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Dispersive Global Solutions to the Time-Dependent Hartree-Fock Type Equation with a Long-Range Potential

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Dedicated to Professor Daisuke Fujiwara on his seventieth birthday

Abstract. We study the time decay estimate for L^p-norm ($2 < p \le \infty$) of a solution to the time-dependent Hartree-Fock type equation with a long-range potential, which decays more slowly than the Coulomb potential as $|x| \to \infty$, for small initial data.

1. Introduction

We study time decay properties (precisely, dispersive estimates) of solutions to the initial value problem of the time-dependent Hartree-Fock type equation

$$i\partial_t u = -\frac{1}{2}\Delta u + F(u)u, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.1}$$

$$u(0,x) = \phi(x), \qquad x \in \mathbb{R}^n. \tag{1.2}$$

Here the space dimension $n \ge 3$, $i = \sqrt{-1}$, $u = (u_1, \ldots, u_N)$ is a $\mathbb{C}^N$-valued unknown function ($N \ge 2$), ϕ is $\mathbb{C}^N$-valued given initial data, $\partial_t = \partial/\partial t$, Δ is the Laplace operator for the space variable x, $F(u) = (F_{jk}(u))_{1 \le j,k \le N}$ is an $N \times N$ Hermitian matrix defined by

$$F_{jk}(u) = V * (|u|^2_{\mathbb{C}^N}\delta_{jk} - u_j\bar{u}_k),$$

"$*$" denotes the convolution for the space variable, $|\cdot|_{\mathbb{C}^N}$ is the norm of $\mathbb{C}^N$, δ_{jk} is Kronecker's delta, the function V is a potential given by

$$V(x) = \lambda|x|^{-\gamma}, \qquad (x \in \mathbb{R}^n \setminus \{0\})$$

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λ is a non-zero real constant, and γ is a constant such that $0 < \gamma < 1$. (Thus V is a long-range potential decaying more slowly than the Coulomb potential as $|x| \to \infty$.) The system (1.1) appears in the quantum mechanics as an approximation to a Fermionic N-body system. In this paper, we show the existence of a solution u to the initial value problem (1.1)–(1.2) satisfying the dispersive estimate, that is, $\|u(t)\|_{L^\infty} = O(|t|^{-n/2})$ as $t \to \pm\infty$, for small initial data ϕ. In the main result (Theorem 1.1 below), we will assume that $1 - 1/(m+2) < \gamma < 1$, where m is the integer defined by (1.7).

There are many works on the Cauchy problem and the large time behavior of solutions for the Hartree equation

$$i\partial_t u = -\frac{1}{2}\Delta u + (V * |u|^2)u, \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.3}$$

where u is a complex-valued unknown function, and there are some works for the Hartree-Fock equation (1.1). For the Cauchy problem of the Hartree equation (1.3), see, e.g., Cazenave [1], Ginibre-Velo [3] and Hayashi-Ozawa [16]. We are interested in properties of solutions for large time. The potential V is called "short-range" when $\gamma > 1$, on the other hand, it is called "long-range" when $0 < \gamma \leq 1$. When $\gamma = 1$, V is called the Coulomb potential. In the short-range case, contribution of the potential is negligible for large time. Roughly speaking, when $1 < \gamma < n$, the solution to the equations (1.1) and (1.3) approaches some solution to the free Schrödinger equation

$$i\partial_t u = -\frac{1}{2}\Delta u, \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.4}$$

as $t \to \pm\infty$. On the other hand, in the long-range case ($0 < \gamma \leq 1$), contribution of the potential is not negligible for large time, and hence solutions do not approach any non-trivial free solutions as $t \to \pm\infty$. Hence to investigate the large time behavior of solutions to the equations (1.1) or (1.3), the long-range case is more difficult than the short-range one. In this paper, we concentrate on the long-range case. (For the large time behavior of solutions to the Hartree equation (1.3) with a short-range potential, see, e.g., Hayashi-Ozawa [15], Hayashi-Y. Tsutsumi [17] and Nawa-Ozawa [24], and to the Hartree-Fock equation, see Wada [28].)

We recall several known results on the large time behavior of solutions to the Hartree equation (1.3) with a long-range potential, namely, the case

$0 < \gamma \leq 1$. The existence of the modified wave operators (the final value problem) for the equation (1.3) was studied by, e.g., Ginibre and Ozawa [2], Ginibre and Velo [4, 5, 6] and Nakanishi [22, 23]. On the other hand, the time decay and large time asymptotics of solutions to the equation (1.3) for small initial data were studied, e.g., by Hayashi and Naumkin [11] and Hayashi, Naumkin and Ozawa [14] when $\gamma = 1$, and by Hayashi, Kaikina and Naumkin [10], Hayashi and Naumkin [12, 13] and Wada [29] when $\gamma < 1$. To study properties of solutions to the equation (1.3) for large time, the case $0 < \gamma < 1$ is more difficult than the critical case $\gamma = 1$. According to their results, for small initial data, the solution to the initial value problem of the equation (1.3) with $\gamma \leq 1$ decays like $t^{-n(1/2-1/p)}$ in L^p as $t \to \infty$, where $2 < p \leq \infty$. This decay rate is the same as the free solution (see Remark 1.3 below). Furthermore there exists some function u_+, which is called a modified scattering state, such that the solution to (1.3) approaches a modified free solution $U(t)e^{-iS(t,-i\nabla)}u_+$ as $t \to \infty$, where $U(t)$ is the free evolution operator defined by (1.6) below, and $S(t,\xi) = (V * |\hat{u}_+|^2)(\xi) \int_1^t \tau^{-\gamma}\, d\tau$. Here $\hat{u}_+$ is the Fourier transform of u_+. The modifier $e^{-iS(t,-i\nabla)}$ is represented explicitly. The nonlinearity $(V * |u|^2)u$ with $\gamma = 1$ as in the equation (1.3) also appears in the Maxwell-Schrödinger system with the Coulomb gauge condition in three space dimensions (see, e.g., [7, 8, 9, 20, 21, 25, 26]).

We return to the Hartree-Fock equation (1.1). To investigate the large time behavior of solutions, the Hartree-Fock equation (1.1) is more complicated than the the Hartree equation (1.3), because (1.1) is a system and $F(u)$ is not a diagonal matrix (though it is Hermite). According to author's knowledge, there is only one result (Wada [30]) on the large time behavior of solutions to the equation (1.1) with a long-range potential. In the case $\gamma = 1$ (the critical case), Wada [30] studied the existence of modified wave operators to the equation (1.1) for small final states, and the time decay and large time asymptotics of solutions to the initial value problem (1.1)–(1.2) for small initial data. He proved that when $\gamma = 1$, there exists a global solution u to the initial value problem (1.1)–(1.2) satisfying the time decay estimate $\|u(t)\|_{L^p} = O(t^{-n(1/2-1/p)})$ as $t \to \infty$ for small initial data, where $2 < p \leq 2n/(n-2\sigma)$ and $1/2 < \sigma < n/2$. Moreover he showed that there exists a unique modified scattering sate u_+ such that the solution u to the equation (1.1) approaches the modified free dynamics $U(t)A(t,-i\nabla)u_+$ as

$t \to \infty$, where $U(t)$ is the free evolution operator defined by (1.6) below, $A = A(t,\xi)$ is an $N \times N$ matrix satisfying the Cauchy problem

$$\begin{cases} i\partial_t A = t^{-1}F(A\hat{u}_+)A, & t \geq 1, \quad \xi \in \mathbb{R}^n, \\ A(1,\xi) = I_N, & \xi \in \mathbb{R}^n, \end{cases} \tag{1.5}$$

and I_N is the $N \times N$ unit matrix. It is easy to see that the modifier $A(t,\xi)$ in [30] is a unitary matrix, but it can not be expressed explicitly. So the modifier for the Hartree-Fock equation (1.1) is more complicated than that of the Hartree equation (1.3).

According to author's knowledge, in the case $0 < \gamma < 1$, there is no result on properties in large time of solutions to the Hartree-Fock equation (1.1). (To investigate properties of solutions to the equation (1.1) for large time, the case $0 < \gamma < 1$ is more difficult than the critical one $\gamma = 1$.) In the present paper, when $\gamma < 1$, we study the time decay of the solution u to the initial value problem (1.1)–(1.2) in L^p $(2 < p \leq \infty)$ for small initial data when the space dimension $n \geq 3$. In the main result (Theorem 1.1 below), we will assume $1 - 1/(m+2) < \gamma < 1$, where m is the integer defined by (1.7) below.

Here we introduce several notations. For $k, s \geq 0$, the weighted Sobolev space based on L^2 is defined as follows:

$$H^{k,s} \equiv \{\psi \in L^2(\mathbb{R}^n;\mathbb{C}^N) : \|\psi\|_{H^{k,s}} \equiv \|(1+|x|^2)^{s/2}(1-\Delta)^{k/2}\psi\|_{L^2} < \infty\}.$$

For $t \in \mathbb{R}$, we define $U(t)$ by

$$U(t) \equiv e^{it\Delta/2} = \mathcal{F}^{-1}e^{-it|\xi|^2/2}\mathcal{F}, \tag{1.6}$$

where $\mathcal{F}$ is the Fourier transform with respect to the space variable. $U(t)$ is the free evolution operator for the Schrödinger equation. (See Remark 1.3.) For $s \geq 0$, we define the operator $|J|^s$ by

$$|J|^s = |J(t)|^s \equiv U(t)|x|^s U(-t), \qquad (t \in \mathbb{R}).$$

The main result in this paper is the following theorem.

THEOREM 1.1. *Let the space dimension $n \geq 3$, and let*

$$m = \begin{cases} \frac{n+1}{2}, & \text{when } n \text{ is odd}, \\ \frac{n}{2}+1, & \text{when } n \text{ is even}. \end{cases} \tag{1.7}$$

Let $1 - 1/(m+2) < \gamma < 1$. *Assume that* $\phi \in H^{0,m}$ *and* $\|\phi\|_{H^{0,m}}$ *is sufficiently small. Then there exists a unique global solution* u *to the initial value problem* (1.1)–(1.2) *satisfying*

$$u \in C(\mathbb{R}; L^2), \qquad |J|^m u \in C(\mathbb{R}; L^2), \tag{1.8}$$

$$\|u(t)\|_{L^2} = \|\phi\|_{L^2}, \tag{1.9}$$

$$\| \, |J|^m u(t)\|_{L^2} \le C\|\phi\|_{H^{0,m}}(1+|t|)^{m(1-\gamma)}, \tag{1.10}$$

$$\|u(t)\|_{L^p} \le C\|\phi\|_{H^{0,m}}(1+|t|)^{-n(1/2-1/p)} \tag{1.11}$$

for $t \in \mathbb{R}$, *where* $2 \le p \le \infty$.

Remark 1.1. Under the assumptions of Theorem 1.1 except the smallness condition on the initial data ϕ, the existence and uniqueness of the global solution u to the initial value problem (1.1)–(1.2) satisfying (1.8), and the equality (1.9) are well-known. (See Proposition 4.1 below.) The main purpose of this paper is to prove the time decay (1.11) of the solution u to that initial value problem for small initial data ϕ.

Remark 1.2. By the definition (1.7) of m, we see that m is the smallest integer such that $m > n/2$. Note that the condition $m > n/2$ implies the embedding $H^m \hookrightarrow L^\infty$.

Remark 1.3. The solution to the Cauchy problem of the free Schrödinger equation (1.4) with initial condition $u(0,x) = \varphi(x)$ is given by

$$u(t,\cdot) = U(t)\varphi.$$

By the definition of $U(t)$, $U(t)$ is unitary in L^2. When $2 \le p \le \infty$, the following estimate is well-known:

$$\|U(t)\varphi\|_{L^p} \le (2\pi|t|)^{-n(1/2-1/p)}\|\varphi\|_{L^{p'}}, \qquad (t \ne 0), \tag{1.12}$$

where p' is the Hölder conjugate exponent of p, i.e., $1/p + 1/p' = 1$. According to the estimates (1.11) and (1.12), the time decay rate (as $t \to \infty$) of the solution to the equation (1.1) is the same as that of the free solution.

Remark 1.4. In this paper, large time asymptotics of the solution to the initial value problem (1.1)–(1.2) is not obtained, and that problem is still open. (See Section 5.)

For convenience to readers, we briefly explain idea of the proof of Theorem 1.1. The proof of the main theorem depends on the spirit of the proof in Wada [29] on the long-range scattering for the Hartree equation (1.3). Because of difficulties caused by the long range potential, it is difficult to obtain desired estimate for u by using the equation (1.1) directly. To avoid this difficulty, we introduce the function w by (4.3) and (4.7), and then w satisfies the equation (4.8). By using the equation (4.8), we derive estimates for w in H^m, which imply the desired estimate for the solution u and Theorem 1.1.

This paper is organized as follows. In Section 2, we introduce several notations which will be used below. In Section 3, we collect several lemmas which will be used in the proof of Theorem 1.1. In Section 4, we prove Theorem 1.1. In Section 5, we remark on the large time asymptotics of solutions.

2. Notation

We introduce several notations used below.

Let $\partial_k = \partial/\partial x_k$ for $k = 1, \ldots, n$, $\partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, and let $\omega = (-\Delta)^{1/2} = \mathcal{F}^{-1}|\xi|\mathcal{F}$. For linear operators P and Q, $[P, Q] = PQ - QP$ is the commutator of them.

For $\mathbb{C}^N$-valued functions, we use the following function spaces. $|\cdot|_{\mathbb{C}^N}$ and $(\cdot,\cdot)_{\mathbb{C}^N}$ denote the norm and the scalar product in $\mathbb{C}^N$, respectively. For a $\mathbb{C}^N$-valued measurable function $\psi = (\psi_1, \ldots \psi_N)$ on $\mathbb{R}^n$ and $1 \le p \le \infty$, $\psi \in L^p$ means that $\psi_j \in L^p$ for $j = 1, \ldots, N$, which is equivalent to $|\psi|_{\mathbb{C}^N} \in L^p$. Its norm is defined by

$$\|\psi\|_{L^p} \equiv \| \, |\psi(\cdot)|_{\mathbb{C}^N} \|_{L^p}.$$

For $\mathbb{C}^N$-valued measurable functions $\varphi = (\varphi_1, \ldots, \varphi_N)$ and $\psi = (\psi_1, \ldots, \psi_N)$ on $\mathbb{R}^n$, their L^2 scalar product is defined by

$$\langle \varphi, \psi \rangle_{L^2} \equiv \int_{\mathbb{R}^n} (\varphi(x), \psi(x))_{\mathbb{C}^N} \, dx = \sum_{j=1}^N \langle \varphi_j, \psi_j \rangle_{L^2}$$

Let $W^{k,p}$ and $\dot{W}^{k,p}$ be the k-th order Sobolev and the homogeneous Sobolev spaces based on L^p, respectively. Namely, for $k \ge 0$ and $1 \le p < \infty$, we denote

$$\begin{aligned} W^{k,p} &\equiv \{\psi \in L^p : \|\psi\|_{W^{k,p}} \equiv \|(1-\Delta)^{k/2}\psi\|_{L^p} < \infty\}, \\ \dot{W}^{k,p} &\equiv \{\psi \in \mathcal{S}' : \|\psi\|_{\dot{W}^{k,p}} \equiv \|\omega^k \psi\|_{L^p} < \infty\}. \end{aligned}$$

For a non-negative integer k, we define

$$\begin{aligned} W^{k,\infty} &\equiv \left\{ \psi \in L^\infty : \|\psi\|_{W^{k,\infty}} \equiv \sum_{|\alpha|\le k} \|\partial^\alpha \psi\|_{L^\infty} < \infty \right\}, \\ \dot{W}^{k,\infty} &\equiv \left\{ \psi \in \mathcal{S}' : \|\psi\|_{\dot{W}^{k,\infty}} \equiv \sum_{|\alpha| = k} \|\partial^\alpha \psi\|_{L^\infty} < \infty \right\}. \end{aligned}$$

For $k \ge 0$, H^k and $\dot{H}^k$ denote $W^{k,2}$ and $\dot{W}^{k,2}$, respectively. Then $H^k = H^{k,0}$.

For matrix-valued functions, we introduce the following notations and function spaces. We denote by M_N the set of $N \times N$ matrices with complex elements. For $A = (a_{j,k})_{1\le j,k\le N} \in M_N$, $|A|_{M_N}$ denotes the operator norm in $\mathbb{C}^N$ of A. Then there exist constants $C_1, C_2 > 0$ such that

$$C_1 \left(\sum_{j,k=1}^N |a_{j,k}|^2 \right)^{1/2} \le |A|_{M_N} \le C_2 \left(\sum_{j,k=1}^N |a_{j,k}|^2 \right)^{1/2}.$$

For an M_N-valued function $A = A(x) = (a_{j,k}(x))$ on $\mathbb{R}^n$ ($A : \mathbb{R}^N \to M_N$), we use the following notations. For $1 \le p \le \infty$, $A \in L^p$ means that $a_{j,k} \in L^p$ for $1 \le j,k \le N$, which is equivalent to $|A|_{M_N} \in L^p$. Its norm is defined by

$$\|A\|_{L^p} \equiv \| \, |A(\cdot)|_{M_N} \|_{L^p}.$$

$A \in \mathcal{S}'$ means that $a_{j,k} \in \mathcal{S}'$ for $1 \le j,k \le N$. For a Fourier multiplier K, KA denotes the matrix whose (j,k) element is $(Ka_{j,k})(x)$. For an M_N-valued function $A = A(x)$, we introduce the Sobolev and homogeneous Sobolev spaces as follows. For $k \ge 0$ and $1 \le p < \infty$, we denote

$$\begin{aligned} W^{k,p} &\equiv \{A \in L^p : \|A\|_{W^{k,p}} \equiv \| \, |(1-\Delta)^{k/2} A(\cdot)|_{M_N} \|_{L^p} < \infty\}, \\ \dot{W}^{k,p} &\equiv \{A \in \mathcal{S}' : \|A\|_{\dot{W}^{k,p}} \equiv \| \, |\omega^k A(\cdot)|_{M_N} \|_{L^p} < \infty\}. \end{aligned}$$

For a non-negative integer k, we define

$$W^{k,\infty} \equiv \left\{ A \in L^\infty : \|A\|_{W^{k,\infty}} \equiv \sum_{|\alpha| \le k} \| \, |\partial^\alpha A(\cdot)|_{M_N} \|_{L^\infty} < \infty \right\},$$
$$\dot{W}^{k,\infty} \equiv \left\{ A \in \mathcal{S}' : \|A\|_{\dot{W}^{k,\infty}} \equiv \sum_{|\alpha| = k} \| \, |\partial^\alpha A(\cdot)|_{M_N} \|_{L^\infty} < \infty \right\}.$$

We also use the following notations for matrix-valued functions. For $k \ge 0$, H^k and $\dot{H}^k$ denote $W^{k,2}$ and $\dot{W}^{k,2}$, respectively.

C denotes various constants, and they may differ from line to line, when it does not cause any confusion.

3. Preliminaries

In this section, we collect several lemmas, which will be used for the proof of Theorem 1.1.

LEMMA 3.1. *Let $n \ge 3$, $0 < \beta < n$, $2 < p < 2n/(n-\beta) < q$ and $1/p + 1/q = 1 - \beta/n$. Then there exists a constant $C > 0$ such that*

$$\| \, |\cdot|^{-\beta} * (\psi_1 \bar{\psi}_2) \|_{L^\infty} \le C (\|\psi_1\|_{L^p} \|\psi_2\|_{L^p} \|\psi_1\|_{L^q} \|\psi_2\|_{L^q})^{1/2}.$$

We can prove Lemma 3.1 exactly in the same way as in the proof of Lemma 2.4 in Wada [27].

LEMMA 3.2. *Let $0 < \beta < n$, and let p and q satisfy $1 < p, q < \infty$ and $1 + 1/q = 1/p + \beta/n$. Then there exists a constant $C > 0$ such that*

$$\| \, |\cdot|^{-\beta} * \psi \|_{L^q} = c \|\omega^{-(n-\beta)} \psi\|_{L^q} \le C \|\psi\|_{L^p}.$$

(*Here c is some positive constant independent of ψ.*)

Lemma 3.2 follows from the embedding $\dot{W}^{p,n-\beta}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$.

Next we introduce the Leibniz rule and the commutator estimate for fractional derivatives. (See, e.g., Kato [18], Kato-Ponce [19] or Ginibre-Velo [7, 9].)

LEMMA 3.3. *Let $s > 0$, $1 < p, q_1, q_2 < \infty$, $1 < r_1, r_2 \le \infty$ and $1/p = 1/q_1 + 1/r_1 = 1/q_2 + 1/r_2$. Then the following estimates hold:*

$$\|\omega^s(\varphi\psi)\|_{L^p} \le C(\|\omega^s\varphi\|_{L^{q_1}}\|\psi\|_{L^{r_1}} + \|\omega^s\psi\|_{L^{q_2}}\|\varphi\|_{L^{r_2}}),$$
$$\|\omega^s(\varphi\psi) - \varphi\omega^s\psi\|_{L^p} \le C(\|\omega^s\varphi\|_{L^{q_1}}\|\psi\|_{L^{r_1}} + \|\omega^{s-1}\psi\|_{L^{q_2}}\|\nabla\varphi\|_{L^{r_2}}).$$

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Our proof of the theorem is mainly based on the spirit of the proof in Wada [29] on the long-range scattering for the Hartree equation (1.3).

On the initial value problem (1.1)–(1.2), the following proposition holds.

PROPOSITION 4.1. *Let $n \ge 3$, $0 < \gamma < 2$ and $\phi \in L^2$. Then there exists a unique solution u to the initial value problem (1.1)–(1.2) in $C(\mathbb{R}; L^2) \cap L^{8/\gamma}_{loc}(\mathbb{R}; L^{4n/(2n-\gamma)})$. Moreover the following hold.*

- *The solution u satisfies*

$$\|u(t)\|_{L^2} = \|\phi\|_{L^2}$$

  for any $t \in \mathbb{R}$.

- *If $\phi \in H^{0,k}$ for $k \in \mathbb{N}$, then $|J|^k u \in C(\mathbb{R}; L^2)$. Furthermore, the solution $u \in C(\mathbb{R}; L^2)$ with $|J|^k u \in C(\mathbb{R}; L^2)$ is unique.*

- *There exists a $\delta > 0$ such that if*

$$\|\phi\|_{H^{0,k}} \le \delta \tag{4.1}$$

  for $k \in \mathbb{N}$, then

$$\sup_{t\in[0,2]} (\|u(t)\|_{L^2} + \|\,|J|^k u(t)\|_{L^2}) \le C_0\|\phi\|_{H^{0,k}}.$$

We can prove Proposition 4.1 in the same say as in the case of the Hartree equation (1.3) (see, e.g., Cazenave [1], Ginibre-Velo [3], Hayashi-Ozawa [16], Wada [29, 30]). Hence in this paper, we omit the proof of Proposition 4.1.

REMARK 4.1. In Proposition 4.1, growth rate of $\| |J|^m u(t)\|_{L^2}$, decay rate of $\|u(t)\|_{L^p}$ for $2 < p \leq \infty$ and asymptotics of the solution u are not obtained.

Throughout this section, let the space dimension $n \geq 3$, m be the positive integer defined by (1.7) and $\phi \in H^{0,m}$. Let $0 < \gamma < 1$.

For simplicity, we only consider the case that t is positive and large. So we may consider the time interval $[1, \infty)$.

It is well-known that the free evolution operator $U(t) = e^{it\Delta/2}$ is decomposed as

$$U(t) = M(t)D(t)\mathcal{F}M(t), \qquad (t \in \mathbb{R} \setminus \{0\}), \tag{4.2}$$

where the operators $M(t)$ and $D(t)$ are defined by

$$\begin{aligned} (M(t)\psi)(x) &= e^{i|x|^2/2t}\psi(x), \\ (D(t)\psi)(x) &= \frac{1}{(it)^{n/2}}\psi\left(\frac{x}{t}\right), \qquad (t \in \mathbb{R} \setminus \{0\}). \end{aligned}$$

The identity (4.2) is called the Dollard decomposition.

Let $\phi \in H^{0,m}$ satisfy the condition (4.1) and $u \in C(\mathbb{R}; L^2)$ with $|J|^m u \in C(\mathbb{R}; L^2)$ be the unique solution to the initial value problem (1.1)–(1.2) obtained in Proposition 4.1 with $k = m$. For $t \geq 1$ and $x \in \mathbb{R}^n$, we put

$$v(t,x) = \mathcal{F}M(t)U(-t)u(t) = (it)^{n/2}e^{-it|x|^2/2}u(t,tx). \tag{4.3}$$

In the second equality in (4.3), we have used the identity (4.2). Then the equation (1.1) is equivalent to

$$i\partial_t v = -\frac{1}{2t^2}\Delta v + t^{-\gamma}F(v)v, \tag{4.4}$$

when $t \geq 1$ and $x \in \mathbb{R}^n$. To overcome the difficulty caused by the long-range potential V, (as in the case of the Coulomb potential in Wada [30]), we introduce the solution B of the following initial value problem of an $N \times N$ matrix-valued ordinary differential equation

$$i\partial_t B = t^{-\gamma}F(v)B, \qquad t \geq 1, \quad x \in \mathbb{R}^n, \tag{4.5}$$

$$B(1,x) = I_N, \qquad x \in \mathbb{R}^n, \tag{4.6}$$

where I_N is the $N \times N$ unit matrix.

LEMMA 4.1. *When* $t \geq 1$ *and* $x \in \mathbb{R}^n$*, the solution* $B(t,x)$ *to the Cauchy problem* (4.5)–(4.6) *is an* $N \times N$ *unitary matrix.*

PROOF. Let $z \in \mathbb{C}^N$. Since the matrix $F(v)$ is Hermite, by the equation (4.5), we see that

$$\begin{aligned}\frac{d}{dt}|B(t,x)z|^2_{\mathbb{C}^N} =& 2\operatorname{Re}(\partial_t B(t,x)z, B(t,x)z)_{\mathbb{C}^N} \\ =& 2\operatorname{Re}(-it^{-\gamma}F(v(t,x))B(t,x)z, B(t,x)z)_{\mathbb{C}^N} \\ =& 0.\end{aligned}$$

Hence the above equality and the initial condition (4.6) yield

$$|B(t,x)z|_{\mathbb{C}^N} = |B(1,x)z|_{\mathbb{C}^N} = |z|_{\mathbb{C}^N}.$$

Therefore $B(t,x)$ is a unitary matrix when $t \geq 1$ and $x \in \mathbb{R}^N$. □

Let

$$w = B^* v. \tag{4.7}$$

Then w satisfies the following equation:

$$i\partial_t w = -\frac{1}{2t^2}\Delta w - \frac{1}{t^2}\sum_{k=1}^{n} B^*(\partial_k B)(\partial_k w) - \frac{1}{2t^2}B^*(\Delta B)w \tag{4.8}$$

for $t \geq 1$ and $x \in \mathbb{R}^n$.

To prove the estimates (1.10) and (1.11), it is sufficient to show that $\|w(t)\|_{H^m}$ is bounded in $t \in [1,\infty)$. (See the estiamtes (4.31) and (4.32) below.) We shall estimate the function w in H^m.

Let $0 < \rho \leq 1$ be fixed. We fix a constant $T > 1$ such that

$$\sup_{1 \leq t \leq T} \|w(t)\|_{H^m} \leq \rho \tag{4.9}$$

arbitrarily. (By Proposition 4.1, if $\|\phi\|_{H^{0,m}}$ is sufficiently small, then we can take a constant $T > 1$ satisfying the inequality (4.9). For example, it is sufficient to take ϕ sufficiently small such that $\|\phi\|_{H^{0,m}} \leq \min\{\delta, \frac{\rho}{C_0}\}$, where δ and C_0 are the constants appearing in Proposition 4.1.)

First we consider $\|w(t)\|_{L^2}$. Since $F(v)$ is an Hermitian matrix, the equation (4.4) implies that

$$
\begin{aligned}
\frac{d}{dt}\|w(t)\|_{L^2}^2 &= \frac{d}{dt}\|v(t)\|_{L^2}^2 \\
&= 2\,\mathrm{Re}\langle \partial_t v(t), v(t)\rangle_{L^2} \\
&= 2\,\mathrm{Re}\left\langle \frac{i}{2t^2}\Delta v - it^{-\gamma}F(v)v, v(t)\right\rangle_{L^2} \\
&= 0
\end{aligned} \tag{4.10}
$$

for $t \in [1, T]$.

Next, we estimate $\|w(t)\|_{\dot{H}^m} = \|\omega^m w(t)\|_{L^2}$. By the equation (4.8), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w(t)\|_{\dot{H}^m}^2 \\
= {}& \mathrm{Re}\langle \omega^m \partial_t w(t), \omega^m w(t)\rangle_{L^2} \\
= {}& \mathrm{Re}\bigg\langle \frac{i}{2t^2}\Delta\omega^m w(t) + \frac{i}{t^2}\omega^m \sum_{k=1}^{n} B(t)^*(\partial_k B(t))\partial_k w(t) \\
& \qquad + \frac{i}{2t^2}\omega^m (B(t)^*(\Delta B(t))w(t)), \omega^m w(t)\bigg\rangle_{L^2} \\
= {}& -\frac{1}{t^2}\sum_{k=1}^{n} \mathrm{Im}\langle B(t)^*(\partial_k B(t))\omega^m \partial_k w(t), \omega^m w(t)\rangle_{L^2} \\
& -\frac{1}{t^2}\sum_{k=1}^{n} \mathrm{Im}\langle [\omega^m, B(t)^*\partial_k B(t)]\partial_k w(t), \omega^m w(t)\rangle_{L^2} \\
& -\frac{1}{2t^2}\mathrm{Im}\langle \omega^m (B(t)^*(\Delta B(t))w(t)), \omega^m w(t)\rangle_{L^2}.
\end{aligned} \tag{4.11}
$$

To overcome loss of derivative in the first term of the right hand side of the equation (4.11), we calculate that term. Since $B^*B = I_N$, $B^*\partial_k B = -(\partial_k B^*)B$ for $k = 1, \ldots, n$. So we see that

$$
\begin{aligned}
&\langle B^*(\partial_k B)\omega^m \partial_k w, \omega^m w\rangle_{L^2} \\
= &- \langle (\partial_k B^*)B\partial_k \omega^m w, \omega^m w\rangle_{L^2} \\
= &\langle \omega^m w, \partial_k (B^*(\partial_k B)\omega^m w)\rangle_{L^2} \\
= &\langle \omega^m w, (\partial_k B^*)(\partial_k B)\omega^m w\rangle_{L^2} + \langle \omega^m w, B^*(\partial_k^2 B)\omega^m w\rangle_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\langle \omega^m w, B^*(\partial_k B)\omega^m \partial_k w\rangle_{L^2} \\
=&\langle (\partial_k B^*)(\partial_k B)\omega^m w, \omega^m w\rangle_{L^2} + \langle (\partial_k^2 B^*)B\omega^m w, \omega^m w\rangle_{L^2} \\
&+\overline{\langle B^*(\partial_k B)\omega^m \partial_k w, \omega^m w\rangle_{L^2}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\operatorname{Im}\langle B^*(\partial_k B)\omega^m \partial_k w, \omega^m w\rangle_{L^2} \\
= &-\frac{i}{2}\langle (\partial_k B^*)(\partial_k B)\omega^m w, \omega^m w\rangle_{L^2} - \frac{i}{2}\langle (\partial_k^2 B^*)B\omega^m w, \omega^m w\rangle_{L^2}.
\end{aligned}
\tag{4.12}
$$

By the equations (4.11) and (4.12), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w(t)\|_{\dot{H}^m}^2 \\
= &\frac{i}{2t^2}\sum_{k=1}^{n}\langle (\partial_k B(t)^*)(\partial_k B(t))\omega^m w(t), \omega^m w(t)\rangle_{L^2} \\
&+\frac{i}{2t^2}\langle (\Delta B(t)^*)B(t)\omega^m w(t), \omega^m w(t)\rangle_{L^2} \\
&-\frac{1}{t^2}\sum_{k=1}^{n}\operatorname{Im}\langle [\omega^m, B(t)^*\partial_k B(t)]\partial_k w(t), \omega^m w(t)\rangle_{L^2} \\
&-\frac{1}{2t^2}\operatorname{Im}\langle \omega^m (B(t)^*(\Delta B(t))w(t)), \omega^m w(t)\rangle_{L^2} \\
\equiv &\ Q_1(t)+Q_2(t)+Q_3(t)+Q_4(t).
\end{aligned}
\tag{4.13}
$$

To estimate the right hand side of the equality (4.13), (that is, $Q_1, \dots, Q_4$), we have to estimate $B(t)$ in appropriate spaces.

LEMMA 4.2. *Let l be an integer satisfying $0 \le l \le n-1$, and let p and q satisfy $2 < p < 2n/(n-\gamma-l) < q < 2n/(n-2)$ and $1/p+1/q = 1-(\gamma+l)/n$. Then there exists a constant $C_1 > 0$ independent of T such that*

$$
\|F(v(t))\|_{\dot{W}^{l,\infty}} \le C_1\|v(t)\|_{L^p}\|v(t)\|_{L^q} = C_1\|w(t)\|_{L^p}\|w(t)\|_{L^q} \tag{4.14}
$$

for $t \in [1, T]$. Furthermore, there exists a constant $C_2 > 0$ independent of T such that

$$
\|F(v(t))\|_{\dot{W}^{l,\infty}} \le C_2\|w(t)\|_{H^m}^2 \le C_2\rho^2 \tag{4.15}
$$

for $t \in [1,T]$*, where* m *is defined by* (1.7).

Proof. Noting $0 < \gamma + l < 1 + l \leq n$, by Lemma 3.1 and the unitarity of the matrix B, we have

$$\begin{aligned}\|F(v(t))\|_{\dot{W}^{l,\infty}} \leq& C\sum_{j,k=1}^{n} \| |\cdot|^{-\gamma-l} * (v_j(t)\overline{v_k(t)})\|_{L^\infty} \\ \leq& C\sum_{j,k=1}^{n} (\|v_j(t)\|_{L^p}\|v_k(t)\|_{L^p}\|v_j(t)\|_{L^q}\|v_k(t)\|_{L^q})^{1/2} \\ \leq& C_1\|v(t)\|_{L^p}\|v(t)\|_{L^q} \\ =& C_1\|w(t)\|_{L^p}\|w(t)\|_{L^q}.\end{aligned}$$

Therefore we obtain the estimate (4.14).

Next we show the estimate (4.15). Since $2 < p < 2n/(n-\gamma-l)$, $H^{(\gamma+l)/2} \hookrightarrow L^p$. In the estimate (4.14), for any $\varepsilon > 0$, we choose q (and p) sufficiently close to $2n/(n-\gamma-l)$ such that $H^{(\gamma+l)/2+\varepsilon} \hookrightarrow L^q$ holds. Note that $(\gamma+l)/2 < (1+l)/2 \leq n/2 < m$, since $0 \leq l \leq n-1$ and $0 < \gamma < 1$. We take the above ε such that $0 < \varepsilon < m - (\gamma+l)/2$. Then we have the embedding $H^m \hookrightarrow L^p \cap L^q$, if p and q are sufficiently close to $2n/(n-\gamma-l)$. The estimate (4.14) and the above embedding imply the estimate (4.15). □

Lemma 4.3. *Let* l *be an integer satisfying* $1 \leq l \leq n-1$*. Then there exists a constant* $C > 0$ *independent of* T *such that*

$$\|B(t)\|_{L^\infty} = 1, \tag{4.16}$$

$$\|B(t)\|_{\dot{W}^{l,\infty}} \leq C\rho^2 t^{l(1-\gamma)} \tag{4.17}$$

for $t \in [1,T]$.

Proof. Since the matrix $B(t,x)$ is unitary for $t \in [1,T]$ and $x \in \mathbb{R}^n$, $|B(t,x)|_{M_N} = 1$. Hence for any $t \in [1,T]$, we have $\|B(t)\|_{L^\infty} = 1$. The identity (4.16) is proved.

We prove the estimate (4.17) by the induction in l. Let $z \in \mathbb{C}^N$ and $\alpha \in \mathbb{Z}_+^n$ be a multi-index satisfying $|\alpha| \geq 1$. Since the matrix $F(v)$ is Hermite, we have

$$\frac{d}{dt}|\partial_x^\alpha B(t,x)z|_{\mathbb{C}^N}^2 \tag{4.18}$$

$$
\begin{aligned}
&= 2\,\mathrm{Re}(\partial_x^\alpha \partial_t B(t,x)z, \partial_x^\alpha B(t,x)z)_{\mathbb{C}^N} \\
&= 2\,\mathrm{Re}(-it^{-\gamma}\partial_x^\alpha (F(v(t,x))B(t,x))z, \partial_x^\alpha B(t,x)z)_{\mathbb{C}^N} \\
&= 2t^{-\gamma}\,\mathrm{Im}\left(\left\{\sum_{\beta\le\alpha}\binom{\alpha}{\beta}(\partial_x^{\alpha-\beta}F(v(t,x)))(\partial_x^\beta B(t,x))\right\}z,\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\left.\partial_x^\alpha B(t,x)z\right)_{\mathbb{C}^N} \\
&= 2t^{-\gamma}\sum_{\beta\le\alpha,\ \beta\ne\alpha}\binom{\alpha}{\beta} \\
&\qquad \times \mathrm{Im}\left(\{(\partial_x^{\alpha-\beta}F(v(t,x)))(\partial_x^\beta B(t,x))\}z, \partial_x^\alpha B(t,x)z\right)_{\mathbb{C}^N}.
\end{aligned}
$$

Note that if $|\alpha| \ge 1$, then $\partial_x^\alpha B(1,x) = 0$ for any $x \in \mathbb{R}^n$ by the initial condition (4.6). Since $|\alpha| \ge 1$, the equality (4.18) implies

$$
\begin{aligned}
&|\partial_x^\alpha B(t,x)z|_{\mathbb{C}^N} \\
\le\ & C\sum_{\beta\le\alpha,\ \beta\ne\alpha}\int_1^t \tau^{-\gamma}|(\partial_x^{\alpha-\beta}F(v(t,x))|_{M_N}|\partial_x^\beta B(t,x)|_{M_N}\,d\tau|z|_{\mathbb{C}^N},
\end{aligned}
$$

and hence

$$
\begin{aligned}
&|\partial_x^\alpha B(t,x)|_{M_N} \\
\le\ & C\sum_{\beta\le\alpha,\ \beta\ne\alpha}\int_1^t \tau^{-\gamma}|\partial_x^{\alpha-\beta}F(v(\tau,x))|_{M_N}|\partial_x^\beta B(\tau,x)|_{M_N}\,d\tau.
\end{aligned}
\tag{4.19}
$$

Therefore we have

$$
\begin{aligned}
&\|\partial_x^\alpha B(t)\|_{L^\infty} \\
\le\ & C\sum_{\beta\le\alpha,\ \beta\ne\alpha}\int_1^t \tau^{-\gamma}\|\partial_x^{\alpha-\beta}F(v(\tau))\|_{L^\infty}\|\partial_x^\beta B(\tau)\|_{L^\infty}\,d\tau.
\end{aligned}
\tag{4.20}
$$

First we show the estimate (4.17) for $l = 1$. By the estimate (4.20) with $|\alpha| = 1$ and (4.16), and Lemma 4.2, we have

$$
\|B(t)\|_{\dot W^{1,\infty}} \le C\int_1^t \tau^{-\gamma}\|F(v(\tau))\|_{\dot W^{1,\infty}}\|B(\tau)\|_{L^\infty}\,d\tau
$$

$$\begin{aligned} &\le C\int_1^t \tau^{-\gamma}\rho^2\,d\tau \\ &\le C\rho^2 t^{1-\gamma}. \end{aligned}$$

Thus the estimate (4.17) with $l=1$ is proved.

Next assume that the inequality (4.17) holds for $l=1,\dots,k$. (k is a some integer satisfying $1\le k\le n-2$.) We prove the inequality (4.17) for $l=k+1$. Noting $2\le k+1\le n-1$, by the estimate (4.20) with $|\alpha|=k+1$, Lemma 4.2 and the assumption of the induction, we have

$$\begin{aligned} \|B(t)\|_{\dot W^{k+1,\infty}} &\le C\int_1^t \tau^{-\gamma}\|F(v(\tau))\|_{\dot W^{1,\infty}\cap\dot W^{k+1,\infty}}\|B(\tau)\|_{W^{k,\infty}}\,d\tau \\ &\le C\int_1^t \tau^{-\gamma}\rho^2\times\rho^2\tau^{k(1-\gamma)}\,d\tau \\ &\le C\rho^2 t^{(k+1)(1-\gamma)}. \end{aligned}$$

Thus the estimate (4.17) with $l=k+1$ is proved. Therefore this completes the proof of the estimate (4.17). □

LEMMA 4.4. *If $1\le p\le\infty$, then*

$$\|v(t)\|_{L^p}=\|w(t)\|_{L^p}$$

for $t\in[1,T]$. Moreover, let l be an integer satisfying $1\le l\le n-1$, then there exists a constant $C>0$ independent of T such that

$$\|v(t)\|_{\dot H^l}\le C\rho^2\|w(t)\|_{H^l}t^{l(1-\gamma)}$$

for $t\in[1,T]$.

PROOF. Since B is a unitary matrix, we have $\|v(t)\|_{L^p}=\|w(t)\|_{L^p}$ for $1\le p\le\infty$.

Next we show the second estimate. Since $1\le l\le n-1$. by the Leibniz rule and Lemma 4.3, we have

$$\begin{aligned} \|v(t)\|_{\dot H^l} =&\|B(t)w(t)\|_{\dot H^l} \\ \le& C\sum_{|\alpha|=l}\|\partial_x^\alpha(B(t)w(t))\|_{L^2} \\ \le& C\|B(t)\|_{W^{l,\infty}}\|w(t)\|_{H^l} \\ \le& C\rho^2 t^{l(1-\gamma)}\|w(t)\|_{H^l}. \end{aligned}$$

Therefore the second inequality is shown. □

LEMMA 4.5. *Let l be an integer satisfying $1 \le l \le n$, and let m be the integer defined by* (1.7)*. Then there exists a constant $C > 0$ independent of T such that*

$$\begin{aligned}\|F(v(t))\|_{\dot{W}^{l,2n/\gamma}} &\le C\|w(t)\|_{H^{l-1}}\|w(t)\|_{\dot{H}^{\gamma/2+1}} t^{(l-1)(1-\gamma)} \\ &\le C\|w(t)\|_{H^{l-1}}\|w(t)\|_{H^m} t^{(l-1)(1-\gamma)}\end{aligned}$$

for $t \in [1, T]$.

PROOF. Since $0 \le l-1 \le n-1$ and $\gamma/2+1 < 3/2 < m$, by Lemmas 3.2, 3.3 and 4.4, we have

$$\begin{aligned}
& \|F(v(t))\|_{\dot{W}^{l,2n/\gamma}} \\
\le\ & C \sum_{j,k=1}^{n} \|\omega^{-(n-\gamma-1)}\omega^{l-1}(v_j(t)\overline{v_k(t)})\|_{L^{2n/\gamma}} \\
\le\ & C \sum_{j,k=1}^{n} \|\omega^{l-1}(v_j(t)\overline{v_k(t)})\|_{L^{2n/(2n-\gamma-2)}} \\
\le\ & C \sum_{j,k=1}^{n} (\|\omega^{l-1}v_j(t)\|_{L^2}\|v_k(t)\|_{L^{2n/(n-\gamma-2)}} \\
& \qquad + \|v_j(t)\|_{L^{2n/(n-\gamma-2)}}\|\omega^{l-1}v_k(t)\|_{L^2}) \\
\le\ & C\|\omega^{l-1}v(t)\|_{L^2}\|v(t)\|_{L^{2n/(n-\gamma-2)}} \\
=\ & C\|v(t)\|_{\dot{H}^{l-1}}\|w(t)\|_{L^{2n/(n-\gamma-2)}} \\
\le\ & Ct^{(l-1)(1-\gamma)}\|w(t)\|_{H^{l-1}}\|w(t)\|_{\dot{H}^{\gamma/2+1}} \\
\le\ & C\|w(t)\|_{H^{l-1}}\|w(t)\|_{H^m} t^{(l-1)(1-\gamma)}.
\end{aligned}$$

This completes the proof of the lemma. □

LEMMA 4.6. *Let m be the integer defined by* (1.7)*. There exists a constant $C > 0$ independent of T such that*

$$\|F(v(t))\|_{\dot{W}^{1,2n/(\gamma+2)}} \le C\|w(t)\|^2_{\dot{H}^{\gamma/4}} \le C\|w(t)\|^2_{H^m}$$

for $t \in [1, T]$. *Furthermore, let* l *be an integer satisfying* $2 \leq l \leq n+1$. *Then there exists a constant* $C > 0$ *independent of* T *such that*

$$\begin{aligned} \|F(v(t))\|_{\dot{W}^{l,2n/(\gamma+2)}} &\leq C\|w(t)\|_{H^{l-2}}\|w(t)\|_{\dot{H}^{\gamma/2+1}} t^{(l-2)(1-\gamma)} \\ &\leq C\|w(t)\|_{H^{l-2}}\|w(t)\|_{H^m} t^{(l-2)(1-\gamma)} \end{aligned}$$

for $t \in [1, T]$.

PROOF. We prove the first estimate. By Lemma 3.2 and the fact $\gamma/4 < 1/4 < m$, we obtain

$$\begin{aligned} \|F(v(t))\|_{\dot{W}^{1,2n/(\gamma+2)}} &\leq C \sum_{j,k=1}^{n} \|\omega^{-(n-\gamma-1)}(v_j(t)\overline{v_k(t)})\|_{L^{2n/(\gamma+2)}} \\ &\leq C \sum_{j,k=1}^{n} \|v_j(t)\overline{v_k(t)}\|_{L^{2n/(2n-\gamma)}} \\ &\leq C\|v(t)\|^2_{L^{4n/(2n-\gamma)}} \\ &= C\|w(t)\|^2_{L^{4n/(2n-\gamma)}} \\ &\leq C\|w(t)\|^2_{\dot{H}^{\gamma/4}} \\ &\leq C\|w(t)\|^2_{H^m}. \end{aligned}$$

Next we show the second estimate. Since $0 \leq l-2 \leq n-1$ and $\gamma/2+1 < 3/2 < m$, by Lemmas 3.2, 3.3 and 4.4, we have

$$\begin{aligned} &\|F(v(t))\|_{\dot{W}^{l,2n/(\gamma+2)}} \\ \leq\ & C \sum_{j,k=1}^{n} \|\omega^{-(n-\gamma-2)}\omega^{l-2}(v_j(t)\overline{v_k(t)})\|_{L^{2n/(\gamma+2)}} \\ \leq\ & C \sum_{j,k=1}^{n} \|\omega^{l-2}(v_j(t)\overline{v_k(t)})\|_{L^{2n/(2n-\gamma-2)}} \\ \leq\ & C \sum_{j,k=1}^{n} (\|\omega^{l-2}v_j(t)\|_{L^2}\|v_k(t)\|_{L^{2n/(n-\gamma-2)}} \\ & \qquad + \|v_j(t)\|_{L^{2n/(n-\gamma-2)}}\|\omega^{l-2}v_k(t)\|_{L^2}) \\ \leq\ & C\|\omega^{l-2}v(t)\|_{L^2}\|v(t)\|_{L^{2n/(n-\gamma-2)}} \\ =\ & C\|v(t)\|_{\dot{H}^{l-2}}\|w(t)\|_{L^{2n/(n-\gamma-2)}} \end{aligned}$$

$$
\begin{aligned}
&\le Ct^{(l-2)(1-\gamma)}\|w(t)\|_{H^{l-2}}\|w(t)\|_{\dot{H}^{\gamma/2+1}} \\
&\le C\|w(t)\|_{H^{l-2}}\|w(t)\|_{H^m}t^{(l-2)(1-\gamma)}.
\end{aligned}
$$

This completes the proof of the lemma. □

Lemma 4.7. *Let m be the integer defined by* (1.7). *Then there exists a constant $C>0$ independent of T such that*

$$\|B(t)\|_{\dot{W}^{m,2n/\gamma}} \le C\rho^2 t^{m(1-\gamma)}, \tag{4.21}$$

$$\|B(t)\|_{\dot{W}^{m+1,2n/\gamma}} \le C\rho^2 t^{(m+1)(1-\gamma)}, \tag{4.22}$$

$$\|B(t)\|_{\dot{W}^{m,2n/(\gamma+2)}} \le C\rho^2 t^{m(1-\gamma)}, \tag{4.23}$$

$$\|B(t)\|_{\dot{W}^{m+1,2n/(\gamma+2)}} \le C\rho^2 t^{(m+1)(1-\gamma)}, \tag{4.24}$$

$$\|B(t)\|_{\dot{W}^{m+2,2n/(\gamma+2)}} \le C\rho^2 t^{(m+2)(1-\gamma)} \tag{4.25}$$

for $t \in [1,T]$.

Proof. First we show the estimates (4.21) and (4.22). By the definition (1.7) of m, we see that $2 \le m \le n-1$ and that $3 \le m+1 \le n$. Let $l = m$ or $m+1$, and let j be an integer satisfying $0 \le j \le l-1$. Then $1 \le l-j \le l \le m+1 \le n$ and $0 \le j \le l-1 \le m \le n-1$, and hence we see that $l-j$ and j satisfy the assumptions of Lemmas 4.5 and 4.3, respectively. By the estimate (4.19), the Hölder inequality, and Lemmas 4.3 and 4.5, we have

$$
\begin{aligned}
\|B(t)\|_{\dot{W}^{l,2n/\gamma}} &\le C\sum_{j=0}^{l-1}\int_1^t \tau^{-\gamma}\|F(v(\tau))\|_{\dot{W}^{l-j,2n/\gamma}}\|B(\tau)\|_{\dot{W}^{j,\infty}}\,d\tau \\
&\le C\sum_{j=0}^{l-1}\int_1^t \tau^{-\gamma}\rho^2\tau^{(l-j-1)(1-\gamma)}\tau^{j(1-\gamma)}\,d\tau \\
&\le C\rho^2 t^{l(1-\gamma)}.
\end{aligned}
$$

Therefore the estimates (4.21) and (4.22) are proved.

Next we show the estimates (4.23) and (4.24). Since $1 \le l-j \le m+1 \le n$ and $0 \le j \le l-1 \le m \le n-1$ as above, we see that $l-j$ and j satisfy the assumptions of Lemmas 4.6 and 4.3, respectively. By the estimate (4.19), the Hölder inequality, and Lemmas 4.3 and 4.6, we have

$$\begin{aligned}
\|B(t)\|_{\dot{W}^{l,2n/(\gamma+2)}} &\leq C\sum_{j=0}^{l-1}\int_1^t \tau^{-\gamma}\|F(v(\tau))\|_{\dot{W}^{l-j,2n/(\gamma+2)}}\|B(\tau)\|_{\dot{W}^{j,\infty}}\,d\tau \\
&= C\sum_{j=0}^{l-2}\int_1^t \tau^{-\gamma}\|F(v(\tau))\|_{\dot{W}^{l-j,2n/(\gamma+2)}}\|B(\tau)\|_{\dot{W}^{j,\infty}}\,d\tau \\
&\quad +C\int_1^t \tau^{-\gamma}\|F(v(\tau))\|_{\dot{W}^{1,2n/(\gamma+2)}}\|B(\tau)\|_{\dot{W}^{l-1,\infty}}\,d\tau \\
&\leq C\sum_{j=0}^{l-2}\int_1^t \rho^2\tau^{-\gamma}\tau^{(l-j-2)(1-\gamma)}\tau^{j(1-\gamma)}\,d\tau \\
&\quad +C\rho^2\int_1^t \tau^{-\gamma}\tau^{(l-1)(1-\gamma)}\,d\tau \\
&\leq C\rho^2 t^{l(1-\gamma)}.
\end{aligned}$$

Hence the estimates (4.23) and (4.24) are shown.

Finally we prove the estimate (4.25). Let j be an integer satisfying $0 \leq j \leq m$. Then $2 \leq m+2-j \leq m+2 \leq n+1$ and $0 \leq j \leq m \leq n-1$, and hence we see that $l-j$ and j satisfy the assumptions of Lemmas 4.6 and 4.3, respectively. By the estimate (4.19), the Hölder inequality, Lemmas 4.2, 4.3 and 4.6, and the inequality (4.24), we have

$$\begin{aligned}
&\|B(t)\|_{\dot{W}^{m+2,2n/(\gamma+2)}} \\
&\leq C\sum_{j=0}^{m}\int_1^t \tau^{-\gamma}\|F(v(\tau))\|_{\dot{W}^{m+2-j,2n/(\gamma+2)}}\|B(\tau)\|_{\dot{W}^{j,\infty}}\,d\tau \\
&\quad + C\int_1^t \tau^{-\gamma}\|F(v(\tau))\|_{\dot{W}^{1,\infty}}\|B(\tau)\|_{\dot{W}^{m+1,2n/(\gamma+2)}}\,d\tau \\
&\leq C\sum_{j=0}^{m}\int_1^t \rho^2\tau^{-\gamma}\tau^{(m+2-j-2)(1-\gamma)}\tau^{j(1-\gamma)}\,d\tau \\
&\quad + C\rho^2\int_1^t \tau^{-\gamma}\tau^{(m+1)(1-\gamma)}\,d\tau \\
&\leq C\rho^2 t^{(m+2)(1-\gamma)}.
\end{aligned}$$

Hence the estimate (4.25) is proved. □

LEMMA 4.8. *Let m be the integer defined by (1.7), and let Q_1, Q_2, Q_3 and Q_4 be the functions defined in the equation (4.13). Then there exists a constant $C > 0$ independent of T such that*

$$\begin{aligned}
|Q_1(t)| &\leq C\rho^2 t^{-2\gamma}\|w(t)\|_{\dot{H}^m}^2,\\
|Q_2(t)| &\leq C\rho^2 t^{-2\gamma}\|w(t)\|_{\dot{H}^m}^2,\\
|Q_3(t)| &\leq C\rho^2 t^{(m+1)(1-\gamma)-2}\|w(t)\|_{H^m}^2,\\
|Q_4(t)| &\leq C\rho^2 t^{(m+2)(1-\gamma)-2}\|w(t)\|_{H^m}^2
\end{aligned}$$

for $t \in [1, T]$

PROOF. First we note that $n/2 < m \leq n-1$.

We begin with the functions Q_1 and Q_2. By Lemma 4.3, we have

$$\begin{aligned}
|Q_1(t)| =& \frac{1}{2t^2}\left|\sum_{k=1}^{n}\langle(\partial_k B(t)^*)(\partial_k B(t))\omega^m w(t), \omega^m w(t)\rangle_{L^2}\right|\\
\leq& \frac{1}{2t^2}\sum_{k=1}^{n}\|\partial_k B(t)^*\|_{L^\infty}\|\partial_k B(t)\|_{L^\infty}\|\omega^m w(t)\|_{L^2}^2\\
\leq& C\rho^2 t^{-2\gamma}\|w(t)\|_{\dot{H}^m}^2
\end{aligned}$$

and

$$\begin{aligned}
|Q_2(t)| =& \frac{1}{2t^2}|\langle(\Delta B(t)^*)B(t)\omega^m w(t), \omega^m w(t)\rangle_{L^2}|\\
\leq& \frac{1}{2t^2}\|\Delta B(t)^*\|_{L^\infty}\|B(t)\|_{L^\infty}\|\omega^m w(t)\|_{L^2}^2\\
\leq& C\rho^2 t^{-2\gamma}\|w(t)\|_{\dot{H}^m}^2
\end{aligned}$$

for $t \in [1, T]$. Therefore the first and second inequalities are proved.

Next we estimate Q_3 and Q_4. By Lemmas 3.3, 4.3 and 4.7, the Sobolev embedding theorem and the fact $\gamma/2 + 1 < 3/2 < m$, we have

$$\begin{aligned}
& |Q_3(t)|\\
=\ & \frac{1}{t^2}\left|\sum_{k=1}^{n}\mathrm{Im}\langle[\omega^m, B(t)^*\partial_k B(t)]\partial_k w(t), \omega^m w(t)\rangle_{L^2}\right|\\
\leq\ & t^{-2}\sum_{k=1}^{n}\|[\omega^m, B(t)^*\partial_k B(t)]\partial_k w(t)\|_{L^2}\|\omega^m w(t)\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&\leq Ct^{-2}\left[\sum_{k=1}^{n}\left\{\|\omega^m(B(t)^*\partial_k B(t))\|_{L^{2n/\gamma}}\|\partial_k w(t)\|_{L^{2n/(n-\gamma)}}\right.\right.\\
&\quad \left.\left.+\left(\sum_{j=1}^{n}\|\partial_j(B(t)^*\partial_k B(t))\|_{L^\infty}\right)\|\omega^{m-1}\partial_k w(t)\|_{L^2}\right\}\right]\|\omega^m w(t)\|_{L^2}\\
&\leq Ct^{-2}\left[\sum_{k=1}^{n}(\|\omega^m B(t)^*\|_{L^{2n/\gamma}}\|\partial_k B(t)\|_{L^\infty}\right.\\
&\quad +\|B(t)^*\|_{L^\infty}\|\omega^m\partial_k B(t)\|_{L^{2n/\gamma}})\|w(t)\|_{H^{\gamma/2+1}}\\
&\quad +(\|B(t)^*\|_{\dot W^{1,\infty}}\|B(t)\|_{\dot W^{1,\infty}}\\
&\quad \left.+\|B(t)^*\|_{L^\infty}\|B(t)\|_{\dot W^{2,\infty}})\|w(t)\|_{\dot H^m}\right]\|\omega^m w(t)\|_{L^2}\\
&\leq C\rho^2 t^{-2}(t^{(m+1)(1-\gamma)}\|w(t)\|_{H^{\gamma/2+1}}+t^{2(1-\gamma)}\|w(t)\|_{\dot H^m})\|w(t)\|_{\dot H^m}\\
&\leq C\rho^2 t^{(m+1)(1-\gamma)-2}\|w(t)\|_{H^m}^2.
\end{aligned}
$$

and

$$
\begin{aligned}
&|Q_4(t)|\\
&= \frac{1}{2t^2}|\mathrm{Im}\langle\omega^m(B(t)^*(\Delta B(t))w(t)),\omega^m w(t)\rangle_{L^2}|\\
&\leq \frac{1}{2t^2}|\langle\omega^m(B(t)^*(\Delta B(t))w(t)),\omega^m w(t)\rangle_{L^2}|\\
&\leq \frac{1}{2t^2}\|\omega^m(B(t)^*(\Delta B(t))w(t))\|_{L^2}\|\omega^m w(t)\|_{L^2}\\
&\leq Ct^{-2}(\|\omega^m(B(t)^*\Delta B(t))\|_{L^{2n/(\gamma+2)}}\|w(t)\|_{L^{2n/(n-\gamma-2)}}\\
&\qquad +\|B(t)^*\Delta B(t)\|_{L^\infty}\|\omega^m w(t)\|_{L^2})\|\omega^m w(t)\|_{L^2}\\
&\leq Ct^{-2}\{(\|\omega^m B(t)^*\|_{L^{2n/(\gamma+2)}}\|\Delta B(t)\|_{L^\infty}\\
&\qquad +\|B(t)^*\|_{L^\infty}\|\omega^{m+2}B(t)\|_{L^{2n/(\gamma+2)}})\|w(t)\|_{L^{2n/(n-\gamma-2)}}\\
&\qquad +\|B(t)^*\|_{L^\infty}\|\Delta B(t)\|_{L^\infty}\|\omega^m w(t)\|_{L^2}\}\|\omega^m w(t)\|_{L^2}\\
&\leq C\rho^2 t^{-2}(t^{(m+2)(1-\gamma)}\|w(t)\|_{H^{\gamma/2+1}}+t^{2(1-\gamma)}\|w(t)\|_{\dot H^m})\|w(t)\|_{\dot H^m}\\
&\leq C\rho^2 t^{(m+2)(1-\gamma)-2}\|w(t)\|_{H^m}^2.
\end{aligned}
$$

Hence the third and fourth inequalities are proved. □

By the equalities (4.10) and (4.13), and Lemma 4.8, we obtain the following proposition.

PROPOSITION 4.2. *Let $\rho \in (0,1]$ and $T > 1$ be the constants fixed above, $\phi \in H^{0,m}$, $u \in C(\mathbb{R}; L^2)$ with $|J|^m u \in C(\mathbb{R}; L^2)$ be the unique solution to the initial value problem* (1.1)–(1.2) *obtained in Proposition 4.1 with $k = m$, and let $w \in C([0,T]; H^m)$ be the function defined by* (4.3) *and* (4.7), *where m be the integer defined by* (1.7). *Assume that the condition* (4.9) *holds. Then there exists a constant $C > 0$ independent of T such that*

$$\frac{d}{dt}\|w(t)\|_{H^m}^2 \le C\rho^2 t^{(m+2)(1-\gamma)-2}\|w(t)\|_{H^m}^2$$

for $t \in [1,T]$.

Now we prove Theorem 1.1.

PROOF OF THEOREM 1.1. Suppose that the assumptions of Theorem 1.1 are satisfied. In particular, assume that $1 - 1/(m+2) < \gamma < 1$, where m is defined by (1.7).

Let $\phi \in H^{0,m}$, and it satisfy the condition (4.1), and $u \in C(\mathbb{R}; L^2)$ with $|J|^m u \in C(\mathbb{R}; L^2)$ be the unique solution to the initial value problem (1.1)–(1.2) obtained in Proposition 4.1 with $k = m$. By Proposition 4.1, the equality (1.9) holds.

We show the estimate (1.10). Let $\rho \in (0,1]$, and

$$T^* = \sup\left\{T : T > 1 \text{ and } \sup_{t\in[1,T]} \|w(t)\|_{H^m} \le \rho\right\},$$

where w is the function defined by (4.3) and (4.7). It is sufficient to show $T^* = \infty$. Let $T \in [1, T^*)$ be arbitrary. By Proposition 4.2 and the Gronwall inequality, we have

$$
\begin{aligned}
(4.26)\qquad & \|w(t)\|_{H^m} \\
\le\ & \|w(1)\|_{H^m} \exp\left(C\rho^2 \int_1^t \tau^{(m+2)(1-\gamma)-2}\, d\tau\right) \\
\le\ & C'\|U(-1)u(1)\|_{H^{0,m}} \exp\left(C\rho^2 \int_1^t \tau^{(m+2)(1-\gamma)-2}\, d\tau\right) \\
\le\ & C'\|\phi\|_{H^{0,m}} \exp\left(C\rho^2 \int_1^t \tau^{(m+2)(1-\gamma)-2}\, d\tau\right)
\end{aligned}
$$

for $t \in [1, T]$. Note that $(m+2)(1-\gamma)-2 < -1$, since $1-1/(m+2) < \gamma < 1$. Hence, by the estimate (4.26), there exist positive constants C_1, C_2 and $C_3 > 0$ independent of T such that

$$\|w(t)\|_{H^m} \leq C_1 e^{C_2\rho^2} \|\phi\|_{H^{0,m}} \leq C_3 \|\phi\|_{H^{0,m}} \tag{4.27}$$

for $t \in [1, T]$. Here we have noted that $0 < \rho \leq 1$. Now we choose the initial data ϕ sufficiently small such that

$$C_3 \|\phi\|_{H^{0,m}} \leq \frac{\rho}{2}. \tag{4.28}$$

By the inequalities (4.27) and (4.28), we have

$$\|w(t)\|_{H^m} \leq \frac{\rho}{2}$$

for $t \in [1, T]$. Hence for any $T \in [1, T^*)$,

$$\sup_{t\in[1,T]} \|w(t)\|_{H^m} \leq \frac{\rho}{2}. \tag{4.29}$$

The estimate (4.29) implies $T^* = \infty$. (Indeed, if $T^* < \infty$, then the inequality (4.29) contradicts the definition of T^*.) Therefore under the condition (4.28), we see that $w \in C([1, \infty); H^m)$ and

$$\|w(t)\|_{H^m} \leq \rho$$

for $t \geq 1$, and moreover by the estimate (4.27), we have

$$\|w(t)\|_{H^m} \leq C \|\phi\|_{H^{0,m}} \tag{4.30}$$

for any $t \geq 1$. This implies that if $\|\phi\|_{H^{0,m}}$ is sufficiently small (precisely speaking, if the conditions (4.1) and (4.28) hold), then the unique solution u to the initial value problem (1.1)–(1.2) satisfies the estimate (1.10). In fact, by the definitions of v and w (see (4.3) and (4.7)), Lemma 4.4 and the estimate (4.30), we see that for $t \geq 1$,

$$\begin{aligned} \| \, |J|^m u(t)\|_{L^2} &= \| \, |J|^m U(t) M^{-1} \mathcal{F}^{-1} v(t)\|_{L^2} \\ &= \|\{U(t)|x|^m U(-t)\} U(t) M^{-1} \mathcal{F}^{-1} v(t)\|_{L^2} \\ &= \| |x|^m \mathcal{F}^{-1} v(t)\|_{L^2} \end{aligned} \tag{4.31}$$

$$
\begin{aligned}
&= \|v(t)\|_{\dot{H}^m} \\
&\leq C\|w(t)\|_{H^m} t^{m(1-\gamma)} \\
&\leq C\|\phi\|_{H^{0,m}} t^{m(1-\gamma)}.
\end{aligned}
$$

This implies the estimate (1.10).

Finally we prove the time decay estimate (1.11) as follows. Let $2 \leq p \leq \infty$. By the definitions of v and w (see (4.3) and (4.7)), the unitarity of the matrix B, the estimate (4.27) and the embedding $H^m \hookrightarrow L^2 \cap L^\infty$ (since $m > n/2$), we have

$$
\begin{aligned}
\|u(t)\|_{L^p} &= t^{-n/2} \left\| v\left(t, \frac{\cdot}{t}\right) \right\|_{L^p} \\
&= t^{-n(1/2-1/p)} \|v(t, \cdot)\|_{L^p} \\
&= t^{-n(1/2-1/p)} \|w(t)\|_{L^p} \\
&\leq Ct^{-n(1/2-1/p)} \|w(t)\|_{H^m} \\
&\leq C\|\phi\|_{H^{0,m}} t^{-n(1/2-1/p)}
\end{aligned}
\tag{4.32}
$$

for any $t \geq 1$. Therefore the estimate (1.11) is proved. This completes the proof of Theorem 1.1. □

5. Concluding Remark

Finally we remark about the asymptotics in large time of the solution to the initial value problem (1.1)–(1.2) in the case $0 < \gamma < 1$.

Suppose that the assumptions of Theorem 1.1 are satisfied. Let u be the unique solution to that initial value problem obtained in Theorem 1.1, and let v and w be the function defined by (4.3) and (4.7), respectively. To investigate the large time behavior of the solution u, it is sufficient to obtain asymptotics of the function v. To do so, first we investigate asymptotics of the function w. By the equation (4.8), the estimate (4.30) and Lemma 4.3, it is easy to see that for $t \geq 1$,

$$
\begin{aligned}
&\|\partial_t w(t)\|_{L^2} \\
&\leq Ct^{-2} \left(\|\Delta w(t)\|_{L^2} + \sum_{k=1}^{n} \|\partial_k B(t)\|_{L^\infty} \|\partial_k w(t)\|_{L^2} \right. \\
&\qquad\qquad \left. + \|\Delta B(t)\|_{L^\infty} \|w(t)\|_{L^2} \right)
\end{aligned}
$$

$$\leq \quad Ct^{-2\gamma}.$$

Therefore for $1 \leq t < t'$,

$$\begin{aligned}\|w(t) - w(t')\|_{L^2} &\leq \int_t^{t'} \|\partial_\tau w(\tau)\|_{L^2}\, d\tau \\ &\leq C \int_t^{t'} \tau^{-2\gamma}\, d\tau \\ &\leq Ct^{1-2\gamma},\end{aligned}$$

since $-2\gamma < -2(1 - 1/(m+2)) < -1$. Noting that $1 - 2\gamma < 0$, we see that there exists a unique $\Phi_+ \in L^2$ such that

$$\|w(t) - \Phi_+\|_{L^2} \leq Ct^{1-2\gamma}$$

for $t \geq 1$. In particular,

$$\text{(5.1)} \qquad \operatorname*{s\text{-}lim}_{t\to\infty} w(t) = \Phi_+ \qquad \text{in } L^2.$$

Since $\|w(t)\|_{H^m}$ is bounded in t, the fact (5.1) implies w-$\lim_{t\to\infty} w(t) = \Phi_+$ in H^m, and hence $\Phi_+ \in H^m$. Therefore we obtain the asymptotics $\Phi_+ \in H^m$ of the function w in L^2. To get asymptotics of v from that of w, we have to obtain asymptotics of the matrix B. Unfortunately, $F(v) = F(Bw) \neq F(w)$, and hence it is not easy to obtain asymptotics of B by using Φ_+ directly.

As mentioned in Section 1, the asymptotics in large time of the solution to the initial value problem of the Hartree equation (1.3) with $\gamma < 1$ was obtained by, e.g., Hayashi, Kaikina and Naumkin [10], Hayashi and Naumkin [12, 13] and Wada [29]. For the Hartree equation (1.3), they used $\tilde{w} = \exp(i \int_1^t V * |v(\tau)|^2\, d\tau)v$ instead of our w. Fortunately, for the Hartree equation (1.3), $\exp(i \int_1^t V * |v(\tau)|^2\, d\tau) = \exp(i \int_1^t V * |\tilde{w}(\tau)|^2\, d\tau)$. So it is possible to investigate asymptotics of $\exp(i \int_1^t V * |v(\tau)|^2\, d\tau)$ by using the asymptotics of $\tilde{w}$ directly.

We return to the Hartree-Fock equation (1.1). As mentioned in Section 1, in the case $\gamma = 1$, Wada [30] obtained the asymptotics in large time of the solution to the initial value problem (1.1)–(1.2). In Lemma 4.5 in [30], he overcame the above difficulty by constructing a M_N-valued function $\widetilde{A} = \widetilde{A}(t, \xi)$ such that

$$\text{(5.2)} \qquad i\partial_t \widetilde{A} = t^{-1} F(\widetilde{A}\widetilde{\Phi}_+)\widetilde{A}, \qquad t \geq 1, \quad \xi \in \mathbb{R}^n$$

$$\|\widetilde{A}(t) - B(t)\|_{L^\infty} = O(t^{-a}), \qquad \text{as } t \to \infty \tag{5.3}$$

for suitable $a > 0$, by the contraction method, where $\widetilde{\Phi}_+ = \text{s-lim}_{t\to\infty} B(t)^* \mathcal{F} U(-t)u(t)$ in a suitable Sobolev space. He put $u_+ = \mathcal{F}^{-1}\widetilde{A}(1)\widetilde{\Phi}_+$ and $A(t) = \widetilde{A}(t)\widetilde{A}(1)^*$, and proved that asymptotics of u is $U(t)A(t, -i\nabla)u_+$. (The M_N-valued function $A = A(t,\xi)$ satisfies the Cauchy problem (1.5), and it is a unitary matrix.) In the construction of the function $\widetilde{A}$ satisfying (5.2)–(5.3), he essentially used the coefficient of the right hand side of (5.2) is t^{-1}. Precisely he essentially used the fact

$$\int_t^\infty \tau^{-1}\tau^{-a}\, d\tau = \frac{1}{a} t^{-a} \tag{5.4}$$

for $t \geq 1$.

In our case $\gamma < 1$, the equation corresponding to (5.2) is

$$i\partial_t \widetilde{A} = t^{-\gamma} F(\widetilde{A}\Phi_+)\widetilde{A}, \qquad t \geq 1, \quad \xi \in \mathbb{R}^n. \tag{5.5}$$

Note that the coefficient of the right hand side of (5.5) is $t^{-\gamma}$ with $\gamma < 1$. Since $\gamma < 1$,

$$\int_t^\infty \tau^{-\gamma}\tau^{-a}\, d\tau \neq C t^{-a}.$$

Therefore the idea of the proof in the case $\gamma = 1$ (Lemma 4.5 in Wada [30]) is not available for our case $\gamma < 1$. Hence, in this paper, large time asymptotics of the solution to the initial value problem (1.1)–(1.2) can not be obtained, and that problem is still open.

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