J. Math. Sci. Univ. Tokyo
16 (2009), 461–500.

A Construction of Pure Solutions for Degenerate Hyperbolic Operators

By Yasuo Chiba

Abstract. For weakly hyperbolic operators whose characteristic roots degenerate only on the initial hypersurface, we construct solutions whose singularities are only just one of the characteristic roots.

1. Introduction

We consider the following analytic ordinary differential equation on $\mathbb{R}$ with a large parameter τ which has an m-th turning point at $x = 0$:

$$P(x, \partial_x, \tau)u(x, \tau) = \left(\sum_{j=0}^{m} a_j(x, \tau)\partial_x^{m-j} \right) u(x, \tau) = 0, \tag{1.1}$$

where $\partial_x = d/dx$, $a_j(x, \tau) = \sum_{k=0}^{j} a_{jk}(x)\tau^k$, $a_{00}(x) = 1$ and each $a_{jk}(x)$ ($k = 0, 1, \cdots, j$; $j = 1, \cdots, m$) is analytic in a neighborhood of $x = 0$.

The principal symbol $\sigma(P)$ of this operator P is decomposed as

$$\sigma(P)(x, \xi, \tau) = \prod_{j=1}^{m} (\xi - \sqrt{-1}x^\lambda \alpha_j(x)\tau) \tag{1.2}$$

at the origin. Here λ is a positive integer and each $\alpha_j(x)$ ($j = 1, 2, \cdots, m$) is analytic in a neighborhood of the origin $x = 0$.

For such an equation with a large parameter, exact WKB analysis is a prevailing method ([AKKT] etc). However, we will construct solutions for such an equation according to microlocal analysis, that is, regarding a large parameter τ as an operator $\partial_t (= \partial/\partial t)$, we consider the following partial differential equation with two variables on $\mathbb{R}_x \times \mathbb{R}_t$:

$$P(x, \partial_x, \partial_t)u(x, t) = \left(\sum_{j=0}^{m} a_j(x, \partial_t)\partial_x^{m-j} \right) u(x, t) = 0. \tag{1.3}$$

2000 *Mathematics Subject Classification.* Primary 35A27; Secondary 35L80, 35S99.

We impose this equation on some hypotheses.

Assumption 1 (Hyperbolicity). *Each* $\alpha_j(x)$ $(j = 1, 2, \cdots, m)$ *is a purely imaginary-valued function on* x *and* $\alpha_j(0)$ *are mutually distinct.*

Assumption 2 (Levi condition). *For* $k = 0, 1, \cdots, j$*;* $j = 1, 2, \cdots, m$*,*

$$\partial_x^s a_{jk}(0) = 0, \quad 0 \leq s < k(\lambda + 1) - j. \tag{1.4}$$

For the equation (1.3), microlocal solutions of boundary value problems on $\{x \geq 0\}$ are said to be microfunctional solutions on $x > 0$ whose Neumann data $\partial_x^k u(+0, t)$ $(k = 0, 1, \cdots, m-1)$ become microfunctions. Our aim is to construct microfunctional solutions for a boundary value problem as follows:

$$\begin{cases} P(x, \partial_x, \partial_t) u(x, t) = 0, & 0 < x < \varepsilon, |t| < \varepsilon, \\ \mathrm{SS}(u) \cap \{x > 0\} \subset H_j, & (*) \end{cases} \tag{1.5}$$

where $H_j := \{(x, *; \sqrt{-1}\xi, \sqrt{-1}\tau); \xi - \sqrt{-1}x^\lambda \alpha_j(x)\tau = 0\}$. After [Y], we call $u(x, t)$ satisfying the condition $(*)$ j-pure solutions. In other words, our main goal is to construct j-pure solutions concretely and to get boundary values $\partial_x^k u(+0, t)$ $(k = 0, 1, \cdots, m-1)$.

There are many researches about such hyperbolic equations, for example, Alinhac [Al], Takasaki [T], Amano–Nakamura [AN], etc. Among them, Amano–Nakamura [AN] is the closest to ours. They analyze the Stokes phenomena of complex ordinary differential equations for Cauchy problems around $x = 0$ and construct asymptotic solutions. In the case of order $m = 2$, they succeed to make solutions by essentially using second order ordinary differential equations with irregular singularities.

Yamane [Y] treats a concrete example of $m = 3$, $\lambda = 1$. Uchikoshi [U] gives some solvable conditions for boundary value problems in the case that the index λ varies with j and that the Levi condition is not assumed on the operator P. His construction needs pseudodifferential operators, while the solutions are not j-pure.

As for ours, in the case of general order m, not using the Stokes analysis of complex differential equations, we construct solutions with a fractional coordinate transform and a quantized Legendre transform based on Kataoka [Kt3].

Specifically speaking, our construction is as follows.

To begin with, u is represented by a natural hyperfunction $\widetilde{u}(x,t)$ with $\operatorname{supp}(u) \subset \{x \geq 0\}$. We identify $\widetilde{u}(x,t)$ as a solution of the equation

$$x^m P\widetilde{u}(x,t) = 0$$

in a neighborhood of $x = 0$.

Secondly, by a fractional coordinate transform

$$y = \frac{x^{\lambda+1}}{\lambda+1},$$

$\widetilde{u}(x,t)$ corresponds to a solution $v(y,t)$ of the equation

$$Q(y, \partial_y, \partial_t)v(y,t) = 0,$$

where Q is a partial differential operator whose coefficients have fractional power singularities with respect to y and $v(y,t)$ is a microfunction which is represented by a hyperfunction with support in $\{y \geq 0\}$.

Lastly, we modify the operator Q by the quantized Legendre transform at $\{\tau > 0\}$ (τ is a dual of t):

$$\beta \circ * \circ \beta^{-1} : \begin{cases} \partial_y \mapsto -\sqrt{-1}w\partial_t, & \partial_t \mapsto \partial_t, \\ y \mapsto -\sqrt{-1}\partial_w(\partial_t)^{-1}, & t \mapsto t + \partial_w w(\partial_t)^{-1}. \end{cases} \tag{1.6}$$

Then $\beta[v](w,t)$ becomes a microfunction with a holomorphic parameter w which can be analytically extended to a domain $\{w \in \mathbb{CP}^1; \operatorname{Im} w < 0\}$. On the other hand, the operator Q is transformed to

$$\begin{aligned} &(1.7) \quad (\beta \circ Q \circ \beta^{-1})(w, \partial_w, \partial_t) \\ &= \sum_{j=0}^{m} \sum_{k=0}^{j} \{-\sqrt{-1}(\lambda+1)\partial_w\partial_t^{-1}\}^{\frac{j}{\lambda+1}} a_{jk} \left(\{-\sqrt{-1}(\lambda+1)\partial_w\partial_t^{-1}\}^{\frac{1}{\lambda+1}}\right) \partial_t^k \\ &\qquad \times \prod_{l=0}^{m-j-1} (-(\lambda+1)\partial_w w - l)\,. \end{aligned}$$

Here we note that the operator $\beta \circ Q \circ \beta^{-1}$ has fractional powers of ∂_w. Such an operator is not defined locally for the sections of microfunctions with holomorphic parameter w, but is defined as an integral operator concerning w; here we take a path of integration with $w = \infty$ as the initial point. More precisely, we consider only sections of microfunctions with holomorphic parameter w over open sets with connected fibers in $\mathbb{CP}^1$ up to $w = \infty$. Moreover, an admissible microfunction $\beta[v](w,t)$ for these operators should have the following form at $w = \infty$:

$$\beta[v](w,t) = w^{-1} V(w^{-\frac{1}{\lambda+1}}, t),$$

where $V(z,t)$ is a microfunction with respect to (z,t) which is holomorphic at $z = 0$. Hence a fractional derivative $\partial_w^{1/(\lambda+1)}$ naturally operates on a global section of such a microfunction with a holomorphic parameter. For instance, in a neighborhood of $w = \infty \in \mathbb{CP}^1$, we have

$$\partial_w^{\frac{m}{\lambda+1}}(w^{-1-\frac{l}{\lambda+1}} f(t)) = e^{\frac{m\pi\sqrt{-1}}{\lambda+1}} \frac{\Gamma(1 + (l+m)/(\lambda+1))}{\Gamma(1 + l/(\lambda+1))} w^{-1-\frac{l+m}{\lambda+1}} f(t)$$

for $\lambda = 1, 2, \cdots$, $l, m = 0, 1, 2, \cdots$, where $\Gamma(\cdot)$ is a gamma function. Indeed this formula reduces a usual one when $m/(\lambda+1)$ is a positive integer.

Furthermore, in Section 6 we will show that the Taylor coefficients $(\partial_z^n V(0,t))_{n=0}^{m-1}$ for $\beta[v](w,t)$ give the boundary values $(\partial_x^n u(+0,t))_{n=0}^{m-1}$ at (1.5).

By the Taylor expansion of $a_{jk}(x)$ at $x = 0$, what we consider becomes

$$\beta \circ Q \circ \beta^{-1} = \sum_{s=0}^{\infty} \sum_{\substack{0 \le j \le m \\ 0 \le k \le (s+j)/(\lambda+1)}} \frac{a_{jk}^{(s)}(0)}{s!} \{-\sqrt{-1}(\lambda+1)\}^{\frac{s+j}{\lambda+1}} \partial_w^{\frac{s+j}{\lambda+1}} \partial_t^{k-\frac{s+j}{\lambda+1}} \times \prod_{l=0}^{m-j-1} (-(\lambda+1)\partial_w w - l).$$

For the sake of brevity, setting

$$\widetilde{a}_{jk}(x) = \sum_{l=0}^{\infty} \frac{a_{jk}^{(l)}(0)}{l!} \{-\sqrt{-1}(\lambda+1)\}^{\frac{l+j}{\lambda+1}} x^{j+l-(\lambda+1)k}$$

$$= \sum_{l'=0}^{\infty} \widetilde{a}_{jk}^{l'} x^{l'}$$

with $\widetilde{a}_{jk}^{l'} = (-\sqrt{-1}(\lambda+1))^{\frac{(\lambda+1)k+l'}{\lambda+1}} a_{jk}^{(l'+(\lambda+1)k-j)}(0)/(l'+(\lambda+1)k-j)!$ and

$$E_j = \prod_{l=0}^{m-j-1} \left(-(\lambda+1)\partial_w w - l\right),$$

we have

$$\beta \circ Q \circ \beta^{-1} = \sum_{\substack{l' \geq 0 \\ 0 \leq k \leq j \leq m}} \widetilde{a}_{jk}^{l'} \partial_w^{\frac{l'}{\lambda+1}+k} \partial_t^{-\frac{l'}{\lambda+1}} E_j.$$

The dominant part of $\beta \circ Q \circ \beta^{-1}$ becomes an m-th ordinary differential operator with polynomial coefficients which does not include neither ∂_t nor fractional derivatives:

$$\begin{aligned} L &= \sum_{\substack{0 \leq j \leq m \\ j/(\lambda+1) \leq k \leq j}} \frac{a_{jk}^{(k(\lambda+1)-j)}(0)}{\{k(\lambda+1)-j\}!} \{-\sqrt{-1}(\lambda+1)\partial_w\}^k E_j \\ &= \sum_{0 \leq k \leq j \leq m} \widetilde{a}_{jk}^{0} \partial_w^k E_j. \end{aligned}$$

The coefficients of the maximal term of the operator L is equal to

$$(\text{constant}) \cdot \prod_{j=1}^{m} (w + \alpha_j(0)).$$

Therefore the operator $\beta \circ Q \circ \beta^{-1}$ has regular singularities at

$$w = -\alpha_1(0), -\alpha_2(0), \cdots, -\alpha_m(0), \infty.$$

Assumption 3. *The characteristic exponents of the operator L at each $w = -\alpha_j(0)$ $(j = 1, 2, \cdots, m)$ are not integers.*

Under the assumptions above, we have the following main theorems.

Main Theorem 1. *We can construct j-pure solutions for* (1.5)*. Precisely, for any $j = 1, \cdots, m$ and any microfunction $u_0(t)$ at a point $\overset{\circ}{p} = (0, 0; \pm\sqrt{-1}) \in \mathbb{R}_x \times \sqrt{-1}T^*\mathbb{R}_t$, we have a unique mild microfunction solution $u(x,t) \in \overset{\circ}{\mathscr{C}}_{\{x=0\}|\{x\geq 0\}}$ of a microlocal boundary value problem at $\overset{\circ}{p}$:*

$$(1.8)\qquad \begin{cases} P(x, \partial_x, \partial_t)u(x,t) = 0, \quad x > 0 \ \textit{(in the sense of } \overset{\circ}{\mathscr{C}}_{\{x=0\}|\{x\geq 0\}}\textit{)}, \\ u(+0, t) = u_0(t), \\ \operatorname{supp}(\operatorname{ext}(u)(x,t)) \cap \{x > 0\} \\ \qquad \subset \{(x, t; \sqrt{-1}(\xi, \tau)); \xi - \sqrt{-1}x^\lambda \alpha_j(x)\tau = 0\}. \end{cases}$$

Further, we have the equations

$$\partial_x^k u(+0, t) = R_{jk}(\partial_t)u_0(t)$$

($j = 1, 2, \cdots, m; k = 0, 1, 2, \cdots, m-1$), where $R_{jk}(\partial_t)$ is a microdifferential operator with fractional order at most $k/(\lambda+1)$.

Here $\overset{\circ}{\mathscr{C}}_{\{x=0\}|\{x\geq 0\}}$ is a sheaf on $\{x = 0\} \times \sqrt{-1}T^*\mathbb{R}_t$ of mild microfunctions ([Kt1]) and $\operatorname{ext} : \overset{\circ}{\mathscr{C}}_{\{x=0\}|\{x\geq 0\}} \ni u(x,t) \mapsto u(x,t)Y(x) \in \mathscr{C}_{\mathbb{R}_x\times\mathbb{R}_t}$ is the canonical extension to $x \leq 0$.

For an arbitrary solution u of the boundary value problem, we obtain the following theorem as an application of Main Theorem 1.

Main Theorem 2. *An arbitrary solution $u(x,t) \in \overset{\circ}{\mathscr{C}}_{\{x=0\}|\{x\geq 0\}}$ of the boundary value problem at a point* $(0, 0; \pm\sqrt{-1})$

$$(1.9)\qquad P(x, \partial_x, \partial_t)u(x,t) = 0, \quad x > 0 \ \textit{(in the sense of } \overset{\circ}{\mathscr{C}}_{\{x=0\}|\{x\geq 0\}}\textit{)}$$

can be uniquely decomposed as a sum of j-pure solutions.

Example 1.1. In the case that

$$P = \partial_x^2 - x^2\partial_t^2,$$

the operator P is transformed to $\beta \circ Q \circ \beta^{-1} = (w^2+1)\partial_w^2 + (7/2)w\partial_w + 3/2$, which is the Gauß hypergeometric operator. Furthermore, the operator

$$P = \partial_x^3 - x^2\partial_t^2\partial_x + 2(a-b)\partial_t\partial_x + \{2(a+b) - 3\}x\partial_t^2$$

becomes the Jordan–Pochhammer operator

$$\beta \circ Q \circ \beta^{-1} = (w^3 + w)\partial_w^3 + \left\{ \frac{15}{2} w^2 - \sqrt{-1}(a-b)w + a + b + \frac{3}{2} \right\} \partial_w^2 \\ + \{12w - 2\sqrt{-1}(a-b)\}\partial_w + 3$$

(a and b satisfy suitable conditions). See more details in [Y].

2. Preliminaries

In this section, we construct a subsheaf $\mathscr{C}\mathscr{O}_+^\infty$ of a sheaf $\mathscr{C}\mathscr{O}$ of microfunctions with holomorphic parameters. Furthermore, we define fractional derivatives and microdifferential operators with fractional filtration in advance.

2.1. Microfunctions

Let $\mathbb{R}_x \times \mathbb{R}_t \subset \mathbb{R}_x \times \mathbb{C}_r \subset \mathbb{C}_z \times \mathbb{C}_r$ with coordinates $z = x + \sqrt{-1}\widetilde{x} \in \mathbb{C}$ and $r = t + \sqrt{-1}\,\widetilde{t} \in \mathbb{C}$.

To start with, we define a sheaf of hyperfunctions with holomorphic parameters and of microfunctions with holomorphic parameters respectively by

$$\mathscr{B}_x\mathscr{O}_r := \mathscr{H}^1_{\mathbb{R}_x \times \mathbb{C}_r}(\mathscr{O}_{\mathbb{C}_z \times \mathbb{C}_r}) \otimes or,$$
$$\mathscr{C}_x\mathscr{O}_r := \mu_{\mathbb{R}_x \times \mathbb{C}_r}(\mathscr{O}_{\mathbb{C}_z \times \mathbb{C}_r})[1] \otimes or,$$

where $\mathscr{O}$ is a sheaf of holomorphic functions, $\mu_{\mathbb{R}_x \times \mathbb{C}_r}(\cdot)$ is a microlocalization functor and or is an orientation sheaf defined in Kashiwara–Schapira [KS]. We often denote $\mathscr{B}\mathscr{O}$ and $\mathscr{C}\mathscr{O}$ instead of $\mathscr{B}_x\mathscr{O}_r$ and $\mathscr{C}_x\mathscr{O}_r$ respectively.

Analytically speaking, the sheaf $\mathscr{C}\mathscr{O}$ is represented as follows:

$$\mathscr{C}_x\mathscr{O}_r = \{f(t, \widetilde{t}, x) \in \mathscr{C}_{\mathbb{R}_x \times \mathbb{R}_t \times \mathbb{R}_{\widetilde{t}}}; \overline{\partial}_r f = 0\},$$

where $\mathscr{C}$ is a sheaf of microfunctions.

We can show the isomorphism of the sheaf of microfunctions under the quantized Legendre transform. This fact leads to the legitimacy of the transform of the operator considered in Section 1. The following proposition is valid ([SKK], [Kt1]).

PROPOSITION 2.1. *Let W^ε be a set $\{(y,t;\sqrt{-1}\eta,\sqrt{-1}\tau) \in \sqrt{-1}T^*(\mathbb{R}_y \times \mathbb{R}_t); \varepsilon\tau > 0\}$ for $\varepsilon = \pm 1$. We get the following isomorphism as to the sheaf of microfunctions:*

$$\beta^\varepsilon : \mathscr{C}_{\mathbb{R}_y \times \mathbb{R}_t}|_{W^\varepsilon} \xrightarrow{\sim} (b^\varepsilon)^{-1}\mathscr{C}_{\mathbb{R}_w \times \mathbb{R}_s}|_{W^\varepsilon},$$

where b^ε is a contact transform associated with β^ε. Then the induced isomorphism of pseudodifferential operators are given as the following quantized contact transform:

$$\begin{cases} \beta^\varepsilon \circ \partial_y \circ (\beta^\varepsilon)^{-1} = -\sqrt{-1}\varepsilon w\partial_s, & \beta^\varepsilon \circ y \circ (\beta^\varepsilon)^{-1} = -\sqrt{-1}\varepsilon\partial_w(\partial_s)^{-1}, \\ \beta^\varepsilon \circ \partial_t \circ (\beta^\varepsilon)^{-1} = \partial_s, & \beta^\varepsilon \circ t \circ (\beta^\varepsilon)^{-1} = s + \partial_w w(\partial_s)^{-1}. \end{cases}$$

REMARK 2.2. In this paper, we only treat $\varepsilon = +1$. We therefore abbreviate β^{+1} to β. Furthermore, since terms with respect to s do not appear in our equation, we regard $s = t$.

For the reduced microdifferential equation in Section 1, we shall construct solutions which are microfunctions with holomorphic parameters. Because of Proposition 2.1, it is possible to make such solutions. Precisely explaining, we show it as follows.

Set spaces

$$V = \mathbb{CP}^1_w \times \mathbb{R}_t \subset \mathscr{V} = \mathbb{CP}^1_w \times \mathbb{C}_r$$

and a projection α as

$$\widetilde{V} = T^*_V\mathscr{V} = \mathbb{CP}^1_w \times \sqrt{-1}T^*\mathbb{R}_t \ni (w,t;\sqrt{-1}\tau) \\ \overset{\alpha}{\mapsto} (t;\sqrt{-1}\tau) \in \sqrt{-1}T^*\mathbb{R}_t,$$

where $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. Then the sheaf $\mathscr{C}\mathscr{O}_V$ of microfunctions with a holomorphic parameter w is a sheaf on $\widetilde{V}$ of relative microfunctions:

$$\mathscr{C}\mathscr{O}_V = \mathscr{C}_{V|\mathscr{V}} (= \mu_V(\mathscr{O}_\mathscr{V})[1] \otimes or).$$

Hence a function $\beta^\varepsilon[v](w,t)$ is regarded as a section in $\Gamma(\alpha^{-1}(W') \cap \{\operatorname{Im} w < 0\}; \mathscr{C}\mathscr{O}_V)$ with a small neighborhood W' of $(0;\sqrt{-1}\overset{\circ}{\tau})$ in $\sqrt{-1}T^*\mathbb{R}_t$.

However there is an essential difficulty in treating sections of $\mathscr{C}\mathbb{O}_V$. A section $g(w,t) \in \Gamma(\alpha^{-1}(W') \cap \{\operatorname{Im} w < 0\}; \mathscr{C}\mathbb{O}_V)$ does not always have a boundary value

$$g(u - \sqrt{-1}0, t) \in \Gamma(\alpha^{-1}(W') \cap \{\operatorname{Im} w = 0\}; q^{-1}\mathscr{C}_{\mathbb{S}^1_w \times \mathbb{R}_t}),$$

where $\mathbb{S}^1_w = \mathbb{CP}^1_w \cap \{\operatorname{Im} w = 0\}$ and

$$q : \widetilde{V} \cap \{\operatorname{Im} w = 0\} \ni (u, t; \sqrt{-1}\tau) \hookrightarrow (u, t; 0, \sqrt{-1}\tau) \in \sqrt{-1}T^*(\mathbb{S}^1 \times \mathbb{R}_t).$$

We therefore give a definition with respect to microfunctions with parameters so as to make them have boundary values.

Definition 2.3. A section

$$g(w,t) \in \Gamma\left(\{(w, t; \sqrt{-1}\tau); \operatorname{Im} w < 0, |w - \mathring{u}| + |t - \mathring{t}| + |\tau - \mathring{\tau}| < \delta\}; \mathscr{C}\mathbb{O}_V\right)$$

is said to have a boundary value at $(\mathring{u}, \mathring{t}; \sqrt{-1}\mathring{\tau})$ if and only if there exists a section

$$G(w,t) \in \Gamma\left(\{(w,t); \operatorname{Im} w < 0, |w - \mathring{u}| + |t - \mathring{t}| < \delta'\}; \mathscr{B}\mathbb{O}_V\right)$$

with a smaller $\delta' > 0$ than δ such that we have $[G(w,t)] = g(w,t)$ in a domain

$$\{(w, t; \sqrt{-1}\tau); \operatorname{Im} w < 0, |w - \mathring{u}| + |t - \mathring{t}| + |\tau - \mathring{\tau}| < \delta'\}.$$

Further, it is well-known that such a boundary value $g(u - \sqrt{-1}0, t)$ defined by $[G(u - \sqrt{-1}0, t)]$ is uniquely determined only by a section $g(w,t)$.

Thus a section $\beta[v](w,t) =: g(w,t)$ is regarded as a section of $\Gamma(\alpha^{-1}(W') \cap \{\operatorname{Im} w < 0\}; \mathscr{C}\mathbb{O}_V)$ with a boundary value on $\mathbb{S}^1_w \times \{(\mathring{t}; \sqrt{-1}\mathring{\tau})\}$.

Moreover, we define a sheaf $\mathscr{C}\mathbb{O}^\infty_+$ of microfunctions with holomorphic parameters with boundary values, which plays an important role in the following sections.

Set the following spaces as follows;

$$
\begin{aligned}
V &= \mathbb{CP}^1 \times \mathbb{R}_t = \{(w,t) \in (\mathbb{C} \cup \{\infty\}) \times \mathbb{R}\},\\
V_+ &= \mathbb{CP}^1_+ \times \mathbb{R}_t = \{(w,t) \in V; \mathrm{Re}\, w \geq 0 \text{ or } w = \infty\},\\
\widetilde{V} &= \mathbb{CP}^1 \times \sqrt{-1}\dot{T}^*\mathbb{R}_t\\
&\qquad = \{(w,t;\sqrt{-1}\tau) \in (\mathbb{C} \cup \{\infty\}) \times \mathbb{R} \times (\sqrt{-1}\mathbb{R}\backslash\{0\})\},\\
\widetilde{V_+} &= \mathbb{CP}^1_+ \times \sqrt{-1}\dot{T}^*\mathbb{R}_t = \{(w,t;\sqrt{-1}\tau) \in \widetilde{V}; \mathrm{Re}\, w \geq 0 \text{ or } w = \infty\}.
\end{aligned}
$$

Furthermore, we set the morphisms as follows:

$$
\begin{tikzcd}
\widetilde{V_+} \arrow[r, hook] \arrow[d, "\pi"'] & \widetilde{V} \arrow[d, "\pi"] \\
V_+ \arrow[r, hook] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Int}(\widetilde{V_+}) \arrow[r, hook, "\lambda"] \arrow[d, "\pi"'] & \widetilde{V_+} \arrow[d, "\pi"] \\
\mathrm{Int}(V_+) \arrow[r, hook, "\lambda"'] & V_+,
\end{tikzcd}
$$

where $\mathrm{Int}(\widetilde{V_+}) = \widetilde{V_+} \cap \{\mathrm{Re} w > 0, w \neq \infty\}$ and $\mathrm{Int}(V_+) = V_+ \cap \{\mathrm{Re} w > 0, w \neq \infty\}$.

Then a sheaf $\mathscr{C}\mathbb{O}^\infty_+$ on $\widetilde{V_+}$ is defined by

$$
\mathscr{C}\mathbb{O}^\infty_+ := Im\left(\pi^{-1}\lambda_*(\mathscr{B}\mathbb{O}|_{\mathrm{Int}(V_+)}) \to \lambda_*(\mathscr{C}\mathbb{O}|_{\mathrm{Int}(\widetilde{V_+})})\right), \tag{2.1}
$$

where $Im(*)$ stands for the image. A sheaf $\mathscr{C}\mathbb{O}^\infty_+$ coincides with a sheaf $\mathscr{C}\mathbb{O}$ on $\mathrm{Int}(\widetilde{V_+})$, but we have $\mathscr{C}\mathbb{O}^\infty_+ \subsetneq \lambda_*(\mathscr{C}\mathbb{O}|_{\mathrm{Int}(\widetilde{V_+})})$ on $\partial\widetilde{V_+}$. As a matter of fact, the sections of $\mathscr{C}\mathbb{O}^\infty_+$ have boundary values.

In conclusion, we emphasize that a section of $\mathscr{C}\mathbb{O}$ doesn't have a boundary value in the defined domain but have it in a smaller domain. We however consider our equation with respect to $\beta[v](w,t)$, where $v(w,t)$ is a section of $\mathscr{C}\mathbb{O}_V$ (details are due to Kataoka [Kt3]).

2.2. Definition of fractional derivatives

Fractional derivation calculus is studied by many mathematicians used by the Riemann–Liouville integral (referred to Oldham–Spanier [OS], Samko–Kilbas–Marichev [SKM]). We later extend formal norms, which are introduced by Boutet de Monvel and Krée [BK], to the case of fractional derivation.

We define a derivation of fractional order $\alpha = p/q$ ($p, q \in \mathbb{N}$, $0 < \alpha < 1$) as the Riemann–Liouville integral for a function $f(w)$ which is holomorphic in $\{w \in \mathbb{C}; \mathrm{Re}w > c\}$ for some $c \in \mathbb{R}$ with a finite limit $\lim_{\mathrm{Re}w \to +\infty} f(w)$. Concretely, we define

$$\left(\frac{d}{dw}\right)^{\alpha} f(w) := \frac{\Gamma(1+\alpha)}{2\pi\sqrt{-1}} \int_{\gamma} \frac{f(s)}{(s-w)^{1+\alpha}} ds, \tag{2.2}$$

where $\Gamma(\cdot)$ is a gamma function. Here the path γ is a proper integral contour as

$$s(t) - w = \begin{cases} \sqrt{-1}\delta - t + \pi/2 & (-\infty < t < \pi/2), \\ \delta e^{\sqrt{-1}t} & (\pi/2 \le t \le 3\pi/2), \\ -\sqrt{-1}\delta + t - 3\pi/2 & (3\pi/2 < t < +\infty). \end{cases}$$

Here we take $0 < \arg(s(t) - w) < 2\pi$. Furthermore, by changing the integral variable s to $\widetilde{s} = s - w$, it is easy to see that $(d/dw)^{\alpha} f(w)$ is holomorphic in $\{w \in \mathbb{C}; \mathrm{Re}w > c\}$ with a finite limit $\lim_{\mathrm{Re}w \to +\infty} (d/dw)^{\alpha} f(w)$. We often use the notation ∂_w^{α} instead of $(d/dw)^{\alpha}$.

Proposition 2.4. *Let $0 < \alpha, \beta < 1$, $0 < \alpha + \beta < 1$ and $f(w)$ be holomorphic in $\{w \in \mathbb{C}; \mathrm{Re}w > c\}$ with a finite limit $\lim_{\mathrm{Re}w \to +\infty} f(w)$. Then the relation*

$$\partial_w^{\alpha} \partial_w^{\beta} f(w) = \partial_w^{\alpha+\beta} f(w) \tag{2.3}$$

is valid.

Proof. The formula (2.3) directly follows from the Fubini theorem and the formula

$$\int_{\gamma} \frac{d\tau}{(\tau - w)^{\alpha+1} (s - \tau)^{\beta+1}} = \frac{2\pi\sqrt{-1}\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \frac{1}{(s-w)^{\alpha+\beta+1}}. \qquad \square$$

3. Construction of Solutions by Iteration Scheme

In this section, we give how to construct solutions for the equation $(\beta \circ Q \circ \beta^{-1})\beta[v] = 0$. To begin with, we decompose $\beta \circ Q \circ \beta^{-1}$ as $L + R \circ \mathrm{mod}\ \mathscr{E}^{\mathbb{R}}_{\mathbb{R}_w \times \mathbb{R}_t} \cdot \partial_w$, where L is an ordinary differential operator of the dominant part of $\beta \circ Q \circ \beta^{-1}$, which is got rid of fractional derivatives as in Section 1. We introduce a scheme for formal symbol solutions of microdifferential operators.

3.1. Iteration scheme

We shall construct a formal symbol type solution $U(w,\partial_t)f(t)$ for an arbitrary microfunction $f(t)$, where $U(w,\partial_t)$ is a pseudodifferential operator with the following formal symbol:

$$U(w,\tau)=\sum_{j=0}^{\infty}U_j(w)\tau^{-\frac{j}{\lambda+1}}. \tag{3.1}$$

Here the formal symbol $U(w,\tau)$ can be expressed as

$$U(w,\tau)=\sum_{j=0}^{\infty}u_j w^{-1-\frac{j}{\lambda+1}}\tau^{-\frac{j}{\lambda+1}}$$

at $w=\infty$. We note that this $U(w,\tau)$ can be regarded as a WKB solution with respect to a small parameter τ^{-1}. Furthermore, we emphasize that $U(w,\partial_t)f(t)$ becomes a section of $\mathscr{C}\mathbb{O}_{+}^{\infty}$.

The equation we consider therefore reduces as follows:

$$(\beta\circ Q\circ\beta^{-1})U(w,\partial_t)=\sum_{\substack{l'\geq 0\\ 0\leq k'\leq k\leq m}}\widetilde{a}_{kk'}^{l'}\partial_w^{\frac{l'}{\lambda+1}+k'}\partial_t^{-\frac{l'}{\lambda+1}}E_kU(w,\partial_t)=0. \tag{3.2}$$

Here we recall that the dominant part of $\beta\circ Q\circ\beta^{-1}$ becomes an ordinary differential operator of m-th order with polynomial coefficients which does not include neither ∂_t nor fractional derivatives:

$$L=\sum_{0\leq k'\leq k\leq m}\widetilde{a}_{kk'}^{0}\partial_w^{k'}E_k. \tag{3.3}$$

The coefficient of the maximal term of the operator L is

$$(\text{constant})\cdot\prod_{j=1}^{m}(w+\alpha_j(0)).$$

Hence we know that the j-th component of the formal sum LU is

$$(LU)_j=LU_j=\sum_{0\leq k'\leq k\leq m}\widetilde{a}_{kk'}^{0}\partial_w^{k'}(E_kU_j). \tag{3.4}$$

We evaluate the rest of the equation defined above. Set

$$(3.5)\qquad \begin{aligned} R(w,\partial_w,\partial_t) &= (\beta\circ Q\circ\beta^{-1})(w,\partial_w,\partial_t) - L(w,\partial_w) \\ &= \sum_{\substack{l'\geq 1\\ 0\leq k'\leq k\leq m}} \widetilde{a}^{l'}_{kk'}\partial_w^{k'+\frac{l'}{\lambda+1}}\partial_t^{-\frac{l'}{\lambda+1}}E_k \end{aligned}$$

with $|\widetilde{a}^{l'}_{kk'}|\sim C^{l'+1}$ (C is a constant).

Then the j-th term $(R\circ U)_j$ of $R\circ U$ becomes

$$(3.6)\qquad (R\circ U)_j = \sum_{\substack{0\leq k'\leq k\leq m\\ j=j'+l',\ l'\geq 1}} \widetilde{a}^{l'}_{kk'}\partial_w^{k'+\frac{l'}{\lambda+1}}(E_kU_{j'}) \mod \mathscr{E}^{\mathbb{R}}_{\mathbb{R}_w\times\mathbb{R}_t}\cdot\partial_w.$$

It is important that this $R\circ U$ becomes a formal symbol of order $\leq -1/(\lambda+1)$.

We shall give an iteration scheme of successive approximation process for the formal symbol $U=\sum_{j=0}^{\infty}U_j(w)\tau^{-\frac{j}{\lambda+1}}$ as follows:

$$(3.7)\qquad \begin{cases} LU_0 = 0, \\ LU_{k+1} = -R\circ U_k \mod \mathscr{E}^{\mathbb{R}}_{\mathbb{R}_w\times\mathbb{R}_t}\cdot\partial_w \qquad (k=0,1,2,\cdots). \end{cases}$$

This process is reduced to the equation

$$(3.8)\qquad (L+R\circ)U = 0 \mod \mathscr{E}^{\mathbb{R}}_{\mathbb{R}_w\times\mathbb{R}_t}\cdot\partial_w.$$

4. Global Estimates as to Ordinary Differential Equations

As is seen in Section 3, fractional derivatives appear in our solution-scheme. Fractional derivatives are globally determined by the path including $w=\infty\in\mathbb{CP}^1$. For the aim of arguing convergence of the scheme, we need global estimates as to the differential equations in the scheme.

4.1. Construction of holomorphic solutions

We fix a path γ in the definition of fractional derivatives which passes through ∞ enclosing $s=w$ on $\mathbb{CP}^1$ once in the positive sense as avoiding the singularities of the microdifferential equation (3.2).

Before estimating solutions globally for the equation, we give a concrete construction of the solutions of $LU = F$ by using the homogeneous equation $LU = 0$.

We recall that $-\alpha_j(0)$ $(j = 1, 2, \cdots, m)$ and a point at infinity ∞ are singularities of the ordinary differential operator L. Furthermore, a solution u of $Lu = 0$ is supposed to form $O(|w|^{-1})$.

Definition 4.1. We define α_j-pure solutions of a homogeneous equation $Lu = 0$ by the solutions satisfying that u can be holomorphically extended to each $-\alpha_l(0)$ $(l \neq j)$ (u generally bifurcates at $-\alpha_j(0)$).

Theorem 4.2. *There exists a basis* $\{u_1, u_2, \cdots, u_m\}$ *of a homogeneous equation* $Lu = 0$ *such that each* u_j $(j = 1, 2, \cdots, m)$ *is an* α_j*-pure solution.*

Proof. Take fundamental solutions $v_1, v_2, \cdots, v_m$ of the equation $Lu = 0$. At each $-\alpha_j(0)$, the solution space $\{u; Lu = 0\}$ is spanned by $m - 1$ regular solutions and 1 non-regular solution by the local theory of Fuchsian equations because the characteristic exponent is not an integer at $-\alpha_j(0)$. Hence we have some constants $(\beta_1^j, \cdots, \beta_m^j) \in \mathbb{C}^m \backslash \{0\}$ such that $c_1v_1 + c_2v_2 + \cdots c_mv_m$ is holomorphic at $w = -\alpha_j(0) \Longleftrightarrow \beta_1^j c_1 + \cdots + \beta_m^j c_m = 0$. Therefore we can find a non-zero α_j-pure solution $u_j = c_1^j v_1 + \cdots + c_m^j v_m$ with coefficients $(c_1^j, \cdots, c_m^j) \neq 0$ satisfying the equations $\beta_1^k c_1^j + \cdots \beta_m^k c_m^j = 0$ (for any $k \neq j$).

Take the α_j-pure solutions $u_1, u_2, \cdots, u_m$. A remaining problem is linear independence. If a relation $\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_m u_m = 0$ holds, $\lambda_1 u_1 = -\lambda_2 u_2 - \cdots - \lambda_m u_m$ becomes holomorphic at singular points $-\alpha_1(0), -\alpha_2(0), \cdots, -\alpha_m(0)$. On the other hand, at $w = \infty$, any solution of $Lu = 0$ has a growth order $O(|w|^{-1})$ (see the beginning of Section 4.2). Hence we have $\lambda_1 = 0$ by Liouville's theorem. In the same way, we conclude that $\lambda_2 = \cdots = \lambda_m = 0$. $\square$

Theorem 4.3. *For the ordinary differential equation* $L(w, D_w)U(w) = F(w)$*, if* $F(w)$ *is holomorphic at regular singular points* $w = -\alpha_2(0), -\alpha_3(0), \cdots, -\alpha_m(0)$*, then there exists a solution* $U(w)$*, which is also holomorphic at* $w = -\alpha_2(0), -\alpha_3(0), \cdots, -\alpha_m(0)$.

Proof. By Theorem 4.2, we have a basis $\{\omega_1, \cdots, \omega_m\}$ of $\mathfrak{X} = \{u; Lu = 0\}$ such that each ω_j is an α_j-pure solution. Since the characteristic exponent at $-\alpha_j(0)$ for L is not integer, by the local theory of Fuchsian equations, we can find a holomorphic solution U^j at $w = -\alpha_j(0)$ to $LU^j = F$ for any $j \neq 1$. Therefore we have

$$U^i - U^j \in \mathfrak{X}, \quad i, j = 2, \cdots, m.$$

Hence there exist some coefficients c_l^{ij} $(i, j = 2, \cdots, m;\ l = 1, \cdots, m)$ such that

$$U^i - U^j = \sum_{l=1}^{m} c_l^{ij} \omega_l, \quad i, j = 2, \cdots, m.$$

Note that $c_l^{ii} = 0$, $c_l^{ij} + c_l^{jk} + c_l^{ki} = 0$ for $i, j, k = 2, \cdots, m$ and $l = 1, \cdots, m$. We set

$$\begin{cases} b_1^i = \dfrac{1}{m-1} \displaystyle\sum_{q=2}^{m} c_1^{iq}, \\ b_l^i = \dfrac{1}{m-1} \displaystyle\sum_{q=2}^{m} (c_l^{iq} - c_l^{lq}) \quad (l \neq 1). \end{cases}$$

Therefore we have $b_i^i = 0$ and

$$b_l^i - b_l^j = \frac{1}{m-1} \left(\sum_{q=2}^{m} c_l^{iq} - \sum_{q=2}^{m} c_l^{jq} \right) = c_l^{ij}.$$

Since $b_i^i = 0$ $(i = 2, \cdots, m)$, $U := U^i - \sum_{l=1}^{m} b_l^i \omega_l$ does not depend on i. Hence U is a holomorphic solution of $LU = F$ at every $w = -\alpha_i(0)$ $(i = 2, \cdots, m)$. $\square$

4.2. Global estimates

We will prove Main Theorem 1 by constructing a 1-pure solution. Then, without loss of generality, we may assume $\alpha_1(x) \equiv 0$, namely $a_{mm}(x) \equiv 0$, after a suitable coordinate change $\widetilde{x} = x$, $\widetilde{t} = t + h(x)$ with $h(x) = \int_0^x \sqrt{-1} s^\lambda \alpha_1(s)\, ds$ (note that $\alpha_1(x)$ is a purely imaginary-valued function). Indeed we have

$$\partial_{\widetilde{x}} = \partial_x - h'(x) \partial_t, \quad \partial_{\widetilde{t}} = \partial_t.$$

Hence L has the following form:

$$L = \ell(w) \left(w \partial^m + \sum_{k=1}^{m} \gamma_k \partial_w^{m-k} \right),$$

where $\ell(w)$ is a non-zero holomorphic function and $\gamma_k(w)$ is holomorphic at $w=0$ for every $k=1,\cdots,m$

A fractional derivative is a non-local operator defined by an integral with a path including a point at a infinity. We therefore need to evaluate the solutions of ordinary differential equation in our scheme globally. To start with, we introduce our solution space.

For a sufficiently small $\delta_0, \varepsilon > 0$, we set

$$D_j := \{w \in \mathbb{C}; |w+\alpha_j(0)| \le \delta_0\} \quad (j=2,3,\cdots,m), \tag{4.1}$$

$$\Omega^1 := \{w \in \mathbb{C}; w \neq 0, |\arg w| \le \pi - \varepsilon\}. \tag{4.2}$$

In view of circumstances at a point at infinity, we transform the microdifferential operator $\beta \circ Q \circ \beta^{-1}$ by $w = z^{-(\lambda+1)}$.

Since

$$\begin{aligned} z^{-(\lambda+1)} \cdot \prod_{l=0}^{m-j-1} (-(\lambda+1)\partial_w w - l) \cdot z^{\lambda+1} &= \prod_{l=0}^{m-j-1} (-(\lambda+1) w \partial_w - l) \\ &= \prod_{l=0}^{m-j-1} (z\partial_z - l) = z^{m-j}\partial_z^{m-j}, \end{aligned}$$

we have

$$\begin{aligned} &z^{-(\lambda+1)} \circ (\beta \circ Q \circ \beta^{-1}) \circ z^{\lambda+1} \\ =&\sum_{s=0}^{\infty}\sum_{j=0}^{m}\sum_{k=0}^{j} \frac{a_{jk}^{(s)}(0)}{s!} z^{-(\lambda+1)} \{-\sqrt{-1}(\lambda+1)\partial_w \partial_t^{-1}\}^{\frac{s+j}{\lambda+1}} z^{\lambda+1} \partial_t^k \cdot z^{m-j}\partial_z^{m-j} \\ =&\sum_{s=0}^{\infty}\sum_{j=0}^{m}\sum_{k=0}^{j} \frac{a_{jk}^{(s)}(0)}{s!} \{-\sqrt{-1}(\lambda+1)\}^{\frac{s+j}{\lambda+1}} z^{-(\lambda+1)} \partial_w^{\frac{s+j}{\lambda+1}} z^{\lambda+1+m-j} \partial_t^{k-\frac{s+j}{\lambda+1}} \partial_z^{m-j}. \end{aligned}$$

We show that the term $z^{-(\lambda+1)}\partial_w^{(s+j)/(\lambda+1)}(z^{\lambda+1+m-j}v(z))$ with a holomorphic function $v(z)$ can be divided by z^m at $z=0$. In fact, by the definition of fractional derivatives,

$$\begin{aligned} &z^{-(\lambda+1)}\partial_w^{\frac{s+j}{\lambda+1}}(z^{\lambda+1+m-j}v(z)) \\ &= w\partial_w^{\frac{s+j}{\lambda+1}}(w^{-1-\frac{m-j}{\lambda+1}} v(w^{-\frac{1}{\lambda+1}})) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma(1+(s+j)/(\lambda+1))}{2\pi\sqrt{-1}} w \int_{\gamma_1} \frac{\tau^{-1-\frac{m-j}{\lambda+1}} v(\tau^{-\frac{1}{\lambda+1}})}{(\tau-w)^{\frac{s+j}{\lambda+1}+1}} d\tau \\
&= \frac{\Gamma(1+(s+j)/(\lambda+1))}{2\pi\sqrt{-1}} z^{m+s} \int_{\gamma_2} \frac{\theta^{-1-\frac{m-j}{\lambda+1}}}{(\theta-1)^{\frac{s+j}{\lambda+1}+1}} v((w\theta)^{-\frac{1}{\lambda+1}}) d\theta \\
&= -\frac{\Gamma(1+(s+j)/(\lambda+1))}{2\pi\sqrt{-1}} z^{m+s} \int_{\gamma_3} \frac{p^{\frac{m+s}{\lambda+1}}}{(1-p)^{\frac{s+j}{\lambda+1}+1}} v(zp^{\frac{1}{\lambda+1}}) dp,
\end{aligned}
$$

where we use transforms $\tau = w\theta$ and $p = \theta^{-1}$ in this calculation and set suitable integral paths γ_1, γ_2 and γ_3. Since $s \geq 0$, this proves the divisibility by z^m. Therefore the equation

$$(\beta \circ Q \circ \beta^{-1})U(w, \partial_t) = (L + R\circ)U(w, \partial_t) = 0$$

can be transformed to

$$z^m(M + N\circ)V(z, \partial_t) = 0$$

with $w = z^{-(\lambda+1)}$ and $V(z, \partial_t) = wU(w, \partial_w)$. Here $M = z^{-(\lambda+1+m)}Lz^{(\lambda+1)}$ is an m-th order non-characteristic operator at $z = 0$ with holomorphic coefficients in z and N is a remaining integral operator as above which preserves the analyticity at $z = 0$. In particular, any solution $U(w)$ of $LU = 0$ has the form

$$U(w) = w^{-1}V(w^{-\frac{1}{\lambda+1}})$$

at $w = \infty$ with a holomorphic function $V(z)$ at $z = 0$.

With these domains, we introduce a solution space

$$
\begin{aligned}
(4.3) \quad X &:= \{(F(w), G(z)) \in \mathcal{O}_w(\Omega^1) \times \mathcal{O}_z(B(0; \varepsilon_0^{1/(\lambda+1)})); \\
&wF(w) = G(z) \text{ with } w = z^{-(\lambda+1)}, 0 < |z| < \varepsilon_0^{1/(\lambda+1)}, |\arg z| < \frac{\pi}{2(\lambda+1)}\}
\end{aligned}
$$

for a sufficiently small $\varepsilon_0 > 0$ and its subspace

$$(4.4) \quad \overset{\circ}{X} := \{(F(w), G(z)) \in X; G^{(l)}(0) = 0, \quad l = 0, 1, 2, \cdots, m-1\} \subset X.$$

For $F \in \overset{\circ}{X}$, we shall solve the equation $LU = F$ in the space X.

Proposition 4.4. *Operators $\partial_w, w\partial_w, \partial_w^{l/(\lambda+1)}$ ($l = 1, 2, \cdots$) naturally act on elements of X. Further, these operators preserve $\overset{\circ}{X}$ in X and $L(X) \subset \overset{\circ}{X}$, $(R\circ)(X) \subset \overset{\circ}{X}$.*

Proof. Take one $(F, G) \in X$. Since we have

$$F'(w) = z^{\lambda+1}\left(-\frac{1}{\lambda+1}z^{\lambda+2}G'(z) - z^{\lambda+1}G(z)\right),$$

it follows that

$$\left(F'(w), -\frac{z^{\lambda+2}}{\lambda+1}G'(z) - z^{\lambda+1}G(z)\right) \in X.$$

In a similar manner,

$$\left(wF'(w), -\frac{z}{\lambda+1}G'(z) - G(z)\right) \in X.$$

We lastly prove this proposition as to fractional derivatives. By the definition,

$$\partial_w^{\frac{l}{\lambda+1}}F(w) = \frac{\Gamma(1 + l/(\lambda+1))}{2\pi\sqrt{-1}}\int_{\gamma_1}\frac{F(w')}{(w'-w)^{l/(\lambda+1)+1}}dw',$$

where γ_1 is a contour from $w' = \infty$ enclosing $w' = w$. As $F(w') = w'^{-1}G(w'^{-\frac{1}{\lambda+1}})$ in a neighborhood of $[w, +\infty)$ for a large positive $w \in \mathbb{R}$, we have

$$\partial_w^{\frac{l}{\lambda+1}}F(w) = C\int_{\gamma_1}\frac{w'^{-1}G(w'^{-\frac{1}{\lambda+1}})}{(w'-w)^{l/(\lambda+1)+1}}\ dw'$$

with a suitable C. Transforming as $w' = z^{-(\lambda+1)}w''$ for any small positive $z = w^{-1/(\lambda+1)} \in \mathbb{R}$, we finally obtain

$$\partial_w^{\frac{l}{\lambda+1}}F(w) = Cz^{l+\lambda+1}\int_{\gamma_2}\frac{w''^{-1}G(zw''^{-\frac{1}{\lambda+1}})}{(w''-1)^{l/(\lambda+1)+1}}\ dw'',$$

where γ_2 is a contour enclosing $[1, \infty)$. Setting

$$\widetilde{G}(z) = Cz^{l}\int_{\gamma_2}\frac{w''^{-1}G(zw''^{-\frac{1}{\lambda+1}})}{(w''-1)^{l/(\lambda+1)+1}}\ dw'',$$

we have $(\partial_w^{\frac{l}{\lambda+1}} F(w), \widetilde{G}(z)) \in X$. It is clear from the above explicit forms that these operators preserve $\overset{\circ}{X}$ in X. $\square$

For estimating solutions for the equation $LU = F$, we introduce several norms.

Definition 4.5. We define the following norms for $F(w) \in \mathcal{O}_w(\Omega^1)$ and $G(z) \in \mathcal{O}_z(B(0;\delta_0))$: for $\mu \in \mathbb{R}$ and $k = 0, 1, 2, \cdots$,

$$\|F\|_{\mu,k} := \max_{0 \le l \le k} \sup_{w \in \Omega^1} \left(\frac{|w|}{1+|w|} \right)^{\mu + l - (k-1)_+} |\partial_w^l F(w)|, \tag{4.5}$$

$$\|G\|'_k := \max_{0 \le l \le k} \sup_{z \in B(0;\delta_0)} |[\partial_w^l] G(z)|, \tag{4.6}$$

where

$$(k-1)_+ = \begin{cases} k-1, & k \ge 1, \\ 0, & k = 0 \end{cases}$$

and $[\partial_w^k] G(z)$ stands for an operation on $G(z)$ with respect to $z = w^{-1/(\lambda+1)}$ in a neighborhood of $z = 0$, that is, with a viewpoint of w-plane, we shall use $[\partial_w^k] G(z)$ for estimations at a point at infinity $w = \infty$.

Definition 4.6. We define norms for a pair $(F(w), G(z)) \in X$:

$$\|(F,G)\|_\mu := \|F\|_{\mu,0} + \|G\|'_0, \tag{4.7}$$

$$\|(F,G)\|_{\mu,k} := \|F\|_{\mu,k} + \|G\|'_k. \tag{4.8}$$

In addition, we define a norm with fractional derivatives as

$$\|(F,G)\|_{\mu, l + \frac{\lambda}{\lambda+1}} := \sum_{k=0}^{(\lambda+1)l+\lambda} \|\partial_w^{\frac{k}{\lambda+1}} (F,G)\|_{\mu,k}. \tag{4.9}$$

When we abbreviate $\|(F,G)\|_*$ to $\|F\|_*$, this $\|F\|_*$ stands only for a term with respect to F.

Remark 4.7. We sometimes identify (F, G) with F because G is uniquely determined by F.

We use the following estimates in Kataoka–Sato [KtS].

Lemma 4.8 ([KtS]). *Let* $S = w\partial_w^m + \sum_{k=1}^m \gamma_k(w)\partial_w^{m-k}$. *Set* $D = \{w \in \mathbb{C}; |w| \le 1\}$ *and* $\Omega = \{w \in \mathbb{C}; 0 < |w| \le 1, |\arg w| \le \pi - \varepsilon\}$ *for an* $\varepsilon > 0$. *Assume that* γ_k *is holomorphic on* D *with* $\gamma_1(0) \ne 0, -1, -2, \cdots$ *and set*

$$A = 1 + \sup_{w \in D} \sum_{k=1}^m |\gamma_k(w)| < \infty,$$
$$B = \min\{|p + \gamma_1(0)|; p = 0, 1, 2, \cdots\} > 0.$$

Then there exists a positive constant C *depending only on* A *and* B *such that the following equations hold:*
(1) regular case: for a function $F(w) \in \mathcal{O}(D)$, *any solution* $U(w) \in \mathcal{O}(D)$ *of the equation* $SU = F$ *satisfies*

$$\sup_{\substack{0 \le j \le m \\ w \in D}} |U^{(j)}(w)| \le C \left\{ \sup_{w \in D} |F(w)| + \sum_{j=0}^{m-2} |U^{(j)}(0)| \right\}. \tag{4.10}$$

(2) non-regular case: for a function $F(w) \in \mathcal{O}(\Omega)$, *any solution* $U(w) \in \mathcal{O}(\Omega)$ *of the equation* $SU = F$ *satisfies*

$$\sup_{\substack{0 \le j \le m \\ w \in \Omega}} |w|^{\mu - m + 1 + j} |U^{(j)}(w)| \le C \left\{ \sup_{w \in \Omega} |w|^{\mu} |F(w)| + \sum_{j=0}^{m-1} |U^{(j)}(1)| \right\}, \tag{4.11}$$

with any $\mu \ge A + m + 1$.

Here we calculate a fractional derivative of $G(z)$ defined in a neighborhood of $w = \infty$. We have

$$\begin{aligned}
\partial_w^{\frac{l}{\lambda+1}} F(w) &= \frac{\Gamma(1 + l/(\lambda+1))}{2\pi\sqrt{-1}} \int_{\gamma_1} \frac{F(\widetilde{w})}{(\widetilde{w} - w)^{1 + l/(\lambda+1)}} d\widetilde{w} \\
&= -(\lambda+1) \frac{\Gamma(1 + l/(\lambda+1))}{2\pi\sqrt{-1}} \int_{\gamma_2} \frac{\widetilde{z}^{l+\lambda} G(\widetilde{z})}{(1 - w\widetilde{z}^{\lambda+1})^{1 + l/(\lambda+1)}} d\widetilde{z} \\
&= -(\lambda+1) \frac{\Gamma(1 + l/(\lambda+1))}{2\pi\sqrt{-1}} \int_{\gamma_3} \frac{z^{l+\lambda+1} \widetilde{z}^{l+\lambda} G(\widetilde{z})}{(z^{\lambda+1} - \widetilde{z}^{\lambda+1})^{1 + l/(\lambda+1)}} d\widetilde{z},
\end{aligned}$$

where γ_1 is a contour enclosing ∞ and w, γ_2 is contour from z enclosing $z^{-(\lambda+1)}$ and γ_3 is from 0 enclosing 1.

Transforming $\widetilde{z} = z\theta$, we get a fractional derivative at $z = 0$ (i.e. $w = \infty$)

$$\begin{aligned}(4.12)\qquad & [\partial_w^{\frac{l}{\lambda+1}}]G(z) \\ & = -(\lambda+1)\frac{\Gamma(1+l/(\lambda+1))}{2\pi\sqrt{-1}} z^l \int_{\gamma_3} \frac{\theta^{l+\lambda}G(z\theta)}{(1-\theta^{\lambda+1})^{1+l/(\lambda+1)}} d\theta\end{aligned}$$

for $l = 1, 2, \cdots$. In the case of $l = 0$, we define $\partial_w^{l/(\lambda+1)}G(z) = G(z)$.

LEMMA 4.9. *For the equation $LU = F \in \overset{\circ}{X}$, there exists a pair $(U, V) \in X$ such that we have*

$$\|(U, V)\|_{\mu,m} \leq C\|(F, G)\|_\mu$$

with a positive constant $C > 0$ for any $\mu \geq A + m + 1$.

PROOF. For each $j = 2, 3, \cdots, m$, take one solution $U^j \in \mathcal{O}(D_j)$ for the initial value problem

$$\begin{cases} LU^j = F, \\ \partial_w^l U^j(-\alpha_j(0)) = 0, \quad l = 0, 1, 2, \cdots, m-2. \end{cases}$$

By virtue of Lemma 4.8, we have

$$\sup_{\substack{w \in D_j \\ 0 \leq l \leq m}} |\partial_w^l U^j(w)| \leq C_1 \sup_{w \in D_j} |F(w)|$$

with a constant $C_1 > 0$. By the Gronwall inequality, an analytic continuation to a domain $K := \{w \in \mathbb{C}; |w - 1| \leq 1/2\} \subset \Omega^1$ leads the following inequality:

$$\sup_{\substack{w \in K \\ 0 \leq l \leq m}} |\partial_w^l U^j(w)| \leq C_2 \|F(w)\|_{\mu,0}$$

with a constant $C_2 > 0$. As is seen in Theorem 4.3,

$$U(w) := \widetilde{U}(w) = U^j(w) - \sum_{j=1}^m b_l^j u_l(w)$$

with an α_l-pure solution $u_l(w)$ becomes independent of j and to be holomorphic on Ω^1. Since b_l^j's are estimated by $\sup_{2\le j\le m, 0\le l\le m}|\partial_w^l U^j(1)|$, by (2) of Lemma 4.8 we get a positive constant $C_3>0$ such that

$$\sup_{\substack{0\le l\le m\\ w\in\Omega^1\\ |w|\le(2/\delta_0)^{\lambda+1}}}\left(\frac{|w|}{1+|w|}\right)^{\mu+l-m+1}|\partial_w^l U(w)|\le C_3\|F(w)\|_{\mu,0}.$$

Secondly, we see the estimate at a point at infinity.

In a neighborhood of $w=(2/\delta_0)^{\lambda+1}$,

$$M(z,\partial_z)V(z)=z^{-m}G(z)\in\mathcal{O}(B(0;\delta_0)),$$

where $M(z,\partial_z)$ is a non-characteristic operator in $|z|<\delta_0$ and $V(z)=wU(w)$ is holomorphic in a neighborhood of $z=\delta_0/2$. There exist constants $C_4,C_5>0$ such that

$$\max_{0\le l\le m}|\partial_z^l V(\frac{\delta_0}{2})|\le C_4\max_{0\le l\le m}|\partial_w^l U(\left(\frac{2}{\delta_0}\right)^{\lambda+1})|\le C_5\|F(w)\|_{\mu,0}.$$

Using the maximum principle and the Gronwall inequality for $M(z,\partial_z)$, we have

$$\begin{aligned}\max_{0\le l\le m}\sup_{z\in B(0;\delta_0)}|\partial_z^l V(z)|&\le C_6\left(\max_{0\le l\le m}|\partial_z^l V(\frac{\delta_0}{2})|+\sup_{z\in B(0;\delta_0)}|z^{-m}G(z)|\right)\\&\le C_7\left(\|F(w)\|_{\mu,0}+\frac{1}{\delta_0^m}\sup_{z\in B(0;\delta_0)}|G(z)|\right)\end{aligned}$$

with constants $C_6,C_7>0$. It follows that we have

$$\|(U,V)\|_{\mu,m}\le C\|(F,G)\|_{\mu}$$

with a positive constant $C>0$. □

Proposition 4.10. *For the equation* $LU=F\in\mathring{X}$ *with* $\|(F,G)\|_{\mu,\frac{\lambda}{\lambda+1}}<\infty$, *there exists a pair* $(U,V)\in X$ *such that*

$$\|(U,V)\|_{\mu,m+\frac{\lambda}{\lambda+1}}\le C\|(F,G)\|_{\mu,\frac{\lambda}{\lambda+1}}$$

with a positive constant $C > 0$ for any $\mu \geq A + m + 1$.

Remark 4.11. We can uniquely determine U more explicitly as follows. For an α_1-pure solution $u_1(w)$, we take a sufficiently large $w_0 \in \mathbb{R}$ which satisfies a condition $u_1(w_0) \neq 0$. As there exists $\widetilde{U} = U^j - \sum_{l=1}^{m} b_l^j u_l(w) \in X$ $(j \neq l)$, we set $U - \widetilde{U} = cu_1$ with some constant c for $LU = F$. We set U with additional condition $U(w_0) = c$ for $LU = F$ as

$$U(w) = \widetilde{U}(w) + \frac{c - \widetilde{U}(w_0)}{u_1(w_0)} u_1(w).$$

Therefore we have the following estimate

$$\|(U, V)\|_{\mu,m} \leq C(\|(F, G)\|_{\mu} + |U(w_0)|)$$

for any similar solution $(U, V) \in X$ of $LU = F$.

Proof of Proposition 4.10. We have

$$L(\partial_w^{\frac{l}{\lambda+1}} U) = [L, \partial_w^{\frac{l}{\lambda+1}}]U + \partial_w^{\frac{l}{\lambda+1}} F,$$

where $[\cdot, \cdot]$ stands for a commutator. By virtue of Lemma 4.9, it follows that

$$\begin{aligned} \text{(4.13)} \quad & \|\partial_w^{\frac{l}{\lambda+1}} (U, V)\|_{\mu,m} \\ & \leq C_1 \left(\|\partial_w^{\frac{l}{\lambda+1}} (F, G)\|_{\mu,0} + \|[L, \partial_w^{\frac{l}{\lambda+1}}](U, V)\|_{\mu,0} + |\partial_w^{\frac{l}{\lambda+1}} U(w_0)| \right) \end{aligned}$$

with a positive constant $C_1 > 0$ because $\partial_w^{\frac{l}{\lambda+1}} (F, G) \in \overset{\circ}{X}$.

Lemma 4.12. *For $0 \leq l \leq \lambda$, there is a positive constant C' such that*

$$|\partial_w^{\frac{l}{\lambda+1}} U(w_0)| \leq C' \|(U, V)\|_{\mu,1}.$$

Proof. In virtue of integration by parts, we obtain

$$\frac{\Gamma(1 + l/(\lambda + 1))}{2\pi\sqrt{-1}} \int_{\gamma_1} \frac{U(\widetilde{w})}{(\widetilde{w} - w)^{1+l/(\lambda+1)}} d\widetilde{w}$$

$$
\begin{aligned}
&= \frac{\Gamma(l/(\lambda+1))}{2\pi\sqrt{-1}} \int_{\gamma_1} \frac{U'(\widetilde{w})}{(\widetilde{w}-w)^{l/(\lambda+1)}} d\widetilde{w} \\
&= \frac{\Gamma(l/(\lambda+1))}{2\pi\sqrt{-1}} \int_{\gamma_2} \frac{U'(t+w)}{t^{l/(\lambda+1)}} dt \quad (\widetilde{w} = t+w) \\
&= \frac{\Gamma(l/(\lambda+1))}{2\pi\sqrt{-1}} \int_0^{\infty} (-1+e^{-2\pi\sqrt{-1}l/(\lambda+1)}) t^{-l/(\lambda+1)} U'(t+w) dt,
\end{aligned}
$$

where γ_1 is a contour from ∞ enclosing w and γ_2 is also a contour from ∞ enclosing 0.

In particular, we have

$$
|\partial_w^{\frac{l}{\lambda+1}} U(w_0)| \le C_2 \left\{ \left(\int_0^N t^{-\frac{l}{\lambda+1}} \right) \|U\|_1 + \int_N^{\infty} N^{-\frac{l}{\lambda+1}} |U'(w+t)| dt \right\}
$$

with a constant $C_2 > 0$. Since $U'(w) = -w^{-2} V(w^{-\frac{1}{\lambda+1}}) - w^{-2-\frac{1}{\lambda+1}} V'(w^{-\frac{1}{\lambda+1}})/(\lambda+1)$, there is a constant $C_3 > 0$ such that

$$
\int_N^{\infty} N^{-\frac{l}{\lambda+1}} |U'(w+t)| dt \le C_3 \|V\|_1'.
$$

This completes the proof of this lemma. □

A sequel to the proof of Proposition 4.10. From now on, we take a solution $U \in X$ satisfying $U(w_0) = 0$.

In order to evaluate a part of a commutator, we may calculate $[w^p \partial_w^q, \partial_w^{l/(\lambda+1)}]U$ $(p \le q)$ because the operator L consists of $w^p \partial_w^q$.

Here we note that

$$
\begin{aligned}
& w^p \partial_w^q \left(\frac{\Gamma(1+l/(\lambda+1))}{2\pi\sqrt{-1}} \int_{\gamma_1} \frac{U(\widetilde{w})}{(\widetilde{w}-w)^{1+l/(\lambda+1)}} d\widetilde{w} \right) \\
&= w^p \frac{\Gamma(1+l/(\lambda+1))}{2\pi\sqrt{-1}} \int_{\gamma_1} U(\widetilde{w}) \frac{d^q}{dw^q} \left((\widetilde{w}-w)^{-1-\frac{l}{\lambda+1}} \right) d\widetilde{w} \\
&= w^p \frac{\Gamma(1+l/(\lambda+1))}{2\pi\sqrt{-1}} \int_{\gamma_1} U^{(q)}(\widetilde{w}) \frac{d\widetilde{w}}{(\widetilde{w}-w)^{1+l/(\lambda+1)}}.
\end{aligned}
$$

Hence we have

$$
|[w^p \partial_w^q, \partial_w^{\frac{l}{\lambda+1}}]U| = \left| \frac{\Gamma(1+l/(\lambda+1))}{2\pi\sqrt{-1}} \int_{\gamma_1} U^{(q)}(\widetilde{w}) \frac{w^p - \widetilde{w}^p}{(\widetilde{w}-w)^{1+l/(\lambda+1)}} d\widetilde{w} \right|
$$

$$= \left| \frac{\Gamma(1 + l/(\lambda+1))}{2\pi\sqrt{-1}} \int_{\gamma_1} U^{(q)}(\widetilde{w}) \frac{w^{p-1} + w^{p-2}\widetilde{w} + \cdots + \widetilde{w}^{p-1}}{(\widetilde{w} - w)^{l/(\lambda+1)}} d\widetilde{w} \right|$$

$$\leq C(l) \int_0^\infty |U^{(q)}(w+t)| t^{-\frac{l}{\lambda+1}} \sum_{r=0}^{p-1} |w|^{p-1-r} |w+t|^r dt$$

with a positive constant $C(l)$. We show the boundedness of the integral in this right-hand side in three cases as follows:

(i) in the case of $|w| < N$, $0 \leq t \leq 2N$. The $\| \cdot \|_{\mu,0}$-norm of the corresponding part of this integral is easily estimated by $\|U^{(q)}(w)\|_{\mu,0}$ since $\sup_{t>0, w\in\Omega^1} |w|/|w+t| < \infty$.

(ii) in the case of $|w| \geq N$ or $t \geq 2N$. Since $\sup_{t>0,w\in\Omega^1}(|w|+t)/|w+t| = \sup_{w\in\Omega^1}(|w|+1)/|w+1| < \infty$, we have $|w+t| \to +\infty$ as $t \to +\infty$. Therefore

$$|U^{(q)}(w+t)| = \left| \left(\frac{d}{dw}\right)^q (w'^{-1}V) \,|_{w'=w+t} \right| \leq C_4 \sup_{\substack{z\in B(0;\delta_0)\\ 0\leq l\leq q}} |V^{(l)}(z)| \cdot |t+w|^{-q-1}$$

with a constant $C_4 > 0$. Hence we obtain an estimation of the $\| \cdot \|_{\mu,0}$-norm of the corresponding part of the integral by

$$C_4 \int_{2N}^\infty |t+w|^{-q-1} \sup_{\substack{z\in B(0;\delta_0)\\ 0\leq l\leq q}} |V^{(l)}(z)| t^{-\frac{l}{\lambda+1}} \sum_{r=0}^{p-1} |w|^{p-1-r}|w+t|^r dt$$

$$= C_4 \sup_{\substack{z\in B(0;\delta_0)\\ 0\leq l\leq q}} |V^{(l)}(z)| \int_{2N}^\infty \sum_{r=0}^{p-1} t^{-\frac{l}{\lambda+1}} \frac{|w|^{p-r-1}}{|t+w|^{q-r+1}} dt$$

$$\leq C_5 \sup_{\substack{z\in B(0;\delta_0)\\ 0\leq l\leq q}} |V^{(l)}(z)| \int_{2N}^\infty \sum_{r=0}^{p-1} t^{-\frac{l}{\lambda+1}-2} dt < \infty$$

with a positive constant $C_5 > 0$.

Since $0 \leq p \leq q \leq m$, we have

$$\|[w^p\partial_w^q, \partial_w^{\frac{l}{\lambda+1}}]U\|_{\mu,0} \leq C_6 \left(\sup_{\substack{z\in B(0;\delta_0)\\ 0\leq l\leq q}} |V^{(l)}(z)| + \|U^{(q)}(w)\|_{\mu,0} \right)$$

$$\leq C_7 \|(U,V)\|_{\mu,m} \leq C_8 \|(F,G)\|_\mu$$

with some constants $C_6, C_7, C_8 > 0$.

A remaining problem in this proposition is an estimate with respect to the commutator at a point at infinity. This term is calculated as follows:

$$[L\partial_w^{\frac{l}{\lambda+1}} - \partial_w^{\frac{l}{\lambda+1}}L]V = [L]([\partial_w^{\frac{l}{\lambda+1}}]V) - [\partial_w^{\frac{l}{\lambda+1}}]([L]V).$$

By the same arguments above, all we have to calculate here is as follows:

$$\begin{aligned}&[(w\partial_w)^p]([\partial_w^{q+\frac{l}{\lambda+1}}]V) - [\partial_w^{\frac{l}{\lambda+1}}]([(w\partial_w)^p][\partial_w^q]V)\\ &= [(w\partial_w)^p]([\partial_w^{\frac{l}{\lambda+1}}]([\partial_w^q]V)) - [\partial_w^{\frac{l}{\lambda+1}}]([(w\partial_w)^p][\partial_w^q]V).\end{aligned}$$

Since $[(w\partial_w)^p] = z^{-(\lambda+1)}\{-z\partial_z/(\lambda+1)\}^p z^{\lambda+1} = \{-z\partial_z/(\lambda+1) - 1\}^p$, setting $G = [\partial_w^q]V$, we have only to evaluate $z^k\partial_z^k([\partial_w^{l/(\lambda+1)}]G) - [\partial_w^{l/(\lambda+1)}](z^k\partial_z^k G)$.

By the definition (4.12) of fractional derivation on G, we obtain

$$\begin{aligned}&\frac{2\pi\sqrt{-1}}{\Gamma(1+l/(\lambda+1))}\left\{z^k\partial_z^k([\partial_w^{l/(\lambda+1)}]G) - [\partial_w^{l/(\lambda+1)}](z^k\partial_z^k G)\right\}\\
&= C(l)\left(z^k\partial_z^k \cdot z^l\int_{\gamma_3}\frac{\theta^{l+\lambda}G(z\theta)}{(1-\theta^{\lambda+1})^{1+l/(\lambda+1)}}d\theta\right.\\
&\qquad\qquad\left. - z^l\int_{\gamma_3}\frac{\theta^{l+\lambda}(z\theta)^kG^{(k)}(z\theta)}{(1-\theta^{\lambda+1})^{1+l/(\lambda+1)}}d\theta\right)\\
&= C(l)\int_{\gamma_3}\frac{\theta^{l+\lambda}}{(1-\theta^{\lambda+1})^{1+l/(\lambda+1)}}\\
&\qquad\times\left\{\sum_{s=0}^{k}\binom{k}{s}\frac{l!}{(l-k+s)!}z^{l+s}\theta^sG^{(s)}(z\theta) - z^{l+k}\theta^kG^{(k)}(z\theta)\right\}d\theta\\
&= C(l)\int_{\gamma_3}\frac{\theta^l}{l}\frac{d}{d\theta}((1-\theta^{\lambda+1})^{-l/(\lambda+1)})\sum_{s=0}^{k-1}\binom{k}{s}\frac{l!}{(l-k+s)!}z^{l+s}\theta^sG^{(s)}(z\theta)d\theta\\
&= -C(l)\sum_{s=0}^{k-1}\binom{k}{s}\frac{(l-1)!}{(l-k+s)!}z^{l+s}\int_{\gamma_3}(1-\theta^{\lambda+1})^{-l/(\lambda+1)}\frac{d}{d\theta}(\theta^{l+s}G^{(s)}(z\theta))d\theta\\
&= -C(l)\sum_{s=0}^{k-1}\binom{k}{s}\frac{(l-1)!}{(l-k+s)!}z^{l+s}\end{aligned}$$

$$\times \int_0^1 (1 - e^{-2\pi\sqrt{-1}l/(\lambda+1)}) \frac{(1-\theta)^{-l/(\lambda+1)}}{(1+\theta+\cdots+\theta^\lambda)^{l/(\lambda+1)}} \frac{d}{d\theta}(\theta^{l+s}G^{(s)}(z\theta))d\theta.$$

As the last term is integrable, there exists a constant, all of which is on behalf of $C > 0$, such that

$$\sup_{z\in B(0;\delta_0)} |[L\partial_w^{\frac{l}{\lambda+1}} - \partial_w^{\frac{l}{\lambda+1}}L]V| \le C\|G\|'_k \le C\|G\|'_p \le C\|V\|'_{p+q} \le C\|V\|'_m$$
$$\le C\|(F,G)\|_\mu.$$

This completes the proof of this proposition. □

5. Estimations by Formal Norms

This section gives the proof of the convergence of the solution operator appears in a successive approximation process. For this aim, we define some formal norms similar to [BK], which determine microdifferential operators.

5.1. Formal norms

To prove the convergence of the scheme, we introduce our formal norms.

Definition 5.1. Let $U = \sum_{j=0}^\infty U_j(w)\tau^{-j/(\lambda+1)}$ be a formal symbol such that $U_j(w)$ is a holomorphic function on Ω^1 with $(U_j(w), V_j(z)) \in X$ for $j = 0, 1, 2, \cdots$ $(V_j(z) = wU_j(w))$. Then a formal norm $N^\mu_{m'}(U, V; T)$ with respect to indefinite T for each $m' = 0, 1, 2, \cdots$ for $\mu \ge A + m' + 1$ (A is defined in Lemma 4.8) is defined by

$$(5.1) \qquad N^\mu_{m'}(U,V;T) := \sum_{j,l=0}^\infty \frac{T^{2j+l}}{(\frac{j+l}{\lambda+1})!} \left\{ \|\partial_w^{\frac{l}{\lambda+1}} U_j(w)\|_{\mu+\frac{j+l}{\lambda+1},m'} + \|[\partial_w^{\frac{l}{\lambda+1}}]V_j(z)\|'_{m'} \right\},$$

where $(\frac{j+l}{\lambda+1})! = \Gamma(\frac{j+l}{\lambda+1}+1)$. When we write $N^\mu_{m'}(U;T)$, this formal norm stands for a norm only for a term with respect to U of $N^\mu_{m'}(U,V;T)$.

Remark 5.2. In Boutet de Monvel-Krée [BK], the monstrous coefficients of the formal norms defined above further require the term with respect to j because of justification of the composition of microdifferential operators. However we only treat convergence of microdifferential operators with fractional order and hence we do not give such terms with respect to j.

5.2. Estimations of the dominant part

Taking these formal norms, we are going to estimate the operators which appear in the approximation process in the previous section.

Before evaluating the formal norm, we prepare the following lemma for the estimates of some commutators between L and fractional derivatives of higher order.

Lemma 5.3. *We obtain the following estimate as to* $E_k = \prod_{i=0}^{m-k-1}(-(\lambda+1)\partial_w w - i)$*: for* $k' = 0,1,2,\cdots$ *and* $l = 0,1,2,\cdots,\lambda$,

$$(5.2) \qquad |\partial_w^{k'+\frac{l}{\lambda+1}}(E_k U_j)| \le M_m \sum_{0\le p'\le p\le q\le m} (l+1)^{p'}|w|^{p-p'}|\partial_w^{\frac{l}{\lambda+1}+q-p'}U_j|$$

with a positive constant $M_m > 0$*, which does not depend on* $j, l = 0,1,2,\cdots$.

Proof. Since $\partial_w^{k'}E_k$ has a form

$$\partial_w^{k'}E_k = \sum_{0\le p\le q\le m} \alpha_{pq}^{k'k} w^p \partial_w^q$$

with some $\alpha_{pq}^{k'k}$, it follows that

$$\begin{aligned}
|\partial_w^{k'+\frac{l}{\lambda+1}}(E_k U_j)| &= \left|\sum_{0\le p\le q\le m} \alpha_{pq}^{k'k}\partial_w^{\frac{l}{\lambda+1}}(w^p\partial_w^q U_j)\right| \\
&= \left|\sum_{0\le p'\le p\le q\le m} \alpha_{pq}^{k'k}\frac{1}{p'!}\frac{p!}{(p-p')!}w^{p-p'}\prod_{i=0}^{p'-1}\left(\frac{l}{\lambda+1}-i\right)\partial_w^{\frac{l}{\lambda+1}-p'+q}U_j\right| \\
&\le \sum_{0\le p'\le p\le q\le m} |\alpha_{pq}^{k'k}|\binom{p}{p'}(l+1)^{p'}|w|^{p-p'}|\partial_w^{\frac{l}{\lambda+1}-p'+q}U_j|. \ \square
\end{aligned}$$

(1) Estimates of higher order derivatives of U.

We shall evaluate the ordinary differential equation $LU = F$.

(1-1) Estimates on Ω^1.

To begin with, we estimate the equation $LU = F$ on Ω^1. Setting $U = \sum_{j=0}^{\infty} U_j(w)\tau^{-\frac{j}{\lambda+1}}$ and $F = \sum_{j=0}^{\infty} F_j(w)\tau^{-\frac{j}{\lambda+1}}$ with $U_j, F_j \in \mathring{X}$, we consider the equation $LU_j = F_j$ for their components.

Let w_0 be a large positive number chosen in Remark 4.11 concerning the 1-pure solution $u_1(w)$ of $LU = 0$.

LEMMA 5.4. *If $U(w_0) = 0$, we have an estimate with respect to the equation $LU_j = F_j \in \overset{\circ}{X}$ in Ω^1 as follows:*

$$\begin{aligned}(5.3)\qquad &\|\partial_w^{\frac{l}{\lambda+1}} U_j(w)\|_{\mu+\frac{j+l}{\lambda+1},m}\\ &\le C\Bigg(\|\partial_w^{\frac{l}{\lambda+1}} F_j(w)\|_{\mu+\frac{j+l}{\lambda+1},0} + |\partial_w^{\frac{l}{\lambda+1}} U_j(w_0)|\\ &\qquad + M\sum_{q=1}^{m}(l+1)^q \|\partial_w^{\frac{l-(\lambda+1)q}{\lambda+1}} U_j(w)\|_{\mu+\frac{j+l-(\lambda+1)q}{\lambda+1},0}\Bigg)\end{aligned}$$

with some constants $C, M > 0$, for $l \ge 1$.

PROOF. Since

$$L(\partial_w^{\frac{l}{\lambda+1}} U_j) = [L, \partial_w^{\frac{l}{\lambda+1}}]U_j + \partial_w^{\frac{l}{\lambda+1}} F_j,$$

we have the following estimate by Lemma 4.9 (and Remark 4.11) for some constant $C_1 > 0$;

$$\begin{aligned}\|\partial_w^{\frac{l}{\lambda+1}} U_j(w)\|_{\mu+\frac{j+l}{\lambda+1},m} \le C_1\Bigg(&\|\partial_w^{\frac{l}{\lambda+1}} F_j(w)\|_{\mu+\frac{j+l}{\lambda+1},0} + |\partial_w^{\frac{l}{\lambda+1}} U_j(w_0)|\\ &+ \|[L, \partial_w^{\frac{l}{\lambda+1}}]U_j(w)\|_{\mu+\frac{j+l}{\lambda+1},0}\Bigg).\end{aligned}$$

Continuing to estimate, we have

$$\begin{aligned}&\|\partial_w^{\frac{l}{\lambda+1}} U_j(w)\|_{\mu+\frac{j+l}{\lambda+1},m}\\ &\le C_1\Bigg\{\|\partial_w^{\frac{l}{\lambda+1}} F_j(w)\|_{\mu+\frac{j+l}{\lambda+1},0} + |\partial_w^{\frac{l}{\lambda+1}} U_j(w_0)|\\ &+ \sup_{w\in\Omega^1\cap\{|w|\le w_0\}} \sum_{\substack{1\le p'\le p\le q\le m\\ 0\le k'\le k\le m}} |\widetilde{a}^0_{k'k}||\alpha^{k'k}_{pq}|\binom{p}{p'}|w|^{p-p'}\\ &\qquad\times\prod_{i=0}^{p'-1}\left(\frac{l}{\lambda+1}-i\right)\|\partial_w^{\frac{l}{\lambda+1}-p'+q} U_j\|_0\Bigg\}\end{aligned}$$

$$\leq C_1 \left\{ \|\partial_w^{\frac{l}{\lambda+1}} F_j\|_{\mu+\frac{j+l}{\lambda+1},0} + |\partial_w^{\frac{l}{\lambda+1}} U_j(w_0)| \right.$$
$$\left. + M \sum_{q=0}^{m-1} (l+1)^{m-q} \|\partial_w^{\frac{l}{\lambda+1}+q} U_j\|_{\mu+\frac{j+l}{\lambda+1}-q,0} \right\}$$

with a positive constant $M > 0$. $\square$

From now on, we often abbreviate weight terms of the norms for brevity's sake.

In the case of $l/(\lambda+1)-q \geq m$, this lemma leads the following inequality with respect to our formal norm:

$$N_m^\mu(U;T) \ll \sum_{j\geq 0,\ l\geq 0} \frac{T^{2j+l}}{(\frac{j+l}{\lambda+1})!} C\{\|\partial_w^{\frac{l}{\lambda+1}} F_j\|_0 + |\partial_w^{\frac{l}{\lambda+1}} U_j(w_0)|$$
$$+ M \sum_{k=1}^{m} (l+1)^k \max_{0\leq k'\leq m} \|\partial_w^{\frac{l-(\lambda+1)k}{\lambda+1}} U_j^{(k')}\|_0\}$$
$$\ll CN_0^\mu(F;T) + C \sum_{j\geq 0,\ l\geq 0} \frac{T^{2j+l}}{(\frac{j+l}{\lambda+1})!} \max_{0\leq k'\leq m} \|\partial_w^{\frac{l-m(\lambda+1)}{\lambda+1}+k'} U_j\|_0$$
$$+ C_1 \sum_{\substack{j,l\geq 0\\ 1\leq k\leq m}} \frac{T^{2j+l}}{(\frac{j+l-(\lambda+1)k}{\lambda+1})!} \max_{0\leq k'\leq m} \|\partial_w^{\frac{l-(\lambda+1)k}{\lambda+1}} U_j^{(k')}\|_0\}$$
$$\ll CN_0^\mu(F;T) + CN_m^\mu(U;T)T^{m(\lambda+1)} + C_1 N_m^\mu(U;T) \sum_{k=1}^{m} T^{(\lambda+1)k}$$

with some constants $C, C_1 > 0$.

In the case of $l/(\lambda+1) - q < m$, we have

$$\sup_{w\in\Omega^1\cap\{|w|\leq w_0\}} |[\partial_w^{\frac{l}{\lambda+1}}, L]U_j|$$
$$\leq C \sup_{w\in\Omega^1\cap\{|w|\leq w_0\}} \sum_{\substack{1\leq p'\leq p\leq q\leq m-k+k'\\ l\geq 1}} (l+1)^{p'} |w|^{p-p'} |\partial_w^{\frac{l}{\lambda+1}-p'+q} U_j|$$
$$\leq C_2 \sup_{w\in\Omega^1\cap\{|w|\leq w_0\}} \sum_{\substack{1\leq p'\leq p\leq q\leq m-k+k'\\ l\geq 1}} |\partial_w^{\frac{l}{\lambda+1}-p'+q} U_j| \leq M\|\partial_w^0 U_j\|_m$$

with positive constants $C_2, M > 0$ because $(l+1)^{p'}$ is bounded and $0 \le l/(\lambda+1) - p' + q \le m$. As regards a formal norm, we have

$$N_m^\mu(U;T) \ll \sum_{j\ge 0,\ l\ge 0} \frac{T^{2j+l}}{(\frac{j+l}{\lambda+1})!} C\{\|\partial_w^{\frac{l}{\lambda+1}} F_j\|_0 + |\partial_w^{\frac{l}{\lambda+1}} U_j(w_0)| + M\|\partial_w^0 U_j\|_m\}$$

$$\ll CN_0^\mu(F;T) + CN_m^\mu(U;T)T^{m(\lambda+1)} + C_3 N_m^\mu(U;T) \sum_{l=1}^{m(m-1)(\lambda+1)} T^l$$

with a constant $C_3 > 0$.

(1-2) Estimates at a point at infinity.

As is seen in Section 4.2, we have obtained

$$\begin{aligned}&\|\partial_w^{\frac{l}{\lambda+1}}(U,V)\|_m \\ &\le C_4 \left(\|\partial_w^{\frac{l}{\lambda+1}}(F,G)\|_0 + \|[L, \partial_w^{\frac{l}{\lambda+1}}](U,V)\|_0 + |\partial_w^{\frac{l}{\lambda+1}} U(w_0)| \right)\end{aligned}$$

with a positive constant $C_4 > 0$. At a point at infinity, we only estimate the commutative term.

In the case of $|w| > w_0$, there exists a positive $C_5 > 0$ such that

$$\begin{aligned}&\sup_{w\in\Omega^1\cap\{|w|>w_0\}} |[\partial_w^{\frac{l}{\lambda+1}}, L]U_j| \\ &= \sup_{w\in\Omega^1\cap\{|w|>w_0\}} \Bigg| w^{-1} \sum_{\substack{0\le k'\le k\le m-1\\ 1\le p'\le p\le q\le m}} \binom{p}{p'} \tilde{a}^0_{k'k} \alpha^{k'k}_{pq} \prod_{i=0}^{p'-1} \left(\frac{l}{\lambda+1} - i \right) \\ &\qquad\qquad \times [w^{p-p'} \partial_w^m]([\partial_w^{\frac{l}{\lambda+1}+q-p-m}] V_j(z)) \Bigg| \\ &\le C_5 \sum_{\tilde{q}=1}^{m} (l+1)^{\tilde{q}} \|[\partial_w^{\frac{l-(\lambda+1)q}{\lambda+1}}] V_j(z)\|'_m.\end{aligned}$$

In addition, at a point at infinity, we get

$$\sup_{\substack{w\in\Omega^1\cap\{|w|>w_0\}\\ |z|<w_0^{-1/(\lambda+1)}}} |[[\partial_w^{\frac{l}{\lambda+1}}, L]]V_j(z)| = |w[\partial_w^{\frac{l}{\lambda+1}}, L]U_j(w)|$$

$$= \sup_{\substack{w\in\Omega^1\cap\{|w|>w_0\}\\ |z|<w_0^{-1/(\lambda+1)}}} \left| w \sum_{\substack{0\le k'\le k\le m-1\\ 1\le p'\le p\le q\le m}} \widetilde{a}^{0}_{k'k}\alpha^{k'k}_{pq}\binom{p}{p'}\prod_{i=0}^{p'-1}\left(\frac{l}{\lambda+1}-i\right) \times w^{p-p'}\partial_w^{\frac{l}{\lambda+1}+q-p'}U_j \right|$$

$$= \sup_{\substack{w\in\Omega^1\cap\{|w|>w_0\}\\ |z|<w_0^{-1/(\lambda+1)}}} \left| \sum_{\substack{0\le k'\le k\le m-1\\ 1\le p'\le p\le q\le m}} \widetilde{a}^{0}_{k'k}\alpha^{k'k}_{pq}\binom{p}{p'}\prod_{i=0}^{p'-1}\left(\frac{l}{\lambda+1}-i\right) \times w(w^{p-p'}\partial_w^{m})w^{-1}[\partial_w^{\frac{l}{\lambda+1}+q-p'-m}]V_j(z) \right|$$

$$\le C_6\sum_{\widetilde{q}=1}^{m}(l+1)^{\widetilde{q}}\|[\partial_w^{\frac{l-(\lambda+1)q}{\lambda+1}}]V_j(z)\|'_m$$

with a positive constant $C>0$ because $w(w^{p-p'}\partial_w^m)w^{-1}$ is a bounded operator.

Proposition 5.5. *If each component F_j and U_j of $F = \sum_{j=0}^{\infty}F_j(w)\tau^{-\frac{j}{\lambda+1}}$ and $U = \sum_{j=0}^{\infty}U_j(w)\tau^{-\frac{j}{\lambda+1}}$ is holomorphic on Ω^1 and $U_j, V_j\in X$, $F_j, G_j\in \overset{\circ}{X}$, we have*

$$N^{\mu}_{m}(U,V;T)\ll \Psi(T)N^{\mu}_{0}(F,G;T),$$

where $\Psi(T)$ is a convergent power series of T with non-negative coefficients independent of F and U.

5.3. Estimations of the operator with fractional derivatives

In this subsection, we lastly estimate the rest of the scheme $R\circ U$.

We recall that

$$(5.4)\qquad (R\circ U)_j = \sum_{\substack{0\le k'\le k\le m\\ j=j'+l',\ l'\ge 1}} \widetilde{a}^{l'}_{k'k}\partial_w^{k'+\frac{l'}{\lambda+1}}(E_kU_{j'}) \mod \mathscr{E}^{\mathbb{R}}_{\mathbb{R}_w\times\mathbb{R}_t}\cdot\partial_w.$$

Proposition 5.6. *There exists a convergent power series $\phi(T)$ of T with positive coefficients with value 0 at $T = 0$ such that we have the following estimates:*

$$N_0^\mu(R \circ (U, V); T) \ll \phi(T) N_m^\mu(U, V; T). \tag{5.4}$$

Proof. In the case of a finite domain $|w| \le w_0$, we get the following inequality by Lemma 5.3:

$$\begin{aligned}
&\sup_{w \in \Omega^1 \cap \{|w| \le w_0\}} |\partial_w^{k' + \frac{l}{\lambda+1}} (E_k U_j)| \\
&\le \sup_{w \in \Omega^1 \cap \{|w| \le w_0\}} \sum_{0 \le p' \le p \le q \le m} M'(l+1)^{p'} |w|^{p-p'} \\
&\qquad\qquad \times \max_{0 \le k' \le m} \|\partial_w^{\frac{l-(m-q+p')(\lambda+1)}{\lambda+1}} U_j^{(k')}\|_0 \\
&\le M' w_0^m \sum_{0 \le K \le m} \|\partial_w^{\frac{l-(\lambda+1)K}{\lambda+1}} U_j^{(K)}\|_0
\end{aligned}$$

with a positive constant M'.

For $(l+l')/(\lambda+1) - k \ge m$, the formal norm of order 0 can be estimated as follows:

$$\begin{aligned}
N_0^\mu(R \circ U; T) &\ll \sum_{\substack{j = j'+l, j' \ge 0 \\ l \ge 1}} \frac{T^{2j' + (l+l') + l'}}{(\frac{j'+l+l'}{\lambda+1})!} C^{l'+1} M' w_0^m \sum_{k=0}^m (l+l'+1)^k \\
&\qquad\qquad \times \max_{0 \le k' \le m} \|\partial_w^{\frac{l+l'-(\lambda+1)k}{\lambda+1}} U_{j'}^{(k')}\|_0 \\
&\ll M' w_0^m \sum_{\substack{l' \ge 1 \\ 0 \le k \le m}} C^{l'+1} \frac{T^{2j' + (l+l') - (\lambda+1)k + l' + (\lambda+1)k}}{(\frac{j' + (l+l' - (\lambda+1)k)}{\lambda+1})!} \\
&\qquad\qquad \times \max_{0 \le k' \le m} \|\partial_w^{\frac{l+l'-(\lambda+1)k}{\lambda+1}} U_{j'}^{(k')}\|_0 \\
&\ll N_m^\mu(U; T) \cdot M' w_0^m \sum_{\substack{l' \ge 1 \\ 0 \le k \le m}} C^{l'+1} T^{l' + (\lambda+1)k}
\end{aligned}$$

with a positive constant $C > 0$.

On the other hand, in the case of $(l + l')/(\lambda + 1) - k < m$, the same calculation of the dominant part leads the following estimate:

$$\|\partial_w^{\frac{l+l'}{\lambda+1}-k} U_j\|_0 \leq C_1 \|\partial_w^0 U_j\|_m$$

with a positive constant $C_1 > 0$.

Introducing a formal norm, we have

$$N_0^\mu(R \circ U; T) \ll N_m^\mu(U; T) \cdot M \sum_{l'=1}^{(\lambda+1)m(m-1)} T^{2l'+l}$$

with a positive constant $M > 0$.

At a point at infinity, there is a constant $C_2 > 0$ such that

$$\begin{aligned}
&\sup_{|z|<w_0^{-1/(\lambda+1)}} |([\partial_w^{\frac{l}{\lambda+1}}][R\circ]V)_j| = \sup_{w\in\Omega^1\cap\{|w|\leq w_0\}} |w((\partial_w^{\frac{l}{\lambda+1}} R\circ)U)_j| \\
&\leq \sup_{w\in\Omega^1\cap\{|w|\leq w_0\}} \sum_{\substack{0\leq\tilde{q}\leq m \\ l'\geq 1}} \left| C_2^{l'+1}(l+l'+1)^{\tilde{q}} w^{p-p'+1} \partial_w^m \partial_w^{\frac{l+l'-(\lambda+1)q}{\lambda+1}} U_{j'} \right| \\
&\leq \sup_{|z|<w_0^{-1/(\lambda+1)}} \sum_{\substack{0\leq\tilde{q}\leq m \\ l'\geq 1}} \left| C_2^{l'+1}(l+l'+1)^{\tilde{q}} [w^{p-p'+1} \partial_w^m w^{-1}][\partial_w^{\frac{l+l'-(\lambda+1)q}{\lambda+1}}] V_{j'} \right| \\
&\leq \sum_{\substack{0\leq\tilde{q}\leq m \\ l'\geq 1}} C_2^{l'+1}(l+l'+1)^{\tilde{q}} \|[\partial_w^{\frac{l+l'-(\lambda+1)q}{\lambda+1}}] V_{j'}\|'_0.
\end{aligned}$$

In the case of $|w| \leq w_0$, we have an estimate with a positive constant $C_3 > 0$:

$$\begin{aligned}
&\sup_{w\in\Omega^1\cap\{|w|\leq w_0\}} |w|^{\mu+\frac{j+l}{\lambda+1}} |\partial_w^{\frac{l}{\lambda+1}} (R \circ U)_j| \\
&\leq \sup_{w\in\Omega^1\cap\{|w|\leq w_0\}} \sum_{\substack{0\leq p'\leq p\leq q\leq m \\ l'\geq 1}} |\tilde{a}_{k'k}^l \alpha_{pq}^k| \binom{p}{p'} (l+1)^{p'} \\
&\qquad\qquad \times |w|^{\mu+\frac{j+l}{\lambda+1}+p-p'} |\partial_w^{\frac{l+l'-(\lambda+1)(m-(q-p'))}{\lambda+1}+m} U_{j'}|
\end{aligned}$$

$$\leq \sup_{w\in\Omega^1\cap\{|w|\leq w_0\}} \sum_{\substack{l'\geq 1\\ 0\leq\tilde{q}\leq m}} (l+l'+1)^{\tilde{q}} C_3^{l'+1} |w|^{\mu+\frac{j'+l+l'}{\lambda+1}} |\partial_w^{\frac{l+l'-(\lambda+1)q}{\lambda+1}+m} U_{j'}|$$

$$\leq \sum_{\substack{l'\geq 1\\ 0\leq\tilde{q}\leq m}} (l+l'+1)^{\tilde{q}} C_3^{l'+1} \|\partial_w^{\frac{l+l'-(\lambda+1)q}{\lambda+1}} U_j\|_{\mu+\frac{j+l+l'-(\lambda+1)q}{\lambda+1},m}.$$

On the other hand, in the case of $|w| > w_0$, there is a constant $C > 0$ such that

$$\begin{aligned}
&\sup_{w\in\Omega^1\cap\{|w|>w_0\}} |\partial_w^{\frac{l}{\lambda+1}} (R\circ U)_j| \\
&\leq \sup_{w\in\Omega^1\cap\{|w|>w_0\}} \sum_{\substack{0\leq p'\leq p\leq q\leq m\\ l'\geq 1}} C_4^{l'+1} \binom{p}{p'} (l+l'+1)^{p'} \\
&\qquad\qquad \times w^{p-p'} \partial_w^m |\partial_w^{\frac{l+l'-(\lambda+1)(m-(q-p'))}{\lambda+1}} U_{j'}| \\
&\leq \sup_{\substack{w\in\Omega^1\cap\{|w|>w_0\}\\ |z|<w_0^{-1/(\lambda+1)}}} \sum_{\substack{0\leq\tilde{q}\leq m\\ l'\geq 1}} C_4^{l'+1} \binom{p}{p'} (l+l'+1)^{p'} \\
&\qquad\qquad \times \left| w^{-1} [w^{p-p'} \partial_w^m] ([\partial_w^{\frac{l+l'-(\lambda+1)(m-(q-p'))}{\lambda+1}}] V_{j'}) \right| \\
&\leq \sum_{\substack{0\leq\tilde{q}\leq m\\ l'\geq 1}} C^{l'+1} (l+l'+1)^{\tilde{q}} \|[\partial_w^{\frac{l+l'-(\lambda+1)q}{\lambda+1}}] V_{j'}\|'_m.
\end{aligned}$$

Adding these two inequalities, we have

$$\begin{aligned}
&\|\partial_w^{\frac{l}{\lambda+1}} (R\circ U)_j\| \\
&\leq \sum_{\substack{0\leq\tilde{q}\leq m\\ l'\geq 1}} C_5^{l'+1} (l+l'+1)^{\tilde{q}} \\
&\quad \times \left(\|\partial_w^{\frac{l+l'-(\lambda+1)q}{\lambda+1}} U_j\|_{\mu+\frac{j+l+l'-(\lambda+1)q}{\lambda+1},m} + \|[\partial_w^{\frac{l+l'-(\lambda+1)q}{\lambda+1}}] V_{j'}\|'_m \right)
\end{aligned}$$

with a positive constant $C_5 > 0$.

Lastly, there is a constant $C_6 > 0$ such that

$$\begin{aligned}
&\sup_{|z|<w_0^{-1/(\lambda+1)}} |([\partial_w^{\frac{l}{\lambda+1}}][R\circ]V)_j| = |w((\partial_w^{\frac{l}{\lambda+1}} R\circ)U)_j| \\
&\leq \sup_{w\in\Omega^1\cap\{|w|>w_0\}} \sum_{\substack{0\leq\tilde{q}\leq m \\ l'\geq 1}} \left| C_6^{l'+1}(l+l'+1)^{\tilde{q}} w^{p-p'+1}\partial_w^m \partial_w^{\frac{l+l'-(\lambda+1)q}{\lambda+1}} U_{j'} \right| \\
&\leq \sup_{|z|<w_0^{-1/(\lambda+1)}} \sum_{\substack{0\leq\tilde{q}\leq m \\ l'\geq 1}} \left| C_6^{l'+1}(l+l'+1)^{\tilde{q}} [w^{p-p'+1}\partial_w^m w^{-1}][\partial_w^{\frac{l+l'-(\lambda+1)q}{\lambda+1}}]V_{j'} \right| \\
&\leq \sum_{\substack{0\leq\tilde{q}\leq m \\ l'\geq 1}} C_6^{l'+1}(l+l'+1)^{\tilde{q}} \|[\partial_w^{\frac{l+l'-(\lambda+1)q}{\lambda+1}}]V_{j'}\|_0'.
\end{aligned}$$

Introducing our formal norm, we have a desired estimate. This completes the proof of this proposition. □

5.4. Convergences of the solutions

Because of the estimations in the previous propositions, we get the following theorem.

Theorem 5.9. *Let $U_0 \equiv U_{00}$ be an arbitrary holomorphic solution of $LU_0 = 0$ in Ω^1. We have a series U_k $(k = 1, 2, \cdots)$ of a successive process such that each $U_k = \sum_{j=-\infty}^{0} U_{jk}$ is a formal symbol satisfying*

$$\partial_w^l U_k|_{w=0} = 0 \quad (l = 0, 1, 2, \cdots, m-1; k \geq 1).$$

Then $U = \sum_{k=0}^{\infty} U_k$ converges in $N_m^\mu(U, V; T)$-norm uniformly and it is a solution of the microdifferential equation

$$(L + R\circ)U = 0 \mod \mathscr{E}_{\mathbb{R}_w\times\mathbb{R}_t}^{\mathbb{R}} \cdot \partial_w.$$

Proof. By the argument above, we finally obtain the following inequality:

$$N_m^\mu(U_{k+1}, V_{k+1}; T)$$

$$\begin{aligned}&\ll \Phi(T)N_0^{\mu}(R\circ(U_k,V_k);T)\\&\ll \Phi(T)\phi(T)N_m^{\mu}(U_k,V_k;T)\\&\ll\cdots\ll\{\Psi(T)\phi(T)\}^{k+1}N_m^{\mu}(U_0,V_0;T).\end{aligned}$$

Hence the infinite sum $\sum_{k=0}^{\infty}N_m^{\mu}(U_k,V_k;T)$ converges

$$(1-\Psi(T)\phi(T))^{-1}N_m^{\mu}(U_0,V_0;T).\ \square$$

6. Solutions for the Boundary Value Problem

In the last of this paper, we give j-pure solutions of the boundary value problem (1.5). We obtain the following main result.

THEOREM 6.1. *We can construct j-pure solutions for* (1.5). *Precisely, for any $j=1,\cdots,m$ and any microfunction $u_0(t)$ at a point $\mathring{p}=(0,0;\pm\sqrt{-1})\in\mathbb{R}_x\times\sqrt{-1}T^*\mathbb{R}_t$, we have a unique mild microfunction solution $u(x,t)\in\mathring{\mathscr{C}}_{\{x=0\}|\{x\geq 0\}}$ of a microlocal boundary value problem at $\mathring{p}$:*

$$(6.1)\quad\begin{cases}P(x,\partial_x,\partial_t)u(x,t)=0,\quad x>0\ \textit{(in the sense of } \mathring{\mathscr{C}}_{\{x=0\}|\{x\geq 0\}}\textit{)},\\ u(+0,t)=u_0(t),\\ \operatorname{supp}(\operatorname{ext}(u)(x,t))\cap\{x>0\}\\ \qquad\qquad\subset\{(x,t;\sqrt{-1}(\xi,\tau));\xi-\sqrt{-1}x^{\lambda}\alpha_j(x)\tau=0\}.\end{cases}$$

Further, we have the equations

$$\partial_x^k u(+0,t)=R_{jk}(\partial_t)u_0(t)$$

($j=1,2,\cdots,m;k=0,1,2,\cdots,m-1$), where $R_{jk}(\partial_t)$ is a microdifferential operator with fractional order at most $k/(\lambda+1)$.

PROOF. By virtue of Section 5.4, we can construct a microdifferential equation $(L+R\circ)U=0\mod \mathscr{E}^{\mathbb{R}}_{\mathbb{R}_w\times\mathbb{R}_t}\cdot\partial_w$. Furthermore, the solutions by the iteration scheme (3.7) become α_j-pure by means of Theorems 4.2 and 4.3. Hence we can construct j-pure solutions for (1.5) in the (x,t)-plane.

Secondly, we give a relationship between boundary values and the coefficients of w in the space after the two transforms.

According to Kataoka [Kt1], we have the following properties of a quantized Legendre transform:

$$\beta(f(t)\delta(y)) = \frac{1}{2\pi}\partial_t f(t),$$
$$\beta \circ \partial_y \circ \beta^{-1} = -\sqrt{-1}w\partial_t.$$

The trace map

$$\overset{\circ}{\mathscr{C}}_{\{x=0\}|\{x\geq 0\}} \ni u(x,t) \mapsto u(+0,t) \in \mathscr{C}_{\{x=0\}}$$

is represented as follows:

$$\begin{aligned}\beta(u(+0,t)\frac{y_+^{\frac{k}{\lambda+1}}}{\Gamma(\frac{k}{\lambda+1}+1)}) &= \beta(u(+0,t)\partial_y^{-(\frac{k}{\lambda+1}+1)}\delta(y))\\ &= \beta \circ \partial_y^{-(\frac{k}{\lambda+1}+1)} \circ \beta^{-1} \circ \beta(u(+0,t)\delta(y))\\ &= (-\sqrt{-1}w\partial_t)^{-(\frac{k}{\lambda+1}+1)} \cdot \frac{1}{2\pi}\partial_t u(+0,t).\end{aligned}$$

Transforming $y = x^{\lambda+1}/(\lambda+1)$, we obtain

$$\begin{aligned}&\beta(u(+0,t)\frac{(\lambda+1)^{-\frac{k}{\lambda+1}}}{\Gamma(\frac{k}{\lambda+1}+1)}(x_+)^k)\\ &= \frac{1}{2\pi}(-\sqrt{-1})^{-(\frac{k}{\lambda+1}+1)}w^{-(\frac{k}{\lambda+1}+1)}\partial_t^{-\frac{k}{\lambda+1}}u(+0,t).\end{aligned}$$

It follows that boundary values $u(+0,t), \partial_x u(+0,t), \cdots, \partial_x^{m-1}u(+0,t)$ become coefficients of $w^{-1}, w^{-1-\frac{1}{\lambda+1}}, \cdots, w^{-1-\frac{m-1}{\lambda+1}}$ after the quantized Legendre transform. This completes the proof. □

We obtain the following theorem as an application of Theorem 6.1.

Theorem 6.2. *An arbitrary solution* $u(x,t) \in \overset{\circ}{\mathscr{C}}_{\{x=0\}|\{x\geq 0\}}$ *of the boundary value problem at a point* $(0,0;\pm\sqrt{-1})$

$$P(x,\partial_x,\partial_t)u(x,t) = 0, \quad x > 0 \quad \text{(in the sense of } \overset{\circ}{\mathscr{C}}_{\{x=0\}|\{x\geq 0\}}) \tag{6.2}$$

can be uniquely decomposed as a sum of j-pure solutions.

Proof. By virtue of Theorem 4.2, we can take $U_0 = c_1u_1 + c_2u_2 + \cdots + c_mu_m$ as a linear sum of α_j-pure solutions $(j = 1, 2, \cdots, m)$ for the equation $LU_0 = 0$. Using Theorem 4.3, a solution U_1 of the scheme $LU_1 = -R \circ U_0$ becomes α_j-pure. Similarly to this procedure, each U_k of $U = \sum_{k=0}^{\infty} U_k(w)\tau^{-\frac{k}{\lambda+1}}$ can be decomposed as a sum of α_j-pure. This implies that the solution $u(x,t)$ can be decomposed as a sum of j-pure in the (x,t)-plane. □

Acknowledgment. The author would like to express his gratitude to Professor Kiyoomi Kataoka for insightful comments.

References

[Al] Alinhac, S., Branching of singularities for a class of hyperbolic operators, Indiana Univ. Math. J. **27** (1978), 1027–1037.

[AN] Amano, K. and G. Nakamura, Branching of singularities for degenerate hyperbolic operators, Publ. Res. Inst. Math. Sci. **20** (1984), 225–275.

[AKKT] Aoki, T., Kawai, T., Koike, T. and Y. Takei, On the exact WKB analysis of operators admitting infinitely many phases Adv. Math. **181** (2004), 165–189.

[BK] Boutet de Monvel, L. and P. Krée, Pseudo-differential operators and Gevrey classes, Ann. Inst. Fourier (Grenoble) **17** (1967), 295–323.

[H] Hanges, N., Parametrices and propagation of singularities for operators with noninvolutive characteristics, Indiana Univ. Math. J. **28** (1979), 87–97.

[KS] Kashiwara, M. and P. Schapira, Sheaves on manifolds, Grundlehren der Math. Wiss. **292**, Springer (1990).

[Kt1] Kataoka, K., Micro-local theory of boundary value problems. I. Theory of mild hyperfunctions and Green's formula, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), 355–399.

[Kt2] Kataoka, K., Microlocal theory of boundary value problems. II. Theorems on regularity up to the boundary for reflective and diffractive operators, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), 31–56.

[Kt3] Kataoka, K., Microlocal analysis of boundary value problems with regular or fractional power singularities, Structure of solutions of differential equations (Katata/Kyoto, 1995), World Sci. Publishing, River Edge, NJ (1996), 215–225.

[KtS] Kataoka, K. and Y. Sato, Formal symbol type solutions of Fuchsian microdifferential equations, J. Math. Sci. Univ. Tokyo **9** (2002), 565–626.

[N] Nakane, S., Propagation of singularities and uniqueness in the Cauchy problem at a class of doubly characteristic points, Comm. Partial Differential Equations **6** (1981), 917–927.

[O1] Oaku, T., A canonical form of a system of microdifferential equations with noninvolutory characteristics and branching of singularities, Invent. Math. **65** (1981/82), 491–525.

[O2] Oaku, T., Microlocal boundary value problem for Fuchsian operators. I. F-mild microfunctions and uniqueness theorem. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **32** (1985), 287–317.

[OS] Oldham, K. and J. Spanier, The fractional calculus. Theory and applications of differentiation and integration to arbitrary order, Academic Press, NewYork-London, (1974).

[SKM] Samko, S. G., Kilbas, A. A. and O. I. Marichev, Fractional integrals and derivatives. Theory and applications., Gordon and Breach Science Publishers, Yverdon, (1993).

[SKK] Sato, M., Kawai, T. and M. Kashiwara, Microfunctions and pseudodifferential equations, Lecture Notes in Math. **287** (1973), Springer-Verlag, 265–529.

[T] Takasaki, K., Singular Cauchy problems for a class of weakly hyperbolic differential operators, Comm. Partial Differential Equations **7** (1982), 1151–1188.

[TT] Taniguchi, K. and Y. Tozaki, A hyperbolic equation with double characteristics which has a solution with branching singularities, Math. Japon. **25** (1980), 279–300.

[U] Uchikoshi, K., Cauchy problems for mixed-type operators, Publ. Res. Inst. Math. Sci. **36** (2000), 191–230.

[Y] Yamane, H., Branching of singularities for some second or third order microhyperbolic operators, J. Math. Sci. Univ. Tokyo **2** (1995), 671–749.

(Received November 13, 2006)

School of Computer Science
Tokyo University of Technology
1404-1 Katakura, Hachioji
Tokyo, 192-0982 JAPAN
E-mail: ychiba@cs.teu.ac.jp