \in K} \left\{ \int_1^\infty \frac{m_f(r, [\zeta^0 : \ldots : \zeta^N])}{\varphi(r)} dr \right\} d\nu$$

$$= \int_1^\infty \frac{1}{\varphi(r)} \left\{ \int_{z \in \Gamma(r)} \left\{ \int_K \log \frac{1}{S_{f(z)}([\zeta^0 : \ldots : \zeta^N])} d\nu \right\} \eta \right\} dr$$

$$\leq \int_1^\infty \frac{1}{\varphi(r)} \left\{ \int_{\Gamma(r)} \sup_{x \in \mathbf{P}^N(\mathbf{C})} u_\nu(x) \eta \right\} dr$$

$$= \int_1^\infty \frac{1}{\varphi(r)} \sup_{x \in \mathbf{P}^N(\mathbf{C})} u_\nu(x) dr < \infty.$$

On the other hand, by the definition of F we have

$$\int_{[\zeta^0:\ldots:\zeta^N] \in K} \left\{ \int_1^\infty \frac{m_f(r, [\zeta^0 : \ldots : \zeta^N])}{\varphi(r)} dr \right\} d\nu = \infty.$$

This is a contradiction. Hence $C(F) = 0$. For an arbitrary divisor $D \in P(\Gamma(M, L))$ we set

$$J(D) = \left\{ r \in [1, \infty);\ \frac{m_f(r, D)}{\varphi(r)} > 1 \right\}.$$

If $D \notin F$, then we see

$$\int_{J(D)} dr < \int_{r \in J(D)} \frac{m_f(r, D)}{\varphi(r)} dr \leq \int_1^\infty \frac{m_f(r, D)}{\varphi(r)} dr < \infty.$$

Therefore for $D \in P(\Gamma(M, L)) \setminus F$

$$m_f(r, D) \leq \varphi(r) + O(1)||. \ \square$$

6. The General Case

In this section we deal with the growth of the proximity function with respect to an effective divisor $D \in P(E)$, where $L \to M$ be a holomorphic line bundle and E is a linear subspace of $\Gamma(M, L)$, and complete the proof of the Main Theorem.

Let M be a compact complex manifold and $\mathcal{I}$ a coherent ideal sheaf of the structure sheaf $\mathcal{O}_M$ over M. Let $\{V_\lambda\}$ be a finite open covering of M and $\eta_{\lambda j} \in \Gamma(V_\lambda, \mathcal{I})$, $j = 1, 2, \ldots$, be finitely many sections of which germs $\underline{\eta_{\lambda 1}}_x, \underline{\eta_{\lambda 2}}_x, \ldots$, generate the fiber $\mathcal{I}_x$ for all $x \in V_\lambda$. Following to [5], Chap. 2 or [7], §2, we let $\{\rho_\lambda\}$ be a partition of unity associated with $\{V_\lambda\}$ and set

$$d_{\mathcal{I}}(x) = \sum_\lambda \rho_\lambda(x) \left(\sum_j |\eta_{\lambda j}(x)|^2 \right)^{1/2}, \quad x \in M.$$

Let f be a meromorphic mapping from $\mathbf{C}^n$ into M such that

$$f(\mathbf{C}^n) \not\subset \operatorname{supp} \mathcal{O}_M/\mathcal{I}.$$

We define the proximity function of f for $\mathcal{I}$ by

$$m_f(r, \mathcal{I}) = \int_{z \in \Gamma(r)} - \log d_{\mathcal{I}} \circ f(z) \eta(z).$$

Next let $L \to M$ be a holomorphic line bundle and $\dim \Gamma(M, L) = N+1$. Let E be an $(l+1)$-dimensional linear subspace of $\Gamma(M, L)$. We take a base $\{\sigma_0, \ldots, \sigma_N\}$ of $\Gamma(M, L)$ and we identify $\Gamma(M, L) \cong \mathbf{C}^{N+1}$ by $\{\sigma_0, \ldots, \sigma_N\}$. Moreover we assume that E is spanned by $\{\sigma_0, \ldots, \sigma_l\}$. Let $\mathcal{I}$ denote the coherent ideal sheaf of $\mathcal{O}_M$ of which fiber over $x \in M$ is generated by $\{\underline{\sigma}_x;\ \sigma \in E\}$. Then the base of E is defined by $B(E) = \mathcal{O}_M/\mathcal{I}$. Thus we write $\mathcal{I} = \mathcal{I}_{B(E)}$.

Let $f : \mathbf{C}^n \to M$ be a meromorphic mapping. Suppose that

$$f(\mathbf{C}^n) \not\subset \operatorname{supp} B(E).$$

Let $(\{U_\lambda\}, \{s_\lambda\})$ be a local trivialization covering of L. We define a meromorphic mapping $\Phi_L : M \to \mathbf{P}^N(\mathbf{C})$ by

$$\Phi_L(x) = [\sigma_{0\lambda}(x) : \ldots : \sigma_{N\lambda}(x)], \quad x \in U_\lambda,$$

where $\sigma_{j\lambda}$ is a holomorphic function on U_λ such that $\sigma_j|U_\lambda = \sigma_{j\lambda}s_\lambda$. Let $(f^0, \ldots, f^N)$ be a reduced representation of $\Phi_L \circ f$. We denote by f_E the meromorphic mapping from $\mathbf{C}^n$ into $\mathbf{P}^l(\mathbf{C})$ represented by $(f^0, \ldots, f^l)$. For $z \in (f|(\mathbf{C}^n \setminus I(f)))^{-1}(U_\lambda \setminus \mathrm{supp}\ B(E))$

$$f_E(z) = [\sigma_{0\lambda} \circ f(z) : \ldots : \sigma_{l\lambda} \circ f(z)].$$

We denote by H_l hyperplane bundle over $\mathbf{P}^l(\mathbf{C})$. The following is known.

Proposition 5. *Let the notation be as above. We have the following.* (i) *If $B(\Gamma(M, L)) = \phi$, then*

$$T_f(r, L) \geq T_{f_E}(r, H_l) + O(1).$$

(ii) (Cf. Noguchi [5].) *For $[\zeta^0 : \ldots : \zeta^l] \in P(E)$*

$$m_f\left(r, \left(\textstyle\sum_{j=0}^l \zeta^j \sigma_j\right)\right) - m_f(r, \mathcal{I}_{B(E)}) = m_{f_E}(r, [\zeta^0 : \ldots : \zeta^l]) + O(1),$$

where $m_{f_E}(r, [\zeta^0 : \ldots : \zeta^l])$ is the proximity function of f_E with respect to a hyperplane $\{[z^0 : \ldots : z^l] \in \mathbf{P}^l(\mathbf{C});\ \sum_{j=0}^l \zeta^j z^j = 0\}$.

Proof. (i) We assume that $B(\Gamma(M, L)) = \phi$. Let $(g^0, \ldots, g^l)$ be a reduced representation of f_E. Then there is a holomorphic function g on $\mathbf{C}^n$ such that $(f^0, \ldots, f^l) = (gg^0, \ldots, gg^l)$. Since $L = \Phi_L^* H_N$ it follows that

$$T_f(r, L) = \int_{z \in \Gamma(r)} \log \left(\sum_{j=0}^N |f^j(z)|^2 \right)^{1/2} \eta + O(1)$$

$$\geq \int_{z \in \Gamma(r)} \log \left(\sum_{j=0}^l |f^j(z)|^2 \right)^{1/2} \eta + O(1)$$

$$\geq \int_{z \in \Gamma(r)} \log \left(\sum_{j=0}^l |g^j(z)|^2 \right)^{1/2} \eta + \int_{z \in \Gamma(1)} \log |g|\ \eta + O(1)$$

$$\geq T_{f_E}(r, H_l) + O(1).$$

(ii) Let h be a Hermitian metric in L and $||\cdot||$ denote the norms on L. Let $\{\tau_\lambda\}$ be a partition of unity associated with $\{U_\lambda\}$. For $x \in U_\nu$ we set

$$k(x) = \log \frac{\left(\sum_{j=0}^{l} |\zeta^j|^2\right)^{1/2}}{||\sum_{j=0}^{l} \zeta^j \sigma_j(x)||} - \log \frac{\left(\sum_{j=0}^{l} |\sigma_{j\nu}(x)|^2\right)^{1/2} \left(\sum_{j=0}^{l} |\zeta^j|^2\right)^{1/2}}{|\sum_{j=0}^{l} \sigma_{j\nu}(x)\zeta^j|}$$

$$+ \log \sum_{\lambda} \tau_\lambda(x) \left(\sum_{j=0}^{l} |\sigma_{j\lambda}(x)|^2\right)^{1/2}.$$

Since

$$||\sum_{j=0}^{l} \zeta^j \sigma_j(x)|| = |\sum_{j=0}^{l} \sigma_{j\nu}(x)\zeta^j| ||s_\nu(x)||,$$

we see

$$k(x) = \log \frac{\sum_\lambda \tau_\lambda(x) \left(\sum_{j=0}^{l} |\sigma_{j\lambda}(x)|^2\right)^{1/2}}{||s_\nu(x)|| \left(\sum_{j=0}^{l} |\sigma_{j\nu}(x)|^2\right)^{1/2}}.$$

We take an arbitrary point $y \in M$ and ν such that $\tau_\nu(y) > 0$. Then there are a relatively compact neighborhood $V \subset U_\nu$ of y and positive constant $C_1, C_2, C_3 > 0$ such that for $x \in V$

$$k(x) \leq \log \frac{\sum_\lambda C_1 \tau_\lambda(x) \left(\sum_{j=0}^{l} |\sigma_{j\nu}(x)|^2\right)^{1/2}}{||s_\nu(x)|| \left(\sum_{j=0}^{l} |\sigma_{j\nu}(x)|^2\right)^{1/2}} = \log \frac{C_1}{||s_\nu(x)||} \leq \log C_2,$$

and

$$k(x) \geq \log \frac{\tau_\nu(x)}{||s_\nu(x)||} \geq \log C_3.$$

Since M is compact there exists a positive constant C such that for an arbitrary $x \in M$

$$|k(x)| < C.$$

This finishes the proof of (ii). □

Let μ_E denote the positive measure induced by Fubini-Study metric on $P(E) = \mathbf{P}^l(\mathbf{C})$.

THEOREM 6. *Let M be a compact complex manifold and $L \to M$ a holomorphic line bundle. Let $1 \leq l \leq N$ be an integer and E an (l+1)-dimensional linear subspace of $\Gamma(M, L)$. Let $f : \mathbf{C}^n \to M$ be a meromorphic mapping such that $f(\mathbf{C}^n) \not\subset \operatorname{supp} B(E)$. If $T_{f_E}(r, H_l) \to \infty \ \ (r \to \infty)$, then for almost all divisor $D \in P(E)$*

$$\limsup_{r\to\infty} \frac{m_f(r, D) - m_f(r, \mathcal{I}_{B(E)})}{\log T_{f_E}(r, H_l)} \leq \frac{1}{2}.$$

Otherwise for almost all divisor $D \in P(E)$

$$m_f(r, D) - m_f(r, \mathcal{I}_{B(E)}) = O(1).$$

PROOF. Set

$$I = \left\{ [\zeta^0 : \ldots : \zeta^l] \in P(E); \right.$$
$$\left. \limsup_{r\to\infty} \frac{m_f(r, (\sum_{j=0}^{l} \zeta^j \sigma_j)) - m_f(r, \mathcal{I}_{B(E)})}{\log T_{f_E}(r, H_l)} > \frac{1}{2} \right\}.$$

Because of Proposition 5 we have that for $[\zeta^0 : \ldots : \zeta^l] \in I$

$$\frac{1}{2} < \limsup_{r\to\infty} \frac{m_{f_E}(r, [\zeta^0 : \ldots : \zeta^l])}{\log T_{f_E}(r, H_l)}.$$

Hence, if $T_{f_E}(r, H_l) \to \infty \ \ (r \to \infty)$, then we have $\mu_E(I) = 0$ by Theorem 2. We assume that $T_{f_E}(r, H_l) = O(1)$. Then f_E is a constant mapping. Hence by Proposition 5 (ii)

$$m_f(r, D) - m_f(r, \mathcal{I}_{B(E)}) = O(1). \ \square$$

By making use of the methods in the proofs of Proposition 5 and Theorem 4 one may also deduce the following:

THEOREM 7. *Let M be a compact complex manifold and $L \to M$ a holomorphic line bundle. Let $1 \leq l \leq N$ be an integer and E an (l+1)-dimensional linear subspace of $\Gamma(M, L)$. Let $f : \mathbf{C}^n \to M$ be a meromorphic* *mapping. Let* $\varphi(r) > 0$ *be a Borel measurable function on* $[1, \infty)$ *which satisfies*

$$\int_1^\infty \frac{dr}{\varphi(r)} < \infty.$$

Then there exists a subset F *of* $P(E)$ *such that* $C(F) = 0$ *and that for all* $D \in P(E) \setminus F$

$$m_f(r, D) - m_f(r, \mathcal{I}_{B(E)}) \leq \varphi(r) + O(1)||.$$

Remark. S. Mori [4] proved that for a non-constant meromorphic mapping $f : \mathbf{C}^n \to \mathbf{P}^N(\mathbf{C})$, the set

$$\left\{ H \in \mathbf{P}^N(\mathbf{C})^*;\ \limsup_{r\to\infty} \frac{m_f(r, D)}{\sqrt{T_f(r, H_N)} \log T_f(r, H_N)} > 0 \right\}$$

is of projective logarithmic capacity zero. Moreover, A. Sadullaev [8] showed that this set forms a polar set.

Note the differences between these results and our Theorems 2 and 7.

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Mathematical Systems, Inc.
10F Four Seasons Bldg.
2-4-3 Shinjuku, Shinjuku-ku
Tokyo 160-0022, Japan
E-mail: nitanda_atsushi@msi.co.jp