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On an Optimal Control Problem for the Wave Equation with Input on an Unknown Surface

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Abstract. An optimal control problem for the wave equation with Dirichlet boundary conditions, initial data in $L^2(\Omega) \times H^{-1}(\Omega)$ and input μ on an unknown interior surface,is studied.Using control techniques and the generalized gradients, feedback laws for an approximating system yielding the support of the Radon measure μ from observed values of the solution in a fixed subregion, are established.

1. Introduction

The purpose of this paper is to study an optimal control problem for the wave equation with an input on an unknown interior surface,i.e. one wishes to identify an obstacle from the observations of far away reflected waves. The problems have applications in geophysical explorations, oceanography, medical diagnosis and reconnaissance. For known point sources, the anti-noise problem for the wave equation was treated by J.L.Lions in [7]. For unknown point sources, the one dimensional initial boundary value problem has been treated by G.Bruckner and M.Yamamoto in [3], estimations of the point sources have been studied by V.Komornik and M.Yamamoto in [5,6] and the general case has been considered by A.El.Badia and T.Ha Duong [4].

For unknown surface source the exact controllability of an initial boundary problem for a nonlinear wave equation has been studied by the author in [10]. In this paper we shall consider the case when the surface-source is the support of a Radon measure.

The controllability of the Laplace equation observed on an interior curve has been treated by A.Osses and J.Puel in [9] .

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Let Ω be a bounded open subset of R^3, let $u \in \mathcal{K}$

$$\mathcal{K} = \left\{u : \, u \geq 0, \, \|u\|_{H^2(\Omega)\cap H_0^1(\Omega)} \leq M\right\}$$

with M defined in Section 2. Consider the problem

$$\begin{aligned} y'' - \Delta y + \mu \quad &= \quad f \text{ in } \Omega \times (0,T), \\ y = 0 \text{ on } \partial\Omega \times (0,T) \quad \text{and} \quad &\{y, \, y'\} \mid_{t=0} = \boldsymbol{\alpha} \text{ in } \Omega. \end{aligned} \tag{1.1}$$

Here μ is a measure defined by

$$< \mu, \varphi >= \int_{\partial\Omega_a} g\varphi d\sigma \;\; \forall \varphi \in C_0(\Omega)$$

and

$$\partial\Omega_a = \{\xi : \xi \in \Omega, u(\xi) = a\}$$

where g is a given function with

$$g \in C(\Omega), g \geq 1,$$

with $a \in (0, \max_{\overline{\Omega}} u)$. Since u is positive and in $C^\lambda(\Omega)$ the boundary $\partial\Omega_a$ is non empty and Ω_a is an interior subset of Ω as a is in the range of u.

Let χ be given and y be a solution with u, a, g and let $J(\mu, u, a)$ be the cost function

$$J(\mu, u, a) = \int_0^T \int_{\tilde{\Omega}} \mid y - \chi \mid dxdt$$

where $\tilde{\Omega}$ is an interior subset of Ω having an empty intersection with supp(μ).

Given $\boldsymbol{\alpha}$ in $L^2(\Omega) \times H^{-1}(\Omega)$, one wishes to find

$$\{\tilde{y}, u, \, \tilde{\mu}, \tilde{a}\} \in C(0,T; L^2(\Omega)) \times \{H^2(\Omega) \cap H_0^1(\Omega)\} \times M_b(\Omega) \times R^+$$

with

$$\begin{aligned} &\int_0^T \int_{\tilde{\Omega}} \mid \tilde{y} - \chi \mid dxdt \\ &\quad = \inf \left\{ J(\mu, u, a) : \; \forall \{y, u, \mu, a\} \text{ such that} \right. \\ &\qquad \left. \operatorname{supp} \mu \cap \operatorname{supp} \chi = \emptyset \,, \, \forall a \in [a_0, \max_{\overline{\Omega}} u], \, u \in \mathcal{K} \right\} \end{aligned} \tag{1.2}$$

The set of all bounded Radon measures in Ω is denoted by $M_b(\Omega)$.

An approximating system for (1.1) is studied in Section 2. Feedback laws for the approximating system are established in Section 4 and the main result of the paper is proved in Section 5. Initial boundary value problems for parabolic equations with *given* Radon measure sources have been studied by H.Amman and P.Quinter [1], by L.Boccardo and T.Gallouet [2]. Elliptic problems with a solution-dependent Radon measure were considered by the author in [11].

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2. An Approximating System

In this section we shall

(i) construct functionals ψ_k, uniformly bounded in $L^1(\Omega)$ with

$$\psi_k \to \mu \text{ in } \mathcal{D}'(\Omega), \ \operatorname{supp}(\mu) = \partial\Omega_a$$

(ii) construct the unique weak transpose solution of an initial boundary problem for the wave equation with $L^1(\Omega)$-source and show that it is the sum of two functions, one being time-independent,

(iii) study the value function associated with the problem in (ii) and a given cost functional.

Let F_k be a positive Lipschitz continuous function on R^+ with

$$F_k(s) = \begin{cases} k & \text{if } s = a \\ 0 & \text{if } s \leq a - k^{-1} \text{ or if } s \geq a + k^1 \end{cases}$$

and

$$\int_{-\infty}^{a} F_k(s,a)ds = 1 = \int_{a}^{\infty} F_k(s,a)ds.$$

The $F_k(s,a)$ approximate the Dirac delta function with mass at a. Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$, $u \geq 0$, then u is in $C^\lambda(\Omega)$ and

$$\Omega_a = \left\{\xi : \xi \in \Omega, 0 < u(\xi) < a, \, 0 < a < \max_{\overline{\Omega}} u\right\}$$

is an interior open subset of Ω with $\partial\Omega_a \in C^\lambda$.

We shall denote by $\mathcal{E}(f, \boldsymbol{\alpha})$ the expression

$$\mathcal{E}(f,\boldsymbol{\alpha}) = 1 + \mid \Omega \mid + \|f\|_{L^2(Q)} + \|\boldsymbol{\alpha}\|_{L^2(\Omega)\times H^{-1}(\Omega)} \tag{2.1}$$

where $Q = \Omega \times (0, T)$.

Step 1. Let M be a constant with $M \geq \mathcal{E}(f, \boldsymbol{\alpha})$ and let

$$\mathcal{K} = \left\{v : \|v\|_{H^2(\Omega)\cap H_0^1(\Omega)} \leq M,\ v = v^+ \geq 0 \text{ in } \Omega\right\} \tag{2.2}$$

then $\mathcal{K}$ is a compact convex subset of $L^2(\Omega)$.

Consider the initial boundary problem

$$\begin{aligned} u_k' - \Delta u_k + gF_k((u_k + v)^+, a) &= 1 \text{ in } Q, \\ u_k = 0 \text{ on } \{\partial\Omega \cup \partial\Omega_a\} \times (0,T) \quad &, \quad u_k(\,.,0) = 0 \text{ in } \Omega. \end{aligned} \tag{2.3}$$

LEMMA 2.1. *Let $\{g, v\}$ be in $C(\Omega) \times \mathcal{K}$ with $\inf_{\overline{\Omega}} g \geq 1$, then there exists a unique solution u_k of (2.3) with*

$$\begin{aligned} \|u_k\|_{L^2(0,T;H_0^1(\Omega))} &+ \|u_k'\|_{L^1(0,T;(H_0^1(\Omega)\cap H^2(\Omega))^*)} + \|u_k\|_{L^\infty(0,T;L^2(\Omega))} \\ &+ \|gF_k((u_k+v)^+, a)\|_{L^1(Q)} \leq 2c_\omega \|g\|_{C(\Omega)} \end{aligned}$$

where c_ω is the Poincare constant.

PROOF. 1) First consider the problem

$$\begin{aligned} \tilde{u}_k' - \Delta \tilde{u}_k + gF_k((\tilde{u}_k + v)^+, a) &= 1 \text{ in } \Omega_a \times (0,T), \\ \tilde{u}_k = 0 \text{ on } \{\partial\Omega \cup \partial\Omega_a\} \times (0,T) \quad &, \quad \tilde{u}_k(\,.,0) = 0 \text{ in } \Omega_a. \end{aligned}$$

Since for fixed k, F_k is Lipschitz continuous there exists a unique solution of the problem and

$$\{\tilde{u}_k, \tilde{u}_k'\} \in L^2(0,T; H_0^1(\Omega_a)) \times L^2(0,T; H^{-1}(\Omega_a)).$$

2) We now consider the initial boundary problem

$$\begin{array}{rcl} \hat{u}_k' - \Delta \hat{u}_k + gF_k((\hat{u}_k + v)^+, a) & = & 1 \text{ in } \Omega/\overline{\Omega}_a \times (0,T), \\ \hat{u}_k = 0 \text{ on } (\partial\Omega/\overline{\Omega}_a) \times (0,T) & , & \hat{u}_k(\,.,0) = 0 \text{ in } \Omega/\overline{\Omega}_a. \end{array}$$

As above there exists a unique solution $\hat{u}_k$ of the problem with

$$\{\hat{u}_k,\, \hat{u}_k'\} \in L^2(0,T; H_0^1(\Omega/\overline{\Omega}_a)) \times L^2(0,T; H^{-1}(\Omega/\overline{\Omega}_a)).$$

3) Set

$$u_k(\xi,t) = \tilde{u}_k \text{ in } \Omega_a \times (0,T)\ ,\ u_k = \hat{u}_k \quad \text{in } (\Omega/\overline{\Omega}_a) \times (0,T).$$

Then u_k is a solution of (2.3) and

$$\{u_k, u_k'\} \in L^2(0,T; H_0^1(\Omega)) \times L^2(0,T; H_0^{-1}(\Omega)).$$

Since F_k is Lipschitz continuous the solution is unique.

We now establish the estimate of the lemma.

We have

$$\begin{array}{rcl} \dfrac{d}{dt}\|u_k + v\|^2_{L^2(\Omega)} & + & 2\|\nabla(u_k+v)\|^2_{L^2(\Omega)} + 2(gF_k((u_k+v)^+, a),\, u_k + v) \\ & \leq & 2 \mid \Omega \mid^{1/2} \|u_k + v\|_{L^2(\Omega)} \\ & \leq & 2 \mid \Omega \mid^{1/2} c_\omega \|\nabla(v_k + v)\|_{L^2(\Omega)} \\ & \leq & \|\nabla(u_k+v)\|^2_{L^2(\Omega)} + c_\omega^2 \mid \Omega \mid . \end{array}$$

Thus,

$$\begin{array}{rcl} \|u_k + v\|^2_{L^\infty(0,T;L^2(\Omega))} & + & \|\nabla(u_k+v)\|^2_{L^2(0,T;L^2(\Omega))} \\ & + & \displaystyle\int_0^T (gF_k((u_k+v),a), u_k^+ + v)dt \leq c_\omega^2 \mid \Omega \mid . \end{array}$$

From the definition of the functional F_k we obtain

$$\int_0^T (gF_k((u_k+v)^+,\, a),\, u_k + v)dt = \int_0^T (gF_k((u_k+v)^+, a),\, (u_k+v)^+)dt.$$

We have only to consider the set where

$$a - k^{-1} \leq (u_k + v)(\xi, t) \leq a + k^{-1}.$$

Take $k > a/2 \geq a_0/2 > 0$ and we get

$$\begin{aligned} \frac{a_0}{2}\|gF_k((u_k+v)^+, a)\|_{L^1(Q)} &\leq \int_0^T (gF_k((u_k+v)^+, a), u_k+v)dt \\ &\leq T \mid \Omega \mid c_\omega^2 \end{aligned}$$

and the lemma is proved □

Set

$$\psi_k((u_k+v)^+, a) = \int_0^T gF_k((u_k+v)^+, a)dt \tag{2.4}$$

then,

$$\|\psi_k((u_k+v)^+, a)\|_{L^1(\Omega)} \leq \frac{2c_\omega^2}{a_0} T \mid \Omega \mid . \tag{2.5}$$

With ψ_k as in (2.5) we now consider the elliptic boundary problem

$$-\Delta z_k + \psi_k((u_k+v)^+, a) = 1 + \Delta v \text{ in } \Omega,\ z_k = 0 \text{ on } \partial\Omega. \tag{2.6}$$

LEMMA 2.2. *Let u_k, v be as in Lemma 2.1 and let ψ_k be as in (2.5). There exists a unique solution z_k in $H_0^1(\Omega)$ of (2.6) with*

$$\|z_k + v\|_{W_0^{1,r}(\Omega)} \leq C(1 + \mid \Omega \mid + \frac{2c_\omega^2}{a_0} T \mid \Omega \mid)\ \forall r \in (6/5, 3/2)$$

where C is a constant independent of u_k, v, k, a.

PROOF. The existence of a unique solution z_k in $H_0^1(\Omega)$ is known. We now establish the estimate of the lemma. Since ψ_k is in $L^1(\Omega)$,we obtain by applying a result due to L.Boccardo and T.Gallouet [2]

$$\begin{aligned} \|z_k + v\|_{W_0^{1,r}(\Omega)} &\leq C\{1 + \mid \Omega \mid + \|\psi_k((u_k+v)^+, a)\|_{L^1(\Omega)}\} \\ &\leq C\{1 + \mid \Omega \mid + \frac{2c_\omega^2}{a_0} T \mid \Omega \mid\} \end{aligned}$$

for $r \in (6/5, 3/2)$ and where C is a constant independent of k, v, u_k, a $\square$

Step 2. We now construct a weak transpose solution of a wave equation. Consider the problem

$$\begin{aligned} x_k'' - \Delta x_k &= f - 1 \text{ in } Q, \\ x_k = 0 \text{ on } \partial\Omega \times (0,T) \quad &, \quad \{x_k,\, x_k'\}\mid_{t=0} = \boldsymbol{\alpha} - \{z_k + v, 0\} \text{ in } \Omega. \end{aligned} \tag{2.7}$$

With $\boldsymbol{\alpha} - \{z_k,\, 0\}$ in $L^2(\Omega) \times H^{-1}(\Omega)$ we are led to the notion of weak transpose solution of the wave equation.

Definition 2.1. Let $\{h, \boldsymbol{\alpha}\}$ be in $L^2(Q) \times L^2(\Omega) \times H^{-1}(\Omega)$, then

$$\{u,\, u'\} \in C(0,T; L^2(\Omega)) \times C(0,T; H^{-1}(\Omega))$$

is said to be a weak transpose solution of

$$u'' - \Delta u = h \text{ in } Q,\; u = 0 \text{ on } \partial\Omega \times (0,T),\; \{u,\, u'\}\mid_{t=0} = \boldsymbol{\alpha}$$

if

$$\begin{aligned} (u'(\,.,t), \varphi(\,.,t)) - (\alpha_1, \varphi(\,.,0)) \quad &- \quad \int_0^t (u(\,.,s), \varphi'(\,.,s))ds \\ &= \quad \int_0^t (u,\, \Delta\varphi)ds + \int_0^t (h,\, \varphi)ds \end{aligned}$$

for all φ in $C^2(0,T; H_0^1(\Omega) \cap H^2(\Omega))$.

Lemma 2.3. *Let z_k be as in Lemma 2.2 and let*

$$\{f, \boldsymbol{\alpha},\, v\} \in L^2(Q) \times \{L^2(\Omega) \times H^{-1}(\Omega)\} \times \mathcal{K}.$$

Then there exists a unique transpose solution x_k of (2.7) with

$$\|x_k\|_{C(0,T;L^2(\Omega))} + \|x_k'\|_{C(0,T;H^{-1}(\Omega))} \le C\{\mathcal{E}(f,\, \boldsymbol{\alpha}) + \frac{2c_\omega^2}{a_0} T \mid \Omega \mid \}$$

where C is a constant independent of k, v, u_k, x_k.

Proof. Since z_k is in $W_0^{1,r}(\Omega)$ and $6/5 < r < 3/2$, it follows from the Sobolev imbedding theorem that $W^{1,r}(\Omega) \subset L^2(\Omega)$. Thus,

$$\boldsymbol{\alpha} - \{z_k + v, 0\} \in L^2(\Omega) \times H^{-1}(\Omega).$$

The existence of a unique weak transpose solution x_k of (2.7) has been shown by J.L.Lions and E.Magenes [8].The estimate is an immediate consequence of those of Lemmas 2.1, 2.2 □

Step 3. The main result of the section is the following theorem.

THEOREM 2.1. *Let* $\{f,\, g,\, v,\, \boldsymbol{\alpha}\}$ *be in*

$$L^2(Q) \times C(\Omega) \times \mathcal{K} \times \{L^2(\Omega) \times H^{-1}(\Omega)\},\ \inf_{\overline{\Omega}} g \geq 1.$$

Let u_k *be as in Lemma 2.1 and let* $\psi_k((u_k+v)^+,\, a)$ *be as in (2.4), then there exists a unique weak transpose solution of the initial boundary problem*

$$\begin{aligned} (2.8) \qquad y_k'' - \Delta y_k + \psi_k((u_k+v)^+,\, a) \ &= \ f \ \ in\ Q, \\ y_k = 0 \ \ on\ \partial\Omega \times (0,T) \quad , \quad &\{y_k,\, y_k'\}\ |_{t=0} = \boldsymbol{\alpha}\ \ in\ \Omega. \end{aligned}$$

Moreover

$$\|y_k\|_{C(0,T;L^2(\Omega))} + \|y_k'\|_{C(0,T;H^{-1}(\Omega))} \leq C\{\mathcal{E}(f,\, \boldsymbol{\alpha}) + \frac{2c_\omega^2}{a_0} T \mid \Omega \mid \}$$

and

$$\|\psi_k((u_k+v)^+,\, a)\|_{L^1(\Omega)} \leq \frac{2c_\omega^2}{a_0} T \mid \Omega \mid$$

where C *is a constant independent of* $k,\, u_k,\, v,\, a$.

PROOF. Let $x_k,\, z_k$ be as in Lemmas 2.1, 2.2 and set

$$y_k = x_k(\xi, t) + z_k(\xi) + v(\xi).$$

It is trivial to check that y_k is indeed a weak transpose solution of (2.8). As in [8] the transpose solution is unique and with the estimates on $x_k,\, z_k,\, v$ we obtain the stated result □

We associate with (2.8) the cost functional

$$(2.9) \qquad J(v,\, \boldsymbol{\alpha}, a) = \int_0^T \int_{\tilde{\Omega}} \mid y_k - \chi \mid dxdt$$

where χ is a given function in $L^1(0,T;L^1(\tilde{\Omega}))$, representing the observed values of y_k in $\tilde{\Omega}\times(0,T)$with $\tilde{\Omega}=\{\xi:\ \xi\in\Omega,\ 0<v(\xi)<a_0\}$.

We consider the value function associated with (2.8)-(2.9).

THEOREM 2.2. *Suppose all the hypotheses of Theorem 2.1 are satisfied. There exists* $\{\tilde{y}_k,\ \tilde{y}_k',\ \tilde{v},\ \tilde{a}\}$ *in*

$$C(0,T;L^2(\Omega))\times C(0,T;H^{-1}(\Omega))\times\mathcal{K}\times[a_0,\ \inf_{v\in\mathcal{K}}(\max_{\overline{\Omega}} v)]$$

such that

$$\begin{aligned}(2.10)\qquad V_k(\boldsymbol{\alpha},0) &= J(\tilde{v},\ \boldsymbol{\alpha},\tilde{a})\\ &= \inf\big\{J(v,\ \boldsymbol{\alpha},\ a):\ \{y_k,\ v\}\ \textit{solution of}\ (2.8)\\ &\quad\ \forall v\in\mathcal{K},\ \forall a\in[a_0,\ \inf_{v\in\mathcal{K}}(\max_{\overline{\Omega}} v)]\big\}.\end{aligned}$$

Moreover

$$|\ V_k(\boldsymbol{\alpha},0)-V_k(\boldsymbol{\beta},0)\ |\le C\|\boldsymbol{\alpha}-\boldsymbol{\beta}\|_{L^2(\Omega)\times H^{-1}(\Omega)}\ \ \forall\,\boldsymbol{\alpha},\ \boldsymbol{\beta}\in L^2(\Omega)\times H^{-1}(\Omega)$$

where C is a constant independent of k.

PROOF. 1) Let $\{y_{k,n},\ v_n,a_n\}$ be a minimizing sequence of the optimal problem (2.10) with

$$V_k(\boldsymbol{\alpha},0)\le J(v_n,\ \boldsymbol{\alpha},a_n)\le V_k(\boldsymbol{\alpha},0)+n^{-1}.$$

For simplicity of notations we write y_n for $y_{k,n}$. With $v_n\in\mathcal{K}$, we get by taking subsequences

$$v_k\to\tilde{v}\ \text{ in } H_0^1(\Omega)\cap(H^2(\Omega))_{weak},\ \tilde{v}=\tilde{v}^+.$$

From the estimates of Lemma 2.1 we obtain for fixed k

$$\{u_{k,n},\ u_{k,n}'\}=\{u_n,u_n'\}\to\{\tilde{u},\ \tilde{u}'\}$$

in

$$\{(L^2(0,T;H_0^1(\Omega))_{weak}\cap L^2(0,T;L^2(\Omega))\}\times(L^2(0,T;H^{-1}(\Omega)))_{weak}.$$

Since

$$\|gF_k((u_n+v_n)^+,\,a_n)\|_{L^1(Q)} \le \frac{2c_\omega^2}{a_0}T\mid\Omega\mid$$

and for fixed k the function F_k is Lipschitz continuous, we get

$$gF_k((u_n+v_n)^+,a_n) \to gF_k((\tilde u+\tilde v)^+;\,\tilde a) \text{ in } L^1(Q).$$

We obtain

$$\begin{aligned}\|\psi_k((\tilde u+\tilde v)^+,\tilde a)\|_{L^1(\Omega)} &\le \int_0^T \|gF_k((\tilde u+\tilde v)^+,\tilde a)\|_{L^1(\Omega)}dt\\ &\le \frac{2c_\omega^2}{a_0}T\mid\Omega\mid.\end{aligned}$$

It follows from the estimates of Theorem 2.1 and from Aubin's theorem that there exists a subsequence such that

$$\{y_n,\,y_n'\} \to \{\tilde y,\tilde y'\}$$

in

$$\{C(0,T;H^{-1}(\Omega))\cap(L^\infty(0,T;L^2(\Omega)))_{weak^*}\}\times(L^\infty(0,T;H^{-1}(\Omega)))_{weak^*}.$$

It is trivial to check that $\tilde y_k$ is the unique transpose solution of (2.10) and that

$$V_k(\boldsymbol{\alpha},0)=J(\tilde v,\boldsymbol{\alpha},\tilde a).$$

2) Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ be in $L^2(\Omega)\times H^{-1}(\Omega)$, then we have

$$\begin{aligned}V_k(\boldsymbol{\beta},0)-V_k(\boldsymbol{\alpha},0) &\le J_k(\tilde v,\boldsymbol{\beta},\tilde a)-J_k(\tilde v,\boldsymbol{\alpha},\tilde a)\\ &\le \|y_k-\tilde y_k\|_{L^1(Q)}.\end{aligned}$$

On the other hand we have

$$\begin{aligned}(y_k-\tilde y_k)''-\Delta(y_k-\tilde y_k) &= \psi_k((\tilde u_k+\tilde v_k)^+,\tilde a)-\psi_k((\tilde u_k+\tilde v_k)^+,\tilde a)=0\\ y_k-\tilde y_k=0 \text{ on } \partial\Omega\times(0,T) \quad &, \quad \{y_k-\tilde y_k,\,y_k'-\tilde y_k'\}\mid_{t=0}=\boldsymbol{\alpha}-\boldsymbol{\beta}.\end{aligned}$$

It follows that

$$\|y_k-\tilde y_k\|_{C(0,T;L^2(\Omega))}\le C\|\boldsymbol{\alpha}-\boldsymbol{\beta}\|_{L^2(\Omega)\times H^{-1}(\Omega)}.$$

Hence

$$V_k(\boldsymbol{\beta},0)-V_k(\boldsymbol{\alpha},0)\le C\|\boldsymbol{\alpha}-\boldsymbol{\beta}\|_{L^2(\Omega)\times H^{-1}(\Omega)}$$

Reversing the role played by $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and we get the stated result □

3. A Nonlinear Approximating System

In this section we shall establish the existence of a solution of a nonlinear initial boundary problem for the wave equation. The result will play a critical role in the feedback laws for the system (2.10).

Let S be the linear continuous mapping of $H^{-1}(\Omega)$ into $C(0,T;L^2(\Omega))$ given by

$$Sh = \varphi \tag{3.1}$$

where φ is the unique solution of the initial boundary problem

$$\varphi' - \Delta\varphi = h \text{ in } Q,\ \varphi = 0 \text{ on } \partial\Omega \times (0,T),\ \varphi(\,.\,,0) = 0 \text{ in } \Omega.$$

Let V_k be as in Theorem 2.2. Since

$$|\, V_k(S\alpha_0, S\alpha_1, 0) - V_k(S\beta_0, S\alpha_1, 0)\,| \leq C\|S\alpha_0 - S\beta_0\|_{C(0,T;L^2(\Omega))}$$

for all $\{\alpha_0, \beta_0, \alpha_1\}$ in $L^2(\Omega) \times H^{-1}(\Omega)$, the generalized subgradient $\partial_{S\alpha_0} V_k(S\boldsymbol{\alpha}, 0)$ exists and is a set-valued mapping of $L^2(Q)$ into the closed convex subsets of $L^2(Q)$. For simplicity of notations we shall write $\partial_0 V_k(S\boldsymbol{\alpha}, 0)$ for $\partial_{S\alpha_0} V_k(S\boldsymbol{\alpha}, 0)$ when there is no confusion possible.

We have

$$\|p(\boldsymbol{\alpha})\|_{L^2(Q)} \leq C \ \ \forall p(\boldsymbol{\alpha}) \in \partial_0 V_k(S\boldsymbol{\alpha}, 0)$$

Since the image of $\partial_0 V_k(S\boldsymbol{\alpha}, 0)$ is a closed bounded convex subset of $L^2(Q)$, there exists a unique element of minimum $L^2(Q)$-norm

$$\|p_*(\boldsymbol{\alpha})\|_{L^2(Q)} \leq \|p(\boldsymbol{\alpha})\|_{L^2(Q)} \ \ \forall p(\boldsymbol{\alpha}) \in \partial_0 V_k(S\boldsymbol{\alpha}, 0).$$

Let

$$\gamma^*(h) = \sup \left\{ (h, a)_{L^2(\Omega):\ \forall a \in [a_0, \inf_{[\ v \in \mathcal{K}} (\max_{\overline{\Omega}} v)]} \right\} \tag{3.2}$$

Then γ^* is a l.s.c. convex mapping of $L^2(\Omega)$ into R and its subgradient $\partial\gamma^*$ exists and is a set valued mapping of $L^2(\Omega)$ into the closed subsets of $[a_0,\, \inf_{v\in\mathcal{K}}(\max_{\overline{\Omega}} v)]$. Using a maximizing sequence we obtain

$$\gamma^*(h) = (h, q_*(h))_{L^2(\Omega)}$$

where $q_*(h)$ is the unique element of minimum $L^2(\Omega)$-norm of the closed convex set $\partial\gamma^*(h)$. It is easy to check that $\partial\gamma^*$ is continuous from $(L^2(\Omega))_{weak}$ to $L^2(\Omega)$.

Let P be the projection of $H^{-1}(\Omega)$ onto the compact convex subset $\mathcal{K}$. Then

$$\|v - Pv\|_{H^{-1}(\Omega)} = \inf\{\|v-u\|_{H^{-1}(\Omega)} : \ \forall u \in \mathcal{K}\}$$

and Pv is uniquely defined. Moreover P is a non expansive mapping. Let w be in $L^2(Q)$ and set

$$\mathcal{P}w = P(\int_0^T w(\xi,t)dt). \tag{3.3}$$

The main result of the section is the following theorem.

THEOREM 3.1. *Suppose all the hypotheses of Theorem 2.1 are satisfied. Then there exists a weak transpose solution $\tilde{y}_k$ of the initial boundary problem*

$$\begin{aligned} \tilde{y}_k'' - \Delta\tilde{y}_k \quad &+ \quad \psi_k((\tilde{u}_k + \mathcal{P}p_*(\tilde{y}_k, \tilde{y}_k'))^+,\, q_*(\tilde{y}_k)) = f, \\ \{\tilde{y}_k,\, \tilde{y}_k'\}\,|_{t=0} = \boldsymbol{\alpha} \quad &, \quad \tilde{y}_k = 0 \ \ on\ \partial\Omega \times (0,T) \end{aligned} \tag{3.4}$$

with

$$\begin{aligned} \tilde{u}_k' - \Delta\tilde{u}_k \quad &+ \quad gF_k((\tilde{u}_k + \mathcal{P}p_*(\tilde{y}_k, \tilde{y}_k'))^+,\, q_*(\tilde{y}_k)) = 1, \\ \tilde{u}_k(\,.,0) = 0 \ \ in\ \Omega \quad &, \quad \tilde{u}_k = 0 \ \ on\ \{\partial\Omega \cup \partial\Omega_{q_*(\tilde{y}_k)}\} \times (0,T). \end{aligned} \tag{3.5}$$

Moreover

$$\begin{aligned} &\|\tilde{y}_k\|_{C(0,T;L^2(\Omega))} + \|\tilde{y}_k'\|_{C(0,T;H^{-1}(\Omega))} \\ &\qquad + \|\psi_k((\tilde{u}_k + \mathcal{P}p_*(\tilde{y}_k,\, \tilde{y}_k'))^+,\, q_*(\tilde{y}_k))\|_{L^1(\Omega)} \le M \end{aligned}$$

with M as in Section 2.

Let

$$\mathcal{B} = \{x : \ \|x\|_{C(0,T;L^2(\Omega))} + \|x'\|_{C(0,T;H^{-1}(\Omega))} \le M\}$$

with M as in Step 1 of Section 2. Consider the initial boundary problem

$$(3.6)\qquad \begin{aligned} &\hat{u}_k - \Delta\hat{u}_k + gF_k((\hat{u}_k + \mathcal{P}p_*(x))^+,\, q_*(x)) = 1 \ \text{ in } Q,\\ &\hat{u}_k = 0 \ \text{ on } \{\partial\Omega \cup \partial\Omega_{q_*(x)}\} \times (0,T),\ \hat{u}_k(\,.,0) = 0 \ \text{ in } \ \Omega \end{aligned}$$

where for simplicity of notations we write $p(x)$ for $p(x,x')$. From Lemma 2.1 we know that there exists a unique solution of (3.6) and that

$$\begin{aligned} \|\hat{u}_k + \mathcal{P}p_*(x)\|_{L^2(),T;H_0^1(\Omega))} &+ \|(\hat{u}_k + \mathcal{P}p_*(x))'\|_{L^1(0,T;(H_0^1(\Omega)\cap H^2(\Omega))^*)}\\ &+ \|gF_k((\hat{u}_k + \mathcal{P}p_*(x))^+, q_*(x))\|_{L^1(Q)} \le M. \end{aligned}$$

Consider the initial boundary problem

$$(3.7)\qquad \begin{aligned} &\hat{y}_k'' - \Delta\hat{y}_k + \psi_k((\hat{u}_k + \mathcal{P}p_*(x))^+, q_*(x)) = f \text{ in } Q,\\ &\hat{y}_k = 0 \ \text{ on } \partial\Omega \times (0,T),\ \{\hat{y}_k,\, \hat{y}_k'\}_{|t=0} = \boldsymbol{\alpha}. \end{aligned}$$

From Theorem 2.1 we know that there exists a unique weak transpose solution of (3.7) and that $\hat{y}_k \in \mathcal{B}$. Let

$$(3.8)\qquad \mathcal{A}(x) = \hat{y}_k.$$

The nonlinear mapping $\mathcal{A}$ of $\mathcal{B}$, considered as a compact convex subset of $L^2(0,T;H^{-1}(\Omega))$ into $L^2(0,T;H^{-1}(\Omega))$ is well defined. To prove the theorem it suffices to show that $\mathcal{A}$ has a fixed point.

Lemma 3.1. *Let $x_n \in \mathcal{B}$ and suppose that $x_n \to x$ in $L^2(0,T;$ $H^{-1}(\Omega))$, then*

$$\{p_*(x_n), q_*(x_n)\} \to \{p_*(x),\, q_*(x)\} \ \textit{in } L^2(Q) \times R.$$

Proof. 1) Since $x_n \in \mathcal{B}$ and since $p_*(x_n)$ is uniformly bounded in $L^2(Q)$-norm, we get by taking subsequences

$$\{x_n, x_n',\, p_*(x_n)\} \to \{x,\, x', \tilde{p}\}$$

in

$$\begin{aligned} \{C(0,T;H^{-1}(\Omega)) \cap (L^\infty(0,T;L^2(\Omega))))_{weak^*}\} &\times (L^2(0,T;H^{-1}(\Omega)))_{weak}\\ &\times (L^2(Q))_{weak} \end{aligned}$$

with $x \in \mathcal{B}$. From the definition of subdifferentials we have

$$V_k(S\alpha_0, Sx_n', 0) - V_k(Sx_n, Sx_n', 0) \geq \int_0^T (p_*(x_n),\ S\alpha_0 - Sx_n)dt.$$

Since

$$| V_k(S\alpha_0, Sx_n', 0) - V_k(S\alpha_0, Sx', 0) | \leq C\|Sx_n' - Sx'\|_{L^2(0,T;L^2(\Omega))}$$

we deduce that

$$V_k(S\alpha_0, Sx', 0) - V_k(Sx, Sx', 0) \geq \int_0^T (\tilde{p},\ S\alpha_0 - Sx)dt \quad \forall \alpha_0 \in L^2(\Omega)$$

and hence $\tilde{p} \in \partial S_x V_k(Sx, Sx', 0)$.

2) We now show that $\tilde{p} = p_*(x, x')$, i.e. is the unique element of minimum $L^2(Q)$-norm of $\partial_{Sx} V_k(Sx, Sx', 0)$. Let

$$\mathcal{B}_\varepsilon = \left\{ x_\varepsilon : \|x_\varepsilon - x\|_{C(0,T;L^2(\Omega))} + \|x_\varepsilon' - x'\|_{C(0,T;H^{-1}(\Omega))} \leq \varepsilon \right\}$$

Then

$$\bigcap_\varepsilon \partial_{Sx_\varepsilon} V_k(Sx_\varepsilon, Sx_\varepsilon', 0) \subset \partial_{Sx} V_k(Sx, Sx', 0)$$

as $x_n \in \mathcal{B}_\varepsilon(x)$ for all $n \geq n_0$. Thus we have

$$\|p(x_n\|_{L^2(Q)} \leq \|p(x)\|_{L^2(Q)} \ \ \forall p(x) \in \partial_{Sx} V_k(Sx, Sx', 0).$$

Therefore

$$\|\tilde{p}\|_{L^2(Q)} \leq \|p(x)\|_{L^2(Q)} \ \ \forall p(x) \in \partial_{Sx} V_k(Sx, Sx', 0)$$

and thus, $p_*(x) = \tilde{p}$. Similarly for q_*.

LEMMA 3.2. *Suppose all the hypotheses of Lemma 3.1 are satisfied. Then*

$$\Omega_{q_*(x)} = \bigcap_n \Omega_{q_*(x_n)}$$

with

$$\Omega_{q_*(x_n)} = \left\{ \xi : \xi \in \Omega,\ \mathcal{P} p_*(x_n) < q_*(x_n) \right\}$$

PROOF. 1) Since $\mathcal{P}$ is a non expansive mapping of $L^2(0,T;H^{-1}(\Omega))$ into $H^{-1}(\Omega)$ we get

$$\|\mathcal{P}p_*(x_n) - \mathcal{P}p_*(x)\|_{H^{-1}(\Omega)} \leq \|p_*(x_n) - p_*(x)\|_{L^2(0,T;H^{-1}(\Omega))}.$$

Since $\mathcal{P}p_*(x_n) \in \mathcal{K}$ it follows from Lemma 3.1 that

$$\mathcal{P}p_*(x_n) \to \mathcal{P}p_*(x) \text{ in } H_0^1(\Omega) \cap H^{2-\varepsilon}(\Omega) \cap C^\lambda(\Omega).$$

2) We have

$$\Omega_{\{\mathcal{P}p_*(x)<q_*(x)-2\varepsilon\}} \subset \Omega_{\{\mathcal{P}p_*(x_n)<q_*(x_{+n})\}} \subset \Omega_{\{\mathcal{P}p_*(x)<q_*(x)+2\varepsilon\}}$$

The $\Omega_{\{\mathcal{P}p_*(x)<q_*(x)\}}$ are decreasing and we obtain

$$\begin{aligned} \Omega_{q_*(x)} &= \bigcup_\varepsilon \Omega_{\{\mathcal{P}p_*(x)<q_*(x)-2\varepsilon\}} \subset \bigcap_n \Omega_{q_*(x_n)} \subset \bigcap_\varepsilon \Omega_{\{\mathcal{P}p_*(x)<q_*(x)+2\varepsilon\}} \\ &= \Omega_{q_*(x)}. \end{aligned}$$

LEMMA 3.3. *The mapping $\mathcal{A}$ of $\mathcal{B}$, considered as a subset of $L^2(0,T;H^{-1}(\Omega))$ into $L^2(0,T;H^{-1}(\Omega))$ is continuous.*

PROOF. Let $x_n \in \mathcal{B}$, $y_n = \mathcal{A}x_n$ with $\mathcal{A}$ be as in (3.8).

1) Let u_n be the solution of (3.6) given by Lemma 2.1, then we have

$$u_k \to u \text{ in } (L^2(0,T;H_0^1(\Omega))_{weak} \cap (L^\infty(0,T;L^2(\Omega)))_{weak^*}$$

and for fixed k

$$\{u_n,\, u_n'\} \to \{u,\, u'\} \text{ in} C(0,T;L^2(\Omega)) \times (L^2(0,T;H^{-1}(\Omega)))_{weak}.$$

Since $F_k(s)$ has mass at $q_*(x)$ we have

$$\begin{aligned} &F_k((u_n + \mathcal{P}p_*(x_n))^+, q_*(x_n)) \\ &\qquad = F_k((u_n + \mathcal{P}p_*(x_n))^+ + q_*(x) - q_*(x_n),\ q_*(x)). \end{aligned}$$

For fixed k, F_k is Lipschitz continuous and thus

$$\begin{aligned} &F_k((u_n + \mathcal{P}p_*(x_n))^+, q_*(x_n)) \\ &\qquad \to F_k((u + \mathcal{P}p_*(x))^+,\ q_*(x)) \text{ in } L^2(Q) \cap (L^\infty(Q))_{weak^*}. \end{aligned}$$

Hence

$$\psi_k((u_n + \mathcal{P}p_*(x_n))^+, q_*(x_n)) \to \psi_k((u + \mathcal{P}p_*(x))^+,\ q_*(x)) \text{ in } L^2(\Omega).$$

2) Since $\Omega_{q_*(x)} = \bigcap \Omega_{q_*(x_n)}$ we get

$$\begin{aligned}\lim \int_{\Omega_{q_n}/\Omega_{q_*}} (u_n D_j\varphi - \varphi D_j u_n)dx &= 0 = \lim \int_{\partial\Omega_{q_*}} u_n \nu_j \varphi d\sigma \\ &= \int_{\partial\Omega_{q_*}} u\nu_j\varphi d\sigma \ \forall\varphi \in H_0^1(\Omega).\end{aligned}$$

Take $\varphi = \nu_j \Phi / \sum_{k=1}^3 \nu_k^2$ with $\Phi \in H_0^1(\Omega) \cap H^2(\Omega)$, then

$$\int_{\partial\Omega_{q_*}} u\Phi d\sigma = 0 \ \ \forall\Phi \in H_0^1(\Omega) \cap H^2(\Omega).$$

We deduce that $u = 0$ on $\partial\Omega_{q_*}$ and u is the solution of (3.6).

3) Let y_n be the solution of (3.7). From the estimate of Theorem 2.1 we obtain subsequences such that

$$\{y_n, y_n'\} \to \{y,\, y'\}$$

in

$$\{L^\infty(0,T; L^2(\Omega)))_{weak^*} \cap C(0,T; H^{-1}(\Omega))\} \times (L^\infty(0,T; H^{-1}(\Omega)))_{weak^*}.$$

It is trivial to check that y is the unique transpose solution of (3.7) and hence $\mathcal{A}x = y$ $\square$

PROOF OF THEOREM 3.1. Since $\mathcal{A}$ satisfies all the hypotheses of the Schauder fixed point theorem there exists $\hat{y}_k$ such that $\mathcal{A}\hat{y}_k = \hat{y}_k$ $\square$

4. Feedback Laws

We shall now give the feedback laws for (2.8).

THEOREM 4.1. *Suppose all the hypotheses of Theorem 2.1 are satisfied and let $V_k(\boldsymbol{\alpha}, 0)$ be as in Theorem 2.2. Then*

$$V_k(\boldsymbol{\alpha}, 0) = \int_0^T \int_{\tilde{\Omega}} \mid \hat{y}_k - \chi \mid d\xi dt$$

where $\hat{y}_k$ is a solution of (3.4) given by Theorem 3.1.

Proof. Let $\hat{y}_k$ be a solution of the nonlinear problem (3.4) given by Theorem 3.1.Let $v \in \mathcal{K}$ and consider the initial boundary problem

$$(4.1)\quad \begin{aligned} y_k'' - \Delta y_k + \psi_k((u_k+v)^+, a) &= f \text{ in } \Omega \times (t,T), \\ y_k = 0 \text{ on } \partial\Omega \times (t,T) \quad , &\quad \{y_k, y_k'\} \mid_{s=t} = \{\hat{y}_k(\,.,t),\, \hat{y}_k'(\,.,t)\} \end{aligned}$$

with

$$(4.2)\quad \begin{aligned} u_k' - \Delta u_k + gF_k((u_k+v)^+, a) &= 1 \text{ in } \Omega \times (t,T), \\ u_k = 0 \text{ on } \{\partial\Omega \cup \partial\Omega_a\} \times (t,T) \quad , &\quad u_k(\,.,t) = 0 \text{ in } \Omega \end{aligned}$$

where $a \in [a_0, \inf_[v \in \mathcal{K}(\max_{\overline{\Omega}} v)]$.

Since $\{\hat{y}_k,\, \hat{y}_k'\}$ is in $C(0,T;L^2(\Omega)) \times C(0,T;H^{-1}(\Omega))$ there exists a unique solution $\{y_k, u_k\}$ of (4.1)-(4.2). From Theorem 2.2 we get

$$V_k(\hat{y}_k,\, \hat{y}_k', t) = J(v_*, \{\hat{y}_k,\, \hat{y}_k'\},\, a_*, t) = \int_t^T \int_{\tilde{\Omega}} \mid y_* - \chi \mid d\xi ds$$

with $\{y_*, u_*, v_*, a_*\}$ being the solution of (4.1)-(4.2). The dynamic programming principle gives

$$\begin{aligned} V_k(\hat{y}_k,\, \hat{y}_k', t) &= \inf \Big\{ V_k(y_k,\, y_k', t+h) + \int_t^{t+h} \int_{\tilde{\Omega}} \mid y - \chi \mid d\xi ds \\ &= \forall \{y,u\} \text{ solution of} \, (4.1) - (4.2),\ \ \forall v \in \mathcal{K}, \\ &\quad \forall a \in [a_0,\, \inf_{v \in \mathcal{K}} (\max_{\overline{\Omega}} v)] \Big\} \end{aligned}$$

Thus,

$$\begin{aligned} V_k(\hat{y}_k,\, \hat{y}_k', t) &= J(v_*, \{\hat{y}_k,\, \hat{y}_k'\} \mid_{s=t}, t) \\ &\le J(\, v_*, \{\hat{y}_k, \hat{y}_k'\} \mid_{s=t}, a_*,\, t+h) \\ &+ \int_t^{t+h} \int_{\tilde{\Omega}} \mid y_* - \chi \mid d\xi ds \end{aligned}$$

From the definition of the cost function we deduce that

$$\int_{t+h}^T \int_{\tilde{\Omega}} \mid y_* - \chi \mid d\xi ds \le V_k(y_*(\,.,t+h), y_*'(\,.,t+h), t+h)$$

and hence

$$V_k(y_*(\,.,t+h), y_*'(\,.,t+h),\, t+h) = \int_{t+h}^{T}\int_{\tilde{\Omega}} \mid y_* - \chi \mid d\xi ds.$$

We have

$$V_k(y_*(\,.,t),\, y_*'(\,.,t),t) = V_k(\hat{y}_k(\,,t), \hat{y}_k'(\,.,t),t)$$
$$\text{as } \{\hat{y}_k, \hat{y}_k'\} \mid_{s=t} = \{y_*, y_*'\} \mid_{s=t} \text{ in } \Omega.$$

Therefore

$$\begin{aligned} &V_k(y_*(\,.,t+h), y_*'(\,.,t+h), t+h) - V_k(y_*(\,.,t),\, y_*'(\,.,t),t) \\ &\quad = -\int_{t}^{t+h}\int_{\tilde{\Omega}} \mid y_* - \chi \mid d\xi ds. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{d}{ds}\{V_k(y_*(\,.,s), y_*'(\,.,s), s)\} &= -\int_{\tilde{\Omega}} \mid y_*(\,.,t) - \chi \mid d\xi \\ &= -\int_{\tilde{\Omega}} \mid \hat{y}_k(\,.,t) - \chi \mid d\xi. \end{aligned}$$

An integration yields

$$V_k(y_*(\,.,0), y_*'(\,.,0), 0) = V_k(\boldsymbol{\alpha}, 0) = \int_0^T \int_{\tilde{\Omega}} \mid \hat{y}_k - \chi \mid d\xi dt$$

and the theorem is proved □

5. Main Results

In this section we shall prove the existence of the input control μ and its support. First we have

THEOREM 5.1. *Suppose all the hypotheses of Theorem 2.1 are satisfied and let $v \in \mathcal{K}$, then there exists a solution*

$$\{y,\, y',\, \mu\} \in C(0,T; L^2(\Omega)) \times C(0,T; H^{-1}(\Omega)) \times M_b(\Omega)$$

of the initial boundary problem

$$
(5.1) \qquad \begin{aligned} y'' - \Delta y + \mu = f \ \ in\ Q \quad &, \quad y = 0 \ \ on\ \partial\Omega \times (0,T), \\ \{y,\, y'\} \mid_{s=0} = \boldsymbol{\alpha}\,,\ <\mu, \varphi> \quad &= \quad \int_{\partial\Omega_a} g\varphi d\sigma \ \forall\varphi \in C_0(\Omega) \end{aligned}
$$

with $\partial\Omega_a = \{\xi : \xi \in \Omega,\ v(\xi) = a\}$. *Furthermore*

$$
\|y\|_{C(0,T;L^2(\Omega))} + \|y'\|_{C(0,T;H^{-1}(\Omega))} + \|\mu\|_{M_b(\Omega)} \leq C
$$

where C *is a constant independent of* v, a.

Remark. The key assertion of the theorem is that the estimate is independent of both v and a. It is crucial as we wish to consider μ as an input control.

The main result of the paper is the following theorem.

Theorem 5.2. *Suppose all the hypotheses of Theorem 2.1 are satisfied and let* $V_k(\boldsymbol{\alpha}, 0)$ *be as in Theorem 4.1*

$$
V_k(\boldsymbol{\alpha}, 0) = J(\hat{y}_k, \mathcal{P}p_*^k,\, \boldsymbol{\alpha}, q_*^k).
$$

Then

$$
\{\hat{y}_k,\, \hat{y}_k', p_*^k, q_*^k\} \to \{\hat{y},\, \hat{y}', \hat{p},\, \hat{q}\}
$$

in

$$
\begin{aligned} \{C(0,T;H^{-1}(\Omega)) \cap (L^\infty(0,T;L^2(\Omega)))_{weak^*}\} \quad &\times \quad (L^\infty(0,T;H^{-1}(\Omega)))_{weak} \\ &\times \quad (L^2(Q))_{weak} \times L^\infty(R) \end{aligned}
$$

with $\hat{q} \in [a_0, \inf_{v \in \mathcal{K}}(\max_{\overline{\Omega}} v)]$. *Moreover* $\{\hat{y}, \hat{y}',\, \hat{\mu}\}$ *is a solution of the initial boundary problem*

$$
(5.2) \qquad \begin{aligned} \hat{y}'' - \Delta\hat{y} + \hat{\mu} = f \ \ in\ Q \quad &, \quad \hat{y} = 0 \ \ on\ \partial\Omega \times (0,T), \\ \{\hat{y},\, \hat{y}'\} \mid_{t=0} = \boldsymbol{\alpha}\,,\ <\hat{\mu},\, \varphi> \quad &= \quad \int_{\partial\Omega_{\hat{q}}} g\varphi d\sigma \ \forall\varphi \in C_0(\Omega) \end{aligned}
$$

with $\partial\Omega_{\hat{q}} = \{\xi : \xi \in \Omega,\, (\mathcal{P}\hat{p}))\xi) = \hat{q}\}$. *Furthermore*

$$
V(\boldsymbol{\alpha}, 0) = \inf\{V_k(\boldsymbol{\alpha}, 0) : \forall k \geq k_0\} = \int_0^T \int_{\tilde{\Omega}} \mid \hat{y} - \chi \mid d\xi dt.
$$

Thus the unknown surface is $\partial\Omega_{\hat{q}}$.

PROOF OF THEOREM 5.1. 1) Let $\{y_k, u_k\}$ be as in Theorem 2.1. From the estimates of the theorem we obtain by taking subsequences

$$\{y_k, y_k', \psi_k((u_k+v)^+, a)\} \to \{y, y', \mu\}$$

in

$$\{C(0,T;H^{-1}(\Omega)) \cap (L^\infty(0,T;L^2(\Omega)))_{weak^*}\} \\ \times (L^\infty(0,T;H^{-1}(\Omega)))_{weak^*} \times \mathcal{D}'(\Omega)$$

with $\mu \in M_b(\Omega)$. It is clear that

$$y'' - \Delta y + \mu = f \text{ in } Q,\ y = 0 \text{ on } \partial\Omega \times (0,T), \{y, y'\} \mid_{t=0} = \boldsymbol{\alpha}.$$

2) We now show that $\operatorname{supp}(\mu) = \partial\Omega_a$.

• We prove that

$$\operatorname{supp}(\mu) \subset \{\xi : \xi \in \Omega,\ \mathcal{P}v(\xi) < a\}.$$

Let $\varphi \in C_0(\Omega)$ with $\operatorname{supp}(\varphi) \subset \{\xi : \xi \in \Omega, \mathcal{P}v(\xi) < a\}$. Since $\mathcal{P}v \in \mathcal{K}$, it belongs to $C^\lambda(\Omega)$ and $v_{sup} = \max_{\overline{\Omega}} \mathcal{P}v$ exists. Let

$$\eta > \{a - v_{sup}\}/2 > 0.$$

From Lemma 2.1 we get

$$u_k + \mathcal{P}v \to u + \mathcal{P}v \text{ in } C(0,T;L^2(\Omega)).$$

Thus we have for all t and almost all ξ

$$(u + \mathcal{P}v)(\xi,t) - \eta \le (u_k + \mathcal{P}v)(\xi,t) \le (u + \mathcal{P}v)(\xi,t) + \eta.$$

Since $u + \mathcal{P}v = u^+ + \mathcal{P}v - u^-$, we get $(u + \mathcal{P}v)^+ = u^+ + \mathcal{P}v$ and we have

$$\begin{aligned} &\{\xi : \xi \in \operatorname{supp}(\varphi),\ 0 < u + \mathcal{P}v < a + k^{-1}\} \\ &\subset \{\xi : \xi \in \operatorname{supp}(\varphi), 0 < u_k + \mathcal{P}v \le a + k^{-1} - \{a - v_{sup}\}/2\} \\ &\subset \{\xi : \xi \in \Omega, 0 < u_k + \mathcal{P}v \le k^{-1} + \{a + v_{sup}\}/2\} \\ &\subset \{\xi : \xi \in \Omega,\ u_k + \mathcal{P}v \le a - k^{-1}\} \end{aligned}$$

with $0 < 4k^{-1} < a - v_{sup}$.

We have used the fact that $-\eta < -\{a+v_{sup}\}/2$ in the above calculations. By construction

$$F_k(s,a) = 0 \text{ for } s < a - k^{-1}$$

and hence

$$\int_\Omega \psi_k((u_k + \mathcal{P}v)^+, a)\varphi d\xi = \int_0^T \int_\Omega F_k((u_k + \mathcal{P}v)^+, a)\varphi d\xi dt = 0$$

It follows that

$$< \mu, \varphi >= 0 \ \forall \varphi \in C_0(\Omega), \ \mathrm{supp}(\varphi) \subset \big\{\xi : \xi \in n\Omega, \ \mathcal{P}v < a\big\}$$

and we get

$$\mathrm{supp}(\mu) \subset \big\{\xi : \xi \in \Omega, \mathcal{P}v > a\big\}.$$

- Let $\varphi \in C_0(\Omega)$, $\mathrm{supp}(\varphi) \subset \big\{\xi : \xi \in \Omega, \ \mathcal{P}v > a\big\}$. Set

$$v_{inf} = \min\big\{\mathcal{P}v(\xi) : \xi \in \mathrm{supp}(\varphi)\big\} \geq a - \varepsilon.$$

We have for all t and almost all ξ

$$\begin{aligned}
&\big\{\xi : \xi \in \mathrm{supp}(\varphi), \ u + \mathcal{P}v > v_{inf} - k^{-1}\big\} \\
&\subset \big\{\xi : \xi \in \mathrm{supp}(\varphi), \ u_k + \mathcal{P}v > (a + v_{inf})/2 - k^{-1}\big\} \\
&\subset \big\{\xi : \xi \in \Omega, \ u_k + \mathcal{P}v > a + k^{-1}\big\}
\end{aligned}$$

for $2k^{-1} < v_{inf} - a$. From the definition of F_k we obtain

$$\int_\Omega \psi_k((u_k + \mathcal{P}v)^+, a)\varphi d\xi = \int_\Omega \int_0^T gF_k((u_k + \mathcal{P}v)^+, a)g\varphi d\xi dt = 0.$$

Therefore $\mathrm{supp}(\mu) \subset \big\{\xi : \xi \in \Omega, \mathcal{P}v < a\big\}$. Combining the two parts and we obtain

$$\mathrm{supp}(\mu) \subset \big\{\xi : \xi \in \Omega, \ \mathcal{P}v = a\big\}.$$

• Suppose that $S = \{\xi : \xi \in \Omega, \mathcal{P}v = a\}/\text{supp}(\mu)$ is non-empty. Let φ be in $C_0(\Omega)$ with $\text{supp}(\varphi) \subset S$. Then

$$\begin{aligned} < \mu, \varphi >= 0 &= \lim_k \int_0^T \int_\Omega gF_k((u_k + \mathcal{P}v)^+, a)\varphi d\xi dt \\ &= \int_{\{\xi:\, \xi\in\Omega, u+\mathcal{P}v=a\}} g\varphi d\sigma \end{aligned}$$

and we have a contradiction. It is clear that

$$< \mu, \varphi >= \int_{\partial\Omega_a} g\varphi d\sigma \ \ \forall \varphi \in C_0(\Omega)$$

and the theorem is proved □

PROOF OF THEOREM 5.2. 1) Let $\{\hat{y}_k,\, \hat{y}'_k, p^k_*, q^k_*\}$ be as in Theorem 4.1. From the estimates of the theorem we obtain a subsequence such that

$$\{\hat{y}_k,\, \hat{y}'_k,\, p^k_*, q^k_*\} \to \{\hat{y},\, \hat{y}', p_*, q_*\}$$

in

$$\{C(0,T; H^{-1}(\Omega)) \cap (L^\infty(0,T; L^2(\Omega)))_{weak^*}\} \times (L^\infty(0,T; H^{-1}(\Omega)))_{weak^*} \\ \times (L^2(Q))_{weak} \times L^\infty(R).$$

Furthermore

$$\psi_k((u_k + \mathcal{P}p^k_*)^+, q^k_*) \to \hat{\mu} \ \text{ in } \mathcal{D}'(\Omega)$$

with $\hat{\mu} \in M_b(\Omega)$ and $q_* \in [a_0, \inf_{v\in\mathcal{K}}(\max_{\overline{\Omega}} v)]$. A proof as done in Theorem 5.1 shows that

$$(5.3) \qquad \begin{aligned} &\hat{y}'' - \Delta\hat{y} + \hat{\mu} = f \text{ in } Q \quad , \quad \hat{y} = 0 \text{ on } \partial\Omega \times (0,T), \\ &\{\hat{y},\, \hat{y}'\}\mid_{t=0} = \boldsymbol{\alpha} \quad , \quad < \hat{\mu},\, \varphi >= \int_{\partial\Omega_{q*}} g\varphi d\sigma \end{aligned}$$

for all $\varphi \in C_0(\Omega)$ with $\partial\Omega_{q_*} = \{\xi : \xi \in \Omega,\, \mathcal{P}p_* = q_*\}$.

2) From Theorem 4.1 we get

$$V_k(\boldsymbol{\alpha}) = \int_0^T \int_{\tilde{\Omega}} \mid \hat{y}_k - \chi \mid d\xi dt$$

and thus,

$$V(\boldsymbol{\alpha}) = \inf\big\{V_k(\boldsymbol{\alpha}) :, \forall k\big\} \geq \int_0^T \int_{\tilde{\Omega}} \mid \hat{y} - \chi \mid d\xi dt.$$

On the other hand

$$V_k(\boldsymbol{\alpha}) \leq J(y_k, v, \boldsymbol{\alpha}, a) \;\; \forall v \in \mathcal{K},\, \forall a \in [a_0,\, \inf_{v\in\mathcal{K}}(\max_{\overline{\Omega}} v)]$$

where $\big\{y_k, v, a\big\}$ is the solution of (2.8) given by Theorem 2.1. From the estimate of Theorem 2.1 and as in Theorem 5.1 we have

$$\big\{y_k, y_k', \psi_k((u_k + v)^+, a)\big\} \to \big\{y, y', \mu\big\}$$

in

$$\begin{aligned} &\{(L^\infty(0,T;L^2(\Omega)))_{weak^*} \cap C(0,T;H^{-1}(\Omega))\} \\ &\quad \times (L^\infty(0,T;H^{-1}(\Omega)))_{weak^*} \times \mathcal{D}'(\Omega) \end{aligned}$$

and $\big\{y,\, \mu\big\}$ is a solution of (5.1). We have

$$\begin{aligned} V(\boldsymbol{\alpha}) &\leq V_k(\boldsymbol{\alpha}) \;\; \forall k \\ &\leq J(y, v, \boldsymbol{\alpha}, a) \; \forall v \in \mathcal{K},\, \forall a \in [a_0,\, \inf_{v\in\mathcal{K}}(\max_{\overline{\Omega}} v)]. \end{aligned}$$

Hence

$$\begin{aligned} V(\boldsymbol{\alpha}) &= \inf\big\{J(y, v, \boldsymbol{\alpha}, a) :\; \{y,\, \mu\} \text{ solution of (5.1)}, \\ &\quad \forall v \in \mathcal{K}, \forall a \in [a_0,\, \inf_{v\in\mathcal{K}}(\max_{\overline{\Omega}} v)]\big\}. \end{aligned}$$

It follows that

$$\begin{aligned} V(\boldsymbol{\alpha}) &= J(\hat{y}, \mathcal{P}p_*, \boldsymbol{\alpha}, q_*) \\ &= \inf\big\{J(y, v, \boldsymbol{\alpha}, a) :\; \{y,\, \mu\} \text{ solution of (5.1) } \forall v \in \mathcal{K}, \\ &\quad \forall a \in [a_0,\, \inf_{v\in\mathcal{K}}(\max_{\overline{\Omega}} v)]\big\} \end{aligned}$$

and the theorem is proved □

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