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A Lower Bound for Dilatations of Certain Class of Pseudo-Anosov Maps of Riemann Surfaces

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Abstract. Let S be a Riemann surface of type (p, n) with $3p + n > 4$ that contains at least one puncture a. Let $\mathscr{S}_{p,n}$ denote the set of pseudo-Anosov maps of S that are isotopic to products of two Dehn twists and are isotopic to the identity map on $\tilde{S} = S \cup \{a\}$. In this article, we give a lower bound for dilatations of elements of $\mathscr{S}_{p,n}$. We also estimate for any hyperbolic structure of $\tilde{S}$ the hyperbolic lengths of those filling closed geodesics of $\tilde{S}$ stemming from the elements of $\mathscr{S}_{p,n}$.

1. Introduction

Let S be a Riemann surface of type (p, n), where p is the genus and n is the number of punctures of S. Assume that $3p + n > 4$. An orientation-preserving self-homeomorphism f_0 of S is called pseudo-Anosov if there is a pair $(\mathscr{F}_h, \mathscr{F}_v)$ of transverse measured foliations on S and an algebraic integer $\lambda = \lambda(f_0) > 1$ such that $f_0(\mathscr{F}_h) = \lambda \mathscr{F}_h$ and $f_0(\mathscr{F}_v) = \lambda^{-1} \mathscr{F}_v$. The number $\lambda = \lambda(f_0)$ is called the dilatation of f_0. By abuse of language, throughout this article a mapping class is called pseudo-Anosov if one of its representative is pseudo-Anosov. Let f be a pseudo-Anosov mapping class with its pseudo-Anosov representative f_0. The dilatation $\lambda(f)$ of f is defined by $\lambda(f_0)$. It was shown in Penner [7] that for any pseudo-Anosov mapping class f of S,

$$\log \lambda(f) > \tfrac{\log 2}{12p - 12 + 4n}.$$

In [6] Leininger showed that for any pseudo-Anosov mapping class f of S represented by a product of two Dehn twists,

$$\log \lambda(f) > \log \lambda_L \approx 0.162.., \tag{1.1}$$

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where λ_L is the Lehmer's number that is the largest real root of the polynomial $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$.

We assume that $n \geq 1$ and let a be a fixed puncture of S. Denote $\tilde{S} = S \cup \{a\}$. In this article we consider those pseudo-Anosov mapping classes of S that are trivial on $\tilde{S}$ and are represented by finite products:

$$\prod_i (t_1^{m_i} \circ t_2^{n_i}), \tag{1.2}$$

where m_i and n_i are non-zero integers, and t_1 and t_2 are positive Dehn twists along any two simple closed geodesics α and β, respectively. Denote $\mathscr{S}_{p,n}$ the collection of these pseudo-Anosov mapping classes.

THEOREM 1.1. *Let S be a Riemann surface of type (p,n) with $3p+n > 4$ and $n \geq 1$. Then*

(1) *for any $f \in \mathscr{S}_{1,3} \cup \mathscr{S}_{0,5} \cup \mathscr{S}_{2,1}$, $\log \lambda(f) > 2.88727$, and*

(2) *for any $f \in \mathscr{S}_{p,n}$ with $(p,n) \neq (0,5)$, $(1,3)$ or $(2,1)$,*

$$\log \lambda(f) > \log h_0(2p + n - 2),$$

where $h_0(x) = 1 + x^2 + x\sqrt{2 + x^2}$.

REMARK 1.1. Theorem 1.1 gives a lower bound for dilatations of elements in $\mathscr{S}$ that is the union of $\mathscr{S}_{p,n}$ for all (p,n) with $3p + n > 4$ and $n \geq 1$. That is, for any $f \in \mathscr{S}$, we have $\log \lambda(f) > 2.29243$ which occurs when $(p,n) = (1,2)$.

One reason to study the set $\mathscr{S}_{p,n}$ of pseudo-Anosov mapping classes is that it is intimately linked to the length estimations of filling closed geodesics of Riemann surfaces. To illustrate, we let $f \in \mathscr{S}_{p,n}$ be a pseudo-Anosov mapping class that is represented by a pseudo-Anosov map also denoted by f, then there is an isotopy $I : \tilde{S} \times [0,1] \to \tilde{S}$ such that $I(\cdot,0) = f$ and $I(\cdot,1) = \mathrm{id}$. Since $f(a) = a$, $\{I(a,t) : 0 \leq t \leq 1\}$ traces out a closed (self-intersecting) curve c'. By Theorem 2 of Kra [5], the curve c' fills $\tilde{S}$.

COROLLARY 1.1. *For any hyperbolic structure on $\tilde{S}$, we let $c \in \tilde{S}$ denote the filling closed geodesic freely homotopic to c'. Then the hyperbolic length $l_{\tilde{S}}(c) > 2K$, where K is the lower bound obtained from Theorem 1.1.*

REMARK 1.2. In [10] we considered those special filling geodesics c generated by two parabolic curves, and gave a better estimations for the hyperbolic lengths of c for any hyperbolic structure on $\tilde{S}$.

The plan of this article is as follows. In Section 2 we introduce the background material we shall need. In Section 3, we study elements of $\mathscr{S}_{p,n}$ through pairs of filling simple closed geodesics. In Section 4, we give some estimates of lower bounds for dilatations of elements in various subsets of $\mathscr{S}_{p,n}$. In Section 5, we estimate the minimal number of intersections of the curves α and β that determine an element of $\mathscr{S}_{p,n}$. In Section 6, we prove Theorem 1.1 and Corollary 1.1.

2. Pseudo-Anosov Maps Represented by Dehn Twists

To establish notation and terminology, we refer to [6, 9]. Let Homeo(S) be the group of orientation-preserving homeomorphisms of S onto itself, and Homeo$_0(S)$ the subgroup of Homeo(S) consisting of elements isotopic to the identity. The group Homeo(S) naturally acts on the space $\mathscr{I}(S)$ of conformal structures on S via pullbacks. The quotient space

$$\mathscr{I}(S)/\mathrm{Homeo}_0(S)$$

is called the Teichmüller space $T(S)$. The quotient group

$$\mathrm{Homeo}(S)/\mathrm{Homeo}_0(S),$$

denoted by Mod_S, is the mapping class group of S and acts on $T(S)$. The subgroup Mod_S^a of Mod_S that consists of mapping classes fixing the puncture a is called the a-pointed mapping class group. When a is filled in, there defines a natural projection $i : \mathrm{Mod}_S^a \to \mathrm{Mod}_{\tilde{S}}$, where $\mathrm{Mod}_{\tilde{S}}$ is the mapping class group of $\tilde{S}$.

The mapping class group Mod_S is identified with the group of holomorphic automorphisms of $T(S)$ when $3p + n > 4$. $T(S)$ is equipped with a Teichmüller metric $d_{T(S)}$ so that Mod_S acts as a group of isometries.

Every quadratic differential ϕ on S defines a flat structure on S. That is, away from each zero of ϕ we write $\phi = dw^2$ to obtain a local parameter w up to a translation $w \mapsto \pm w + c$ for a constant c. Note that for each complex number z in the unit disk $\Delta = \{z : \ |z| < 1\}$, the form

$$\nu_z = z\ \bar{\phi}/|\phi|$$

determines an equivalent class $[\nu_z]$ in $T(S)$. We see that

$$\Delta \ni z \longmapsto [\nu_z] \in T(S)$$

is an isometry of Δ into $T(S)$ with respect to the hyperbolic metric on Δ and the Teichmüller metric $d_{T(S)}$ on $T(S)$.

Let $\mathbf{H}$ denote the hyperbolic plane $\{z = x + iy \in \mathbf{C} : \ y > 0\}$ equipped with the hyperbolic metric $\frac{|dz|}{\text{Im } z}$. We thus obtain an isometry

$$\mathbf{H} \hookrightarrow T(S). \tag{2.1}$$

The image of (2.1) is called a Teichmüller disk and is denoted by D_ϕ. For each such D_ϕ, we can consider its stabilizer $\text{Stab}(D_\phi)$ in Mod_S. Each element $f \in \text{Stab}(D_\phi)$ determines a Möbius transformation $\mathscr{D}(f)$ and the collection of all $\mathscr{D}(f)$, where $f \in \text{Stab}(D_\phi)$, form a Fuchsian group V_ϕ. See Veech [9] for more information about the group V_ϕ.

It is important to note, see Leininger [6] for example, that hyperbolic elements $\mathscr{D}(f)$ in V_ϕ correspond to pseudo-Anosov elements f in Mod_S. The isometry (2.1) yields that

$$\lambda(f) = \exp\left(\frac{T}{2}\right), \tag{2.2}$$

where T denotes the translation length of $\mathscr{D}(f)$.

Assume that $\mathscr{A} = \{\alpha_1, \ldots, \alpha_u\}$ and $\mathscr{B} = \{\beta_1, \ldots, \beta_v\}$, $u, v \geq 1$, are collections of disjoint and homotopically independent simple closed geodesics on S that fills S in the sense that $S - \{\mathscr{A}, \mathscr{B}\}$ consists of topological disks and once punctured disks. Let t_1 and t_2 denote the positive multi twists along some curves in $\mathscr{A}$ and some curves in $\mathscr{B}$, respectively. Observe that $\mathscr{A} \cup \mathscr{B}$ is regarded as a graph whose dual graph defines a complex $\mathscr{C}$. From the argument of Thurston [8] (see also Veech [9] and Leininger [6] for an exposition), $\mathscr{C}$ can be used to define a Euclidean cone metric on S. In this way, we obtain a quadratic differential ϕ and a Teichmüller disc D_ϕ on which the multi-twists t_1 and t_2 act invariantly. This means that, if we denote by $\mathscr{D}_1$ and $\mathscr{D}_2$ the corresponding Möbius transformations on $\mathbf{H}$ through the isometry (2.1), then $\mathscr{D}_1$ and $\mathscr{D}_2 \in V_\phi$ are parabolic elements. Note that most elements in the subgroup $\langle \mathscr{D}_1, \mathscr{D}_2 \rangle$ generated by $\mathscr{D}_1$ and $\mathscr{D}_2$ are hyperbolic. We see that most elements in $\langle t_1, t_2 \rangle$ are pseudo-Anosov.

Following Leininger [6], we let N denote the $u \times v$ matrix whose (i, j) entry is the minimal geometric intersection number $i(\alpha_i, \beta_j)$ of α_i and β_j

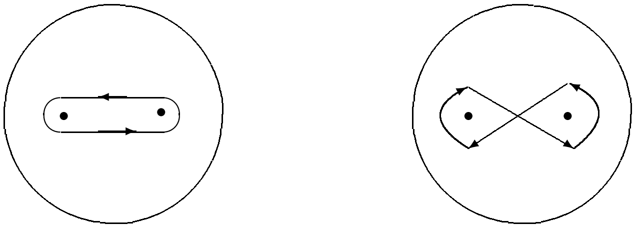

(a) (b)

Figure 1.

for $\alpha_i \in \mathscr{A}$ and $\beta_j \in \mathscr{B}$. Since $(\mathscr{A}, \mathscr{B})$ fills S, the $u \times u$ matrix NN^t is irreducible. Denote $\mu(NN^t)$ the Perron-Frobenius eigenvalue of NN^t, and set $\mu = \sqrt{\mu(NN^t)}$. Then $\mathscr{D}_1$ and $\mathscr{D}_2$ can be represented by the following 2×2 matrices:

$$\mathscr{D}_1 = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathscr{D}_2 = \begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix}. \tag{2.3}$$

Denote $\Gamma = \langle \mathscr{D}_1, \ \mathscr{D}_2 \rangle$. If $\mu > 2$, then Γ is a torsion free discrete subgroup of V_ϕ. By Lemma 6.3 of Leininger [6], $\mathbf{H}/\Gamma$ is a twice punctured disk endowed with a hyperbolic structure, and the smallest translation length T_0 among all hyperbolic elements of Γ is realized by the hyperbolic element

$$(\mathscr{D}_1 \mathscr{D}_2)^{\pm 1}.$$

See Figure 1 (a) for an illustration.

More precisely, the translation length T_0 is given by $\log \varepsilon^2$, where ε is the larger root of the equation $x^2 + (2 - \mu^2)x + 1 = 0$. By applying Corollary 6.7 of [6], we assert that for any pseudo-Anosov map $f \in \langle t_1, t_2 \rangle$, its dilatation $\lambda(f) \geq \varepsilon$ and the equality holds if and only if $f = (t_1 \circ t_2)^{\pm 1}$ up to a conjugacy.

3. Elements of $\mathscr{S}_{p,n}$ and Their Projections to $\tilde{S}$

In what follows we assume that $u = v = 1$; that is, $\mathscr{A} = \{\alpha\}$, $\mathscr{B} = \{\beta\}$, and t_1 and t_2 are simple positive Dehn twists along α and β, respectively. In this case, $\mu = i(\alpha, \beta)$ and every $f \in \mathscr{S}_{p,n}$ can be generated by t_1 and t_2

for certain pair (α, β) of simple closed geodesics. This implies that $\{\alpha, \beta\}$ fills the surface S. Write f in the form (1.2). Let $\tilde{\alpha}$ and $\tilde{\beta}$ denote the simple closed geodesics on $\tilde{S}$ homotopic to α and β, respectively if α and β are viewed as curves on $\tilde{S}$. Recall that there is a group epimorphism $i : \mathrm{Mod}_S^a \to \mathrm{Mod}_{\tilde{S}}$. From Theorem 1.2 of [12], either (i) $\tilde{\alpha}$ and $\tilde{\beta}$ are trivial, or (ii) $\tilde{\alpha}$ and $\tilde{\beta}$ are both nontrivial. If $\tilde{\alpha}$ and $\tilde{\beta}$ are trivial, then $i(t_1) = t_{\tilde{\alpha}}$ and $i(t_2) = t_{\tilde{\beta}}$ are trivial mapping classes. Thus the projection $i\left(\prod_i (t_1^{m_i} \circ t_2^{n_i})\right)$ is trivial for any integers m_i and n_i.

We consider the case that both $\tilde{\alpha}$ and $\tilde{\beta}$ are nontrivial.

LEMMA 3.1. *Let $f \in \mathscr{F}_{p,n}$ be of form* (1.2). *Assume that $\tilde{\alpha}$ and $\tilde{\beta}$ are nontrivial. Then there are integers m_i and n_j that take alternate signs.*

PROOF. We assume that all m_i and n_i are positive (the negative case can be handled similarly). If $\tilde{\alpha} = \tilde{\beta}$, then

$$i(f) = t_{\tilde{\alpha}}^{\sum_i (m_i + n_i)}$$

is nontrivial. This is a contradiction. If $\tilde{\alpha}$ and $\tilde{\beta}$ are disjoint, then it is easy to see that

$$\prod_i (t_{\tilde{\alpha}}^{m_i} \circ t_{\tilde{\beta}}^{n_i}) = t_{\tilde{\alpha}}^{\sum_i m_i} \circ t_{\tilde{\beta}}^{\sum_i n_i}$$

is nontrivial. This again contradicts that $i(f)$ is trivial. So $\tilde{\alpha}$ and $\tilde{\beta}$ must intersect.

The Dehn twists $t_{\tilde{\alpha}}$ and $t_{\tilde{\beta}}$ can be lifted to $\tau_1, \tau_2 : \mathbf{H} \to \mathbf{H}$ so that for $i = 1, 2$, τ_i determines a simply connected region K_i whose complement $\mathbf{H} - K_i$ is a disjoint union of half-planes U_i (called maximal elements in the sequel) each of which is an invariant region under the lift τ_i.

The lift τ_i also determines a mapping class (denoted by τ_i^*) of S under a Bers isomorphism φ (Theorem 9 of Bers [2]). By Lemma 3.3 of [11], τ_i^* is represented by the Dehn twist t_i. See [11] and [12] for more detailed information on the lifts of Dehn twists obtained in this way.

Consider the map

$$\zeta = \prod_i \left(\tau_1^{m_i} \tau_2^{n_i}\right). \tag{3.1}$$

Let ζ^* denote the corresponding element of Mod_S^a under the isomorphism φ.

From (3.1) and (1.2) we obtain

$$\zeta^* = f. \tag{3.2}$$

From (1.2) again,

$$\prod_i \left(t_{\tilde{\alpha}}^{m_i} \circ t_{\tilde{\beta}}^{n_i} \right) = i \left(\prod_i (t_1^{m_i} \circ t_2^{n_i}) \right) = i(f).$$

That is to say, ζ is a lift of $i(f)$. By hypothesis, $f \in \mathscr{F}_{p,n}$, i.e., $i(f)$ is trivial. If we denote by $\varrho : \mathbf{H} \to \tilde{S}$ a universal covering with a covering group G, then this is equivalent to saying that ζ defined as (3.1) satisfies $\zeta|_{\mathbf{S}^1} = h|_{\mathbf{S}^1}$ for an element $h \in G$. A contradiction will be derived once Lemma 3.2 below is established. □

LEMMA 3.2. *Let ζ be defined as* (3.1). *Assume that all m_i and n_i are positive integers. Then $\zeta|_{\mathbf{S}^1} \neq h|_{\mathbf{S}^1}$ for any element $h \in G$.*

PROOF. It is trivial that $\zeta|_{\mathbf{S}^1} \neq \mathrm{id}$. Recall that $\mathbf{H} - K_i$, $i = 1, 2$, is a disjoint union of maximal elements of τ_i. There are two cases to consider.

CASE 1. $K_1 \cap K_2 \neq \emptyset$. In this case, we note that every element of G is either parabolic or hyperbolic; it has at most two fixed points and at least one fixed point on $\mathbf{S}^1$. Since $\zeta|_{K_1 \cap K_2} = \mathrm{id}$, if $(K_1 \cap K_2) \cap \mathbf{S}^1$ contains more than 3 points, then $\zeta|_{\mathbf{S}^1} \neq h$ for any nontrivial element $h \in G$.

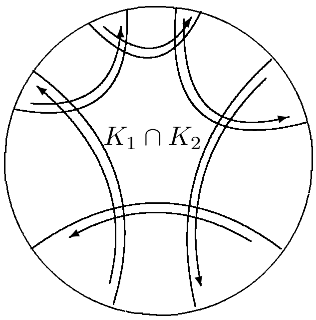


Figure 2.

Suppose that $(K_1 \cap K_2) \cap \mathbf{S}^1$ contains no points. That is, $K_1 \cap K_2$ stays away from $\mathbf{S}^1$. See Figure 2. As $m_i, n_i > 0$, we observe that the motion $\zeta|_{\mathbf{S}^1}$ is in the clockwise direction without any fixed points. This implies that $\zeta|_{\mathbf{S}^1} \neq h|_{\mathbf{S}^1}$ for any nontrivial element $h \in G$.

The remaining cases are handled similarly. If $(K_1 \cap K_2) \cap \mathbf{S}^1$ contains only one point z, and if $\zeta|_{\mathbf{S}^1} = h|_{\mathbf{S}^1}$ for a nontrivial element $h \in G$, then h is parabolic with fixed point z. Now we choose a maximal element U_1 of τ_1 and a maximal element U_2 of τ_2 so that $\partial U_1 \cap \partial U_2 \neq \emptyset$. Then for any point $z' \in (U_1 \cap U_2) \cap \mathbf{S}^1$, it is easily seen that the action of $\zeta|_{(U_1 \cap U_2) \cap \mathbf{S}^1}(z')$ is hyperbolic. So $\zeta|_{\mathbf{S}^1} \neq h|_{\mathbf{S}^1}$. This leads to a contradiction. If $(K_1 \cap K_2) \cap \mathbf{S}^1$ contains only two points z_1, z_2, and if $\zeta|_{\mathbf{S}^1} = h|_{\mathbf{S}^1}$ for a nontrivial element $h \in G$, then h is hyperbolic that takes z_1, say, as its attracting fixed point and z_2 as its repelling fixed point. Since all $n_i, m_i > 0$, the action of $\zeta|_{\mathbf{S}^1}$ is in the clockwise direction, whereas on the one side of z_1, h is in the clockwise direction, and on the other side of z_1, h is in the counter clockwise direction. It follows that $\zeta|_{\mathbf{S}^1} \neq h|_{\mathbf{S}^1}$.

CASE 2. $K_1 \cap K_2 = \emptyset$. In this case, we can write $\zeta = g \circ \lambda$ for some $g \in G$ and some λ of form (3.1) with $K_1^* \cap K_2^* \neq \emptyset$, where K_1^* and K_2^* are complements of all maximal elements determined by λ. It follows that

$$\zeta|_{\mathbf{S}^1} = g\lambda|_{\mathbf{S}^1}. \tag{3.3}$$

From the discussion of Case 1, $\lambda|_{\mathbf{S}^1} \neq h|_{\mathbf{S}^1}$ for any $h \in G$. Assume that $\zeta|_{\mathbf{S}^1} = h_0|_{\mathbf{S}^1}$ for some element $h_0 \in G$. From (3.3), $\lambda|_{\mathbf{S}^1} = g^{-1}h_0|_{\mathbf{S}^1}$. This leads to a contradiction by the discussion in Case 1. Hence the lemma is proved. □

If (1.2) contains only one factor, we have the following result.

LEMMA 3.3. *Let m, n be arbitrary nonzero integers and let $f = t_1^m \circ t_2^n$ be an element of $\mathscr{S}_{p,n}$. Then $\tilde{\alpha} = \tilde{\beta}$ and hence $n + m = 0$.*

PROOF. Suppose that $\tilde{\alpha} \neq \tilde{\beta}$. Then either $\tilde{\alpha}, \tilde{\beta}$ are disjoint, or they intersect. The former case leads to that $i(f)$ is a nontrivial multi-twist, contradicting that $f \in \mathscr{S}_{p,n}$. In later case, we use the assumption of $i(f) = \mathrm{id}$ to calculate that $i(t_1^{-m} \circ f) = t_{\tilde{\alpha}}^{-m}$. On the other hand, we also have

$$i(t_1^{-m} \circ f) = i(t_2^n) = t_{\tilde{\beta}}^n,$$

which leads to that $t_{\tilde{\alpha}}^{-m} = t_{\tilde{\beta}}^{n}$. But this is impossible since $\tilde{\alpha}$ and $\tilde{\beta}$ intersect.

We conclude that $\tilde{\alpha} = \tilde{\beta}$. But then $\mathrm{id} = i(f) = t_{\tilde{\alpha}}^{m+n}$, which occurs if and only if $m + n = 0$. □

4. Some Estimates with Respect to Intersection Numbers

We continue to assume that $3p + n > 4$ and $n \geq 1$. Let $f \in \mathscr{S}_{p,n}$. Then f is of form (1.2). The aim of this section is to present some lower bounds for dilatations $\lambda(f)$ of elements f in various subsets of $\mathscr{S}_{p,n}$ in terms of the intersection number $i(\alpha, \beta)$. We assume that $\mu = i(\alpha, \beta) > 2$.

Lemma 4.1. *With the conditions above, assume that $\tilde{\alpha}$ and $\tilde{\beta}$ are trivial loops. Then*

$$\lambda(f) > h_1(i(\alpha, \beta)),$$

where $h_1(x) = \frac{1}{2}\left(x^2 - 2 + x\sqrt{x^2 - 4}\right)$.

Proof. Since $\tilde{\alpha}$ and $\tilde{\beta}$ are trivial, for arbitrary integers m_i and n_i, $t_{\tilde{\alpha}}^{m}$ and $t_{\tilde{\beta}}^{n}$ are trivial. So the shortest closed geodesic on $\mathbf{H}/\Gamma$ is drawn in Figure 1 (a) which can be achieved when $m = n = 1$.

By Corollary 6.7 of [6], $\lambda(f) \geq \varepsilon$. Hence

$$\lambda(f) \geq h_1(\mu) \text{ for } \mu = i(\alpha, \beta). \ \square \tag{4.1}$$

Lemma 4.2. *Assume that $f \in \mathscr{S}_{p,n}$ is of form* (1.2) *with $i \geq 2$ and $\tilde{\alpha}, \tilde{\beta}$ are both nontrivial. Then*

$$\lambda(f) > h_0(i(\alpha, \beta)),$$

where $h_0(x) = 1 + x^2 + x\sqrt{2 + x^2}$.

Proof. Let $f \in \mathscr{S}_{p,n}$ be of form (1.2). Then (α, β) fills S and thus it determines a quadratic differential ϕ, which in turn defines a Teichmüller disk D_ϕ in $T(S)$. Recall that the Dehn twists t_1 and t_2 determines two parabolic elements $\mathscr{D}_1$ and $\mathscr{D}_2$ that have representations (2.3) for $\mu = i(\alpha, \beta)$. By Lemma 6.3 of [6], for $\mu > 2$, Γ is a torsion free Fuchsian group so that $\mathbf{H}/\Gamma$ is a twice punctured disk.

Since $i \geq 2$, by Lemma 3.1, there are at least one m_i and n_j that take alternate signs. For $\xi, \eta = 1, 2$ and $\eta \neq \xi$, we let $\mathcal{M}$ denote the finite set consisting of elements

$$\mathcal{D}_\xi^{-1}\mathcal{D}_\eta\mathcal{D}_\xi\mathcal{D}_\eta, \quad \mathcal{D}_\xi^{-1}\mathcal{D}_\eta^{-1}\mathcal{D}_\xi\mathcal{D}_\eta, \quad \mathcal{D}_\xi^{-1}\mathcal{D}_\eta\mathcal{D}_\xi^{-1}\mathcal{D}_\eta$$

and their inverses. A simple calculation shows that

$$\left|\text{trace } \mathcal{D}_\xi^{-1}\mathcal{D}_\eta^{-1}\mathcal{D}_\xi\ \mathcal{D}_\eta\right| = 2 + \mu^4,$$
$$\left|\text{trace } \mathcal{D}_\xi^{-1}\mathcal{D}_\eta\ \mathcal{D}_\xi\ \mathcal{D}_\eta\right| = \mu^4 - 2,$$

and

$$\left|\text{trace } \mathcal{D}_\xi^{-1}\mathcal{D}_\eta\mathcal{D}_\xi^{-1}\mathcal{D}_\eta\right| = \mu^4 + 4\mu^2 + 2.$$

If $\mu > 2$, all these elements are hyperbolic and hence they define pseudo-Anosov maps of S. Note that the value $\mu^4 - 2$, which is larger than $2 + 2\mu^2$ for all $\mu \geq 2$, is the minimum value among all traces of elements in $\mathcal{M}$, and hence is the minimum value among all traces of elements in $\mathcal{S}_{p,n}$ with $i \geq 2$. Let T_1 denote the translation length of $\mathcal{D}_\xi^{-1}\mathcal{D}_\eta\mathcal{D}_\xi\mathcal{D}_\eta$. From Beardon [1], the translation length T_1 of the hyperbolic element trace $\mathcal{D}_\xi^{-1}\mathcal{D}_\eta\mathcal{D}_\xi\mathcal{D}_\eta$ satisfies

$$\cosh\left(\tfrac{T_1}{2}\right) = \tfrac{1}{2}\ \left|\text{trace}\left(\mathcal{D}_\xi^{-1}\mathcal{D}_\eta\mathcal{D}_\xi\mathcal{D}_\eta\right)\right|.$$

Then by the process of cutting and pasting, for any element f in $\mathcal{S}_{p,n}$ with $i \geq 2$, the translation length T of $\mathcal{D}(f)$ satisfies the inequality:

$$\cosh\left(\tfrac{T}{2}\right) \geq 1 + \mu^2.$$

From the isometry,

$$\lambda(f) = \exp\left(\tfrac{T}{2}\right).$$

It follows that

$$\lambda(f) > \text{the larger root of } x^2 - (2 + 2\mu^2)x + 1 = 0.$$

Hence

$$\lambda(f) > h_0(\mu).$$

This proves the lemma. □

A similar argument of Lemma 4.2 yields

Lemma 4.3. *Let $3p+n>4$ and $n\geq 1$. Let $f\in \mathscr{S}_{p,n}$ be of form* (1.2). *Assume that $\tilde{\alpha},\tilde{\beta}$ are both nontrivial and the expression* (1.2) *contains only one single factor, that is, $i=1$. Then*

$$\lambda(f) > h(i(\alpha,\beta)),$$

where $h(x)=\frac{1}{2}\left(2+x^2+x\sqrt{4+x^2}\right)$.

Proof. Let f be as in Lemma 3.3. From Lemma 3.3, $\tilde{\alpha}=\tilde{\beta}$ and thus $m+n=0$. By the same argument as in Lemma (4.2), we see that $\mathscr{D}_1{\mathscr{D}_2}^{-1}$ is hyperbolic and its axis projects to a geodesic c that is a closed self-intersecting geodesic and takes the shortest length among closed self-intersecting geodesics on $\mathbf{H}/\Gamma$. See Figure 1 (b). Let T_1 denote the translation length of $\mathscr{D}_1{\mathscr{D}_2}^{-1}$. Note that the absolute value of trace of $\mathscr{D}_1{\mathscr{D}_2}^{-1}$ is $2+\mu^2$, where we continue to denote $\mu=i(\alpha,\beta)$. We can then prove the lemma by using the same argument of Lemma 4.2. Details are omitted. □

5. Intersections of Two Filling Geodesics

We first consider the case that $\tilde{\alpha}$ and $\tilde{\beta}$ are both trivial on $\tilde{S}$ and (α,β) fills S. Then α and β are boundaries of twice punctured disks Δ_1 and Δ_2 with the puncture $a\in\Delta_1\cap\Delta_2$. The deformation retracts of Δ_1 and Δ_2 are two paths γ_1 and γ_2, where γ_i connects a and another puncture b_i of $\tilde{S}$ without passing through any other punctures of $\tilde{S}$. Note that b_1 may be equal to b_2.

By fattening a path we are able to reverse the above procedure to produce two filling simple curves α and β on S with minimum intersections. To illustrate, we use the construction of Lemma 5 of [10], which assets that there are two paths γ_1 and γ_2 connecting a and a puncture b_i $i=1,2$, with minimum intersection numbers such that $S-\{\gamma_1,\gamma_2\}$ consists of disks and once punctured disks.

Observe that γ_1 and γ_2 define two twice punctured disks Δ_1 and Δ_2 so that $a\in\Delta_1\cap\Delta_2$. The two boundary curves $\partial\Delta_1$ and $\partial\Delta_2$ have the properties that (i) $(\partial\Delta_1,\partial\Delta_2)$ fills S and (ii) $i(\partial\Delta_1,\partial\Delta_2)$ is the minimum among all curves α and β with $\tilde{\alpha}$ and $\tilde{\beta}$ trivial. More specifically, the following lemma was proved in [10]:

LEMMA 5.1. *With the above conditions,* $\mu = i(\alpha, \beta) \geq 8p + 4n - 10$ *if* $n \geq 3$*;* $\mu \geq 8p + 2$ *if* $n = 2$*; and* $\mu \geq 8p + 1$ *if* $n = 1$.

Now we consider that $\tilde{\alpha}$ and $\tilde{\beta}$ are both nontrivial on $\tilde{S}$. We have

LEMMA 5.2. *With the above hypothesis,* (1) $i(\alpha, \beta) \geq 2p + n - 2$*, and* (2) *if* $\tilde{\alpha} = \tilde{\beta}$*, then*

$$\mu = i(\alpha, \beta) \geq \max\{4,\ 4p + 2n - 6\}.$$

PROOF. (1) Observe that the union $\alpha \cup \beta$ is regarded as a 4-valence graph on S. Let $\overline{S}$ denote the comtactification of S, and let V, E, F denote the vertices, edges and faces of the graph, respectively. Then its Euler characteristic

$$2 - 2p = \chi(\overline{S}) = V + F - E \tag{5.1}$$

Since $\alpha \cup \beta$ is of 4-valence, $V = i(\alpha, \beta)$ and $E = 2V$. Note that $\alpha \cup \beta$ fills S, $F \geq n$. Hence

$$i(\alpha, \beta) \geq 2p - 2 + n.$$

This proves (1).

(2) Assume now that $\tilde{\alpha} = \tilde{\beta}$. This implies that α intersects β in an even number of intersections. Since (α, β) fills S, $i(\alpha, \beta) > 0$.

Our first goal is to prove

$$i(\alpha, \beta) \geq 4. \tag{5.2}$$

Indeed, if $i(\alpha, \beta) = 2$, then α and β bound an a-punctured bigon B as shown in Figure 3.

In Figure 3, β is the union of two smaller arcs c_0 and c_1. Let $P_1, P_2 \in \beta$ be two points near B. By replacing c_1 with a segment $\overline{P_1 P_2}$ connecting P_1 and P_2, we obtain a curve $\beta' = c_0 \cup \overline{P_1 P_2}$.

Then α and β' bound a cylinder $\mathscr{P}$. Let $\mathscr{P}_0 = \mathscr{P} \cup B$. Observe that

$$S - \mathscr{P}_0 \cong \tilde{S} - \{\tilde{\alpha}\}. \tag{5.3}$$

If S is of type (p, n) with $3p + n > 4$, $n \geq 1$, then $\tilde{S}$ is of type $(p, n - 1)$ and $\tilde{S} - \{\tilde{\alpha}\}$ always contains at least one nontrivial loop. It follows from

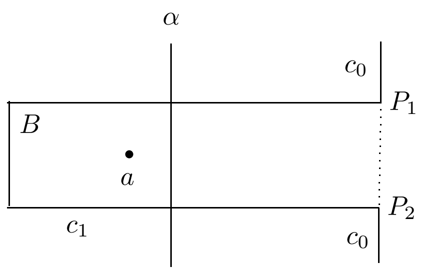


Figure 3.

(5.3) that $S - \mathscr{P}_0$ contains at least one nontrivial loop. Hence (α, β) does not fill S, which leads to a contradiction. We see that if (α, β) fills S and $\tilde{\alpha} = \tilde{\beta}$, then (5.2) holds.

We conclude that $i(\alpha, \beta) \geq 4$. Let B_0 be the innermost a-punctured bigon formed by α and β , and let P_1, P_2 be the vertices of B_0. Let δ_1 and δ_0 be the boundary curves of B_0, where δ_1 is the segment of α connecting P_1 and P_2. Since $i(\alpha, \beta) > 3$, the segment σ_1 of β starting from P_1 must also intersect α at a point Q_1 that is different from P_1 and P_2. Likewise, the segment σ_2 of β starting from P_2 must also intersect α at a point Q_2 that is different from all P_1, P_2, and Q_1. The segment of α connecting Q_1 and Q_2 is denoted by δ_2.

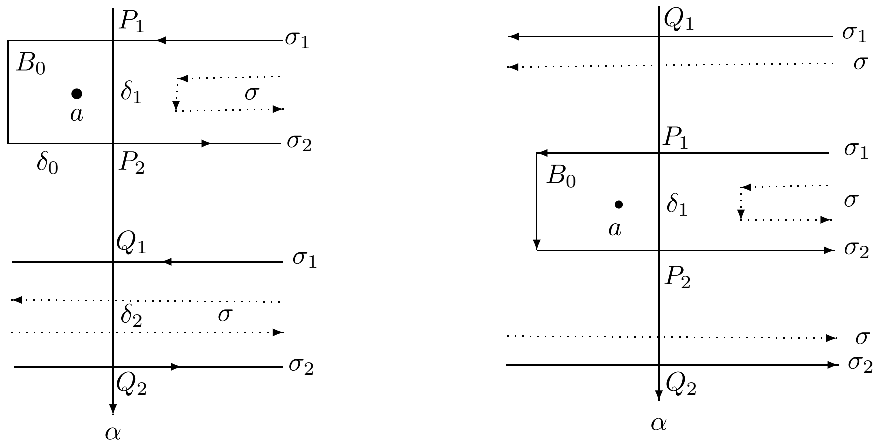
Let $\mathfrak{Q}$ denote the quadrilateral formed by $\{\delta_1, \delta_2; \sigma_1, \sigma_2\}$. Since β is simple, either δ_1 and δ_2 are disjoint or or $\delta_1 \subset \delta_2$. The two cases are depicted in Figure 4(a) and Figure 4(b) depending on whether $\delta_1 \cap \delta_2 = \emptyset$ or $\delta_1 \subset \delta_2$.

In particular, when α bounds a disk that contains not only a but also more than one punctures of $\tilde{S}$, Figure 4 (a) can be drawn as Figure 5.

If $\mathfrak{Q}$ contains a segment σ of β, as shown by dotted lines in Figure 4(a), Figure 4(b) and Figure 5, then since β is simple, it intersects δ_2 at two points.

Let B_1 denote the bigon formed by σ and δ_2 (B_1 is shown but is not labeled in these figures). Then B_1 does not include any puncture of $\tilde{S}$. Otherwise, β can not be deformed to α, and this would be a contradiction. So without loss of generality we may assume by pushing σ to leave $\mathfrak{Q}$ that $\mathfrak{Q}$ does not contain any segment of β.

We also claim that $\mathfrak{Q}$ does not contain any puncture of $\tilde{S}$. Indeed, when



(a) (b)

Figure 4.

a is filled in, S becomes $\tilde{S}$ and δ_0 is pushed to δ_1, and thus β is deformed through B_0 to a curve β'. If $\mathfrak{Q}$ contains punctures of $\tilde{S}$, then β' cannot be deformed to α. This again contradicts that $\tilde{\alpha} = \tilde{\beta}$.

From the above discussion, a pair $\{P_1, P_2\}$ of vertices of $\alpha \cup \beta$ determines a quadrilateral $\mathfrak{Q}$ that does not contain any punctures of $\tilde{S}$. Then we can push $\mathfrak{Q}$ to the next quadrilateral $\mathfrak{Q}'$ that is determined by the pair $\{Q_1, Q_2\}$. From the same argument as above, $\mathfrak{Q}'$ does not contain any punctures of $\tilde{S}$. Since $\mathfrak{Q}' \neq \mathfrak{Q}$, $\mathfrak{Q}' - \mathfrak{Q}$ consists of some quadrilaterals that do not contain any punctures of $\tilde{S}$.

All intersections of α and β are grouped in terms of vertices of quadrilaterals all of which do not include any punctures of $\tilde{S}$. By induction, one proves that there are at least $\frac{i(\alpha,\beta)}{2} - 1$ quadrilaterals that do not contain any punctures of $\tilde{S}$.

Recall that $\alpha \cup \beta$ can be thought of as a 4-valence graph on $\overline{S}$. The Euler characteristic calculation yields that (5.1) holds.

Since $\alpha \cup \beta$ fills $\tilde{S}$, each face must contain at most one puncture of $\tilde{S}$. As discussed above, there are at least $\frac{i(\alpha,\beta)}{2} - 1$ quadrilaterals that do not contain any punctures of $\tilde{S}$. Hence from (5.1),

$$F \geq \frac{i(\alpha,\beta)}{2} + n - 1.$$

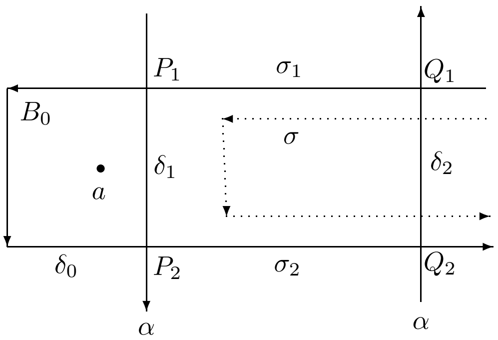


Figure 5.

It follows that

$$i(\alpha, \beta) \geq 4p + 2n - 6.$$

This proves Lemma 5.2. □

6. Proof of the Results

We need several elementary calculations.

LEMMA 6.1. *Assume that* $2p+n < 6$ *and* $3p+n > 4$ *with* $n \geq 1$. *Then* $(p, n) = (1, 2), (0, 5), (1, 3)$ *or* $(2, 1)$.

In what follows, we set

$$\nu = 4p + 2n - 6, \quad \sigma = 2p + n - 2,$$

and

$$\mu = 8p + 4n - 10 \ \text{ if } \ n \geq 3; \quad \text{and} \quad \mu = 8p + n \ \text{ if } \ 1 \leq n \leq 2.$$

Then $\nu = 2\sigma - 2$.

LEMMA 6.2. *Let* $h(x)$ *and* $h_0(x)$ *be defined as in Lemmas* 4.3 *and* 4.2, *respectively. Then*

$$h(\nu) > h_0(\sigma) \tag{6.1}$$

for $(p,n) \neq (1,2),(0,5),(1,3)$ *or* $(2,1)$.

PROOF. Since $\nu < \sqrt{4+\nu^2}$ and $\sigma < \sqrt{2+\sigma^2}$, the inequality (6.1) follows from

$$1+\nu^2 > 3+2\sigma^2. \tag{6.2}$$

Notice that $\sigma^2-4\sigma+1>0$ for $\sigma \geq 4$ or $2p+n \geq 6$. It follows from Lemma 6.1 that for $(p,n) \neq (1,2),(0,5),(1,3)$ or $(2,1)$, the inequality (6.1) holds. □

LEMMA 6.3. *Let* $h(x)$ *and* $h_1(x)$ *be defined as in Lemmas* 4.3 *and* 4.1, *respectively.*

(1) *If* $(p,n) = (1,2)$, *then* $\mu = 8p+n = 10$ *and*

$$h_1(\mu) = h_1(10) > h(4) \approx 17.94427...$$

(2) *If* $3p+n>4$, $n \geq 1$, *then*

$$h_1(\mu) > h(\nu). \tag{6.3}$$

PROOF. (1) The proof that $h_1(10) > h(4)$ is a direct calculation.

(2) Notice that for any positive real number x, we have $x < \sqrt{4+x^2}$; and for any $x \geq 4$, $x-4 < \sqrt{x^2-4}$. From these inequalities, we assert that (6.3) follows from the inequality

$$\mu^2 - 2\mu > 4+\nu^2. \tag{6.4}$$

So it suffices to establish (6.4) in various cases.

CASE 1. $n \geq 3$. Then $\mu = 8p+4n-10$ and $\nu = 4p+2n-6$. Set $u = 2p+n$. By hypothesis, $3p+n>4$ and $n \geq 1$. This implies that

$$u = 2p+n > \tfrac{n+8}{3} \geq 3.$$

Hence $u \geq 4$. Denote

$$h_2(x) = (2x-5)^2 - (x-3)^2 - 2x + 4.$$

When $x \geq 3$, $h_2'(x) = 6x - 16 > 0$, and the function $h_2(x)$ is increasing when $x \geq 3$. But $h_2(4) = 4$. Hence $h_2(u) > 0$ for $u \geq 4$. It follows that

$$(4u - 10)^2 - 8u + 20 > 4 + 4\,(u - 3)^2 \quad \text{for } u \geq 4.$$

This says that (6.4) is satisfied.

Case 2. $1 \leq n \leq 2$. In this case, $\mu = 8p + n$ and $\nu = 4p + 2n - 6$. If $n = 1$, then since $48p^2 + 32p - 21 > 0$ for all $p \geq 1$, which is equivalent to

$$(8p + 1)^2 - 2\,(8p + 1) > 4 + 16\,(p - 1)^2\,.$$

It follows that (6.4) is satisfied. If $n = 2$, then $\mu = 8p + 2$ and $\nu = 4p - 2$. Observe that $6p^2 + 4p - 1 > 0$ for all $p \geq 1$, we see that

$$(8p + 2)^2 - 2\,(8p + 2) > 4 + 4(2p - 1)^2.$$

That is, (6.4) holds.

From the discussions of the two cases, we conclude that (6.4) holds for all pairs (p, n) with $3p + n > 4$ and $n \geq 1$. □

Proof of Theorem 1.1. Every element $f \in \mathscr{S}_{p,n}$ can be written in the form (1.2) for m_i, n_i being nonzero integers and (α, β) being a pair of filling simple closed geodesics of S. Then by Theorem 1.2 of [12], either $\tilde{\alpha}, \tilde{\beta}$ are trivial loops on $\tilde{S}$, or $\tilde{\alpha}, \tilde{\beta}$ are both nontrivial on $\tilde{S}$. First we assume that $(p, n) \neq (1, 2),\ (0, 5),\ (1, 3)$ or $(2, 1)$.

If $\tilde{\alpha}$ and $\tilde{\beta}$ are trivial, then Lemma 4.1 and Lemma 5.1 yield

$$\lambda(f) > h_1(8p + 4n - 10)$$

if $n \geq 3$; and

$$\lambda(f) > h_1(8p + n)$$

if $1 \leq n \leq 2$. From Lemma 6.3 and Lemma 6.2, we obtain the following inequalities:

$$h_1(8p + 4n - 10) > h_0(2p - 2 + n)$$

and

$$h_1(8p + n) > h_0(2p - 2 + n).$$

It follows immediately that

$$\lambda(f) > h_0(2p-2+n) \tag{6.5}$$

so long as f is represented as (1.2) for $\tilde{\alpha}$ and $\tilde{\beta}$ being trivial.

It remains to consider the possibility that $\tilde{\alpha}$ and $\tilde{\beta}$ are nontrivial. If in the expression (1.2) $i = 1$, that is, $f = t_1^m \circ t_2^n$ for some integers m and n. Then by Lemma 3.3, we must have that $\tilde{\alpha} = \tilde{\beta}$ and $m + n = 0$. From Lemma 5.2 and Lemma 4.3, we get that

$$\lambda(f) > h(4p+2n-6). \tag{6.6}$$

From Lemma 6.2, we conclude that

$$h(4p+2n-6) > h_0(2p+n-2).$$

Together with (6.6) we see that

$$\lambda(f) > h_0(2p+n-2). \tag{6.7}$$

Next, we assume that $i \geq 2$. It is not clear whether $\tilde{\alpha} = \tilde{\beta}$. In this situation we apply Lemma 4.2 and Lemma 5.2 to obtain

$$\lambda(f) > h_0(2p-2+n). \tag{6.8}$$

Finally we consider some special cases.

(a) $(p,n) = (1,2)$. In this case, $\nu = 2$, $\sigma = 2$, $\mu = 10$. We have

$$\lambda(f) > \min\ \{h(4),\ h_1(\mu),\ h_0(\sigma)\} = h_0(2) = 5 + 2\sqrt{6} \approx 9.89898.$$

(b) $(p,n) = (1,3)$. In this case, $\nu = 4$, $\sigma = 3$, $\mu = 10$. We have

$$\lambda(f) > \min\ \{h(\nu),\ h_1(\mu),\ h_0(\sigma)\} = h(4) = 9 + 4\sqrt{5} \approx 17.9443.$$

(c) $(p,n) = (0,5)$. In this case, $\nu = 4$, $\sigma = 3$, $\mu = 10$. We have

$$\lambda(f) > \min\ \{h(\nu),\ h_1(\mu),\ h_0(\sigma)\} = h(4) = 9 + 4\sqrt{5} \approx 17.9443.$$

(d) $(p,n) = (2,1)$. In this case, $\nu = 4$, $\sigma = 3$, $\mu = 17$. We have

$$\lambda(f) > \min\ \{h(\nu),\ h_1(\mu),\ h_0(\sigma)\} = h(4) = 9 + 4\sqrt{5} \approx 17.9443.$$

This completes the proof of Theorem 1.1.□

To prove Corollary 1.1, we recall that

$$i : \mathrm{Mod}_S^a \to \mathrm{Mod}_{\tilde{S}}$$

is the natural projection defined by forgetting the puncture a. From Theorem 4.1 and Theorem 4.2 of Birman [3] (see also Theorem 10 of Bers [2]), $\ker(i)$ is a normal subgroup of Mod_S^a and is isomorphic to the covering group G (for the covering map $\varrho : \mathbf{H} \to \tilde{S}$). For every element $h \in G$, the corresponding element in $\ker(i)$ is denoted by h^*.

Kra showed, see [5], that $\ker(i)$ contains infinitely many pseudo-Anosov maps which form a subset $\mathscr{S}_{p,n}^*$ of $\ker(i)$. Note that

$$\mathscr{S}_{p,n} \subset \mathscr{S}_{p,n}^*.$$

Although by a theorem of Hubert–Lanneau [4], there exist pseudo-Anosov maps of S that can not be represented by any finite products of two Dehn twists along filling simple closed geodesics, it is not known whether $\mathscr{S}_{p,n} = \mathscr{S}_{p,n}^*$.

Proof of Corollary 1.1. Let $c \subset \tilde{S}$ be a filling geodesic that stems from an element $g^* \in \mathscr{S}_{p,n}$. This means that, if we denote by c_g the axis of the corresponding essential hyperbolic element g of G (under the Bers isomorphism), then $c = \varrho(c_g)$, where $\varrho : \mathbf{H} \to \tilde{S}$ is the universal covering map with the covering group G.

Under the isometry (2.1), the pseudo-Anosov mapping class g^* corresponds to a Möbius transformation $\mathfrak{D}(g^*)$ in the Veech group V_ϕ. Let T_1 denote the translation length of $\mathfrak{D}(g^*)$. From (2.2),

$$\log \lambda(f)^2 = T_1. \tag{6.9}$$

Let T_g denote the translation length of g. Then Proposition 7 of Kra [5] and (6.9) yield

$$T_g \geq T_{1/2} = \log \lambda(f). \tag{6.10}$$

From (6.10), we obtain

$$l_{\tilde{S}}(c) = T_g \geq K,$$

where K is the lower bound obtained from Theorem 1.1.

This proves the corollary. □

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