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Spectra of Domains with Partial Degeneration

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Abstract. We consider the eigenvalue problem of the Laplacian (Neumann B.C.) in the domain which has a very thin subregion. We give a detailed characterization of the asymptotics of the eigenvalues and the eigenfunctions. The perturbation formulas take various forms depending on the type of the eigenvalue and geometric situations (dimension, shape).

§1. Introduction

In this paper, we deal with a detailed characterization of the eigenvalues of the Laplace operator (with Neumann B.C.) in the domain $\Omega(\zeta)$, which has an extremely thin subregion $Q(\zeta)$ (see Fig.1-Fig.5 for typical examples). We also analyze the behaviors of the corresponding eigenfunctions. The domain $\Omega(\zeta) \subset \mathbb{R}^n$ takes the following form

$$\Omega(\zeta) = D \cup Q(\zeta) \quad (\zeta > 0) \tag{1.1}$$

such that D is a bounded (fixed) domain and $Q(\zeta)$ is a region which shrinks to a lower dimensional set Q ($\dim Q = \ell$) for $\zeta \to 0$. We suppose that $Q(\zeta)$ is almost equal to the set $Q \times B^{(m)}(\zeta)$. Here $B^{(m)}(\zeta)$ is a $m-$dimensional ball of radius ζ and $\ell + m = n$. $\partial\Omega(\zeta)$ is sufficiently regular for each $\zeta > 0$. The precise definition of $\Omega(\zeta)$ is to be given in §2. For the domain $\Omega(\zeta)$, we consider the eigenvalue problem

$$\begin{aligned} &\Delta\Phi + \mu\Phi = 0 \quad \text{in} \quad \Omega(\zeta), \\ &\partial\Phi/\partial\boldsymbol{\nu} = 0 \quad \text{on} \quad \partial\Omega(\zeta) \quad \text{(Neumann B.C.)} \end{aligned} \tag{1.2}$$

where Δ is the Laplace operator in $\mathbb{R}^n$ and $\boldsymbol{\nu}$ is the unit outward normal vector on $\partial\Omega(\zeta)$. It is known that the spectral set of the Laplacian (with Neumann B.C.) in a bounded domain with a smooth boundary is

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a discrete sequence of nonnegative real numbers, which are eigenvalues of finite multiplicity (cf.Courant-Hilbert [12], Edmunds-Evans [15], Mizohata [39]). So we can denote the eigenvalues of (1.2) by $\mu_k(\zeta)$ ($k \geqq 1$), which are arranged in increasing order (with counting multiplicity). We will investigate the limitting behavior of $\mu_k(\zeta)$ for $\zeta \to 0$. We will show in later sections (cf. Proposition 2.1 in §2 and its proof in §3) that, for each k, the eigenvalue $\mu_k(\zeta)$ has a limit value (denoted by μ_k) for $\zeta \to 0$ and the set of all such values μ_k ($k = 1, 2, 3, \cdots$) is equal to the union of two sets $\{\omega_k\}_{k=1}^{\infty}$ and $\{\lambda_k\}_{k=1}^{\infty}$. Here ω_k is the $k-$th eigenvalue of the Laplacian in D (with Neumann B.C.) and λ_k is the k-th eigenvalue of the (low-dimensional) Laplacian in Q with Dirichlet B.C. on ∂Q. The main purpose of this paper is to investigate detailed behaviors of the convergence

$$\lim_{\zeta \to 0} \mu_k(\zeta) = \mu_k \tag{1.3}$$

in relation to the limit eigenvalue problems in D and Q. We get a perturbation formula of $\mu_k(\zeta) - \mu_k$ (for $\zeta \to 0$). Due to this result, we see the accurate convergence rates and understand what kind of quantities of the geometric situations are involved in the asymptotic behaviors of the eigenvalues. Here we introduce a notation to classify the limit set of eigenvalues, by which we give more detailed statement of results. We decompose the set $\{\mu_k\}_{k=1}^{\infty}$ into the following three subsets.

$$E_I = \{\omega_k\}_{k=1}^{\infty} \setminus \{\lambda_k\}_{k=1}^{\infty}, \quad E_{II} = \{\lambda_k\}_{k=1}^{\infty} \setminus \{\omega_k\}_{k=1}^{\infty},$$
$$E_{III} = \{\omega_k\}_{k=1}^{\infty} \cap \{\lambda_k\}_{k=1}^{\infty}$$

We call elements of $E_I \cup E_{II}$ **non-resonant** eigenvalues and those of E_{III} **resonant** eigenvalues.

We describe several previous results on this problem. There have been many important works on eigenvalue problems on such singularly perturbed domains in these several decades. Among them, the pioneering work Beale [7] studied the domain with a thin handle (like a Dumbbell shaped domain but unbounded) from point of view of wave phenomena and gave a characterization of the eigenfrequencies and scattering frequencies. This work gives a method which is applicable to the proof of (1.3). Later there have been several works on the analysis of the details of the convergence (1.3) for the case of the Dumbbell shaped domain (cf. Fig.1, Fig.2, Fig.4). Arrieta

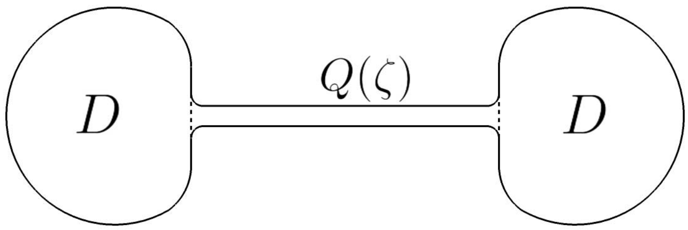

Fig. 1. Dumbbell-shaped domain (2D)

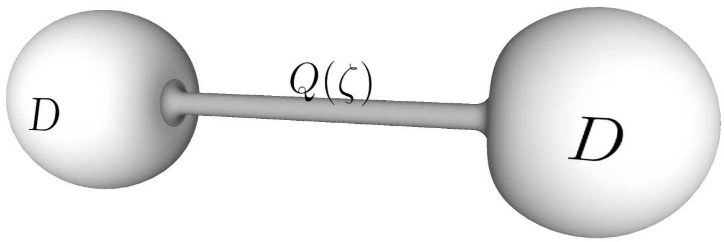

Fig. 2. Dumbbell-shaped domain (3D)

[4], Fang [16], Jimbo -Morita [29], Jimbo [25], Gadylshin [21], Nazarov-Plamenevskii [41] gave perturbation formulas of eigenvalues for this domain (but in situations which are mutually different). We mention some of these results.

(I) For the case $n \geqq 3, \ell = 1$ and $\mu_k \in E_I$, Jimbo [25] obtained

$$\mu_k(\zeta) - \mu_k = \boldsymbol{\alpha}(k)\zeta^{n-1} + o(\zeta^{n-1}) \tag{1.4}$$

(see also Arrieta [4], Fang [16], Jimbo and Morita [29], Kozlov-Maz'ya-Movchan [34]).

(II-1) For the case $n = 2, \ell = 1$ and $\mu_k \in E_{II}$, Arrieta [4] obtained

$$\mu_k(\zeta) - \mu_k = \boldsymbol{\beta}(k)\zeta\log(1/\zeta) + o(\zeta\log(1/\zeta)). \tag{1.5}$$

(II-2) For the case $n = 3, \ell = 1$ and $\mu_k \in E_{II}$, Gadylshin [21] obtained

$$\mu_k(\zeta) - \mu_k = \boldsymbol{\beta}'(k)\zeta + o(\zeta). \tag{1.6}$$

The coefficients $\boldsymbol{\alpha}(k)$, $\boldsymbol{\beta}(k)$, $\boldsymbol{\beta}'(k)$ in the above formulas are given in terms of the eigenvalue problems of the limit domains (See §2 for details). On the other hand, there has not been a result for the case of $\mu_k \in E_{III}$ (even in the case of Dumbbell shaped domains) as far as the authors know. We deal with the non-resonant case E_I and E_{II} for $\ell = \dim Q \geqq 2$ as well as $\ell = 1$ and we also study the resonant eigenvalue $\mu_k(\zeta)$ (the case $\mu_k \in E_{III}$) and prove perturbation formulas. For $\mu_k \in E_{III}$, the corresponding eigenfunction $\Phi_{k,\zeta}$ behaves like a mixture (or linear combination) of several modes coming from both of $\{\omega_k\}_{k=1}^{\infty}$ and $\{\lambda_k\}_{k=1}^{\infty}$ (this is why we call it resonant) and so it makes difficulty for which we need harder analysis. We present the main results (perturbation formula of eigenvalues) in Theorem 2.5 and Theorem 2.10-2.14 in §2 (cf. the list in Fig.8). As an important special case, we have a perturbation formula for $\mu_k(\zeta)$ for the domain (Doughnut+Pancake shaped domain) in Fig.3 (i.e. $n = 3, \ell = 2, m = 1$) (as a special case) as follows.

$$\mu_k(\zeta) - \mu_k = \begin{cases} \boldsymbol{\alpha}(k)\zeta + o(\zeta) & \text{if} \quad \mu_k \in E_I \\ \boldsymbol{\beta}_1(k)\zeta\log(1/\zeta) + o(\zeta\log(1/\zeta)) & \text{if} \quad \mu_k \in E_{II} \\ \boldsymbol{\gamma}_1(k)\zeta^{1/2} + o(\zeta^{1/2}) & \text{if} \quad \mu_k \in E_{III} \end{cases}$$

In this example, we can see that the properties of convergence of the eigenvalue are seriously dependent on situations (particularly on which of three sets E_I, E_{II}, E_{III}, μ_k belongs to). This is one of the most interesting parts of our main results. We can also deal with the case that $Q(\zeta)$ and D are not connected (cf. Fig.2, Fig.5).

There have been also analysis on the spectrum of totally thin domain (cf. Ramm [49], Schatzman [55]). In [55], a similar perturbation formula of the eigenvalue on a thin tubular domain around a manifold is studied. On the other hand, there are other kinds of singularly perturbed domains (besides thin or partially thin domains). One typical example of them is a domain with a small hole (or impurities, cracks). That is the domain like $\Omega'(\zeta) = \Omega \setminus B(\zeta)$ where a thin set $B(\zeta)$ shrinks to a lower dimensional set. Note that this domain $\Omega'(\zeta)$ increases at the limit $\zeta \to 0$ and so it is an interesting contrast to that our domain $\Omega(\zeta)$ decreases. This

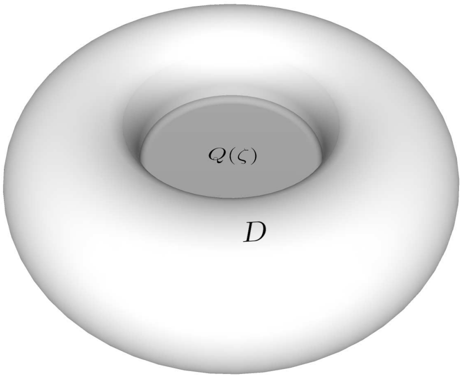

Fig. 3. Doughnut+Pancake domain (3D)

type of domains are also related with real physical phenomena and there are a lot of works on their elaborate spectral analysis and perturbation formulas of eigenvalues have been studied in many situations. See Bérard-Gallot [8], Chavel-Feldman [9,11], Rauch-Taylor [50], Courtois [13], Flucher [17], Ozawa [43,44,45,46], Swanson [56,57] and other literature in the references. In either cases ($\Omega(\zeta)$ or $\Omega'(\zeta)$), the domain perturbation is singular in the sense that the domain can not be parametrized by a map depending smoothly up to the limit $\zeta = 0$. Hence it can not be dealt with a usual perturbation technique (fixing the domain by change of the variable) and so we need to invent a method to overcome the difficulty. This makes the problem more challenging and interesting. We refer to Jimbo-Kosugi [30], Panasenko [47] for other type of singular perturbation problem of elliptic eigenvalue problem. They study operators with the variable coefficients which degenerate in a subregion of the domain and there is a difficulty of singular phenomena at the limit. They obtain the elaborate characterization of the asymptotics of the eigenvalues. The results and difficulty of analysis are similar to those of the present paper. We briefly mention the background of the research. Many mathematical subjects of PDEs in singularly perturbed domains arise from phenomena of physics and engineering, where singular properties of shape of materials or spaces lead to the feature of the behavior of mathematical objects. Thus properties of solutions (or other quantities of PDEs) are studied with emphasis on each context (e.g. heat equations, wave equations, pattern formation in reaction-diffusion equations, Ginzburg-Landau equations in complicated domains). See Dancer [14], Kosugi [32,33], Matano [36], Matano-Mimura [37], Jimbo [26,27,28], Jimbo-Morita-Zhai [31], Morita [40], Vegas [58], Hale-Vegas [23], Raugel [51], Rubinstein-Sternberg-Wolansky [53], Rubinstein-Schatzman [52], Yanagida [59] and the references for related topics. We think that our results are related with the sound wave phenomena in the space with a sound hard boundary and we hope that they are applicable to

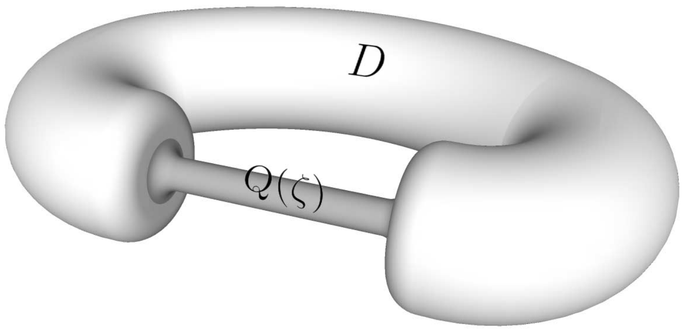

Fig. 4. Another Dumbbell-Shaped domain

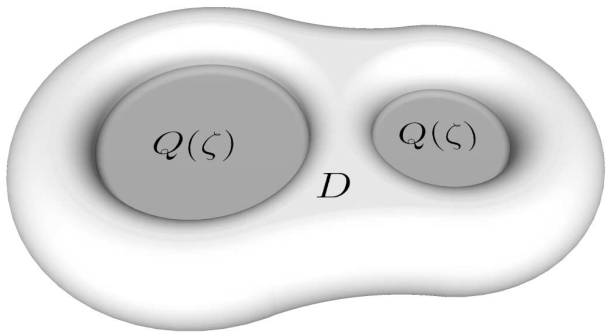

Fig. 5. Doubledoughnut+2Pancakes domain

elaborate analysis on many nonlinear PDE problems in future.

In the proof of the results (§3-§7) we often several notation including the Landau's symbols O and o ("large O" and "small O"). We give some of them.

Definition.
$g_1(\zeta) = O(g_2(\zeta)) \quad (\zeta \to 0) \Longleftrightarrow \limsup_{\zeta\to 0} |g_1(\zeta)/g_2(\zeta)| < \infty$
$g_1(\zeta) = o(g_2(\zeta)) \quad (\zeta \to 0) \Longleftrightarrow \lim_{\zeta\to 0} |g_1(\zeta)/g_2(\zeta)| = 0$
$\nabla = (\partial/\partial x_1, \cdots, \partial/\partial x_n), \quad \nabla' = (\partial/\partial x_1, \cdots, \partial/\partial x_\ell)$
$\nabla_z = (\partial/\partial s, \nabla_\eta) = (\partial/\partial s, \partial/\partial\eta_1, \partial/\partial\eta_2, \cdots, \partial/\partial\eta_m) \quad (z = (s, \eta))$
$\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2 \ \Delta' = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_\ell^2$
$\Delta_z = \partial^2/\partial s^2 + \partial^2/\partial\eta_1^2 + \cdots + \partial^2/\partial\eta_m^2$

§2. Formulation and the Main Results

We first formulate the domain $\Omega(\zeta) \subset \mathbb{R}^n$ ($\zeta > 0:$ small parameter). $n \geqq 2$ is a natural number. Let ℓ be a natural number such that $1 \leqq \ell < n$ and put $m = n - \ell$. For $x \in \mathbb{R}^n$, we can express it as $x = (x', x'') \in \mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}^m$, where $x' = (x_1, x_2, \cdots, x_\ell) \in \mathbb{R}^\ell$ and $x'' = (x_{\ell+1}, \cdots, x_n) \in \mathbb{R}^m$. Denote the origins of $\mathbb{R}^n, \mathbb{R}^\ell, \mathbb{R}^m$ by o, o', o'', respectively. $B^{(N)}(t) = \{y \in \mathbb{R}^N \mid |y| < t\}$ which is the ball in $\mathbb{R}^N$ of radius t.

The domain $\Omega(\zeta)$ has the following form:

$$\Omega(\zeta) = D \cup Q(\zeta) \tag{2.1}$$

where D and $Q(\zeta)$ are specified below. Let $D \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^\ell$ be bounded domains (or finite disjoint union of bounded domains) with C^4 boundaries, respectively. We impose the following conditions on D and Q.

Assumption. There exists $\zeta_0 > 0$ such that

$$\left\{\overline{Q} \times B^{(m)}(3\zeta_0)\right\} \cup \overline{D} \text{ is connected,}$$
$$\left\{\overline{Q} \times B^{(m)}(3\zeta_0)\right\} \cap \overline{D} = \partial Q \times B^{(m)}(3\zeta_0) \subset \partial D.$$

Although the first condition follows from the second one, we put it for comprehensibility. We note that ∂Q and $\overline{Q}$ are the boundary and the closure of Q in $\mathbb{R}^\ell$, respectively and so $\dim(\partial Q) = \ell - 1$ and denote the outward unit normal vector at $\xi \in \partial Q$ in $\mathbb{R}^\ell$ by $\boldsymbol{n} = \boldsymbol{n}(\xi)$. We define the subset $Q_t \subset Q$ for $t > 0$ by $Q_t = \{x' \in Q \mid \mathrm{dist}(x', \partial Q) > t\}$. For small $t \geqq 0$ it is written also as

$$Q_t = Q \setminus \{\xi + s\,\boldsymbol{n}(\xi) \in \mathbb{R}^\ell \mid \xi \in \partial Q,\ -t \leqq s < 0\}.$$

For the definition of the shrinking region $Q(\zeta)$, we prepare a positive continuous function $\boldsymbol{q} = \boldsymbol{q}(s) \in C^4((-\infty, 0)) \cap C^0((-\infty, 0])$ such that

$$\boldsymbol{q}(s) = 1 \quad (-\infty < s \leqq -1), \quad d\boldsymbol{q}/ds > 0 \quad (-1 < s < 0), \quad \boldsymbol{q}(0) = 2$$

and the inverse function $\boldsymbol{q}^{-1}(t)$ (defined in the interval $1 \leqq t \leqq 2$) satisfies

$$\lim_{t \uparrow 2} (d^k \boldsymbol{q}^{-1}(t)/dt^k) = 0 \quad \text{for} \quad 1 \leqq k \leqq 4.$$

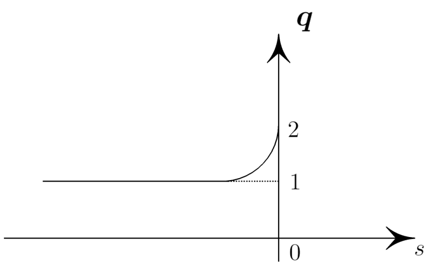

Fig. 6. Graph of $\boldsymbol{q} = \boldsymbol{q}(s)$

We define $Q(\zeta)$ in the following form:

$$Q(\zeta) = Q^{(1)}(\zeta) \cup Q^{(2)}(\zeta)$$

where

$$Q^{(1)}(\zeta) = Q_{2\zeta} \times B^{(m)}(\zeta) \subset \mathbb{R}^\ell \times \mathbb{R}^m,$$
$$Q^{(2)}(\zeta) = \{(\xi + s\,\boldsymbol{n}(\xi), \eta) \in \mathbb{R}^\ell \times \mathbb{R}^m \mid -2\zeta \leqq s \leqq 0,$$
$$|\eta| < \zeta \boldsymbol{q}(s/\zeta),\ \xi \in \partial Q\}.$$

From the above conditions, $\Omega(\zeta)$ is a bounded domain in $\mathbb{R}^n$ with a C^4 boundary $\partial\Omega(\zeta)$ for $\zeta \in (0, \zeta_0)$.

We consider the eigenvalue problem (1.1) for the domain $\Omega(\zeta)$ defined above.

Definition. Let $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ be the set of the eigenvalues of (1.1), which are arranged in increasing order (with counting multiplicity) and let $\{\Phi_{k,\zeta}\}_{k=1}^{\infty}$ be the corresponding complete system of eigenfunctions orthonormalized in $L^2(\Omega(\zeta))$. Obviously, we have the following equation (of weak form)

$$\int_{\Omega(\zeta)} (\nabla\Phi_{k,\zeta}\nabla\Psi - \mu_k(\zeta)\Phi_{k,\zeta}\Psi)dx = 0 \quad (\Psi \in H^1(\Omega(\zeta))). \tag{2.2}$$

It is easy to see $\mu_1(\zeta) = 0$ and the corresponding eigenfunction $\Phi_{1,\zeta}(x)$ is a constant function since $\Omega(\zeta)$ is connected. Moreover we can put $\Phi_{1,\zeta}(x) = 1/|\Omega(\zeta)|^{1/2}$.

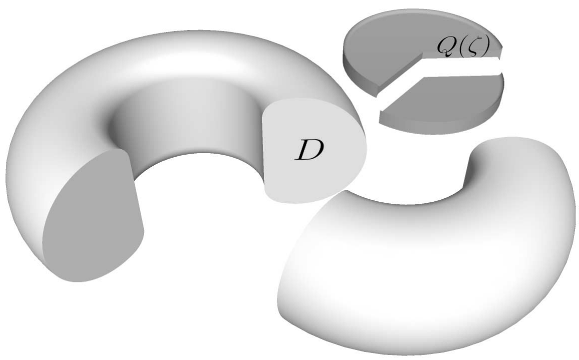

Fig. 7. Decomposition of Doughnut+Pancake Domain $n = 3, \ell = 2$

The purpose of this paper is to obtain an elaborate characterization of the asymptotic behavior of $\mu_k(\zeta)$ for $\zeta \to 0$. The following result is so called the 0−th order approximation and it is the first step toward our main results in this paper.

Proposition 2.1. *For any $k \in \mathbb{N}$, there exists $\mu_k \geqq 0$ such that*

$$\lim_{\zeta\to 0} \mu_k(\zeta) = \mu_k \quad (k \geqq 1) \tag{2.3}$$

and the set of all μ_k $(k \geqq 1)$ *is characterized as*

$$\{\mu_k\}_{k=1}^{\infty} = \{\omega_d\}_{d=1}^{\infty} \cup \{\lambda_r\}_{r=1}^{\infty}.$$

Here $\{\omega_d\}_{d=1}^{\infty}$ *and* $\{\lambda_r\}_{r=1}^{\infty}$ *are the eigenvalues which are arranged in increasing order (with counting multiplicity) of the following eigenvalue problems (2.4) and (2.5), respectively,*

$$\Delta\phi + \omega\phi = 0 \quad in \quad D, \qquad \partial\phi/\partial\boldsymbol{\nu} = 0 \quad on \quad \partial D, \tag{2.4}$$

$$\Delta'\psi + \lambda\psi = 0 \quad in \quad Q, \qquad \psi = 0 \quad on \quad \partial Q, \tag{2.5}$$

where $\Delta' = \sum_{i=1}^{\ell} \partial^2/\partial x_i^2$. *More precisely,* $\{\mu_k\}_{k=1}^{\infty}$ *is the sequence which is obtained by rearranging* $\{\omega_d\}_{d=1}^{\infty} \cup \{\lambda_r\}_{r=1}^{\infty}$ *in increasing order with counting multiplicity. We can express* μ_k *explicitly as follows,*

$$\mu_k = \max\left\{\min(\omega_i, \lambda_{k-i+1}) \mid 1 \leqq i \leqq k\right\}.$$

By the property $0 = \omega_1 < \lambda_1$ *(in (2.4),(2.5)), we can also express* μ_k *as*

$$\mu_1 = 0, \quad \mu_k = \min\left\{\max(\omega_i, \lambda_{k-i}), \omega_k \mid 1 \leqq i \leqq k-1\right\} \ (k \geqq 2).$$

It is easy to see that these two characterization of μ_k *coincide. We use the second one in the proof (in §3).*

As well as the limit value of the eigenvalue $\mu_k(\zeta)$, we can characterize the behavior of the corresponding eigenfunction $\Phi_{k,\zeta}$.

PROPOSITION 2.2. *For any positive sequence* $\{\zeta_p\}_{p=1}^{\infty}$ *with* $\lim_{p\to\infty} \zeta_p = 0$, *there exist a subsequence* $\{\sigma_p\}_{p=1}^{\infty}$ *and* $\Phi_k \in C^2(\overline{D})$, $\widehat{\Phi}_k \in C^2(\overline{Q})$ *such that*

$$\begin{cases} \lim\limits_{p\to\infty} \|\Phi_{k,\sigma_p} - \Phi_k\|_{L^2(D)} = 0, \\ \lim\limits_{p\to\infty} \sup\limits_{x=(x',x'')\in Q(\sigma_p)} |S(m)^{1/2}\sigma_p^{m/2}\Phi_{k,\sigma_p}(x) - \widehat{\Phi}_k(x')| = 0, \\ \lim\limits_{p\to\infty} \sup\limits_{x\in D} |S(m)^{1/2}\sigma_p^{m/2}\Phi_{k,\sigma_p}(x)| = 0, \end{cases} \tag{2.6}$$

$$\begin{cases} \Delta\Phi_k + \mu_k\Phi_k = 0 \quad in \quad D, \quad \partial\Phi_k/\partial\boldsymbol{\nu} = 0 \quad on \quad \partial D, \\ \Delta'\widehat{\Phi}_k + \mu_k\widehat{\Phi}_k = 0 \quad in \quad Q, \quad \widehat{\Phi}_k = 0 \quad on \quad \partial Q, \end{cases} \tag{2.7}$$

$$(\Phi_k, \Phi_{k'})_{L^2(D)} + (\widehat{\Phi}_k, \widehat{\Phi}_{k'})_{L^2(Q)} = \delta(k, k') \quad (k, k' \geqq 1), \tag{2.8}$$

where $\delta(k, k')$ is the Kronecker delta symbol define by

$$\delta(k, k') = \begin{cases} 1 & (k = k') \\ 0 & (k \neq k') \end{cases}$$

and $S(m)$ is the m-dimensional measure of the unit ball in $\mathbb{R}^m$. That is, $S(1) = 2$, $S(2) = \pi$, $S(3) = 4\pi/3$, $S(4) = \pi^2/2, \ldots, S(m) = \pi^{m/2}/\mathbf{\Gamma}((m/2) + 1)$. Here $\mathbf{\Gamma}$ is the Gamma function, which is defined as $\mathbf{\Gamma}(s) = \int_0^\infty t^{s-1} e^{-t} dt$.

The proofs of these propositions are to be given in §3.

Remark 2.3. (i) From (2.8), either Φ_k or $\widehat{\Phi}_k$ is non-zero for each k. It is easy to see that if $\mu_k \in \{\omega_d\}_{d=1}^\infty \setminus \{\lambda_r\}_{r=1}^\infty$, then $\widehat{\Phi}_k \equiv 0$ in Q and if $\mu_k \in \{\lambda_r\}_{r=1}^\infty \setminus \{\omega_d\}_{d=1}^\infty$, then $\Phi_k \equiv 0$ in D. However, for the case $\mu_k \in \{\omega_d\}_{d=1}^\infty \cap \{\lambda_r\}_{r=1}^\infty$, both $\Phi_k \not\equiv 0$ (in D) and $\widehat{\Phi}_k \not\equiv 0$ (in Q) may hold simultaneously. So $\Phi_{k,\zeta}$ plays an important role in both D and $Q(\zeta)$ in this case.
(ii) In Proposition 2.2, the sequence $\{\zeta_p\}_{p=1}^\infty$ was arbitrary and so we can assert

$$\begin{aligned} &\limsup_{\zeta \to 0} \left(\zeta^{m/2} \sup_{x \in \Omega(\zeta)} |\Phi_{k,\zeta}(x)| \right) < \infty, \\ &\lim_{\zeta \to 0} \left(\zeta^{m/2} \sup_{x \in D} |\Phi_{k,\zeta}(x)| \right) = 0 \end{aligned} \tag{2.9}$$

holds for any $k \geqq 1$ (cf. (2.6)).
(iii) In addition to (2.6) we have the following property

$$\lim_{p \to \infty} \|\Phi_{k,\sigma_p} - \Phi_k\|_{C^{3,\theta}(\overline{D}_t)} = 0, \quad \limsup_{\zeta \to 0} \|\Phi_{k,\zeta}\|_{C^{3,\theta}(\overline{D}_t)} < \infty$$

for any $t > 0$ and $0 \leqq \theta < 1$. Here $D_t = \{x \in D \mid \mathrm{dist}(x, \partial Q \times \{o''\}) \geqq t\}$. This property is proved with the aid of elliptic regularity argument (cf. Gilbarg-Trudinger [22]). Recall that ∂D is C^4.

We need to prepare several notation, constants and functions for the statement of the results.

DEFINITION. Let $\{\phi_d\}_{d=1}^{\infty} \subset L^2(D)$, $\{\psi_r\}_{r=1}^{\infty} \subset L^2(Q)$ be the corresponding eigenfunctions of (2.4) and (2.5) to ω_d and λ_r, respectively. We can assume, without loss of generality that they are real valued and orthonormal in $L^2(D)$ and $L^2(Q)$, respectively, i.e.

$$(2.10) \qquad \begin{aligned} &(\phi_d, \phi_{d'})_{L^2(D)} = \delta(d, d'), \\ &(\psi_r, \psi_{r'})_{L^2(Q)} = \delta(r, r') \quad (d, d', r, r' \geqq 1). \end{aligned}$$

Notation. From Proposition 2.1, we can decompose the sequence $\{\mu_k\}_{k=1}^{\infty}$ into the following three subsets

$$\begin{aligned} &E_I = \{\omega_d\}_{d=1}^{\infty} \setminus \{\lambda_r\}_{r=1}^{\infty}, \quad E_{II} = \{\lambda_r\}_{r=1}^{\infty} \setminus \{\omega_d\}_{d=1}^{\infty}, \\ &E_{III} = \{\omega_d\}_{d=1}^{\infty} \cap \{\lambda_r\}_{r=1}^{\infty}. \end{aligned}$$

According to these sets, we decompose $\mathbb{N}$ into (mutually disjoint) three sets

$$\begin{aligned} &\mathbb{N}_I = \{k \in \mathbb{N} \mid \mu_k \in E_I\}, \quad \mathbb{N}_{II} = \{k \in \mathbb{N} \mid \mu_k \in E_{II}\}, \\ &\mathbb{N}_{III} = \{k \in \mathbb{N} \mid \mu_k \in E_{III}\}. \end{aligned}$$

We need to introduce a system for numbering the eigenvalues, which we use to define several quantities for the main theorems.

DEFINITION. We define three increasing sequences of natural numbers $\{d(j)\}_{j=1}^{\infty}$, $\{r(j)\}_{j=1}^{\infty}$, $\{k(j)\}_{j=1}^{\infty}$ by induction.

$$\begin{aligned} &d(1) = 1, \quad d(j+1) = \min\{d \in \mathbb{N} \mid \omega_d > \omega_{d(j)}\}. \\ &r(1) = 1, \quad r(j+1) = \min\{r \in \mathbb{N} \mid \lambda_r > \lambda_{r(j)}\}. \\ &k(1) = 1, \quad k(j+1) = \min\{k \in \mathbb{N} \mid \mu_k > \mu_{k(j)}\}. \end{aligned}$$

Let $\widehat{d}(j)$, $\widehat{r}(j)$, $\widehat{k}(j)$ be the multiplicity of $\omega_{d(j)}$, $\lambda_{r(j)}$, $\mu_{k(j)}$, respectively. Namely,

$$\widehat{d}(j) = d(j+1) - d(j), \quad \widehat{r}(j) = r(j+1) - r(j), \quad \widehat{k}(j) = k(j+1) - k(j).$$

It is easy to see the following properties.

If $\mu_{k(j)} = \omega_{d(j')} \in E_I$, then $\widehat{k}(j) = \widehat{d}(j')$.

If $\mu_{k(j)} = \lambda_{r(j'')} \in E_{II}$, then $\widehat{k}(j) = \widehat{r}(j'')$.
If $\mu_{k(j)} = \omega_{d(j')} = \lambda_{r(j'')} \in E_{III}$, then $\widehat{k}(j) = \widehat{d}(j') + \widehat{r}(j'')$.

As was mentioned in §1, the properties of the behavior $\mu_k(\zeta)$ mainly depend on the three cases $\mu_k \in E_I$ or $\mu_k \in E_{II}$ or $\mu_k \in E_{III}$. The purpose of this paper is to study elaborately these properties and dependencies. From Proposition 2.1 and Proposition 2.2, the $\widehat{k}(j)$ eigenvalues $\mu_k(\zeta)$ for $k(j) \leqq k < k(j+1)$ approach the value $\mu_{k(j)} = \mu_{k(j)+1} = \mu_{k(j)+2} = \cdots = \mu_{k(j+1)-1}$ as $\zeta \to 0$. We describe their behaviors with the aid of $\widehat{k}(j) \times \widehat{k}(j)$ matrices. The results are to be given mainly in three different cases: $\mu_{k(j)} \in E_I$, $\mu_{k(j)} \in E_{II}$, $\mu_{k(j)} \in E_{III}$ (cf.Fig.8).

For the statement of the main results, we need to prepare several more notation for the statement of the results. We will use three series of matrices

$$\boldsymbol{A}(j) \quad (\text{for } k(j) \in \mathbb{N}_I), \qquad \boldsymbol{B}(j) \quad (\text{for } k(j) \in \mathbb{N}_{II} \cup \mathbb{N}_{III}),$$
$$\boldsymbol{C}(j) \quad (\text{for } k(j) \in \mathbb{N}_{III}).$$

The first one of them is the following.

Definition (The matrix $\boldsymbol{A}(j)$). For $k(j) \in \mathbb{N}_I$, there exists a unique $j' \in \mathbb{N}$ such that $\mu_{k(j)} = \omega_{d(j')}$. We define a real symmetric matrix $\boldsymbol{A}(j)$

<table>
<tr><td>Case I ($k \in \mathbb{N}_I$)
Non-Resonant</td><td colspan="4">$m \geqq 1$: Theorem 2.5</td></tr>
<tr><td>Case II ($k \in \mathbb{N}_{II}$)
Non-Resonant</td><td colspan="2">$m = 1$: Theorem 2.10</td><td colspan="2">$m \geqq 2$: Theorem 2.11</td></tr>
<tr><td rowspan="2">Case III ($k \in \mathbb{N}_{III}$)
Resonant</td><td rowspan="2">$m = 1$:
Theorem 2.12</td><td rowspan="2" colspan="2">$m = 2$:
Theorem 2.13</td><td>$m \geqq 3, T(\boldsymbol{q}, m) > 0$:
Theorem 2.14-(i)</td></tr>
<tr><td>$m \geqq 3, T(\boldsymbol{q}, m) < 0$:
Theorem 2.14-(ii)</td></tr>
</table>

Fig. 8. List of the main results

which is $\widehat{d}(j') \times \widehat{d}(j')$. We put the quantity

$$a(p,q) = \int_{\partial Q} \frac{\partial V_p}{\partial \boldsymbol{n}}(\xi)\, \phi_q(\xi, o'')\, dS'$$

for $d(j') \leqq p, q < d(j'+1)$ where $V_p \in C^2(\overline{Q})$ is the unique solution V of

$$\Delta' V + \omega_p V = 0 \quad \text{in} \quad Q, \quad V(\xi) = \phi_p(\xi, o'') \quad \text{for} \quad \xi \in \partial Q, \tag{2.11}$$

for $d(j') \leqq p < d(j'+1)$. Here dS' is the $\ell - 1$ dimensional measure on ∂Q.

Put the matrix $\boldsymbol{A}(j) = [a(p,q)]_{d(j') \leqq p,q < d(j'+1)}$. Denote the eigenvalues of $S(m)\, \boldsymbol{A}(j)$ by

$$\boldsymbol{\alpha}(k(j)) \leqq \boldsymbol{\alpha}(k(j)+1) \leqq \cdots \leqq \boldsymbol{\alpha}(k(j+1)-1).$$

Recall that $S(m)$ is the volume of the unit ball in $\mathbb{R}^m$.

We note that $\boldsymbol{A}(j)$ depends on the choice of $\{\phi_d\}_{d \in \mathbb{N}}$, but that $\{\boldsymbol{\alpha}(k) \mid k(j) \leqq k < k(j+1)\}$ does not depend on that choice. The characterization of the behaviors of eigenvalues depends only on the geometric situation.

Thus we have the numbers $\boldsymbol{\alpha}(k)$ for all $k \in \mathbb{N}_I$.

REMARK 2.4. The unique existence of V in (2.11) is guaranteed by $\omega_{d(j')} \notin \{\lambda_r\}_{r=1}^{\infty}$. It is easy to see that $a(p,q)$ is symmetric in p, q. Actually it is also expressed (after partial integration) by

$$a(p,q) = \int_Q \left(\nabla' V_p \nabla' V_q - \omega_p\, V_p V_q \right)\, dx'.$$

By (2.11), we defined $V_p \in C^2(\overline{Q})$ for p such that $\omega_p \in E_I$. It is convenient in later sections to define V_p also for the case $\omega_p \in E_{III}$.

DEFINITION. For $p \in \mathbb{N}$ such that $\omega_p \in E_{III}$, let V_p be the unique solution V of the following equation

$$\Delta' V = 0 \quad \text{in} \quad Q, \quad V(\xi) = \phi_p(\xi, o'') \quad \text{for} \quad \xi \in \partial Q. \tag{2.12}$$

Thus the function V_p has been defined for each $p \in \mathbb{N}$.

Now we are ready to present our first main result. Using these quantities, we can describe the following characterization for asymptotics of $\mu_k(\zeta)$ for $k \in \mathbb{N}_I$.

THEOREM 2.5. *Assume $k \in \mathbb{N}_I$, then we have,*

$$\lim_{\zeta \to 0} \frac{\mu_k(\zeta) - \mu_k}{\zeta^m} = \boldsymbol{\alpha}(k). \tag{2.13}$$

REMARK 2.6. For the case of Dumbbell shaped domain, this result agrees to those in Jimbo [25] ($n \geqq 3, m = n-1$), Gadylshin [21] ($n = 3, m = 2$), Arrieta [4] ($n = 2, m = 1$).

Harmonic function $G = G(z)$ and Quantity $T(\boldsymbol{q}, m)$.

In the characterization of the eigenvalues of the latter cases (II),(III), we need a characteristic quantity $T(\boldsymbol{q}, m)$ which is the function of $\boldsymbol{q}$ and $m \geqq 2$.

We define an unbounded domain H in $\mathbb{R} \times \mathbb{R}^m$. An element $z \in \mathbb{R} \times \mathbb{R}^m$ is expressed as $z = (s, \eta) = (s, \eta_1, \eta_2, \cdots, \eta_m)$. H is given by

$$H = H_1 \cup H_2 \subset \mathbb{R} \times \mathbb{R}^m$$

where H_1 and H_2 are given by

$$H_1 = (0, \infty) \times \mathbb{R}^m, \quad H_2 = \{(s, \eta) \in \mathbb{R} \times \mathbb{R}^m \mid -\infty < s \leqq 0, |\eta| < \boldsymbol{q}(s)\}.$$

We will prepare a certain important harmonic function $G(z) = G(s, \eta)$ (which is particular to H) for the statement of the main results (and later discussions and the proofs). Consider the equation,

$$\Delta_z G = 0 \quad \text{in} \quad H, \quad \partial G / \partial \widetilde{\boldsymbol{n}} = 0 \quad \text{on} \quad \partial H, \tag{2.14}$$

where $\Delta_z = \partial^2/\partial s^2 + \partial^2/\partial \eta_1^2 + \cdots + \partial^2/\partial \eta_m^2$ and $\widetilde{\boldsymbol{n}}$ is the unit outward normal vector on ∂H. The existence of a (non-trivial) solution G is given in the following result.

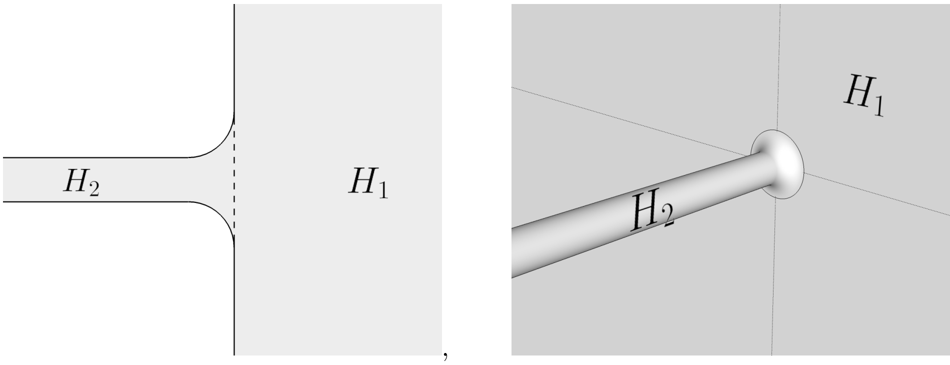

Fig. 9. $H = H_1 \cup H_2 \subset \mathbb{R}^{m+1}$ (Left $m = 1$, Right $m \geqq 2$)

PROPOSITION 2.7.
(i) *For $m = 1$, there exist a non-constant solution $G = G(z) = G(s, \eta)$ of (2.14) and constants $\boldsymbol{\kappa}_1 > 0$, $\delta > 0$, $c_0 > 0$, $\boldsymbol{\kappa}_2 \in \mathbb{R}$ such that*

$$|G(z) - (-\boldsymbol{\kappa}_1 s + \boldsymbol{\kappa}_2)| + |\partial G(z)/\partial s + \boldsymbol{\kappa}_1| + |\nabla_\eta G(z)| \leqq c_0 e^{\delta s} \quad (z = (s, \eta) \in H_2),$$

$$\left| G(z) - (2\boldsymbol{\kappa}_1/\pi) \log \frac{1}{|z|} \right| \leqq \frac{c_0}{|z|} \quad (z \in H_1),$$

$$\left| \nabla_z G(z) - (-2\boldsymbol{\kappa}_1/\pi) \frac{z}{|z|^2} \right| \leqq \frac{c_0}{|z|^2} \quad (z \in H_1).$$

(ii) *For $m \geqq 2$, there exist a non-constant positive solution $G = G(z) = G(s, \eta)$ of (2.14) and constants $\boldsymbol{\kappa}_1 > 0$, $\delta > 0$, $c_0 > 0$, $\boldsymbol{\kappa}_2 \in \mathbb{R}$ such that*

$$|G(z) - (-\boldsymbol{\kappa}_1 s + \boldsymbol{\kappa}_2)| + |\partial G(z)/\partial s + \boldsymbol{\kappa}_1| + |\nabla_\eta G(z)| \leqq c_0 e^{\delta s} \quad (z = (s, \eta) \in H_2),$$

$$\left| G(z) - \frac{2S(m)\boldsymbol{\kappa}_1}{(m-1)(m+1)S(m+1)|z|^{m-1}} \right| \leqq \frac{c_0}{|z|^m} \quad (z \in H_1),$$

$$\left| \nabla_z G(z) - \frac{-2S(m)\boldsymbol{\kappa}_1 z}{(m+1)S(m+1)|z|^{m+1}} \right| \leqq \frac{c_0}{|z|^{m+1}} \quad (z \in H_1).$$

The outline of the proof of this proposition will be given in §8 Appendix.

REMARK 2.8. We can have similar estimates for higher order derivatives $\nabla_z^p G$ for $p \geqq 2$. The following rough estimates for G follow immediately from the above Proposition 2.7. There exists $c_0' > 0$ such that

$$\text{The case } m = 1: \qquad \begin{cases} |G(z)| \leqq c_0'(1 + \log|z|) \\ |\nabla_z G(z)| \leqq c_0'/|z| \end{cases} \quad (z \in H_1, |z| \geqq 1),$$

$$\text{The case } m \geqq 2: \qquad \begin{cases} |G(z)| \leqq c_0'/|z|^{m-1} \\ |\nabla_z G(z)| \leqq c_0'/|z|^m \end{cases} \quad (z \in H_1).$$

DEFINITION (The constant $T(\boldsymbol{q}, m)$). For the case $m \geqq 2$, we put $T(\boldsymbol{q}, m) = \boldsymbol{\kappa}_2/\boldsymbol{\kappa}_1$ for $\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2$ which are obtained in Proposition 2.7-(ii). We should note that $T(\boldsymbol{q}, m)$ depends on only $\boldsymbol{q}$ and m. This quantity was first introduced in Gadylshin [21] for the study in the case of the Dumbbell shaped domain $n = 3, m = 2$.

We prepare several more matrices.

DEFINITION (The matrix $\boldsymbol{B}(j)$). Assume $k(j) \in \mathbb{N}_{II}$ or $k(j) \in \mathbb{N}_{III}$. Then, there exists a unique $j' \in \mathbb{N}$ such that $\mu_{k(j)} = \lambda_{r(j')}$. We put the quantity

$$(2.15) \qquad b(p, q) = \int_{\partial Q} \frac{\partial \psi_p}{\partial \boldsymbol{n}}(\xi) \frac{\partial \psi_q}{\partial \boldsymbol{n}}(\xi)\, dS' \quad (r(j') \leqq p, q < r(j' + 1)).$$

Here dS' is the $\ell - 1$ dimensional measure on ∂Q. We define a real symmetric matrix $\boldsymbol{B}(j)$ by $\boldsymbol{B}(j) = [b(p, q)]_{r(j') \leqq p,q < r(j'+1)}$.

LEMMA 2.9. *In both cases $k(j) \in \mathbb{N}_{II}$ and $k(j) \in \mathbb{N}_{III}$, the matrix $\boldsymbol{B}(j)$ defined above is invertible and all the eigenvalues of $\boldsymbol{B}(j)$ are positive.*

PROOF OF LEMMA 2.9. Put $q = r(j' + 1) - r(j') = \widehat{r}(j')$ and take any non-zero vector

$$\boldsymbol{u} = {}^t(u_1, \cdots, u_q) \in \mathbb{R}^q.$$

By a simple calculation, we have

$$(\boldsymbol{B}(j)\boldsymbol{u}, \boldsymbol{u})_{\mathbb{R}^q} = \int_{\partial Q} \left(\sum_{p=1}^{q} u_p \frac{\partial \psi_{p+r(j')-1}}{\partial \boldsymbol{n}}(\xi) \right)^2 dS' \geqq 0$$

which is non-negative. We will prove that this value is positive for any nonzero vector $\boldsymbol{u}$. Assume it attains zero for some nonzero vector $\boldsymbol{u} \in \mathbb{R}^q$. Then the function

$$\psi(x') = \sum_{p=1}^{q} u_p \, \psi_{p+r(j')-1}(x') \quad (x' \in Q)$$

satisfies the Neumann B.C. on ∂Q. However it is an eigenfunction of (2.5) with respect to the eigenvalue $\lambda_{r(j')}$. Consequently, ψ satisfies both the Dirichlet and the Neumann B.C. on ∂Q and so $\psi \equiv 0$ in Q. Since $\boldsymbol{u}$ is a nonzero vector, this is contrary to that $\{\psi_r\}$ is linearly independent system (actually orthonormal) in $L^2(Q)$. Therefore the quadratic form $(\boldsymbol{B}(j)\boldsymbol{u}, \boldsymbol{u})_{\mathbb{R}^q}$ is positive definite. □

DEFINITION (The matrix $\boldsymbol{C}(j)$). For $k(j) \in \mathbb{N}_{III}$, there exist a unique pair $j' \in \mathbb{N}$ and $j'' \in \mathbb{N}$ such that $\mu_{k(j)} = \omega_{d(j')}$ and $\mu_{k(j)} = \lambda_{r(j'')}$. Put

$$c(p,q) = \int_{\partial Q} \phi_p(\xi, o'') \, \frac{\partial \psi_q}{\partial \boldsymbol{n}}(\xi) \, dS' \tag{2.16}$$

for $d(j') \leqq p < d(j'+1), r(j'') \leqq q < r(j''+1)$. Define the $\widehat{d}(j') \times \widehat{r}(j'')$ matrix by

$$\boldsymbol{C}(j) = [c(p,q)]_{d(j') \leqq p < d(j'+1), r(j'') \leqq q < r(j''+1)}.$$

Now we present a result characterizing $\mu_k(\zeta)$ approaching $\mu_k \in E_{II} \cup E_{III}$.

Concerning the eigenvalues $\mu_k(\zeta)$ approaching an element in E_{II}, we have two subcases $m = 1$ and $m \geqq 2$.

THEOREM 2.10. *Assume $m = 1$ and $k(j) \in \mathbb{N}_{II}$. Then we have,*

$$\lim_{\zeta \to 0} \frac{\mu_k(\zeta) - \mu_k}{\zeta \log(1/\zeta)} = \boldsymbol{\beta}_1(k) \quad (\textit{for} \quad k(j) \leqq k < k(j+1)), \tag{2.17}$$

where $\boldsymbol{\beta}_1(k(j)) \leqq \boldsymbol{\beta}_1(k(j)+1) \leqq \boldsymbol{\beta}_1(k(j)+2) \leqq \cdots \leqq \boldsymbol{\beta}_1(k(j+1)-1)$ are the eigenvalues of the matrix

$$(-2/\pi)\boldsymbol{B}(j).$$

For the case of Dumbbell shaped domain $n = 2, \ell = 1, m = 1$ (Fig.1) and simple eigenvalue, this result agrees to the one in Arrieta [4].

Theorem 2.11. *Assume $m \geqq 2$ and $k(j) \in \mathbb{N}_{II}$. Then we have,*

$$\lim_{\zeta \to 0} \frac{\mu_k(\zeta) - \mu_k}{\zeta} = \boldsymbol{\beta}_m(k) \quad (k(j) \leqq k < k(j+1)), \tag{2.18}$$

where $\boldsymbol{\beta}_m(k(j)) \leqq \boldsymbol{\beta}_m(k(j)+1) \leqq \boldsymbol{\beta}_m(k(j)+2) \leqq \cdots \leqq \boldsymbol{\beta}_m(k(j+1)-1)$ are the eigenvalues of the matrix

$$-T(\boldsymbol{q}, m)\boldsymbol{B}(j).$$

For the case of Dumbbell shaped domain $n = 3, \ell = 1, m = 2$ (Fig.2) and simple eigenvalue, this result agrees to the one in Gadylshin [21].

We note that $\{\boldsymbol{\beta}_m(k) \mid k(j) \leqq k < k(j+1)\}$ does not depend on the choice of $\{\psi_r\}_{r \in \mathbb{N}}$, but on Q and $\boldsymbol{q}$.

Concerning the eigenvalues $\mu_k(\zeta)$ approaching an element in E_{III}, we have three subcases $m = 1$, $m = 2$ and $m \geqq 3$.

Theorem 2.12. *Assume $m = 1$ and $k(j) \in \mathbb{N}_{III}$.*

$$\lim_{\zeta \to 0} \frac{\mu_k(\zeta) - \mu_k}{\zeta^{1/2}} = \boldsymbol{\gamma}_1(k) \quad (k(j) \leqq k < k(j+1)), \tag{2.19}$$

where $\boldsymbol{\gamma}_1(k(j)) \leqq \boldsymbol{\gamma}_1(k(j)+1) \leqq \boldsymbol{\gamma}_1(k(j)+2) \leqq \cdots \leqq \boldsymbol{\gamma}_1(k(j+1)-1)$ are the eigenvalues of the matrix

$$\begin{pmatrix} O & \sqrt{2}\boldsymbol{C}(j) \\ \sqrt{2}\,{}^t\boldsymbol{C}(j) & O \end{pmatrix}. \tag{2.20}$$

Theorem 2.13. *Assume $m = 2$ and $k(j) \in \mathbb{N}_{III}$, then we have*

$$\lim_{\zeta \to 0} \frac{\mu_k(\zeta) - \mu_k}{\zeta} = \boldsymbol{\gamma}_2(k) \quad (k(j) \leqq k < k(j+1)), \tag{2.21}$$

where

$$\boldsymbol{\gamma}_2(k(j)) \leqq \boldsymbol{\gamma}_2(k(j)+1) \leqq \boldsymbol{\gamma}_2(k(j)+2) \leqq \cdots \leqq \boldsymbol{\gamma}_2(k(j+1)-1)$$

are the eigenvalues of the matrix

$$\begin{pmatrix} O & \sqrt{\pi}\boldsymbol{C}(j) \\ \sqrt{\pi}\ {}^t\boldsymbol{C}(j) & -T(\boldsymbol{q},2)\boldsymbol{B}(j) \end{pmatrix}. \tag{2.22}$$

THEOREM 2.14. *Assume $m \geqq 3$ and $k(j) \in \mathbb{N}_{III}$.*
The case (i): *If $T(\boldsymbol{q},m) > 0$, we have*

$$\lim_{\zeta\to 0} \frac{\mu_k(\zeta) - \mu_k}{\zeta} = \boldsymbol{\gamma}_m^+(k) \quad (k(j) \leqq k < k(j) + \widehat{r}(j'')), \tag{2.23}$$

$$\lim_{\zeta\to 0} \frac{\mu_k(\zeta) - \mu_k}{\zeta^{m-1}} = \boldsymbol{\gamma}_m^+(k) \quad (k(j) + \widehat{r}(j'') \leqq k < k(j+1)), \tag{2.24}$$

where
(i-1) $\boldsymbol{\gamma}_m^+(k)$ $(k(j) \leqq k < k(j) + \widehat{r}(j''))$ *are the eigenvalues of the matrix*

$$-T(\boldsymbol{q},m)\boldsymbol{B}(j).$$

(i-2) $\boldsymbol{\gamma}_m^+(k)$ $(k(j) + \widehat{r}(j'') \leqq k < k(j+1))$ *are the eigenvalues of the matrix*

$$(S(m)/T(\boldsymbol{q},m))\,\boldsymbol{C}(j)\,\boldsymbol{B}(j)^{-1}\,{}^t\boldsymbol{C}(j).$$

The case (ii): *If $T(\boldsymbol{q},m) < 0$, we have*

$$\lim_{\zeta\to 0} \frac{\mu_k(\zeta) - \mu_k}{\zeta^{m-1}} = \boldsymbol{\gamma}_m^-(k) \quad (k(j) \leqq k < k(j) + \widehat{d}(j')), \tag{2.25}$$

$$\lim_{\zeta\to 0} \frac{\mu_k(\zeta) - \mu_k}{\zeta} = \boldsymbol{\gamma}_m^-(k) \quad (k(j) + \widehat{d}(j') \leqq k < k(j+1)), \tag{2.26}$$

where
(ii-1) $\boldsymbol{\gamma}_m^-(k)$ $(k(j) \leqq k < k(j) + \widehat{d}(j'))$ *are the eigenvalues of the matrix*

$$(S(m)/T(\boldsymbol{q},m))\,\boldsymbol{C}(j)\,\boldsymbol{B}(j)^{-1}\,{}^t\boldsymbol{C}(j).$$

(ii-2) $\boldsymbol{\gamma}_m^-(k)$ $(k(j) + \widehat{d}(j') \leqq k < k(j+1))$ *are the eigenvalues of the matrix*

$$-T(\boldsymbol{q},m)\,\boldsymbol{B}(j).$$

We note that $\{\boldsymbol{\gamma}^{\pm}_m(k) \mid k(j) \leqq k < k(j+1)\}$ *does not depend on the choice of* $\{\psi_r\}_{r\in\mathbb{N}}$ *and* $\{\phi_d\}_{d\in\mathbb{N}}$*, but on* Q, D *and* $\boldsymbol{q}$*.*

The proofs of Theorem 2.5 and Theorem 2.10 – 2.14 are to be given in § 5 – § 7 after the preparation of some approximate eigenfunctions, comparison functions with properties and a certain characterization of the eigenfunctions.

§3. 0-th Order Approximation (Proof of Proposition 2.1 and Proposition 2.2)

In this section we will prove Proposition 2.1 and Proposition 2.2 with some auxiliary properties of eigenfunctions $\Phi_{k,\zeta}$.

Proof of Proposition 2.1 and Proposition 2.2. We will prove that each $\mu_k(\zeta)$ approaches μ_k which is defined by

$$\begin{aligned} &\mu_1 = 0, \\ &\mu_k = \min\{\max(\omega_i, \lambda_{k-i}),\ \omega_k \mid i = 1, 2, \cdots, k-1\} \quad (k \geqq 2). \end{aligned} \tag{3.1}$$

The case $k = 1$ is trivial because $\mu_1(\zeta) = 0$. So we assume $k \geqq 2$ hereafter.

Recall that ω_d ($d \geqq 1$) and λ_r ($r \geqq 1$) are the eigenvalues in (2.4) and (2.5), respectively. To obtain the behaviors of the eigenvalues $\mu_k(\zeta)$, we will construct (rough) approximate eigenfunctions by using $\{\phi_d\}_{d=1}^{\infty}$ and $\{\psi_r\}_{r=1}^{\infty}$ in (2.4) and (2.5) (introduced in §2). We extend each ϕ_d ($d \geqq 1$) as a C^3 function to $\mathbb{R}^n$ and denote it also by ϕ_d. Let $\widetilde{\phi}_d$ be the restriction of ϕ_d to $\overline{\Omega(\zeta)}$ and it belongs to $C^3(\overline{\Omega(\zeta)})$. For $r \geqq 1$ we define $\widetilde{\psi}_r(x)$ by

$$\widetilde{\psi}_r(x) = 0 \ \text{ for } \ x \in D, \quad \widetilde{\psi}_r(x) = \psi_r(x') \ \text{ for } \ x = (x', x'') \in Q(\zeta).$$

Then $\widetilde{\psi}_r \in H^1(\Omega(\zeta)) \cap C^0(\overline{\Omega(\zeta)})$ and it is C^3 except at $\partial D \cap \partial Q(\zeta)$. It is known that the eigenvalue is characterized through the following Max-Min principle.

Lemma 3.1 (Courant-Hilbert [12]). *For any natural number* $k \geqq 1$

$$\mu_k(\zeta) = \sup_{dim W = k-1, W \subset L^2(\Omega(\zeta))} \inf\{\mathcal{R}_\zeta[\Phi] \mid \Phi \in H^1(\Omega(\zeta)), \\ \Phi \perp W \ in\, L^2(\Omega(\zeta))\}. \tag{3.2}$$

Here the functional $\mathcal{R}_\zeta[\Phi]$ is defined by

$$(3.3) \qquad \mathcal{R}_\zeta[\Phi] = \int_{\Omega(\zeta)} |\nabla\Phi|^2 dx \Big/ \int_{\Omega(\zeta)} |\Phi|^2 dx \quad (\textbf{Rayleigh quotient}).$$

W is a linear subspace of $L^2(\Omega(\zeta))$. In the expression (3.2), $\Phi \perp W$ in $L^2(\Omega(\zeta))$ implies $(\Phi, \Psi)_{L^2(\Omega(\zeta))} = 0$ for any $\Psi \in W$. See also Edmunds-Evans [15;Chap.XI].

Upper estimate of $\mu_k(\zeta)$

First we prove that the limit-sup of $\mu_k(\zeta)$ is not greater than μ_k by Lemma 3.1. From the definition of μ_k, the following (i) or (ii) holds.

(i) there exist integers $d, r \geqq 1$ such that $d + r = k$ and $\mu_k = \max(\omega_d, \lambda_r)$

(ii) $\mu_k = \omega_k$.

We deal with only the case (i) because (ii) is similar. We take an arbitrary $(k-1)$-dimensional subspace $W \subset L^2(\Omega(\zeta))$. Consider the following finite dimensional subspace

$$\widetilde{W} = \text{L.H.}[\widetilde{\phi}_1, \widetilde{\phi}_2, \cdots, \widetilde{\phi}_d, \widetilde{\psi}_1, \cdots, \widetilde{\psi}_r] \subset L^2(\Omega(\zeta)).$$

Here L.H.$[X]$ is the linear subspace generated by the set X. Using (2.10), we can easily show that $\dim\widetilde{W} = k$ for small $\zeta > 0$. From the dimension theorem and the decomposition

$$\widetilde{W} = (\widetilde{W} \cap W) \oplus (\widetilde{W} \cap W^\perp),$$

we can take a non-zero element $\Phi \in \widetilde{W}$ such that $\Phi \perp W$ in $L^2(\Omega(\zeta))$. We can put

$$\Phi = c_1\widetilde{\phi}_1 + c_2\widetilde{\phi}_2 + \cdots + c_d\widetilde{\phi}_d + (S(m)^{-1/2}\zeta^{-m/2})(c'_1\widetilde{\psi}_1 + \cdots + c'_r\widetilde{\psi}_r)$$

(all the coefficients c_i, c'_j depend on $\zeta > 0$ and W).

By multiplying a positive constant, we can assume $\|\Phi\|_{L^2(\Omega(\zeta))} = 1$ without changing the value $\mathcal{R}_\zeta[\Phi]$. Using the properties of ϕ_i, ψ_j (cf. (2.4),(2.5), (2.10)), we calculate

$$(3.4) \qquad 1 = \|\Phi\|^2_{L^2(\Omega(\zeta))} = c_1^2 + \cdots + c_d^2 + (c'_1)^2 + \cdots + (c'_r)^2 + \sum_{1 \leqq i,j \leqq d} \kappa(1, i, j, \zeta)\, c_i\, c_j$$

$$+ \sum_{1 \leqq i \leqq d, 1 \leqq j \leqq r} \kappa(2,i,j,\zeta)\, c_i\, c_j' + \sum_{1 \leqq i,j \leqq r} \kappa(3,i,j,\zeta)\, c_i'\, c_j'.$$

Here we notice that $\kappa(1,i,j,\zeta) = O(\zeta^m)$, $\kappa(2,i,j,\zeta) = O(\zeta^{m/2})$, $\kappa(3,i,j,\zeta) = O(\zeta)$ and these coefficients do not depend on the choice of W and c_i, c_j', but they depend on $\widetilde{\phi}_1, \cdots, \widetilde{\phi}_d$, $\widetilde{\psi}_1, \cdots, \widetilde{\psi}_r$. For example $\kappa(1,i,j,\zeta) = \int_{Q(\zeta)} \widetilde{\phi}_i\, \widetilde{\phi}_j\, dx$. Substituting Φ into $\mathcal{R}_\zeta$ we obtain

$$\begin{aligned}\mathcal{R}_\zeta[\Phi] = &\int_D |\nabla(c_1\, \widetilde{\phi}_1 + \cdots + c_d\, \widetilde{\phi}_d)|^2\, dx \\ &+ \int_{Q(\zeta)} |\nabla(c_1\, \widetilde{\phi}_1 + \cdots + c_d\, \widetilde{\phi}_d) \\ &\qquad + (S(m)^{-1/2} \zeta^{-m/2}) \nabla(c_1'\, \widetilde{\psi}_1 + \cdots + c_r'\, \widetilde{\psi}_r)|^2\, dx.\end{aligned}$$

We use the properties of ϕ_i and ψ_j.

$$\begin{aligned}\mathcal{R}_\zeta[\Phi] = &\ \omega_1\, c_1^2 + \cdots + \omega_d\, c_d^2 + \lambda_1 (c_1')^2 + \cdots + \lambda_r (c_r')^2 \\ &+ \sum_{1 \leqq i,j \leqq d} \kappa'(1,i,j,\zeta)\, c_i\, c_j + \sum_{1 \leqq i \leqq d, 1 \leqq j \leqq r} \kappa'(2,i,j,\zeta)\, c_i\, c_j' \\ &+ \sum_{1 \leqq i,j \leqq r} \kappa'(3,i,j,\zeta)\, c_i'\, c_j'.\end{aligned}$$

Here $\kappa'(1,i,j,\zeta) = O(\zeta^m)$, $\kappa'(2,i,j,\zeta) = O(\zeta^{m/2})$, $\kappa'(3,i,j,\zeta) = O(\zeta)$ which are also independent of W. We get

$$\begin{aligned}\mathcal{R}_\zeta[\Phi] \leqq &\max(\omega_d, \lambda_r) \left(c_1^2 + \cdots + c_d^2 + (c_1')^2 + \cdots + (c_r')^2\right) \\ &+ \sum_{1 \leqq i,j \leqq d} |\kappa'(1,i,j,\zeta)||c_i||c_j| + \sum_{1 \leqq i \leqq d, 1 \leqq j \leqq r} |\kappa'(2,i,j,\zeta)||c_i||c_j'| \\ &+ \sum_{1 \leqq i,j \leqq r} |\kappa'(3,i,j,\zeta)||c_i'||c_j'|.\end{aligned} \tag{3.5}$$

Estimating the right hand side of (3.5) under the constraint (3.4), we get

$$\mathcal{R}_\zeta[\Phi] \leqq \max(\omega_d, \lambda_r) + K\, \zeta^{\min(1, m/2)}$$

for each $k \geqq 1$. K does not depend on choice of W and the coefficients c_i, c_j' $(1 \leqq i \leqq d, 1 \leqq j \leqq r)$. Taking supremum for all choices W with $\dim W = k - 1$ in the above inequality with the aid of (3.2), we get

$$\mu_k(\zeta) \leqq \max(\omega_d, \lambda_r) + K\, \zeta^{\min(1, m/2)} = \mu_k + K\, \zeta^{\min(1, m/2)} \tag{3.6}$$

for each $k \geqq 1$ from (3.2). The second case (ii) can be dealt with similarly. Actually, putting $\widetilde{W} = L.H.[\widetilde{\phi}_1, \cdots, \widetilde{\phi}_k]$ and applying the similar argument as in the proof of (3.6), we get

$$\mu_k(\zeta) \leqq \mu_k + K' \zeta^{\min(1,m/2)}.$$

Here K' is a constant which depends only on k. In both cases (i) and (ii), we have

$$\limsup_{\zeta\to 0} \mu_k(\zeta) \leqq \mu_k \quad (k \geqq 1). \tag{3.7}$$

Lower estimate of $\mu_k(\zeta)$ and characterization of $\Phi_{k,\zeta}$

We take an arbitrary sequence of positive numbers $\{\zeta_p\}_{p=1}^{\infty}$ with $\lim_{p\to\infty} \zeta_p = 0$ and then we have (from (3.7))

$$0 \leqq \liminf_{p\to\infty} \mu_k(\zeta_p) \leqq \limsup_{p\to\infty} \mu_k(\zeta_p) \leqq \mu_k \quad (k \geqq 1).$$

To deal with the lower bound of $\mu_k(\zeta)$ with a characterization of the corresponding eigenfunction $\Phi_{k,\zeta}$, we recall

$$\begin{cases} \mu_k(\zeta) = \displaystyle\int_D |\nabla \Phi_{k,\zeta}|^2\,dx + \int_{Q(\zeta)} |\nabla \Phi_{k,\zeta}|^2\,dx, \\ \displaystyle\int_D \Phi_{k,\zeta}\,\Phi_{k',\zeta}\,dx + \int_{Q(\zeta)} \Phi_{k,\zeta}\,\Phi_{k',\zeta}\,dx = \delta(k,k') \quad (k,k' \geqq 1). \end{cases} \tag{3.8}$$

We will consider the limit of $\Phi_{k,\zeta}$ for $\zeta \to 0$ with the aid of an argument of compactness. Using the upper estimate of $\mu_k(\zeta)$ in (3.7) and (3.8), we have the boundedness of $\{\Phi_{k,\zeta|D}\}_{\zeta>0}$ in $H^1(D)$ for each k. We put

$$\widehat{Q} = Q \times B^{(m)}(1) \subset \mathbb{R}^n, \quad \widehat{Q}(\zeta) = \{(x', (1/\zeta)x'') \mid (x', x'') \in Q(\zeta)\} \subset \mathbb{R}^n.$$

$\widehat{Q}(\zeta) \supset \widehat{Q}$ for $\zeta > 0$ and $\widehat{Q}$ is expressed as

$$\begin{aligned} \widehat{Q}(\zeta) = {} & (Q_{2\zeta} \times B^{(m)}(1)) \\ & \cup \{(\xi + s\boldsymbol{n}(\xi), \eta) \in \mathbb{R}^n \mid \xi \in \partial Q, -2\zeta \leqq s \leqq 0, |\eta| < \boldsymbol{q}(s/\zeta)\}. \end{aligned}$$

We change the variable by $y' = x'$, $\zeta y'' = x''$ and put

$$\widehat{\Phi}_{k,\zeta}(y', y'') = S(m)^{1/2}\zeta^{m/2}\Phi_{k,\zeta}(y', \zeta y'') \quad \text{for} \quad (y', y'') \in \widehat{Q}(\zeta). \tag{3.9}$$

From (3.8) we have

$$\begin{aligned} &S(m)^{-1}\int_{\widehat{Q}} |\widehat{\Phi}_{k,\zeta}|^2 dy'dy'' \leqq 1, \\ &S(m)^{-1}\int_{\widehat{Q}} \left(|\nabla_{y'}\widehat{\Phi}_{k,\zeta}|^2 + \frac{1}{\zeta^2}|\nabla_{y''}\widehat{\Phi}_{k,\zeta}|^2 \right) dy'dy'' \leqq \mu_k(\zeta) \leqq \mu_k + 1, \end{aligned} \tag{3.10}$$

for small $\zeta > 0$.

Applying the Rellich's theorem to the family $\{\Phi_{k,\zeta_p}\}_{p=1}^{\infty} \subset H^1(D)$ and $\{\widehat{\Phi}_{k,\zeta_p}\}_{p=1}^{\infty} \subset H^1(\widehat{Q})$ with the Cantor's diagonal argument, we can take a subsequence $\{\sigma_p\}_{p=1}^{\infty} \subset \{\zeta_p\}_{p=1}^{\infty}$ and a non-negative value μ'_k, $\Phi_k \in H^1(D)$, $\widehat{\Phi}_k \in H^1(\widehat{Q})$ for all $k \geqq 1$ such that

$$\begin{cases} \lim_{p\to\infty} \mu_k(\sigma_p) = \mu'_k \leqq \mu_k, \\ \Phi_{k,\sigma_p} \text{ weakly converges to } \Phi_k \text{ in } H^1(D) \text{ for } p \to \infty, \\ \widehat{\Phi}_{k,\sigma_p} \text{ weakly converges to } \widehat{\Phi}_k \text{ in } H^1(\widehat{Q}) \text{ for } p \to \infty, \\ \lim_{p\to\infty} \|\Phi_{k,\sigma_p} - \Phi_k\|_{L^2(D)} = 0, \quad \lim_{p\to\infty} \|\widehat{\Phi}_{k,\sigma_p} - \widehat{\Phi}_k\|_{L^2(\widehat{Q})} = 0. \end{cases} \tag{3.11}$$

From (3.10) we have

$$\int_{\widehat{Q}} |\nabla_{y''}\widehat{\Phi}_{k,\zeta}|^2 dy'dy'' = O(\zeta^2),$$

and that $\widehat{\Phi}_k(y', y'')$ does not depend on the variable y'' and so it can be regarded as a function in $H^1(Q)$. Applying Prop.8.3 to the family $\{\Phi_{k,\sigma_p}\}_{p=1}^{\infty}$, we conclude that $\Phi_k \in C^2(\overline{D})$ and Φ_k satisfies the equation

$$\Delta\Phi_k + \mu'_k\Phi_k = 0 \quad \text{in} \quad D, \quad \partial\Phi_k/\partial\nu = 0 \quad \text{on} \quad \partial D. \tag{3.12}$$

(Note that if $\Phi_k \not\equiv 0$ in D, $\mu'_k \in \{\omega_d\}_{d=1}^{\infty}$. But it is not always the case).

Next we consider the asymptotic behavior of Φ_{k,σ_p} in $Q(\sigma_p)$. For this purpose we put

$$\Psi_{k,\zeta}(x) = \Phi_{k,\zeta}(x)/\|\Phi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))} \quad (x \in \Omega(\zeta))$$

and consider its behavior in the sense of uniform convergence. Applying Prop.8.1 (cf. Jimbo [26;Theorem 1]) with the aid of the Cantor's diagonal argument, we have a subsequence of $\{\sigma_p\}_{p=1}^\infty$ (we still denote it by $\{\sigma_p\}_{p=1}^\infty$) and $\Phi'_k \in C^2(\overline{D})$, $\widehat{\Phi}'_k \in C^2(\overline{Q})$ such that

$$(3.13)\quad \begin{aligned} &\lim_{p\to\infty}\sup_{x\in D}|\Psi_{k,\sigma_p}(x)-\Phi'_k(x)|=0,\\ &\lim_{p\to\infty}\sup_{(x',x'')\in Q(\sigma_p)}|\Psi_{k,\sigma_p}(x',x'')-\widehat{\Phi}'_k(x')|=0,\end{aligned}$$

$$(3.14)\quad \Delta\Phi'_k+\mu'_k\,\Phi'_k=0 \quad \text{in} \quad D, \quad \partial\Phi'_k/\partial\nu=0 \quad \text{on} \quad \partial D,$$

$$(3.15)\quad \Delta'\widehat{\Phi}'_k+\mu'_k\,\widehat{\Phi}'_k=0 \quad \text{in} \quad Q, \quad \widehat{\Phi}'_k(\xi)=\Phi'_k(\xi,o'') \quad \text{for} \quad \xi\in\partial Q,$$

for all $k \geqq 1$. From $\|\Psi_{k,\sigma_p}\|_{L^\infty(\Omega(\sigma_p))}=1$ and (3.13), we have

$$(3.16)\quad \max\left(\|\Phi'_k\|_{L^\infty(D)},\,\|\widehat{\Phi}'_k\|_{L^\infty(Q)}\right)=1.$$

The second line of (3.8) yields

$$(3.17)\quad \|\Phi_{k,\sigma_p}\|^2_{L^\infty(\Omega(\sigma_p))}\left(\|\Psi_{k,\sigma_p}\|^2_{L^2(D)}+\sigma_p^m\int_{\widehat{Q}}|\Psi_{k,\sigma_p}(y',\sigma_p y'')|^2dy'dy''\right) \leqq 1$$

and the following conditions immediately follow.

$$\begin{aligned} &\|\Phi_{k,\sigma_p}\|^2_{L^\infty(\Omega(\sigma_p))}\|\Psi_{k,\sigma_p}\|^2_{L^2(D)}\leqq 1,\\ &\|\Phi_{k,\sigma_p}\|^2_{L^\infty(\Omega(\sigma_p))}\sigma_p^m\int_{\widehat{Q}}|\Psi_{k,\sigma_p}(y',\sigma_p y'')|^2dy'dy''\leqq 1\end{aligned}$$

Using (3.13), we get

$$(3.18)\quad \begin{aligned} &\|\Phi_{k,\sigma_p}\|^2_{L^\infty(\Omega(\sigma_p))}\left(\|\Phi'_k\|^2_{L^2(D)}+g_1(\sigma_p)\right)\leqq 1,\\ &\|\Phi_{k,\sigma_p}\|^2_{L^\infty(\Omega(\sigma_p))}S(m)\sigma_p^m\left(\|\widehat{\Phi}'_k\|^2_{L^2(Q)}+g_2(\sigma_p)\right)\leqq 1,\end{aligned}$$

where $g_1 = g_1(\zeta)$, $g_2 = g_2(\zeta)$ are functions with $\lim_{p\to\infty} g_1(\sigma_p) = 0$, $\lim_{p\to\infty} g_2(\sigma_p) = 0$ as $p \to \infty$. Using (3.16), we have $\|\Phi'_k\|^2_{L^2(D)} \neq 0$ or $\|\widehat{\Phi}'_k\|^2_{L^2(Q)} \neq 0$ and so in any case we conclude, for any $k \geqq 1$,

$$(3.19)\quad \|\Phi_{k,\sigma_p}\|^2_{L^\infty(\Omega(\sigma_p))}\sigma_p^m=O(1) \quad \text{as} \quad p\to\infty.$$

Rewriting the second line of (3.8), we get

$$\begin{aligned}\delta(k,k') &= (\Phi_{k,\sigma_p},\Phi_{k',\sigma_p})_{L^2(D)} + S(m)^{-1}(\widehat{\Phi}_{k,\sigma_p},\widehat{\Phi}_{k',\sigma_p})_{L^2(\widehat{Q})}\\ &\quad + \sigma_p^m\|\Phi_{k,\sigma_p}\|^2_{L^\infty(\Omega(\sigma_p))}\\ &\quad \times \int_{\widehat{Q}(\sigma_p)\setminus\widehat{Q}} \Psi_{k,\sigma_p}(y',\sigma_p y'')\Psi_{k,\sigma_p}(y',\sigma_p y'')dy'dy''\end{aligned}$$

for $k,k' \geqq 1$. As the measure of $\widehat{Q}(\sigma_p)\setminus\widehat{Q}$ goes to 0 as $p\to\infty$ and $\|\Psi_{k,\zeta}\|_{L^\infty(Q(\zeta))} \leqq 1$ and so we get

$$(3.20)\qquad (\Phi_k,\Phi_{k'})_{L^2(D)} + (\widehat{\Phi}_k,\widehat{\Phi}_{k'})_{L^2(Q)} = \delta(k,k') \quad \text{for} \quad k,k'\geqq 1.$$

Using $\Psi_{k,\zeta}$, we can rewrite (3.9) as

$$(3.21)\qquad \begin{aligned}\widehat{\Phi}_{k,\sigma_p}(y',y'') = S(m)^{1/2}\sigma_p^{m/2}\|\Phi_{k,\sigma_p}\|_{L^\infty(\Omega(\sigma_p))}\Psi_{k,\sigma_p}(y',\sigma_p y''),\\ (y',y'')\in\widehat{Q}(\sigma_p).\end{aligned}$$

We consider $L^2(\widehat{Q})$-norm of the both sides of (3.21) for the limit $p\to\infty$. There are two possibility of cases (i) and (ii).

$$\text{(i)}\quad \|\widehat{\Phi}_k\|_{L^2(Q)} > 0, \quad \text{(ii)}\quad \|\widehat{\Phi}_k\|_{L^2(Q)} = 0.$$

First we consider the case (i) to prove that $\widehat{\Phi}_k$ satisfies the Dirichlet B.C. on ∂Q.

We assume (i) $\|\widehat{\Phi}_k\|_{L^2(Q)} > 0$.

From (3.13), (3.19) and (3.21), we have $\|\widehat{\Phi}'_k\|_{L^2(Q)} > 0$ and

$$S(m)^{1/2}\sigma_p^{m/2}\|\Phi_{k,\sigma_p}\|_{L^\infty(\Omega(\sigma_p))} \to \|\widehat{\Phi}_k\|_{L^2(Q)}/\|\widehat{\Phi}'_k\|_{L^2(Q)} > 0 \quad (p\to\infty).$$

Hence we get $\|\Phi_{k,\sigma_p}\|_{L^\infty(\Omega(\sigma_p))}\to\infty$ and we conclude $\Phi'_k\equiv 0$ in D by using (3.13) and (3.17). From (3.15), $\widehat{\Phi}'_k$ vanishes on ∂Q. Simultaneously we have

$$\widehat{\Phi}_k(x') = (\|\widehat{\Phi}_k\|_{L^2(Q)}/\|\widehat{\Phi}'_k\|_{L^2(Q)})\,\widehat{\Phi}'_k(x') \quad (x'\in Q).$$

Hence, $\widehat{\Phi}_k$ vanishes on ∂Q. From (3.13), (3.15) and (3.17), we have

$$(3.22)\qquad \Delta'\widehat{\Phi}_k + \mu'_k\widehat{\Phi}_k = 0 \quad \text{in} \quad Q, \quad \widehat{\Phi}_k = 0 \quad \text{on} \quad \partial Q,$$

$$\lim_{p\to\infty} \sup_{(y',y'')\in\hat{Q}(\sigma_p)} |\widehat{\Phi}_{k,\sigma_p}(y',y'') - \widehat{\Phi}_k(y')| = 0. \tag{3.23}$$

On the other hand, for the case (ii), $\|\widehat{\Phi}_k\|_{L^2(Q)} = 0$ implies that $\widehat{\Phi}_k \equiv 0$ in Q and so (3.22) trivially holds. From (3.12) and (3.22) with (3.20), we conclude that

$$\mu'_k \in \{\omega_d\}_{d=1}^{\infty} \cup \{\lambda_r\}_{r=1}^{\infty} \tag{3.24}$$

for each k. Using the orthonormality (3.20), we get $\mu'_k \geqq \mu_k$ for all k. Since the choice $\{\zeta_p\}_{p=1}^{\infty}$ was arbitrary and μ'_k depends only on D, Q, we obtain

$$\liminf_{\zeta\to 0} \mu_k(\zeta) \geqq \mu_k \quad (k \geqq 1).$$

Using the inequality (3.7), we have

$$\mu'_k = \mu_k, \quad \lim_{\zeta\to 0} \mu_k(\zeta) = \mu_k. \tag{3.25}$$

(3.25) agrees to the assertion of Prop.2.1. Now that we know (3.25) and Prop.2.1, the assertions of Prop.2.2 follow from the arguments above. Actually (3.12), (3.20), (3.23),(3.25) imply (2.7), (2.9) and (2.8). The first and second properties of (2.6) follow from (3.11) and (3.23), respectively. We have completed the proof of Proposition 2.2. □

Now we have established Proposition 2.1 and Proposition 2.2. Using the above proof, we can deduce several properties for the eigenfunctions.

PROPOSITION 3.2. (i) *If* $\mu_k \in E_I$, *then*

$$\limsup_{\zeta\to 0} \|\Phi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))} < +\infty. \tag{3.26}$$

(ii) *If* $\mu_k \in E_{II}$, *then*

$$\begin{cases} 0 < \liminf_{\zeta\to 0} \zeta^{m/2} \|\Phi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))} \\ \quad \leqq \limsup_{\zeta\to 0} \zeta^{m/2} \|\Phi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))} < +\infty, \\ \lim_{\zeta\to 0} \|\Phi_{k,\zeta}\|_{L^2(D)} = 0. \end{cases} \tag{3.27}$$

Proof of Proposition 3.2. (i) Assume $\mu_k \in E_I$ and that there exists a sequence of positive values $\{\zeta_p\}_{p=1}^{\infty}$ with

$$\lim_{p\to\infty} \zeta_p = 0, \quad \lim_{p\to\infty} \|\Phi_{k,\zeta_p}\|_{L^\infty(\Omega(\zeta_p))} = +\infty. \tag{3.28}$$

We carry out a similar argument for (3.13), (3.14), (3.15) on the function

$$\Psi_{k,\zeta}(x) = \Phi_{k,\zeta}(x)/\|\Phi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}.$$

From Prop.8.1, there exists a subsequence $\{\sigma_p\}$ and $\Phi'_k \in C^2(\overline{D})$ and $\widehat{\Phi}'_k \in C^2(\overline{Q})$ with (3.13), (3.14), (3.15), (3.16). We already know that $\mu'_k = \mu_k$. From (3.16), (3.17), (3.28), we have $\|\Phi'_k\|_{L^2(D)} = 0$ and so $\widehat{\Phi}'_k \not\equiv 0$ in Q. This means $\mu_k \in \{\lambda_r\}_{r=1}^{\infty}$. This is contrary to $\mu_k \in E_I$ and concludes (i).

(ii) Assume $\mu_k \in E_{II}$. As we have $\mu'_k = \mu_k$ in (3.12) and $\mu_k \notin \{\omega_d\}_{d=1}^{\infty}$, $\Phi_k \equiv 0$ in D and it implies the second line of the assertion. From (3.18), $\|\Phi_{k,\sigma_p}\|^2_{L^\infty(\Omega(\sigma_p))}\sigma_p^m$ is bounded when $p \to \infty$. Recall that the sequence $\{\zeta_p\}_{p=1}^{\infty}$ was taken arbitrarily and so we can assert that $\|\Phi_{k,\zeta}\|^2_{L^\infty(\Omega(\zeta))}\zeta^m$ is bounded for small $\zeta > 0$. On the other hand, if, there exists a sequence of positive values $\{\zeta_p\}_{p=1}^{\infty}$ with $\|\Phi_{k,\zeta_p}\|^2_{L^\infty(\Omega(\zeta_p))}\zeta_p^m \to 0$ for $p \to \infty$. We apply the same argument as in the proof of Prop.2.1 (for the subsequence $\{\sigma_p\}_{p=1}^{\infty}$ and the conditions (3.13)-(3.23)). Taking the limit $p \to \infty$ for $\zeta = \sigma_p$ in (3.21) we have $\widehat{\Phi}_k \equiv 0$ in Q and hence $\Phi_k \not\equiv 0$ in D (from (3.20)) which implies $\mu_k \in \{\omega_d\}_{d=1}^{\infty}$. It is a contradiction and concludes (ii). □

Proposition 3.3. *Assume that there exists a positive sequence $\{\zeta_p\}_{p=1}^{\infty}$ and $k \in \mathbb{N}$ such that*

$$\lim_{p\to\infty} \zeta_p = 0, \quad \liminf_{p\to\infty} \|\Phi_{k,\zeta_p}\|_{L^2(D)} > 0.$$

Then we have

$$\mu_k \in \{\omega_d\}_{d=1}^{\infty} \quad \textit{and} \quad \mu_k(\zeta_p) - \mu_k = O(\zeta_p^{m/2}) \quad (p \to \infty).$$

Proof of Proposition 3.3. Assume $\mu_k \notin \{\omega_d\}_{d=1}^{\infty}$ and then $\mu_k \in E_{II}$. From Prop.3.2-(ii), we get the first assertion. For the second assertion, we use the argument in the proof of Prop.2.1, Prop.2.2. Take any subsequence $\{\epsilon_p\}_{p=1}^{\infty} \subset \{\zeta_p\}_{p=1}^{\infty}$, there exist a subsequence $\{\sigma_p\}_{p=1}^{\infty}$ and $\Phi_k \in C^2(\overline{D})$, $\widehat{\Phi}_k \in C^2(\overline{Q})$ with the same condition as (3.10), (3.11) and (3.12). From the assumption and (3.11), $\Phi_k \not\equiv 0$ in D. Φ_k is an eigenfunction in (3.12) corresponding to the eigenvalue $\mu_k \in \{\omega_d\}_{d=1}^{\infty}$. Extend $\Phi_k \in C^2(\overline{D})$ as a C^2 function in $\mathbb{R}^n$ with a compact support. We denote this function by $\widetilde{\Phi}_k$. Put $\Psi = \widetilde{\Phi}_k$ in (2.2) and carry out partial integration and we get

$$(3.29)\quad (\mu_k(\zeta) - \mu_k)\int_D \Phi_{k,\zeta}\widetilde{\Phi}_k\,dx = \int_{Q(\zeta)} (\nabla\Phi_{k,\zeta}\nabla\widetilde{\Phi}_k - \mu_k(\zeta)\Phi_{k,\zeta}\widetilde{\Phi}_k)dx,$$

$$\left|\int_{Q(\zeta)} \nabla\Phi_{k,\zeta}\nabla\widetilde{\Phi}_k\,dx\right| \leqq \|\nabla\Phi_{k,\zeta}\|_{L^2(Q(\zeta))}\|\nabla\widetilde{\Phi}_k\|_{L^2(Q(\zeta))}$$

$$\leqq \mu_k(\zeta)^{1/2}\|\nabla\widetilde{\Phi}_k\|_{L^\infty(\mathbb{R}^n)}|Q(\zeta)|^{1/2} = O(\zeta^{m/2}),$$

$$\left|\int_{Q(\zeta)} \Phi_{k,\zeta}\widetilde{\Phi}_k\,dx\right| \leqq \|\Phi_{k,\zeta}\|_{L^2(Q(\zeta))}\|\widetilde{\Phi}_k\|_{L^\infty(\mathbb{R}^n)}|Q(\zeta)|^{1/2} = O(\zeta^{m/2}),$$

$$\int_D \Phi_{k,\sigma_p}\widetilde{\Phi}_k\,dx \to \|\Phi_k\|_{L^2(D)}^2 > 0 \quad (p \to \infty).$$

Put $\zeta = \sigma_p$ in (3.29), we see that $(\mu_k(\sigma_p) - \mu_k)/\sigma_p^{m/2}$ is bounded for $p \to \infty$. As $\{\epsilon_p\}_{p=1}^{\infty}$ was taken arbitrarily as a subsequence of $\{\zeta_p\}_{p=1}^{\infty}$, we conclude that $(\mu_k(\zeta_p) - \mu_k)/\zeta_p^{m/2}$ is bounded for $p \to \infty$. The second assertion of the proposition is proved. □

§4. Approximate Eigenfunctions $\widetilde{\phi}_{d,\zeta}$ $\widetilde{\psi}_{r,\zeta}$ and Comparison Functions $\varphi_{1,\zeta}$ $\varphi_{2,\zeta}$

In this section we will define several auxiliary functions (which are approximate eigenfunctions and some comparison functions) for investigating the behavior of $\Phi_{k,\zeta}$. We prepare notation for several subregions of $\Omega(\zeta)$ for later arguments through this paper.

Notation. For positive parameters t_1, t_2, ζ, we define the following sets ($\subset \mathbb{R} \times \mathbb{R}^m$).

$$\widetilde{\Sigma}^+(t_1) = \{(s,\eta) \in \mathbb{R} \times \mathbb{R}^m \mid 0 < s,\, s^2 + |\eta|^2 < t_1^2\},$$

$$\begin{aligned}
\widetilde{\Sigma}^-(\zeta,t_2) &= \{(s,\eta)\in\mathbb{R}\times\mathbb{R}^m \mid -t_2 < s \leqq 0,\ |\eta| < \zeta\,\boldsymbol{q}(s/\zeta)\},\\
\widetilde{\Gamma}^+(t_1) &= \{(s,\eta)\in\mathbb{R}\times\mathbb{R}^m \mid 0 < s,\ s^2+|\eta|^2 = t_1^2\},\\
\widetilde{\Gamma}^-(\zeta,t_2) &= \{(s,\eta)\in\mathbb{R}\times\mathbb{R}^m \mid s = -t_2,\ |\eta| < \zeta\},\\
\widetilde{\Lambda}(t_1) &= \widetilde{\Sigma}^-(t_1,2t_1)\cup\widetilde{\Sigma}^+(2t_1).
\end{aligned}$$

Notation. For positive parameters t_1, t_2, ζ, we define the following sets ($\subset \mathbb{R}^\ell\times\mathbb{R}^m = \mathbb{R}^n$).

$$\begin{aligned}
\Sigma^+(t_1) &= \{(\xi+s\,\boldsymbol{n}(\xi),\eta)\in\mathbb{R}^\ell\times\mathbb{R}^m \mid (s,\eta)\in\widetilde{\Sigma}^+(t_1), \xi\in\partial Q\},\\
\Sigma^-(\zeta,t_2) &= \{(\xi+s\,\boldsymbol{n}(\xi),\eta)\in\mathbb{R}^\ell\times\mathbb{R}^m \mid (s,\eta)\in\widetilde{\Sigma}^-(\zeta,t_2), \xi\in\partial Q\},\\
\Gamma^+(t_1) &= \{(\xi+s\,\boldsymbol{n}(\xi),\eta)\in\mathbb{R}^\ell\times\mathbb{R}^m \mid (s,\eta)\in\widetilde{\Gamma}^+(t_1), \xi\in\partial Q\},\\
\Gamma^-(\zeta,t_2) &= \{(\xi+s\,\boldsymbol{n}(\xi),\eta)\in\mathbb{R}^\ell\times\mathbb{R}^m \mid (s,\eta)\in\widetilde{\Gamma}^-(\zeta,t_2), \xi\in\partial Q\},\\
\Lambda(t_1) &= \Sigma^-(t_1,2t_1)\cup\Sigma^+(2t_1).
\end{aligned}$$

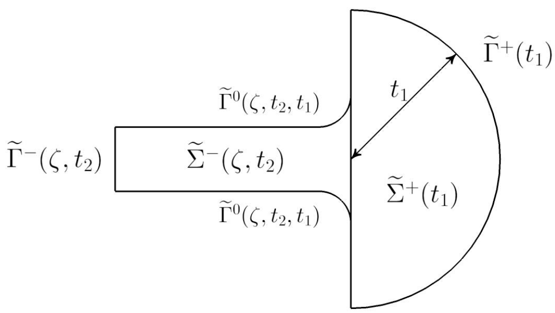

Fig. 10. Section of the junction part $\subset \mathbb{R}\times\mathbb{R}^m$

4.1. Construction of the approximate eigenfunctions $\widetilde{\phi}_{d,\zeta}$ and $\widetilde{\psi}_{r,\zeta}$

We will construct the approximate eigenfunction $\widetilde{\phi}_{d,\zeta} \in H^1(\Omega(\zeta))\cap C^0(\overline{\Omega(\zeta)})$ for $d \geqq 1$.

DEFINITION. For $d \geqq 1$ we put

$$
(4.1) \qquad \widetilde{\phi}_{d,\zeta}(x) = \begin{cases} \phi_d(x) & \text{for } x \in D \setminus \Sigma^+(2\zeta), \\ N_{d,\zeta}(\xi, s, \eta) & \text{for } x = (\xi + s\,\boldsymbol{n}(\xi), \eta) \in \Lambda(\zeta) \\ & \qquad = \Sigma^-(\zeta, 2\zeta) \cup \Sigma^+(2\zeta), \\ V_d(x') & \text{for } x = (x', x'') \in Q(\zeta) \setminus \Sigma^-(\zeta, 2\zeta). \end{cases}
$$

Here $N = N_{d,\zeta}(\xi, s, \eta) = N_{d,\xi,\zeta}(s, \eta) \in C^2(\widetilde{\Lambda}(\zeta))$ is the unique solution (for the parameter $\xi \in \partial Q, \zeta \in (0, \zeta_0)$) of

$$
(4.2) \quad \begin{cases} \dfrac{\partial^2 N}{\partial s^2} + \displaystyle\sum_{j=1}^{m} \dfrac{\partial^2 N}{\partial \eta_j^2} = 0 & \text{for} (s, \eta) \in \widetilde{\Lambda}(\zeta) = \widetilde{\Sigma}^-(\zeta, 2\zeta) \cup \widetilde{\Sigma}^+(2\zeta), \\ N(s, \eta) = \phi_d(\xi + s\,\boldsymbol{n}(\xi), \eta) & \text{for} (s, \eta) \in \widetilde{\Gamma}^+(2\zeta), \\ N(s, \eta) = V_d(\xi + s\,\boldsymbol{n}(\xi)) & \text{for} (s, \eta) \in \widetilde{\Gamma}^-(\zeta, 2\zeta), \\ \partial N / \partial \widetilde{\boldsymbol{n}} = 0 & \text{on}\, \partial\widetilde{\Lambda}(\zeta) \setminus (\widetilde{\Gamma}^+(2\zeta) \cup \widetilde{\Gamma}^-(\zeta, 2\zeta)). \end{cases}
$$

Here $\widetilde{\boldsymbol{n}}$ is the unit outward normal vector on $\partial\widetilde{\Lambda}(\zeta)$. Recall that ϕ_d was given in §2-(2.4), (2.10). Since we assumed that $\partial Q, \partial D$ are C^4, we have $\phi_d \in C^{3,\theta}(\overline{D})$ $(0 < \theta < 1)$. V_d was defined through (2.11) (in case $\omega_d \in E_I$) and (2.12) (in case $\omega_d \in E_{III}$). In the case $\omega_d \in E_{III}$, $\widetilde{\phi}_{d,\zeta}$ is harmonic in $Q(\zeta) \setminus \Sigma^-(\zeta, 2\zeta)$. It is also true that $V_d \in C^{3,\theta}(\overline{Q})$ $(0 < \theta < 1)$ and $N_{d,\xi,\zeta}(s, \eta) \in C^{3,\theta}(\widetilde{\Lambda}(\zeta))$.

We will construct the approximate eigenfunction $\widetilde{\psi}_{r,\zeta} \in H^1(\Omega(\zeta)) \cap C^0(\overline{\Omega(\zeta)})$ for $r \geqq 1$, which are given separately for $m = 1$ and $m \geqq 2$.

DEFINITION. For the harmonic function $G = G(z)$ in H in Proposition 2.7, we put

$$
(4.3) \qquad \widehat{G}(z) = \begin{cases} \boldsymbol{\kappa}_1\, s + G(z) & \text{for} \quad z = (s, \eta) \in H,\ s \leqq 0, \\ G(z) & \text{for} \quad z = (s, \eta) \in H,\ s > 0. \end{cases}
$$

We recall that $\boldsymbol{\kappa}_1 > 0, \boldsymbol{\kappa}_2$ are the constants in Proposition 2.7-(i) $(m = 1)$ and Proposition 2.7-(ii) $(m \geqq 2)$.

Definition. We prepare a positive (parameter) function $h = h(\zeta)$ as follows.

$$h(\zeta) = (\log \zeta)^2 \tag{4.4}$$

This function $h(\zeta)$ will appear in the definition of $\widetilde{\psi}_{r,\zeta}$ later. It is easy to see that

$$\lim_{\zeta \downarrow 0} h(\zeta) = \infty, \quad \lim_{\zeta \downarrow 0} h(\zeta)^{\theta_1} \zeta^{\theta_2} = 0 \quad \text{for any} \quad \theta_1 > 0, \theta_2 > 0.$$

The case $m = 1$.

First we define the sets $\widetilde{\Sigma}^*(\zeta, t), \widetilde{\Gamma}^*(\zeta, t) \subset \mathbb{R} \times \mathbb{R}$ by

$$\widetilde{\Sigma}^*(\zeta, t) = \{z = (s, \eta) \in \mathbb{R} \times \mathbb{R} \mid G(\frac{s}{\zeta}, \frac{\eta}{\zeta}) + \frac{2\boldsymbol{\kappa}_1}{\pi} \log \frac{t}{\zeta} > 0,\ s > 0\},$$

$$\widetilde{\Gamma}^*(\zeta, t) = \{z = (s, \eta) \in \mathbb{R} \times \mathbb{R} \mid G(\frac{s}{\zeta}, \frac{\eta}{\zeta}) + \frac{2\boldsymbol{\kappa}_1}{\pi} \log \frac{t}{\zeta} = 0,\ s > 0\}.$$

We can see that, for $t > 0$, the set $\widetilde{\Gamma}^*(\zeta, t)$ is a C^∞ simple curve whose endpoints are located in the line $\{0\} \times \mathbb{R}$. This set $\widetilde{\Gamma}^*(\zeta, t)$ smoothly approaches $\widetilde{\Gamma}^+(t)$ for $\zeta \to 0$ and the tangent vectors of $\widetilde{\Gamma}^*(\zeta, t)$ at the endpoints are perpendicular to the vertical line (cf. Fig.11). We can prove this property by investigating the asymptotic behavior of G and $\nabla_z G$ by Proposition 2.7-(i) with the aid of the implicit function theorem.

Definition. For $m = 1$, we define

$$\Sigma^*(\zeta, t) = \{(\xi + s\,\boldsymbol{n}(\xi), \eta) \in \mathbb{R}^\ell \times \mathbb{R} \mid \xi \in \partial Q,\ (s, \eta) \in \widetilde{\Sigma}^*(\zeta, t)\} \subset \mathbb{R}^n. \tag{4.5}$$

For $r \in \mathbb{N}$, we define functions $v^{(1)}_{r,\zeta}(x)$, $v^{(2)}_{r,\zeta}(x)$ in $\Sigma^-(\zeta, h\zeta) \cup \Sigma^*(\zeta, t)$, $Q(\zeta) \setminus \Sigma^-(\zeta, h\zeta)$, respectively. First we put

$$\begin{cases} v^{(1)}_{r,\zeta}(x) = (-1/\boldsymbol{\kappa}_1)(\partial \psi_r / \partial \boldsymbol{n})(\xi) \left(\widehat{G}(s/\zeta, \eta/\zeta) + (2\boldsymbol{\kappa}_1/\pi) \log(t/\zeta)\right) \\ \qquad\qquad \text{for} \quad x = (\xi + s\,\boldsymbol{n}(\xi), \eta) \in \Sigma^-(\zeta, h\zeta) \cup \Sigma^*(\zeta, t),\ \xi \in \partial Q. \end{cases} \tag{4.6}$$

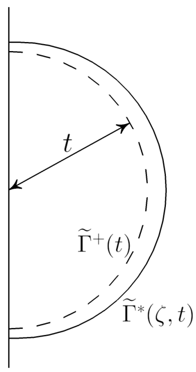

Fig. 11. $\lim_{\zeta\to 0}\widetilde{\Gamma}^*(\zeta,t)=\widetilde{\Gamma}^+(t)$

Here $h=h(\zeta)$ is defined in (4.4). Let $v=v_r^{(2)}(x)\in C^2(\overline{Q(\zeta)\setminus\Sigma^-(\zeta,h\zeta)})$ be the unique solution of the equation,

$$\text{(4.7)} \qquad \begin{cases} \Delta v(x)=0 & \text{in } Q(\zeta)\setminus\Sigma^-(\zeta,h\zeta),\\ v(x)=v_{r,\zeta}^{(1)}(x)\text{for} x\in\Gamma^-(\zeta,h\zeta), \\ \partial v/\partial\nu=0 & \text{on}\,\partial(Q(\zeta)\setminus\Sigma^-(\zeta,h\zeta))\setminus\Gamma^-(\zeta,h\zeta).\end{cases}$$

We define an approximate eigenfunction $\widetilde{\psi}_{r,\zeta}\in H^1(\Omega(\zeta))\cap C^0(\overline{\Omega(\zeta)})$, as follows. By putting

$$\text{(4.8)} \qquad v_{r,\zeta}(x)=\begin{cases} v_{r,\zeta}^{(1)}(x) & \text{for} \quad x\in\Sigma^-(\zeta,h\zeta)\cup\Sigma^*(\zeta,t),\\ v_{r,\zeta}^{(2)}(x) & \text{for} \quad x\in Q(\zeta)\setminus\Sigma^-(\zeta,h\zeta),\\ 0 & \text{for} \quad x\in D\setminus\Sigma^*(\zeta,t),\end{cases}$$

we define

$$\text{(4.9)} \qquad \widetilde{\psi}_{r,\zeta}(x)=\widetilde{\psi}_r(x)+\zeta\, v_{r,\zeta}(x) \quad (x\in\Omega(\zeta)).$$

Recall

$$\widetilde{\psi}_r(x)=\begin{cases}\psi_r(x') & x=(x',x'')\in Q(\zeta),\\ 0 & x\in D,\end{cases}$$

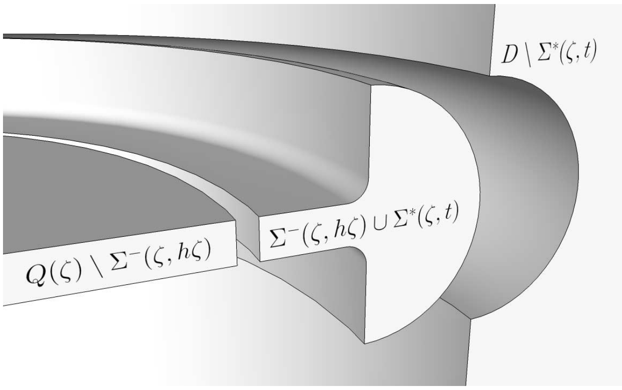

Fig. 12. Decomposition of the junction part ($n = 3, \ell = 2, m = 1$)

and

$$\Omega(\zeta) = \big(\Sigma^-(\zeta, h\zeta) \cup \Sigma^*(\zeta, t)\big) \cup \big(Q(\zeta) \setminus \Sigma^-(\zeta, h\zeta)\big) \cup (D \setminus \Sigma^*(\zeta, t)) .$$

The case $m \geqq 2$

For $r \in \mathbb{N}$, we define functions $v^{(1)}_{r,\zeta}(x)$, $v^{(2)}_{r,\zeta}(x)$, $v^{(3)}_{r,\zeta}(x)$ in $\Sigma^-(\zeta, h\zeta) \cup \Sigma^+(t)$, $Q(\zeta) \setminus \Sigma^-(\zeta, h\zeta)$, $D \setminus \Sigma^+(t)$, respectively. First we put

$$(4.10) \qquad \begin{cases} v^{(1)}_{r,\zeta}(x) = (-1/\boldsymbol{\kappa}_1)(\partial\psi_r/\partial\boldsymbol{n})(\xi)\widehat{G}(s/\zeta, \eta/\zeta) \\ \qquad\qquad \text{for} \quad x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Sigma^-(\zeta, h\zeta) \cup \Sigma^+(t), \xi \in \partial Q. \end{cases}$$

We define $v = v^{(2)}_{r,\zeta} \in C^2(\overline{Q(\zeta) \setminus \Sigma^-(\zeta, h\zeta)})$, $w = v^{(3)}_{r,\zeta} \in C^2(\overline{D \setminus \Sigma^+(t)})$ by the following equations,

$$(4.11) \qquad \begin{cases} \Delta v(x) = 0 \qquad \text{in } Q(\zeta) \setminus \Sigma^-(\zeta, h\zeta), \\ v(x) = v^{(1)}_{r,\zeta}(x) \text{for} x \in \Gamma^-(\zeta, h\zeta), \\ \partial v/\partial\nu = 0 \qquad \text{on}\, \partial(Q(\zeta) \setminus \Sigma^-(\zeta, h\zeta)) \setminus \Gamma^-(\zeta, h\zeta), \end{cases}$$

$$(4.12) \qquad \begin{cases} \Delta w(x) = 0 \qquad \text{in } D \setminus \Sigma^+(t), \\ w(x) = v^{(1)}_{r,\zeta}(x) \text{for} x \in \Gamma^+(t), \\ \partial w/\partial\nu = 0 \qquad \text{on}\, \partial(D \setminus \Sigma^+(t)) \setminus \Gamma^+(t). \end{cases}$$

$t > 0$ is a parameter. We put

$$v_{r,\zeta}(x) = \begin{cases} v^{(1)}_{r,\zeta}(x) & \text{for} \quad x \in \Sigma^-(\zeta, h\zeta) \cup \Sigma^+(t), \\ v^{(2)}_{r,\zeta}(x) & \text{for} \quad x \in Q(\zeta) \setminus \Sigma^-(\zeta, h\zeta), \\ v^{(3)}_{r,\zeta}(x) & \text{for} \quad x \in D \setminus \Sigma^+(t). \end{cases} \tag{4.13}$$

We define an approximate eigenfunction $\widetilde{\psi}_{r,\zeta} \in H^1(\Omega(\zeta)) \cap C^0(\overline{\Omega(\zeta)})$ which is piecewise smooth, as follows,

$$\widetilde{\psi}_{r,\zeta}(x) = \widetilde{\psi}_r(x) + \zeta\, v_{r,\zeta}(x) \quad (x \in \Omega(\zeta)). \tag{4.14}$$

We recall

$$\Omega(\zeta) = \left(\Sigma^-(\zeta, h\zeta) \cup \Sigma^+(t)\right) \cup \left(Q(\zeta) \setminus \Sigma^-(\zeta, h\zeta)\right) \cup \left(D \setminus \Sigma^+(t)\right).$$

We remark that the function $\widetilde{\psi}_{r,\zeta}$ involves positive parameters $t > 0$ and $h = h(\zeta) > 0$.

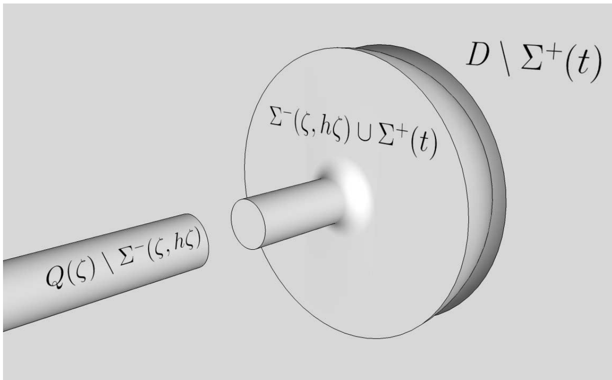

Fig. 13. Decomposition of the junction part $(n = 3, \ell = 1, m = 2)$

4.2. Coordinate system (ξ, s, η) around $\partial Q \times \{o''\}$ in $\mathbb{R}^n$

For calculation and estimation of auxiliary functions, approximate eigenfunctions, we need a coordinate system (ξ, s) in some neighborhood $\mathcal{O}$

which is located around ∂Q in $\mathbb{R}^\ell$. A point $x' \in \mathcal{O}$ is related to $(\xi, s) \in \partial Q \times (-\delta_0, \delta_0)$ by the relation $x' = \xi + s\,\boldsymbol{n}(\xi)$.

As ∂Q is a $\ell - 1$ dimensional compact manifold in $\mathbb{R}^\ell$, it is a union of finite number of local patches, each of which has a local coordinate $(\xi_1, \xi_2, \cdots, \xi_{\ell-1})$ [1]. By taking a small $\delta_0 > 0$, we choose

$$\mathcal{O} = \{\xi + s\,\boldsymbol{n}(\xi) \mid \xi \in \partial Q,\ -\delta_0 < s < \delta_0\} \subset \mathbb{R}^\ell.$$

$\mathcal{O}$ is C^3- diffeomorphic to $\partial Q \times (-\delta_0, \delta_0)$. Using the local coordinate $(\xi_1, \cdots, \xi_{\ell-1})$ of a patch in ∂Q, we introduce a local coordinate $(\xi_1, \cdots, \xi_{\ell-1}, s)$. Denote the metric tensor of $\mathbb{R}^\ell$ with respect to the coordinate $(\xi_1, \cdots, \xi_{\ell-1}, s)$ by $g = (g_{ij}(\xi, s))_{ij}$. Here s corresponds to the component $i = \ell$. We have the following properties

$$\begin{aligned} &g_{i\ell}(\xi, s) \equiv g_{\ell i}(\xi, s) \equiv 0 \quad (1 \leqq i \leqq \ell - 1), \\ &g_{\ell\ell}(\xi, s) \equiv 1 \quad (x' = \xi + s\,\boldsymbol{n}(\xi) \in \mathcal{O}). \end{aligned}$$

Under this coordinate system in $\mathcal{O}$ and the metric tensor $(g_{ij}(\xi, s))$, we can express the Laplacian Δ in terms of $(\xi_1, \cdots, \xi_{\ell-1}, \xi_\ell, \eta)$ as follows,

$$\begin{aligned} \Delta\Psi(x) &= \frac{1}{\sqrt{g}} \sum_{i,j=1}^{\ell-1} \frac{\partial}{\partial \xi_i}\left(\sqrt{g} g^{ij} \frac{\partial \Psi}{\partial \xi_j}\right) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial s}\left(\sqrt{g} \frac{\partial \Psi}{\partial s}\right) + \sum_{j=1}^{m} \frac{\partial^2 \Psi}{\partial \eta_j^2} \\ &= \frac{1}{\sqrt{g}} \sum_{i,j=1}^{\ell-1} \frac{\partial}{\partial \xi_i}\left(\sqrt{g} g^{ij} \frac{\partial \Psi}{\partial \xi_j}\right) + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial s} \frac{\partial \Psi}{\partial s} + \frac{\partial^2 \Psi}{\partial s^2} \\ &\quad + \sum_{j=1}^{m} \frac{\partial^2 \Psi}{\partial \eta_j^2}. \end{aligned} \tag{4.15}$$

We used the properties of the coefficients g_{ij} in (4.15). Here $\eta = (\eta_1, \cdots, \eta_m) \in \mathbb{R}^m$, $g = \det(g_{ij}(\xi, s))$. $(g^{ij}(\xi, s))$ is the inverse matrix of $(g_{ij}(\xi, s))$. Remark that the functions $g^{ij} = g^{ij}(\xi, s)$, $g_{ij} = g_{ij}(\xi, s)$, $g = g(\xi, s)$ depend on the choice of the local coordinate $(\xi_1, \cdots, \xi_{\ell-1})$ on ∂Q.

For calculation and estimation of integration, we need some formula concerning integration of functions (defined in $\Sigma^-(\zeta, t_2)$ and $\Sigma^+(t_1)$) with respect to the variable (ξ, s, η).

[1] A point ξ in a local patch of ∂Q corresponds to $(\xi_1, \cdots, \xi_{\ell-1}) \in \mathbb{R}^{\ell-1}$ through a local coordinate map.

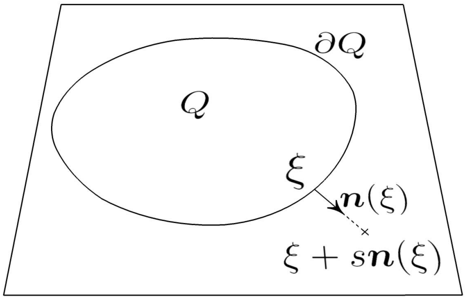

Fig. 14. Coordinate (ξ, s) around $\partial Q \subset \mathbb{R}^\ell$

Lemma 4.1. *There exist continuous functions* $\rho_1 = \rho_1(\xi, s)$*,* $\rho_2 = \rho_2(\xi, s)$ *defined in* $\partial Q \times (-\delta_0, \delta_0)$ *with the conditions* $\rho_1(\xi, 0) \equiv 1$*,* $\rho_2(\xi, 0) \equiv 1$ *(for* $\xi \in \partial Q$*) such that*

$$\int_\Sigma \Psi(x)dx = \int_{\partial Q} \int_{\widetilde{\Sigma}} \Psi(\xi + s\boldsymbol{n}(\xi), \eta)\, \rho_1(\xi, s)\, ds\, d\eta\, dS', \tag{4.16}$$

for a function Ψ *in* Σ*. Here*

$$(\Sigma, \widetilde{\Sigma}) = (\Sigma^-(\zeta, t_2), \widetilde{\Sigma}^-(\zeta, t_2)) \quad or$$
$$(\Sigma^+(t_1), \widetilde{\Sigma}^+(t_1)) \quad or \quad (\Sigma^*(\zeta, t_1), \widetilde{\Sigma}^*(\zeta, t_1)).$$

$$\int_\Gamma \Psi(x)dS = \int_{\partial Q} \int_{\widetilde{\Gamma}} \Psi(\xi + s\boldsymbol{n}(\xi), \eta)\, \rho_2(\xi, s)\, d\widetilde{S}\, dS', \tag{4.17}$$

for a function Ψ *in* Γ*. Here*

$$(\Gamma, \widetilde{\Gamma}) = (\Gamma^-(\zeta, t_2), \widetilde{\Gamma}^-(\zeta, t_2)) \quad or$$
$$(\Gamma^+(t_1), \widetilde{\Gamma}^+(t_1)) \quad or \quad (\Gamma^*(\zeta, t_1), \widetilde{\Gamma}^*(\zeta, t_1)).$$

The functions $\rho_1 = \rho_1(\xi, s)$*,* $\rho_2 = \rho_2(\xi, s)$ *are well-defined and* C^2*. They are expressed as*

$$\rho_1(\xi, s) = (g(\xi, s)/g(\xi, 0))^{1/2},$$

$$\rho_2(\xi,s)=(1+s\tau_1(\xi))(1+s\tau_2(\xi))\cdots(1+s\tau_{\ell-1}(\xi)).$$

Here $\tau_1(\xi),\tau_2(\xi),\cdots,\tau_{\ell-1}(\xi)$ are principal curvature of ∂Q at ξ with respect to the outward unit normal vector $\boldsymbol{n}(\xi)$. dS denotes the measure on a hypersurface Γ (Γ is $\Gamma^+(t_1)$ or $\Gamma^-(\zeta,t_2)$ or $\Gamma^*(\zeta,t_1)$) in $\mathbb{R}^n$, dS' denotes the $\ell-1$ dimensional measure on ∂Q, $d\widetilde{S}$ denotes the measure on a hypersurface $\widetilde{\Gamma}$ ($\widetilde{\Gamma}$ is $\widetilde{\Gamma}^+(t_1)$ or $\widetilde{\Gamma}^-(\zeta,t_2)$ or $\widetilde{\Gamma}^*(\zeta,t_1)$). This result can be proved by a simple calculation in geometry.

4.3. Some estimates for $\widetilde{\phi}_{d,\zeta}$ and $\widetilde{\psi}_{r,\zeta}$

We will prove several properties of $\widetilde{\phi}_{d,\zeta}$ and $\widetilde{\psi}_{r,\zeta}$ which are necessary in later sections (§5-§7). First we deal with $\widetilde{\phi}_{d,\zeta}$.

Lemma 4.2 (Estimates for $\widetilde{\phi}_{d,\zeta}$). *$\widetilde{\phi}_{d,\zeta}\in C^0(\overline{\Omega(\zeta)})\cap H^1(\Omega(\zeta))$, $\nabla\widetilde{\phi}_{d,\zeta}\in L^\infty(\Omega(\zeta))$ for $d\geqq 1$ and there exists $c_1>0$ which is independent of $\zeta\in(0,\zeta_0]$ such that*

$$\sup_{x\in\Omega(\zeta)}|\widetilde{\phi}_{d,\zeta}(x)|\leqq c_1,\quad \|\nabla\widetilde{\phi}_{d,\zeta}\|_{L^\infty(\Omega(\zeta))}\leqq c_1,$$
$$\sup_{x\in\Lambda(\zeta)}|\Delta\widetilde{\phi}_{d,\zeta}(x)|\leqq c_1 \tag{4.18}$$

for $0<\zeta\leqq\zeta_0$.

Proof of Lemma 4.2. Recall that $\widetilde{\phi}_{d,\zeta}(x)=\phi_d(x)$ in $D\setminus\Sigma^+(2\zeta)$ and $\widetilde{\phi}_{d,\zeta}(x',x'')=V_d(x')$ for $(x',x'')\in Q(\zeta)\setminus\Sigma^-(\zeta,2\zeta)$. In view of the definition of $\widetilde{\phi}_{d,\zeta}$ in $\Sigma^-(\zeta,2\zeta)\cup\Sigma^+(2\zeta)$, we apply the maximum principle to (4.2) (harmonic function with the boundary condition) and get

$$\sup_{x\in\Sigma^-(\zeta,2\zeta)\cup\Sigma^+(2\zeta)}|\widetilde{\phi}_{d,\zeta}(x)|\leqq\sup_{x\in\Gamma^-(\zeta,2\zeta)\cup\Gamma^+(2\zeta)}|\widetilde{\phi}_{d,\zeta}(x)|$$
$$\leqq\max(\|V_d\|_{L^\infty(Q)},\|\phi_d\|_{L^\infty(D)})$$

These estimates yield the first inequality of (4.18). From the definition of $\widetilde{\phi}_{d,\zeta}$, it is easy to see

$$\sup_{x\in D\setminus\Sigma^+(2\zeta)}|\nabla\widetilde{\phi}_{d,\zeta}(x)|\leqq\sup_{x\in D}|\nabla\phi_d(x)|,$$
$$\sup_{x\in Q(\zeta)\setminus\Sigma^-(\zeta,2\zeta)}|\nabla\widetilde{\phi}_{d,\zeta}(x)|\leqq\sup_{x'\in Q}|\nabla' V_d(x')| \tag{4.19}$$

We estimate $\nabla\widetilde{\phi}_{d,\zeta}$ and $\Delta\widetilde{\phi}_{d,\zeta}$ in $\Lambda(\zeta) = \Sigma^+(2\zeta)\cup\Sigma^-(\zeta, 2\zeta)$. For this purpose we use the coordinate system in a neighborhood around $\partial Q \times \{o''\}$. For $x = (\xi + s\,\boldsymbol{n}(\xi), \eta) \in \Lambda(\zeta)$, $\widetilde{\phi}_{d,\zeta}(x) = N_{d,\zeta}(\xi, s, \eta)$. To estimate $\nabla N_{d,\zeta}(\xi, s, \eta)$ in $\Lambda(\zeta)$, we change the scale of the variable. Define the function

$$N^*_\zeta(\xi, s, \eta) = (1/\zeta)(N_{d,\zeta}(\xi, \zeta s, \zeta\eta) - \phi_d(\xi, o'')) \tag{4.20}$$

which satisfies the following equation for $\xi \in \partial Q$, $(s, \eta) \in \widetilde{\Lambda}(1) \subset \mathbb{R}^{1+m}$.

$$\begin{cases} \dfrac{\partial^2 N^*_\zeta}{\partial s^2} + \displaystyle\sum_{j=1}^m \dfrac{\partial^2 N^*_\zeta}{\partial \eta_j^2} = 0 \quad \text{for} \quad (s, \eta) \in \widetilde{\Lambda}(1) = \widetilde{\Sigma}^-(1, 2) \cup \widetilde{\Sigma}^+(2), \\ N^*_\zeta(\xi, s, \eta) = (\phi_d(\xi + \zeta s\,\boldsymbol{n}(\xi), \zeta\eta) - \phi_d(\xi, o''))/\zeta \\ \qquad\qquad \text{for} \quad (s, \eta) \in \widetilde{\Gamma}^+(2), \\ N^*_\zeta(\xi, s, \eta) = (V_d(\xi + \zeta\, s\,\boldsymbol{n}(\xi)) - V_d(\xi))/\zeta \\ \qquad\qquad \text{for} \quad (s, \eta) \in \widetilde{\Gamma}^-(1, 2), \\ \dfrac{\partial N^*_\zeta}{\partial \widetilde{\boldsymbol{n}}} = 0 \quad \text{on} \quad \partial\widetilde{\Lambda}(1) \setminus (\widetilde{\Gamma}^+(2) \cup \widetilde{\Gamma}^-(1, 2)). \end{cases} \tag{4.21}$$

Here we recall $\phi_d(\xi, o'') = V_d(\xi)$ for $\xi \in \partial Q$ and $V_d \in C^{3,\theta}(\overline{Q})$, $\phi_d \in C^{3,\theta}(\overline{D})$ for any $\theta \in [0, 1)$. $\widetilde{\boldsymbol{n}}$ is outward unit normal vector on $\partial\widetilde{\Lambda}(1)$. We can regard (4.21) as an equation with parameters $\zeta > 0, \xi \in \partial Q$. We apply the regularity argument of elliptic BVP (cf. Gilbarg-Trudinger [22]). Note that the quantities

$$\begin{aligned} &\|(V_d(\xi + \zeta\, s\,\boldsymbol{n}(\xi)) - V_d(\xi))/\zeta\|_{C^{2,\theta}(\tilde{\Gamma}^-(1,2))}, \\ &\|(\phi_d(\xi + \zeta\, s\,\boldsymbol{n}(\xi), \zeta\eta) - \phi_d(\xi, o''))/\zeta\|_{C^{2,\theta}(\tilde{\Gamma}^+(2))} \end{aligned} \tag{4.22}$$

are bounded by a constant which does not depend on the parameter $\zeta > 0$ and $\xi \in \partial Q$. So we have an estimate $\|N^*_\zeta\|_{C^{2,\theta}(\tilde{\Lambda}(1))} \leqq c$ for a constant c which does not depend on ζ, ξ. Using the relation

$$\nabla_\eta N_{d,\zeta}(\xi, s, \eta) = (\nabla_\eta N^*_\zeta)(\xi, \frac{s}{\zeta}, \frac{\eta}{\zeta}), \quad \frac{\partial}{\partial s}(N_{d,\zeta}(\xi, s, \eta)) = \frac{\partial N^*_\zeta}{\partial s}(\xi, \frac{s}{\zeta}, \frac{\eta}{\zeta}),$$

we conclude that

$$\sup_{x=(\xi+s\boldsymbol{n}(\xi),\eta)\in\Lambda(\zeta)} \left(\left| \frac{\partial N_{d,\zeta}}{\partial s} \right| + |\nabla_\eta N_{d,\zeta}| \right) \text{ is bounded in } 0 < \zeta \leqq \zeta_0. \tag{4.23}$$

To prove the estimate of the derivatives of N^*_ζ with respect to the parameter $\xi \in \partial Q$, we consider the following quantities

$$\sup_{\xi\in\partial Q,(s,\eta)\in\widetilde{\Gamma}^-(1,2)} |\nabla_\xi(V_d(\xi+\zeta\, s\,\boldsymbol{n}(\xi)) - V_d(\xi))|,$$
$$\sup_{\xi\in\partial Q,(s,\eta)\in\widetilde{\Gamma}^-(1,2)} |\nabla^2_\xi(V_d(\xi+\zeta\, s\,\boldsymbol{n}(\xi)) - V_d(\xi))|,$$
$$\sup_{\xi\in\partial Q,(s,\eta)\in\widetilde{\Gamma}^+(2)} |\nabla_\xi(\phi_d(\xi+\zeta\, s\,\boldsymbol{n}(\xi),\zeta\eta) - \phi_d(\xi,o'')|,$$
$$\sup_{\xi\in\partial Q,(s,\eta)\in\widetilde{\Gamma}^+(2)} |\nabla^2_\xi(\phi_d(\xi+\zeta\, s\,\boldsymbol{n}(\xi),\zeta\eta) - \phi_d(\xi))|.$$

From the regularity of V_d and ϕ_d, these quantities are bounded by a constant which does not depend on the parameter $\zeta > 0$. Applying the maximum principle to the equations which $\zeta\,\nabla_\xi N^*_\zeta$ and $\zeta\,\nabla^2_\xi N^*_\zeta$ (harmonic in $\widetilde{\Lambda}(1)$ with Neumann B.C. on $\partial\widetilde{\Lambda}(1)\setminus(\widetilde{\Gamma}^-(1,2)\cup\widetilde{\Gamma}^+(2)))$ satisfy, we have that the derivatives up to 2nd order of $\zeta\,N^*_\zeta$ with respect to ξ are uniformly bounded when $\zeta > 0$ is small. Therefore, by the aid of the relations

$$\begin{cases} \nabla_\xi N_{d,\zeta}(\xi,s,\eta) = \nabla_\xi(\phi_d(\xi,o'') + \zeta N^*_\zeta(\xi,s/\zeta,\eta/\zeta)), \\ \nabla^2_\xi N_{d,\zeta}(\xi,s,\eta) = \nabla^2_\xi(\phi_d(\xi,o'') + \zeta N^*_\zeta(\xi,s/\zeta,\eta/\zeta)), \end{cases}$$

we conclude that

$$\sup_{x=(\xi+s\boldsymbol{n}(\xi),\eta)\in\Lambda(\zeta)} (|\nabla_\xi N_{d,\zeta}(\xi,s,\eta)| + |\nabla^2_\xi N_{d,\zeta}(\xi,s,\eta)|) \quad \text{is bounded in } 0<\zeta\leqq\zeta_0. \tag{4.24}$$

(4.23) and (4.24) imply the second estimate of (4.18).

Next we consider the behavior of $\Delta N_{d,\zeta}(\xi,s,\eta)$ in $\Lambda(\zeta)$. Putting $\Psi = N_{d,\zeta}(\xi,s,\eta)$ in (4.2) and using (4.15), we get

$$\Delta N_{d,\zeta} = \frac{1}{\sqrt{g}}\sum_{i,j=1}^{\ell-1}\frac{\partial}{\partial\xi_i}\left(\sqrt{g}g^{ij}\frac{\partial N_{d,\zeta}}{\partial\xi_j}\right) + \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial s}\frac{\partial N_{d,\zeta}}{\partial s}.$$

We estimate the right hand side. From the estimates (4.23) and (4.24), it follows that $\sup_{x\in\Lambda(\zeta)}|\Delta N_{d,\zeta}(x)|$ is bounded for $0<\zeta\leqq\zeta_0$. This concludes the proof of (4.18). □

We deal with the estimates for the $\widetilde{\psi}_{r,\zeta}$ in the following two lammas (Lemma 4.3-(i) :$m=1$ and Lemma 4.3-(ii): $m \geqq 2$). We use same notation c_2, $c_3(t)$ for constants which appear in two lemmas, for simplicity, because the results of two cases ($m=1$ and $m \geqq 2$) are used seperately and so there will be no confusion.

LEMMA 4.3-(i) (Estimates for $\widetilde{\psi}_{r,\zeta}$, $m=1$). *For $m=1$, $r \geqq 1$, there exist constants $c_2 > 0$, $c_3(t) > 0$ (for $t > 0$) which are independent of $\zeta \in (0, \zeta_0]$ such that the following conditions hold.*

$\widetilde{\psi}_{r,\zeta} \in C^0(\overline{\Omega(\zeta)}) \cap H^1(\Omega(\zeta))$, $\nabla\widetilde{\psi}_{r,\zeta} \in L^\infty(\Omega(\zeta))$ and $\nabla\widetilde{\psi}_{r,\zeta}$ is continuous at $\partial D \cap \partial Q(\zeta)$.

$$\begin{aligned} &\frac{\partial \widetilde{\psi}_{r,\zeta}}{\partial \nu} = 0 \quad (x \in \partial\Omega(\zeta) \setminus \partial\Sigma^-(\zeta, 2\zeta)), \\ &\left| \frac{\partial \widetilde{\psi}_{r,\zeta}}{\partial \nu} \right| \leqq c_2\,\zeta \quad (x \in \partial\Omega(\zeta) \cap \partial\Sigma^-(\zeta, 2\zeta)) \end{aligned} \tag{4.25}$$

$$|v_{r,\zeta}(x)| \leqq c_2 \begin{cases} 1 + \log(t/\zeta) \ \textit{for}\ x \in Q(\zeta) \cup \Sigma^+(2\zeta) \\ 1 + \log \frac{2t}{\sqrt{s^2+|\eta|^2}} \\ \qquad \textit{for}\ x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Sigma^*(\zeta, t) \setminus \Sigma^+(2\zeta) \end{cases} \tag{4.26}$$

for $0 < 2\zeta \leqq t$.

$$\begin{cases} \left| \dfrac{\partial}{\partial s} \left(v^{(1)}_{r,\zeta}(\xi + s\,\boldsymbol{n}(\xi), \eta) \right)_{|s=-h\zeta+0} \right| \leqq c_2 \\ \left| \dfrac{\partial}{\partial s} \left(v^{(2)}_{r,\zeta}(\xi + s\,\boldsymbol{n}(\xi), \eta) \right)_{|s=-h\zeta-0} \right| \leqq c_2\,(1 + \log(t/\zeta)) \end{cases} \tag{4.27}$$

for $x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Gamma^-(\zeta, h\zeta)$, $0 < \zeta \leqq \zeta_0$, $0 < 2\zeta \leqq t$.

$$\left| \frac{\partial}{\partial s} \left(v^{(1)}_{r,\zeta}(\xi + s\,\boldsymbol{n}(\xi), \eta) \right) \right| \leqq c_2, \qquad \left| \nabla_\eta \left(v^{(1)}_{r,\zeta}(\xi + s\,\boldsymbol{n}(\xi), \eta) \right) \right| \leqq c_2 \tag{4.28}$$

for $x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Gamma^*(\zeta, t)$, $0 < \zeta \leqq \zeta_0$.

$$|\nabla v^{(1)}_{r,\zeta}(x)| + |\Delta v^{(1)}_{r,\zeta}(x)| \leqq c_2(\log(t/\zeta) + (1/\zeta)e^{\delta s/\zeta}) \quad (x \in \Sigma^-(\zeta, h\zeta)) \tag{4.29}$$

$$(4.30)\qquad |\nabla v^{(1)}_{r,\zeta}(x)| + |\Delta v^{(1)}_{r,\zeta}(x)| \leqq c_2 \begin{cases} (1/\zeta) + \log(t/\zeta) & (x \in \Sigma^+(2\zeta)) \\ \frac{1}{\sqrt{s^2+|\eta|^2}} + \log\frac{2t}{\sqrt{s^2+|\eta|^2}} & (x \in \Sigma^*(\zeta,t)\setminus\Sigma^+(2\zeta)) \end{cases}$$

LEMMA 4.3-(ii) (Estimates for $\widetilde{\psi}_{r,\zeta}$, $m \geqq 2$). *For $m \geqq 2$, $r \geqq 1$, there exist constants $c_2 > 0$, $c_3(t) > 0$ (for $t > 0$) which are independent of $\zeta \in (0, \zeta_0]$ such that the following conditions hold.*

$\widetilde{\psi}_{r,\zeta} \in C^0(\overline{\Omega(\zeta)}) \cap H^1(\Omega(\zeta))$, $\nabla\widetilde{\psi}_{r,\zeta} \in L^\infty(\Omega(\zeta))$ and $\nabla\widetilde{\psi}_{r,\zeta}$ is continuous at $\partial D \cap \partial Q(\zeta)$.

$$(4.31)\qquad \begin{aligned} &\frac{\partial\widetilde{\psi}_{r,\zeta}}{\partial\nu} = 0 \ (x \in \partial\Omega(\zeta)\setminus\partial\Sigma^-(\zeta,2\zeta)), \\ &\left|\frac{\partial\widetilde{\psi}_{r,\zeta}}{\partial\nu}\right| \leqq c_2\,\zeta \ (x \in \partial\Omega(\zeta)\cap\partial\Sigma^-(\zeta,2\zeta)) \end{aligned}$$

$$(4.32)\qquad |v_{r,\zeta}(x)| \leqq c_2 \begin{cases} 1 \text{ for } x \in Q(\zeta)\cup\Sigma^+(2\zeta) \\ \zeta^{m-1}/(s^2+|\eta|^2)^{(m-1)/2} \\ \qquad \text{for } x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Sigma^+(t)\setminus\Sigma^+(2\zeta) \end{cases}$$

for $0 < 2\zeta \leqq t$.

$$(4.33)\qquad \begin{aligned} &\left|\frac{\partial}{\partial s}\left(v^{(1)}_{r,\zeta}(\xi + s\,\boldsymbol{n}(\xi),\eta)\right)_{|s=-h\zeta+0}\right| \leqq c_2 \\ &\left|\frac{\partial}{\partial s}\left(v^{(2)}_{r,\zeta}(\xi + s\,\boldsymbol{n}(\xi),\eta)\right)_{|s=-h\zeta-0}\right| \leqq c_2 \end{aligned}$$

for $x = (\xi + s\boldsymbol{n}(\xi),\eta) \in \Gamma^-(\zeta, h\zeta)$, $0 < \zeta \leqq \zeta_0$.

$$(4.34)\qquad \left|\frac{\partial}{\partial s}\left(v^{(1)}_{r,\zeta}(\xi + s\,\boldsymbol{n}(\xi),\eta)\right)\right| \leqq c_2, \quad \left|\nabla_\eta\left(v^{(1)}_{r,\zeta}(\xi + s\,\boldsymbol{n}(\xi),\eta)\right)\right| \leqq c_2$$

for $x = (\xi + s\boldsymbol{n}(\xi),\eta) \in \Gamma^+(t)$, $0 < \zeta \leqq \zeta_0$.

$$(4.35)\qquad |\nabla v^{(1)}_{r,\zeta}(x)| + |\Delta v^{(1)}_{r,\zeta}(x)| \leqq c_2(1 + (1/\zeta)e^{\delta s/\zeta}) \quad (x \in \Sigma^-(\zeta, h\zeta))$$

$$(4.36)\qquad |\nabla v^{(1)}_{r,\zeta}(x)| + |\Delta v^{(1)}_{r,\zeta}(x)|$$

$$\leqq c_2 \begin{cases} 1/\zeta & (x \in \Sigma^+(2\zeta)) \\ \zeta^{m-1}/(s^2+|\eta|^2)^{m/2} & (x \in \Sigma^+(t)\setminus\Sigma^+(2\zeta)) \end{cases}$$

$$|v^{(3)}_{r,\zeta}(x)| + |\nabla v^{(3)}_{r,\zeta}(x)| \leqq c_3(t)\zeta^{m-1} \quad (x \in D\setminus\Sigma^+(t)) \tag{4.37}$$

Here $\delta > 0$ *is the constant in Proposition 2.7.*

From (4.27), (4.33), we know the estimates of normal derivative of $v^{(1)}_{r,\zeta}, v^{(2)}_{r,\zeta}$ at $\Gamma^-(\zeta, h\zeta)$.

PROOF OF LEMMA 4.3-(i): $m = 1$. The continuity and piecewise C^1 of $\widetilde{\psi}_{r,\zeta}$ in $\overline{\Omega(\zeta)}$ is clear from the definition. We check the continuity of $\nabla\widetilde{\psi}_{r,\zeta}$ at $\partial D \cap \partial Q(\zeta)$. From the definition, $\widetilde{\psi}_{r,\zeta}$ satisfies the Neumann B.C. on $\partial\Omega(\zeta)\setminus\partial\Sigma^-(\zeta,2\zeta)$. So we deal with the normal derivative of $\widetilde{\psi}_{r,\zeta}$ on $\partial\Omega(\zeta)\cap\partial\Sigma^-(\zeta,2\zeta)$. From $m = 1$ and the definition of $\widehat{G}$, we have

$$\begin{aligned} \widetilde{\psi}_{r,\zeta}(x) &= \widetilde{\psi}_r(x) + \zeta v^{(1)}_{r,\zeta}(x) \\ &= \begin{cases} \frac{-\zeta}{\boldsymbol{\kappa}_1}\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\left(G(\frac{s}{\zeta},\frac{\eta}{\zeta}) + \frac{2\boldsymbol{\kappa}_1}{\pi}\log\frac{t}{\zeta}\right) \text{ for } x \in \Sigma^*(\zeta,t), \\ \psi_r(\xi + s\boldsymbol{n}(\xi)) - s\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi) - \frac{\zeta}{\boldsymbol{\kappa}_1}\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\left(G(\frac{s}{\zeta},\frac{\eta}{\zeta}) + \frac{2\boldsymbol{\kappa}_1}{\pi}\log\frac{t}{\zeta}\right) \\ \qquad \text{for } x \in \Sigma^-(\zeta,h\zeta). \end{cases} \end{aligned}$$

From the Dirichlet B.C. of ψ_r, $\psi_r(\xi) = 0$ for $\xi \in \partial Q$ and from the Taylor's theorem,

$$\begin{aligned} &\psi_r(\xi + s\boldsymbol{n}(\xi)) - s\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi) = O(s^2), \\ &\frac{\partial}{\partial s}\left(\psi_r(\xi + s\boldsymbol{n}(\xi)) - s\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\right) = O(s), \end{aligned} \tag{4.38}$$

uniformly in $\xi \in \partial Q$. Hence $\nabla\widetilde{\psi}_{r,\zeta}$ are continuous across $\partial D \cap \partial Q(\zeta)$. Note that the unit normal vector ν on $\partial\Omega(\zeta)\cap\partial\Sigma^-(\zeta,2\zeta)$ has only (s,η)-component. From (2.14), the term $(\partial\psi_r/\partial\boldsymbol{n})(\xi)G(s/\zeta,\eta/\zeta)$ satisfies the Neumann B.C. on $\partial\Omega(\zeta)\cap\partial\Sigma^-(\zeta,2\zeta)$. On the other hand, from (4.38), we have

$$\sup_{\partial\Omega(\zeta)\cap\partial\Sigma^-(\zeta,2\zeta)}\left|\frac{\partial}{\partial\nu}\left(\psi_r(\xi + s\boldsymbol{n}(\xi)) - s\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\right)\right| = O(\zeta).$$

This estimate implies (4.25). We deal with (4.26). We have only to prove the estimates of (4.26) in the region $\Sigma^-(\zeta, h\zeta) \cup \Sigma^+(t)$, because $v_{r,\zeta}^{(2)}$ is harmonic in $Q(\zeta)\backslash\Sigma^-(\zeta, h\zeta)$ with the Neumann B.C. on $\partial(Q(\zeta)\backslash\Sigma^-(\zeta, h\zeta))\backslash\Gamma^-(\zeta, h\zeta)$ and the inequality (which follows from the maximum principle),

$$(4.39) \qquad \sup_{Q(\zeta)\backslash\Sigma^-(\zeta,h\zeta)} |v_{r,\zeta}^{(2)}(x)| \leqq \sup_{\Gamma^-(\zeta,h\zeta)} |v_{r,\zeta}^{(2)}(x)| = \sup_{\Gamma^-(\zeta,h\zeta)} |v_{r,\zeta}^{(1)}(x)|.$$

Eventually the proof of the estimate in $Q(\zeta) \setminus \Sigma^-(\zeta, h\zeta)$ reduces to the following (4.40). Using the explicit expression, we estimate $v_{r,\zeta}^{(1)}$ on the set $\Sigma^-(\zeta, h\zeta)$,

$$\begin{aligned}(4.40) \quad |v_{r,\zeta}^{(1)}(x)| &\leqq \frac{1}{\boldsymbol{\kappa}_1} \sup_{\xi\in\partial Q} \left|\frac{\partial\psi_r}{\partial \boldsymbol{n}}(\xi)\right| \\ &\quad \times \left(\sup_{s<0,|\eta|<\zeta} |\widehat{G}(s/\zeta, \eta/\zeta)| + \frac{2\boldsymbol{\kappa}_1}{\pi} \log(t/\zeta) \right) \\ &\leqq \frac{1}{\boldsymbol{\kappa}_1} \|\nabla'\psi_r\|_{L^\infty(Q)} \left(\sup_{(\tilde{s},\tilde{\eta})\in H_2} |\widehat{G}(\widetilde{s}, \widetilde{\eta})| + \frac{2\boldsymbol{\kappa}_1}{\pi} \log(t/\zeta) \right).\end{aligned}$$

For $x \in \Sigma^+(2\zeta)$, $0 < 2\zeta \leqq t$,

$$\begin{aligned}(4.41) \quad |v_{r,\zeta}^{(1)}(x)| &\leqq \frac{1}{\boldsymbol{\kappa}_1} \sup_{\xi\in\partial Q} \left|\frac{\partial\psi_r}{\partial \boldsymbol{n}}(\xi)\right| \left(\sup_{(\tilde{s},\tilde{\eta})\in\tilde{\Sigma}^+(2)} |G(\widetilde{s}, \widetilde{\eta})| + \frac{2\boldsymbol{\kappa}_1}{\pi} \log(t/\zeta) \right) \\ &\leqq \frac{\|\nabla'\psi_r\|_{L^\infty(Q)}}{\boldsymbol{\kappa}_1} \left(\sup_{(\tilde{s},\tilde{\eta})\in\tilde{\Sigma}^+(2)} |G(\widetilde{s}, \widetilde{\eta})| + \frac{2\boldsymbol{\kappa}_1}{\pi} \log(t/\zeta) \right).\end{aligned}$$

Note that $G(\widetilde{s}, \widetilde{\eta}) = \widehat{G}(\widetilde{s}, \widetilde{\eta})$ for $(\widetilde{s}, \widetilde{\eta}) \in H_1$ and also that $\widehat{G}$ is bounded in H_2. For $x \in \Sigma^*(\zeta, t) \setminus \Sigma^+(2\zeta)$, $0 < 2\zeta \leqq t$,

$$\begin{aligned}(4.42) \quad |v_{r,\zeta}^{(1)}(x)| &= \frac{1}{\boldsymbol{\kappa}_1} \left|\frac{\partial\psi_r}{\partial \boldsymbol{n}}(\xi)\right| \Bigg| \left(G(\frac{s}{\zeta}, \frac{\eta}{\zeta}) - \frac{2\boldsymbol{\kappa}_1}{\pi} \log\frac{\zeta}{\sqrt{s^2+|\eta|^2}} \right) \\ &\qquad + \frac{2\boldsymbol{\kappa}_1}{\pi} \log\frac{t}{\sqrt{s^2+|\eta|^2}} \Bigg| \\ &\leqq \frac{1}{\boldsymbol{\kappa}_1} \left|\frac{\partial\psi_r}{\partial \boldsymbol{n}}(\xi)\right| \left| \frac{c_0\zeta}{\sqrt{s^2+|\eta|^2}} + \frac{2\boldsymbol{\kappa}_1}{\pi} \log\frac{t}{\sqrt{s^2+|\eta|^2}} \right|\end{aligned}$$

$$\leqq \frac{\|\nabla' \psi_r\|_{L^\infty(Q)}}{\boldsymbol{\kappa}_1} \left(\frac{c_0}{2} + \frac{2\boldsymbol{\kappa}_1}{\pi} \log \frac{2t}{\sqrt{s^2+|\eta|^2}} \right).$$

In the above calculation, Prop.2.7-(i) was used. These inequalities (4.38), (4.39), (4.40), (4.41), (4.42) imply (4.26).

We deal with (4.27). From the definition of $v_{r,\zeta}^{(1)}$, we have

$$\begin{aligned} \text{(4.43)} \qquad & \frac{\partial}{\partial s} v_{r,\zeta}^{(1)}(\xi + s\,\boldsymbol{n}(\xi), \eta)_{|s=-h(\zeta)\zeta+0} \\ &= -\frac{1}{\boldsymbol{\kappa}_1 \zeta} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)(\partial_s \widehat{G})(s/\zeta, \eta/\zeta))_{|s=-h(\zeta)\zeta+0} \\ &= -\frac{1}{\boldsymbol{\kappa}_1 \zeta} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)(\partial_s \widehat{G})(-h(\zeta), \eta/\zeta). \end{aligned}$$

Here $\partial_s \widehat{G} = \partial \widehat{G}/\partial s$.

From Prop.2.7-(i) and (4.3), we have $|(\partial_s \widehat{G})(s,\eta)| \leqq c_0\, e^{\delta\, s}$ $(s < 0, \eta \in B^{(m)}(1))$. Thus we get

$$\begin{aligned} |\partial_s \widehat{G}(-h(\zeta), \eta/\zeta)| &\leqq c_0\, e^{-\delta h(\zeta)} = c_0\, e^{-\delta(\log\zeta)^2} \\ &= c_0\, \zeta^{\delta \log(1/\zeta)} \quad (\eta \in B^{(m)}(\zeta)). \end{aligned}$$

Substituting this estimate into (4.43), we get the first estimate of (4.27). Next we consider the second part of (4.27). We estimate the boundary derivative of $v_{r,\zeta}^{(2)}$ at $s = -h(\zeta)\zeta - 0$ ($x = (\xi - h\zeta\boldsymbol{n}(\xi), \eta) \in \Gamma^-(\zeta, h\zeta)$) by the aid of a barrier function. For this purpose we consider the domain

$$\mathcal{U}(\zeta,\tau) \equiv \Sigma^-(\zeta,\tau) \setminus \Sigma^-(\zeta, h(\zeta)\zeta)$$

in which we make barrier functions to bound $v_{r,\zeta}^{(2)}(x)$. $\tau > 0$ is a constant to be fixed later. Define the functions

$$\text{(4.44)} \quad \Theta_{\pm M}(x) = v_{r,\zeta}^{(1)}(\xi - h(\zeta)\zeta\boldsymbol{n}(\xi), \eta) \pm M\left\{(\tau - h(\zeta)\zeta)^2 - (\tau + s)^2\right\}$$

for $x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \mathcal{U}(\zeta, \tau)$. It is easy to see

$$\text{(4.45)} \qquad \Theta_{\pm M}(x) = v_{r,\zeta}^{(2)}(x) = v_{r,\zeta}^{(1)}(x) \quad \text{for} \quad x \in \Gamma^-(\zeta, h(\zeta)\zeta)$$

since $x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Gamma^-(\zeta, h(\zeta)\zeta)$ implies $s = -h\zeta$. Calculate

$$\begin{aligned}\Delta\Theta_{\pm M}(x) = &\frac{1}{\sqrt{g}} \sum_{i,j=1}^{\ell-1} \frac{\partial}{\partial \xi_i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial \xi_j} v_{r,\zeta}^{(1)}(\xi - h(\zeta)\zeta\boldsymbol{n}(\xi), \eta) \right) \\ &\pm 2M \left(1 + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial s}(s+\tau) \right)\end{aligned}$$

for $x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \mathcal{U}(\zeta, \tau)$ $\quad(\xi \in \partial Q,\ -\tau < s < -h(\zeta)\zeta,\ \eta \in B^{(m)}(\zeta))$. We can take $\tau > 0$ small so that

$$\tau \left| \frac{1}{\sqrt{g(\xi,s)}} \frac{\partial \sqrt{g(\xi,s)}}{\partial s} \right| \leqq \frac{1}{4} \quad (\xi \in \partial Q, -\delta_0 \leqq s \leqq 0) \tag{4.46}$$

and fix. Remark that $g = g(\xi, s)$ is a function in ξ, s and τ depends only on the geometry of ∂Q. We will prove that if $M > 0$ is large then Θ_{-M} and Θ_M are a lower solution and an upper solution for $v_{r,\zeta}^{(2)}$, respectively. We calculate

$$\begin{aligned}\Delta\Theta_{\pm M} = &\frac{-1}{\boldsymbol{\kappa}_1\sqrt{g}} \sum_{i,j=1}^{\ell-1} \frac{\partial}{\partial \xi_i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial \xi_j} \left(\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \right) \right) \\ &\times \left(\widehat{G}(-h(\zeta), \frac{\eta}{\zeta}) + \frac{2\boldsymbol{\kappa}_1}{\pi} \log \frac{t}{\zeta} \right) \\ &\pm 2M \left(1 + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial s}(s+\tau) \right).\end{aligned} \tag{4.47}$$

In view of (4.47), we can take a large $M_1(\zeta, t) > 0$ such that

$$(\pm 1)\Delta\Theta_{\pm M}(x) < 0 \quad \text{in} \quad \mathcal{U}(\zeta, \tau) \tag{4.48}$$

for $M \geqq M_1(\zeta, t)$. In the above calculation we used a local coordinate $(\xi_1, \cdots, \xi_{\ell-1})$ of a patch of ∂Q. So we take the maximum of M after the arguments for all patches (there are only finite numbers of local coordinate patches). We can choose $M_1(\zeta, t) > 0$ such that $M_1(\zeta, t) = O(\log(t/\zeta))$. On the other hand we define $M_2(\zeta, t) > 0$ by

$$M_2(\zeta, t) = 2(\tau/2)^{-2} \frac{\|\nabla' \psi_r\|_{L^\infty(Q)}}{\boldsymbol{\kappa}_1} \left(\sup_{(s,\eta) \in H_2} |\widehat{G}(s, \eta)| + \frac{2\boldsymbol{\kappa}_1}{\pi} \log(t/\zeta) + 1 \right)$$

A simple estimate on $\Gamma^{-}(\zeta,\tau)$ yields

$$\Theta_{-M}(x) \leqq v_{r,\zeta}^{(2)}(x) \leqq \Theta_{M}(x) \quad \text{on} \quad \Gamma^{-}(\zeta,\tau) \tag{4.49}$$

for $M \geqq M_2(\zeta,t)$. Put $M(\zeta,t) = \max(M_1(\zeta,t), M_2(\zeta,t))$. We have (4.45), (4.48) and (4.49) for $\Theta_{\pm M(\zeta,t)}$ and so we apply the maximum principle to $\Theta_{\pm M(\zeta,t)}(x), v_{r,\zeta}^{(2)}$ in the region $\mathcal{U}(\zeta,\tau)$ and get

$$\Theta_{-M(\zeta,t)}(x) \leqq v_{r,\zeta}^{(2)}(x) \leqq \Theta_{M(\zeta,t)}(x) \quad \text{in} \quad \mathcal{U}(\zeta,\tau)$$

for small $\zeta > 0$. Therefor we have

$$\begin{aligned}
-2M(\zeta,t)(\tau - h(\zeta)\zeta) = \frac{\partial \Theta_{M(\zeta,t)}}{\partial s}_{|s=-h(\zeta)\zeta-0} &\leqq \frac{\partial v_{r,\zeta}^{(2)}}{\partial s} \\
&\leqq \frac{\partial \Theta_{-M(\zeta,t)}}{\partial s}_{|s=-h(\zeta)\zeta-0} = 2M(\zeta,t)(\tau - h(\zeta)\zeta).
\end{aligned}$$

From this estimate, we have the second estimate of (4.27). (4.28) is a part of (4.30). So we deal with (4.29) and (4.30).

$$\begin{aligned}
\partial_s v_{r,\zeta}^{(1)}(x) &= \frac{-1}{\zeta \boldsymbol{\kappa}_1} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)(\partial_s \widehat{G})(\frac{s}{\zeta}, \frac{\eta}{\zeta}), \\
\nabla_\eta v_{r,\zeta}^{(1)}(x) &= \frac{-1}{\zeta \boldsymbol{\kappa}_1} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)(\nabla_\eta \widehat{G})(\frac{s}{\zeta}, \frac{\eta}{\zeta}), \\
\nabla_\xi v_{r,\zeta}^{(1)}(x) &= \nabla_\xi \left(\frac{-1}{\boldsymbol{\kappa}_1} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \right) (\widehat{G}(\frac{s}{\zeta}, \frac{\eta}{\zeta}) + \frac{2\boldsymbol{\kappa}_1}{\pi} \log(t/\zeta)), \\
\Delta v_{r,\zeta}^{(1)} &= \frac{1}{\sqrt{g}} \sum_{1 \leqq i,j \leqq l-1} \frac{\partial}{\partial \xi_i} \left(\sqrt{g} g^{ij} \frac{\partial v_{r,\zeta}^{(1)}}{\partial \xi_j} \right) + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial s} \frac{\partial v_{r,\zeta}^{(1)}}{\partial s} \\
&\quad + \frac{\partial^2 v_{r,\zeta}^{(1)}}{\partial s^2} + \sum_{i=1}^{m} \frac{\partial^2 v_{r,\zeta}^{(1)}}{\partial \eta_i^2} \\
&= \frac{1}{\sqrt{g}} \sum_{1 \leqq i,i \leqq \ell-1} \frac{\partial}{\partial \xi_i} \left(\sqrt{g} g^{ij} \frac{\partial v_{r,\zeta}^{(1)}}{\partial \xi_j} \right) + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial s} \frac{\partial v_{r,\zeta}^{(1)}}{\partial s} \\
&= \frac{-1}{\boldsymbol{\kappa}_1 \sqrt{g}} \sum_{1 \leqq i,j \leqq \ell-1} \frac{\partial}{\partial \xi_i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial \xi_j} \left(\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \right) \right)
\end{aligned}$$

$$\times \left(\widehat{G}(\frac{s}{\zeta}, \frac{\eta}{\zeta}) + \frac{2\boldsymbol{\kappa}_1}{\pi} \log \frac{t}{\zeta} \right)$$
$$- \frac{1}{\zeta \boldsymbol{\kappa}_1} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial s} \frac{\partial \widehat{G}}{\partial s}(\frac{s}{\zeta}, \frac{\eta}{\zeta})$$

In $\Sigma^-(\zeta, h\zeta) \cup \Sigma^*(\zeta, t)$, we have the estimate

$$\begin{aligned} |\nabla v_{r,\zeta}^{(1)}(x)| &\leqq c \frac{\|\nabla' \psi_r\|_{L^\infty(Q)}}{\zeta \boldsymbol{\kappa}_1} \left(|(\partial_s \widehat{G})(\frac{s}{\zeta}, \frac{\eta}{\zeta})| + |(\nabla_\eta \widehat{G})(\frac{s}{\zeta}, \frac{\eta}{\zeta})| \right) \\ &\quad + c \frac{\|(\nabla')^2 \psi_r\|_{L^\infty(Q)}}{\boldsymbol{\kappa}_1} \left(|\widehat{G}(\frac{s}{\zeta}, \frac{\eta}{\zeta})| + \frac{2\boldsymbol{\kappa}_1}{\pi} \log \frac{t}{\zeta} \right), \\ |\Delta v_{r,\zeta}^{(1)}(x)| &\leqq c \frac{\|(\nabla')^3 \psi_r\|_{L^\infty(Q)}}{\boldsymbol{\kappa}_1} \left(|\widehat{G}(\frac{s}{\zeta}, \frac{\eta}{\zeta})| + \frac{2\boldsymbol{\kappa}_1}{\pi} \log \frac{t}{\zeta} \right) \\ &\quad + c \frac{\|\nabla' \psi_r\|_{L^\infty(Q)}}{\zeta \boldsymbol{\kappa}_1} |(\partial_s \widehat{G})(\frac{s}{\zeta}, \frac{\eta}{\zeta})|. \end{aligned}$$

c is a constant which is independent of ζ and t. Using the properties of G in Prop.2.7-(i), we have the estimates for $|\nabla v_{r,\zeta}(x)|$ and $|\Delta v_{r,\zeta}(x)|$ in $\Sigma^-(\zeta, h(\zeta)\zeta)$, $\Sigma^-(\zeta, h\zeta)$ and $\Sigma^*(\zeta, t)$. Thus (4.29) and (4.30) are proved. We complete the proof of Lemma 4.3-(i). □

Proof of Lemma 4.3-(ii): $m \geqq 2$. All the properties ((4.31)-(4.36)) except for (4.37) can be proved similarly as the case Lemma 4.3-(i). We need to be careful about that $v_{r,\zeta}^{(1)}$ takes a little different form (actually it does not have log-term) and we use Prop.2.7-(ii) ($m \geqq 2$) in place of Prop.2.7-(i) ($m = 1$). So we briefly give the proof to each of (4.31)-(4.36). The continuity and piecewise C^1 of $\widetilde{\psi}_{r,\zeta}$ is clear. Note the expression of $\widetilde{\psi}_{r,\zeta}$ in $\Sigma^-(\zeta, h\zeta) \cup \Sigma^+(t)$ is given by

$$\begin{aligned} \widetilde{\psi}_{r,\zeta}(x) &= \widetilde{\psi}_r(x) + \zeta v_{r,\zeta}^{(1)}(x) \\ &= \begin{cases} (-\zeta/\boldsymbol{\kappa}_1) \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) G(s/\zeta, \eta/\zeta) \text{ for } \quad x \in \Sigma^+(t), \\ \psi_r(\xi + s\boldsymbol{n}(\xi)) - s \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) - \frac{\zeta}{\boldsymbol{\kappa}_1} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) G(s/\zeta, \eta/\zeta) \\ \qquad \text{for} \quad x \in \Sigma^-(\zeta, h\zeta). \end{cases} \end{aligned} \tag{4.50}$$

Due to (4.38) and the property of G, the continuity of $\nabla \widetilde{\psi}_{r,\zeta}$ across $\partial D \cap \partial Q(\zeta)$ and (4.31) are true. We can carry out a similar argument as in (4.26) with the aid of Prop.2.7-(ii) to $|v_{r,\zeta}(x)|$. Actually we have

$$\sup_{Q(\zeta)\setminus\Sigma^-(\zeta,h\zeta)}|v^{(2)}_{r,\zeta}(x)| = \sup_{\Gamma^-(\zeta,h\zeta)}|v^{(1)}_{r,\zeta}(x)| \leqq \sup_{\Sigma^-(\zeta,h\zeta)}|v^{(1)}_{r,\zeta}(x)|. \tag{4.51}$$

In $\Sigma^-(\zeta,h\zeta)$,

$$\begin{aligned}|v^{(1)}_{r,\zeta}(x)| &\leqq \frac{1}{\boldsymbol{\kappa}_1}\sup_{\xi\in\partial Q}\left|\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\right|\sup_{s<0,|\eta|<\zeta}|\widehat{G}(\frac{s}{\zeta},\frac{\eta}{\zeta})| \\ &\leqq \frac{\|\nabla'\psi_r\|_{L^\infty(Q)}}{\boldsymbol{\kappa}_1}\sup_{(\tilde{s},\tilde{\eta})\in H_2}|\widehat{G}(\tilde{s},\tilde{\eta})|.\end{aligned} \tag{4.52}$$

In $\Sigma^+(2\zeta)$,

$$|v^{(1)}_{r,\zeta}(x)| \leqq \frac{1}{\boldsymbol{\kappa}_1}\left|\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\right| G(\frac{s}{\zeta},\frac{\eta}{\zeta}) \leqq \frac{\|\nabla'\psi_r\|_{L^\infty(Q)}}{\boldsymbol{\kappa}_1}\sup_{(\tilde{s},\tilde{\eta})\in\tilde{\Sigma}^+(2)} G(\tilde{s},\tilde{\eta}) \tag{4.53}$$

for $0<2\zeta\leqq t$. In $\Sigma^+(t)\setminus\Sigma^+(2\zeta)$,

$$|v^{(1)}_{r,\zeta}(x)| \leqq \frac{1}{\boldsymbol{\kappa}_1}\left|\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\right| G(\frac{s}{\zeta},\frac{\eta}{\zeta}) \leqq \frac{\|\nabla'\psi_r\|_{L^\infty(Q)}}{\boldsymbol{\kappa}_1}\frac{c_0\zeta^{m-1}}{(s^2+\eta^2)^{(m-1)/2}} \tag{4.54}$$

for $0<2\zeta\leqq t$. These inequalities imply (4.32). The first inequality of (4.33) and (4.34) follow from (4.35) and (4.36). The second inequality of (4.33) can be proved in similar way as the second one of (4.27). For that proof we use the (upper and lower) barrier functions (as in (4.44))

$$\Theta_{\pm M}(x) = v^{(1)}_{r,\zeta}(\xi-h(\zeta)\zeta\boldsymbol{n}(\xi),\eta) \pm M\left\{(\tau-h(\zeta)\zeta)^2-(\tau+s)^2\right\}$$

for $x=(\xi+s\boldsymbol{n}(\xi),\eta)\in\mathcal{U}(\zeta,\tau)=\Sigma^-(\zeta,\tau)\setminus\Sigma^-(\zeta,h\zeta)$. Note that

$$\begin{aligned}&\Theta_{\pm M}(x) = v^{(2)}_{r,\zeta}(x) = v^{(1)}_{r,\zeta}(x) &&\text{on}\quad \Gamma^-(\zeta,h\zeta),\\ &\frac{\partial}{\partial\nu}\Theta_{\pm M} = \frac{\partial}{\partial\nu}v^{(2)}_{r,\zeta} = 0 &&\text{on}\quad \partial\mathcal{U}(\zeta,\tau)\setminus(\Gamma^-(\zeta,h\zeta)\cup\Gamma^-(\zeta,\tau)).\end{aligned}$$

We can take an adequately small $\tau>0$ and a large number $M>0$ such that

$$\Theta_{-M}(x) \leqq v^{(2)}_{r,\zeta}(x) \leqq \Theta_M(x) \quad \text{on} \quad \Gamma^-(\zeta,\tau),$$

$$(\pm 1)\Delta\Theta_{\pm M}(x) < 0 \quad \text{in} \quad \mathcal{U}(\zeta,\tau).$$

From this condition we have

$$\Theta_{-M}(x) \leqq v^{(2)}_{r,\zeta}(x) \leqq \Theta_M(x) \quad \text{in} \quad \mathcal{U}(\zeta,\tau).$$

Consequently we have the estimate

$$\begin{aligned}\left(\frac{\partial\Theta_M}{\partial s}\right)_{|s=-h\zeta-0} &\leqq \left(\frac{\partial v^{(2)}_{r,\zeta}}{\partial s}\right)_{|s=-h\zeta-0}\\ &\leqq \left(\frac{\partial\Theta_{-M}}{\partial s}\right)_{|s=-h\zeta-0} \quad \text{on} \quad \Gamma^-(\zeta,h\zeta).\end{aligned}$$

Calculate the values of the right and left sides of the above inequalities and get an estimate for $|(\partial/\partial s)(v^{(2)}_{r,\zeta}(\xi+s\boldsymbol{n}(\xi),\eta)_{|s=-h\zeta-0}|$ which leads to the second inequality of (4.33). For the proof of (4.35) and (4.36), we need the following explicit expressions of $\nabla v^{(1)}_{r,\zeta}$, $\Delta v^{(1)}_{r,\zeta}$ in $\Sigma^-(\zeta,h\zeta)\cup\Sigma^+(t)$.

$$\begin{aligned}\partial_s v^{(1)}_{r,\zeta}(x) &= \frac{-1}{\zeta\boldsymbol{\kappa}_1}\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)(\partial_s\widehat{G})(\frac{s}{\zeta},\frac{\eta}{\zeta}),\\ \nabla_\eta v^{(1)}_{r,\zeta}(x) &= \frac{-1}{\zeta\boldsymbol{\kappa}_1}\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)(\nabla_\eta\widehat{G})(\frac{s}{\zeta},\frac{\eta}{\zeta})\\ \nabla_\xi v^{(1)}_{r,\zeta}(x) &= \nabla_\xi\left(\frac{-1}{\boldsymbol{\kappa}_1}\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\right)\widehat{G}(\frac{s}{\zeta},\frac{\eta}{\zeta}),\\ \Delta v^{(1)}_{r,\zeta} &= \frac{-1}{\boldsymbol{\kappa}_1\sqrt{g}}\sum_{1\leqq i,j\leqq \ell-1}\frac{\partial}{\partial\xi_i}\left(\sqrt{g}g^{ij}\frac{\partial}{\partial\xi_j}(\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi))\right)\widehat{G}(\frac{s}{\zeta},\frac{\eta}{\zeta})\\ &\quad -\frac{1}{\zeta\boldsymbol{\kappa}_1}\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial s}(\partial_s\widehat{G})(\frac{s}{\zeta},\frac{\eta}{\zeta}).\end{aligned}$$

Estimating these expressions with the aid of Prop.2.7-(ii), we get the inequalities (4.35) and (4.36).

For the proof of (4.37), we define the function

$$\widetilde{v}^{(3)}_{r,\zeta}(x) = \zeta^{-(m-1)}v^{(3)}_{r,\zeta}(x) \tag{4.55}$$

which is a harmonic function in $D\setminus\Sigma^+(t)$. This function $\widetilde{v}^{(3)}_{r,\zeta}(x)$ satisfies for the Neumann B.C. on $\partial(D\setminus\Sigma^+(t))\setminus\Gamma^+(t)$. On $\Gamma^+(t)$, we have

$$\widetilde{v}^{(3)}_{r,\zeta}(x) = \zeta^{-(m-1)}v^{(1)}_{r,\zeta}(x) = \frac{-1}{\zeta^{m-1}\boldsymbol{\kappa}_1}\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\widehat{G}(\frac{s}{\zeta},\frac{\eta}{\zeta}) \tag{4.56}$$

$$\text{for} \quad x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Gamma^+(t).$$

From Prop.2.7-(ii), the function in the right hand side smoothly converges to

$$-\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \frac{2S(m)}{(m-1)(m+1)S(m+1)t^{m-1}} \quad \text{on} \quad \Gamma^+(t)$$

when $\zeta \to 0$. Due to this fact, we can prove that $\widetilde{v}^{(3)}_{r,\zeta}(x)$ approaches the smooth function $\widetilde{v}^{(3)}_r(x)$ (in $D \setminus \Sigma^+(t)$), which satisfies

$$\Delta \widetilde{v}^{(3)}_r(x) = 0 \quad \text{in} \quad D \setminus \Sigma^+(t), \quad \partial \widetilde{v}^{(3)}_r / \partial \nu = 0 \quad \text{on} \quad \partial(D \setminus \Sigma^+(t)) \setminus \Gamma^+(t)$$

with

$$\widetilde{v}^{(3)}_r(x) = -\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \frac{2S(m)}{(m-1)(m+1)S(m+1)t^{m-1}}$$
$$(x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Gamma^+(t)).$$

From this convergence, the estimate (4.37) follows. We complete the proof of Lemma 4.3-(ii). □

4.4. Comparison functions $\varphi_{1,\zeta}$, $\varphi_{2,\zeta}$ and some estimates for $\Phi_{k,\zeta}$

We recall some auxiliary functions $\varphi_{1,\zeta}$, $\varphi_{2,\zeta}$ which were constructed in Jimbo [26;Lem.2.7]. These functions play roles of barrier functions to bound the behaviors of the eigenfunctions around the junction part. Define the set

$$J(\zeta, t_2, t_1) \equiv \Sigma^-(\zeta, t_2) \cup \Sigma^+(t_1)$$

for $t_1, t_2 > 0$. We give a statement for the existence and properties for these auxiliary (barrier) functions in the following proposition.

PROPOSITION 4.4 ([26]). *For $M > 0$, there exist $t_1, t_2 > 0$, $\zeta_1 > 0$, $c_4 > 0$, $c_5 > 0$ and the positive functions*

$$\varphi_{1,\zeta},\ \varphi_{2,\zeta} \in H^1(J(\zeta, t_2, t_1)) \cap C^0(\overline{J(\zeta, t_2, t_1)})$$

with the following properties (4.57)-(4.60).

$$c_4 \leqq \varphi_{1,\zeta}(x) \leqq c_5 \quad (x \in J(\zeta, t_2, t_1)) \tag{4.57}$$

(i) *For* $m = 1$,

$$(4.58)\quad \begin{cases} c_4\,(-s+\zeta\,\log(1/\zeta)) \leqq \varphi_{2,\zeta}(x) \leqq c_5\,(-s+\zeta\log(1/\zeta)) \\ \qquad\qquad (x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Sigma^-(\zeta, t_2)), \\ 0 < \varphi_{2,\zeta}(x) \leqq c_5\,\zeta\log(1/\zeta) \\ \qquad\qquad (x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Sigma^+(2\zeta)), \\ 0 < \varphi_{2,\zeta}(x) \leqq c_5\,\zeta\left(1 + \log(t_1/\sqrt{s^2+|\eta|^2})\right) \\ \qquad\qquad (x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Sigma^+(t_1) \setminus \Sigma^+(2\zeta)). \end{cases}$$

(ii) *For* $m \geqq 2$,

$$(4.59)\quad \begin{cases} c_4\,(-s+\zeta) \leqq \varphi_{2,\zeta}(x) \leqq c_5\,(-s+\zeta) \\ \qquad\qquad (x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Sigma^-(\zeta, t_2)), \\ 0 < \varphi_{2,\zeta}(x) \leqq c_5\zeta \\ \qquad\qquad (x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Sigma^+(2\zeta)), \\ 0 < \varphi_{2,\zeta}(x) \leqq c_5 \dfrac{\zeta^m}{(s^2+|\eta|^2)^{(m-1)/2}} \\ \qquad\qquad (x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Sigma^+(t_1) \setminus \Sigma^+(2\zeta)), \end{cases}$$

for $0 < \zeta \leqq \zeta_1$ *(The constant* ζ_1 *depends on only* D *and function* $\boldsymbol{q} = \boldsymbol{q}(s)$*). Moreover,* $\varphi_{1,\zeta}$*,* $\varphi_{2,\zeta}$ *satisfy the following differential inequalities,*

$$(4.60)\quad \begin{cases} \Delta\varphi_{i,\zeta} + (M+1)\varphi_{i,\zeta} \leqq 0 & in \quad J(\zeta, t_2, t_1), \\ \partial\varphi_{i,\zeta}/\partial\nu = 0 & on \quad \partial J(\zeta, t_2, t_1) \setminus (\Gamma^+(t_1) \cup \Gamma^-(\zeta, t_2)), \end{cases}$$

for $i = 1, 2$*. The meaning of the inequality in (4.60) is taken in the generalized sense (cf. Gilbarg-Trudinger [22;Chap.8]).*

In later sections, we make use of the above comparison functions to estimate the behavior of the eigenfunctions.

Lemma 4.5. *For* $\varphi_{i,\zeta}$ *(*$i = 1, 2$*) for* $M \geqq \mu_k$ *in Proposition 4.4, there exist* $c_6 > 0, c_7 > 0$ *such that*

$$(4.61)\qquad |\Phi_{k,\zeta}(x)| \leqq c_6 \left(\sup_{y \in D\setminus\Sigma^+(t_1)} |\Phi_{k,\zeta}(y)| \right) \varphi_{1,\zeta}(x)$$

$$+ c_7 \left(\sup_{y \in \Gamma^-(\zeta,t_2)} |\Phi_{k,\zeta}(y)| \right) \varphi_{2,\zeta}(x)$$

for $x \in J(\zeta, t_2, t_1) = \Sigma^-(\zeta, t_2) \cup \Sigma^+(t_1)$ *for* $0 < \zeta < \zeta_1$. *Here* $t_1 > 0, t_2 > 0, \zeta_1 > 0$ *are constants given in Proposition 4.4 for* $M = \mu_k$.

Combining Proposition 4.4 and Lemma 4.5, we can prove some estimates for $\Phi_{k,\zeta}$ in $J(\zeta, t_2, t_1)$.

REMARK 4.6. By replacing the constants c_6, c_7 by larger ones in Lemma 4.5, we also have the following estimates.
(i) For $m = 1$, $x = (\xi + s\boldsymbol{n}(\xi), \eta) \in J(\zeta, t_2, t_1) = \Sigma^-(\zeta, t_2) \cup \Sigma^+(t_1)$,

$$|\Phi_{k,\zeta}(x)| \leqq c_6 \sup_{y \in D \setminus \Sigma^+(t_1)} |\Phi_{k,\zeta}(y)| + c_7 \sup_{y \in \Gamma^-(\zeta,t_2)} |\Phi_{k,\zeta}(y)| \, (-s + \zeta \log(1/\zeta)) \quad \text{in} \quad \Sigma^-(\zeta, t_2),$$

$$|\Phi_{k,\zeta}(x)| \leqq c_6 \sup_{y \in D \setminus \Sigma^+(t_1)} |\Phi_{k,\zeta}(y)| + c_7 \sup_{y \in \Gamma^-(\zeta,t_2)} |\Phi_{k,\zeta}(y)| \, \zeta \log(1/\zeta) \quad \text{in} \quad \Sigma^+(2\zeta),$$

$$|\Phi_{k,\zeta}(x)| \leqq c_6 \sup_{y \in D \setminus \Sigma^+(t_1)} |\Phi_{k,\zeta}(y)| + c_7 \sup_{y \in \Gamma^-(\zeta,t_2)} |\Phi_{k,\zeta}(y)| \, \zeta \left(1 + \log(t_1/\sqrt{s^2 + |\eta|^2})\right) \quad \text{in} \quad \Sigma^+(t_1) \setminus \Sigma^+(2\zeta).$$

(ii) For $m \geqq 2$, $x = (\xi + s\boldsymbol{n}(\xi), \eta) \in J(\zeta, t_2, t_1) = \Sigma^-(\zeta, t_2) \cup \Sigma^+(t_1)$,

$$|\Phi_{k,\zeta}(x)| \leqq c_6 \sup_{y \in D \setminus \Sigma^+(t_1)} |\Phi_{k,\zeta}(y)| + c_7 \sup_{y \in \Gamma^-(\zeta,t_2)} |\Phi_{k,\zeta}(y)| \, (-s + \zeta) \quad \text{in } \Sigma^-(\zeta, t_2),$$

$$|\Phi_{k,\zeta}(x)| \leqq c_6 \sup_{y \in D \setminus \Sigma^+(t_1)} |\Phi_{k,\zeta}(y)| + c_7 \sup_{y \in \Gamma^-(\zeta,t_2)} |\Phi_{k,\zeta}(y)| \, \zeta \quad \text{in } \Sigma^+(2\zeta),$$

$$|\Phi_{k,\zeta}(x)| \leqq c_6 \sup_{y \in D \setminus \Sigma^+(t_1)} |\Phi_{k,\zeta}(y)| + c_7 \sup_{y \in \Gamma^-(\zeta,t_2)} |\Phi_{k,\zeta}(y)| \, \frac{\zeta^m}{(s^2 + |\eta|^2)^{(m-1)/2}} \quad \text{in} \quad \Sigma^+(t_1) \setminus \Sigma^+(2\zeta).$$

In the inequalities of Lemma 4.5 and Remark 4.6, we can replace $\Phi_{k,\zeta}$ by $\varepsilon(\zeta)\Phi_{k,\zeta}$ by the homogeneity. Here $\epsilon(\zeta)$ is an arbitrary constant which depends on ζ.

As an application of these estimates, we can prove that if $k \in \mathbb{N}_{II}$, $\Phi_{k,\zeta}$ decays to zero in D at a certain rapid rate. We state the estimates in the forms which are useful in later sections. We put

$$\widehat{\Phi}_{k,\zeta}(x) = S(m)^{1/2}\zeta^{m/2}\Phi_{k,\zeta}(x)$$

for which we have the following estimates.

Corollary 4.7.
(i) *For any $k \in \mathbb{N}_{II}$ and $m = 1$. Then there exists $c_8 > 0$ such that, for $x \in \Sigma^-(\zeta, t_2) \cup D$*

$$(4.62)\quad |\widehat{\Phi}_{k,\zeta}(x)| \leqq c_8 \begin{cases} \zeta \log(2t_1/\zeta) + |s| \\ \qquad \text{for} \quad x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Sigma^-(\zeta, t_2), \\ \zeta \log(2t_1/\zeta) \\ \qquad \text{for} \quad x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Sigma^+(2\zeta), \\ \zeta \log \frac{2t_1}{(s^2+|\eta|^2)^{1/2}} \\ \qquad \text{for} \quad x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Sigma^*(\zeta, t_1) \setminus \Sigma^+(2\zeta), \\ \zeta \quad \text{for} \quad x \in D \setminus \Sigma^*(\zeta, t_1). \end{cases}$$

(ii) *For any $k \in \mathbb{N}_{II}$ and $m \geqq 2$, there exists $c_8 > 0$ such that, for $x \in \Sigma^-(\zeta, t_2) \cup D$*

$$(4.63)\quad |\widehat{\Phi}_{k,\zeta}(x)| \leqq c_8 \begin{cases} \zeta + |s| \\ \qquad \text{for} \quad x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Sigma^-(\zeta, t_2), \\ \zeta \qquad \text{for} \quad x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Sigma^+(2\zeta), \\ \frac{\zeta^m}{(s^2+|\eta|^2)^{(m-1)/2}} \\ \qquad \text{for} \quad x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Sigma^+(t_1) \setminus \Sigma^+(2\zeta), \\ \zeta^m \quad \text{for} \quad x \in D \setminus \Sigma^+(t_1). \end{cases}$$

for $0 < 2\zeta \leqq t_1$. t_1, t_2 are constants given in Proposition 4.4 and Lemma 4.5 for $M = \mu_k$.

PROOF OF COROLLARY 4.7. From Prop.3.2-(ii), $\Phi_{k,\zeta}$ decays to 0 in D for $\zeta \to 0$ in L^2-sense. This corollary are concerned with the pointwise decay rate. We prove this corollary with the aid of Lem.4.5.

(i) First we prove

$$\limsup_{\zeta\to 0}\left(\sup_{D\setminus\Sigma^+(t_1)}|\Phi_{k,\zeta}(x)|\right)/\zeta^{1/2}<\infty. \tag{4.64}$$

Assume the contrary. Then there exists a positive sequence $\{\zeta_p\}_{p=1}^{\infty}$ such that $\zeta_p \to 0$ for $p \to \infty$ and

$$\lim_{p\to\infty}\left(\sup_{D\setminus\Sigma^+(t_1)}|\Phi_{k,\zeta_p}(x)|\right)/\zeta_p^{1/2}=\infty. \tag{4.65}$$

Define $\widetilde{\Phi}_{k,\zeta_p}(x)=\Phi_{k,\zeta_p}(x)/\left(\sup_{y\in D\setminus\Sigma^+(t_1)}|\Phi_{k,\zeta_p}(y)|\right)$ and then we have

$$\sup_{x\in D\setminus\Sigma^+(t_1)}|\widetilde{\Phi}_{k,\zeta_p}(x)|=1. \tag{4.66}$$

On the other hand, from Lem.4.5-(4.61), we have

$$|\widetilde{\Phi}_{k,\zeta}(x)|\leqq c_6\varphi_{1,\zeta}(x)+c_7\left(\sup_{y\in\Gamma^-(\zeta,t_2)}|\Phi_{k,\zeta}(y)|/\sup_{y\in D\setminus\Sigma^+(t_1)}|\Phi_{k,\zeta}(y)|\right)\varphi_{2,\zeta}(x) \quad \text{in } J(\zeta,t_2,t_1).$$

From Rem.2.3-(ii), there exists c such that $|\Phi_{k,\zeta}(y)|\leqq c\zeta^{-1/2}$ in $\Omega(\zeta)$ (as $m=1$). Using this estimate we have

$$|\widetilde{\Phi}_{k,\zeta}(x)|\leqq c_6\varphi_{1,\zeta}(x)+c_7\left(c\zeta^{1/2}/\sup_{D\setminus\Sigma^+(t_1)}|\Phi_{k,\zeta}|\right)(1/\zeta)\varphi_{2,\zeta}(x) \quad \text{in} \quad J(\zeta,t_2,t_1). \tag{4.67}$$

Put $\zeta=\zeta_p$ and take $p\to\infty$ in (4.67), we have, from (4.57), (4.58), (4.65), (4.66) that

$$\limsup_{p\to\infty}\sup_{x\in D\setminus\Sigma^+(\epsilon)}|\widetilde{\Phi}_{k,\zeta_p}(x)|\leqq \max(c_6\,c_5,1) \quad \text{for any } \epsilon>0,$$

$$\limsup_{p\to\infty} |\widetilde{\Phi}_{k,\zeta_p}(x)| \leqq \max(c_6\, c_5, 1) \quad \text{for} \quad x \in D.$$

Using these estimates, we apply Prop.8.2 and we assert that a subsequence of $\widetilde{\Phi}_{k,\zeta_p}$ is convergent to a limit $\widetilde{\Phi}_k \in C^2(\overline{D})$ which satisfies

$$\text{(4.68)} \qquad \begin{cases} \Delta\widetilde{\Phi}_k + \mu_k\widetilde{\Phi}_k = 0 \quad \text{in} \quad D, \quad \partial\widetilde{\Phi}_k/\partial\nu = 0 \quad \text{on} \quad \partial D, \\ \sup_{x\in D\setminus\Sigma^+(t_1)} |\widetilde{\Phi}_k(x)| = 1, \quad \sup_{x\in D} |\widetilde{\Phi}_k(x)| \leqq \max(c_6 c_5, 1). \end{cases}$$

This implies $\mu_k \in \{\omega_d\}_{d=1}^{\infty}$ which leads to a contradiction because $\mu_k \in E_{II}$. Using (4.64) in Rem.4.6-(i) and Rem.2.3-(2.9), we conclude (i).

(ii) can be proved similarly as (4.62). The only difference is that we prove the following property

$$\text{(4.69)} \qquad \limsup_{\zeta\to 0} \left(\sup_{D\setminus\Sigma^+(t_1)} |\Phi_{k,\zeta}(x)| \right) /\zeta^{m/2} < \infty.$$

in place of (4.64) by the completely same argument. Using (4.69) and Rem.2.3-(ii) in Rem.4.6-(ii), we get (4.63). □

Corollary 4.8. *For any $k \in \mathbb{N}_{II}$, there exists $c_9 > 0$ such that*

$$\text{(4.70)} \qquad \sup_{x\in\Sigma^-(\zeta,h(\zeta)\zeta)} |\widehat{\Phi}_{k,\zeta}(x)| \leqq c_9\, h(\zeta)\zeta \quad (m \geqq 1).$$

Proof of Corollary 4.8. This result directly follows from Cor.4.7. □

§5. The Case I : $\mu_k \in E_I$ (Proof of Theorem 2.5)

We will study the behavior of $\mu_k(\zeta)$ when $\mu_k \in E_I$ (i.e. $k \in \mathbb{N}_I$) for the proof of Theorem 2.5. Assume $k(j) \in \mathbb{N}_I$, then there exists a unique natural number j' such that $\mu_{k(j)} = \omega_{d(j')}$. Then we have $\widehat{k}(j) = \widehat{d}(j')$ and $\mu_k = \omega_d$ for $d(j') \leqq d < d(j'+1)$, $k(j) \leqq k < k(j+1)$. From Proposition 2.1, the eigenvalues $\mu_k(\zeta)$ $(k(j) \leqq k < k(j+1))$ approach the value

$$\mu_{k(j)} = \mu_{k(j)+1} = \mu_{k(j)+2} = \cdots = \mu_{k(j+1)-1}$$

for $\zeta \to 0$. From Proposition 3.1, we have

$$\limsup_{\zeta \to 0} \left(\sup_{x \in \Omega(\zeta)} |\Phi_{k,\zeta}(x)| \right) < \infty \quad \text{for} \quad k \in \mathbb{N}_I. \tag{5.1}$$

We apply Proposition 2.2 in combination with Proposition 8.1 and see that for an arbitrary sequence of positive values $\{\zeta_p\}_{p=1}^\infty$ converging to 0 for $p \to \infty$, there exist a subsequence $\{\sigma_p\}_{p=1}^\infty \subset \{\zeta_p\}_{p=1}^\infty$ and $\{\Phi_k\}_{k=1}^\infty \subset C^2(\overline{D})$ and $\{\widehat{\Phi}_k\}_{k=1}^\infty \subset C^2(\overline{Q})$ satisfying (2.6), (2.7), (2.8), (2.9). Moreover there exists $W_k \in C^2(\overline{Q})$ (for $k(j) \leqq k < k(j+1)$) such that

$$\begin{cases} \lim_{p \to \infty} \sup_{x \in D} |\Phi_{k,\sigma_p}(x) - \Phi_k(x)| = 0, \\ \lim_{p \to \infty} \sup_{x=(x',x'') \in Q(\sigma_p)} |\Phi_{k,\sigma_p}(x) - W_k(x')| = 0, \end{cases} \tag{5.2}$$

$$\Delta' W_k + \mu_k W_k = 0 \quad \text{in} \quad Q, \ W_k(\xi) = \Phi_k(\xi, o'') \quad \text{for} \quad \xi \in \partial Q. \tag{5.3}$$

From the properties (5.1) and (2.6), we have $\widehat{\Phi}_k \equiv 0$ in Q (for $k(j) \leqq k < k(j+1)$) and consequently, we see that $\{\Phi_k\}_{k(j) \leqq k < k(j+1)}$ is orthonormal in $L^2(D)$ (cf.(2.8)). So it spans the same subspace as $\{\phi_d\}_{d(j') \leqq d < d(j'+1)}$ and it holds that

$$\Phi_k = \sum_{d(j') \leqq q < d(j'+1)} (\Phi_k, \phi_q)_{L^2(D)} \phi_q.$$

Hence we have

$$W_k(x') = \sum_{d(j') \leqq q < d(j'+1)} (\Phi_k, \phi_q)_{L^2(D)} V_q(x') \quad (x' \in Q) \tag{5.4}$$

for $k(j) \leqq k < k(j+1)$. This is obtained by comparing (2.11) and (5.3) with the aid of the uniqueness of W_k in (5.3). The uniqueness is guaranteed by $\mu_k \notin \{\omega_d\}_{d=1}^\infty$.

We start the analysis on $\mu_k(\zeta)$. The eigenfunction $\Phi_{k,\zeta}$ satisfies (2.2) which immediately leads to

$$\begin{aligned} &\int_{D \setminus \Sigma^+(2\zeta)} (\nabla \Phi_{k,\zeta} \nabla \Psi - \mu_k(\zeta)\, \Phi_{k,\zeta}\, \Psi)\, dx \\ &+ \int_{\Lambda(\zeta)} (\nabla \Phi_{k,\zeta} \nabla \Psi - \mu_k(\zeta)\, \Phi_{k,\zeta}\, \Psi)\, dx \end{aligned} \tag{5.5}$$

$$+\int_{Q(\zeta)\setminus\Sigma^-(\zeta,2\zeta)} (\nabla\Phi_{k,\zeta}\nabla\Psi - \mu_k(\zeta)\,\Phi_{k,\zeta}\,\Psi)\,dx = 0 \qquad \text{for} \quad \forall\Psi \in H^1(\Omega(\zeta)).$$

We put $\Psi = \widetilde{\phi}_{d,\zeta}$ for $d(j') \leqq d < d(j'+1)$ in (5.5) and denote the three terms by $I_1(\zeta), I_2(\zeta), I_3(\zeta)$. That is,

$$I_1(\zeta) = \int_{D\setminus\Sigma^+(2\zeta)} (\nabla\Phi_{k,\zeta}\nabla\widetilde{\phi}_{d,\zeta} - \mu_k(\zeta)\,\Phi_{k,\zeta}\,\widetilde{\phi}_{d,\zeta})\,dx,$$
$$I_2(\zeta) = \int_{\Lambda(\zeta)} (\nabla\Phi_{k,\zeta}\nabla\widetilde{\phi}_{d,\zeta} - \mu_k(\zeta)\,\Phi_{k,\zeta}\,\widetilde{\phi}_{d,\zeta})\,dx,$$
$$I_3(\zeta) = \int_{Q(\zeta)\setminus\Sigma^-(\zeta,2\zeta)} (\nabla\Phi_{k,\zeta}\nabla\widetilde{\phi}_{d,\zeta} - \mu_k(\zeta)\,\Phi_{k,\zeta}\,\widetilde{\phi}_{d,\zeta})\,dx.$$

We evaluate each term $I_j(\sigma_p)$ after rewriting them in comprehensive forms. By partial integration and the definition of $\widetilde{\phi}_{d,\zeta}$, we get

$$(5.6)\quad I_1(\zeta) = \int_{\partial(D\setminus\Sigma^+(2\zeta))} \Phi_{k,\zeta}\,\frac{\partial\phi_d}{\partial\nu_3}\,dS + \int_{D\setminus\Sigma^+(2\zeta)} (\mu_k - \mu_k(\zeta))\,\Phi_{k,\zeta}\,\phi_d\,dx.$$

We used $\Delta\phi_d = -\mu_k\,\phi_d$ in D (with the Neumann B.C. on ∂D) for $d(j') \leqq d < d(j'+1)$, $k(j) \leqq k < k(j+1)$. ν_3 and ν_4 are the outward unit vectors on $D \setminus \Sigma^+(2\zeta)$ and $\Sigma^+(2\zeta)$, respectively. Note that $\nu_3 = -\nu_4$ on $\Gamma^+(2\zeta)$. We calculate the first term of the above expression,

$$\begin{aligned}
&\int_{\partial(D\setminus\Sigma^+(2\zeta))} \Phi_{k,\zeta}\,\frac{\partial\phi_d}{\partial\nu_3}\,dS \\
&= \int_{\Gamma^+(2\zeta)} \Phi_{k,\zeta}\,\frac{\partial\phi_d}{\partial\nu_3}\,dS = -\int_{\Gamma^+(2\zeta)} \Phi_{k,\zeta}\,\frac{\partial\phi_d}{\partial\nu_4}\,dS = -\int_{\partial\Sigma^+(2\zeta)} \Phi_{k,\zeta}\,\frac{\partial\phi_d}{\partial\nu_4}\,dS \\
&= -\int_{\partial\Sigma^+(2\zeta)} (\Phi_{k,\zeta}(x) - \Phi_k(\xi,o''))\,\frac{\partial\phi_d}{\partial\nu_4}\,dS - \int_{\partial\Sigma^+(2\zeta)} \Phi_k(\xi,o'')\,\frac{\partial\phi_d}{\partial\nu_4}\,dS \\
&= -\int_{\partial\Sigma^+(2\zeta)} (\Phi_{k,\zeta}(x) - \Phi_k(\xi,o''))\,\frac{\partial\phi_d}{\partial\nu_4}\,dS - \int_{\Sigma^+(2\zeta)} \mathrm{div}(\Phi_k(\xi,o'')\nabla\phi_d)\,dx.
\end{aligned}$$

We used the relation of the variables x, ξ, η which is $x = (\xi + s\,\boldsymbol{n}(\xi), \eta)$, $\xi \in \partial Q$ in a neighborhood of $\partial Q \times \{o''\}$. Consequently, we have

$$(5.7)\qquad I_1(\zeta) = -\int_{\partial\Sigma^+(2\zeta)} (\Phi_{k,\zeta}(x) - \Phi_k(\xi,o''))\frac{\partial\phi_d}{\partial\nu_4}\,dS$$

$$- \int_{\Sigma^+(2\zeta)} \operatorname{div}(\Phi_k(\xi, o'')\nabla\phi_d)\, dx$$
$$+ \int_{D\setminus\Sigma^+(2\zeta)} (\mu_k - \mu_k(\zeta))\, \Phi_{k,\zeta}\, \phi_d\, dx.$$

About $I_2(\zeta)$, $I_3(\zeta)$, we calculate

$$
\begin{aligned}
(5.8)\qquad I_2(\zeta) &= \int_{\Lambda(\zeta)} \nabla(\Phi_{k,\zeta}(x) - \Phi_k(\xi, o''))\nabla\widetilde{\phi}_{d,\zeta}\, dx \\
&\quad + \int_{\Lambda(\zeta)} \nabla\Phi_k(\xi, o'')\nabla\widetilde{\phi}_{d,\zeta}\, dx - \int_{\Lambda(\zeta)} \mu_k(\zeta)\, \Phi_{k,\zeta}\, \widetilde{\phi}_{d,\zeta}\, dx \\
&= \int_{\partial\Lambda(\zeta)} (\Phi_{k,\zeta}(x) - \Phi_k(\xi, o''))\, \frac{\partial\widetilde{\phi}_{d,\zeta}}{\partial\nu_1}\, dS \\
&\quad - \int_{\Lambda(\zeta)} (\Phi_{k,\zeta}(x) - \Phi_k(\xi, o''))\, \Delta\widetilde{\phi}_{d,\zeta}\, dx \\
&\quad + \int_{\Lambda(\zeta)} \nabla\Phi_k(\xi, o'')\nabla\widetilde{\phi}_{d,\zeta}\, dx - \int_{\Lambda(\zeta)} \mu_k(\zeta)\, \Phi_{k,\zeta}\, \widetilde{\phi}_{d,\zeta}\, dx,
\end{aligned}
$$

$$
\begin{aligned}
(5.9)\qquad I_3(\zeta) &= \int_{\Gamma^-(\zeta,2\zeta)} \Phi_{k,\zeta}\, \frac{\partial\widetilde{\phi}_{d,\zeta}}{\partial\nu_2}\, dS \\
&\quad + \int_{Q(\zeta)\setminus\Sigma^-(\zeta,2\zeta)} (\mu_k - \mu_k(\zeta))\, \Phi_{k,\zeta}\, V_d\, dx.
\end{aligned}
$$

Here ν_1 and ν_2 are the outward unit normal vector on $\partial\Lambda(\zeta)$ and $\partial(Q(\zeta) \setminus \Sigma^-(\zeta, 2\zeta))$, respectively. V_d is regarded as a function in $Q(\zeta) \setminus \Sigma^-(\zeta, 2\zeta)$ by putting $V_d(x', x'')$ to be $V_d(x')$.

We take $\zeta = \sigma_p$ and evaluate each $I_j(\sigma_p)$ $(j = 1, 2, 3)$. With the aid of (5.1), (5.2) and Lemma 4.2, we get

$$
(5.10)\qquad \begin{cases} I_1(\sigma_p) = (\mu_k - \mu_k(\sigma_p)) \displaystyle\int_D \Phi_{k,\sigma_p}\phi_d\, dx + o(\sigma_p^m), \\ I_2(\sigma_p) = o(\sigma_p^m), \end{cases}
$$

$$(5.11)\qquad I_3(\sigma_p) = \int_{\Gamma^-(\sigma_p,2\sigma_p)} \Phi_{k,\sigma_p}\, \frac{\partial\widetilde{\phi}_{d,\sigma_p}}{\partial\nu_2}\, dS + o(\sigma_p^m).$$

To evaluate the right hand side of $I_3(\sigma_p)$, we prepare the following lemma.

LEMMA 5.1. *For $k(j) \leqq k < k(j+1)$, $d(j') \leqq d < d(j'+1)$, we have*

$$\int_{\Gamma^-(\sigma_p, 2\sigma_p)} \Phi_{k,\sigma_p} \frac{\partial \widetilde{\phi}_{d,\sigma_p}}{\partial \nu_2}\, dS = S(m) \int_{\partial Q} W_k(\xi) \frac{\partial V_d}{\partial \boldsymbol{n}}(\xi)\, dS'\, \sigma_p^m + o(\sigma_p^m). \tag{5.12}$$

PROOF OF LEMMA 5.1. We calculate the left hand side of (5.12).

$$\begin{aligned} &\int_{\Gamma^-(\sigma_p, 2\sigma_p)} \Phi_{k,\sigma_p} \frac{\partial \widetilde{\phi}_{d,\sigma_p}}{\partial \nu_2}\, dS \\ &= \int_{\Gamma^-(\sigma_p, 2\sigma_p)} (\Phi_{k,\sigma_p}(x) - W_k(x')) \frac{\partial \widetilde{\phi}_{d,\sigma_p}}{\partial \nu_2}\, dS \\ &\quad + \int_{\Gamma^-(\sigma_p, 2\sigma_p)} \{W_k(\xi + s\boldsymbol{n}(\xi)) \langle \nabla' V_d(\xi + s\boldsymbol{n}(\xi)), \boldsymbol{n}(\xi)\rangle \\ &\qquad\qquad - W_k(\xi)\langle \nabla' V_d(\xi), \boldsymbol{n}(\xi)\rangle\}\, dS \\ &\quad + \int_{\partial Q} \int_{|\eta|<\sigma_p} W_k(\xi) \langle \nabla' V_d(\xi), \boldsymbol{n}(\xi)\rangle (\rho_2(\xi, -2\sigma_p) - 1)\, d\eta\, dS' \\ &\quad + \int_{\partial Q} \int_{|\eta|<\sigma_p} W_k(\xi) \langle \nabla' V_d(\xi), \boldsymbol{n}(\xi)\rangle\, d\eta\, dS' \end{aligned} \tag{5.13}$$

Applying (5.2) and Lem.4.2, we see that the right hand side of (5.13) equals to

$$S(m)\, \sigma_p^m \int_{\partial Q} W_k(\xi)\, \langle (\nabla' V_d)(\xi), \boldsymbol{n}(\xi)\rangle\, dS' + o(\sigma_p^m).$$

Recall $\Phi_k(\xi, o'') = W_k(\xi)$ and $\rho_2(\xi, 0) = 1$ for $\xi \in \partial Q$ and so the above expression coincides with the right hand side of (5.12). □

Substitute the result in Lemma 5.1 into $I_1(\sigma_p) + I_2(\sigma_p) + I_3(\sigma_p) = 0$ and we conclude

$$(\mu_k(\sigma_p) - \mu_k) \int_D \Phi_{k,\sigma_p} \phi_d\, dx = S(m) \int_{\partial Q} W_k(\xi) \frac{\partial V_d}{\partial \boldsymbol{n}}(\xi)\, dS'\, \sigma_p^m + o(\sigma_p^m) \tag{5.14}$$

for $d(j') \leqq d < d(j'+1)$, $k(j) \leqq k < k(j+1)$. We notice

$$\lim_{p\to\infty} \int_D \Phi_{k,\sigma_p} \phi_d \, dx = \int_D \Phi_k \phi_d \, dx$$

and that for any k such that $k(j) \leqq k < k(j+1)$, there exists d such that $d(j') \leqq d < d(j'+1)$ and $(\Phi_k, \phi_d)_{L^2(D)} \neq 0$, we conclude that there exists the limit value

$$\lim_{p\to\infty} (\mu_k(\sigma_p) - \mu_k)/\sigma_p^m \quad (\text{for} \quad k(j) \leqq k < k(j+1)).$$

We denote this value by $\boldsymbol{\alpha}'(k)$. (5.14) is rewritten as

$$\boldsymbol{\alpha}'(k)\, (\Phi_k, \phi_d)_{L^2(D)} = S(m) \int_{\partial Q} W_k(\xi) \frac{\partial V_d}{\partial \boldsymbol{n}}(\xi)\, dS' \quad (x' \in Q)$$

for $d(j') \leqq d < d(j'+1), k(j) \leqq k < k(j+1)$. From (5.4), the above equality leads to

$$\begin{aligned} (5.15) \qquad & \boldsymbol{\alpha}'(k)\, (\Phi_k, \phi_d)_{L^2(D)} \\ &= \sum_{d(j') \leqq q < d(j'+1)} S(m) \int_{\partial Q} V_q(\xi) \frac{\partial V_d}{\partial \boldsymbol{n}}(\xi)\, dS' \, (\Phi_k, \phi_q)_{L^2(D)}. \end{aligned}$$

We define the matrix

$$U = \left(\int_D \phi_d \, \Phi_k \, dx \right)_{d(j') \leqq d < d(j'+1), k(j) \leqq k < k(j+1)}$$

and then from (5.15), we get

$$S(m)\boldsymbol{A}(j)U = U \begin{pmatrix} \boldsymbol{\alpha}'(k(j)) & & & \\ & \boldsymbol{\alpha}'(k(j)+1) & & \\ & & \ddots & \\ & & & \boldsymbol{\alpha}'(k(j+1)-1) \end{pmatrix}.$$

This implies the conclusion $\boldsymbol{\alpha}(k) = \boldsymbol{\alpha}'(k)$ for $k(j) \leqq k < k(j+1)$ (See the definition of $\boldsymbol{\alpha}(k)$ in §2). Since the sequence $\{\zeta_p\}_{p=1}^{\infty}$ (which approaches zero) is arbitrary and $\boldsymbol{\alpha}(k)$ depends only on D, Q and not depend on such choice of sequence (see the definition of $\boldsymbol{\alpha}(k)$ in §2), we get

$$\lim_{\zeta\to 0} (\mu_k(\zeta) - \mu_k)/\zeta^m = \boldsymbol{\alpha}(k) \quad (\text{for} \quad k(j) \leqq k < k(j+1)).$$

It agrees to the assertion of Theorem 2.5. □

§6. The Case II : $\mu_k \in E_{II}$ (Proof of Theorems 2.10, 2.11)

In this section we prove Theorem 2.10 and Theorem 2.11 which are concerned with the eigenvalues $\mu_k(\zeta)$ for $k \in \mathbb{N}_{II}$. Assume $k(j) \in \mathbb{N}_{II}$ and then there exists a unique natural number j' such that $\mu_{k(j)} = \lambda_{r(j')}$. Then we have $\widehat{k}(j) = \widehat{r}(j')$ and $\mu_k = \lambda_r$ if $k(j) \leqq k < k(j+1)$, $r(j') \leqq r < r(j'+1)$.

Definition. We put

$$\widehat{\Phi}_{k,\zeta}(x) = S(m)^{1/2}\zeta^{m/2}\Phi_{k,\zeta}(x) \quad (x \in \Omega(\zeta)).$$

$\widehat{\Phi}_{k,\zeta}$ is also a $k-$th eigenfunction of (1.1). From Remark 2.3 and Proposition 3.2, we have

$$0 < \liminf_{\zeta\to 0}\left(\sup_{x\in\Omega(\zeta)} |\widehat{\Phi}_{k,\zeta}(x)|\right) \leqq \limsup_{\zeta\to 0}\left(\sup_{x\in\Omega(\zeta)} |\widehat{\Phi}_{k,\zeta}(x)|\right) < \infty. \tag{6.1}$$

Later we use these estimates for $\widehat{\Phi}_{k,\zeta}$ $(k \in \mathbb{N}_{II})$.

Now we use Proposition 2.1 and Proposition 2.2. Take any positive sequence $\{\zeta_p\}_{p=1}^\infty$ with $\lim_{p\to\infty}\zeta_p = 0$. Then, there exist a subsequence $\{\sigma_p\}_{p=1}^\infty$ and $\{\Phi_k\}_{k=1}^\infty \subset C^2(\overline{D})$, $\{\widehat{\Phi}_k\}_{k=1}^\infty \subset C^2(\overline{Q})$ satisfying (2.6), (2.7), (2.8), (2.9). Note that $\Phi_k \equiv 0$ in D for $k \in \mathbb{N}_{II}$ and that $\{\widehat{\Phi}_k \mid k \in \mathbb{N}_{II}\}$ is orthonormal in $L^2(Q)$.

$$\begin{aligned} &\lim_{p\to\infty}\sup_{x=(x',x'')\in Q(\sigma_p)} |\widehat{\Phi}_{k,\sigma_p}(x) - \widehat{\Phi}_k(x')| = 0, \\ &\widehat{\Phi}_k(x') = \sum_{r(j')\leqq r<r(j'+1)} (\widehat{\Phi}_k, \psi_r)_{L^2(Q)}\psi_r(x') \end{aligned} \tag{6.2}$$

for $k(j) \leqq k < k(j+1)$.

We deal with the two cases $m = 1$ and $m \geqq 2$, separately, because there are technical differences. In the proofs below, the parameter t is taken so that $0 < t \leqq \zeta_0$ and fixed. $\widehat{\Phi}_k$ is a function defined in Q. We need to regard it as a function on $\Omega(\zeta)$ which is equal to 0 in D and to $\widehat{\Phi}_k(x')$ for $x = (x', x'') \in Q(\zeta)$. We denote it by same notation $\widehat{\Phi}_k$. Under this notation $\Delta\widehat{\Phi}_k + \mu_k\widehat{\Phi}_k = 0$ in $Q(\zeta)$ and $\widehat{\Phi}_k \equiv 0$ in D.

6.1. Proof of Theorem 2.10 ($m = 1$)

In this case $m = 1$, $\widehat{\Phi}_{k,\zeta}(x) = (2\zeta)^{1/2}\Phi_{k,\zeta}(x)$. By putting $\Psi = \widetilde{\psi}_{r,\zeta}$ in (2.2) after multiplying by $(2\zeta)^{1/2}$, we have

$$\begin{aligned}(6.3)\qquad &\int_{Q(\zeta)\setminus\Sigma^-(\zeta,h\zeta)} \nabla\widehat{\Phi}_{k,\zeta}\nabla\widetilde{\psi}_{r,\zeta}\,dx + \int_{\Sigma^-(\zeta,h\zeta)\cup\Sigma^*(\zeta,t)} \nabla\widehat{\Phi}_{k,\zeta}\nabla\widetilde{\psi}_{r,\zeta}\,dx \\ &\quad - \int_{\Omega(\zeta)} \mu_k(\zeta)\widehat{\Phi}_{k,\zeta}\widetilde{\psi}_{r,\zeta}\,dx = 0 \quad (\text{for} \quad r(j') \leqq r < r(j'+1)).\end{aligned}$$

We note $\widetilde{\psi}_{r,\zeta} \equiv 0$ in $D \setminus \Sigma^*(\zeta,t)$. Recall $h = h(\zeta) = (\log\zeta)^2$. By partial integration we have

$$\begin{aligned}&\int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\frac{\partial}{\partial\nu_2}\,\widetilde{\psi}_{r,\zeta}\,dS - \int_{Q(\zeta)\setminus\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\Delta\widetilde{\psi}_{r,\zeta}\,dx \\ &+ \int_{\partial(\Sigma^-(\zeta,h\zeta)\cup\Sigma^*(\zeta,t))} \widehat{\Phi}_{k,\zeta}\,\frac{\partial\widetilde{\psi}_{r,\zeta}}{\partial\nu_1}\,dS \\ &- \int_{\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\Delta\widetilde{\psi}_{r,\zeta}\,dx - \int_{\Sigma^*(\zeta,t)} \widehat{\Phi}_{k,\zeta}\,\Delta\widetilde{\psi}_{r,\zeta}\,dx \\ &- \int_{Q(\zeta)\cup\Sigma^*(\zeta,t)} \mu_k(\zeta)\,\widehat{\Phi}_{k,\zeta}\,\widetilde{\psi}_{r,\zeta}\,dx = 0.\end{aligned}$$

We used that $\nabla\widetilde{\psi}_{r,\zeta}$ is continuos across $\partial D \cap \partial Q(\zeta)$ and $v_{r,\zeta}$ is harmonic in $Q(\zeta) \setminus \Sigma^-(\zeta,h\zeta)$. From $\partial\widetilde{\psi}_r/\partial\nu_2 + \partial\widetilde{\psi}_r/\partial\nu_1 = 0$ ($x \in \Gamma^-(\zeta,h\zeta)$) and $\widetilde{\psi}_{r,\zeta} = \widetilde{\psi}_r + \zeta v_{r,\zeta}$, we get

$$\begin{aligned}&\int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\frac{\partial}{\partial\nu_2}(\zeta v_{r,\zeta})\,dS - \int_{Q(\zeta)\setminus\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\Delta\widetilde{\psi}_r\,dx \\ &+ \int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\frac{\partial}{\partial\nu_1}(\zeta v_{r,\zeta})\,dS \\ &+ \int_{\Gamma^*(\zeta,t)} \widehat{\Phi}_{k,\zeta}\,\frac{\partial}{\partial\nu_1}(\zeta v_{r,\zeta})\,dS \\ &+ \int_{\partial(\Sigma^-(\zeta,h\zeta)\cup\Sigma^*(\zeta,t))\setminus(\Gamma^-(\zeta,h\zeta)\cup\Gamma^*(\zeta,t))} \widehat{\Phi}_{k,\zeta}\,\frac{\partial}{\partial\nu_1}\widetilde{\psi}_{r,\zeta}\,dS \\ &- \int_{\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\Delta(\widetilde{\psi}_r + \zeta v_{r,\zeta})\,dx - \int_{\Sigma^*(\zeta,t)} \widehat{\Phi}_{k,\zeta}\,\Delta(\widetilde{\psi}_r + \zeta v_{r,\zeta})\,dx \\ &- \int_{Q(\zeta)\cup\Sigma^*(\zeta,t)} \mu_k(\zeta)\,\widehat{\Phi}_{k,\zeta}(\widetilde{\psi}_r + \zeta v_{r,\zeta})\,dx = 0.\end{aligned}$$

Using $\Delta\widetilde{\psi}_r = -\mu_k \widetilde{\psi}_r$ in $Q(\zeta)$, we have

$$
\begin{aligned}
(6.4) \qquad & (\mu_k(\zeta) - \mu_k) \int_{Q(\zeta)} \widehat{\Phi}_{k,\zeta}\, \widetilde{\psi}_r \, dx \\
&= \zeta \int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta} \frac{\partial v_{r,\zeta}}{\partial \nu_2}\, dS + \zeta \int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta} \frac{\partial v_{r,\zeta}}{\partial \nu_1} dS \\
&\quad + \zeta \int_{\Gamma^*(\zeta,t)} \widehat{\Phi}_{k,\zeta} \frac{\partial v_{r,\zeta}}{\partial \nu_1} dS \\
&\quad + \int_{\partial(\Sigma^-(\zeta,h\zeta)\cup\Sigma^*(\zeta,t))\setminus(\Gamma^-(\zeta,h\zeta)\cup\Gamma^*(\zeta,t))} \widehat{\Phi}_{k,\zeta} \frac{\partial \widetilde{\psi}_{r,\zeta}}{\partial \nu_1}\, dS \\
&\quad - \zeta \int_{\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta} (\Delta v_{r,\zeta} + \mu_k(\zeta) v_{r,\zeta})\, dx \\
&\quad - \zeta \int_{\Sigma^*(\zeta,t)} \widehat{\Phi}_{k,\zeta} \,(\Delta v_{r,\zeta} + \mu_k(\zeta)\, v_{r,\zeta})\, dx \\
&\quad - \mu_k(\zeta)\, \zeta \int_{Q(\zeta)\setminus\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\, v_{r,\zeta}\, dx.
\end{aligned}
$$

To deal with the terms of the right hand side of (6.4), we prepare the following estimates.

LEMMA 6.1. *There exists a constant $c_{10} > 0$ such that*

$$
(6.5) \quad \begin{cases} \left| \displaystyle\int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta} \frac{\partial v_{r,\zeta}}{\partial \nu_2}\, dS \right| \leqq c_{10}\, \zeta^2 (1 + \log(t/\zeta))(\log(1/\zeta))^2, \\ \left| \displaystyle\int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta} \frac{\partial v_{r,\zeta}}{\partial \nu_1}\, dS \right| \leqq c_{10}\, \zeta^2 (\log\zeta)^2, \end{cases}
$$

$$
(6.6) \quad \left| \int_{\Gamma^*(\zeta,t)} \widehat{\Phi}_{k,\zeta} \frac{\partial v_{r,\zeta}}{\partial \nu_1}\, dS \right| \leqq c_{10}\, \zeta,
$$

$$
(6.7) \quad \left| \int_{\partial(\Sigma^-(\zeta,h\zeta)\cup\Sigma^*(\zeta,t))\setminus(\Gamma^-(\zeta,h\zeta)\cup\Gamma^*(\zeta,t))} \widehat{\Phi}_{k,\zeta} \frac{\partial \widetilde{\psi}_{r,\zeta}}{\partial \nu_1}\, dS \right| \leqq c_{10}\, \zeta^3 \log(1/\zeta),
$$

$$
\begin{aligned}
(6.8) \quad & \left| \int_{\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta} \,(\Delta v_{r,\zeta} + \mu_k(\zeta)\, v_{r,\zeta})\, dx \right| \\
& \leqq c_{10}\, \zeta^2 (\log\zeta)^2 (1 + \zeta(\log\zeta)^2 \log(t/\zeta)),
\end{aligned}
$$

(6.9) $$\left| \int_{\Sigma^*(\zeta,t)} \widehat{\Phi}_{k,\zeta} \left(\Delta v_{r,\zeta} + \mu_k(\zeta)\, v_{r,\zeta} \right) dx \right| \leqq c_{10} \left(t + \zeta \log(t/\zeta) \right) \zeta$$

for $0 < \zeta \leqq \zeta_0$, $0 < 2\zeta \leqq t \leqq t_1$.

Proof of Lemma 6.1. (6.5) follows from Lem.4.3-(i)-(4.27) and Cor.4.8-(4.70). (6.6) follows from Lem.4.3-(i)-(4.28) and Cor.4.7-(4.62). (6.7) follows from Lem.4.3-(i)-(4.25) and Cor.4.6-(4.62). From Cor.4.8-(4.70) and Lem.4.3-(i)-(4.29), we have

$$\begin{aligned}
&\int_{\Sigma^-(\zeta,h\zeta)} |\widehat{\Phi}_{k,\zeta}(x)|\, |\Delta v_{r,\zeta}(x)|\, dx \\
&\quad \leqq \int_{\Sigma^-(\zeta,h\zeta)} (c_9\, h(\zeta)\zeta)\, c_2 \left(\log \frac{t}{\zeta} + \frac{e^{\delta s/\zeta}}{\zeta} \right) dx \\
&\quad \leqq \int_{\partial Q} \int_{\widetilde{\Sigma}^-(\zeta,h\zeta)} (c_9 c_2\, h\zeta) \left(\log \frac{t}{\zeta} + \frac{e^{\delta s/\zeta}}{\zeta} \right) \rho_1(\xi,s)\, ds\, d\eta\, dS' \\
&\quad \leqq c(h^2\zeta^3 \log(t/\zeta) + h\zeta^2).
\end{aligned}$$

Similarly, from (4.26) and Cor.4.8-(4.70), we have

$$\int_{\Sigma^-(\zeta,h\zeta)} |\widehat{\Phi}_{k,\zeta}(x)|\, |v_{r,\zeta}(x)|\, dx \leqq c' h^2 \zeta^3 (1 + \log(t/\zeta)).$$

Here c, c' are positive constants which are independent of the parameters. These estimates imply (6.8). From Lem.4.3-(i)-(4.30), Cor.4.7, we have

$$\begin{aligned}
&\int_{\Sigma^*(\zeta,t)} |\widehat{\Phi}_{k,\zeta}(x)|\, |\Delta v_{r,\zeta}(x)|\, dx \\
&= \int_{\Sigma^*(\zeta,t)\setminus\Sigma^+(2\zeta)} |\widehat{\Phi}_{k,\zeta}|\, |\Delta v_{r,\zeta}|\, dx + \int_{\Sigma^+(2\zeta)} |\widehat{\Phi}_{k,\zeta}|\, |\Delta v_{r,\zeta}|\, dx \\
&\leqq \int_{\Sigma^*(\zeta,t)\setminus\Sigma^+(2\zeta)} c_2\, c_8\, \zeta \left(\log \frac{2t_1}{(s^2+|\eta|^2)^{1/2}} \right) \\
&\quad \times \left(\frac{1}{(s^2+|\eta|^2)^{1/2}} + \log \frac{2t}{(s^2+|\eta|^2)^{1/2}} \right) dx \\
&\quad + \int_{\Sigma^+(2\zeta)} c_2\, c_8 \left(\zeta \log \frac{2t_1}{\zeta} \right) \left(\frac{1}{\zeta} + \log \frac{t}{\zeta} \right) dx
\end{aligned}$$

$$\leqq c''(t\zeta + \zeta^2 \log(t/\zeta)),$$

$$\int_{\Sigma^*(\zeta,t)} |\widehat{\Phi}_{k,\zeta}|\,|v_{r,\zeta}|\,dx = \int_{\Sigma^*(\zeta,t)\setminus\Sigma^+(2\zeta)} |\widehat{\Phi}_{k,\zeta}|\,|v_{r,\zeta}|\,dx + \int_{\Sigma^+(2\zeta)} |\widehat{\Phi}_{k,\zeta}|\,|v_{r,\zeta}|\,dx$$
$$\leqq \int_{\Sigma^*(\zeta,t)\setminus\Sigma^+(2\zeta)} c_2\,c_8\,\zeta \left(\log \frac{2t_1}{(s^2+|\eta|^2)^{1/2}}\right)\left(1+\log\frac{t}{(s^2+|\eta|^2)^{1/2}}\right) dx$$
$$+ \int_{\Sigma^+(2\zeta)} \zeta\,c_2\,c_8\,(\log(1/\zeta))^2 dx$$
$$\leqq c'''(t\zeta + \zeta^2 \log(t/\zeta)).$$

Here c'', c''' are positive constants which are independent of the parameters. These estimates imply (6.9). □

Applying Lemma 6.1 to (6.4), we get

$$(\mu_k(\zeta) - \mu_k) \int_{Q(\zeta)} \widehat{\Phi}_{k,\zeta} \widetilde{\psi}_r\,dx = -\mu_k(\zeta)\,\zeta \int_{Q(\zeta)\setminus\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta} v_{r,\zeta} dx + o(\zeta^2 \log(1/\zeta)). \tag{6.10}$$

Put $\zeta = \sigma_p$ ($p \geqq 1$) and $h = h(\sigma_p)$, we evaluate the both sides of the above expression (6.10).

Lemma 6.2. *We have*

$$\int_{Q(\sigma_p)} \widehat{\Phi}_{k,\sigma_p}\,\widetilde{\psi}_r\,dx = 2\sigma_p \int_Q \widehat{\Phi}_k\,\psi_r\,dx' + o(\sigma_p), \tag{6.11}$$

$$\int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h(\sigma_p)\sigma_p)} \widehat{\Phi}_{k,\sigma_p}\,v_{r,\sigma_p}\,dx = \frac{4\sigma_p}{\pi\mu_k}\log\frac{1}{\sigma_p}\int_{\partial Q} \frac{\partial\widehat{\Phi}_k}{\partial \boldsymbol{n}}\frac{\partial\psi_r}{\partial\boldsymbol{n}}\,dS' + o(\sigma_p \log\frac{1}{\sigma_p}), \tag{6.12}$$

for $r(j') \leqq r < r(j'+1)$, $k(j) \leqq k < k(j+1)$.

Proof of Lemma 6.2. We first consider (6.11). From (6.2) we have

$$\int_{Q(\sigma_p)} \widehat{\Phi}_{k,\sigma_p}\widetilde{\psi}_r\,dx = \int_{Q(\sigma_p)} \widehat{\Phi}_k\,\widetilde{\psi}_r\,dx + \int_{Q(\sigma_p)} (\widehat{\Phi}_{k,\sigma_p} - \widehat{\Phi}_k)\,\widetilde{\psi}_r\,dx$$

$$= 2\sigma_p \int_Q \widehat{\Phi}_k \, \psi_r \, dx' + O(\sigma_p^2) + o(\sigma_p)$$

which implies (6.11). Next we estimate

$$\begin{aligned} (6.13) \qquad & \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h(\sigma_p)\sigma_p)} \widehat{\Phi}_{k,\sigma_p} \, v_{r,\sigma_p} \, dx \\ & = \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h(\sigma_p)\sigma_p)} (\widehat{\Phi}_{k,\sigma_p} - \widehat{\Phi}_k) \, v_{r,\sigma_p} \, dx \\ & \quad + \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h(\sigma_p)\sigma_p)} \widehat{\Phi}_k \, v_{r,\sigma_p} \, dx. \end{aligned}$$

Denote the two terms of the right hand side by $I_1(\sigma_p)$ and $I_2(\sigma_p)$.

$$\begin{aligned} |I_1(\sigma_p)| & \leqq \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h(\sigma_p)\sigma_p)} |\widehat{\Phi}_{k,\sigma_p}(x) - \widehat{\Phi}_k(x)| \, |v_{r,\sigma_p}(x)| \, dx \\ & \leqq |Q(\sigma_p)| \sup_{x\in Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h(\sigma_p)\sigma_p)} |\widehat{\Phi}_{k,\sigma_p}(x) - \widehat{\Phi}_k(x)| \, |v_{r,\sigma_p}(x)| \end{aligned}$$

Here $|Q(\sigma_p)|$ is the volume of $Q(\sigma_p)$ and so $|Q(\sigma_p)| = O(\sigma_p)$. From (6.2) and

$$\sup_{x\in Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h(\sigma_p)\sigma_p)} |v_{r,\sigma_p}(x)| \leqq c_2 \, (1+\log(t/\sigma_p)) \quad \text{(cf. Lem.4.3-(i)-(4.26))},$$

we have

$$(6.14) \qquad \lim_{p\to\infty} I_1(\sigma_p)/(\sigma_p \log(1/\sigma_p)) = 0.$$

Next we deal with $I_2(\sigma_p)$. Using $\Delta\widehat{\Phi}_k = (-1/\mu_k)\widehat{\Phi}_k$ in $Q(\sigma_p)$ and carrying out partial integration, we have

$$\begin{aligned} I_2(\sigma_p) & = -\frac{1}{\mu_k} \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h\sigma_p)} \Delta\widehat{\Phi}_k \, v_{r,\sigma_p} \, dx \\ & = -\frac{1}{\mu_k} \int_{\Gamma^-(\sigma_p,h\sigma_p)} \left(\frac{\partial\widehat{\Phi}_k}{\partial\nu_2} \, v_{r,\sigma_p} - \widehat{\Phi}_k \, \frac{\partial v_{r,\sigma_p}}{\partial\nu_2} \right) dS \\ & \quad - \frac{1}{\mu_k} \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h(\sigma_p)\sigma_p)} \widehat{\Phi}_k \, \Delta v_{r,\sigma_p} \, dx \end{aligned}$$

$$= -\frac{1}{\mu_k}\int_{\Gamma^-(\sigma_p, h\sigma_p)} \left\{ \frac{\partial \widehat{\Phi}_k}{\partial \nu_2} \frac{(-1)}{\boldsymbol{\kappa}_1} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \left(\widehat{G}(\frac{s}{\sigma_p}, \frac{\eta}{\sigma_p}) + \frac{2\boldsymbol{\kappa}_1}{\pi} \log \frac{t}{\sigma_p} \right) - \widehat{\Phi}_k \frac{\partial v_{r,\sigma_p}}{\partial \nu_2} \right\} dS$$

We used $\Delta v_{r,\zeta} = 0$ in $Q(\zeta) \setminus \Sigma^-(\zeta, h\zeta)$ and the definition of $v_{r,\zeta}$ in the above calculation. We note that $\widehat{\Phi}_k$ is a function in Q (with Dirichlet B.C. on ∂Q) and it can be regarded as a function in $Q(\zeta)$ by $\widehat{\Phi}_k(x', x'') = \widehat{\Phi}_k(x')$. We also denote it by $\widehat{\Phi}_k$.

$$\begin{aligned} (6.15) \qquad I_2(\sigma_p) = {} & \frac{2}{\mu_k \pi} \log(t/\sigma_p) \int_{\Gamma^-(\sigma_p, h(\sigma_p)\sigma_p)} \frac{\partial \widehat{\Phi}_k}{\partial \nu_2} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\, dS \\ & - \frac{1}{\mu_k} \int_{\Gamma^-(\sigma_p, h(\sigma_p)\sigma_p)} \left\{ \frac{\partial \widehat{\Phi}_k}{\partial \nu_2} \frac{(-1)}{\boldsymbol{\kappa}_1} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\, \widehat{G}(-h(\sigma_p), \frac{\eta}{\sigma_p}) - \widehat{\Phi}_k \frac{\partial v_{r,\sigma_p}}{\partial \nu_2} \right\} dS \end{aligned}$$

For $x = (\xi - h(\sigma_p)\sigma_p \boldsymbol{n}(\xi), \eta) \in \Gamma^-(\sigma_p, h(\sigma_p)\sigma_p)$, $\xi \in \partial Q$, we see that $\nu_2(x) = (\boldsymbol{n}(\xi), 0)$ and

$$\begin{aligned} (6.16) \qquad \frac{\partial \widehat{\Phi}_k}{\partial \nu_2}(x) &= \langle \nabla \widehat{\Phi}_k(x), \nu_2 \rangle = \langle (\nabla' \widehat{\Phi}_k)(\xi - h(\sigma_p)\sigma_p \boldsymbol{n}(\xi)), \boldsymbol{n}(\xi) \rangle \\ &= \langle \nabla' \widehat{\Phi}_k(\xi), \boldsymbol{n}(\xi) \rangle \\ &\quad + \langle (\nabla' \widehat{\Phi}_k)(\xi - h(\sigma_p)\sigma_p \boldsymbol{n}(\xi)) - \nabla' \widehat{\Phi}_k(\xi), \boldsymbol{n}(\xi) \rangle, \end{aligned}$$

$$(6.17) \qquad |\widehat{\Phi}_k(x)| = |\widehat{\Phi}_k(\xi - h\sigma_p \boldsymbol{n}(\xi)) - \widehat{\Phi}_k(\xi)| \leqq \sup_{x' \in Q} |\nabla' \widehat{\Phi}_k(x')|\, h(\sigma_p)\sigma_p,$$

$$\begin{aligned} (6.18) \qquad & |\langle (\nabla' \widehat{\Phi}_k)(\xi - h(\sigma_p)\sigma_p \boldsymbol{n}(\xi)) - \nabla' \widehat{\Phi}_k(\xi), \boldsymbol{n}(\xi) \rangle| \\ & \leqq \sup_{x' \in Q} |(\nabla')^2 \widehat{\Phi}_k(x')|\, h(\sigma_p)\, \sigma_p, \end{aligned}$$

$$(6.19) \qquad \sup_{x \in \Gamma^-(\sigma_p, h(\sigma_p)\sigma_p)} \left| \frac{\partial \widehat{\Phi}_k}{\partial \nu_2}(x) \right| \leqq \sup_{x' \in Q} |\nabla' \widehat{\Phi}_k(x')|.$$

With the aid of (6.16) and (6.18), we calculate the first term in the right hand side of $I_2(\sigma_p)$.

$$\int_{\Gamma^-(\sigma_p,h(\sigma_p)\sigma_p)} \frac{\partial\widehat{\Phi}_k}{\partial\nu_2}\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,dS = \int_{\partial Q}\int_{\widetilde{\Gamma}^-(\sigma_p,h(\sigma_p)\sigma_p)} \frac{\partial\widehat{\Phi}_k}{\partial\nu_2}\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,d\widetilde{S}\,dS'$$

$$= \int_{\partial Q}\int_{\widetilde{\Gamma}^-(\sigma_p,h(\sigma_p)\sigma_p)} \langle\nabla'\widehat{\Phi}_k(\xi),\boldsymbol{n}(\xi)\rangle\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,\rho_2(\xi,-h(\sigma_p)\sigma_p)\,d\widetilde{S}dS'$$

$$+\int_{\Gamma^-(\sigma_p,h(\sigma_p)\sigma_p)} \langle(\nabla'\widehat{\Phi}_k)(\xi-h(\sigma_p)\sigma_p\boldsymbol{n}(\xi))-\nabla'\widehat{\Phi}_k(\xi),\boldsymbol{n}(\xi)\rangle\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,dS$$

$$= 2\sigma_p\int_{\partial Q}\langle\nabla'\widehat{\Phi}_k(\xi),\boldsymbol{n}(\xi)\rangle\,\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,dS'$$

$$+\,2\sigma_p\int_{\partial Q}\langle\nabla'\widehat{\Phi}_k(\xi),\boldsymbol{n}(\xi)\rangle\,\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,\left(\rho_2(\xi,-h(\sigma_p)\sigma_p)-1\right)\,dS'$$

$$+\,O(h(\sigma_p)\sigma_p^2)$$

$$= 2\sigma_p\int_{\partial Q}\langle\nabla'\widehat{\Phi}_k(\xi),\boldsymbol{n}(\xi)\rangle\,\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,dS' + O(h(\sigma_p)\sigma_p^2).$$

On the other hand, using Lem.4.3-(i)-(4.27), (6,17) and (6.19), we can prove that the second term of the right hand side of $I_2(\sigma_p)$ in (6.15) is $O(\sigma_p^2)$. Summing up these estimates, we get

$$\text{(6.20)}\quad I_2(\sigma_p) = \frac{4\sigma_p}{\mu_k\pi}\log(t/\sigma_p)\int_{\partial Q}\frac{\partial\widehat{\Phi}_k}{\partial\boldsymbol{n}}\frac{\partial\psi_r}{\partial\boldsymbol{n}}\,dS' + O(\sigma_p^2(\log\sigma_p)^2\log(t/\sigma_p)).$$

Using $I_1(\sigma_p) = o(\sigma_p\log(1/\sigma_p))$ (cf. (6.14)) in (6.13), we conclude (6.12). □

Consequently, we substitute (6.11), (6.12) into (6.10) and we get

$$\text{(6.21)}\quad (\mu_k(\sigma_p)-\mu_k)\int_Q\widehat{\Phi}_k\,\psi_r\,dx' = -\frac{2}{\pi}\left(\int_{\partial Q}\frac{\partial\widehat{\Phi}_k}{\partial\boldsymbol{n}}\frac{\partial\psi_r}{\partial\boldsymbol{n}}\,dS'\right)\sigma_p\log\frac{1}{\sigma_p} + o(\sigma_p\log\frac{1}{\sigma_p})$$

for $k(j) \leqq k < k(j+1)$, $r(j') \leqq r < r(j'+1)$. Here we note that both of $\{\widehat{\Phi}_k\}_{k(j)\leqq k<k(j+1)}$ and $\{\psi_r\}_{r(j')\leqq r<r(j'+1)}$ are orthonormal systems in $L^2(Q)$ and span the same subspace (see (6.2)). So we can define the following square matrix (orthogonal) of size $\widehat{k}(j) = \widehat{r}(j')$

$$U = \left(\int_Q\psi_r\,\widehat{\Phi}_k\,dx'\right)_{r(j')\leqq r<r(j'+1),k(j)\leqq k<k(j+1)}.$$

Using that the matrix U is invertible in (6.21), we see that the limit value

$$\lim_{p\to\infty}(\mu_k(\sigma_p)-\mu_k)/(\sigma_p\log(1/\sigma_p))$$

exists for $k(j)\leqq k<k(j+1)$. We denote this limit value by $\boldsymbol{\beta}'(k)$. Using (6.21), we have

$$\boldsymbol{\beta}'(k)\int_Q \psi_r\,\widehat{\Phi}_k\,dx' = -\frac{2}{\pi}\int_{\partial Q}\frac{\partial\psi_r}{\partial\boldsymbol{n}}\frac{\partial\widehat{\Phi}_k}{\partial\boldsymbol{n}}\,dS'$$

for $k(j)\leqq k<k(j+1)$, $r(j')\leqq r<r(j'+1)$. Using (6.2) we get the following matrix relation,

$$-\frac{2}{\pi}\boldsymbol{B}(j)U = U\begin{pmatrix}\boldsymbol{\beta}'(k(j)) & & & \\ & \boldsymbol{\beta}'(k(j)+1) & & \\ & & \ddots & \\ & & & \boldsymbol{\beta}'(k(j+1)-1)\end{pmatrix}.$$

This matrix equation implies $\boldsymbol{\beta}'(k)=\boldsymbol{\beta}(k)$ for $k(j)\leqq k<k(j+1)$ (See the definition of $\boldsymbol{\beta}(k)$ in § 2). The choice of $\{\zeta_p\}_{p=1}^{\infty}$ was arbitrary and $\boldsymbol{\beta}(k)$ depends only on Q, D. So we conclude

$$\lim_{\zeta\to 0}(\mu_k(\zeta)-\mu_k)/(\zeta\log(1/\zeta))=\boldsymbol{\beta}(k)\quad(\text{for}\quad k(j)\leqq k<k(j+1))$$

which is the assertion of Theorem 2.10.

6.2. Proof of Theorem 2.11 ($m\geqq 2$)

We carry out a similar argument as **6.1**. Let $k(j)\leqq k<k(j+1)$, $r(j')\leqq r<r(j'+1)$ and put $\Psi=\widetilde{\psi}_{r,\zeta}$ in (2.2), we have

$$\int_{\Omega(\zeta)}(\nabla\Phi_{k,\zeta}\nabla\widetilde{\psi}_{r,\zeta}-\mu_k(\zeta)\,\Phi_{k,\zeta}\,\widetilde{\psi}_{r,\zeta})\,dx=0$$

which leads to

$$\int_{Q(\zeta)\setminus\Sigma^-(\zeta,h\zeta)}\nabla\Phi_{k,\zeta}\nabla\widetilde{\psi}_{r,\zeta}\,dx+\int_{\Sigma^-(\zeta,h\zeta)\cup\Sigma^+(t)}\nabla\Phi_{k,\zeta}\nabla\widetilde{\psi}_{r,\zeta}\,dx$$
$$+\int_{D\setminus\Sigma^+(t)}\nabla\Phi_{k,\zeta}\nabla\widetilde{\psi}_{r,\zeta}\,dx-\int_{\Omega(\zeta)}\mu_k(\zeta)\,\Phi_{k,\zeta}\,\widetilde{\psi}_{r,\zeta}\,dx=0.$$

Using $\widehat{\Phi}_{k,\zeta} = S(m)^{1/2}\zeta^{m/2}\Phi_{k,\zeta}$ and carrying out partial integration, we have

$$\begin{aligned}
&\int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\frac{\partial\widetilde{\psi}_{r,\zeta}}{\partial\nu_2}\,dS - \int_{Q(\zeta)\setminus\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\Delta\widetilde{\psi}_{r,\zeta}\,dx \\
&+ \int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\frac{\partial\widetilde{\psi}_{r,\zeta}}{\partial\nu_1}\,dS + \int_{\Gamma^+(t)} \widehat{\Phi}_{k,\zeta}\frac{\partial\widetilde{\psi}_{r,\zeta}}{\partial\nu_1}\,dS \\
&+ \int_{(\Sigma^-(\zeta,h\zeta)\cup\Sigma^+(t))\setminus(\Gamma^-(\zeta,h\zeta)\cup\Gamma^+(t))} \widehat{\Phi}_{k,\zeta}\frac{\partial\widetilde{\psi}_{r,\zeta}}{\partial\nu_1}\,dS - \int_{\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\Delta\widetilde{\psi}_{r,\zeta}\,dx \\
&- \int_{\Sigma^+(t)} \widehat{\Phi}_{k,\zeta}\,\Delta\widetilde{\psi}_{r,\zeta}\,dx + \int_{\Gamma^+(t)} \widehat{\Phi}_{k,\zeta}\frac{\partial\widetilde{\psi}_{r,\zeta}}{\partial\nu_3}\,dS - \int_{\Omega(\zeta)} \mu_k(\zeta)\widehat{\Phi}_{k,\zeta}\,\widetilde{\psi}_{r,\zeta}\,dx = 0.
\end{aligned}$$

We used $\Delta\widetilde{\psi}_{r,\zeta} = 0$ in $D\setminus\Sigma^+(t)$. Using $\widetilde{\psi}_{r,\zeta} = \widetilde{\psi}_r + \zeta v_{r,\zeta}$, $\Delta\widetilde{\psi}_r = -\mu_k\widetilde{\psi}_r$ in $Q(\zeta)$, $\partial\widetilde{\psi}_r/\partial\nu_1 + \partial\widetilde{\psi}_r/\partial\nu_2 = 0$ on $\Gamma^-(\zeta,h\zeta)$ and $\Delta v_{r,\zeta} = 0$ in $Q(\zeta)\setminus\Sigma^-(\zeta,h\zeta)$ and that $\nabla\widetilde{\psi}_{r,\zeta}$ is continuous across $\partial D\cap\partial Q(\zeta)$, we get

$$\begin{aligned}
&\zeta\int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\frac{\partial v_{r,\zeta}}{\partial\nu_2}\,dS - \int_{Q(\zeta)\setminus\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\Delta\widetilde{\psi}_r\,dx \\
&+ \zeta\int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\frac{\partial v_{r,\zeta}}{\partial\nu_1}\,dS \\
&+ \zeta\int_{\Gamma^+(t)} \widehat{\Phi}_{k,\zeta}\,\frac{\partial v_{r,\zeta}}{\partial\nu_1}\,dS + \int_{\partial(\Sigma^-(\zeta,h\zeta)\cup\Sigma^+(t))\setminus(\Gamma^-(\zeta,h\zeta)\cup\Gamma^+(t))} \widehat{\Phi}_{k,\zeta}\,\frac{\partial\widetilde{\psi}_{r,\zeta}}{\partial\nu_1}\,dS \\
&- \int_{\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\Delta(\widetilde{\psi}_r + \zeta v_{r,\zeta})\,dx - \int_{\Sigma^+(t)} \widehat{\Phi}_{k,\zeta}\,\Delta(\widetilde{\psi}_r + \zeta v_{r,\zeta})\,dx \\
&+ \zeta\int_{\Gamma^+(t)} \widehat{\Phi}_{k,\zeta}\,\frac{\partial v_{r,\zeta}}{\partial\nu_3}\,dS - \int_{\Omega(\zeta)} \mu_k(\zeta)\,\widehat{\Phi}_{k,\zeta}\,\widetilde{\psi}_{r,\zeta}\,dx = 0
\end{aligned}$$

and then we have

$$\begin{aligned}
&\zeta\int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\frac{\partial v_{r,\zeta}}{\partial\nu_2}\,dS + \zeta\int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\frac{\partial v_{r,\zeta}}{\partial\nu_1}\,dS + \zeta\int_{\Gamma^+(t)} \widehat{\Phi}_{k,\zeta}\,\frac{\partial v_{r,\zeta}}{\partial\nu_1}\,dS \\
&+ \int_{\partial(\Sigma^-(\zeta,h\zeta)\cup\Sigma^+(t))\setminus(\Gamma^-(\zeta,h\zeta)\cup\Gamma^+(t))} \widehat{\Phi}_{k,\zeta}\,\frac{\partial\widetilde{\psi}_{r,\zeta}}{\partial\nu_1}\,dS - \zeta\int_{\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\Delta v_{r,\zeta}\,dx \\
&- \zeta\int_{\Sigma^+(t)} \widehat{\Phi}_{k,\zeta}\,\Delta v_{r,\zeta}\,dx + \zeta\int_{\Gamma^+(t)} \widehat{\Phi}_{k,\zeta}\,\frac{\partial v_{r,\zeta}}{\partial\nu_3}\,dS
\end{aligned}$$

$$- (\mu_k(\zeta) - \mu_k) \int_{Q(\zeta)} \widehat{\Phi}_{k,\zeta}\, \widetilde{\psi}_r\, dx - \zeta\mu_k(\zeta) \int_{\Omega(\zeta)} \widehat{\Phi}_{k,\zeta}\, v_{r,\zeta}\, dx = 0.$$

Consequently, we get

$$\begin{aligned}
(6.22) \qquad & (\mu_k(\zeta) - \mu_k) \int_{Q(\zeta)} \widehat{\Phi}_{k,\zeta}\, \widetilde{\psi}_r\, dx \\
&= \zeta \int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\, \frac{\partial v_{r,\zeta}}{\partial \nu_2}\, dS + \zeta \int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\, \frac{\partial v_{r,\zeta}}{\partial \nu_1}\, dS \\
&\quad + \zeta \int_{\Gamma^+(t)} \widehat{\Phi}_{k,\zeta}\, \frac{\partial v_{r,\zeta}}{\partial \nu_1}\, dS \\
&\quad + \int_{\partial(\Sigma^-(\zeta,h\zeta)\cup\Sigma^+(t))\setminus(\Gamma^-(\zeta,h\zeta)\cup\Gamma^+(t))} \widehat{\Phi}_{k,\zeta}\, \frac{\partial \widetilde{\psi}_{r,\zeta}}{\partial \nu_1}\, dS \\
&\quad - \zeta \int_{\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\, \Delta v_{r,\zeta}\, dx - \zeta \int_{\Sigma^+(t)} \widehat{\Phi}_{k,\zeta}\, \Delta v_{r,\zeta}\, dx \\
&\quad + \zeta \int_{\Gamma^+(t)} \widehat{\Phi}_{k,\zeta}\, \frac{\partial v_{r,\zeta}}{\partial \nu_3}\, dS - \zeta\, \mu_k(\zeta) \int_{D\cup\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\, v_{r,\zeta}\, dx \\
&\quad - \mu_k(\zeta)\, \zeta \int_{Q(\zeta)\setminus\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\, v_{r,\zeta}\, dx.
\end{aligned}$$

We prepare some estimates for terms which appear in (6.22).

LEMMA 6.3. *There exists $c_{11} > 0$ and $c_{12}(t) > 0$ such that*

$$(6.23) \qquad \begin{cases} |\int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\, \frac{\partial v_{r,\zeta}}{\partial \nu_2}\, dS| \leqq c_{11} (\log\zeta)^2 \zeta^{m+1}, \\ |\int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\, \frac{\partial v_{r,\zeta}}{\partial \nu_1}\, dS| \leqq c_{11} (\log\zeta)^2 \zeta^{m+1}, \end{cases}$$

$$(6.24) \qquad \left| \int_{\Gamma^+(t)} \widehat{\Phi}_{k,\zeta}\, \frac{\partial v_{r,\zeta}}{\partial \nu_1}\, dS \right| \leqq c_{11} \zeta^{2m-1}/t^{m-1},$$

$$(6.25) \qquad \left| \int_{\partial(\Sigma^-(\zeta,h\zeta)\cup\Sigma^+(t))\setminus(\Gamma^-(\zeta,h\zeta)\cup\Gamma^+(t))} \widehat{\Phi}_{k,\zeta}\, \frac{\partial \widetilde{\psi}_{r,\zeta}}{\partial \nu_1}\, dS \right| \leqq c_{11}\, \zeta^{m+2},$$

$$(6.26) \qquad \begin{cases} \left| \int_{\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta} \Delta v_{r,\zeta}\, dx \right| \leqq c_{11}\, (\log\zeta)^2\, \zeta^{m+1}, \\ \left| \int_{\Sigma^+(t)} \widehat{\Phi}_{k,\zeta}\, \Delta v_{r,\zeta}\, dx \right| \leqq c_{11} \begin{cases} \zeta^3 \log(t/\zeta) & (m = 2), \\ \zeta^{m+1} & (m \geqq 3), \end{cases} \end{cases}$$

(6.27) $$\left|\int_{D\setminus\Sigma^+(t)} \widehat{\Phi}_{k,\zeta}\, v_{r,\zeta}\, dx\right| \leqq c_{12}(t)\, \zeta^{2m-1},$$

(6.28) $$\left|\int_{\Gamma^+(t)} \widehat{\Phi}_{k,\zeta}\, \frac{\partial v_{r,\zeta}}{\partial \nu_3}\, dS\right| \leqq c_{12}(t)\, \zeta^{2m-1},$$

for $0 < \zeta \leqq \zeta_0$, $0 < 2\zeta < t$.

Proof of Lemma 6.3. From Lem.4.3-(ii)-(4.33), Cor.4.8-(4.70), we have

$$\left|\widehat{\Phi}_{k,\zeta}\, \frac{\partial v_{r,\zeta}}{\partial \nu_2}\right| \leqq c_9\, h(\zeta)\, \zeta \times c_2 \quad \text{on} \quad \Gamma^-(\zeta, h\zeta)$$
$$\left|\widehat{\Phi}_{k,\zeta}\, \frac{\partial v_{r,\zeta}}{\partial \nu_1}\right| \leqq c_9\, h(\zeta)\, \zeta \times c_2 \quad \text{on} \quad \Gamma^-(\zeta, h\zeta)$$

and $|\Gamma^-(\zeta, h\zeta)| = O(\zeta^m)$ and we get (6.23). From Lem.4.3-(ii)-(4.34), Cor.4.7-(4.63), we have

$$\left|\widehat{\Phi}_{k,\zeta}\, \frac{\partial v_{r,\zeta}}{\partial \nu_1}\right| \leqq c_8\, \frac{\zeta^m}{t^{m-1}} \times \frac{c_2 \zeta^{m-1}}{t^m} \quad \text{on} \quad \Gamma^+(t)$$

and $|\Gamma^+(t)| = O(t^m)$ and so we get (6.24). (6.25) follows from Lem.4.3-(ii)-(4.31), Cor.4.7-(4.63). We deal with (6.26). From Cor.4.8-(4.71), Lem.4.3-(ii)-(4.35), we have

$$\begin{aligned}\int_{\Sigma^-(\zeta,h\zeta)} |\widehat{\Phi}_{k,\zeta}|\, |\Delta v_{r,\zeta}|\, dx &\leqq \int_{\Sigma^-(\zeta,h\zeta)} c_8\, (\zeta + h(\zeta)\, \zeta)\, c_2\, (1 + \frac{1}{\zeta} e^{\delta s/\zeta})\, dx \\ &\leqq c\, h(\zeta)\, \zeta^{m+1}.\end{aligned}$$

c is a constant which is independent of the parameters. This implies the first estimate of (6.26). From Cor.4.7-(4.63), Lem.4.3-(ii)-(4.36), we have

$$\begin{aligned}&\int_{\Sigma^+(t)} |\widehat{\Phi}_{k,\zeta}|\, |\Delta v_{r,\zeta}|\, dx = \int_{\Sigma^+(t)\setminus\Sigma^+(2\zeta)} |\widehat{\Phi}_{k,\zeta}|\, |\Delta v_{r,\zeta}|\, dx \\ &\quad + \int_{\Sigma^+(2\zeta)} |\widehat{\Phi}_{k,\zeta}|\, |\Delta v_{r,\zeta}|\, dx \\ &\leqq \int_{\Sigma^+(t)\setminus\Sigma^+(2\zeta)} \frac{c_8 \zeta^m}{(s^2 + |\eta|^2)^{(m-1)/2}}\, \frac{c_2 \zeta^{m-1}}{(s^2 + |\eta|^2)^{m/2}}\, dx + \int_{\Sigma^+(2\zeta)} c_8\, \zeta\, (c_2/\zeta)\, dx\end{aligned}$$

$$\leqq c' \begin{cases} \zeta^{m+1} & (m \geqq 3), \\ \zeta^3 \log(t/\zeta) & (m = 2). \end{cases}$$

The constant $c' > 0$ is independent of the parameters. These inequalities imply the second estimate of (6.26). (6.27) and (6.28) follow from Lem.4.3-(ii)-(4.37) and Cor.4.7-(4.63). □

Using the estimates in Lemma 6.3 and $m \geqq 2$ in (6.22), we have

$$(6.29) \qquad (\mu_k(\zeta) - \mu_k) \int_{Q(\zeta)} \widehat{\Phi}_{k,\zeta} \, \widetilde{\psi}_r \, dx = -\mu_k(\zeta) \, \zeta \int_{Q(\zeta) \setminus \Sigma^-(\zeta, h\zeta)} \widehat{\Phi}_{k,\zeta} \, v_{r,\zeta} \, dx + o(\zeta^{m+1}).$$

We evaluate both sides of the above expression.

Lemma 6.4. *We have*

$$(6.30) \qquad \int_{Q(\sigma_p)} \widehat{\Phi}_{k,\sigma_p} \, \widetilde{\psi}_r \, dx = S(m) \, \sigma_p^m \int_Q \widehat{\Phi}_k \, \psi_r \, dx' + o(\sigma_p^m),$$

$$(6.31) \qquad \int_{Q(\sigma_p) \setminus \Sigma^-(\sigma_p, h(\sigma_p)\sigma_p)} \widehat{\Phi}_{k,\sigma_p} v_{r,\sigma_p} \, dx = \frac{1}{\mu_k} T(\boldsymbol{q}, m) \, S(m) \, \sigma_p^m \int_{\partial Q} \frac{\partial \widehat{\Phi}_k}{\partial \boldsymbol{n}} \, \frac{\partial \psi_r}{\partial \boldsymbol{n}} \, dS' + o(\sigma_p^m),$$

for $k(j) \leqq k < k(j+1)$, $r(j') \leqq r < r(j'+1)$.

Proof of Lemma 6.4. The left hand side of (6.30) is

$$\int_{Q(\sigma_p)} \widehat{\Phi}_{k,\sigma_p} \, \widetilde{\psi}_r \, dx = \int_{Q(\sigma_p)} (\widehat{\Phi}_{k,\sigma_p}(x', x'') - \widehat{\Phi}_k(x')) \, \widetilde{\psi}_r \, dx' dx'' + \int_{Q(\sigma_p)} \widehat{\Phi}_k \, \widetilde{\psi}_r \, dx' dx''.$$

From (6.2), it is easy to see that the first term of the above expression is $o(\sigma_p^m)$ and the second term is

$$S(m) \, \sigma_p^m \left(\int_Q \widehat{\Phi}_k \, \psi_r \, dx' + o(1) \right)$$

and so we conclude (6.30). Next we consider the left hand side of (6.31). Hereafter we denote $h = h(\sigma_p)$ for simplicity.

$$\begin{aligned}\int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h\sigma_p)} \widehat{\Phi}_{k,\sigma_p}\, v_{r,\sigma_p}\, dx &= \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h\sigma_p)} (\widehat{\Phi}_{k,\sigma_p} - \widehat{\Phi}_k)\, v_{r,\sigma_p}\, dx \\ &\quad + \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h\sigma_p)} \widehat{\Phi}_k\, v_{r,\sigma_p}\, dx.\end{aligned} \tag{6.32}$$

Here we note $h = h(\sigma_p)$. Denote the two terms of the right hand side of (6.32) by $I_1(\sigma_p)$ and $I_2(\sigma_p)$.

$$|I_1(\sigma_p)| \leqq \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h\sigma_p)} |\widehat{\Phi}_{k,\sigma_p}(x) - \widehat{\Phi}_k(x)|\, |v_{r,\sigma_p}(x)|\, dx. \tag{6.33}$$

From Lem.4.3-(ii)-(4.32) and (6.2), we have

$$\lim_{p\to\infty} I_1(\sigma_p)/\sigma_p^m = 0. \tag{6.34}$$

Next we evaluate $I_2(\sigma_p)$. Using the equalities $\widehat{\Phi}_k = (-1/\mu_k)\Delta\widehat{\Phi}_k$ and $\Delta v_{r,\sigma_p} = 0$ in $Q(\sigma_p)\setminus\Sigma^-(\sigma_p, h(\sigma_p)\sigma_p)$, we have

$$\begin{aligned} I_2(\sigma_p) &= -\frac{1}{\mu_k}\int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h\sigma_p)} \Delta\widehat{\Phi}_k\, v_{r,\sigma_p}\, dx \\ &= -\frac{1}{\mu_k}\int_{\Gamma^-(\sigma_p,h\sigma_p)} \left(\frac{\partial\widehat{\Phi}_k}{\partial\nu_2} v_{r,\sigma_p} - \widehat{\Phi}_k \frac{\partial v_{r,\sigma_p}}{\partial\nu_2}\right) dS \\ &\quad - \frac{1}{\mu_k}\int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h\sigma_p)} \widehat{\Phi}_k\, \Delta v_{r,\sigma_p}\, dx \\ &= -\frac{1}{\mu_k}\int_{\Gamma^-(\sigma_p,h\sigma_p)} \left\{\frac{\partial\widehat{\Phi}_k}{\partial\nu_2}\, \frac{(-1)}{\boldsymbol{\kappa}_1}\, \frac{\partial\psi_r}{\partial\boldsymbol{n}}\, \widehat{G}(-h(\sigma_p), \frac{\eta}{\sigma_p}) - \widehat{\Phi}_k\, \frac{\partial v_{r,\sigma_p}}{\partial\nu_2}\right\} dS \end{aligned}$$

and so we get

$$I_2(\sigma_p) = \frac{1}{\boldsymbol{\kappa}_1\mu_k}\int_{\Gamma^-(\sigma_p,h\sigma_p)} \frac{\partial\widehat{\Phi}_k}{\partial\nu_2}\, \frac{\partial\psi_r}{\partial\boldsymbol{n}}\, \widehat{G}(-h(\sigma_p), \frac{\eta}{\sigma_p})\, dS \tag{6.35}$$

$$+ \frac{1}{\mu_k} \int_{\Gamma^-(\sigma_p, h\sigma_p)} \widehat{\Phi}_k \frac{\partial v_{r,\sigma_p}}{\partial \nu_2} \, dS.$$

From Lem.4.3-(ii)-(4.33) we have

$$\sup_{x \in \Gamma^-(\sigma_p, h(\sigma_p)\sigma_p)} \left| \frac{\partial v_{r,\sigma_p}}{\partial \nu_2}(x) \right| \leqq c_2.$$

From the Dirichlet B.C. of $\widehat{\Phi}_k$ on ∂Q, we have

$$\sup_{x \in \Gamma^-(\sigma_p, h(\sigma_p)\sigma_p)} |\widehat{\Phi}_k(x)| \leqq \sup_{x' \in Q} |\nabla' \widehat{\Phi}_k(x')| \, h(\sigma_p) \, \sigma_p.$$

Thus we have

$$(6.36) \quad \left| \int_{x \in \Gamma^-(\sigma_p, h(\sigma_p)\sigma_p)} \widehat{\Phi}_k \frac{\partial v_{r,\sigma_p}}{\partial \nu_2} \, dS \right| \leqq \int_{x \in \Gamma^-(\sigma_p, h(\sigma_p)\sigma_p)} |\widehat{\Phi}_k| \left| \frac{\partial v_{r,\sigma_p}}{\partial \nu_2} \right| dS \leqq c \, h(\sigma_p) \, \sigma_p^{m+1}.$$

Here $c > 0$ is a constant which is independent of the parameters. This gives the estimate for the second term of $I_2(\sigma_p)$. We deal with the first term of $I_2(\sigma_p)$. From Prop.2.7-(ii),

$$(6.37) \quad \sup_{|\eta|<1} |\widehat{G}(s, \eta) - \boldsymbol{\kappa}_2| = \sup_{|\eta|<1} |G(s, \eta) - (-\boldsymbol{\kappa}_1 \, s + \boldsymbol{\kappa}_2)| \leqq c_0 \, e^{\delta s} \quad (s \leqq 0).$$

For $x = (\xi - h(\sigma_p)\sigma_p \boldsymbol{n}(\xi), \eta) \in \Gamma^-(\sigma_p, h(\sigma_p)\sigma_p)$, $\xi \in \partial Q$, we have $\nu_2(x) = (\boldsymbol{n}(\xi), 0)$ and

$$\frac{\partial \widehat{\Phi}_k}{\partial \nu_2}(x) = \langle (\nabla' \widehat{\Phi}_k)(\xi - h(\sigma_p)\sigma_p \boldsymbol{n}(\xi)), \boldsymbol{n}(\xi) \rangle.$$

So we have

$$(6.38) \quad \int_{\Gamma^-(\sigma_p, h\sigma_p)} \frac{\partial \widehat{\Phi}_k}{\partial \nu_2} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \, \widehat{G}(\frac{s}{\sigma_p}, \frac{\eta}{\sigma_p}) \, dS$$
$$= \int_{\partial Q} \int_{(s,\eta) \in \widetilde{\Gamma}^-(\sigma_p, h\sigma_p)} \langle \nabla' \widehat{\Phi}_k(\xi), \boldsymbol{n}(\xi) \rangle$$
$$\times \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \, \widehat{G}(-h(\sigma_p), \frac{\eta}{\sigma_p}) \, \rho_1(\xi, s) \, d\widetilde{S} \, dS'$$

$$
\begin{aligned}
&+\int_{\partial Q}\int_{(s,\eta)\in\tilde\Gamma^-(\sigma_p,h\sigma_p)}\langle(\nabla'\widehat\Phi_k)(\xi-h(\sigma_p)\sigma_p\boldsymbol{n}(\xi))-\nabla'\widehat\Phi_k(\xi),\boldsymbol{n}(\xi)\rangle\\
&\qquad\qquad\times\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,\widehat G(-h(\sigma_p),\frac{\eta}{\sigma_p})\,\rho_1(\xi,s)\,d\widetilde S\,dS'\\
&=\boldsymbol{\kappa}_2\int_{\partial Q}\int_{(s,\eta)\in\tilde\Gamma^-(\sigma_p,h\sigma_p)}\langle\nabla'\widehat\Phi_k(\xi),\boldsymbol{n}(\xi)\rangle\,\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,\rho_1(\xi,s)\,d\widetilde S\,dS'\\
&\quad+\int_{\partial Q}\int_{(s,\eta)\in\tilde\Gamma^-(\sigma_p,h\sigma_p)}\langle\nabla'\widehat\Phi_k(x'),\boldsymbol{n}(\xi)\rangle\\
&\qquad\qquad\times\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\left(\widehat G(-h(\sigma_p),\frac{\eta}{\sigma_p})-\boldsymbol{\kappa}_2\right)\rho_1\,d\widetilde S\,dS'\\
&\quad+O(h(\sigma_p)\sigma_p^{m+1})\\
&=\boldsymbol{\kappa}_2\,S(m)\,\sigma_p^m\int_{\partial Q}\langle\nabla'\widehat\Phi_k(\xi),\boldsymbol{n}(\xi)\rangle\,\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,dS'+o(\sigma_p^m)\\
&\quad+\boldsymbol{\kappa}_2\int_{\partial Q}\int_{(s,\eta)\in\tilde\Gamma^-(\sigma_p,h\sigma_p)}\langle\nabla'\widehat\Phi_k(\xi),\boldsymbol{n}(\xi)\rangle\\
&\qquad\qquad\times\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,(\rho_1(\xi,s)-1)\,d\widetilde S\,dS'\\
&\quad+O(e^{-\delta(\log\sigma_p)^2})+O(h(\sigma_p)\sigma_p^{m+1})\\
&=\boldsymbol{\kappa}_2 S(m)\int_{\partial Q}\frac{\partial\widehat\Phi_k}{\partial\boldsymbol{n}}\frac{\partial\psi_r}{\partial\boldsymbol{n}}\,dS'\sigma_p^m+o(\sigma_p^m).
\end{aligned}
$$

Here $h=h(\sigma_p)$. Therefore, using $T(\boldsymbol{q},m)=\boldsymbol{\kappa}_2/\boldsymbol{\kappa}_1$, we have

$$
\begin{aligned}
&\frac{1}{\boldsymbol{\kappa}_1\mu_k}\int_{\Gamma^-(\sigma_p,h(\sigma_p)\sigma_p)}\frac{\partial\widehat\Phi_k}{\partial\nu_2}\,\frac{\partial\psi_r}{\partial\boldsymbol{n}}\,\widehat G(\frac{s}{\sigma_p},\frac{\eta}{\sigma_p})\,dS\\
&\qquad=T(\boldsymbol{q},m)\,S(m)\int_{\partial Q}\frac{\partial\widehat\Phi_k}{\partial\boldsymbol{n}}\,\frac{\partial\psi_r}{\partial\boldsymbol{n}}\,dS'\,\sigma_p^m+o(\sigma_p^m).
\end{aligned}
$$

Substituting these estimates (6.34), (6.35), (6.36), (6.37), (6.38) into (6.32), we have (6.31). □

From (6.29), (6.30) and (6.31) we get

$$
(6.39)\quad(\mu_k(\sigma_p)-\mu_k)\int_Q\widehat\Phi_k\psi_r\,dx'=-T(\boldsymbol{q},m)\int_{\partial Q}\frac{\partial\widehat\Phi_k}{\partial\boldsymbol{n}}\,\frac{\partial\psi_r}{\partial\boldsymbol{n}}\,dS'\,\sigma_p+o(\sigma_p)
$$

for $k(j) \leqq k < k(j+1)$, $r(j') \leqq r < r(j'+1)$. Here we note that both of $\{\widehat{\Phi}_k\}_{k=k(j)}^{k(j+1)-1}$ and $\{\psi_r\}_{r=r(j')}^{r(j'+1)-1}$ are orthonormal systems in $L^2(Q)$ and they span the common subspace. So we can define the following orthogonal matrix of size $\widehat{k}(j) = \widehat{r}(j')$

$$U = \left(\int_Q \psi_r \, \widehat{\Phi}_k \, dx' \right)_{r(j') \leqq r < r(j'+1), k(j) \leqq k < k(j+1)} .$$

This implies that there exists the limit

$$\lim_{p \to \infty} (\mu_k(\sigma_p) - \mu_k)/\sigma_p$$

for $k(j) \leqq k < k(j+1)$. We denote this value by $\boldsymbol{\beta}'(k)$.

$$\boldsymbol{\beta}'(k) \int_Q \psi_r \, \widehat{\Phi}_k \, dx' = -T(\boldsymbol{q}, m) \int_{\partial Q} \frac{\partial \psi_r}{\partial \boldsymbol{n}} \frac{\partial \widehat{\Phi}_k}{\partial \boldsymbol{n}} dS' \tag{6.40}$$

which leads to

$$-T(\boldsymbol{q}, m)\boldsymbol{B}(j)U = U \begin{pmatrix} \boldsymbol{\beta}'(k(j)) & & & \\ & \boldsymbol{\beta}'(k(j)+1) & & \\ & & \ddots & \\ & & & \boldsymbol{\beta}'(k(j+1)-1) \end{pmatrix}.$$

This implies $\boldsymbol{\beta}'(k) = \boldsymbol{\beta}(k)$ for $k(j) \leqq k < k(j+1)$ (See the definition of $\boldsymbol{\beta}(k)$ in § 2). The choice of $\{\zeta_p\}_{p=1}^{\infty}$ was arbitrary and $\boldsymbol{\beta}(k)$ depends only on Q, D, $\boldsymbol{q}$, m. So we conclude

$$\lim_{\zeta \to 0} (\mu_k(\zeta) - \mu_k)/\zeta = \boldsymbol{\beta}(k) \quad (\text{for} \quad k(j) \leqq k < k(j+1))$$

which is the assertion of Theorem 2.11.

§7. The Case III : $\mu_k \in E_{III}$ (Proof of Theorem 2.12, 2.13, 2.14)

We consider the behavior of the eigenvalue $\mu_k(\zeta)$ which approaches $\mu_k \in E_{III}$. This resonant case (the case III) is more complicated than non-resonant ones (case I, case II), because the corresponding eigenfunction $\Phi_{k,\zeta}$ behaves like a superposition of eigenfunctions of two non-resonant types (I) and (II) (but it is still interesting). Let $k(j) \in \mathbb{N}_{III}$, there exists a unique pair $(j', j'') \in \mathbb{N} \times \mathbb{N}$ such that $\mu_{k(j)} = \omega_{d(j')} = \lambda_{r(j'')}$. Note that $\mu_k = \omega_d = \lambda_r$ for $k(j) \leqq k < k(j+1)$, $d(j') \leqq d < d(j'+1)$, $r(j'') \leqq r < r(j''+1)$ and $\widehat{k}(j) = \widehat{d}(j') + \widehat{r}(j'')$.

We take an arbitrary positive sequence $\{\zeta_p\}_{p=1}^{\infty}$ which approaches 0 for $p \to \infty$. Applying Proposition 2.1 and Proposition 2.2, we have a subsequence $\{\sigma_p\}_{p=1}^{\infty} \subset \{\zeta_p\}_{p=1}^{\infty}$ and $\Phi_k \in C^2(\overline{D})$, $\widehat{\Phi}_k \in C^2(\overline{Q})$ which satisfy (2.6)-(2.9) for $k \geqq 1$. To analyze the eigenvalues $\mu_k(\zeta)$ (for $k(j) \leqq k < k(j+1)$), we need to know more about behaviors of the eigenfunctions $\Phi_{k,\zeta}$. We divide the problem into 3 cases $m = 1, m = 2$ and $m \geqq 3$.

7.1. Proof of Theorem 2.12 ($m = 1$)

First we prepare some estimates for a bound of $\Phi_{k,\zeta}$ around D.

LEMMA 7.1. *For $m = 1$, we have for any $t > 0$ and $k \in \mathbb{N}$,*

$$\limsup_{\zeta \to 0} \left(\sup_{x \in D \cup \Sigma^-(\zeta, t\zeta^{1/2})} |\Phi_{k,\zeta}(x)| \right) < \infty \quad (k \geqq 1). \tag{7.1}$$

PROOF OF LEMMA 7.1. Using $m = 1$ and Rem.2.3 -(iii) and (2.9) in Rem.4.6-(i), we get (7.1). □

We put $\Psi = \widetilde{\phi}_{d,\zeta}$ in (2.2) for $d(j') \leqq d < d(j'+1)$ and carry out integration by parts in each region $Q(\zeta) \setminus \Sigma^-(\zeta, 2\zeta)$, $\Sigma^-(\zeta, 2\zeta) \cup \Sigma^+(2\zeta)$, $D \setminus \Sigma^+(2\zeta)$ and we have,

$$\begin{aligned}
(7.2) \qquad & (\mu_k(\zeta) - \mu_k) \int_{D \setminus \Sigma^+(2\zeta)} \Phi_{k,\zeta}\, \widetilde{\phi}_{d,\zeta}\, dx \\
&= -\mu_k(\zeta) \int_{Q(\zeta) \cup \Sigma^+(2\zeta)} \Phi_{k,\zeta}\, \widetilde{\phi}_{d,\zeta}\, dx + \int_{\Gamma^-(\zeta, 2\zeta)} \Phi_{k,\zeta} \frac{\partial \widetilde{\phi}_{d,\zeta}}{\partial \nu_2}\, dS \\
&\quad + \int_{\partial(\Sigma^-(\zeta,2\zeta) \cup \Sigma^+(2\zeta))} \Phi_{k,\zeta} \frac{\partial \widetilde{\phi}_{d,\zeta}}{\partial \nu_1}\, dS - \int_{\Sigma^-(\zeta,2\zeta)} \Phi_{k,\zeta}\, \Delta \widetilde{\phi}_{d,\zeta}\, dx \\
&\quad - \int_{\Sigma^+(2\zeta)} \Phi_{k,\zeta}\, \Delta \widetilde{\phi}_{d,\zeta}\, dx + \int_{\Gamma^+(2\zeta)} \Phi_{k,\zeta} \frac{\partial \widetilde{\phi}_{d,\zeta}}{\partial \nu_3}\, dS.
\end{aligned}$$

We note that as $\omega_{d(j')} \in E_{III}$, $\widetilde{\phi}_{d,\zeta}$ is harmonic in $Q(\zeta) \setminus \Sigma^-(\zeta, 2\zeta)$ with the Neumann B.C. on $\partial(Q(\zeta) \setminus \Sigma^-(\zeta, 2\zeta)) \setminus \Gamma^-(\zeta, 2\zeta)$ above.

Similarly as in § 6, we put $\Psi = \widetilde{\psi}_{r,\zeta}$ for $r(j'') \leqq r < r(j''+1)$ in (2.2) and carry out integration by parts in each region $Q(\zeta) \setminus \Sigma^-(\zeta, h\zeta)$, $\Sigma^-(\zeta, h\zeta) \cup \Sigma^*(\zeta, t)$, $D \setminus \Sigma^*(\zeta, t)$ and we have

$$
\begin{aligned}
(7.3) \qquad & (\mu_k(\zeta) - \mu_k) \int_{Q(\zeta)} \Phi_{k,\zeta} \widetilde{\psi}_r \, dx \\
&= \zeta \int_{\Gamma^-(\zeta,h\zeta)} \Phi_{k,\zeta} \frac{\partial v_{r,\zeta}}{\partial \nu_2} \, dS + \zeta \int_{\Gamma^-(\zeta,h\zeta)} \Phi_{k,\zeta} \frac{\partial v_{r,\zeta}}{\partial \nu_1} \, dS \\
&\quad + \zeta \int_{\Gamma^*(\zeta,t)} \Phi_{k,\zeta} \frac{\partial v_{r,\zeta}}{\partial \nu_1} dS \\
&\quad + \int_{\partial(\Sigma^-(\zeta,h\zeta) \cup \Sigma^*(\zeta,t)) \setminus (\Gamma^-(\zeta,h\zeta) \cup \Gamma^*(\zeta,t))} \Phi_{k,\zeta} \frac{\partial \widetilde{\psi}_{r,\zeta}}{\partial \nu_1} \, dS \\
&\quad - \zeta \int_{\Sigma^-(\zeta,h\zeta)} \Phi_{k,\zeta} \left(\Delta v_{r,\zeta} + \mu_k(\zeta) \, v_{r,\zeta}\right) dx \\
&\quad - \zeta \int_{\Sigma^*(\zeta,t)} \Phi_{k,\zeta} \left(\Delta v_{r,\zeta} + \mu_k(\zeta) \, v_{r,\zeta}\right) dx \\
&\quad - \zeta \mu_k(\zeta) \int_{Q(\zeta) \setminus \Sigma^-(\zeta,h\zeta)} \Phi_{k,\zeta} \, v_{r,\zeta} \, dx.
\end{aligned}
$$

We used $v_{r,\zeta} \equiv 0$ in $D \setminus \Sigma^*(\zeta, t)$. We prepare several estimates for evaluation of the terms in (7.2).

Lemma 7.2. *There exists $c_{13} > 0$ such that, we have , for $k(j) \leqq k < k(j+1)$, $d(j') \leqq d < d(j'+1)$,*

$$
(7.4) \qquad
\begin{aligned}
&\left| \int_{\Gamma^-(\zeta,2\zeta)} \Phi_{k,\zeta} \frac{\partial \widetilde{\phi}_{d,\zeta}}{\partial \nu_2} \, dS \right| \leqq c_{13} \, \zeta, \\
&\left| \int_{\partial(\Sigma^-(\zeta,2\zeta) \cup \Sigma^+(2\zeta))} \Phi_{k,\zeta} \frac{\partial \widetilde{\phi}_{d,\zeta}}{\partial \nu_1} \, dS \right| \leqq c_{13} \, \zeta,
\end{aligned}
$$

$$
(7.5) \qquad \left| \int_{\Sigma^-(\zeta,2\zeta)} \Phi_{k,\zeta} \, \Delta \widetilde{\phi}_{d,\zeta} dx \right| \leqq c_{13} \, \zeta^2, \qquad \left| \int_{\Sigma^+(2\zeta)} \Phi_{k,\zeta} \Delta \widetilde{\phi}_{d,\zeta} dx \right| \leqq c_{13} \zeta^2,
$$

$$
(7.6)\qquad
\begin{aligned}
&\left| \int_{\Gamma^+(2\zeta)} \Phi_{k,\zeta} \frac{\partial \widetilde{\phi}_{d,\zeta}}{\partial \nu_3} \, dS \right| \leqq c_{13}\, \zeta, \\
&\left| \int_{\Sigma^-(\zeta,2\zeta)\cup\Sigma^+(2\zeta)} \Phi_{k,\zeta} \, \widetilde{\phi}_{d,\zeta} \, dx \right| \leqq c_{13}\, \zeta^2.
\end{aligned}
$$

PROOF OF LEMMA 7.2. The estimates follow from (7.1), $m = 1$ and Lem.4.2-(4.18). □

LEMMA 7.3. *For $k(j) \leqq k < k(j+1)$ and $d(j') \leqq d < d(j'+1)$, it holds that*

$$
(7.7)\qquad \int_{D\setminus\Sigma^+(2\sigma_p)} \Phi_{k,\sigma_p} \, \widetilde{\phi}_{d,\sigma_p} \, dx = \int_D \Phi_k \, \phi_d \, dx + o(1),
$$

$$
(7.8)\qquad
\begin{aligned}
&\int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,2\sigma_p)} \Phi_{k,\sigma_p} \, \widetilde{\phi}_{d,\sigma_p} \, dx \\
&\quad = \frac{-\sqrt{2}}{\mu_k} \int_{\partial Q} \frac{\partial \widehat{\Phi}_k}{\partial \boldsymbol{n}}(\xi) \, \phi_d(\xi, o'') \, dS' \, \sigma_p^{1/2} + o(\sigma_p^{1/2}),
\end{aligned}
$$

for $p \to \infty$.

PROOF OF LEMMA 7.3. At first we have

$$
(7.9)\qquad
\begin{aligned}
\int_{D\setminus\Sigma^+(2\sigma_p)} \Phi_{k,\sigma_p} \, \widetilde{\phi}_{d,\sigma_p} \, dx &= (\Phi_{k,\sigma_p} - \Phi_k, \widetilde{\phi}_{d,\sigma_p})_{L^2(D\setminus\Sigma^+(2\sigma_p))} \\
&\quad + (\Phi_k, \widetilde{\phi}_{d,\sigma_p})_{L^2(D\setminus\Sigma^+(2\sigma_p))}.
\end{aligned}
$$

Using (2.6), Lem.4.2, we have

$$
\lim_{p\to\infty} \|\widetilde{\phi}_{d,\sigma_p} - \phi_d\|_{L^\infty(D)} = 0, \quad \lim_{p\to\infty} \|\Phi_{k,\sigma_p} - \Phi_k\|_{L^2(D)} = 0,
$$

and we see that the right hand side in (7.9) approaches $(\Phi_k, \phi_d)_{L^2(D)}$ for $p \to \infty$ and (7.7) is proved. Next we consider the left hand side of (7.8).

$$
(7.10)\qquad
\begin{aligned}
&\int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,2\sigma_p)} \Phi_{k,\sigma_p} \, \widetilde{\phi}_{d,\sigma_p} \, dx \\
&\quad = (2\sigma_p)^{1/2} \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,2\sigma_p)} (2\sigma_p)^{-1} \, \widehat{\Phi}_k \, \widetilde{\phi}_{d,\sigma_p} \, dx
\end{aligned}
$$

$$
\begin{aligned}
&+ (2\sigma_p)^{1/2} \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,2\sigma_p)} (2\sigma_p)^{-1} (\widehat{\Phi}_{k,\sigma_p} - \widehat{\Phi}_k)\widetilde{\phi}_{d,\sigma_p}\, dx \\
&= (-1/\mu_k)\, (2\sigma_p)^{1/2} \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,2\sigma_p)} (2\sigma_p)^{-1}\, \Delta\widehat{\Phi}_k\, \widetilde{\phi}_{d,\sigma_p}\, dx \\
&+ o(\sigma_p^{1/2})
\end{aligned}
$$

Remark that (2.6) was used above. Carrying out partial integration and using that $\widetilde{\phi}_{d,\sigma_p}$ is harmonic in $Q(\sigma_p) \setminus \Sigma^-(\sigma_p, 2\sigma_p)$ (as we are dealing with the resonant case and $\omega_d \in E_{III}$), we have

$$
\begin{aligned}
(7.11)\quad &\frac{1}{2\sigma_p} \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,2\sigma_p)} \Delta\widehat{\Phi}_k\, \widetilde{\phi}_{d,\sigma_p}\, dx \\
&= \frac{1}{2\sigma_p} \int_{\Gamma^-(\sigma_p,2\sigma_p)} \left(\frac{\partial\widehat{\Phi}_k}{\partial\nu_2} \widetilde{\phi}_{d,\sigma_p} - \widehat{\Phi}_k \frac{\partial\widetilde{\phi}_{d,\sigma_p}}{\partial\nu_2} \right) dS \\
&= \frac{1}{2\sigma_p} \int_{\Gamma^-(\sigma_p,2\sigma_p)} \left(\langle\nabla\widehat{\Phi}_k, \nu_2\rangle\, \widetilde{\phi}_{d,\sigma_p} - \widehat{\Phi}_k \frac{\partial\widetilde{\phi}_{d,\sigma_p}}{\partial\nu_2} \right) dS \\
&= \frac{1}{2\sigma_p} \int_{\partial Q} \int_{|\eta|<\sigma_p} \frac{\partial\widehat{\Phi}_k}{\partial\boldsymbol{n}}(\xi)\, \phi_d(\xi, o'')\, d\eta dS' \\
&\quad + \frac{1}{2\sigma_p} \int_{\partial Q} \int_{|\eta|<\sigma_p} \frac{\partial\widehat{\Phi}_k}{\partial\boldsymbol{n}}(\xi)\, \phi_d(\xi, o'')\, (\rho_2(\xi, -2\sigma_p) - 1)\, d\eta\, dS' \\
&\quad + \frac{1}{2\sigma_p} \int_{\partial Q} \int_{|\eta|<\sigma_p} \langle(\nabla'\widehat{\Phi}_k)(\xi - 2\sigma_p\boldsymbol{n}(\xi)) - (\nabla'\widehat{\Phi}_k)(\xi), \boldsymbol{n}(\xi)\rangle \\
&\qquad\qquad \times \phi_d(\xi, o'')\, \rho_2(\xi, -2\sigma_p)\, d\eta\, dS' \\
&\quad - \frac{1}{2\sigma_p} \int_{\partial Q} \int_{|\eta|<\sigma_p} \left(\widehat{\Phi}_k(\xi - 2\sigma_p\boldsymbol{n}(\xi)) - \widehat{\Phi}_k(\xi) \right) \\
&\qquad\qquad \times \frac{\partial\widetilde{\phi}_{d,\sigma_p}}{\partial\nu_2}\, \rho_2(\xi, -2\sigma_p)\, d\eta\, dS'.
\end{aligned}
$$

Here we used $\nu_2(\xi) = (\boldsymbol{n}(\xi), 0)$ for $x = (\xi - 2\sigma_p\boldsymbol{n}(\xi), \eta) \in \Gamma^-(\sigma_p, 2\sigma_p)$ and the Dirichlet B.C. of $\widehat{\Phi}_k$ on ∂Q. As for the first term it is clear from $m = 1$, that

$$
(7.12)\quad \frac{1}{2\sigma_p} \int_{\partial Q} \int_{|\eta|<\sigma_p} \frac{\partial\widehat{\Phi}_k}{\partial\boldsymbol{n}}(\xi)\, \phi_d(\xi, o'')\, d\eta\, dS' = \int_{\partial Q} \frac{\partial\widehat{\Phi}_k}{\partial\boldsymbol{n}}(\xi)\, \phi_d(\xi, o'')\, dS'.
$$

From Lem.4.2, we see that

$$\sup_{\xi\in\partial Q} |\widehat{\Phi}_k(\xi - 2\sigma_p \boldsymbol{n}(\xi)) - \widehat{\Phi}_k(\xi)| \leqq (2\sigma_p)\|\nabla'\widehat{\Phi}_k\|_{L^\infty(Q)},$$

$$\sup_{\xi\in\partial Q} |(\nabla'\widehat{\Phi}_k)(\xi - 2\sigma_p \boldsymbol{n}(\xi)) - (\nabla'\widehat{\Phi}_k)(\xi)| \leqq (2\sigma_p)\|(\nabla')^2\widehat{\Phi}_k\|_{L^\infty(Q)},$$

$$\sup_{x\in\Gamma^-(\sigma_p,2\sigma_p)} |\frac{\partial\widetilde{\phi}_{d,\sigma_p}}{\partial\nu_2}(x)| \leqq c_1.$$

Using the estimates in (7.11) and (7.12), we substitute them into (7.10) and we have (7.8). □

Applying Lemma 7.2 and Lemma 7.3 to (7.2), we get

$$(7.13)\qquad (\mu_k(\sigma_p) - \mu_k)\int_D \Phi_k\,\phi_d\,dx = (2\sigma_p)^{1/2}\int_{\partial Q}\frac{\partial\widehat{\Phi}_k}{\partial\boldsymbol{n}}(\xi)\,\phi_d(\xi,o'')\,dS' + o(\sigma_p^{1/2})$$

for $p\to\infty$.

Next we prepare several estimates for evaluation of the terms of (7.3).

LEMMA 7.4. *There exists $c_{14}>0$ such that, for $k(j)\leqq k<k(j+1), r(j'')\leqq r<r(j''+1)$, we have*

$$(7.14)\qquad \begin{aligned}\left|\int_{\Gamma^-(\zeta,h\zeta)}\Phi_{k,\zeta}\frac{\partial v_{r,\zeta}}{\partial\nu_2}\,dS\right| &\leqq c_{14}\,\zeta(1+\log(t/\zeta)),\\ \left|\int_{\Gamma^-(\zeta,h\zeta)}\Phi_{k,\zeta}\frac{\partial v_{r,\zeta}}{\partial\nu_1}\,dS\right| &\leqq c_{14}\,\zeta,\end{aligned}$$

$$(7.15)\qquad \left|\int_{\partial(\Sigma^-(\zeta,h\zeta)\cup\Sigma^*(\zeta,t))\setminus(\Gamma^-(\zeta,h\zeta)\cup\Gamma^*(\zeta,t))}\Phi_{k,\zeta}\frac{\partial\widetilde{\psi}_{r,\zeta}}{\partial\nu_1}\,dS\right| \leqq c_{14}\,\zeta^2,$$

$$(7.16)\qquad \left|\int_{\Sigma^-(\zeta,h\zeta)}\Phi_{k,\zeta}\,(\Delta v_{r,\zeta}+\mu_k(\zeta)v_{r,\zeta})\,dx\right| \leqq c_{14}\,\zeta\left(1+\zeta\log(t/\zeta)\,(\log\zeta)^2\right),$$

$$(7.17)\qquad \left|\int_{Q(\zeta)\setminus\Sigma^-(\zeta,h\zeta)}\Phi_{k,\zeta}\,v_{r,\zeta}\,dx\right| \leqq c_{14}\,\zeta^{1/2}\left(1+\log(t/\zeta)\right).$$

Proof of Lemma 7.4. (7.14), (7.15), (7.16) follow from (7.1) and Lem.4.3-(i)-(4.27), (4.25), (4.29), respectively. Remark that $\|\Phi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))} = O(\zeta^{-1/2})$ as $m = 1$ and Rem.2.3-(2.9). Using Lem.4.3-(i)-(4.26), we have (7.17). □

Lemma 7.5. *We have*

$$\int_{Q(\sigma_p)} \Phi_{k,\sigma_p}\, \widetilde{\psi}_r\, dx = (2\sigma_p)^{1/2} \int_Q \widehat{\Phi}_k\, \psi_r\, dx' + o(\sigma_p^{1/2}). \tag{7.18}$$

There exist $c_{15} > 0$ *such that the functions* $\Upsilon_1(k,r,\zeta,t), \Upsilon_2(k,r,\zeta,t)$ *defined through the relations*

$$\int_{\Gamma^*(\zeta,t)} \Phi_{k,\zeta}\, \frac{\partial v_{r,\zeta}}{\partial \nu_1}\, dS = 2\int_{\partial Q} \Phi_k(\xi, o'')\, \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\, dS' + \Upsilon_1(k,r,\zeta,t), \tag{7.19}$$

$$\int_{\Sigma^*(\zeta,t)} \Phi_{k,\zeta}\, (\Delta v_{r,\zeta} + \mu_k(\zeta)\, v_{r,\zeta})\, dx = \Upsilon_2(k,r,\zeta,t), \tag{7.20}$$

for $k(j) \leqq k < k(j+1), r(j'') \leqq r < r(j''+1)$, *satisfy the estimates*

$$\limsup_{p\to\infty} |\Upsilon_1(k,r,\sigma_p,t)| \leqq c_{15}t, \quad \limsup_{\zeta\to 0} |\Upsilon_2(k,r,\zeta,t)| \leqq c_{15}t. \tag{7.21}$$

Proof of Lemma 7.5. From Prop.2.2, (7.18) is clear. We deal with (7.19). We get an expression of $\Upsilon_1(k,r,\sigma_p,t)$ which is comprehensive for estimation. For $x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Sigma^*(\sigma_p, t)$,

$$\begin{aligned} &\int_{\Gamma^*(\sigma_p,t)} \Phi_{k,\sigma_p}(x)\, \frac{\partial v_{r,\sigma_p}}{\partial \nu_1}\, dS \\ &= \int_{\Gamma^*(\sigma_p,t)} \Phi_k(\xi, o'')\, \frac{\partial v_{r,\sigma_p}}{\partial \nu_1}\, dS \\ &\quad + \int_{\Gamma^*(\sigma_p,t)} (\Phi_k(x) - \Phi_k(\xi,o''))\, \frac{\partial v_{r,\sigma_p}}{\partial \nu_1}\, dS \\ &\quad + \int_{\Gamma^*(\sigma_p,t)} (\Phi_{k,\sigma_p}(x) - \Phi_k(x))\, \frac{\partial v_{r,\sigma_p}}{\partial \nu_1}\, dS. \end{aligned} \tag{7.22}$$

We calculate the first term in the right hand side of (7.22) carefully. From the definition of $v_{r,\zeta}$ in $\Sigma^*(\zeta,t)$,

$$v_{r,\zeta}(x) = -\frac{1}{\boldsymbol{\kappa}_1} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \left(\widehat{G}(\frac{s}{\zeta}, \frac{\eta}{\zeta}) + \frac{2\boldsymbol{\kappa}_1}{\pi} \log \frac{t}{\zeta} \right)$$

$$(x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Sigma^*(\zeta, t)).$$

Hereafter we denote $\nabla_z \widehat{G} = (\partial \widehat{G}/\partial s, \nabla_\eta \widehat{G})$. We have

$$\text{(7.23)} \quad \begin{aligned} \frac{\partial v_{r,\zeta}}{\partial \nu_1}(x) &= -\frac{1}{\boldsymbol{\kappa}_1} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \, \langle (\nabla_z \widehat{G})(\frac{s}{\zeta}, \frac{\eta}{\zeta}), \widetilde{\nu}_1 \rangle \, \frac{1}{\zeta} \\ &= \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \, \frac{2}{\pi t} - \frac{1}{\boldsymbol{\kappa}_1} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \left(\langle (\nabla_z \widehat{G})(\frac{s}{\zeta}, \frac{\eta}{\zeta}), \widetilde{\nu}_1 \rangle \, \frac{1}{\zeta} + \frac{2\boldsymbol{\kappa}_1}{\pi t} \right) \end{aligned}$$

for $x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Gamma^*(\zeta, t)$. Here $\widetilde{\nu}_1$ is the outward unit normal vector on $\widetilde{\Gamma}^*(\zeta, t)$. Therefore we have

$$\text{(7.24)} \quad \begin{aligned} &\int_{\Gamma^*(\sigma_p, t)} \Phi_k(\xi, o'') \, \frac{\partial v_{r,\sigma_p}}{\partial \nu_1} \, dS \\ &= \int_{\Gamma^*(\sigma_p, t)} \Phi_k(\xi, o'') \, \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \, \frac{2}{\pi t} \, dS \\ &\quad - \int_{\Gamma^*(\sigma_p, t)} \Phi_k(\xi, o'') \, \frac{1}{\boldsymbol{\kappa}_1} \, \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \\ &\quad \times \left(\langle (\nabla_z \widehat{G})(\frac{s}{\sigma_p}, \frac{\eta}{\sigma_p}), \widetilde{\nu}_1 \rangle \, \frac{1}{\sigma_p} + \frac{2\boldsymbol{\kappa}_1}{\pi t} \right) dS \\ &= \int_{\partial Q} \int_{\widetilde{\Gamma}^*(\sigma_p, t)} \Phi_k(\xi, o'') \, \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \, \frac{2}{\pi t} \, \rho_2 \, d\widetilde{S} \, dS' \\ &\quad - \int_{\Gamma^*(\sigma_p, t)} \Phi_k(\xi, o'') \, \frac{1}{\boldsymbol{\kappa}_1} \, \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \\ &\quad \times \left(\langle (\nabla_z \widehat{G})(\frac{s}{\sigma_p}, \frac{\eta}{\sigma_p}), \widetilde{\nu}_1 \rangle \, \frac{1}{\sigma_p} + \frac{2\boldsymbol{\kappa}_1}{\pi t} \right) dS \\ &= 2 \int_{\partial Q} \Phi_k(\xi, o'') \, \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \, dS' \\ &\quad + \frac{2}{\pi t} \left(|\widetilde{\Gamma}^*(\sigma_p, t)| - |\widetilde{\Gamma}^+(t)| \right) \int_{\partial Q} \Phi_k(\xi, o'') \, \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \, dS' \\ &\quad + \int_{\partial Q} \int_{\widetilde{\Gamma}^*(\sigma_p, t)} \Phi_k(\xi, o'') \, \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \, \frac{2}{\pi t} \, (\rho_2 - 1) \, d\widetilde{S} \, dS' \\ &\quad - \int_{\Gamma^*(\sigma_p, t)} \Phi_k(\xi, o'') \, \frac{1}{\boldsymbol{\kappa}_1} \, \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \\ &\quad \times \left(\langle (\nabla_z \widehat{G})(\frac{s}{\sigma_p}, \frac{\eta}{\sigma_p}), \widetilde{\nu}_1 \rangle \, \frac{1}{\sigma_p} + \frac{2\boldsymbol{\kappa}_1}{\pi t} \right) dS \end{aligned}$$

Here $d\widetilde{S}$ is the measure on $\widetilde{\Gamma}^*(\sigma_p, t)$. Substituting (7.24) into (7.22), we have (7.19) by putting

$$
\begin{aligned}
(7.25)\quad \Upsilon_1(k,r,\zeta,t) &= \frac{2}{\pi t}\left(|\widetilde{\Gamma}^*(\zeta,t)| - |\widetilde{\Gamma}^+(t)|\right)\int_{\partial Q}\Phi_k(\xi,o'')\,\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,dS' \\
&+ \int_{\partial Q}\int_{\widetilde{\Gamma}^*(\zeta,t)}\Phi_k(\xi,o'')\,\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,\frac{2}{\pi t}(\rho_2(\xi,s)-1)\,d\widetilde{S}\,dS' \\
&- \int_{\Gamma^*(\zeta,t)}\Phi_k(\xi,o'')\,\frac{1}{\boldsymbol{\kappa}_1}\,\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\left(\langle(\nabla_z\widehat{G})(\frac{s}{\zeta},\frac{\eta}{\zeta}),\widetilde{\nu}_1\rangle\,\frac{1}{\zeta} + \frac{2\boldsymbol{\kappa}_1}{\pi t}\right)dS \\
&+ \int_{\Gamma^*(\zeta,t)}(\Phi_k(x) - \Phi_k(\xi,o''))\,\frac{\partial v_{r,\zeta}}{\partial\nu_1}\,dS \\
&+ \int_{\Gamma^*(\zeta,t)}(\Phi_{k,\zeta}(x) - \Phi_k(x))\,\frac{\partial v_{r,\zeta}}{\partial\nu_1}\,dS.
\end{aligned}
$$

In view of (2.6) and the inequality

$$
\left|\frac{1}{\zeta}(\nabla_z G)(z/\zeta) + \frac{2\boldsymbol{\kappa}_1 z}{\pi|z|^2}\right| \leqq \frac{c_0\zeta}{|z|^2} \quad (z=(s,\eta)\in H_1) \quad (cf.\ \text{Prop.2.7-(i)})
$$

we have

$$
\begin{aligned}
(7.26)\quad &\limsup_{p\to\infty}|\Upsilon_1(k,r,\sigma_p,t)| \\
&\leqq \limsup_{p\to\infty}\int_{\partial Q}\int_{\widetilde{\Gamma}^*(\sigma_p,t)}|\Phi_k(\xi,o'')|\left|\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\right|\frac{2}{\pi t}\,|\rho_2(\xi,s)-1|\,d\widetilde{S}\,dS' \\
&\quad + \limsup_{p\to\infty}\int_{\Gamma^*(\sigma_p,t)}|\Phi_k(x) - \Phi_k(\xi,o'')|\left|\frac{\partial v_{r,\sigma_p}}{\partial\nu_1}\right|dS \\
&\leqq \limsup_{p\to\infty}\int_{\partial Q}\int_{\widetilde{\Gamma}^*(\sigma_p,t)}\left|\Phi_k(\xi,o'')\,\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\right|\frac{2}{\pi t}\,|\rho_2(\xi,s)-1|\,d\widetilde{S}\,dS' \\
&\quad + c_2\limsup_{p\to\infty}\left(\sup_{x=(\xi+s\boldsymbol{n}(\xi),\eta)\in\Gamma^*(\sigma_p,t)}|\Phi_k(x)-\Phi_k(\xi,o'')|\right)|\Gamma^*(\sigma_p,t)| \\
&= 2\int_{\partial Q}\left|\Phi_k(\xi,o'')\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\right|dS' \sup_{\xi\in\partial Q,(s,\eta)\in\widetilde{\Gamma}^+(t)}|\rho_2(\xi,s)-1| \\
&\quad + c_2\sup_{x=(\xi+s\boldsymbol{n}(\xi),\eta)\in\Gamma^+(t)}|\Phi_k(x)-\Phi_k(\xi,o'')|\,|\Gamma^+(t)| \leqq c\,t.
\end{aligned}
$$

$c > 0$ is a positive constant which is independent of the parameters. This (7.26) implies the first part of (7.21). Next we deal with (7.20).

$$\begin{aligned}(7.27)\quad &|\Upsilon_2(k,r,\zeta,t)| \\ &= \left| \int_{\Sigma^*(\zeta,t)} \Phi_{k,\zeta} \, (\Delta v_{q,\zeta} + \mu_k(\zeta)\, v_{r,\zeta}) \, dx \right| \\ &\leqq \|\Phi_{k,\zeta}\|_{L^\infty(D)} \left(\int_{\Sigma^*(\zeta,t)} |\Delta v_{r,\zeta}| \, dx + \mu_k(\zeta) \int_{\Sigma^*(\zeta,t)} |v_{r,\zeta}| \, dx \right).\end{aligned}$$

We note that $\|\Phi_{k,\zeta}\|_{L^\infty(D)}$ is bounded for small $\zeta > 0$ (cf. (7.1)). We use Lem.4.3-(i) to estimate terms involving $v_{r,\zeta}$.

$$\begin{aligned}(7.28)\quad &\int_{\Sigma^*(\zeta,t)} |\Delta v_{r,\zeta}| \, dx \\ &\leqq \int_{\partial Q} \int_{(s,\eta)\in\widetilde{\Sigma}^+(2\zeta)} (c_2/\zeta)\, \rho_1(\xi,s) \, ds \, d\eta \, dS' \\ &\quad + \int_{\partial Q} \int_{(s,\eta)\in\widetilde{\Sigma}^+(2t)\setminus\widetilde{\Sigma}^+(2\zeta)} c_2 \left(\frac{1}{\sqrt{s^2+|\eta|^2}} + \log \frac{t}{\sqrt{s^2+|\eta|^2}} \right) \\ &\qquad\qquad \times \rho_1(\xi,s) \, ds \, d\eta \, dS' \\ &\leqq c'(\zeta + t^2) \qquad (0 < 2\zeta < t \leqq t_0).\end{aligned}$$

$$\begin{aligned}(7.29)\quad &\int_{\Sigma^*(\zeta,t)} |v_{r,\zeta}| dx \\ &\leqq \int_{\partial Q} \int_{(s,\eta)\in\widetilde{\Sigma}^+(2\zeta)} c_2 \, (1 + \log(t/\zeta))\, \rho_1(\xi,s) \, ds \, d\eta \, dS' \\ &\quad + \int_{\partial Q} \int_{(s,\eta)\in\widetilde{\Sigma}^+(2t)\setminus\widetilde{\Sigma}^+(2\zeta)} c_2 \, (1 + \log(t/\sqrt{s^2+|\eta|^2}))\, \rho_1(\xi,s) \, ds \, d\eta \, dS' \\ &\leqq c''(\zeta^2 + t^2) \quad (0 < 2\zeta < t \leqq t_0).\end{aligned}$$

c' and c'' are positive constants which are independent of the parameters These estimates (7.27), (7.28), (7.29) imply the second part of (7.21). □

Using (7.3) with Lemma 7.4 and Lemma 7.5, we have

$$(7.30)\qquad \limsup_{p\to\infty} \, (\widehat{\Phi}_k, \psi_r)_{L^2(Q)} \left(\frac{\mu_k(\sigma_p) - \mu_k}{\sigma_p^{1/2}} \right)$$

$$
\begin{aligned}
&- \sqrt{2}\int_{\partial Q} \Phi_k(\xi, o'')\,\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\,dS' \\
&= \frac{1}{\sqrt{2}} \limsup_{p\to\infty} (\Upsilon_1(k, r, \sigma_p, t) - \Upsilon_2(k, r, \sigma_p, t)),
\end{aligned}
$$

$$
\begin{aligned}
(7.31)\qquad &\liminf_{p\to\infty}\ (\widehat{\Phi}_k, \psi_r)_{L^2(Q)} \left(\frac{\mu_k(\sigma_p) - \mu_k}{\sigma_p^{1/2}}\right) \\
&- \sqrt{2}\int_{\partial Q} \Phi_k(\xi, o'')\,\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\,dS' \\
&= \frac{1}{\sqrt{2}} \liminf_{p\to\infty} (\Upsilon_1(\sigma_p, t) - \Upsilon_2(\sigma_p, t)).
\end{aligned}
$$

As the left hand side of (7.30) and (7.31) are independent on the variable t and so are the right hand sides. Moreover we know, from (7.21),

$$
\limsup_{p\to\infty} |\Upsilon_1(k, r, \sigma_p, t)| \leqq c_{15}\, t, \quad \limsup_{p\to\infty} |\Upsilon_2(k, r, \sigma_p, t)| \leqq c_{15}\, t.
$$

So the right hand sides of (7.31) and (7.32) are zero and we can conclude

$$
(7.32)\qquad \lim_{p\to\infty} \frac{(\mu_k(\sigma_p) - \mu_k)}{\sigma_p^{1/2}} (\widehat{\Phi}_k, \psi_r)_{L^2(Q)} = \sqrt{2}\int_{\partial Q} \Phi_k(\xi, o'')\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\,dS'.
$$

We consider (7.13) and (7.32) to deduce the conclusion. We recall the orthogonality condition (2.9) and see that

$$
\{(\Phi_k, \widehat{\Phi}_k) \in L^2(D) \times L^2(Q) \mid\ k(j) \leqq k < k(j+1)\}
$$

span the same subspace as

$$
\begin{aligned}
\{(\phi_d, 0), (0, \psi_r) \in L^2(D) \times L^2(Q) \mid\ & d(j') \leqq d < d(j'+1), \\
& r(j'') \leqq r < r(j''+1)\}.
\end{aligned}
$$

Hence, for each k $(k(j) \leqq k < k(j+1))$, at least one of the following two conditions holds.

(i) There exists some d such that $(\Phi_k, \phi_d)_{L^2(D)} \neq 0$, $d(j') \leqq d < d(j'+1)$.

(ii) There exists some r such that $(\widehat{\Phi}_k, \psi_r)_{L^2(Q)} \neq 0$, $r(j'') \leqq r < r(j''+1)$.

From (7.13) and (7.32) we conclude that the limit value

$$
\lim_{p\to\infty} (\mu_k(\sigma_p) - \mu_k)/\sigma_p^{1/2}
$$

exists for $k(j) \leqq k < k(j+1)$. We denote this value by $\boldsymbol{\gamma}_1'(k)$. Simultaneously we have

$$\boldsymbol{\gamma}_1'(k) \int_D \Phi_k(x)\phi_d(x)dx = \sqrt{2}\int_{\partial Q} \frac{\partial \widehat{\Phi}_k}{\partial \boldsymbol{n}}(\xi)\,\phi_d(\xi,o'')\,dS', \tag{7.33}$$

$$\boldsymbol{\gamma}_1'(k) \int_Q \widehat{\Phi}_k(x')\,\psi_r(x')\,dx' = \sqrt{2}\int_{\partial Q} \Phi_k(\xi,o'')\,\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\,dS', \tag{7.34}$$

for $k(j) \leqq k < k(j+1)$, $d(j') \leqq d < d(j'+1)$, $r(j'') \leqq r < r(j''+1)$.

Substituting

$$\Phi_k = \sum_{d(j') \leqq d < d(j'+1)} (\Phi_k, \phi_d)_{L^2(D)}\phi_d, \quad \widehat{\Phi}_k = \sum_{r(j'') \leqq r < r(j''+1)} (\widehat{\Phi}_k, \psi_r)_{L^2(Q)}\psi_r$$

into (7.33) and (7.34), we have

$$\begin{aligned} &\boldsymbol{\gamma}_1'(k) \int_D \Phi_k \phi_d\,dx \\ &\quad = \sqrt{2} \sum_{r(j'') \leqq r < r(j''+1)} (\widehat{\Phi}_k, \psi_r)_{L^2(Q)} \int_{\partial Q} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\,\phi_d(\xi,o'')\,dS', \\ &\boldsymbol{\gamma}_1'(k) \int_Q \widehat{\Phi}_k \psi_r\,dx' \\ &\quad = \sqrt{2} \sum_{d(j') \leqq d < d(j'+1)} (\Phi_k, \phi_d)_{L^2(D)} \int_{\partial Q} \phi_d(\xi,o'')\,\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\,dS'. \end{aligned}$$

Define $\widehat{d}(j') \times \widehat{k}(j)$ matrix E, $\widehat{r}(j'') \times \widehat{k}(j)$ matrix F and $\widehat{k}(j) \times \widehat{k}(j)$ matrix U by and put

$$\begin{aligned} E &= \left((\phi_d, \Phi_k)_{L^2(D)}\right)_{d(j') \leqq d < d(j'+1), k(j) \leqq k < k(j+1)}, \\ F &= \left((\psi_r, \widehat{\Phi}_k)_{L^2(Q)}\right)_{r(j'') \leqq r < r(j''+1), k(j) \leqq k < k(j+1)}, \end{aligned} \qquad U = \begin{pmatrix} E \\ F \end{pmatrix}.$$

Remark that two families

$$\left\{(\Phi_k, \widehat{\Phi}_k) \mid k(j) \leqq k < k(j+1)\right\},$$
$$\left\{(\phi_d, 0), (0, \psi_r) \mid d(j') \leqq d < d(j'+1), r(j'') \leqq r < r(j''+1)\right\}$$

are orthonormal in $L^2(D) \times L^2(Q)$, respectively and span the identical subspace and so U is an orthogonal matrix and

$$\begin{pmatrix} O & \sqrt{2}\boldsymbol{C}(j) \\ \sqrt{2}{}^t\boldsymbol{C}(j) & O \end{pmatrix} U$$
$$= U \begin{pmatrix} \boldsymbol{\gamma}_1'(k(j)) & & & \\ & \boldsymbol{\gamma}_1'(k(j)+1) & & \\ & & \ddots & \\ & & & \boldsymbol{\gamma}_1'(k(j+1)-1) \end{pmatrix}.$$

This implies $\boldsymbol{\gamma}_1'(k) = \boldsymbol{\gamma}_1(k)$ for $k \in \mathbb{N}_{III}$ for $k(j) \leqq k < k(j+1)$ (See the definition of $\boldsymbol{\gamma}_1(k)$ in §2). The choice of the sequence $\{\zeta_p\}_{p=1}^{\infty}$ was arbitrary and $\boldsymbol{\gamma}_1(k)$ depends only on $D, Q, \boldsymbol{q}, m$. So we have

$$\lim_{\zeta \to 0} \frac{\mu_k(\zeta) - \mu_k}{\zeta^{1/2}} = \boldsymbol{\gamma}_1(k) \quad (\text{for} \quad k(j) \leqq k < k(j+1)),$$

which is the conclusion of Theorem 2.12.

7.2. Proof of Theorem 2.13 ($m = 2$)

For the case $m = 2$, we carry out a similar argument as the case $m = 1$ (in **7.1**). First we note the following estimates around D.

LEMMA 7.6. *Assume $m = 2$ and then for any $t > 0$ and $k \in \mathbb{N}$, we have*

$$(7.35) \qquad \begin{cases} \limsup_{\zeta \to 0} \left(\sup_{x \in \Sigma^-(\zeta, t\zeta) \cup D} |\Phi_{k,\zeta}(x)| \right) < \infty & (k \geqq 1), \\ \limsup_{\zeta \to 0} \left(\sup_{x \in \Sigma^-(\zeta, h(\zeta)\zeta)} |\Phi_{k,\zeta}(x)|/h(\zeta) \right) < \infty (k \geqq 1). \end{cases}$$

PROOF OF LEMMA 7.6. Both estimates follow from Rem.4.6-(ii) and the estimates Rem.2.3-(ii), (iii). □

Putting $\Psi = \widetilde{\phi}_{d,\zeta}$ in (2.2) for $d(j') \leqq d < d(j'+1)$ and carrying out a partial integration in each region, $Q(\zeta) \setminus \Sigma^-(\zeta, 2\zeta)$, $\Sigma^-(\zeta, 2\zeta)$, $\Sigma^+(2\zeta)$, $D \setminus \Sigma^+(2\zeta)$, we have

$$(7.36) \qquad (\mu_k(\zeta) - \mu_k) \int_{D \setminus \Sigma^+(2\zeta)} \Phi_{k,\zeta}\, \widetilde{\phi}_{d,\zeta}\, dx$$

$$
\begin{aligned}
&= -\mu_k(\zeta) \int_{Q(\zeta)\cup\Sigma^+(2\zeta)} \Phi_{k,\zeta}\, \widetilde{\phi}_{d,\zeta}\, dx + \int_{\Gamma^-(\zeta,2\zeta)} \Phi_{k,\zeta} \frac{\partial \widetilde{\phi}_{d,\zeta}}{\partial \nu_2} dS \\
&\quad + \int_{\partial(\Sigma^-(\zeta,2\zeta)\cup\Sigma^+(2\zeta))} \Phi_{k,\zeta} \frac{\partial \widetilde{\phi}_{d,\zeta}}{\partial \nu_1} dS - \int_{\Sigma^-(\zeta,2\zeta)\cup\Sigma^+(2\zeta)} \Phi_{k,\zeta}\, \Delta\widetilde{\phi}_{d,\zeta}\, dx \\
&\quad + \int_{\Gamma^+(2\zeta)} \Phi_{k,\zeta} \frac{\partial \widetilde{\phi}_{d,\zeta}}{\partial \nu_3} dS.
\end{aligned}
$$

Putting $\Psi = \widetilde{\psi}_{r,\zeta}$ in (2.2) for $r(j'') \leqq r < r(j''+1)$ and carrying out a partial integration in each region $Q(\zeta)\setminus\Sigma^-(\zeta,h\zeta)$, $\Sigma^-(\zeta,h\zeta)$, $\Sigma^+(t)$, $D\setminus\Sigma^+(t)$, we have

$$
\begin{aligned}
(7.37)\qquad &(\mu_k(\zeta) - \mu_k) \int_{Q(\zeta)} \Phi_{k,\zeta}\, \widetilde{\psi}_r\, dx \\
&= \zeta \int_{\Gamma^+(t)} \frac{\partial \Phi_{k,\zeta}}{\partial \nu_3} v_{r,\zeta}\, dS + \zeta \int_{\Gamma^-(\zeta,h\zeta)} \Phi_{k,\zeta} \frac{\partial v_{r,\zeta}}{\partial \nu_2} dS \\
&\quad + \zeta \int_{\Gamma^-(\zeta,h\zeta)} \Phi_{k,\zeta} \frac{\partial v_{r,\zeta}}{\partial \nu_1} dS + \zeta \int_{\Gamma^+(t)} \Phi_{k,\zeta} \frac{\partial v_{r,\zeta}}{\partial \nu_1} dS \\
&\quad + \int_{\partial(\Sigma^-(\zeta,h\zeta)\cup\Sigma^+(t))\setminus(\Gamma^-(\zeta,h\zeta)\cup\Gamma^+(t))} \Phi_{k,\zeta} \frac{\partial \widetilde{\psi}_{r,\zeta}}{\partial \nu_1} dS \\
&\quad - \zeta \int_{\Sigma^-(\zeta,h\zeta)} \Phi_{k,\zeta}\, (\Delta v_{r,\zeta} + \mu_k(\zeta)\, v_{r,\zeta})\, dx \\
&\quad - \zeta \int_{Q(\zeta)\setminus\Sigma^-(\zeta,h\zeta)} \mu_k(\zeta)\, \Phi_{k,\zeta}\, v_{r,\zeta}\, dx \\
&\quad - \zeta \int_{\Sigma^+(t)} \Phi_{k,\zeta}\, (\Delta v_{r,\zeta} + \mu_k(\zeta)\, v_{r,\zeta})\, dx.
\end{aligned}
$$

Here $h = h(\zeta)$. We prepare the the estimates for the terms in (7.36).

LEMMA 7.7. *There exists a constant $c_{16} > 0$ such that, for $k(j) \leqq k < k(j+1)$, $d(j') \leqq d < d(j'+1)$, we have*

$$
(7.38)\qquad
\begin{aligned}
&\left| \int_{\Gamma^-(\zeta,2\zeta)} \Phi_{k,\zeta} \frac{\partial \widetilde{\phi}_{d,\zeta}}{\partial \nu_2} dS \right| \leqq c_{16}\, \zeta^2, \\
&\left| \int_{\partial(\Sigma^-(\zeta,2\zeta)\cup\Sigma^+(2\zeta))} \Phi_{k,\zeta} \frac{\partial \widetilde{\phi}_{d,\zeta}}{\partial \nu_1} dS \right| \leqq c_{16}\, \zeta^2,
\end{aligned}
$$

$$\left|\int_{\Sigma^-(\zeta,2\zeta)} \Phi_{k,\zeta}\, \Delta\widetilde{\phi}_{d,\zeta}\, dx\right| \leqq c_{16}\, \zeta^3,$$
$$\left|\int_{\Sigma^+(2\zeta)} \Phi_{k,\zeta}\, \Delta\widetilde{\phi}_{d,\zeta} dx\right| \leqq c_{16}\, \zeta^3, \tag{7.39}$$

$$\left|\int_{\Gamma^+(2\zeta)} \Phi_{k,\zeta}\, \frac{\partial\widetilde{\phi}_{d,\zeta}}{\partial\nu_3}\, dS\right| \leqq c_{16}\, \zeta^2. \tag{7.40}$$

Proof of Lemma 7.7. (7.38),(7.39),(7.40) follow from $m = 2$ and Lem.7.6-(7.35), Lem.4.2-(4.18). □

Lemma 7.8. *For $k(j) \leqq k < k(j+1)$, $d(j') \leqq d < d(j'+1)$, we have*

$$\int_{D\setminus\Sigma^+(2\sigma_p)} \Phi_{k,\sigma_p}\, \widetilde{\phi}_{d,\sigma_p}\, dx = \int_D \Phi_k\, \phi_d\, dx + o(1), \tag{7.41}$$

$$\int_{Q(\sigma_p)} \Phi_{k,\sigma_p}\, \widetilde{\phi}_{d,\sigma_p}\, dx = -\frac{\sqrt{\pi}}{\mu_k} \int_{\partial Q} \frac{\partial\widehat{\Phi}_k}{\partial \boldsymbol{n}}(\xi)\, \phi_d(\xi, o'')\, dS'\, \sigma_p + o(\sigma_p). \tag{7.42}$$

Proof of Lemma 7.8. It is easy to see that (7.41) immediately follows from (2.6), Lem.7.6-(7.35) and $\lim_{\zeta\to 0}\sup_{x\in D} |\widetilde{\phi}_{d,\zeta}(x) - \phi_d(x)| = 0$. Next we deal with (7.42).

$$\begin{aligned}
\int_{Q(\sigma_p)} \Phi_{k,\sigma_p}\, \widetilde{\phi}_{d,\sigma_p}\, dx &= (\sqrt{\pi}\sigma_p)^{-1} \int_{Q(\sigma_p)} \widehat{\Phi}_{k,\sigma_p}\, \widetilde{\phi}_{d,\sigma_p}\, dx \\
&= (\sqrt{\pi}\sigma_p)^{-1} \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,2\sigma_p)} \widehat{\Phi}_k\, \widetilde{\phi}_{d,\sigma_p}\, dx \\
&\quad + (\sqrt{\pi}\sigma_p)^{-1} \int_{\Sigma^-(\sigma_p,2\sigma_p)} \widehat{\Phi}_k\, \widetilde{\phi}_{d,\sigma_p}\, dx \\
&\quad + (\sqrt{\pi}\sigma_p)^{-1} \int_{Q(\sigma_p)} (\widehat{\Phi}_{k,\sigma_p} - \widehat{\Phi}_k)\, \widetilde{\phi}_{d,\sigma_p}\, dx \\
&= (-1/\mu_k)(1/\sqrt{\pi}\sigma_p) \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,2\sigma_p)} \Delta\widehat{\Phi}_k\, \widetilde{\phi}_{d,\sigma_p}\, dx \\
&\quad + O(\sigma_p^2) + o(\sigma_p)
\end{aligned} \tag{7.43}$$

$$= (-1)/(\mu_k\sqrt{\pi}\sigma_p)\int_{\Gamma^-(\sigma_p,2\sigma_p)}\left(\frac{\partial\widehat{\Phi}_k}{\partial\nu_2}\,\widetilde{\phi}_{d,\sigma_p} - \widehat{\Phi}_k\,\frac{\partial}{\partial\nu_2}\,\widetilde{\phi}_{d,\sigma_p}\right)dS$$
$$+\, o(\sigma_p).$$

We used $m = 2$ and that $\widetilde{\phi}_{d,\sigma_p}$ is harmonic in $Q(\sigma_p)\setminus\Sigma^-(\sigma_p,2\sigma_p)$ with the Neumann B.C. on $\partial(Q(\sigma_p)\setminus\Sigma^-(\sigma_p,2\sigma_p))\setminus\Gamma^-(\sigma_p,2\sigma_p)$ (because $\omega_d\in E_{III}$, $d(j')\leqq d< d(j'+1)$) above.

We estimate each term of the right hand side of (7.43). We use Lem.4.2 and the Dirichlet B.C. of $\widehat{\Phi}_k$ on ∂Q. For $x=(\xi-2\sigma_p\boldsymbol{n}(\xi),\eta)\in\Gamma^-(\sigma_p,2\sigma_p)$,

$$\sup_{x\in\Gamma^-(\sigma_p,2\sigma_p)}|(\partial\widetilde{\phi}_{d,\sigma_p}/\partial\nu_2)(x)|\leqq c_1\quad(p\geqq 1),$$
$$|\widehat{\Phi}_k(x)|\leqq 2\sigma_p\sup_{x'\in Q}|\nabla'\widehat{\Phi}_k(x')|\quad(x\in\Gamma^-(\sigma_p,2\sigma_p)).$$

Using these estimates we get

$$\left|\int_{\Gamma^-(\sigma_p,2\sigma_p)}\widehat{\Phi}_k\frac{\partial}{\partial\nu_2}\,\widetilde{\phi}_{d,\sigma_p}\,dS\right| \leqq (2\,\pi\,\sigma_p^3)\,c_1\sup_{x'\in Q}|\nabla'\widehat{\Phi}_k(x')|\int_{\partial Q}\rho_2(\xi,-2\sigma_p)\,dS', \tag{7.44}$$

$$\begin{aligned}\int_{\Gamma^-(\sigma_p,2\sigma_p)}\frac{\partial\widehat{\Phi}_k}{\partial\nu_2}\widetilde{\phi}_{d,\sigma_p}\,dS &= \int_{\Gamma^-(\sigma_p,2\sigma_p)}\langle(\nabla'\widehat{\Phi}_k)(\xi-2\sigma_p\boldsymbol{n}(\xi)),\boldsymbol{n}(\xi)\rangle\,\phi_d(\xi,o'')\,dS\\ &= \int_{\partial Q}\int_{|\eta|<\sigma_p}\langle(\nabla'\widehat{\Phi}_k)(\xi-2\sigma_p\boldsymbol{n}(\xi)),\boldsymbol{n}(\xi)\rangle\\ &\qquad\times\phi_d(\xi,o'')\,\rho_2(\xi,-2\sigma_p)\,d\eta\,dS'\\ &= \int_{\partial Q}\int_{|\eta|<\sigma_p}\langle\nabla'\widehat{\Phi}_k(\xi),\boldsymbol{n}(\xi)\rangle\,\phi_d(\xi,o'')\,d\eta dS'\\ &\quad+\int_{\partial Q}\int_{|\eta|<\sigma_p}\langle(\nabla'\widehat{\Phi}_k)(\xi-2\sigma_p\boldsymbol{n}(\xi)),\boldsymbol{n}(\xi)\rangle\\ &\qquad\times\phi_d(\xi,o'')\,(\rho_2(\xi,-2\sigma_p)-1)\,d\eta\,dS'\end{aligned} \tag{7.45}$$

$$+\int_{\partial Q}\int_{|\eta|<\sigma_p}\langle(\nabla'\widehat{\Phi}_k)(\xi-2\sigma_p\boldsymbol{n}(\xi))-\nabla'\widehat{\Phi}_k(\xi),\boldsymbol{n}(\xi)\rangle \times \phi_d(\xi,o'')\,d\eta\,dS'$$
$$=\pi\,\sigma_p^2\int_{\partial Q}\langle\nabla'\widehat{\Phi}_k(\xi),\boldsymbol{n}(\xi)\rangle\,\phi_d(\xi,o'')\,dS'+O(\sigma_p^3).$$

Substituting (7.44), (7.45) into (7.43), we get (7.42). □

We put $\zeta=\sigma_p$ and take $p\to\infty$ in (7.36) with using Lemma 7.7 and Lemma 7.8, we get

$$(\mu_k(\sigma_p)-\mu_k)\int_D\Phi_{k,\sigma_p}\,\phi_d\,dx = \sqrt{\pi}\sigma_p\int_{\partial Q}\frac{\partial\widehat{\Phi}_k}{\partial\boldsymbol{n}}(\xi)\,\phi_d(\xi,o'')\,dS'+o(\sigma_p). \tag{7.46}$$

We prepare some estimates to deal with (7.37).

Lemma 7.9. *There exists a constant $c_{17}>0$ such that for $k(j)\leqq k<k(j+1)$, $r(j'')\leqq r<r(j''+1)$, we have*

$$\left|\int_{\Gamma^-(\zeta,h\zeta)}\Phi_{k,\zeta}\,\frac{\partial v_{r,\zeta}}{\partial\nu_2}\,dS\right|\leqq c_{17}\,\zeta^2(\log(1/\zeta))^2, \quad \left|\int_{\Gamma^-(\zeta,h\zeta)}\Phi_{k,\zeta}\,\frac{\partial v_{r,\zeta}}{\partial\nu_1}\,dS\right|\leqq c_{17}\,\zeta^2(\log(1/\zeta))^2, \tag{7.47}$$

$$\left|\int_{\partial(\Sigma^-(\zeta,h\zeta)\cup\Sigma^+(t))\setminus(\Gamma^-(\zeta,h\zeta)\cup\Gamma^+(t))}\Phi_{k,\zeta}\,\frac{\partial\widetilde{\psi}_{r,\zeta}}{\partial\nu_1}\,dS\right|\leqq c_{17}\,\zeta^3, \tag{7.48}$$

$$\left|\int_{\Sigma^-(\zeta,h\zeta)}\Phi_{k,\zeta}\,(\Delta v_{r,\zeta}+\mu_k(\zeta)\,v_{r,\zeta})\,dx\right|\leqq c_{17}\,\zeta^2(\log\zeta)^2. \tag{7.49}$$

Proof of Lemma 7.9. (7.47) follows from Lem.7.6-(7.35) and Lem.4.3-(ii)-(4.33). (7.48) follows from Lem.7.6-(7.35) and Lem.4.3-(ii)-(4.31). We deal with (7.49). From Lem.4.3-(ii)-(4.35) and (7.35), we estimate

$$\left|\int_{\Sigma^-(\zeta,h\zeta)}\Phi_{k,\zeta}\,(\Delta v_{r,\zeta}+\mu_k(\zeta)\,v_{r,\zeta})\,dx\right|$$

$$\leqq \int_{\Sigma^-(\zeta,h\zeta)} |\Phi_{k,\zeta}|\,|\Delta v_{r,\zeta}|\,dx + \mu_k(\zeta)\int_{\Sigma^-(\zeta,h\zeta)} |\Phi_{k,\zeta}|\,|v_{r,\zeta}|\,dx$$
$$\leqq \int_{\partial Q}\int_{-h(\zeta)\zeta}^{0}\int_{|\eta|<\zeta \boldsymbol{q}(s/\zeta)} h(\zeta)c_2\left(1+(1/\zeta)\,e^{\delta s/\zeta}\right)\rho_1(\xi,s)\,ds\,d\eta\,dS'$$
$$+\,\mu_k(\zeta)\int_{\partial Q}\int_{-h(\zeta)\zeta}^{0}\int_{|\eta|<\zeta \boldsymbol{q}(s/\zeta)} h(\zeta)\,c_2\,\rho_1(\xi,s)\,ds\,d\eta dS' \leqq c\,\zeta^2\,(\log\zeta)^2.$$

$c>0$ is a constant which is independent of the parameters. These estimates imply (7.49). □

LEMMA 7.10. *We have, for $k(j)\leqq k<k(j+1)$, $r(j'')\leqq r<r(j''+1)$,*

$$\int_{Q(\sigma_p)} \Phi_{k,\sigma_p}\,\widetilde{\psi}_r\,dx = \sqrt{\pi}\sigma_p\left(\int_Q \widehat{\Phi}_k\,\psi_r\,dx' + o(1)\right), \tag{7.50}$$

$$\int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h\sigma_p)} \Phi_{k,\sigma_p}\,v_{r,\zeta}\,dx = \frac{1}{\mu_k}\,T(\boldsymbol{q},2)\sqrt{\pi}\,\sigma_p\left(\int_{\partial Q}\frac{\partial\widehat{\Phi}_k}{\partial\boldsymbol{n}}(\xi)\,\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,dS' + o(\sigma_p)\right). \tag{7.51}$$

The functions $\Upsilon_3(k,r,\zeta,t)$, $\Upsilon_4(k,r,\zeta,t)$, $\Upsilon_5(k,r,\zeta,t)$ are defined through the following (7.52),(7.53),(7.54)

$$\Upsilon_3(k,r,\zeta,t) = \frac{1}{\zeta}\int_{\Gamma^+(t)}\frac{\partial\Phi_{k,\zeta}}{\partial\nu_3}\,v_{r,\zeta}\,dS, \tag{7.52}$$

$$\Upsilon_4(k,r,\zeta,t) = \frac{1}{\zeta}\int_{\Sigma^+(t)}\Phi_{k,\zeta}\left(\Delta v_{r,\zeta}+\mu_k(\zeta)\,v_{r,\zeta}\right)dx, \tag{7.53}$$

$$\int_{\Gamma^+(t)}\Phi_{k,\zeta}\,\frac{\partial v_{r,\zeta}}{\partial\nu_1}\,dS = \pi\,\zeta\left(\int_{\partial Q}\Phi_k(\xi,o'')\,\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,dS' + \Upsilon_5(k,r,\zeta,t)\right) \tag{7.54}$$

($h=h(\sigma_p)$) and they satisfy the estimates

$$\begin{aligned}&\limsup_{\zeta\to 0}|\Upsilon_4(k,r,\zeta,t)| \leqq c_{18}t,\\ &\limsup_{p\to\infty}|\Upsilon_j(k,r,\sigma_p,t)| \leqq c_{18}t \quad (j=3,5)\end{aligned} \tag{7.55}$$

for $k(j) \leqq k < k(j+1)$, $r(j'') \leqq r < r(j''+1)$.

Proof of Lemma 7.10. (7.50) follows directly from (2.6). We deal with (7.51).

$$\begin{aligned}
&\int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h\sigma_p)} \Phi_{k,\sigma_p}\, v_{r,\sigma_p}\, dx = (\sqrt{\pi}\sigma_p)^{-1} \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h\sigma_p)} \widehat{\Phi}_{k,\sigma_p}\, v_{r,\sigma_p}\, dx \\
&= (\sqrt{\pi}\sigma_p)^{-1} \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h\sigma_p)} \widehat{\Phi}_k\, v_{r,\sigma_p}\, dx \\
&\quad + (\sqrt{\pi}\sigma_p)^{-1} \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h\sigma_p)} (\widehat{\Phi}_{k,\sigma_p} - \widehat{\Phi}_k)\, v_{r,\sigma_p}\, dx \\
&(\equiv I_1(\sigma_p) + I_2(\sigma_p))
\end{aligned}$$

Note $h = h(\sigma_p) = (\log\sigma_p)^2$ (cf. (4.4)).

$$\begin{aligned}
(7.56)\quad |I_2(\sigma_p)| &\leqq (\sqrt{\pi}\sigma_p)^{-1} \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h\sigma_p)} |\widehat{\Phi}_{k,\sigma_p}(x) - \widehat{\Phi}_k(x)||v_{r,\zeta}(x)|\, dx \\
&\leqq (\sqrt{\pi}\sigma_p)^{-1} |Q(\sigma_p)| c_2 \\
&\qquad \times \sup_{x\in Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h\sigma_p)} |\widehat{\Phi}_{k,\sigma_p}(x) - \widehat{\Phi}_k(x)||v_{r,\sigma_p}(x)|.
\end{aligned}$$

Hence $\lim_{\zeta\to 0} |Q(\zeta)|/\zeta^2 = \pi|Q|$, (2.6) and Lem.4.3-(ii)-(4.32) yield $\lim_{p\to\infty} I_2(\sigma_p)/\sigma_p = 0$.

$$\begin{aligned}
(7.57)\qquad I_1(\sigma_p) &= (-1/\mu_k)\,(\sqrt{\pi}\sigma_p)^{-1} \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h\sigma_p)} \Delta\widehat{\Phi}_k\, v_{r,\sigma_p}\, dx \\
&= (-1/\mu_k)\,(\sqrt{\pi}\sigma_p)^{-1} \\
&\quad \times \int_{\Gamma^-(\sigma_p,h\sigma_p)} \left(\frac{\partial\widehat{\Phi}_k}{\partial\nu_2}\, v_{r,\sigma_p} - \widehat{\Phi}_k\, \frac{\partial v_{r,\sigma_p}}{\partial\nu_2} \right) dS \\
&= (-1/\mu_k)\,(\sqrt{\pi}\sigma_p)^{-1} \int_{\Gamma^-(\sigma_p,h\sigma_p)} \langle \nabla\widehat{\Phi}_k, \nu_2\rangle\, v_{r,\sigma_p}\, dS \\
&\quad + (1/\mu_k)\,(\sqrt{\pi}\sigma_p)^{-1} \\
&\quad \times \int_{\Gamma^-(\sigma_p,h\sigma_p)} \widehat{\Phi}_k\, \frac{\partial v_{r,\sigma_p}}{\partial\nu_2}\, dS \quad (\equiv I_{1,1}(\sigma_p) + I_{1,2}(\sigma_p))
\end{aligned}$$

From Lem.4.3-(ii)-(4.33) and $\sup_{\Gamma^-(\sigma_p,h(\sigma_p)\sigma_p)}|\widehat{\Phi}_k(x)| = O(h(\sigma_p)\sigma_p)$, we have

$$I_{1,2}(\sigma_p) = O(h(\sigma_p)\sigma_p^2) = O((\log\sigma_p)^2\sigma_p^2). \tag{7.58}$$

As $\nu_2(x) = (\boldsymbol{n}(\xi), 0)$ for $x = (\xi - h(\sigma_p)\sigma_p\boldsymbol{n}(\xi), \eta) \in \Gamma^-(\sigma_p, h(\sigma_p)\sigma_p)$

$$\begin{aligned}
I_{1,1}(\sigma_p)
&= \frac{-1}{\mu_k\sqrt{\pi}\sigma_p}\int_{\partial Q}\int_{|\eta|<\sigma_p}\langle(\nabla'\widehat{\Phi}_k)(\xi - h(\sigma_p)\sigma_p\boldsymbol{n}(\xi)), \boldsymbol{n}(\xi)\rangle \\
&\quad\times\left(-\frac{1}{\boldsymbol{\kappa}_1}\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\right)\widehat{G}(-h(\sigma_p), \eta/\sigma_p)\rho_1(\xi, -h(\sigma_p)\sigma_p)\,d\eta\,dS' \\
&= \frac{-1}{\mu_k\sqrt{\pi}\sigma_p}\int_{\partial Q}\int_{|\eta|<\sigma_p}\langle(\nabla'\widehat{\Phi}_k)(\xi), \boldsymbol{n}(\xi)\rangle \\
&\quad\times\left(-\frac{1}{\boldsymbol{\kappa}_1}\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\right)\boldsymbol{\kappa}_2\,d\eta\,dS' \\
&\quad+\frac{-1}{\mu_k\sqrt{\pi}\sigma_p} \\
&\quad\times\int_{\partial Q}\int_{|\eta|<\sigma_p}\langle(\nabla'\widehat{\Phi}_k)(\xi - h(\sigma_p)\sigma_p\boldsymbol{n}(\xi)) - \nabla'\widehat{\Phi}_k(\xi), \boldsymbol{n}(\xi)\rangle \\
&\quad\times\left(-\frac{1}{\boldsymbol{\kappa}_1}\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\right)\widehat{G}(-h(\sigma_p), \eta/\sigma_p)\,\rho_1(\xi, -h(\sigma_p)\sigma_p)\,d\eta\,dS' \\
&\quad+\frac{-1}{\mu_k\sqrt{\pi}\sigma_p}\int_{\partial Q}\int_{|\eta|<\sigma_p}\langle\nabla'\widehat{\Phi}_k(\xi), \boldsymbol{n}(\xi)\rangle\left(-\frac{1}{\boldsymbol{\kappa}_1}\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\right) \\
&\quad\times(\widehat{G}(-h(\sigma_p), \eta/\sigma_p) - \boldsymbol{\kappa}_2)\,\rho_1(\xi, -h(\sigma_p)\sigma_p)\,d\eta\,dS' \\
&\quad+\frac{-1}{\mu_k\sqrt{\pi}\sigma_p}\int_{\partial Q}\int_{|\eta|<\sigma_p}\langle\nabla'\widehat{\Phi}_k(\xi), \boldsymbol{n}(\xi)\rangle\left(-\frac{1}{\boldsymbol{\kappa}_1}\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\right) \\
&\quad\times\widehat{G}(-h(\sigma_p), \eta/\sigma_p)\,(\rho_1(\xi, -h(\sigma_p)\sigma_p) - 1)\,d\eta\,dS'.
\end{aligned} \tag{7.59}$$

Using $T(\boldsymbol{q}, 2) = \boldsymbol{\kappa}_1/\boldsymbol{\kappa}_2$, we have the following expression.

$$I_{1,1}(\sigma_p) = \frac{T(\boldsymbol{q}, 2)\sqrt{\pi}\sigma_p}{\mu_k}\int_{\partial Q}\langle(\nabla'\widehat{\Phi}_k)(\xi), \boldsymbol{n}(\xi)\rangle\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi))\,dS' + o(\sigma_p).$$

From this asymptotic expression for $I_{1,1}(\sigma_p)$ with (7.56), (7.58), we have (7.51).

We deal with (7.53).

$$
\begin{aligned}
(7.60)\qquad |\Upsilon_4(k,r,\zeta,t)| &= \frac{1}{\zeta}\left|\int_{\Sigma^+(t)} \Phi_{k,\zeta}\,(\Delta v_{r,\zeta} + \mu_k(\zeta)\,v_{r,\zeta})\,dx\right| \\
&\leqq \frac{1}{\zeta}\int_{\Sigma^+(t)} |\Phi_{k,\zeta}|\,|\Delta v_{r,\zeta}|\,dx + \frac{1}{\zeta}\mu_k(\zeta)\int_{\Sigma^+(t)} |\Phi_{k,\zeta}|\,|v_{r,\zeta}|\,dx \\
&\leqq \frac{1}{\zeta}\|\Phi_{k,\zeta}\|_{L^\infty(D)}\int_{\Sigma^+(t)} |\Delta v_{r,\zeta}|\,dx \\
&\quad + \frac{1}{\zeta}\,\mu_k(\zeta)\,\|\Phi_{k,\zeta}\|_{L^\infty(D)}\int_{\Sigma^+(t)} |v_{r,\zeta}|\,dx
\end{aligned}
$$

From (4.32) and (4.36), we have

$$
\begin{aligned}
(7.61)\qquad & \int_{\Sigma^+(t)} |\Delta v_{r,\zeta}|\,dx \\
&\leqq \int_{\partial Q}\int_{(s,\eta)\in\tilde{\Sigma}^+(2\zeta)} (c_2/\zeta)\,\rho_1(\xi,s)\,ds\,d\eta dS' \\
&\quad + \int_{\partial Q}\int_{(s,\eta)\in\tilde{\Sigma}^+(t)\setminus\tilde{\Sigma}^+(2\zeta)} \frac{c_2\zeta}{(s^2+|\eta|^2)}\,\rho_1(\xi,s)\,ds\,d\eta\,dS' \\
&\leqq 2\,(c_2/\zeta)\,(2\pi(2\zeta)^3/3)\int_{\partial Q} 1\,dS' + \int_{\partial Q}\int_{2\zeta}^{t} 2\frac{c_2\zeta}{\tau^2}4\,\pi\,\tau^2\,d\tau\,dS' \\
&= \left(\frac{32\pi c_2\zeta^2}{3} + 8\pi c_2\zeta(t-2\zeta)\right)\int_{\partial Q} 1\,dS' \\
&\leqq \zeta\left(\frac{32\,\pi\,\zeta}{3} + 8\pi t\right) c_2\,|\partial Q|
\end{aligned}
$$

$$
\begin{aligned}
(7.62)\qquad & \int_{\Sigma^+(t)} |v_{r,\zeta}|\,dx \\
&\leqq \int_{\partial Q}\int_{(s,\eta)\in\tilde{\Sigma}^+(2\zeta)} c_2\,\rho_1(\xi,s)\,ds\,d\eta\,dS' \\
&\quad + \int_{\partial Q}\int_{(s,\eta)\in\tilde{\Sigma}^+(t)\setminus\tilde{\Sigma}^+(2\zeta)} (c_2\,\zeta/\sqrt{s^2+|\eta|^2})\,\rho_1(\xi,s)\,ds\,d\eta\,dS' \\
&\leqq 2\,c_2\,(2\pi(2\zeta)^3/3)\int_{\partial Q} 1\,dS' + \int_{\partial Q}\int_{2\zeta}^{t} 2\frac{c_2\zeta}{\tau}4\,\pi\,\tau^2\,d\tau\,dS'
\end{aligned}
$$

$$= \zeta \left(\frac{32\,\pi\,\zeta^2}{3} + 4\pi t^2 \right) c_2\,|\partial Q|.$$

Substituting (7.61) and (7.62) into (7.60) we get

$$\limsup_{\zeta \to 0} |\Upsilon_4(k,r,\zeta,t)| \leqq \left(\limsup_{\zeta \to 0} \|\Phi_{k,\zeta}\|_{L^\infty(D)} \right) (8\pi t + 4\mu_k \pi t^2)\, c_2\,|\partial Q|.$$

From Lem.7.6, $\limsup_{\zeta \to 0} \|\Phi_{k,\zeta}\|_{L^\infty(D)}$ is finite. The first part of (7.55) is verified.

Next we deal with (7.52). Recall Prop.2.7-(ii) for the case of $m = 2$ and we know,

$$\text{(7.63)} \qquad \begin{cases} \left| \widehat{G}(z/\zeta) - \dfrac{\boldsymbol{\kappa}_1 \zeta}{2\,|z|} \right| \leqq \dfrac{c_0 \zeta^2}{|z|^2} & (z = (s,\eta) \in H_1), \\ \left| (1/\zeta)\,(\nabla_z \widehat{G})(z/\zeta) - \dfrac{-\boldsymbol{\kappa}_1 \zeta z}{2\,|z|^3} \right| \leqq \dfrac{c_0 \zeta^2}{|z|^3} & (z = (s,\eta) \in H_1). \end{cases}$$

Therefore we know that the function

$$v_{r,\zeta}(x) = -\frac{1}{\boldsymbol{\kappa}_1} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\, \frac{1}{\zeta}\, \widehat{G}(\frac{s}{\zeta}, \frac{\eta}{\zeta}) \quad (\text{on} \quad \Gamma^+(t))$$

smoothly approaches

$$-\frac{1}{\boldsymbol{\kappa}_1} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\, \frac{\boldsymbol{\kappa}_1}{2\sqrt{s^2 + |\eta|^2}} = -\frac{1}{\boldsymbol{\kappa}_1} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\, \frac{\boldsymbol{\kappa}_1}{2t} = -\frac{1}{2t} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \quad (\text{on} \quad \Gamma^+(t))$$

for $\zeta \to 0$. Therefore we have

$$\text{(7.64)} \qquad \begin{aligned} \lim_{p \to \infty} \Upsilon_3(k,r,\sigma_p,t) &= \frac{-1}{2t} \int_{\Gamma^+(t)} \frac{\partial \Phi_k}{\partial \nu_3} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\, dS \\ &= \frac{-1}{2t} \int_{\partial Q} \int_{\widetilde{\Gamma}^+(t)} \frac{\partial \Phi_k}{\partial \nu_3}(\xi + s\boldsymbol{n}(\xi), \eta)\, \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\, \rho_2(\xi,s)\, d\widetilde{S}\, dS. \end{aligned}$$

Here $d\widetilde{S}$ is the measure on $\widetilde{\Gamma}^+(t)$. Estimating the right hand side with $|\Gamma^+(t)| = O(t^2)$, we conclude the second part of (7.55) for $j = 3$. Next we deal with (7.54). First we look into $\partial v_{r,\zeta}/\partial \nu_1$ on $\Gamma^+(t)$.

$$\frac{\partial v_{r,\zeta}}{\partial \nu_1} = -\frac{1}{\boldsymbol{\kappa}_1} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\, \frac{\partial}{\partial \nu_1}(\widehat{G}(\frac{s}{\zeta}, \frac{\eta}{\zeta})) = -\frac{1}{\boldsymbol{\kappa}_1} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\, \frac{1}{\zeta}\, \langle (\nabla_z \widehat{G})(\frac{s}{\zeta}, \frac{\eta}{\zeta}), \widetilde{\boldsymbol{n}} \rangle$$

for $x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Gamma^+(t)$. Using (7.63) we have

$$\begin{aligned}(7.65)\qquad &\left|\frac{\partial v_{r,\zeta}}{\partial \nu_1} - \frac{1}{\boldsymbol{\kappa}_1}\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\,\frac{\boldsymbol{\kappa}_1 \zeta}{2t^2}\right| \\ &= \left|\frac{1}{\boldsymbol{\kappa}_1}\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\right| \left|\langle(\nabla_z \widehat{G})(\frac{s}{\zeta}, \frac{\eta}{\zeta}), \widetilde{\boldsymbol{n}}\rangle\,\frac{1}{\zeta} - \frac{-\boldsymbol{\kappa}_1 \zeta}{2\,t^2}\right| \\ &\leqq \left((1/\boldsymbol{\kappa}_1) \sup_{\xi \in \partial Q} |(\partial \psi_r/\partial \boldsymbol{n})(\xi)|\right)(c_0 \zeta^2/t^3)\ (x \in \Gamma^+(t)).\end{aligned}$$

and consequently we get

$$(7.66)\qquad (1/\zeta)\left|\frac{\partial v_{r,\zeta}}{\partial \nu_1}(x)\right| \leqq \frac{1}{2\,t^2}\left|\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\right| + \frac{c_0\,\zeta}{\boldsymbol{\kappa}_1 t^3}\sup_{\xi \in \partial Q}\left|\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\right|$$

for $x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Gamma^+(t)$.

$$\begin{aligned}&\int_{\Gamma^+(t)} \Phi_{k,\zeta}\,\frac{\partial v_{r,\zeta}}{\partial \nu_1} dS = \int_{\Gamma^+(t)}\left(\Phi_k\,\frac{\partial v_{r,\zeta}}{\partial \nu_1} + (\Phi_{k,\zeta} - \Phi_k)\,\frac{\partial v_{r,\zeta}}{\partial \nu_1}\right) dS \\ &= \int_{\Gamma^+(t)}\left\{\Phi_k\left(\frac{1}{\boldsymbol{\kappa}_1}\,\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\,\frac{\boldsymbol{\kappa}_1 \zeta}{2t^2}\right)\right. \\ &\qquad\left. +\Phi_k\left(\frac{\partial v_{r,\zeta}}{\partial \nu_1} - \frac{1}{\boldsymbol{\kappa}_1}\,\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\,\frac{\boldsymbol{\kappa}_1 \zeta}{2t^2}\right) + (\Phi_{k,\zeta} - \Phi_k)\,\frac{\partial v_{r,\zeta}}{\partial \nu_1}\right\} dS\end{aligned}$$

Therefore we have

$$\begin{aligned}(7.67)\qquad &\frac{1}{\sigma_p}\int_{\Gamma^+(t)} \Phi_{k,\sigma_p}\frac{\partial v_{r,\sigma_p}}{\partial \nu_1}\,dS \\ &= \int_{\Gamma^+(t)} \Phi_k \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\frac{1}{2t^2} dS \\ &\quad + \int_{\Gamma^+(t)} \Phi_k\left(\frac{1}{\sigma_p}\,\frac{\partial v_{r,\sigma_p}}{\partial \nu_1} - \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\,\frac{1}{2t^2}\right) dS \\ &\quad + \int_{\Gamma^+(t)} (\Phi_{k,\sigma_p} - \Phi_k)\,\frac{1}{\sigma_p}\frac{\partial v_{r,\sigma_p}}{\partial \nu_1}\,dS \\ &= \int_{\partial Q}\int_{\widetilde{\Gamma}^+(t)} \Phi_k(\xi, o'')\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\,\frac{1}{2t^2}\,d\widetilde{S} dS' \\ &\quad + \int_{\partial Q}\int_{\widetilde{\Gamma}^+(t)} (\Phi_k(x)\,\rho_2(\xi, s) - \Phi_k(\xi, o''))\,\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\,\frac{1}{2t^2}\,d\widetilde{S}\,dS'\end{aligned}$$

$$+\int_{\Gamma^+(t)} \Phi_k \left(\frac{1}{\sigma_p} \frac{\partial v_{q,\sigma_p}}{\partial \nu_1} - \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \frac{1}{2t^2} \right) dS$$
$$+\int_{\Gamma^+(t)} (\Phi_{k,\sigma_p} - \Phi_k) \frac{1}{\sigma_p} \frac{\partial v_{r,\sigma_p}}{\partial \nu_1} dS.$$

Since $m = 2$ and $\int_{\widetilde{\Gamma}^+(t)} 1 d\widetilde{S} = 2\pi t^2$, we have

$$\int_{\partial Q} \int_{\widetilde{\Gamma}^+(t)} \Phi_k(\xi, o'') \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \frac{1}{2t^2} d\widetilde{S}\, dS' = \pi \int_{\partial Q} \Phi_k(\xi, o'') \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\, dS'.$$

From

$$\Upsilon_5(k, r, \sigma_p, t) = (1/\pi\sigma_p) \int_{\Gamma^+(t)} \Phi_{k,\sigma_p} \frac{\partial v_{r,\sigma_p}}{\partial \nu_1} dS - \int_{\partial Q} \Phi_k(\xi, o'') \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\, dS',$$

we get

$$\begin{aligned}
&\Upsilon_5(k, r, \sigma_p, t) \\
&= \frac{1}{\pi} \int_{\partial Q} \int_{\widetilde{\Gamma}^+(t)} (\Phi_k(\xi + s\boldsymbol{n}(\xi), \eta)\, \rho_2(\xi, s) - \Phi_k(\xi, o'')) \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \frac{1}{2t^2} d\widetilde{S}\, dS' \\
&\quad + \frac{1}{\pi} \int_{\Gamma^+(t)} \Phi_k \left(\frac{1}{\sigma_p} \frac{\partial v_{r,\sigma_p}}{\partial \nu_1} - \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \frac{1}{2t^2} \right) dS \\
&\quad + \frac{1}{\pi} \int_{\Gamma^+(t)} (\Phi_{k,\sigma_p} - \Phi_k) \frac{1}{\sigma_p} \frac{\partial v_{r,\sigma_p}}{\partial \nu_1} dS \equiv I_3(t) + I_4(\sigma_p, t) + I_5(\sigma_p, t).
\end{aligned}$$

We denoted three terms in the right hand side of $\Upsilon_5(k, r, \sigma_p, t)$ by $I_3(t)$, $I_4(\sigma_p, t), I_5(\sigma_p, t)$.

From (7.65) we have

$$\begin{aligned}
|I_4(\sigma_p, t)| &\leqq \frac{1}{\pi} \int_{\Gamma^+(t)} \left| \Phi_k \left(\frac{1}{\sigma_p} \frac{\partial v_{r,\sigma_p}}{\partial \nu_1} - \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \frac{1}{2t^2} \right) \right| dS \\
&\leqq \frac{1}{\pi} \frac{c_0\, \sigma_p}{\boldsymbol{\kappa}_1 t^3} \int_{\Gamma^+(t)} |\Phi_k(x)|\, dS \left(\sup_{\xi \in \partial Q} |\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)| \right).
\end{aligned}$$

We conclude

$$\lim_{p \to \infty} I_4(\sigma_p, t) = 0 \quad \text{for} \quad 0 < t \leqq t_0,$$

$$
\begin{aligned}
|I_5(\sigma_p,t)| &\leqq \frac{1}{\pi}\int_{\Gamma^+(t)} |\Phi_{k,\sigma_p}-\Phi_k|\,\frac{1}{\sigma_p}\left|\frac{\partial v_{r,\sigma_p}}{\partial \nu_1}\right| dS \\
&\leqq \frac{1}{\pi}\,|\Gamma^+(t)| \sup_{x\in\Gamma^+(t)} |\Phi_{k,\sigma_p}(x)-\Phi_k(x)| \left(\sup_{\Gamma^+(t)} \frac{1}{\sigma_p}\left|\frac{\partial v_{r,\sigma_p}}{\partial \nu_1}\right|\right) \\
&\leqq \frac{1}{\pi}\,|\Gamma^+(t)| \sup_{x\in\Gamma^+(t)} |\Phi_{k,\sigma_p}(x)-\Phi_k(x)| \\
&\quad\times\left(\frac{1}{2t^2}+\frac{c_0\,\sigma_p}{\boldsymbol{\kappa}_1 t^3}\right)\sup_{\xi\in\partial Q}\left|\frac{\partial\psi_r}{\partial \boldsymbol{n}}(\xi)\right|.
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
&\lim_{p\to\infty} |I_5(\sigma_p,t)| = 0 \quad \text{for} \quad 0<t\leqq t_0, \\
&|I_3(t)| = \left|\frac{1}{\pi}\int_{\partial Q}\int_{\widetilde{\Gamma}^+(t)} (\Phi_k(\xi+s\boldsymbol{n}(\xi),\eta)\,\rho_2(\xi,s)-\Phi_k(\xi,o''))\right. \\
&\qquad\qquad\qquad\qquad\left.\times\frac{\partial\psi_r}{\partial \boldsymbol{n}}(\xi)\,\frac{1}{2t^2}\,d\widetilde{S}\,dS'\right| \\
&\leqq \frac{1}{\pi}\sup_{\xi\in\partial Q,(s,\eta)\in\widetilde{\Gamma}^+(t)}\left|(\Phi_k(\xi+s\boldsymbol{n}(\xi),\eta)\,\rho_2(\xi,s)-\Phi_k(\xi,o''))\,\frac{\partial\psi_r}{\partial \boldsymbol{n}}(\xi)\right| \\
&\quad\times\int_{\partial Q}\int_{\widetilde{\Gamma}^+(t)}\frac{1}{2t^2}\,d\widetilde{S}\,dS' \\
&= 2|\partial Q| \sup_{\xi\in\partial Q,(s,\eta)\in\widetilde{\Gamma}^+(t)}\left|(\Phi_k(\xi+s\boldsymbol{n}(\xi),\eta)\,\rho_2(\xi,s)-\Phi_k(\xi,o''))\,\frac{\partial\psi_r}{\partial \boldsymbol{n}}(\xi)\right|.
\end{aligned}
$$

This estimate yields $|I_3(t)| = O(t)$.

Summing up all the results for $I_3(t), I_4(\sigma_p,t), I_5(\sigma_p,t)$, we conclude that there exists a constant $c>0$ such that

$$
\lim_{p\to\infty} |\Upsilon_5(k,r,\sigma_p,t)| \leqq c\,t.
$$

We verified the second part of (7.55) for $j=5$. □

We put $\zeta=\sigma_p$ in (7.40) and divide it by $\sqrt{\pi}\sigma_p^2$ and take limsup and liminf for $p\to\infty$. From Lemma 7.9 and Lemma 7.10, we get the following relations.

$$
\limsup_{p\to\infty}\left(\frac{\mu_k(\sigma_p)-\mu_k}{\sigma_p}\right)(\widehat{\Phi}_k,\psi_r)_{L^2(Q)} \tag{7.68}
$$

$$= \sqrt{\pi}\left(\int_{\partial Q} \Phi_k(\xi, o'')\,\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\,dS'\right)$$
$$- T(\boldsymbol{q}, 2)\int_{\partial Q} \frac{\partial \widehat{\Phi}_k}{\partial \boldsymbol{n}}(\xi)\,\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) dS'$$
$$+ \limsup_{p\to\infty}\left(\sqrt{\pi}\Upsilon_5(k, r, \sigma_p, t) - \frac{1}{\sqrt{\pi}}\Upsilon_4(k, r, \sigma_p, t)\right),$$

(7.69)
$$\liminf_{p\to\infty}\left(\frac{\mu_k(\sigma_p) - \mu_k}{\sigma_p}\right)(\widehat{\Phi}_k, \psi_r)_{L^2(Q)}$$
$$= \sqrt{\pi}\left(\int_{\partial Q} \Phi_k(\xi, o'')\,\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\,dS'\right)$$
$$- T(\boldsymbol{q}, 2)\int_{\partial Q} \frac{\partial \widehat{\Phi}_k}{\partial \boldsymbol{n}}(\xi)\,\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\,dS'$$
$$+ \liminf_{p\to\infty}\left(\sqrt{\pi}\Upsilon_5(k, r, \sigma_p, t) - \frac{1}{\sqrt{\pi}}\,\Upsilon_4(k, r, \sigma_p, t)\right).$$

The left hand sides of (7.68) and (7.69) are independent of $t > 0$ and so we can take $t \to 0$ in the right hand side in view of the condition (7.55) and we conclude

$$\lim_{p\to\infty}\frac{\mu_k(\sigma_p) - \mu_k}{\sigma_p}\,(\widehat{\Phi}_k, \psi_r)_{L^2(Q)}$$
$$= \sqrt{\pi}\left(\int_{\partial Q} \Phi_k(\xi, o'')\,\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\,dS'\right) - T(\boldsymbol{q}, 2)\int_{\partial Q} \frac{\partial \widehat{\Phi}_k}{\partial \boldsymbol{n}}(\xi)\,\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\,dS'.$$

It is rewritten as

(7.70)
$$(\mu_k(\sigma_p) - \mu_k)\int_Q \widehat{\Phi}_k\,\psi_r\,dx'$$
$$= \sqrt{\pi}\sigma_p\left(\int_{\partial Q} \Phi_k(\xi, o'')\,\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\,dS'\right)$$
$$- T(\boldsymbol{q}, 2)\sigma_p\int_{\partial Q} \frac{\partial \widehat{\Phi}_k}{\partial \boldsymbol{n}}(\xi)\,\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\,dS' + o(\sigma_p)$$

for $k(j) \leqq k < k(j+1)$, $r(j'') \leqq r < r(j''+1)$. To deduce the conclusion, we consider (7.46) and (7.70). We recall the orthogonality condition (2.8) and that

$$\{(\Phi_k, \widehat{\Phi}_k) \in L^2(D) \times L^2(Q) \mid \quad k(j) \leqq k < k(j+1)\}$$

span the same subspace as

$$\{(\phi_d,0),(0,\psi_r)\in L^2(D)\times L^2(Q)\mid d(j')\leqq d<d(j'+1),\\ r(j'')\leqq r<r(j''+1)\}.$$

Hence, for each k ($k(j)\leqq k<k(j+1)$), at least one of the following two conditions hold,
(i) There exists some d such that $(\Phi_k,\phi_d)_{L^2(D)}\neq 0$ and $d(j')\leqq d<d(j'+1)$,
(ii) There exists some r such that $(\widehat{\Phi}_k,\psi_r)_{L^2(Q)}\neq 0$ and $r(j'')\leqq r<r(j''+1)$.

Therefore, by the aid of (7.46) and (7.70), we conclude

$$\lim_{p\to\infty}\frac{\mu_k(\sigma_p)-\mu_k}{\sigma_p} \tag{7.71}$$

exists for $k(j)\leqq k<k(j+1)$. We denote this value by $\boldsymbol{\gamma}'_2(k)$. Simultaneously, we have

$$\boldsymbol{\gamma}'_2(k)\int_D\Phi_k\,\phi_d\,dx=\sqrt{\pi}\int_{\partial Q}\frac{\partial\widehat{\Phi}_k}{\partial\boldsymbol{n}}(\xi)\,\phi_d(\xi,o'')\,dS' \tag{7.72}$$

for $k(j)\leqq k<k(j+1)$, $d(j')\leqq d<d(j'+1)$.

$$\begin{aligned}\boldsymbol{\gamma}'_2(k)\int_Q\widehat{\Phi}_k\psi_r dx'&=\sqrt{\pi}\int_{\partial Q}\Phi_k(\xi,o'')\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)dS'\\&\quad-T(\boldsymbol{q},2)\int_{\partial Q}\frac{\partial\widehat{\Phi}_k}{\partial\boldsymbol{n}}(\xi)\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)dS'\end{aligned} \tag{7.73}$$

for $k(j)\leqq k<k(j+1), r(j'')\leqq r<r(j''+1)$.

Substituting

$$\Phi_k=\sum_{d(j')\leqq d<d(j'+1)}(\Phi_k,\phi_d)_{L^2(D)}\,\phi_d,\quad \widehat{\Phi}_k=\sum_{r(j'')\leqq q<r(j''+1)}(\widehat{\Phi}_k,\psi_q)_{L^2(Q)}\,\psi_q$$

we get

$$\begin{aligned}&\boldsymbol{\gamma}'_2(k)\int_D\Phi_k\,\phi_d\,dx\\&=\sqrt{\pi}\sum_{r(j'')\leqq r<r(j''+1)}(\widehat{\Phi}_k,\psi_r)_{L^2(Q)}\int_{\partial Q}\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,\phi_d(\xi,o'')\,dS',\end{aligned} \tag{7.74}$$

$$(7.75)\quad \boldsymbol{\gamma}_2'(k)\int_Q \widehat{\Phi}_k\,\psi_r\,dx'$$
$$= \sqrt{\pi}\sum_{d(j')\leqq d<d(j'+1)}(\Phi_k,\phi_d)_{L^2(D)}\int_{\partial Q}\phi_d(\xi,o'')\,\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,dS'$$
$$-\,T(\boldsymbol{q},2)\sum_{r(j'')\leqq q<r(j''+1)}(\widehat{\Phi}_k,\psi_q)_{L^2(Q)}\int_{\partial Q}\frac{\partial\psi_q}{\partial\boldsymbol{n}}(\xi)\,\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,dS'.$$

Define $\widehat{d}(j')\times\widehat{k}(j)$ matrix E, $\widehat{r}(j'')\times\widehat{k}(j)$ matrix F and $\widehat{k}(j)\times\widehat{k}(j)$ matrix U by

$$E=\Big((\phi_d,\Phi_k)_{L^2(D)}\Big)_{d(j')\leqq d<d(j'+1),k(j)\leqq k<k(j+1)},\qquad U=\begin{pmatrix}E\\F\end{pmatrix}$$
$$F=\Big((\psi_r,\widehat{\Phi}_k)_{L^2(Q)}\Big)_{r(j'')\leqq r<r(j''+1),k(j)\leqq k<k(j+1)},$$

It is written in the following simple form.

$$\begin{pmatrix}O & \sqrt{\pi}\boldsymbol{C}(j)\\ \sqrt{\pi}\,{}^t\boldsymbol{C}(j) & -T(\boldsymbol{q},2)\,\boldsymbol{B}(j)\end{pmatrix}U$$
$$=U\begin{pmatrix}\boldsymbol{\gamma}_2'(k(j)) & & & \\ & \boldsymbol{\gamma}_2'(k(j)+1) & & \\ & & \ddots & \\ & & & \boldsymbol{\gamma}_2'(k(j+1)-1)\end{pmatrix}$$

This implies $\boldsymbol{\gamma}_2'(k)=\boldsymbol{\gamma}_2(k)$ for $k\in\mathbb{N}_{III}$ for $k(j)\leqq k<k(j+1)$ (See the definition of $\boldsymbol{\gamma}_2(k)$ in §2). The choice of the sequence $\{\zeta_p\}_{p=1}^{\infty}$ was arbitrary and so we conclude

$$\lim_{\zeta\to 0}\frac{\mu_k(\zeta)-\mu_k}{\zeta}=\boldsymbol{\gamma}_2(k)=\boldsymbol{\gamma}_2'(k).$$

which is the conclusion of Theorem 2.13.

7.3. Proof of Theorem 2.14 ($m\geqq 3$)

We deal with the case $m\geqq 3$ and $k\in\mathbb{N}_{III}$. We use the following scaled eigenfunctions.

DEFINITION.

$$\begin{cases}\widehat{\Phi}_{k,\zeta}(x)=S(m)^{1/2}\zeta^{m/2}\,\Phi_{k,\zeta}(x)\quad(x\in\Omega(\zeta))\\ \widehat{\Phi}^*_{k,\zeta}(x)=\zeta\,\Phi_{k,\zeta}(x)\quad(x\in\Omega(\zeta))\end{cases}$$

We start with the following estimates.

LEMMA 7.11. *There exist* $c_{19} > 0$ *such that*

$$(7.76)\quad \begin{cases} |\widehat{\Phi}_{k,\zeta}(x)| \leqq c_{19}(\zeta - s) \quad (x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Sigma^-(\zeta, t_2)), \\ |\widehat{\Phi}_{k,\zeta}(x)| \leqq c_{19}\zeta \quad (x \in \Sigma^+(2\zeta)), \\ |\widehat{\Phi}_{k,\zeta}(x)| \leqq c_{19}\left(\zeta^{m/2} + \dfrac{\zeta^m}{(s^2 + |\eta|^2)^{(m-1)/2}}\right) \\ \qquad\qquad (x \in \Sigma^+(t_1) \setminus \Sigma^+(2\zeta)). \end{cases}$$

$$(7.77)\quad \begin{cases} \limsup_{\zeta\to 0} \sup_{x\in\Sigma^+(2\zeta)} |\widehat{\Phi}_{k,\zeta}(x)|/\zeta < \infty, \\ \limsup_{\zeta\to 0} \sup_{x\in D\setminus\Sigma^+(t)} |\widehat{\Phi}_{k,\zeta}(x)|/\zeta^{m/2} < \infty \quad (\text{for any} \quad t > 0), \\ \limsup_{\zeta\to 0} \sup_{x\in\Sigma^-(\zeta,h(\zeta)\zeta)} |\widehat{\Phi}_{k,\zeta}(x)|/h(\zeta)\zeta < \infty. \end{cases}$$

PROOF OF LEMMA 7.11. (7.76) follows from Rem.4.6-(ii) and Rem.2.3-(ii). Note $m \geqq 3$. The first estimate of (7.77) follows from Rem.2.3-(iii) and the definition of $\widehat{\Phi}_{k,\zeta}$. The second and third estimates of (7.77) follow from Rem.4.6-(ii). □

We can carry out a similar calculation as in **7.2**. Putting $\Psi = \widetilde{\phi}_{d,\zeta}$ in (2.2) for $d(j') \leqq d < d(j'+1)$, we have

$$(7.78)\quad \begin{aligned} &(\mu_k(\zeta) - \mu_k)\int_{D\setminus\Sigma^+(2\zeta)} \Phi_{k,\zeta}\,\widetilde{\phi}_{d,\zeta}\,dx \\ &= -\mu_k(\zeta)\int_{Q(\zeta)\cup\Sigma^+(2\zeta)} \Phi_{k,\zeta}\,\widetilde{\phi}_{d,\zeta}\,dx + \int_{\Gamma^-(\zeta,2\zeta)} \Phi_{k,\zeta}\,\frac{\partial\widetilde{\phi}_{d,\zeta}}{\partial\nu_2}\,dS \\ &\quad + \int_{\partial(\Sigma^-(\zeta,2\zeta)\cup\Sigma^+(2\zeta))} \Phi_{k,\zeta}\,\frac{\partial\widetilde{\phi}_{d,\zeta}}{\partial\nu_1}\,dS \\ &\quad - \int_{\Sigma^-(\zeta,2\zeta)\cup\Sigma^+(2\zeta)} \Phi_{k,\zeta}\,\Delta\widetilde{\phi}_{d,\zeta}\,dx + \int_{\Gamma^+(2\zeta)} \Phi_{k,\zeta}\,\frac{\partial\widetilde{\phi}_{d,\zeta}}{\partial\nu_3}\,dS. \end{aligned}$$

Putting $\Psi = \widetilde{\psi}_{r,\zeta}$ in (2.2) for $r(j'') \leqq r < r(j''+1)$ and carry out integration by parts in each region $Q(\zeta) \setminus \Sigma^-(\zeta, h\zeta)$, $\Sigma^-(\zeta, h\zeta) \cup \Sigma^+(t)$, $D \setminus \Sigma^+(t)$ and multiply it by $S(m)^{1/2}\zeta^{m/2}$, we have

$$
\begin{aligned}
(7.79)\qquad & (\mu_k(\zeta)-\mu_k)\int_{Q(\zeta)} \widehat{\Phi}_{k,\zeta}\,\widetilde{\psi}_r\,dx \\
&= \zeta\int_{\Gamma^+(t)} \frac{\partial\widehat{\Phi}_{k,\zeta}}{\partial\nu_3}\,v_{r,\zeta}\,dS + \zeta\int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\frac{\partial v_{r,\zeta}}{\partial\nu_2}\,dS \\
&\quad + \zeta\int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\frac{\partial v_{r,\zeta}}{\partial\nu_1}\,dS + \zeta\int_{\Gamma^+(t)} \widehat{\Phi}_{k,\zeta}\,\frac{\partial v_{r,\zeta}}{\partial\nu_1}\,dS \\
&\quad + \int_{\partial(\Sigma^-(\zeta,h\zeta)\cup\Sigma^+(t))\setminus(\Gamma^-(\zeta,h\zeta)\cup\Gamma^+(t))} \widehat{\Phi}_{k,\zeta}\,\frac{\partial\widetilde{\psi}_{r,\zeta}}{\partial\nu_1}\,dS \\
&\quad - \zeta\int_{\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,(\Delta v_{r,\zeta}+\mu_k(\zeta)\,v_{r,\zeta})\,dx \\
&\quad - \mu_k(\zeta)\,\zeta\int_{Q(\zeta)\setminus\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,v_{r,\zeta}\,dx \\
&\quad - \zeta\int_{\Sigma^+(t)} \widehat{\Phi}_{k,\zeta}\,(\Delta v_{r,\zeta}+\mu_k(\zeta)\,v_{r,\zeta})\,dx.
\end{aligned}
$$

We first deal with evaluation of the terms of (7.79). For this purpose, we need the following estimates.

LEMMA 7.12. *There exists a constant* $c_{20}>0$ *and* $c_{21}(t)>0$ *such that*

$$
(7.80)\qquad \begin{cases} \left|\displaystyle\int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\frac{\partial v_{r,\zeta}}{\partial\nu_2}\,dS\right| \leqq c_{20}(\log\zeta)^2\zeta^{m+1}, \\ \left|\displaystyle\int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\frac{\partial v_{r,\zeta}}{\partial\nu_1}\,dS\right| \leqq c_{20}\,(\log\zeta)^2\zeta^{m+1}, \end{cases}
$$

$$
(7.81)\qquad \left|\int_{\Gamma^+(t)} \widehat{\Phi}_{k,\zeta}\,\frac{\partial v_{r,\zeta}}{\partial\nu_1}\,dS\right| \leqq c_{21}(t)\,\zeta^{(3m/2)-1},
$$

$$
(7.82)\qquad \left|\int_{\partial(\Sigma^-(\zeta,h\zeta)\cup\Sigma^+(t))\setminus(\Gamma^-(\zeta,h\zeta)\cup\Gamma^+(t))} \widehat{\Phi}_{k,\zeta}\,\frac{\partial\widetilde{\psi}_{r,\zeta}}{\partial\nu_1}\,dS\right| \leqq c_{20}\,\zeta^{m+2},
$$

$$(7.83)\qquad \begin{cases} \left|\int_{\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}_{k,\zeta}\,\Delta v_{r,\zeta}\,dx\right| \leqq c_{20}\,(\log\zeta)^2\zeta^{m+1}, \\ \left|\int_{\Sigma^+(t)} \widehat{\Phi}_{k,\zeta}\,\Delta v_{r,\zeta}\,dx\right| \leqq c_{20}\,(t\zeta^{(3m/2)-1}+\zeta^{m+1}), \end{cases}$$

$$(7.84)\qquad \left|\int_{\Gamma^+(t)} \frac{\partial\widehat{\Phi}_{k,\zeta}}{\partial\nu_3}\,v_{r,\zeta}\,dS\right| \leqq c_{21}(t)\,\zeta^{(3m/2)-1}.$$

Proof of Lemma 7.12. (7.80) follows from Lem.4.3-(ii)-(4.33) and Lem.7.11-(7.77). (7.81) follows from Lem.7.11-(7.77) and Lem.4.3-(ii)-(4.36). (7.82) follows from Lem.4.3-(ii)-(4.31) and Lem.7.11-(7.77). We estimate the left hand side of (7.83). Using Lem.7.11, we have

$$\begin{aligned} &\left|\int_{\Sigma^-(\zeta,h(\zeta)\zeta)} \widehat{\Phi}_{k,\zeta}\,\Delta v_{r,\zeta}\,dx\right| \leqq \int_{\Sigma^-(\zeta,h(\zeta)\zeta)} |\widehat{\Phi}_{k,\zeta}(x)|\,|\Delta v_{r,\zeta}(x)|\,dx \\ &\leqq c_{19}(\zeta+h(\zeta)\zeta)\int_{\partial Q}\int_{\widetilde{\Sigma}^-(\zeta,h\zeta)} c_2\,(1+(1/\zeta)e^{\delta s/\zeta})\,\rho_1(\xi,s)\,ds\,d\eta\,dS' \\ &\leqq 2c_{19}(\zeta+h(\zeta)\zeta)c_2\int_{\partial Q}\int_{-h\zeta}^{0}\int_{|\eta|<\zeta\boldsymbol{q}(s/\zeta)} (1+(1/\zeta)e^{\delta s/\zeta})\,ds\,d\eta\,dS' \\ &\leqq c(\log\zeta)^2\zeta^{m+1}. \end{aligned}$$

c is a positive constant which is independent of the parameters.

$$\begin{aligned} &\left|\int_{\Sigma^+(t)} \widehat{\Phi}_{k,\zeta}\,\Delta v_{r,\zeta}\,dx\right| \leqq \int_{\Sigma^+(2\zeta)} |\widehat{\Phi}_{k,\zeta}(x)|\,|\Delta v_{r,\zeta}(x)|\,dx \\ &\quad + \int_{\Sigma^+(t)\setminus\Sigma^+(2\zeta)} |\widehat{\Phi}_{k,\zeta}(x)|\,|\Delta v_{r,\zeta}(x)|\,dx \\ &\leqq \int_{\Sigma^+(2\zeta)} (c_{19}\zeta)\,(c_2/\zeta)\,dx \\ &\quad + \int_{\Sigma^+(t)\setminus\Sigma^+(2\zeta)} c_{19}\left(\zeta^{m/2}+\frac{\zeta^m}{(s^2+|\eta|^2)^{(m-1)/2}}\right)\frac{c_2\,\zeta^{(m-1)}}{(s^2+|\eta|^2)^{m/2}}\,dx \\ &\leqq c'(\zeta^{m+1}+t\zeta^{(3m/2)-1}). \end{aligned}$$

We used $m \geqq 3$. c' is a positive constant which is independent of the parameters. From Lem.7.11,

$$\|\widehat{\Phi}_{k,\zeta}\|_{L^\infty(D\setminus\Sigma^+(t))} = O(\zeta^{m/2})$$

for any small $t >$. Using elliptic estimates,

$$\|\nabla\widehat{\Phi}_{k,\zeta}\|_{L^\infty(D\setminus\Sigma^+(t))} = O(\zeta^{m/2})$$

for any small $t > 0$. (7.84) follows from this estimate and Lem.4.3-(ii)-(4.37). □

LEMMA 7.13. *For $k(j) \leqq k < k(j+1)$, $r(j'') \leqq r < r(j''+1)$, it holds that*

$$\int_{Q(\sigma_p)} \widehat{\Phi}_{k,\sigma_p}\, \widetilde{\psi}_r\, dx = S(m) \int_Q \widehat{\Phi}_k\, \psi_r\, dx'\, \sigma_p^m + o(\sigma_p^m), \tag{7.86}$$

$$\begin{aligned}&\int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h(\sigma_p)\sigma_p)} \widehat{\Phi}_{k,\sigma_p}\, v_{r,\sigma_p}\, dx \\ &= \frac{1}{\mu_k} T(\boldsymbol{q}, m)\, S(m) \int_{\partial Q} \frac{\partial\widehat{\Phi}_k}{\partial\boldsymbol{n}}\, \frac{\partial\psi_r}{\partial\boldsymbol{n}}\, dS'\, \sigma_p^m + o(\sigma_p^m),\end{aligned} \tag{7.87}$$

for $p \to \infty$.

PROOF OF LEMMA 7.13. (7.86) follows from Prop.2.2-(2.6). Next we prove (7.87). The proof is similar to the one in Lem.7.4.

$$\begin{aligned}&\int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h(\sigma_p)\sigma_p)} \widehat{\Phi}_{k,\sigma_p}\, v_{r,\sigma_p}\, dx \\ &= \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h(\sigma_p)\sigma_p)} (\widehat{\Phi}_{k,\sigma_p} - \widehat{\Phi}_k)\, v_{r,\sigma_p}\, dx \\ &\quad + \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h(\sigma_p)\sigma_p)} \widehat{\Phi}_k\, v_{r,\sigma_p}\, dx.\end{aligned} \tag{7.88}$$

Denote the two terms in the right hand side of (7.88) by $I_6(\sigma_p), I_7(\sigma_p)$. From Lem.4.3 -(ii) -(4.32) and Prop.2.2, we have

$$|I_6(\sigma_p)| \leqq \int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h\sigma_p)} |\widehat{\Phi}_{k,\sigma_p} - \widehat{\Phi}_k(x)|\, |v_{r,\sigma_p}(x)|\, dx \tag{7.89}$$

$$\leqq c_2\,|Q(\sigma_p)| \sup_{x\in Q(\sigma_p)} |\widehat{\Phi}_{k,\sigma_p}(x) - \widehat{\Phi}_k(x)| = o(\sigma_p^m).$$

Using the definition of v_{r,σ_p} and $\Delta\widehat{\Phi}_k = -\mu_k\widehat{\Phi}_k$ in $Q(\sigma_p)$, we have

$$(7.90)\qquad \begin{aligned} I_7(\sigma_p) &= -\frac{1}{\mu_k}\int_{Q(\sigma_p)\setminus\Sigma^-(\sigma_p,h\sigma_p)} \Delta\widehat{\Phi}_k\, v_{r,\sigma_p}\,dx \\ &= -\frac{1}{\mu_k}\int_{\Gamma^-(\sigma_p,h\sigma_p)} (\frac{\partial\widehat{\Phi}_k}{\partial\nu_2}\,v_{r,\sigma_p} - \widehat{\Phi}_k\,\frac{\partial v_{r,\sigma_p}}{\partial\nu_2})\,dS \\ &= -\frac{1}{\mu_k}\int_{\Gamma^-(\sigma_p,h\sigma_p)} \left\{\frac{\partial\widehat{\Phi}_k}{\partial\nu_2}(\frac{-1}{\boldsymbol{\kappa}_1}\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi))\widehat{G}(-h(\sigma_p),\frac{\eta}{\sigma_p}) \right.\\ &\qquad\qquad\qquad\qquad \left. -\widehat{\Phi}_k\frac{\partial v_{r,\sigma_p}}{\partial\nu_2}\right\} dS \\ &= \frac{1}{\boldsymbol{\kappa}_1\mu_k}\int_{\Gamma^-(\sigma_p,h\sigma_p)} \frac{\partial\widehat{\Phi}_k}{\partial\nu_2}\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\widehat{G}(-h(\sigma_p),\frac{\eta}{\sigma_p})dS \\ &\quad + \frac{1}{\mu_k}\int_{\Gamma^-(\sigma_p,h\sigma_p)} \widehat{\Phi}_k\frac{\partial v_{r,\sigma_p}}{\partial\nu_2}\,dS. \end{aligned}$$

From Lem.4.3-(ii)-(4.33) and the Dirichlet B.C. of $\widehat{\Phi}_k$ on ∂Q, we have

$$\sup_{x\in\Gamma^-(\sigma_p,h(\sigma_p)\sigma_p)} |\widehat{\Phi}_k(x)| \leqq h(\sigma_p)\,\sigma_p\,\|\nabla'\widehat{\Phi}_k\|_{L^\infty(Q)},$$
$$\sup_{x\in\Gamma^-(\sigma_p,h(\sigma_p)\sigma_p)} |(\partial v_{r,\sigma_p}/\partial\nu_2)(x)| \leqq c_2.$$

Due to these inequalities, the second term of the right hand side of (7.90) is estimated as follows.

$$(7.91)\qquad \begin{aligned} &\left|\int_{\Gamma^-(\sigma_p,h(\sigma_p)\sigma_p)} \widehat{\Phi}_k\,\frac{\partial v_{r,\sigma_p}}{\partial\nu_2}\,dS\right| \\ &\leqq S(m)\,\sigma_p^m\,h(\sigma_p)\,\sigma_p\|\nabla'\widehat{\Phi}_k\|_{L^\infty(Q)}\,c_2 = o(\sigma_p^m). \end{aligned}$$

We deal with the first term of the right hand side of (7.90). By the aid of

$$|\widehat{G}(s,\eta) - \boldsymbol{\kappa}_2| = |G(s,\eta) - (-\boldsymbol{\kappa}_1\,s + \boldsymbol{\kappa}_2)| \leqq c_0e^{\delta s}$$
$$(s<0, |\eta|<1)\quad \text{(cf. Prop.2.7-(ii))},$$

$$\frac{\partial \widehat{\Phi}_k}{\partial \nu_2}(x) = \langle (\nabla' \Phi_k)(\xi - h(\sigma_p)\sigma_p \boldsymbol{n}(\xi)), \boldsymbol{n}(\xi)\rangle \qquad (x = (\xi - h(\sigma_p)\sigma_p \boldsymbol{n}(\xi), \eta) \in \Gamma^-(\sigma_p, h(\sigma_p)\sigma_p)),$$

we calculate

$$\begin{aligned}
(7.92)\quad & \frac{1}{\boldsymbol{\kappa}_1 \mu_k} \int_{\Gamma^-(\sigma_p, h(\sigma_p)\sigma_p)} \frac{\partial \widehat{\Phi}_k}{\partial \nu_2} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\, \widehat{G}(\frac{s}{\sigma_p}, \frac{\eta}{\sigma_p})\, dS \\
&= \frac{1}{\boldsymbol{\kappa}_1 \mu_k} \int_{\partial Q} \int_{|\eta|<\sigma_p} \langle (\nabla' \widehat{\Phi}_k)(\xi - h\sigma_p \boldsymbol{n}(\xi)), \boldsymbol{n}(\xi)\rangle \\
&\qquad\qquad \times \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \widehat{G}(-h(\sigma_p), \frac{\eta}{\sigma_p}) \rho_2(\xi, -h\sigma_p)\, d\eta\, dS' \\
&= \frac{1}{\boldsymbol{\kappa}_1 \mu_k} \int_{\partial Q} \int_{|\eta|<\sigma_p} \langle (\nabla' \widehat{\Phi}_k)(\xi - h\sigma_p \boldsymbol{n}(\xi)), \boldsymbol{n}(\xi)\rangle \\
&\qquad\qquad \times \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\, \boldsymbol{\kappa}_2\, \rho_2(\xi, -h\sigma_p)\, d\eta\, dS' \\
&\quad + \frac{1}{\boldsymbol{\kappa}_1 \mu_k} \int_{\partial Q} \int_{|\eta|<\sigma_p} \langle (\nabla' \widehat{\Phi}_k)(\xi - h\sigma_p \boldsymbol{n}(\xi)), \boldsymbol{n}(\xi)\rangle \\
&\qquad\qquad \times \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) (\widehat{G}(-h, \frac{\eta}{\sigma_p}) - \boldsymbol{\kappa}_2) \rho_2(\xi, -h\sigma_p)\, d\eta\, dS' \\
&= \frac{T(\boldsymbol{q}, m)}{\mu_k} S(m)\, \sigma_p^m \int_{\partial Q} \langle \nabla' \widehat{\Phi}_k(\xi), \boldsymbol{n}(\xi)\rangle \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\, dS' \\
&\quad + \frac{T(\boldsymbol{q}, m)}{\mu_k} \int_{\partial Q} \int_{|\eta|<\sigma_p} \Big(\rho_2(\xi, s) \langle (\nabla' \widehat{\Phi}_k)(\xi - h\sigma_p \boldsymbol{n}(\xi)), \boldsymbol{n}(\xi)\rangle \\
&\qquad\qquad - \langle \nabla' \widehat{\Phi}_k(\xi), \boldsymbol{n}(\xi)\rangle \Big) \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\, d\eta\, dS' \\
&\quad + \frac{1}{\boldsymbol{\kappa}_1 \mu_k} \int_{\partial Q} \int_{|\eta|<\sigma_p} \langle (\nabla' \widehat{\Phi}_k)(\xi - h(\sigma_p)\sigma_p \boldsymbol{n}(\xi)), \boldsymbol{n}(\xi)\rangle \\
&\qquad\qquad \times \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\, (\widehat{G}(-h, \frac{\eta}{\sigma_p}) - \boldsymbol{\kappa}_2) \rho_2(\xi, -h\sigma_p)\, d\eta\, dS' \\
&= \frac{T(\boldsymbol{q}, m)}{\mu_k} S(m) \int_{\partial Q} \frac{\partial \widehat{\Phi}_k}{\partial \boldsymbol{n}} \frac{\partial \psi_r}{\partial \boldsymbol{n}}\, dS'\, \sigma_p^m + o(\sigma_p^m).
\end{aligned}$$

Note $h = h(\sigma_p)$. Substituting (7.91) and (7.92) into (7.90) with (7.89), we have (7.87). □

Applying Lemma 7.12 and Lemma 7.13 to (7.79), we get

$$(7.93)\quad (\mu_k(\sigma_p)-\mu_k)\int_Q \widehat{\Phi}_k\psi_r dx' = -T(\boldsymbol{q},m)\,\sigma_p\int_{\partial Q}\frac{\partial\widehat{\Phi}_k}{\partial\boldsymbol{n}}\frac{\partial\psi_r}{\partial\boldsymbol{n}}\,dS' + o(\sigma_p)$$

for $k(j)\leqq k<k(j+1)$, $r(j'')\leqq r<r(j''+1)$. Here the orthogonality condition in Proposition 2.2 implies

$$\begin{aligned}\dim L.H.[(\Phi_{k(j)},\widehat{\Phi}_{k(j)}),(\Phi_{k(j)+1},\widehat{\Phi}_{k(j)+1}),(\Phi_{k(j)+2},\widehat{\Phi}_{k(j)+2}),\\ ...,(\Phi_{k(j+1)-1},\widehat{\Phi}_{k(j+1)-1})],\\ =k(j+1)-k(j)=\widehat{k}(j)\quad\text{in}\quad L^2(D)\times L^2(Q).\end{aligned}$$

It follows that

$$\begin{aligned}&\dim L.H.[\Phi_{k(j)},\Phi_{k(j)+1},\Phi_{k(j)+2},\cdots,\Phi_{k(j+1)-1}]\\ &\quad=d(j'+1)-d(j')=\widehat{d}(j')\quad\text{in}\quad L^2(D),\\ &\dim L.H.[\widehat{\Phi}_{k(j)},\widehat{\Phi}_{k(j)+1},\widehat{\Phi}_{k(j)+2},\cdots,\widehat{\Phi}_{k(j+1)-1}]\\ &\quad=r(j''+1)-r(j'')=\widehat{r}(j'')\quad\text{in}\quad L^2(Q),\end{aligned}$$

because $\widehat{k}(j)=\widehat{d}(j')+\widehat{r}(j'')$.

Now we can take a subset $K\subset\{k\in\mathbb{N}\mid k(j)\leqq k<k(j+1)\}$ such that $\#(K)=\widehat{r}(j'')$ and

$$L.H.[\{\widehat{\Phi}_k\in L^2(Q)\mid k\in K\}]=L.H.[\{\psi_r\in L^2(Q)\mid r(j'')\leqq r<r(j''+1)\}].$$

Hence we see that, for any $k\in K$, there exists r such that $(\widehat{\Phi}_k,\psi_r)_{L^2(Q)}\neq 0$, $r(j'')\leqq r<r(j''+1)$. Therefore, due to (7.93), there exists the limit

$$\lim_{p\to\infty}(\mu_k(\sigma_p)-\mu_k)/\sigma_p\quad(\text{for}\quad k\in K).$$

Denoting this value by $\boldsymbol{\gamma}'_m(k)$ $(k\in K)$, we have,

$$(7.94)\qquad \boldsymbol{\gamma}'_m(k)\int_Q\widehat{\Phi}_k\psi_r dx' = -T(\boldsymbol{q},m)\int_{\partial Q}\frac{\partial\widehat{\Phi}_k}{\partial\boldsymbol{n}}\frac{\partial\psi_r}{\partial\boldsymbol{n}}\,dS'\quad(k\in K)$$

by (7.93). Using

$$\widehat{\Phi}_k=\sum_{r(j'')\leqq q<r(j''+1)}(\widehat{\Phi}_k,\psi_q)_{L^2(Q)}\,\psi_q\qquad(k\in K)$$

we get

$$\begin{aligned}(7.95)\qquad &\boldsymbol{\gamma}_m'(k)(\widehat{\Phi}_k,\psi_r)_{L^2(Q)}\\ &= -T(\boldsymbol{q},m)\sum_{r(j'')\leqq q<r(j''+1)}(\widehat{\Phi}_k,\psi_q)_{L^2(Q)}\int_{\partial Q}\frac{\partial\psi_q}{\partial\boldsymbol{n}}\frac{\partial\psi_r}{\partial\boldsymbol{n}}\,dS'\end{aligned}$$

for $k \in K$. Define the matrix U (invertible) by

$$U = \left(\int_Q \psi_r\,\widehat{\Phi}_k\,dx'\right)_{r(j'')\leqq r<r(j''+1),\,k\in K},$$

we have

$$\begin{aligned}(7.96)\qquad &-T(\boldsymbol{q},m)\,\boldsymbol{B}(j)\,U\\ &= U\begin{pmatrix}\boldsymbol{\gamma}_m'(q(1)) & & & \\ & \boldsymbol{\gamma}_m'(q(2)) & & \\ & & \ddots & \\ & & & \boldsymbol{\gamma}_m'(q(\widehat{r}(j'')))\end{pmatrix}.\end{aligned}$$

where the numbers $q(1) < q(2) < \cdots < q(r(j''))$ are elements of K. This matrix equation (7.96) implies that the values $\boldsymbol{\gamma}_m'(k)$ $(k \in K)$ agree to the eigenvalues of $-T(\boldsymbol{q},m)\boldsymbol{B}(j)$. As $\boldsymbol{B}(j)$ is a positive definite real symmetric matrix, the sign of $-\boldsymbol{\gamma}_m'(k)$ agrees to the sign of the real number $T(\boldsymbol{q},m)$. Particularly, $\boldsymbol{\gamma}_m'(k) \neq 0$ $(k \in K)$ since $T(\boldsymbol{q},m) \neq 0$ (This is one of the assumptions of Thm.2.14).

Next we deal with $\mu_k(\zeta)$ for $k \in K'$ where K' is defined as

$$K' = \{k(j), k(j)+1, \cdots, k(j+1)-1\} \setminus K.$$

LEMMA 7.14. *$\Phi_k(x) \equiv 0$ in D for $k \in K$ and $\widehat{\Phi}_{k'}(x') \equiv 0$ in Q for $k' \in K'$.*

PROOF OF LEMMA 7.14. Assume $k \in K$ and $\|\Phi_k\|_{L^2(D)} > 0$. Then Prop.3.3 implies

$$\mu_k(\sigma_p) - \mu_k = O(\sigma_p^{m/2}).$$

As $m \geqq 3$, $(\mu_k(\sigma_p)-\mu_k)/\sigma_p \to 0$ for $p \to \infty$. This is contrary to $\boldsymbol{\gamma}_m'(k) \neq 0$. Therefore $\Phi_k \equiv 0$ in D for $k \in K$. Hence, $\{\widehat{\Phi}_k \mid k \in K\}$ is orthonormal in

$L^2(Q)$. From these properties combined with (2.9) and that $\{(0, \widehat{\Phi}_k) \mid k \in K\}$ span the $\widehat{r}(j'')$ dimensional subspace, we conclude that $\widehat{\Phi}_{k'} \equiv 0$ in Q for $k' \in K'$. $\square$

We have established the following situation. That is, for $k(j) \leqq k < k(j+1)$,

$$(7.97) \qquad \begin{cases} k \in K' & \Longleftrightarrow \quad \mu_k(\sigma_p) - \mu_k = O(\sigma_p^{m/2}), \\ k \in K & \Longleftrightarrow \quad \limsup_{p\to\infty} \dfrac{|\mu_k(\sigma_p) - \mu_k|}{\sigma_p} < \infty, \\ & \qquad\quad \limsup_{p\to\infty} \dfrac{\sigma_p}{|\mu_k(\sigma_p) - \mu_k|} < \infty. \end{cases}$$

So in this case ($m \geqq 3$, resonant, $T(\boldsymbol{q}, m) \neq 0$), the order of smallness of $\mu_k(\sigma_p) - \mu_k$ clearly depends on whether $k \in K$ or $k \in K'$. As $\boldsymbol{\gamma}'_m(k)$ (for $k \in K$) have a common sign. We also see the following property of K and K'. That is, if $T(\boldsymbol{q}, m) > 0$, then

$$\begin{aligned} K &= \{k(j),\, k(j)+1,\, k(j)+2, \ldots, k(j) + \widehat{r}(j'') - 1\}, \\ K' &= \{k(j) + \widehat{r}(j''),\, k(j) + \widehat{r}(j'') + 1, \ldots, k(j+1) - 1\}. \end{aligned}$$

On the other hand,if $T(\boldsymbol{q}, m) < 0$,

$$\begin{aligned} K' &= \{k(j),\, k(j)+1,\, k(j)+2, \ldots, k(j) + \widehat{d}(j') - 1\}, \\ K &= \{k(j) + \widehat{d}(j'),\, k(j) + \widehat{d}(j') + 1, \ldots, k(j+1) - 1\}. \end{aligned}$$

Thus it has been proved that K and K' are determined uniquely. That is, K, K' do not depend on the choice of $\{\zeta_p\}_{p=1}^{\infty}$ (and $\{\sigma_p\}_{p=1}^{\infty}$), but they are determined only by the sign of $T(\boldsymbol{q}, m)$.

To investigate the detailed asymptotics of the resonant eigenvalues, we need to know more properties of the behaviors of the eigenfunction $\Phi_{k,\zeta}$ for $k \in K$ and $k \in K'$, respectively.

Lemma 7.15. *For $t > 0$, there exists $c_{22}(t) > 0$ such that, for any $k \in K$, it holds that*

$$(7.98) \qquad \|\Phi_{k,\sigma_p}\|_{L^\infty(D \setminus \Sigma^+(t))} \leqq c_{22}(t)\, \sigma_p^{(m/2)-1}.$$

PROOF OF LEMMA 7.15. First we prove

$$\limsup_{p\to\infty} |(\Phi_{k,\sigma_p}, \phi_d)_{L^2(D)}|/\sigma_p^{(m/2)-1} < \infty \qquad (7.99)$$
$$(\text{for} \quad k \in K,\ d(j') \leqq d < d(j'+1)).$$

We extend ϕ_d to be a C^2 function in $\mathbb{R}^n$ (independent of $\zeta > 0$), which we also denote by ϕ_d. Putting $\Psi = \phi_d$ in (2.2) and get

$$\int_D (\nabla\Phi_{k,\sigma_p}\nabla\phi_d - \mu_k(\sigma_p)\,\Phi_{k,\sigma_p}\,\phi_d)\,dx + \int_{Q(\sigma_p)} (\nabla\Phi_{k,\sigma_p}\nabla\phi_d - \mu_k(\sigma_p)\,\Phi_{k,\sigma_p}\,\phi_d)\,dx = 0$$

which leads to

$$(\mu_k(\sigma_p) - \mu_k)\int_D \Phi_{k,\sigma_p}\,\phi_d\,dx = \int_{Q(\sigma_p)} (\nabla\Phi_{k,\sigma_p}\nabla\phi_d - \mu_k(\sigma_p)\,\Phi_{k,\sigma_p}\,\phi_d)\,dx.$$

Therefore we have

$$|(\Phi_{k,\sigma_p}, \phi_d)_{L^2(D)}| \leqq \frac{\|\nabla\Phi_{k,\sigma_p}\|_{L^2(Q(\sigma_p))}\|\nabla\phi_d\|_{L^2(Q(\sigma_p))} + \mu_k(\sigma_p)\|\Phi_{k,\sigma_p}\|_{L^2(Q(\sigma_p))}\|\phi_d\|_{L^2(Q(\sigma_p))}}{|\mu_k(\sigma_p)-\mu_k|}. \qquad (7.100)$$

Using the following properties

$$\lim_{p\to\infty}\frac{\mu_k(\sigma_p)-\mu_k}{\sigma_p} = \gamma'_m(k) \neq 0 \quad \text{for} \quad k \in K,$$
$$\|\nabla\phi_d\|_{L^2(Q(\sigma_p))} = O(\sigma_p^{m/2}), \qquad \|\phi_d\|_{L^2(Q(\sigma_p))} = O(\sigma_p^{m/2}),$$
$$\|\nabla\Phi_{k,\sigma_p}\|_{L^2(Q(\sigma_p))} \leqq \mu_k(\sigma_p)^{1/2}, \qquad \|\Phi_{k,\sigma_p}\|_{L^2(Q(\sigma_p))} \leqq 1$$

in (7.100), we get (7.99). To prove (7.98), we assume that there exists a subsequence $\{\epsilon_p\}_{p=1}^\infty \subset \{\sigma_p\}_{p=1}^\infty$ and $t \in (0, t_0)$ such that

$$\limsup_{p\to\infty}\ \sup_{y\in D\setminus\Sigma^+(t)} |\Phi_{k,\epsilon_p}(y)|/\epsilon_p^{(m/2)-1} = \infty. \qquad (7.101)$$

Define a function $\widetilde{\Phi}_{k,\zeta}$ by

$$\widetilde{\Phi}_{k,\zeta}(x) = \Phi_{k,\zeta}(x)\Big/\left(\sup_{y\in D\setminus\Sigma^+(t)} |\Phi_{k,\zeta}(y)|\right) \quad (x \in \Omega(\zeta)).$$

Then we have $\sup_{x\in D\setminus\Sigma^+(t)}|\widetilde{\Phi}_{k,\zeta}(x)|=1$. This is also an eigenfunction for the eigenvalue $\mu_k(\zeta)$. Using Lem.4.5, we have an estimate for $\widetilde{\Phi}_{k,\zeta}$ which follows from Rem.4.6.

$$\begin{aligned}|\widetilde{\Phi}_{k,\zeta}(x)| &\leqq c_6\left(\sup_{y\in D\setminus\Sigma^+(t_1)}|\widetilde{\Phi}_{k,\zeta}(y)|\right)\\ &\quad + c_7\left(\sup_{y\in\Gamma^-(\zeta,t_2)}|\widetilde{\Phi}_{k,\zeta}(y)|\right)\zeta \quad (x\in\Sigma^+(2\zeta)),\\ |\widetilde{\Phi}_{k,\zeta}(x)| &\leqq c_6\left(\sup_{y\in D\setminus\Sigma^+(t_1)}|\widetilde{\Phi}_{k,\zeta}(y)|\right)\\ &\quad + c_7\left(\sup_{y\in\Gamma^-(\zeta,t_2)}|\widetilde{\Phi}_{k,\zeta}(y)|\right)\frac{\zeta^m}{(s^2+|\eta|^2)^{(m-1)/2}}\end{aligned}$$

$(x=(\xi+s\boldsymbol{n}(\xi),\eta)\in\Sigma^+(t_1)\setminus\Sigma^+(2\zeta))$ where $c_6>0, c_7>0,\ t_1>0, t_2>0$ are constants in Lem.4.5.

Here we assume $0<t\leqq t_1$ without loss of generality, because we can replace t by t_1 if $t>t_1$. Remark that (7.101) still holds when $t>0$ is replaced by a smaller one. $D\setminus\Sigma^+(t_1)\subset D\setminus\Sigma^+(t)$ implies

$$\sup_{y\in D\setminus\Sigma^+(t_1)}|\widetilde{\Phi}_{k,\zeta}(y)|\leqq 1.$$

We have

$$|\widetilde{\Phi}_{k,\zeta}(x)|\leqq c_6+c_7\frac{\sup_{y\in\Gamma^-(\zeta,t_2)}|\Phi_{k,\zeta}(y)|}{\sup_{y\in D\setminus\Sigma^+(t)}|\Phi_{k,\zeta}(y)|}\frac{\zeta^m}{(s^2+|\eta|^2)^{(m-1)/2}}\quad (x\in\Sigma^+(t)).$$

We use the estimate $\|\Phi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}\leqq c\,\zeta^{-m/2}$ for $k(j)\leqq k<k(j+1)$ (cf. Rem.2.3). c is a positive constant which is independent of the parameters. Put $\zeta=\epsilon_p$ in the above inequality and use (7.101) and we see that there exists a constant $c'>0$ such that

$$|\widetilde{\Phi}_{k,\epsilon_p}(x)|\leqq c_6+\frac{c'\epsilon_p}{(s^2+|\eta|^2)^{(m-1)/2}}\qquad \text{in}\quad \Sigma^+(t), \tag{7.102}$$

Applying the regularity argument in the elliptic BVP (cf. Prop. 8.2) with the estimate (7.102), we can show that there exists a subsequence $\{\epsilon_{p'}\}$ such that $\widetilde{\Phi}_{k,\epsilon_{p'}}$ which approaches $\widetilde{\Phi}_k \in C^2(\overline{D})$ in $C^2(\overline{D} \setminus \Sigma^+(\kappa))$ for any $\kappa > 0$ with

$$\Delta \widetilde{\Phi}_k + \mu_k \widetilde{\Phi}_k = 0 \quad \text{in} \quad D, \quad \partial \widetilde{\Phi}_k / \partial \nu = 0 \quad \text{on} \quad \partial D, \qquad \sup_{y \in D \setminus \Sigma^+(t)} |\widetilde{\Phi}_k(y)| = 1.$$

From (7.102) we know that $|\widetilde{\Phi}_{k,\epsilon_p}(x)|$ $(p \geqq 1)$ is uniformly bounded by an integrable function in D from (7.102) and so we apply the Lebesgue's convergence theorem and get

$$\lim_{p' \to \infty} (\widetilde{\Phi}_{k,\epsilon_{p'}}, \phi_d)_{L^2(D)} = (\widetilde{\Phi}_k, \phi_d)_{L^2(D)}.$$

However, from (7.99), (7.101), we have

$$\begin{aligned} &|(\widetilde{\Phi}_{k,\epsilon_p}, \phi_d)_{L^2(D)}| \\ &= \left(|(\widetilde{\Phi}_{k,\epsilon_p}, \phi_d)_{L^2(D)}| / \epsilon_p^{(m/2)-1} \right) \left(\epsilon_p^{(m/2)-1} / \sup_{x \in D \setminus \Sigma^+(t)} |\Phi_{k,\epsilon_p}(x)| \right) \to 0 \end{aligned}$$

for $p \to \infty$ and we obtain

$$(\widetilde{\Phi}_k, \phi_d)_{L^2(D)} = 0 \quad \text{for} \quad d(j') \leqq d < d(j'+1).$$

This property is a contradiction, because $\mu_k \in \{\omega_d\}_{d(j') \leqq d < d(j'+1)}$ and so $\widetilde{\Phi}_k$ belongs to the linear hull of $\{\phi_d\}_{d(j') \leqq d < d(j'+1)}$ and so $\widetilde{\Phi}_k \equiv 0$ in D holds. This is contrary to $\sup_{D \setminus \Sigma^+(t)} |\widetilde{\Phi}_k| = 1$. The estimate (7.98) has been verified. □

Next we look into the behavior of $\Phi_{k,\zeta}$ for $k \in K'$.

LEMMA 7.16. *There exists $c_{23} > 0$ such that, for $k' \in K'$, it holds that*

(7.103) $$\sigma_p \|\Phi_{k',\sigma_p}\|_{L^\infty(\Omega(\sigma_p))} \leqq c_{23},$$

(7.104) $$\begin{cases} \|\Phi_{k',\sigma_p}\|_{L^\infty(D \cup \Sigma^-(\sigma_p, 2\sigma_p))} \leqq c_{23}, \\ (1/h(\sigma_p)) \, \|\Phi_{k',\sigma_p}\|_{L^\infty(\Sigma^-(\sigma_p, h(\sigma_p)\sigma_p))} \leqq c_{23}. \end{cases}$$

Proof of Lemma 7.16. We will prove the estimates (7.103), (7.104) with the aid of the following property (7.105). The orthogonality conditions $(\Phi_{k,\zeta}, \Phi_{k',\zeta})_{L^2(\Omega(\zeta))} = 0$ $(k \in K,\, k' \in K')$ yields

$$\int_{Q(\zeta)} \Phi_{k,\zeta}\, \Phi_{k',\zeta}\, dx = -\int_D \Phi_{k,\zeta}\, \Phi_{k',\zeta}\, dx \quad (k \in K,\, k' \in K'). \tag{7.105}$$

For any $k' \in K'$, put

$$\varrho_\zeta(k') = \sup_{x \in \Omega(\zeta)} |\Phi_{k',\zeta}(x)| > 0.$$

Take any subsequence $\{\epsilon_p\}_{p=1}^{\infty} \subset \{\sigma_p\}_{p=1}^{\infty}$ such that

$$\lim_{p\to\infty} \varrho_{\epsilon_p}(k') = \infty. \tag{7.106}$$

If there is not such subsequence, (7.103) and (7.104) are trivially true and there is nothing to prove. Here we put

$$\widetilde{\Phi}_{k',\zeta}(x) = \Phi_{k',\zeta}(x)/\varrho_\zeta(k')$$

and we have $\sup_{\Omega(\zeta)} |\widetilde{\Phi}_{k',\zeta}(x)| = 1$. From Prop.8.1, there exists a subsequence $\{\tau_p\}_{p=1}^{\infty} \subset \{\epsilon_p\}_{p=1}^{\infty}$ and $\widetilde{\Phi}^*_{k'} \in C^2(\overline{Q})$ such that, for $k' \in K'$

$$\begin{cases} \lim\limits_{p\to\infty} \sup\limits_{x=(x',x'')\in Q(\tau_p)} |\widetilde{\Phi}_{k',\tau_p}(x', x'') - \widetilde{\Phi}^*_{k'}(x')| = 0, \\ \Delta' \widetilde{\Phi}^*_{k'} + \mu_{k'} \widetilde{\Phi}^*_{k'} = 0 \quad \text{in} \quad Q, \quad \widetilde{\Phi}^*_{k'} = 0 \quad \text{on} \quad \partial Q, \\ \lim\limits_{p\to\infty} \sup\limits_{x\in D} |\widetilde{\Phi}_{k',\tau_p}(x)| = 0, \quad \sup\limits_{x'\in Q} |\widetilde{\Phi}^*_{k'}(x')| = 1. \end{cases} \tag{7.107}$$

We used

$$\|\widetilde{\Phi}_{k',\epsilon_p}\|_{L^2(D)} = \|\Phi_{k',\epsilon_p}\|_{L^2(D)}/\varrho_{\epsilon_p}(k') \leqq 1/\varrho_{\epsilon_p}(k') \to 0 \quad (p \to \infty)$$

(due to (7.106)). Since $\widetilde{\Phi}^*_{k'}$ belongs to the linear hull of the orthonormal system $\{\widehat{\Phi}_k\}_{k\in K}$ in $L^2(Q)$ (from (7.107)), we have

$$\widetilde{\Phi}^*_{k'}(x') = \sum_{k\in K} \chi(k', k) \widehat{\Phi}_k(x') \quad (x' \in Q), \quad \text{where}$$
$$\chi(k', k) = (\widetilde{\Phi}^*_{k'}, \widehat{\Phi}_k)_{L^2(Q)} \quad (k \in K).$$

Here note that $\sum_{k\in K}\chi(k',k)^2>0$. (7.105) gives

$$\begin{aligned}\sum_{k\in K}&\chi(k',k)(\Phi_{k,\tau_p}\,\Phi_{k',\tau_p})_{L^2(Q(\tau_p))}\\&=-\sum_{k\in K}\chi(k',k)(\Phi_{k,\tau_p}\,\Phi_{k',\tau_p})_{L^2(D)}.\end{aligned}\tag{7.108}$$

First we denote the left hand side of (7.108) by $I_1(k',\tau_p)$.

$$\begin{aligned}I_1(k',\tau_p)&=\sum_{k\in K}\chi(k',k)\int_{Q(\tau_p)}\Phi_{k,\tau_p}\,\Phi_{k',\tau_p}dx\\&=\varrho_{\tau_p}(k')\sum_{k\in K}\chi(k',k)\int_{Q(\tau_p)}\Phi_{k,\tau_p}\,\widetilde{\Phi}_{k',\tau_p}dx\\&=\varrho_{\tau_p}(k')\sum_{k\in K}\chi(k',k)\int_{Q(\tau_p)}S(m)^{-1/2}\tau_p^{-m/2}\widehat{\Phi}_{k,\tau_p}\,\widetilde{\Phi}_{k',\tau_p}dx\\&=\varrho_{\tau_p}\sum_{k\in K}\chi(k',k)S(m)^{-1/2}\tau_p^{-m/2}\\&\quad\times\int_{Q(\tau_p)}\Big(\widehat{\Phi}_k\widetilde{\Phi}^*_{k'}+(\widehat{\Phi}_{k,\tau_p}-\widehat{\Phi}_k)\,\widetilde{\Phi}^*_{k'}+\widehat{\Phi}_{k,\tau_p}(\widetilde{\Phi}_{k',\tau_p}-\widetilde{\Phi}^*_{k'})\Big)\,dx.\end{aligned}$$

Hence we have

$$\begin{aligned}&I_1(k',\tau_p)-\varrho_{\tau_p}(k')\sum_{k\in K}\chi(k',k)S(m)^{1/2}\tau_p^{m/2}\int_Q\widehat{\Phi}_k\widetilde{\Phi}^*_{k'}dx'\\&=\varrho_{\tau_p}(k')\sum_{k\in K}S(m)^{-1/2}\tau_p^{-m/2}\int_{Q(\tau_p)\setminus(Q\times B^{(m)}(\tau_p))}\widehat{\Phi}_k\widetilde{\Phi}^*_{k'}dx\\&\quad+\varrho_{\tau_p}(k')\sum_{k\in K}\chi(k',k)S(m)^{-1/2}\tau_p^{-m/2}\\&\quad\times\int_{Q(\tau_p)}\Big((\widehat{\Phi}_{k,\tau_p}-\widehat{\Phi}_k)\,\widetilde{\Phi}^*_{k'}+\widehat{\Phi}_{k,\tau_p}(\widetilde{\Phi}_{k',\tau_p}-\widetilde{\Phi}^*_{k'})\Big)\,dx\end{aligned}$$

and we get from (2.6), (7.107)

$$\theta(\tau_p)\equiv I_1(k',\tau_p)/\varrho_{\tau_p}(k')\tau_p^{m/2}-S(m)^{1/2}\sum_{k\in K}\chi(k',k)^2\to 0\quad(p\to\infty)$$

which implies

$$\varrho_{\tau_p}(k')\tau_p^{m/2}=\frac{I_1(k',\tau_p)}{S(m)^{1/2}\sum_{k\in K}\chi(k',k)^2+\theta(\tau_p)}.\tag{7.109}$$

Denote the right hand side of (7.108) by $I_2(k',\tau_p)$. We use Rem.4.6 to estimate $\Phi_{k,\zeta}$ ($k\in K$) and $\Phi_{k',\zeta}$ ($k'\in K'$) by applying Rem.4.6 and Lem.7.15 and

$$\|\Phi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))} \leqq c\zeta^{-m/2}, \quad \|\Phi_{k',\zeta}\|_{L^\infty(D\setminus\Sigma^+(t_1))} \leqq c'(t_1).$$

We have

$$\begin{cases} |\Phi_{k,\sigma_p}(x)| \leqq c_6\, c_{22}(t_1)\, \sigma_p^{(m/2)-1} + c_7\, c\, \sigma_p^{-m/2+1} \quad (x\in\Sigma^+(2\sigma_p)), \\ |\Phi_{k,\sigma_p}(x)| \leqq c_6\, c_{22}(t_1)\, \sigma_p^{(m/2)-1} + \dfrac{c_7\, c\, \sigma_p^{m/2}}{(s^2+|\eta|^2)^{(m-1)/2}} \\ \hfill (x\in\Sigma^+(t_1)\setminus\Sigma^+(2\sigma_p)), \end{cases}$$

for $k\in K$ and

$$(7.110)\qquad \begin{cases} |\Phi_{k',\sigma_p}(x)| \leqq c_6\, c'(t_1) + c_7\, \varrho_{\sigma_p}(k')(-s+\sigma_p) \\ \hfill (x\in\Sigma^-(\sigma_p,h(\sigma_p))), \\ |\Phi_{k',\sigma_p}(x)| \leqq c_6\, c'(t_1) + c_7\, \varrho_{\sigma_p}(k')\, \sigma_p \qquad (x\in\Sigma^+(2\sigma_p)), \\ |\Phi_{k',\sigma_p}(x)| \leqq c_6\, c'(t_1) + \dfrac{c_7\, \varrho_{\sigma_p}(k')\sigma_p^m}{(s^2+|\eta|^2)^{(m-1)/2}} \\ \hfill (x\in\Sigma^+(t_1)\setminus\Sigma^+(2\sigma_p)), \end{cases}$$

for $k'\in K'$.

Here $x=(\xi+s\boldsymbol{n}(\xi),\eta)$ in $\Sigma^-(\sigma_p,h(\sigma_p)\sigma_p)\cup\Sigma^+(t_1)$. Using these estimates, we have, for $k\in K$, $k'\in K'$,

$$\begin{aligned} |(\Phi_{k,\sigma_p}\,\Phi_{k',\sigma_p})_{L^2(D)}| &\leqq \int_{D\setminus\Sigma^+(t_1)} |\Phi_{k,\sigma_p}|\,|\Phi_{k',\sigma_p}|\,dx \\ &\quad + \int_{\Sigma^+(2\sigma_p)} |\Phi_{k,\sigma_p}|\,|\Phi_{k',\sigma_p}|\,dx \\ &\quad + \int_{\Sigma^+(t_1)\setminus\Sigma^+(2\sigma_p)} |\Phi_{k,\sigma_p}|\,|\Phi_{k',\sigma_p}|\,dx \\ &\leqq \int_{D\setminus\Sigma^+(t_1)} c_{22}(t_1)\,\sigma_p^{(m/2)-1}\,c\,dx \\ &\quad + \int_{\Sigma^+(2\sigma_p)} (c_6\,c_{22}(t_1)\,\sigma_p^{(m/2)-1} + c_7\,c\,\sigma_p^{1-(m/2)}) \\ &\qquad\qquad \times (c_6\,c'(t_1) + c_7\,\varrho_{\sigma_p}(k')\,\sigma_p)\,dx \end{aligned}$$

$$+ \int_{\Sigma^+(t_1)\setminus\Sigma^+(2\sigma_p)} \left(c_6 c_{22}(t_1) + \frac{c_7\, c\, \sigma_p^{m/2}}{(s^2+|\eta|^2)^{(m-1)/2}} \right) \times \left(c_6 c'(t_1) + \frac{c_7 \varrho_{\sigma_p}(k')\sigma_p^m}{(s^2+|\eta|^2)^{(m-1)/2}} \right) dx$$
$$\leqq c''(t_1)\,(\sigma_p^{(m/2)-1} + \varrho_{\sigma_p}\,\sigma_p^{(m/2)+1}).$$

As $I_1(k',\tau_p) = I_2(k',\tau_p)$ (i.e. (7.108)) and $\{\tau_p\}$ is a subsequence of $\{\sigma_p\}$, we combine (7.109) and this inequality

$$\varrho_{\tau_p}(k')\,\tau_p^{m/2} = \frac{|I_2(k',\tau_p)|}{S(m)^{1/2}\sum_{k\in K}\chi(k',k)^2 + \theta(\tau_p)} \leqq \frac{\sum_{k\in K} c''(t_1)\,(\tau_p^{(m/2)-1} + \varrho_{\tau_p}\tau_p^{(m/2)+1})}{S(m)^{1/2}\sum_{k\in K}\chi(k',k)^2 + \theta(\tau_p)}.$$

We get

$$\varrho_{\tau_p}(k')\tau_p \leqq \frac{\sum_{k\in K} c''(t_1)\,(1+\varrho_{\tau_p}\tau_p^2)}{S(m)^{1/2}\sum_{k\in K}\chi(k',k)^2 + \theta(\tau_p)}$$

which leads to

$$\limsup_{p\to\infty} \varrho_{\tau_p}(k')\tau_p < \infty \quad (k' \in K').$$

Since $\{\epsilon_p\}$ was an arbitrary subsequence of $\{\sigma_p\}$, the above estimate implies

$$\limsup_{p\to\infty} \varrho_{\sigma_p}(k')\sigma_p < \infty \quad (k' \in K')$$

which implies (7.103). (7.104) follows immediately from (7.103) and (7.110). □

Recall $\widehat{\Phi}^*_{k,\zeta}(x) = \zeta\Phi_{k,\zeta}(x)$. Then, due to (7.77) and Lemma 7.16, we have

$$\limsup_{p\to\infty} \sup_{x\in\Omega(\sigma_p)} |\widehat{\Phi}^*_{k',\sigma_p}(x)| < \infty \quad (k' \in K'), \tag{7.111}$$

$$\lim_{p\to\infty} \sup_{x\in D} |\widehat{\Phi}^*_{k',\sigma_p}(x)| = 0 \quad (k' \in K'). \tag{7.112}$$

About the behavior of $\widehat{\Phi}^*_{k',\sigma_p}(x)$ in $Q(\sigma_p)$, we have the following result.

LEMMA 7.17. *There exists a subsequence* $\{\epsilon_p\}_{p=1}^{\infty} \subset \{\sigma_p\}_{p=1}^{\infty}$ *and* $\widehat{\Phi}^*_{k'} \in C^2(\overline{Q})$ $(k' \in K')$ *such that for* $k' \in K'$

$$\lim_{p\to\infty} \sup_{(x',x'')\in Q(\epsilon_p)} |\widehat{\Phi}^*_{k',\epsilon_p}(x',x'') - \widehat{\Phi}^*_{k'}(x')| = 0, \tag{7.113}$$

$$\Delta'\widehat{\Phi}^*_{k'} + \mu_{k'}\widehat{\Phi}^*_{k'} = 0 \quad in \quad Q, \quad \widehat{\Phi}^*_{k'} = 0 \quad on \quad \partial Q. \tag{7.114}$$

PROOF OF LEMMA 7.17. These results follow from the condition (7.111) and Prop. 8.1. □

We use the equations (7.78) and (7.79) to deduce a relation between $\Phi_{k'}$, $\widehat{\Phi}^*_{k'}$ $(k' \in K')$. Put $\zeta = \epsilon_p$ in (7.78) for $k' \in K', d(j') \leqq d < d(j'+1)$ and use $\Phi_{k',\epsilon_p} = \epsilon_p^{-1}\widehat{\Phi}^*_{k',\epsilon_p}$ in (7.78) and we get

$$\begin{aligned}
&(\mu_{k'}(\epsilon_p) - \mu_{k'}) \int_{D\setminus\Sigma^+(2\epsilon_p)} \Phi_{k',\epsilon_p}\widetilde{\phi}_{d,\epsilon_p} dx \\
&= -\mu_{k'}(\epsilon_p)\epsilon_p^{-1} \int_{Q(\epsilon_p)\setminus\Sigma^-(\epsilon_p,2\epsilon_p)} \widehat{\Phi}^*_{k',\epsilon_p}\widetilde{\phi}_{d,\epsilon_p} dx \\
&\quad - \mu_{k'}(\epsilon_p) \int_{\Sigma^-(\epsilon_p,2\epsilon_p)\cup\Sigma^+(2\epsilon_p)} \Phi_{k',\epsilon_p}\widetilde{\phi}_{d,\epsilon_p} dx \\
&\quad + \int_{\Gamma^-(\epsilon_p,2\epsilon_p)} \Phi_{k',\epsilon_p} \frac{\partial\widetilde{\phi}_{d,\epsilon_p}}{\partial\nu_2} dS \\
&\quad + \int_{\partial(\Sigma^-(\epsilon_p,2\epsilon_p)\cup\Sigma^+(2\epsilon_p))} \Phi_{k',\epsilon_p} \frac{\partial\widetilde{\phi}_{d,\epsilon_p}}{\partial\nu_1} dS \\
&\quad - \int_{\Sigma^-(\epsilon_p,2\epsilon_p)\cup\Sigma^+(2\epsilon_p)} \Phi_{k',\epsilon_p}\Delta\widetilde{\phi}_{d,\epsilon_p} dx \\
&\quad + \int_{\Gamma^+(2\epsilon_p)} \Phi_{k',\epsilon_p} \frac{\partial\widetilde{\phi}_{d,\epsilon_p}}{\partial\nu_3} dS.
\end{aligned} \tag{7.115}$$

To evaluate both sides of this expression, we need several estimates.

LEMMA 7.18. *There exists a constant* $c_{24} > 0$ *such that*

$$\left| \int_{\Sigma^-(\epsilon_p,2\epsilon_p)\cup\Sigma^+(2\epsilon_p)} \Phi_{k',\epsilon_p}\widetilde{\phi}_{d,\epsilon_p}\, dx \right| \leqq c_{24}\, \epsilon_p^{m+1}, \tag{7.116}$$

$$\left|\int_{\Sigma^-(\epsilon_p,2\epsilon_p)\cup\Sigma^+(2\epsilon_p)} \Phi_{k',\epsilon_p}\Delta\widetilde{\phi}_{d,\epsilon_p}\,dx\right| \leqq c_{24}\,\epsilon_p^{m+1}, \tag{7.117}$$

$$\left|\int_{\Gamma^-(\epsilon_p,2\epsilon_p)} \Phi_{k',\epsilon_p}\frac{\partial\widetilde{\phi}_{d,\epsilon_p}}{\partial\nu_2}\,dS\right| \leqq c_{24}\,\epsilon_p^{m}, \tag{7.118}$$

$$\left|\int_{\partial(\Sigma^-(\epsilon_p,2\epsilon_p)\cup\Sigma^+(2\epsilon_p))} \Phi_{k',\epsilon_p}\frac{\partial\widetilde{\phi}_{d,\epsilon_p}}{\partial\nu_1}\,dS\right| \leqq c_{24}\,\epsilon_p^{m}, \tag{7.119}$$

$$\left|\int_{\Gamma^+(2\epsilon_p)} \Phi_{k',\epsilon_p}\frac{\partial\widetilde{\phi}_{d,\epsilon_p}}{\partial\nu_3}\,dS\right| \leqq c_{24}\,\epsilon_p^{m} \tag{7.120}$$

for $k' \in K'$.

PROOF OF LEMMA 7.18. (7.116)-(7.120) follow from Lem.4.3-(ii) and Lem.7.16. □

LEMMA 7.19. *For $k' \in K'$ and $d(j') \leqq d < d(j'+1)$, it holds that*

$$\int_{D\setminus\Sigma^+(2\epsilon_p)} \Phi_{k',\epsilon_p}\widetilde{\phi}_{d,\epsilon_p}dx = \int_D \Phi_{k'}\phi_d\,dx + o(1), \tag{7.121}$$

$$\begin{aligned}\int_{Q(\epsilon_p)\setminus\Sigma^-(\epsilon_p,2\epsilon_p)} &\widehat{\Phi}^*_{k',\epsilon_p}\widetilde{\phi}_{d,\epsilon_p}dx \\ &= (-\frac{1}{\mu_k})S(m)\int_{\partial Q}\frac{\partial\widehat{\Phi}^*_{k'}}{\partial\boldsymbol{n}}(\xi)\,\phi_d(\xi,o'')\,dS'\,\epsilon_p^m + o(\epsilon_p^m),\end{aligned} \tag{7.122}$$

for $p \to \infty$.

PROOF OF LEMMA 7.19. (7.121) is clear from the definition of $\widetilde{\phi}_{d,\epsilon_p}$ and Lem.7.16. We deal with (7.122).

$$\begin{aligned}&\int_{Q(\epsilon_p)\setminus\Sigma^-(\epsilon_p,2\epsilon_p)} \widehat{\Phi}^*_{k',\epsilon_p}\,\widetilde{\phi}_{d,\epsilon_p}\,dx \\ &= \int_{Q(\epsilon_p)\setminus\Sigma^-(\epsilon_p,2\epsilon_p)} \widehat{\Phi}^*_{k}\,\widetilde{\phi}_{d,\epsilon_p}\,dx + \int_{Q(\epsilon_p)\setminus\Sigma^-(\epsilon_p,2\epsilon_p)} (\widehat{\Phi}^*_{k',\epsilon_p} - \widehat{\Phi}^*_{k'})\,\widetilde{\phi}_{d,\epsilon_p}\,dx \\ &= (-\frac{1}{\mu_k})\int_{Q(\epsilon_p)\setminus\Sigma^-(\epsilon_p,2\epsilon_p)} \Delta\widehat{\Phi}^*_{k'}\,\widetilde{\phi}_{d,\epsilon_p}\,dx \\ &\quad + \int_{Q(\epsilon_p)\setminus\Sigma^-(\epsilon_p,2\epsilon_p)} (\widehat{\Phi}^*_{k',\epsilon_p} - \widehat{\Phi}^*_{k'})\,\widetilde{\phi}_{d,\epsilon_p}\,dx \qquad (\equiv I_1(\epsilon_p) + I_2(\epsilon_p)).\end{aligned}$$

In this calculation we used that $\widetilde{\phi}_{d,\epsilon_p}$ is harmonic in $Q(\epsilon_p) \setminus \Sigma^-(\epsilon_p, 2\epsilon_p)$. Due to the property (7.113), we have

$$(7.123) \qquad |I_2(\epsilon_p)| \leqq \sup_{x \in Q(\epsilon_p)} |\widehat{\Phi}^*_{k',\epsilon_p}(x) - \widehat{\Phi}^*_{k'}(x)| \int_{Q(\epsilon_p)} |\widetilde{\phi}_{d,\epsilon_p}|\, dx = o(\epsilon_p^m).$$

$I_1(\epsilon_p)$ is calculated as follows.

$$\begin{aligned}
(7.124) \qquad & I_1(\epsilon_p) \\
&= -\frac{1}{\mu_k} \int_{\Gamma^-(\epsilon_p,2\epsilon_p)} \left(\frac{\partial \widehat{\Phi}^*_{k'}}{\partial \nu_2} \widetilde{\phi}_{d,\epsilon_p} - \widehat{\Phi}^*_{k'} \frac{\partial}{\partial \nu_2} \widetilde{\phi}_{d,\epsilon_p} \right) dS \\
&= -\frac{1}{\mu_k} \int_{\Gamma^-(\epsilon_p,2\epsilon_p)} \left(\langle \nabla \widehat{\Phi}^*_{k'}, \nu_2 \rangle \, \widetilde{\phi}_{d,\epsilon_p} - \widehat{\Phi}^*_{k'} \frac{\partial}{\partial \nu_2} \widetilde{\phi}_{d,\epsilon_p} \right) dS \\
&= -\frac{1}{\mu_k} \int_{\Gamma^-(\epsilon_p,2\epsilon_p)} \Big(\langle (\nabla' \widehat{\Phi}^*_{k'})(\xi - 2\epsilon_p \boldsymbol{n}(\xi)), \boldsymbol{n}(\xi) \rangle \, \phi_d(\xi, o'') \\
&\qquad\qquad - \widehat{\Phi}^*_{k'} \frac{\partial}{\partial \nu_2} \widetilde{\phi}_{d,\epsilon_p} \Big) \, dS \\
&= -\frac{1}{\mu_k} \int_{\Gamma^-(\epsilon_p,2\epsilon_p)} \langle \nabla' \widehat{\Phi}^*_{k'}(\xi), \boldsymbol{n}(\xi) \rangle \, \phi_d(\xi, o'') \, dS \\
&\quad - \frac{1}{\mu_k} \int_{\Gamma^-(\epsilon_p,2\epsilon_p)} \langle (\nabla' \widehat{\Phi}^*_{k'})(\xi - 2\epsilon_p \boldsymbol{n}(\xi)) \\
&\qquad\qquad - \nabla' \widehat{\Phi}^*_{k'}(\xi), \boldsymbol{n}(\xi) \rangle \, \phi_d(\xi, o'') \, dS \\
&\quad + \frac{1}{\mu_k} \int_{\Gamma^-(\epsilon_p,2\epsilon_p)} \widehat{\Phi}^*_{k'} \frac{\partial}{\partial \nu_2} \widetilde{\phi}_{d,\epsilon_p} \, dS \\
&= -\frac{1}{\mu_k} \int_{\Gamma^-(\epsilon_p,2\epsilon_p)} \langle \nabla' \widehat{\Phi}^*_{k'}(\xi), \boldsymbol{n}(\xi) \rangle \, \phi_d(\xi, o'') \, dS + O(\epsilon_p^{m+1})
\end{aligned}$$

$$\begin{aligned}
(7.125) \qquad & \int_{\Gamma^-(\epsilon_p,2\epsilon_p)} \langle \nabla' \widehat{\Phi}^*_{k'}(\xi), \boldsymbol{n}(\xi) \rangle \phi_d(\xi, o'') \, dS \\
&= \int_{\partial Q} \int_{|\eta| < \epsilon_p} \frac{\partial \widehat{\Phi}^*_{k'}}{\partial \boldsymbol{n}}(\xi) \, \phi_d(\xi, o'') \, \rho_2(\xi, -2\epsilon_p) \, d\eta \, dS' \\
&= \int_{\partial Q} \int_{|\eta| < \epsilon_p} \frac{\partial \widehat{\Phi}^*_{k'}}{\partial \boldsymbol{n}}(\xi) \, \phi_d(\xi, o'') \, d\eta \, dS'
\end{aligned}$$

$$+ \int_{\partial Q} \int_{|\eta|<\epsilon_p} \frac{\partial \widehat{\Phi}^*_{k'}}{\partial \boldsymbol{n}}(\xi)\, \phi_d(\xi, o'')\, (\rho_2(\xi, -2\epsilon_p) - 1)\, d\eta\, dS'$$

$$= \int_{\partial Q} S(m)\, \epsilon_p^m \frac{\partial \widehat{\Phi}^*_{k'}}{\partial \boldsymbol{n}}(\xi)\, \phi_d(\xi, o'')\, dS' + O(\epsilon_p^{m+1})$$

From (7.123), (7.124), (7.125), we get (7.122). □

Applying Lemma 7.18 and Lemma 7.19 we have

$$(7.126) \qquad (\mu_{k'}(\epsilon_p) - \mu_{k'}) \int_D \Phi_{k'}\, \phi_d\, dx = S(m) \int_{\partial Q} \frac{\partial \widehat{\Phi}^*_{k'}}{\partial \boldsymbol{n}}(\xi)\, \phi_d(\xi, o'')\, dS'\, \epsilon_p^{m-1} + o(\epsilon_p^{m-1})$$

for $d(j') \leqq d < d(j'+1), k' \in K'$.

As $\{\Phi_{k'}\}_{k'\in K'}$ and $\{\phi_d\}_{d(j')\leqq d<d(j'+1)}$ span the common subspace and hence, for any $k' \in K'$, there exists d with $d(j') \leqq d < d(j'+1)$ and $(\Phi_{k'}, \phi_d)_{L^2(D)} \neq 0$. By using (7.126), we have the existence of the following limit

$$\lim_{p\to\infty} \frac{\mu_{k'}(\epsilon_p) - \mu_{k'}}{\epsilon_p^{m-1}} \quad (\text{for } k' \in K').$$

By denoting this value by $\boldsymbol{\gamma}'_m(k')$ for $k' \in K'$, we have

$$(7.127) \qquad \boldsymbol{\gamma}'_m(k') \int_D \Phi_{k'}\, \phi_d\, dx = S(m) \int_{\partial Q} \frac{\partial \widehat{\Phi}^*_{k'}}{\partial \boldsymbol{n}}(\xi)\, \phi_d(\xi, o'')\, dS'$$

$k' \in K'$, $d(j') \leqq d < d(j'+1)$ from (7.126).

On the other hand, substituting $\Phi_{k',\zeta} = (1/\zeta)\widehat{\Phi}^*_{k',\zeta}$ into the first, second and the last term of the right hand side of (7.79), we have the following relation by a simple calculation.

$$(7.128) \qquad (\mu_k(\zeta) - \mu_k) \int_{Q(\zeta)} \Phi_{k,\zeta}\, \widetilde{\psi}_r\, dx = \int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}^*_{k,\zeta} \frac{\partial v_{r,\zeta}}{\partial \nu_2}\, dS + \int_{\Gamma^-(\zeta,h\zeta)} \widehat{\Phi}^*_{k,\zeta} \frac{\partial v_{r,\zeta}}{\partial \nu_1}\, dS + \zeta \int_{\Gamma^+(t)} \Phi_{k,\zeta} \frac{\partial v_{r,\zeta}}{\partial \nu_1}\, dS$$

$$
\begin{aligned}
&+ \int_{\partial(\Sigma^-(\zeta,h\zeta)\cup\Sigma^+(t))\setminus(\Gamma^-(\zeta,h\zeta)\cup\Gamma^+(t))} \Phi_{k,\zeta}\,\frac{\partial\widetilde{\psi}_{r,\zeta}}{\partial\nu_1}\,dS \\
&- \zeta\int_{\Sigma^-(\zeta,h\zeta)} \Phi_{k,\zeta}\,\Delta v_{r,\zeta}\,dx - \zeta\int_{\Sigma^+(t)} \Phi_{k,\zeta}\Delta v_{r,\zeta}dx \\
&+ \zeta\int_{\Gamma^+(t)} \frac{\partial\Phi_{k,\zeta}}{\partial\nu_3}v_{r,\zeta}\,dS - \mu_k(\zeta)\zeta\int_{\Sigma^+(t)\cup\Sigma^-(\zeta,h\zeta)} \Phi_{k,\zeta}\,v_{r,\zeta}\,dx \\
&- \mu_k(\zeta)\int_{Q(\zeta)\setminus\Sigma^-(\zeta,h\zeta)} \widehat{\Phi}^*_{k,\zeta}\,v_{r,\zeta}\,dx.
\end{aligned}
$$

In the above calculation, we used

$$
\begin{aligned}
\int_{\Gamma^+(t)} \Phi_{k,\zeta}\,\frac{\partial v_{r,\zeta}}{\partial\nu_3}\,dS &= \int_{D\setminus\Sigma^+(t)} \operatorname{div}(\Phi_{k,\zeta}\nabla v_{r,\zeta})\,dx = \int_{D\setminus\Sigma^+(t)} \nabla\Phi_{k,\zeta}\nabla v_{r,\zeta}\,dx \\
&= \int_{\Gamma^+(t)} \frac{\partial\Phi_{k,\zeta}}{\partial\nu_3}\,v_{r,\zeta}\,dS - \int_{D\setminus\Sigma^+(t)} \Delta\Phi_{k,\zeta}\,v_{r,\zeta}\,dx \\
&= \int_{\Gamma^+(t)} \frac{\partial\Phi_{k,\zeta}}{\partial\nu_3}v_{r,\zeta}\,dS + \mu_k(\zeta)\int_{D\setminus\Sigma^+(t)} \Phi_{k,\zeta}\,v_{r,\zeta}\,dx.
\end{aligned}
$$

To evaluate the terms in (7.128) for $\zeta = \epsilon_p$, we need the following estimates.

Lemma 7.20. *There exists a constant $c_{25} > 0$ such that, for $k' \in K'$, we have the following estimates.*

$$
\left|\int_{Q(\sigma_p)} \Phi_{k',\sigma_p}\widetilde{\psi}_r\,dx\right| \leqq c_{25}\,\sigma_p^{m-1}, \tag{7.129}
$$

$$
\begin{cases}
\left|\displaystyle\int_{\Gamma^-(\sigma_p,h(\sigma_p)\sigma_p)} \widehat{\Phi}^*_{k',\sigma_p}\,\frac{\partial v_{r,\sigma_p}}{\partial\nu_2}\,dS\right| \leqq c_{25}\,(\log\sigma_p)^2\sigma_p^{m+1}, \\
\left|\displaystyle\int_{\Gamma^-(\sigma_p,h(\sigma_p)\sigma_p)} \widehat{\Phi}^*_{k',\sigma_p}\,\frac{\partial v_{r,\sigma_p}}{\partial\nu_1}\,dS\right| \leqq c_{25}\,(\log\sigma_p)^2\sigma_p^{m+1},
\end{cases} \tag{7.130}
$$

$$
\begin{aligned}
&\left|\int_{\partial(\Sigma^-(\sigma_p,h(\sigma_p)\sigma_p)\cup\Sigma^+(t))\setminus(\Gamma^-(\sigma_p,h(\sigma_p)\sigma_p)\cup\Gamma^+(t))} \Phi_{k',\sigma_p}\,\frac{\partial\widetilde{\psi}_{r,\sigma_p}}{\partial\nu_1}\,dS\right| \\
&\leqq c_{25}\,\sigma_p^{m+1},
\end{aligned} \tag{7.131}
$$

$$
\left|\int_{\Sigma^-(\sigma_p,h(\sigma_p)\sigma_p)} \Phi_{k',\sigma_p}\Delta v_{r,\sigma_p}\,dx\right| \leqq c_{25}\,(\log\sigma_p)^4\sigma_p^{m+2}, \tag{7.132}
$$

$$\left|\int_{\Sigma^-(\sigma_p,h(\sigma_p)\sigma_p)} \Phi_{k',\sigma_p}\, v_{r,\sigma_p}\, dx\right| \leqq c_{25}\,(\log\sigma_p)^4\,\sigma_p^{m+2}. \tag{7.133}$$

PROOF OF LEMMA 7.20. (7.129) is obvious from Lem.7.16. (7.130) follows from Lem.4.3-(ii) and Lem.7.17. (7.131) follows from Lem.4.3-(ii)-(4.31) and Lem.7.16. We deal with (7.132). From (7.104) in Lem.7.16, we note that

$$|\Phi_{k',\sigma_p}(x)| \leqq ch(\sigma_p) \quad (x\in\Sigma^-(\sigma_p,h(\sigma_p)\sigma_p))$$

for some constant $c>0$. (7.132) is estimated as follows.

$$\begin{aligned}
&\left|\int_{\Sigma^-(\sigma_p,h(\sigma_p)\sigma_p)} \Phi_{k',\sigma_p}\Delta v_{r,\sigma_p}\, dx\right| \leqq \int_{\Sigma^-(\sigma_p,h(\sigma_p)\sigma_p)} |\Phi_{k,\sigma_p}|\,|\Delta v_{r,\sigma_p}|\, dx\\
&\leqq \int_{\partial Q}\int_{\widetilde{\Sigma}^-(\sigma_p,h(\sigma_p)\sigma_p)} c\,h(\sigma_p)\,c_2\,(1+(1/\sigma_p)e^{\delta s/\sigma_p})\,\rho_1(\xi,s)\,ds\,d\eta\,dS'\\
&\leqq 2\,c\,h(\sigma_p)\,c_2\int_{\partial Q}\int_{\widetilde{\Sigma}^-(\sigma_p,h(\sigma_p)\sigma_p)} (1+(1/\sigma_p)e^{\delta s/\sigma_p})\,ds\,d\eta\,dS'\\
&\leqq c'\,h(\sigma_p)^2\,\sigma_p^{m+1}
\end{aligned}$$

for some constant $c'>0$. This estimate leads to (7.132). (7.133) follows from Lem.4.3-(ii)-(4.32) and Lem.7.16-(7.104). □

To estimate other terms in (7.128) we prepare the followings estimates.

LEMMA 7.21. *We have*

$$\begin{aligned}
&\int_{Q(\epsilon_p)\setminus\Sigma^-(\epsilon_p,h(\epsilon_p)\epsilon_p)} \widehat{\Phi}^*_{k',\epsilon_p}\, v_{r,\epsilon_p}\, dx\\
&\qquad = \frac{T(\boldsymbol{q},m)S(m)}{\mu_k}\int_{\partial Q} \frac{\partial\widehat{\Phi}_{k'}}{\partial\boldsymbol{n}}\frac{\partial\psi_r}{\partial\boldsymbol{n}} dS'\,\epsilon_p^m + o(\epsilon_p^m).
\end{aligned} \tag{7.134}$$

There exists a constant $c_{26}>0$ such that the functions $\Upsilon_i(k,r,\sigma_p,t)$ $(i=6,7,8,9)$ defined by

$$\begin{aligned}
\int_{\Gamma^+(t)} \Phi_{k',\sigma_p}\frac{\partial v_{r,\sigma_p}}{\partial\nu_1}\, dS &= S(m)\int_{\partial Q}\Phi_{k'}(\xi,o'')\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,dS'\sigma_p^{m-1}\\
&\qquad + \Upsilon_6(k,r,\sigma_p,t),
\end{aligned} \tag{7.135}$$

$$\Upsilon_7(k,r,\sigma_p,t)=\int_{\Sigma^+(t)}\Phi_{k',\sigma_p}\,\Delta v_{r,\sigma_p}\,dx,$$

(7.136)

$$\Upsilon_8(k,r,\sigma_p,t)=\int_{\Sigma^+(t)}\Phi_{k',\zeta}\,v_{r,\sigma_p}\,dx,$$

(7.137) $$\Upsilon_9(k,r,\sigma_p,t)=\frac{1}{\sigma_p}\int_{\Gamma^+(t)}\frac{\partial\Phi_{k',\sigma_p}}{\partial\nu_3}v_{r,\sigma_p}\,dS$$

satisfy the property

(7.138) $$\limsup_{p\to\infty}\frac{|\Upsilon_i(k,r,\sigma_p,t)|}{\sigma_p^{m-1}}\leqq c_{26}\,t\quad(i=6,7,8,9).$$

Proof of Lemma 7.21. We can deal with (7.134) similarly as Lem.6.4-(6.25). So we omit its proof. We calculate (7.135).

$$\begin{aligned}
&\int_{\Gamma^+(t)}\Phi_{k',\sigma_p}\frac{\partial v_{r,\sigma_p}}{\partial\nu_1}\,dS\\
&=\int_{\partial Q}\int_{\widetilde\Gamma^+(t)}\Phi_{k',\sigma_p}\frac{\partial}{\partial\nu_1}\left(\frac{-1}{\boldsymbol{\kappa}_1}\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\widehat{G}(\frac{s}{\sigma_p},\frac{\eta}{\sigma_p})\right)\rho_1(\xi,s)\,d\widetilde{S}\,dS'\\
&=\frac{-1}{\sigma_p\boldsymbol{\kappa}_1}\int_{\partial Q}\int_{\widetilde\Gamma^+(t)}\Phi_{k',\sigma_p}\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\frac{\partial G}{\partial\widetilde\nu}(\frac{s}{\sigma_p},\frac{\eta}{\sigma_p})\,\rho_1(\xi,s)\,d\widetilde{S}\,d\eta\,dS'\\
&=\frac{-1}{\sigma_p\boldsymbol{\kappa}_1}\int_{\partial Q}\int_{\widetilde\Gamma^+(t)}\Phi_{k'}(\xi,o'')\,\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,\frac{\partial G}{\partial\widetilde\nu}(\frac{s}{\sigma_p},\frac{\eta}{\sigma_p})\,d\widetilde{S}\,dS'\\
&\quad-\frac{1}{\sigma_p\boldsymbol{\kappa}_1}\int_{\partial Q}\int_{\widetilde\Gamma^+(t)}\Phi_{k'}(\xi,o'')\,\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,\frac{\partial G}{\partial\widetilde\nu}(\frac{s}{\sigma_p},\frac{\eta}{\sigma_p})\,(\rho_1(\xi,s)-1)\,d\widetilde{S}\,dS'\\
&\quad-\frac{1}{\sigma_p\boldsymbol{\kappa}_1}\int_{\partial Q}\int_{\widetilde\Gamma^+(t)}(\Phi_{k'}(\xi+s\boldsymbol{n}(\xi),\eta)-\Phi_{k'}(\xi,o''))\\
&\qquad\qquad\times\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,\frac{\partial G}{\partial\widetilde\nu}(\frac{s}{\sigma_p},\frac{\eta}{\sigma_p})\,\rho_1(\xi,s)\,d\widetilde{S}\,dS'\\
&\quad-\frac{1}{\sigma_p\boldsymbol{\kappa}_1}\int_{\partial Q}\int_{\widetilde\Gamma^+(t)}(\Phi_{k',\sigma_p}-\Phi_{k'})\,\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,\frac{\partial G}{\partial\widetilde\nu}(\frac{s}{\sigma_p},\frac{\eta}{\sigma_p})\,\rho_1(\xi,s)\,d\widetilde{S}\,dS'
\end{aligned}$$

Note that $\widehat{G}(\widetilde{s},\widetilde{\eta})=G(\widetilde{s},\widetilde{\eta})$ for $(\widetilde{s},\widetilde{\eta})\in H_1$. From Prop.8.5, we have

(7.139) $$\int_{\widetilde\Gamma^+(t)}\Phi_{k'}(\xi,o'')\,\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,\frac{\partial G}{\partial\widetilde\nu}(\frac{s}{\sigma_p},\frac{\eta}{\sigma_p})\,d\widetilde{S}$$

$$= -\boldsymbol{\kappa}_1\, S(m)\, \sigma_p^m\, \Phi_{k'}(\xi, o'')\, \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi).$$

Using the calculation (7.139) in the first term of the right hand side, we have the relation (7.135) by putting

$$\begin{aligned}
&\Upsilon_6(k, r, \sigma_p, t) \\
&= -\frac{1}{\sigma_p \boldsymbol{\kappa}_1} \int_{\partial Q} \int_{\widetilde{\Gamma}^+(t)} \Phi_{k'}(\xi, o'')\, \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\, \frac{\partial G}{\partial \widetilde{\nu}}(\frac{s}{\sigma_p}, \frac{\eta}{\sigma_p})\, (\rho_1(\xi, s) - 1)\, d\widetilde{S}\, dS' \\
&\quad - \frac{1}{\sigma_p \boldsymbol{\kappa}_1} \int_{\partial Q} \int_{\widetilde{\Gamma}^+(t)} (\Phi_{k'}(\xi + s\boldsymbol{n}(\xi), \eta) - \Phi_{k'}(\xi, o'')) \\
&\qquad\qquad \times \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\, \frac{\partial G}{\partial \widetilde{\nu}}(\frac{s}{\sigma_p}, \frac{\eta}{\sigma_p})\, \rho_1(\xi, s)\, d\widetilde{S}\, dS' \\
&\quad - \frac{1}{\sigma_p \boldsymbol{\kappa}_1} \int_{\partial Q} \int_{\widetilde{\Gamma}^+(t)} (\Phi_{k', \sigma_p} - \Phi_{k'})\, \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\, \frac{\partial G}{\partial \widetilde{\nu}}(\frac{s}{\sigma_p}, \frac{\eta}{\sigma_p})\, \rho_1(\xi, s)\, d\widetilde{S}\, dS'.
\end{aligned}$$

We estimate $\Upsilon_6(k, r, \sigma_p, t)$.

$$\begin{aligned}
&|\Upsilon_6(k, r, \sigma_p, t)|/\sigma_p^{m-1} \\
&\leqq \frac{1}{\sigma_p^m \boldsymbol{\kappa}_1} \int_{\partial Q} \int_{\widetilde{\Gamma}^+(t)} |\Phi_{k'}(\xi, o'')| \left|\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\right| \left|\frac{\partial G}{\partial \widetilde{\nu}}(\frac{s}{\sigma_p}, \frac{\eta}{\sigma_p})\right| |\rho_1(\xi, s) - 1|\, d\widetilde{S}\, dS' \\
&\quad + \frac{1}{\sigma_p^m \boldsymbol{\kappa}_1} \int_{\partial Q} \int_{\widetilde{\Gamma}^+(t)} |\Phi_{k'}(\xi + s\boldsymbol{n}(\xi), \eta) - \Phi_{k'}(\xi, o'')| \\
&\qquad\qquad \times \left|\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\right| \left|\frac{\partial G}{\partial \widetilde{\nu}}(\frac{s}{\sigma_p}, \frac{\eta}{\sigma_p})\right| \rho_1(\xi, s)\, d\widetilde{S}\, dS' \\
&\quad + \frac{1}{\sigma_p^m \boldsymbol{\kappa}_1} \int_{\partial Q} \int_{\widetilde{\Gamma}^+(t)} |\Phi_{k', \sigma_p} - \Phi_{k'}| \left|\frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi)\right| \left|\frac{\partial G}{\partial \widetilde{\nu}}(\frac{s}{\sigma_p}, \frac{\eta}{\sigma_p})\right| \rho_1(\xi, s)\, d\widetilde{S}\, dS'
\end{aligned}$$

From Prop.2.7 we have

$$\text{(7.140)} \qquad \left|\frac{\partial G}{\partial \widetilde{\nu}}(\frac{s}{\sigma_p}, \frac{\eta}{\sigma_p})\right| \leqq \left|(\nabla_z G)(\frac{s}{\sigma_p}, \frac{\eta}{\sigma_p})\right| \leqq \frac{c_0'\, \sigma_p^m}{(s^2 + |\eta|^2)^{m/2}} = \frac{c_0'\, \sigma_p^m}{t^m}$$

for $x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Gamma^+(t)$. Moreover there exists $c > 0$ such that

$$|\rho_1(\xi, s) - 1| \leqq c\, t \qquad\qquad (x = (\xi + s\boldsymbol{n}(\xi), \eta) \in \Gamma^+(t)),$$

$$\begin{aligned}
&|\Phi_{k'}(\xi+s\boldsymbol{n}(\xi),\eta)-\Phi_{k'}(\xi,o'')| \\
&\ \leqq t\|\nabla\Phi_{k'}\|_{L^\infty(D)} \qquad\qquad (x=(\xi+s\boldsymbol{n}(\xi),\eta)\in\Gamma^+(t)),\\
&1/2\leqq\rho_1(\xi,s)\leqq 2 \qquad\qquad (x=(\xi+s\boldsymbol{n}(\xi),\eta)\in\Gamma^+(t)).
\end{aligned}$$

We get

$$\begin{aligned}
&|\Upsilon_6(k,r,\sigma_p,t)|/\sigma_p^{m-1}\\
&\leqq \frac{c_0'\,c\,t}{\boldsymbol{\kappa}_1}\int_{\partial Q}\int_{\widetilde{\Gamma}^+(t)}|\Phi_{k'}(\xi,o'')|\left|\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\right|\frac{1}{t^m}\,d\widetilde{S}\,dS'\\
&\quad+\frac{t}{\boldsymbol{\kappa}_1}\|\nabla\Phi_{k'}\|_{L^\infty(D)}\int_{\partial Q}\int_{\widetilde{\Gamma}^+(t)}\left|\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\right|\frac{2\,c_0'}{t^m}\,d\widetilde{S}\,dS'\\
&\quad+\frac{1}{\boldsymbol{\kappa}_1}\sup_{x\in\Gamma^+(t)}|\Phi_{k',\sigma_p}(x)-\Phi_{k'}(x)|\int_{\partial Q}\int_{\widetilde{\Gamma}^+(t)}\left|\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\right|\frac{2\,c_0'}{t^m}\,d\widetilde{S}\,dS'.
\end{aligned}$$

Due to $|\widetilde{\Gamma}^+(t)|/t^m=(1/2)(m+1)S(m+1)$ for $t>0$ and the property

$$\lim_{p\to\infty}\sup_{x\in\Gamma^+(t)}|\Phi_{k',\sigma_p}(x)-\Phi_{k'}(x)|=0$$

we have

$$\begin{aligned}
&\limsup_{p\to\infty}|\Upsilon_6(k,r,\sigma_p,t)|/\sigma_p^{m-1}\\
&\leqq\frac{c\,t\,(m+1)\,S(m+1)}{2\,\boldsymbol{\kappa}_1}|\partial Q|\|\Phi_{k'}\|_{L^\infty(D)}\|\nabla\psi_r\|_{L^\infty(Q)}\\
&\quad+\frac{2\,t}{\boldsymbol{\kappa}_1}\|\nabla\Phi_{k'}\|_{L^\infty(D)}\|\nabla\psi_q\|_{L^\infty(Q)}|\partial Q|.
\end{aligned}$$

This concludes (7.138) for $i=6$. Next we deal with (7.136).

$$\begin{aligned}
&|\Upsilon_7(k,r,\sigma_p,t)|\\
&\leqq\int_{\Sigma^+(t)}|\Phi_{k',\sigma_p}||\Delta v_{r,\sigma_p}|\,dx\\
&\leqq\|\Phi_{k',\sigma_p}\|_{L^\infty(D)}\left(\int_{\Sigma^+(t)\setminus\Sigma^+(2\sigma_p)}c_2\frac{\sigma_p^{m-1}}{(s^2+|\eta|^2)^{m/2}}dx+\int_{\Sigma^+(2\sigma_p)}(c_2/\sigma_p)dx\right)
\end{aligned}$$

$$\frac{|\Upsilon_7(k,r,\sigma_p,t)|}{\sigma_p^{m-1}}$$
$$\leqq \|\Phi_{k',\sigma_p}\|_{L^\infty(D)}\left(\int_{\Sigma^+(t)\setminus\Sigma^+(2\sigma_p)}\frac{c_2}{(s^2+|\eta|^2)^{m/2}}dx+\int_{\Sigma^+(2\sigma_p)}(c/\sigma_p^m)dx\right)$$
$$\limsup_{p\to\infty}|\Upsilon_7(k,r,\sigma_p,t)|/\sigma_p^{m-1}$$
$$\leqq \|\Phi_{k',\sigma_p}\|_{L^\infty(D)}\int_{\Sigma^+(t)}\frac{c}{(s^2+|\eta|^2)^{m/2}}dx=O(t)$$

(7.138) for $i=7$ is valid. We can deal with $\Upsilon_8(k,r,\sigma_p,t)$ in (7.136) similarly.

$$\begin{aligned}\frac{\Upsilon_9(k,r,\sigma_p,t)}{\sigma_p^{m-1}} &= \frac{1}{\sigma_p^m}\int_{\Gamma^+(t)}\frac{\partial\Phi_{k',\sigma_p}}{\partial\nu_3}\,v_{r,\sigma_p}\,dS \\ &= \int_{\Gamma^+(t)}\frac{\partial\Phi_{k',\sigma_p}}{\partial\nu_3}\left(-\frac{1}{\boldsymbol{\kappa}_1}\right)\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,G(\frac{s}{\sigma_p},\frac{\eta}{\sigma_p})\,\frac{1}{\sigma_p^{m-1}}\,dS \\ &= \int_{\Gamma^+(t)}\frac{\partial\Phi_k}{\partial\nu_3}\left(-\frac{1}{\boldsymbol{\kappa}_1}\right)\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,G(\frac{s}{\sigma_p},\frac{\eta}{\sigma_p})\,\frac{1}{\sigma_p^{m-1}}\,dS \\ &\quad+\int_{\Gamma^+(t)}\frac{\partial(\Phi_{k',\sigma_p}-\Phi_{k'})}{\partial\nu_3}\left(-\frac{1}{\boldsymbol{\kappa}_1}\right)\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi) \\ &\qquad\qquad\times G(\frac{s}{\sigma_p},\frac{\eta}{\sigma_p})\,\frac{1}{\sigma_p^{m-1}}\,dS\end{aligned}$$

$$\begin{aligned}\frac{|\Upsilon_9(k,r,\sigma_p,t)|}{\sigma_p^{m-1}} &\leqq \int_{\Gamma^+(t)}\left|\frac{\partial\Phi_{k'}}{\partial\nu_3}\right|\frac{1}{\boldsymbol{\kappa}_1}\left|\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\right|\frac{c_0'}{t^{m-1}}\,dS \\ &\quad+\int_{\Gamma^+(t)}\left|\frac{\partial(\Phi_{k',\sigma_p}-\Phi_{k'})}{\partial\nu_3}\right|\frac{1}{\boldsymbol{\kappa}_1}\left|\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\right|\frac{c_0'}{t^{m-1}}\,dS\end{aligned}$$

We get

$$\limsup_{p\to\infty}\frac{|\Upsilon_9(k,r,\sigma_p,t)|}{\sigma_p^{m-1}}\leqq\frac{c}{\boldsymbol{\kappa}_1}\|\nabla\Phi_{k'}\|_{L^\infty(D)}\|\nabla\psi_q\|_{L^\infty(Q)}|\Gamma^+(t)|/t^{m-1}$$

for some constant $c>0$. Since $|\Gamma^+(t)|=O(t^m)$, $\Upsilon_9(k,r,\sigma_p,t)$ satisfies the condition (7.138) for $i=9$. □

Applying Lemma 7.20 and Lemma 7.21 to (7.128), we have the following relation

$$\int_{\partial Q}\frac{\partial\psi_r}{\partial\boldsymbol{n}}(\xi)\,\Phi_{k'}(\xi,o'')\,dS'\,\sigma_p^m \tag{7.141}$$

$$- T(\boldsymbol{q}, m) \int_{\partial Q} \frac{\partial \widehat{\Phi}^*_{k'}}{\partial \boldsymbol{n}} \frac{\partial \psi_r}{\partial \boldsymbol{n}} \, dS' \, \sigma_p^m + \Upsilon_{10}(k, r, \sigma_p, t) = 0$$

such that $\limsup_{p\to\infty} |\Upsilon_{10}(k, r, \sigma_p, t)|/\sigma_p^m = O(t)$. Divide (7.141) by σ_p^m and consider the limit $t \downarrow 0$. Eventually we can conclude that $\lim_{p\to\infty} \Upsilon_{10}(k, r, \sigma_p, t)/\sigma_p^m \equiv 0$ for $t > 0$. Therefore we have the following result (recall (7.127)).

Lemma 7.22. *For $k \in K'$, $d(j') \leqq d < d(j'+1)$, $r(j'') \leqq r < r(j''+1)$, we have*

$$\boldsymbol{\gamma}'_m(k) \int_D \Phi_k \, \phi_d \, dx = S(m) \int_{\partial Q} \frac{\partial \widehat{\Phi}^*_k}{\partial \boldsymbol{n}}(\xi) \, \phi_d(\xi, o'') \, dS', \tag{7.142}$$

$$\int_{\partial Q} \frac{\partial \psi_r}{\partial \boldsymbol{n}}(\xi) \, \Phi_k(\xi, o'') \, dS' = T(\boldsymbol{q}, m) \int_{\partial Q} \frac{\partial \widehat{\Phi}^*_k}{\partial \boldsymbol{n}} \frac{\partial \psi_r}{\partial \boldsymbol{n}} \, dS'. \tag{7.143}$$

Substitute

$$\Phi_k = \sum_{d(j') \leqq d < d(j'+1)} (\Phi_k, \phi_d)_{L^2(D)} \, \phi_d$$

into the above equation (7.142) and define the following matrices by

$$E = \big((\phi_d, \Phi_k)_{L^2(D)} \big)_{d(j') \leqq d < d(j'+1), k \in K'},$$
$$F = \Big((\psi_r, \widehat{\Phi}^*_k)_{L^2(Q)} \Big)_{r(j'') \leqq r < r(j''+1), k \in K'}$$

we get

$$E \begin{pmatrix} \ddots & & \\ & \boldsymbol{\gamma}'_m(k) & \\ & & \ddots \end{pmatrix}_{k \in K'} = S(m) \, \boldsymbol{C}(j) \, F, \quad {}^t\boldsymbol{C}(j) \, E = T(\boldsymbol{q}, m) \, \boldsymbol{B}(j) \, F.$$

Since $\boldsymbol{B}(j)$ is invertible, we have $F = (1/T(\boldsymbol{q}, m)) \, \boldsymbol{B}(j)^{-1} \, {}^t\boldsymbol{C}(j) \, E$ and get

$$E \begin{pmatrix} \ddots & & \\ & \boldsymbol{\gamma}'_m(k) & \\ & & \ddots \end{pmatrix}_{k \in K'} = (S(m)/T(\boldsymbol{q}, m)) \, \boldsymbol{C}(j) \, \boldsymbol{B}(j)^{-1} \, {}^t\boldsymbol{C}(j) \, E$$

This implies that the values $\boldsymbol{\gamma}'_m(k)$ $(k \in K')$ are the eigenvalues of the matrix

$$(S(m)/T(\boldsymbol{q},m))\,\boldsymbol{C}(j)\,\boldsymbol{B}(j)^{-1}\,{}^t\boldsymbol{C}(j).$$

Therefore we conclude $\boldsymbol{\gamma}'_m(k) = \boldsymbol{\gamma}_m(k)$ for $k \in K'$.

If $T(\boldsymbol{q},m) > 0$, then $K = \{k(j), k(j)+1, \cdots, k(j)+\widehat{r}(j'')-1\}$ and $K' = \{k(j)+\widehat{r}(j''), k(j)+\widehat{r}(j'')+1, k(j+1)-1\}$.

If $T(\boldsymbol{q},m) < 0$, then $K' = \{k(j), k(j)+1, \cdots, k(j)+\widehat{d}(j')-1\}$ and $K = \{k(j)+\widehat{d}(j'), k(j)+\widehat{d}(j')+1, k(j+1)-1\}$.

The sets K and K' and the $\boldsymbol{\gamma}_m(k)$ do not depend on the choice of $\{\zeta_p\}_{p=1}^{\infty}$. So we have the conclusion of Theorem 2.14.

§8. Appendix

In this section we prepare several auxiliary results which are necessary in the proofs of the main theorems in the paper.

8.1. Several results for convergence of eigenfunctions

First we recall the main results in Jimbo [26], which are concerned with behaviors of solutions of elliptic equations (with Neumann B.C.) in $\Omega(\zeta)$ for $\zeta \to 0$. Let us consider the following elliptic equation in $\Omega(\zeta)$ $(\zeta > 0)$,

$$\Delta u + C_\zeta(x) f(u) = 0 \quad \text{in} \quad \Omega(\zeta), \qquad \partial u/\partial\nu = 0 \quad \text{on} \quad \partial\Omega(\zeta), \tag{8.1}$$

where $f = f(s)$ is a C^1 map from $\mathbb{R}$ into $\mathbb{R}$ and $C_\zeta, C, \widehat{C}$ are given functions in $\overline{\Omega(\zeta)}$, $\overline{D}$, $\overline{Q}$, respectively such that

$$C_\zeta \in C^1(\overline{\Omega(\zeta)}), \quad C \in C^1(\overline{D}), \quad \widehat{C} \in C^1(\overline{Q}),$$

$$\lim_{\zeta\to 0}\sup_{x\in D}|C_\zeta(x) - C(x)| = 0, \quad \lim_{\zeta\to 0}\sup_{x=(x',x'')\in Q(\zeta)}|C_\zeta(x',x'') - \widehat{C}(x')| = 0.$$

For this equation we recall the following result.

Proposition 8.1 ([26]). *Assume that $\{\zeta_p\}_{p=1}^{\infty}$ is a sequence of positive values which converges to 0 for $p \to \infty$ and $u_{\zeta_p} \in C^2(\overline{\Omega(\zeta_p)})$ is a solution of (8.1) for $\zeta = \zeta_p$ with*

$$\sup_{p \geqq 1}\sup_{x\in\Omega(\zeta_p)}|u_{\zeta_p}(x)| < +\infty. \tag{8.2}$$

Then there exist a subsequence $\{\sigma_p\}_{p=1}^{\infty} \subset \{\zeta_p\}_{p=1}^{\infty}$ *and* $w \in C^2(\overline{D})$ *and* $V \in C^2(\overline{Q})$ *such that*

$$\Delta w + C(x)\, f(w) = 0 \quad in \quad D, \qquad \frac{\partial w}{\partial \nu} = 0 \quad on \quad \partial D,$$
$$\Delta' V + \widehat{C}(x')\, f(V) = 0 \quad in \quad Q, \qquad V(x') = w(x', o'') \quad for \quad x' \in \partial Q,$$
$$\lim_{p\to\infty} \sup_{x\in D} |u_{\sigma_p}(x) - w(x)| = 0, \qquad \lim_{p\to\infty} \sup_{(x',x'')\in Q(\sigma_p)} |u_{\sigma_p}(x', x'') - V(x')| = 0.$$

In this paper this result is frequently used in §3-§7. We use it for the case of linear equation such that $f(u) = u$ and $C_\zeta(x) = \mu_k(\zeta)$. We also use similar results for behaviors of eigenfunctions in particular local portion of $\Omega(\zeta)$. We prepare some results for such purposes.

Let $\mathcal{J}$ be a regular manifold in $\mathbb{R}^n$ such that $\dim(\mathcal{J}) \leqq n-2$ and $\mathcal{J} \subset \partial D$. Define a set which is located around $\mathcal{J}$ as follows,

$$\mathcal{J}(t) = \{x \in D \mid \operatorname{dist}(x, \mathcal{J}) < t\}.$$

In the proofs of the main results (§3-§7), we use the following results for the case

$$\mathcal{J} = \partial Q \times \{o''\}$$

which is $(\ell-1)$-dimensional manifold. Note that $\ell - 1 \leqq n-2$. In this case $\mathcal{J} \subset \partial D$ and $\mathcal{J}(t)$ is equal to the set $\Sigma^+(t)$.

Let $\{\zeta_p\}_{p=1}^{\infty}$ be a sequence of positive values which converges to 0 for $p \to \infty$. Assume that $\mu(\zeta)$ is a constant depending on $\zeta > 0$ and $\lim_{p\to\infty} \mu(\zeta_p) = \mu$. Let $u_{\zeta_p} \in C^2(\overline{D})$ satisfy the condition

$$\text{(8.3)} \qquad \begin{aligned} &\Delta u_{\zeta_p} + \mu(\zeta_p)\, u_{\zeta_p} = 0 \quad \text{in} \quad D, \\ &\partial u_{\zeta_p}/\partial \nu = 0 \quad \text{on} \quad \partial D \setminus \overline{\mathcal{J}(2\zeta_p)}, \quad (p \geqq 1). \end{aligned}$$

Then we have the following result.

Proposition 8.2. *Assume*

$$\text{(8.4)} \qquad \sup_{p\geqq 1} \int_{D\setminus \mathcal{J}(t)} |u_{\zeta_p}|^2 dx < +\infty \quad \textit{for any} \quad t > 0.$$

Then there exist a subsequence $\{\sigma_p\}_{p=1}^{\infty} \subset \{\zeta_p\}_{p=1}^{\infty}$ *and* $w \in C^2(\overline{D} \setminus \mathcal{J})$ *such that*

$$\begin{cases} \Delta w + \mu\, w = 0 \quad \textit{in} \quad D, \qquad \dfrac{\partial w}{\partial \nu} = 0 \quad \textit{on} \quad \partial D \setminus \mathcal{J}, \\ \lim_{p\to\infty} \|u_{\sigma_p} - w\|_{C^2(\overline{D}\setminus\mathcal{J}(t))} = 0 \quad \textit{for any} \quad t > 0. \end{cases} \tag{8.5}$$

SKETCH OF THE PROOF OF PROPOSITION 8.2. The proof is carried out by the aid of the compactness due to Schauder estimates in Gilbarg-Trudinger [22] and the Cantor's diagonal argument. □

We remark that w may have singularity on $\mathcal{J}$. If we assume a stronger condition in place of (8.4), we can prove that $\mathcal{J}$ is a removable singular set.

PROPOSITION 8.3. *Assume*

$$\sup_{p \geqq 1} \int_D (|\nabla u_{\zeta_p}|^2 + |u_{\zeta_p}|^2)\, dx < +\infty. \tag{8.6}$$

Then there exist a subsequence $\{\sigma_p\}_{p=1}^{\infty} \subset \{\zeta_p\}_{p=1}^{\infty}$ *and* $w \in C^2(\overline{D})$ *with*

$$\Delta w + \mu\, w = 0 \qquad \textit{in} \quad D, \qquad \frac{\partial w}{\partial \nu} = 0 \quad \textit{on} \quad \partial D. \tag{8.7}$$

SKETCH OF THE PROOF OF PROPOSITION 8.3. Using Prop.8.2, we have a subsequence $\{\sigma_p\}_{p=1}^{\infty}$ and $w \in C^2(\overline{D} \setminus \mathcal{J})$ with the property (8.5). From the condition (8.6) and the Rellich's theorem, u_{σ_p} weakly converges to w in $H^1(D)$ and strongly converges to w in $L^2(D)$ for $p \to \infty$. Take any $t > 0$ and an arbitrary $\Psi \in C^2(\overline{D})$ which vanishes identically in $\mathcal{J}(t)$. Multiplying Ψ by the equation of (8.5) and carrying out partial integration, we get

$$\int_D (\nabla w \nabla \Psi - \mu\, w\, \Psi)\, dx = 0. \tag{8.8}$$

On the other hand, the set

$$\{\Psi \in C^2(\overline{D}) \mid \operatorname{supp}(\Psi) \cap \mathcal{J} = \emptyset\}$$

is dense in $H^1(D)$ as the codimension of $\mathcal{J}$ is equal or greater than 2 (cf. Chavel-Feldman [9]). Therefore (8.8) holds for any $\Psi \in H^1(D)$. This implies that w is a classical solution of the equation means (8.7) (The regularly theory of weak solutions of elliptic equations applies. See Mizohata [39]). w satisfies the Neumann B.C. on the whole boundary ∂D. $\Box$

We can prove another type of removable singularity theorem.

Proposition 8.4. *Assume that $w \in C^2(\overline{D} \setminus \mathcal{J})$ satisfies the equation*

$$\Delta w + \mu\, w = 0 \quad in \quad D, \qquad \frac{\partial w}{\partial \nu} = 0 \quad on \quad \partial D \setminus \mathcal{J}, \tag{8.9}$$

with the condition

$$\sup_{x \in D} |w(x)| < \infty, \tag{8.10}$$

then $w \in C^2(\overline{D})$ and $\partial w / \partial \nu = 0$ on ∂D.

Proof of Proposition 8.4. Since $\mathcal{J} \subset \partial D$ and $\dim \mathcal{J} \leqq n-2$, we can take two sequences of positive values $\{\tau_1(p)\}_{p=1}^{\infty}$, $\{\tau_2(p)\}_{p=1}^{\infty}$ and a family of functions $\{\Psi_p\}_{p=1}^{\infty} \subset C^2(\overline{D})$ such that

$$0 < \tau_1(p) < \tau_2(p) \quad (p \geqq 1), \qquad \lim_{p\to\infty} \tau_1(p) = 0, \qquad \lim_{p\to\infty} \tau_2(p) = 0,$$

$$\begin{aligned} &\Psi_p(x) \equiv 0 \quad \text{in} \quad \mathcal{J}(\tau_1(p)), \\ &\phi_p(x) \equiv 1 \quad \text{in} \quad D \setminus \mathcal{J}(\tau_2(p)), \end{aligned} \qquad 0 \leqq \Psi_p \leqq 1 \quad \text{in} \quad D,$$

$$\lim_{p\to\infty} \int_D |\nabla \Psi_p|^2\, dx = 0.$$

That is, $\mathcal{J}$ is $(1,2)-$ polar set (or the capacity of $\mathcal{J}$ is zero. See Edmunds-Evans [15], Chavel-Feldman [9] for this property.

Multiply the equation (8.9) by $\Psi_p^2 w$ and integrate it on D and get

$$\int_D (\nabla w \nabla(\Psi_p^2 w) - \mu\, w^2\, \Psi_p^2)\, dx = 0,$$

$$\int_D (|\nabla w|^2\, \Psi_p^2 + 2\, w\, \Psi_p \nabla \Psi_p \nabla w - \mu\, w^2\, \Psi_p^2)\, dx = 0,$$

$$\int_D |\nabla w|^2 \, \Psi_p^2 \, dx = \int_D (-2 \, (\Psi_p \nabla w) \, (w \nabla \Psi_p) + \mu \, w^2 \, \Psi_p^2) \, dx$$
$$\leqq \int_D ((1/2) \, |\nabla w|^2 \, \Psi_p^2 + 2 \, w^2 \, |\nabla \Psi_p|^2 + \mu \, w^2 \, \Psi_p^2) \, dx,$$

From this inequality, we get a bound for $\int_D |\nabla w|^2 \, \Psi_p^2 \, dx$ which is

$$\int_D |\nabla w|^2 \, \Psi_p^2 \, dx \leqq \int_D (4 \, w^2 \, |\nabla \Psi_p|^2 + 2 \, |\mu| \, w^2 \, \Psi_p^2) \, dx.$$

The right hand side of the above inequality is bounded by a constant which is independent of $p \geqq 1$. Taking $p \to \infty$ and we conclude $w \in H^1(D)$. Applying the arguments in Prop.8.2, Prop.8.3, we get the conclusion. □

8.2. Harmonic function in H

In this subsection we prove Proposition 2.7. First we prepare several notation for the proofs. For the domain $H = H_1 \cup H_2 \subset \mathbb{R}^{m+1}$ (cf. §2), we define for $R > 0$,

$$H_1(R) = \{(s, \eta) \in H_1 \mid \; s^2 + |\eta|^2 < R^2\},$$
$$H_2(R) = \{(s, \eta) \in H_2 \mid \; s > -R\}$$

and put $H(R) = H_1(R) \cup H_2(R)$. The boundary $\partial H(R)$ is divided into 3 parts,

$$\partial H(R) = T_1(R) \cup T_2(R) \cup T_3(R)$$

where each set is written as follows

$$T_1(R) = \{(s, \eta) \in H \mid s^2 + |\eta|^2 = R^2, \; s > 0\},$$
$$T_2(R) = \{(s, \eta) \in H \mid s = -R\},$$
$$T_3(R) = \{(s, \eta) \in \partial H \mid -R < s \leqq 0\}.$$

OUTLINE OF THE PROOF OF PROPOSITION 2.7-(i) ($m = 1$). For the case $m = 1$, H is a domain in $\mathbb{R}^2$. To construct G, we apply the method of conjugate harmonic function. We construct an approximate conjugate harmonic function in $H(R)$. As $m = 1$, $T_3(R)$ has two components $T_3^{\pm}(R)$ which can be written as follows,

$$T_3^{\pm}(R) = \{(s, \eta) \in T_3(R) \mid \quad \pm\eta > 0\}.$$

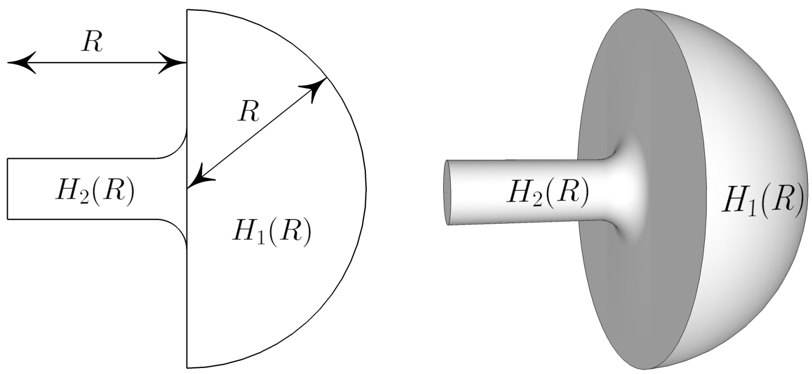

Fig. 15. $H_1(R) \cup H_2(R)$: Left $m = 1$, Right $m = 2$

Let Φ_R be the unique solution of the equation,

$$\begin{aligned}&\Delta_z \Phi = 0 \quad \text{in} \quad H(R), \qquad \Phi = 1 \quad \text{on} \quad T_3^+(R), \qquad \Phi = -1 \quad \text{on} \quad T_3^-(R),\\ &\Phi(s,\eta) = (2\theta/\pi) \quad ((s,\eta) = (R\cos\theta, R\sin\theta) \in T_1(R),\ -\pi/2 \leqq \theta \leqq \pi/2),\\ &\Phi(-R,\eta) = \eta \quad ((-R,\eta) \in T_2(R),\ -1 \leqq \eta \leqq 1).\end{aligned}$$

This is the Laplace equation with a given data on the boundary and so it has the unique solution $\Phi_R(s,\eta)$. Note that $\Phi_R(s,\eta)$ is odd in η. Taking the limit $R \to \infty$, we can argue the convergence of Φ_R and get the unique limit Φ which is a harmonic function with

$$\begin{gathered}-1 \leqq \Phi \leqq 1, \quad \Phi(s,-\eta) = -\Phi(s,\eta) \text{ in } H,\\ \Phi = 1 \text{ on } T_3^+(\infty), \quad \Phi = -1 \text{ on } T_3^-(\infty).\end{gathered}$$

Let $\Psi = \Psi(s,\eta)$ be the conjugate harmonic function to $\Phi = \Phi(s,\eta)$, that is,

$$\left.\begin{aligned}&\partial\Phi/\partial s + \partial\Psi/\partial\eta = 0,\\ &\partial\Phi/\partial\eta - \partial\Psi/\partial s = 0,\end{aligned}\right. \quad \text{in} \quad H \quad \text{(Cauchy-Riemann equation).}$$

This is derived from the condition that $\Psi(s,\eta) + \Phi(s,\eta)\,i$ is holomorphic in the variable $z = s + \eta i$. From this equation we obtain Ψ by

$$\Psi(s,\eta) = \int_0^s \frac{\partial\Phi}{\partial\eta}(s',0)\,ds' - \int_0^\eta \frac{\partial\Phi}{\partial s}(s,\eta')\,d\eta' \quad ((s,\eta) \in H).$$

Ψ is harmonic in H with the Neumann B.C. on ∂H and it is even in the variable η. We can also consider the expansion Φ, Ψ in fundamental harmonic functions in each region H_1 and H_2. Taking into the boundary condition and the behavior at infinity, we have the following expansions.
(i) $(s, \eta) \in H_1$

$$\Phi(s,\eta) = (2/\pi)\theta - \sum_{k=1}^{\infty} c_k' \frac{\sin 2k\theta}{\tau^{2k}}$$
$$((s,\eta) = (\tau\cos\theta, \tau\sin\theta) \in H_1, \tau \geqq 3, s > 0),$$
$$\Psi(s,\eta) = (2/\pi)\log\tau - \boldsymbol{\kappa}' + \sum_{k=1}^{\infty} c_k' \frac{\cos 2k\theta}{\tau^{2k}}$$
$$((s,\eta) = (\tau\cos\theta, \tau\sin\theta) \in H_1, \tau \geqq 3, s > 0),$$

(ii) $(s, \eta) \in H_2$

$$\Phi(s,\eta) = \eta + \sum_{k=1}^{\infty} c_k \sin(k\pi\eta)\exp(k\pi s) \quad ((s,\eta) \in H,\ s \leqq -2),$$
$$\Psi(s,\eta) = s - \boldsymbol{\kappa} + \sum_{k=1}^{\infty} c_k \cos(k\pi\eta)\exp(k\pi s) \quad ((s,\eta) \in H,\ s \leqq -2).$$

Putting $G = -\Psi$, we can prove the desired properties in the conclusion of Proposition 2.7-(i).

OUTLINE OF THE PROOF OF PROPOSITION 2.7-(ii) ($m \geqq 2$).

We make use of the method of an upper-lower solution pair to construct a desired harmonic function G in H. First, we consider the following equations,

$$\begin{cases} \Delta_z N = 0 \quad \text{in} \quad H(2), \\ N = 1 \quad \text{on} \quad T_2(2), \qquad N = 0 \quad \text{on} \quad T_1(2), \\ \partial N/\partial \widetilde{\boldsymbol{n}} = 0 \quad \text{on} \quad T_3(2) \end{cases} \tag{8.11}$$

where $\widetilde{\boldsymbol{n}}$ is the unit outward vector on $\partial H(2)$.

It is easy to see that (8.11) has a unique (smooth) solution $N(z)$ and from the maximum principle that there exist $\delta_2 \geqq \delta_1 > 0$ such that

$$\begin{aligned} &-\delta_2 \leqq \partial N/\partial \widetilde{\boldsymbol{n}} \leqq -\delta_1 \quad \text{on} \quad T_1(2), \\ &\delta_1 \leqq \partial N/\partial \widetilde{\boldsymbol{n}} \leqq \delta_2 \quad \text{on} \quad T_3(2). \end{aligned} \tag{8.12}$$

Define two positive functions $\Psi^{\pm}(z)$ as follows

$$\Psi^+(z) = \begin{cases} \frac{1}{|z|^{m-1}} & \text{for} \quad z \in H_1 \setminus H(2), \\ \frac{m-1}{2^m \delta_2} N(z) + \frac{1}{2^{m-1}} & \text{for} \quad z \in H(2), \\ \frac{-(m-1)\delta_1}{2^m \delta_2}(s+2) + \frac{m-1}{2^m \delta_2} + \frac{1}{2^{m-1}} & \text{for} \quad z \in H_2 \setminus H(2), \end{cases}$$

$$\Psi^-(z) = \begin{cases} \frac{1}{|z|^{m-1}} & \text{for} \quad z \in H_1 \setminus H(2), \\ \frac{m-1}{2^m \delta_1} N(z) + \frac{1}{2^{m-1}} & \text{for} \quad z \in H(2), \\ \frac{-(m-1)\delta_2}{2^m \delta_1}(s+2) + \frac{m-1}{2^m \delta_1} + \frac{1}{2^{m-1}} & \text{for} \quad z \in H_2 \setminus H(2). \end{cases}$$

We easily see that Ψ^- and Ψ^+ are a sub-harmonic function and a super-harmonic function (with Neumann B.C. on ∂H) with the property

$$0 < \Psi^-(z) \leqq \big((\delta_2/\delta_1)^2 + 1\big)\, \Psi^+(z) \quad \text{in} \quad H. \tag{8.13}$$

By applying the comparison existence theorem to the lower-solution Ψ^- and the upper-solution $(\delta_2/\delta_1)^2\Psi^+(z)$, we get a harmonic function $\Psi(z)$ between these functions. That is,

$$\begin{cases} \Psi^-(z) \leqq \Psi(z) \leqq ((\delta_2/\delta_1)^2 + 1)\, \Psi^+(z) \quad \text{in} \quad H, \\ \Delta_z \Psi = 0 \quad \text{in} \quad H, \qquad \partial\Psi/\partial\boldsymbol{n} = 0 \quad \text{on} \quad \partial H. \end{cases} \tag{8.14}$$

From the inequality in (8.14), we have the decay estimate at ∞ in H_1 and grow estimate of $\Psi(z)$ at ∞ in H_2 is verified. That is

$$\begin{aligned} &\Psi(z) = O(1/|z|^{m-1}) \quad \text{at} \quad |z| = \infty \quad \text{in} \quad H_1, \\ &\Psi(z)/|z| = O(1) \quad \text{at} \quad |z| = \infty \quad \text{in} \quad H_2. \end{aligned}$$

We consider more elaborate behavior of $\Psi(z)$ at ∞. We use Fourier's method to express Ψ by the fundamental harmonic functions in H_1. With the aid of the polar coordinate (τ, θ) (z is expressed as i.e. $z = \tau\,\theta$, $\tau > 0$, $\theta \in S^m$), we can consider those solutions in the form of $W(\theta)J(\tau)$. Making the linear combination of such solutions, we can express Ψ in the form of infinite series

$$\Psi(z) = \sum_{k=0}^{\infty} \sum_{p=1}^{\iota(k)} c_{k,p}\, W_{k,p}(\theta)\, \tau^{-m-k+1} \tag{8.15}$$

in H_1. Here $W_{k,p}$ $(k \geqq 0, 1 \leqq p \leqq \iota(k))$ are complete system of eigenfunctions of the following eigenvalue problem,

$$(8.16) \quad \begin{aligned} &\Delta_{S^m}\phi + \Lambda\phi = 0 \quad \text{in} \quad S^m_+, \\ &\partial\phi/\partial\nu = 0 \quad \text{on} \quad \partial S^m_+ \quad \text{(Neumann B.C.)} \end{aligned}$$

where Δ_{S^m} is the standard Laplace-Beltrami operator in S^m and S^m_+ is the half sphere which is defined as

$$S^m_+ = \{(s,\eta) \in \mathbb{R}^{m+1} \mid s^2 + |\eta|^2 = 1,\ s > 0\}.$$

It is known that (8.16) has the eigenvalues $\Lambda_k = k\,(k-1+m)$ $(k \geqq 0)$ with its multiplicity $\iota(k)$. From the smoothness of Ψ, the infinite series is convergent in sufficiently strong topology. On the other hand, we study the behavior of of Ψ in H_2. $H_2 \setminus H(2)$ is a cylinder region and so we also use Fourier's method to to expand Ψ in the fundamental harmonic functions. We consider the a solution of Laplace equation in the form of $J(s)W(\eta)$ and make the linear combination, we have

$$(8.17) \quad \Psi(z) = -\boldsymbol{\kappa}_1\, s + \boldsymbol{\kappa}_2 + \sum_{k=1}^{\infty}\sum_{p=1}^{\iota'(k)} c'_{k,p} \exp\left(\sqrt{\Lambda'_{k,p}}\, s\right) \phi'_{k,p}(\eta) \qquad (z = (s,\eta) \in H_2).$$

We took account of the behavior of Ψ at ∞. Here

$$\{\Lambda'_{k,p} \mid k \geqq 0, 1 \leqq p \leqq \iota'(k)\}, \quad \{\phi'_{k,j}(\eta) \mid k \geqq 0, 1 \leqq p \leqq \iota'(k)\}$$

are the eigenvalues and their corresponding eigenfunctions of

$$(8.18) \quad \Delta_\eta\phi' + \Lambda'\phi' = 0 \quad \text{in} \quad B^{(m)}(1), \qquad \partial\phi'/\partial\nu = 0 \quad \text{on} \quad \partial B^{(m)}(1).$$

Note that the first eigenvalue of (8.18) $\Lambda'_0 = 0$ while other ones are all positive. The infinite series in (8.17) is convergent in sufficiently strong topology. By putting $\Psi = G$, we can prove the properties of Prop.2.7-(ii) with the aid of this expression. □

PROPOSITION 8.5. *For $m \geqq 2$, the solution G of (2.14) in Proposition 2.7-(ii) satisfies*

$$(8.19) \quad \int_{T_1(R)} \frac{\partial G}{\partial \widetilde{\nu}}\, d\widetilde{S} = -\boldsymbol{\kappa}_1\, S(m) \quad (R \geqq 3),$$

where $\widetilde{\nu}(s,\eta) = (s,\eta)/(s^2+|\eta|^2)^{1/2}$.

Proof of Proposition 8.5. By integrating the equation $\Delta_z G = 0$ in the set $H_1(R) \cup H_2(\tau)$, we get

$$\int_{T_1(R)} \frac{\partial G}{\partial \widetilde{\nu}}\, d\widetilde{S} = \int_{|\eta|<1} \frac{\partial G}{\partial s}(-\tau,\eta)\, d\eta.$$

Using that $(\partial G/\partial s)(-\tau,\eta)$ converges to $-\boldsymbol{\kappa}_1$ uniformly in $\eta \in B^{(m)}(1)$ for $\tau \to \infty$, we get (8.19). □

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