J. Math. Sci. Univ. Tokyo
16 (2009), 199–230.

The Product Formula for Local Constants in Torsion Rings

By Seidai YASUDA

Abstract. Let p be a rational prime and K a local field of residue characteristic p. In this paper, we prove the product formula for local ε_0-constants defined in [Y1].

1. Introduction

This paper is a continuation of the author's article [Y1]. Let K be a complete discrete valuation field whose residue field k is finite of characteristic p (such a field will be called a *p-local field*). Let q denote the cardinality of k. Let W_K be the Weil group of K. In [Y1], we defined the local constants $\varepsilon_{0,R}(V, \psi) \in R^\times$, generalizing Deligne's $\varepsilon_0(V, \psi, dx)$, for triples $(R, (\rho, V), \psi)$ where R is a *strict p'-coefficient ring* (cf. loc. cit.), (ρ, V) is an object in $\mathrm{Rep}(W_K, R)$, and $\psi : K \to R^\times$ is a non-trivial continuous additive character.

In this paper we will concentrate on the case where $\mathrm{char}\, K = p$. When R_0 is the ring of integers of a finite extension of $\mathbb{Q}_\ell$ for a prime $\ell \neq p$, the product formula of Deligne-Laumon describes the determinant of Frobenius on the etale cohomologies of a smooth R_0-sheaf on a curve over k as a product of local ε_0-constants. In this paper, we generalize the product formula to the case where R_0 is a pro-finite p'-coefficient ring, giving evidence that our construction provides a good theory of local ε_0-constants.

Let us briefly review the contents of this paper. After recalling in § 3 some basic facts necessary in this paper, we give, in § 4, the statement of the product formula which is the main result of this paper. The next four sections are devoted to the proof of the product formula. In § 9, we give an application of our product formula to Saito's theorem in [Sa1].

Acknowledgements. The author would like to thank my supervisor Prof. Takeshi Saito for many helpful advices and comments. The author

2000 *Mathematics Subject Classification*. Primary 11F80, 11G20; Secondary 14F20, 11S15, 11S40.

also would like to thank the referee for careful reading of the manuscript and valuable comments.

2. Notation

Let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers respectively.

Let $\mathbb{Z}_{>0}$ (resp. $\mathbb{Z}_{\geq 0}$) be the ordered set of positive (resp. non-negative) integers. We also define $\mathbb{Q}_{\geq 0}$, $\mathbb{Q}_{>0}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ in a similar way. For a prime number ℓ, let $\mathbb{F}_\ell$ denote the finite field of ℓ elements. For a ring R, let $R^\times$ denote the group of units in R. For a finite extension L/K of fields, let $[L : K]$ denote the degree of L over K. For a subgroup H of a group G of finite index, its index is denoted by $[G : H]$.

Throughout this paper, we fix once for all a prime number p. We consider a complete discrete valuation field K whose residue field k is finite of characteristic p. We say such a field K is a *p-local field*.

For a p-local field K, let $\mathcal{O}_K$ denote its ring of integers, $\mathfrak{m}_K$ the maximal ideal of $\mathcal{O}_K$, and $v_K : K^\times \to \mathbb{Z}$ the normalized valuation. We also denote by W_K the Weil group of K, by $\mathrm{rec} = \mathrm{rec}_K : K^\times \xrightarrow{\cong} W_K^{\mathrm{ab}}$ the reciprocity map in the local class field theory, which sends a prime element of K to a lift of geometric Frobenius of k.

If L/K is a finite separable extension of p-local fields, let $e_{L/K} \in \mathbb{Z}$, $f_{L/K} \in \mathbb{Z}$, $D_{L/K} \in \mathcal{O}_L/\mathcal{O}_L^\times$, $d_{L/K} \in \mathcal{O}_K/\mathcal{O}_K^{\times 2}$ denotes the ramification index of L/K, the residual degree of L/K, the different of L/K, the discriminant of L/K respectively.

For a topological group (or more generally for a topological monoid) G and a commutative topological ring R, let $\mathrm{Rep}(G, R)$ denote the category whose object is a pair (ρ, V) of a finitely generated free R-module V and a continuous group homomorphism $\rho : G \to GL_R(V)$ (we endow $GL_R(V)$ with the topology induced from the direct product topology of $\mathrm{End}_R(V)$), and whose morphisms are R-linear maps compatible with actions of G.

A sequence

$$0 \to (\rho', V') \to (\rho, V) \to (\rho'', V'') \to 0$$

of morphisms in $\mathrm{Rep}(G, R)$ is called a *short exact sequence* in $\mathrm{Rep}(G, R)$ if $0 \to V' \to V \to V'' \to 0$ is the short exact sequence of R-modules.

In this paper, a noetherian local ring with residue field of characteristic $\neq p$ is called a *p'-coefficient ring*. Any p'-coefficient ring $(R, \mathfrak{m}_R)$ is considered as a topological ring with the $\mathfrak{m}_R$-preadic topology. A *strict p'-coefficient ring* is a p'-coefficient ring R with an algebraically closed residue field such that $(R^\times)^p = R^\times$.

3. Review of Basic Facts

3.1. Ramification subgroups

Let K be a p-local field. We denote its residue field by k. We fix a separable closure $\overline{K}$ of K, and denote by $\overline{k}$ the residue field of the valuation field $\overline{K}$. Let $G = W_K$ denote the Weil group of K. Let $G^v = G \cap \mathrm{Gal}(\overline{K}/K)^v$ and $G^{v+} = G \cap \mathrm{Gal}(\overline{K}/K)^{v+}$ be the *upper numbering ramification subgroups* of G. These subgroups have the following properties:

- G^v and G^{v+} are closed normal subgroups of G.
- $G^v \supset G^{v+} \supset G^w$ for every $v, w \in \mathbb{Q}_{\geq 0}$ with $w > v$.
- G^{v+} is equal to the closure of $\bigcup_{w>v} G^w$.
- $G^0 = I_K$, the inertia subgroup of W_K. $G^{0+} = P_K$, the wild inertia subgroup of W_K. In particular, G^w for $w > 0$ and G^{w+} for $w \geq 0$ are pro p-groups.
- For $w \in \mathbb{Q}$, $w > 0$, G^w/G^{w+} is an abelian group which is killed by p.

3.2. Character sheaves

Let S be a scheme of characteristic p, $(R, \mathfrak{m}_R)$ a complete p'-coefficient ring, and G a commutative group scheme over S. An *invertible character R-sheaf* on G is a smooth invertible R-sheaf (that is, a pro-system of smooth invertible $R/\mathfrak{m}_R^n$-sheaves) $\mathcal{L}$ on G such that $\mathcal{L} \boxtimes \mathcal{L} \cong \mu^* \mathcal{L}$, where $\mu : G \times_S G \to G$ is the group law. We have $i^* \mathcal{L} \cong \mathcal{L}$, where $i : G \to G$ is the inverse morphism. If $\mathcal{L}_1$, $\mathcal{L}_2$ are two invertible character R-sheaf on G, then so is $\mathcal{L}_1 \otimes_R \mathcal{L}_2$.

LEMMA 3.1 (Orthogonality relation). *Suppose that S is quasi-compact and quasi-separated, and that the structure morphism $\pi : G \to S$ is compactifiable. Let $\mathcal{L}$ be an invertible character R-sheaf on G such that $\mathcal{L} \otimes_R R/\mathfrak{m}_R$ is non-trivial. Then we have $R\pi_! \mathcal{L} = 0$.*

Proof. We may assume that R is a field. Since $R\mathrm{pr}_1(\mathcal{L} \boxtimes \mathcal{L}) \cong (\pi^* R\pi_! \mathcal{L}) \otimes \mathcal{L}$ and $R\mathrm{pr}_1(\mu^* \mathcal{L}) \cong \pi^* R\pi_! \mathcal{L}$, we have $(\pi^* R^i \pi_! \mathcal{L}) \otimes \mathcal{L} \cong \pi^* R^i \pi_! \mathcal{L}$ for all i. Hence $R^i \pi_! \mathcal{L} = 0$ for all i. □

The following lemma will be used in the subsequent paper [Y2]:

Lemma 3.2. *Suppose further that S and G are noetherian and connected, and that R is a finite ring. Let $\mathcal{L}$ be a smooth invertible R-sheaf on G. Then $\mathcal{L}$ is an invertible character R-sheaf if and only if there is a finite etale homomorphism $G' \to G$ of commutative S-group schemes with a constant kernel H_S and a homomorphism $\chi : H \to R^\times$ of groups such that $\mathcal{L}$ is the sheaf defined by G' and χ.*

Proof. The if part is easy. We prove the only if part. Let $\mathcal{L}$ be an invertible character R-sheaf on G, $\rho : \pi_1(G) \to R^\times$ a representation of the etale fundamental group $\pi_1(G)$ of G corresponding to $\mathcal{L}$, and $f : X \to G$ the finite etale Galois covering of G corresponding to $\operatorname{Ker} \rho$. Let us define a group law on X. Let $e : S \to G$ be the unit section. Since $(\mathcal{L}|_{e(S)})^{\otimes 2} \cong \mathcal{L}|_{e(S)}$, there exists a section $e' : S \to X$ satisfying $e = f \circ e'$. Since $\mathcal{L} \boxtimes \mathcal{L} \cong \mu^* \mathcal{L}$, there exists a finite etale morphism $X \times_S X \to X \times_{G,\mu} (G \times_S G)$ over $G \times_S G$. Then there exists a unique morphism $\mu' : X \times_S X \to X$ over S which is the composition of the above morphism, the projection $X \times_{G,\mu} (G \times_S G) \to X$, and an automorphism $X \to X$ over G such that $\mu' \circ (e' \times e') = e'$. Since $i^* \mathcal{L} \cong \mathcal{L}$, there exists a unique isomorphism $i' : X \times_{G,i} G \xrightarrow{\cong} X$ over G. It is a routine to check that (e', μ', i') defines a structure of a commutative S-group scheme on X. Hence the assertion follows. □

4. Product Formula (Statement)

Let k be a finite field of characteristic p with q elements, X_0 a proper smooth connected curve over k, $U_0 \subset X_0$ a non-empty open subscheme of X_0, $j_0 : U_0 \hookrightarrow X_0$ the inclusion, $X = X_0 \otimes_k \overline{k}$, $U = U_0 \otimes_k \overline{k}$, $j = j_0 \times id : U \hookrightarrow X$, R a strict p'-coefficient ring, $R_0 \subset R$ a finite subring, and $\mathcal{F}$ a smooth R_0-flat R_0-sheaf on U_0.

Define the global ε-constant $\varepsilon_{R_0}(U_0, \mathcal{F})$ as

$$\varepsilon_{R_0}(U_0, \mathcal{F}) = \det(-\mathrm{Fr}_q; R\Gamma_c(U, \mathcal{F}))^{-1} = \det(-\mathrm{Fr}_q; R\Gamma_c(X, j_{0,!}\mathcal{F}))^{-1}.$$

Let $\omega \in \Gamma(U_0, \Omega^1_{U_0/k})$ be a non-zero differential on U. Fix a non-trivial additive character $\psi : k \to R_0^\times$. For a closed point $x \in X_0$, we denote by $\kappa(x)$ the residue field at x, by $q_x = \sharp\kappa(x)$ the cardinality of $\kappa(x)$, by K_x the completion of the function field of X_0 at x, by $\mathcal{F}_x$ the isomorphism class in $\mathrm{Rep}(W_{K_x}, R)$ corresponding to the pull-back of $\mathcal{F}$ by the canonical morphism $\mathrm{Spec}\,(K_x) \to U$, and by $\psi_{\omega,x} : K_x \to R^\times$ the additive character given by

$$\psi_{\omega,x}(a) = \psi(\mathrm{Tr}_{\kappa(x)/k}(\mathrm{Res}_x(a\omega)))$$

for $a \in K_x$. Here Res_x is the residue homomorphism at x.

THEOREM 4.1 (Product formula for $(U_0, \mathcal{F}, \omega)$). *In the above notation, we have*

$$\varepsilon_{R_0}(U_0, \mathcal{F}) = q^{\frac{1}{2}\chi(X)\mathrm{rank}\,(\mathcal{F})} \prod_{x \in X_0 - U_0} \varepsilon_{0,R}(\mathcal{F}_x, \psi_{\omega,x}),$$

where $\chi(X)$ is the Euler number of X.

LEMMA 4.2.

(1) *Let $V_0 \subset U_0$ be a open dense subscheme. Then the product formula for $(U_0, \mathcal{F}, \omega)$ is equivalent to that for $(V_0, \mathcal{F}|_{V_0}, \omega|_{V_0})$.*

(2) *Let $\omega' \in \Gamma(U_0, \Omega^1_{U_0/k})$ be another non-zero differential on U_0. Then the product formula for $(U_0, \mathcal{F}, \omega)$ is equivalent to that for $(U_0, \mathcal{F}, \omega')$.*

(3) *Let Y_0 be another proper smooth connected curve over k, $f : Y_0 \to X_0$ a finite morphism such that the restriction $V_0 = U_0 \times_{X_0} Y_0 \to U_0$ of f to V_0 is etale, $\mathcal{G}$ a smooth R_0-flat R_0-sheaf on V_0. Then the product formula for $(V_0, \mathcal{G}, f^*\omega)$ is equivalent to that for $(U_0, (f|_{V_0})_*\mathcal{G}, \omega)$.*

(4) *Let R'_0 be another finite subring of R containing R_0. Then the product formula for $(U_0, \mathcal{F}, \omega)$ is equivalent to that for $(U_0, \mathcal{F} \otimes_{R_0} R'_0, \omega)$.*

PROOF. (1) Let $x \in U_0 - V_0$. Since $\mathcal{F}$ is unramified at x and $\psi_{\omega,x}$ has conductor 0,

$$\varepsilon_{0,R}(\mathcal{F}_x, \psi_{\omega,x}) = (-1)^{\mathrm{rank}\,\mathcal{F}_x} \det(\mathcal{F}_x)(\mathrm{Fr}_x).$$

Let $j' : V_0 \hookrightarrow U_0$ be the canonical inclusion. For $x \in U_0$, let $i_x : x \hookrightarrow U_0$ be the canonical morphism. By the short exact sequence

$$0 \to j'_! j'^* \mathcal{F} \to \mathcal{F} \to \bigoplus_{x \in U_0 - V_0} i_{x,*} i_x^* \mathcal{F} \to 0,$$

we have

$$\varepsilon_{R_0}(U_0, \mathcal{F}) = \varepsilon_{R_0}(V_0, \mathcal{F}|_{V_0}) \cdot \prod_{x \in U_0 - V_0} \det(-\mathrm{Fr}_x\, ; \mathcal{F}_x).$$

Hence the lemma follows.

(2) By shrinking U_0 if necessary, we may assume that there exists an invertible element $f \in \Gamma(U_0, \mathcal{O}_{X_0}^\times)$ satisfying $\omega' = f\omega$. For $x \in X_0$, we have

$$\varepsilon_{0,R}(\mathcal{F}_x, \psi_{\omega',x}) = \det(\mathcal{F}_x)(\mathrm{rec}_{K_x}(f)) q_x^{\mathrm{ord}_x(f)} \varepsilon_{0,R}(\mathcal{F}_x, \psi_{\omega,x}).$$

Since $\det(\mathcal{F}_x) = (\det \mathcal{F})_x$ is unramified at $x \in U$, we have, by global class field theory,

$$\prod_{x \in X_0 - U_0} \det(\mathcal{F}_x)(\mathrm{rec}_{K_x}(f)) = \prod_{x \in X_0} \det(\mathcal{F}_x)(\mathrm{rec}_{K_x}(f)) = 1.$$

Hence the lemma follows from

$$\prod_{x \in X_0 - U_0} q_x^{\mathrm{ord}_x(f)} = \prod_{x \in X_0} q_x^{\mathrm{ord}_x(f)} = 1.$$

(3) Since $(f|_{V_0})_* = (f|_{V_0})_!$, we have

$$R\Gamma_c(V_0 \otimes_k \overline{k}, \mathcal{G}) = R\Gamma_c(U, (f|_{V_0})_* \mathcal{G}).$$

Hence $\varepsilon_{R_0}(U_0, (f|_{V_0})_* \mathcal{G}) = \varepsilon_{R_0}(V_0, \mathcal{G})$.

On the other hand, for $x \in X_0$, we have

$$((f|_{V_0})_* \mathcal{G})_x \cong \bigoplus_{f(y) = x} \mathrm{Ind}_{W_{K_y}}^{W_{K_x}} \mathcal{G}_y.$$

By [Y1, Thm. 5.6] we have

$$\prod_{x \in X_0 - U_0} \varepsilon_{0,R}(((f|_{V_0})_* \mathcal{G})_x, \psi_{\omega,x}) = \prod_{y \in Y_0 - V_0} \varepsilon_{0,R}(\mathrm{Ind}_{W_{K_y}}^{W_{K_{f(x)}}} \mathcal{G}_y, \psi_{\omega,f(y)})$$
$$= \prod_{y \in Y_0 - V_0} \varepsilon_{0,R}(\mathcal{G}_y, \psi_{\omega,f(y)} \circ \mathrm{Tr}_{K_y/K_{f(y)}}) \cdot \lambda_R(K_y/K_{f(y)}, \psi_{\omega,f(y)})^{\mathrm{rank}\, \mathcal{F}_y}.$$

Since $\psi_{\omega,f(y)} \circ \mathrm{Tr}_{K_y/K_{f(y)}} = \psi_{f^*\omega,y}$, we have

$$\begin{aligned}&\prod_{x\in X_0-U_0} \varepsilon_{0,R}(((f|_{V_0})_*\mathcal{G})_x, \psi_{\omega,x})\\ &= \prod_{y\in Y_0-V_0} \varepsilon_{0,R}(\mathcal{G}_y, \psi_{f^*\omega,y}) \cdot \lambda_R(K_y/K_{f(y)}, \psi_{\omega,f(y)})^{\mathrm{rank}\,\mathcal{F}_y}.\end{aligned}$$

To prove the lemma, it suffices to prove

$$q^{\frac{1}{2}\chi(X)} \prod_{y\in Y_0-V_0} \lambda_R(K_y/K_{f(y)}, \psi_{\omega,f(y)}) = q^{\frac{1}{2}\chi(Y)}.$$

which follows from similar computation for trivial $\overline{\mathbb{Q}}_\ell$-sheaf on V_0, where ℓ is the residue characteristic of R.

(4) is obvious. $\square$

Therefore, to prove Theorem 4.1, we may assume that $X_0 = \mathbb{P}^1_k$, $U_0 \subset \mathbb{A}^1_k$, and that $\mathcal{F}$ is unramified at ∞.

5. Key Proposition

Let K be a p-local field of characteristic p whose residue field is k. We fix a prime element π_K in K. We identify $X = \mathrm{Spec}\,(\bigoplus_{n\in\mathbb{Z}} \mathfrak{m}_K^n/\mathfrak{m}_K^{n+1})$ with $\mathbb{G}_{m,k}$ via the canonical isomorphism $\bigoplus_{n\in\mathbb{Z}} \mathfrak{m}_K^n/\mathfrak{m}_K^{n+1} \cong k[\pi_K^{\pm 1}]$. Let $i_0 : \mathrm{Spec}\,(K) \to X$ be the morphism whose associated homomorphism of coordinate rings is given by the canonical inclusion

$$\bigoplus_{n\in\mathbb{Z}} \mathfrak{m}_K^n/\mathfrak{m}_K^{n+1} \cong k[\pi_K^{\pm 1}] \hookrightarrow k((\pi_K)) \cong K.$$

Let R_0 be a finite p'-coefficient ring which contains a primitive p-th root of unity. For an object V in $\mathrm{Rep}(W_K, R_0)$, let $\mathcal{F}_V$ be the canonical extension of V (cf. [Kz1, p. 76, Thm. 1.4.1]). We exchange the roles of 0 and ∞ in [Kz1]; the sheaf $\mathcal{F}_V$ is a smooth R_0-flat R_0-sheaf on $X = \mathbb{G}_{m,k}$ which is tame at ∞ such that $i_0^*\mathcal{F}_V$ is the sheaf on $\mathrm{Spec}\,(K)$ corresponding to V.

Fix a non-trivial additive character $\phi_0 : \mathfrak{m}_K^{-1}/\mathcal{O}_K \to R_0^\times$ and let $\mathcal{L} = \mathcal{L}_{\phi_0}$ be the Artin-Schreier sheaf on $\mathrm{Spec}\,(\bigoplus_{n\geq 0} \mathfrak{m}_K^n/\mathfrak{m}_K^{n+1})$, and $\mathcal{L}'$ its restriction to X. For an object V in $\mathrm{Rep}(W_K, R_0)$, set $\mathcal{F}'_V = \mathcal{F}_V \otimes_{R_0} \mathcal{L}'$.

PROPOSITION 5.1. *Suppose that V is totally wildly ramified. Let $r = \operatorname{rank} V + \mathrm{sw}(V)$. For $r' \in \mathbb{Z}_{\geq 0}$, let $\widetilde{s}_{r'} : \mathrm{Sym}^{r'} \mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$ be the morphism induced by the product map $s_{r'} : \mathbb{G}_{m,k}^{r'} \to \mathbb{G}_{m,k}$. Let $\mathfrak{S}_{r'}$ be the r'-th symmetric group and let $\Gamma^{r'}_{\mathrm{ext}}(\mathcal{F}'_V)$ denote the $\mathfrak{S}_{r'}$-invariant part of the direct image of ${\mathcal{F}'_V}^{\boxtimes r'}$ under the quotient morphism $\mathbb{G}_{m,k}^{r'} \to \mathrm{Sym}^{r'} \mathbb{G}_{m,k}$. Then*

(1) *For $r' > r$, we have $R\widetilde{s}_{r',!}\Gamma^{r'}_{\mathrm{ext}}\mathcal{F}'_V = 0$.*

(2) *The complex $R\widetilde{s}_{r,!}\Gamma^{r}_{\mathrm{ext}}\mathcal{F}'_V$ is supported on a closed point of $\mathbb{G}_{m,k}$.*

LEMMA 5.2. *Let $(R, \mathfrak{m})$ be a local ring, $K^\bullet$ a complex of finitely generated free R-modules which is bounded above. Suppose that $K \otimes_R R/\mathfrak{m}$ is acyclic. Then K is also acyclic.*

PROOF. Assume that $K^\bullet$ is not acyclic. Let i be the maximal integer such that $K^i \neq 0$. If $d^{i-1} : K^{i-1} \to K^i$ is surjective, then $\operatorname{Ker} d^{i-1}$ is a finitely generated free R-module and the complex

$$\cdots \to K^{i-2} \to \operatorname{Ker} d^{i-1} \to 0 \to \cdots$$

is quasi-isomorphic to $K^\bullet$. Hence we may assume that d^{i-1} is not surjective. Since

$$0 = \operatorname{Coker}(K^{i-1} \otimes_R R/\mathfrak{m} \to K^i \otimes_R R/\mathfrak{m}) \cong (\operatorname{Coker} d^{i-1}) \otimes_R R/\mathfrak{m},$$

we have $\operatorname{Coker} d^{i-1} = 0$. This is a contradiction. $\square$

Since the geometric stalks of $R\widetilde{s}_{r',!}\Gamma^{r'}_{\mathrm{ext}}\mathcal{F}'_V$ are bounded above and have constructible cohomologies, we may assume, by the above lemma, that R_0 is a finite field in proving Proposition 5.1.

PROOF OF PROPOSITION 5.1 (1). We may assume that R_0 is a field. Assume that $R\widetilde{s}_{r',!}\Gamma^{r'}_{\mathrm{ext}}\mathcal{F}'_V \neq 0$. Take the minimum i satisfying $R^i\widetilde{s}_{r',!}\Gamma^{r'}_{\mathrm{ext}}\mathcal{F}'_V \neq 0$ and put $\mathcal{H} = R^i\widetilde{s}_{r',!}\Gamma^{r'}_{\mathrm{ext}}\mathcal{F}'_V$. Since $R\Gamma_c(\mathbb{G}_{m,\overline{k}}, R\widetilde{s}_{r',!}\Gamma^{r'}_{\mathrm{ext}}\mathcal{F}'_V) = \bigwedge^{r'} H^1_c(\mathbb{G}_{m,\overline{k}}, \mathcal{F}'_V) = 0$, we have $H^0_c(\mathbb{G}_{m,\overline{k}}, \mathcal{H}) = H^1_c(\mathbb{G}_{m,\overline{k}}, \mathcal{H}) = 0$. We see that $\mathcal{H}$ is a smooth R_0-sheaf on $\mathbb{G}_{m,k}$ which is tame both at 0 and ∞. In fact, there exists a non-empty open subscheme $U \subset \mathbb{G}_{m,k}$ such that $\mathcal{H}|_U$ is smooth. Take the maximum such U and let

R be the rank of $\mathcal{H}|_U$. By the Grothendieck-Ogg-Shafarevich formula, we have

$$\dim_{R_0} H_c^2(\mathbb{G}_{m,\overline{k}}, \mathcal{H}) = -\sum_{x \in \mathbb{G}_{m,k} - U} (R - \dim_{R_0} \mathcal{H}_{\overline{x}}) - \sum_{x \in \mathbb{P}^1_k - U} \mathrm{sw}_x(\mathcal{H}|_U)).$$

Hence $U = \mathbb{G}_{m,k}$ and $\mathrm{sw}_0(\mathcal{H}) = \mathrm{sw}_\infty(\mathcal{H}) = 0$. By replacing k by its finite extension, we may assume that the representation of $\pi_1(\mathbb{G}_{m,k})$ corresponding to $\mathcal{H}$ is abelian. By replacing R_0 by its finite extension, there exists an invertible smooth R_0-subsheaf $\mathcal{W}$ of $\mathcal{H}$. Let $\mathcal{M} = \mathcal{W}^{-1}$. Since $s_{r'}^* \mathcal{M} \cong \mathcal{M}^{\boxtimes r}$, we have $\widetilde{s}_{r'}^* \mathcal{M} \cong \Gamma^r_{\mathrm{ext}} \mathcal{M}$. Hence, by replacing $\mathcal{F}'_V$ by $\mathcal{F}'_V \otimes \mathcal{M}$, we have $R\Gamma_c(\mathbb{G}_{m,\overline{k}}, R\widetilde{s}_{r',!} \Gamma^{r'}_{\mathrm{ext}}(\mathcal{F}'_V \otimes \mathcal{M})) \neq 0$. This is a contradiction. $\square$

The following lemma is proved in the same way as in the proof of [DH, p. 101, Prop. 2.2].

LEMMA 5.3. *Let K be a p-local field, R_0 a finite field of characteristic $\neq p$, V a totally wild object in* $\mathrm{Rep}(W_K, R_0)$. *Then there exist a finite extension R'_0 of R_0, finitely many finite separable extensions $L_1, \ldots, L_m$ of K, integers $n_1, \ldots, n_m \in \mathbb{Z}$, and wild object χ_i in* $\mathrm{Rep}(W_{L_i}, R'_0)$ *of rank one such that*

$$V = \sum_i n_i \mathrm{Ind}_{W_{L_i}}^{W_K} \chi_i$$

in the Grothendieck group of objects in $\mathrm{Rep}(W_K, R'_0)$.

LEMMA 5.4. *Let $k = \overline{k}$ be an algebraically closed field of characteristic p, Y a smooth irreducible affine curve over k, $Y \hookrightarrow \overline{Y}$ the smooth completion of Y. Suppose that $\overline{Y} - Y$ consists of more than or equal to two points. We set $\overline{Y} - Y = \{q_1, \ldots, q_m\}$. Let R_0 be a finite p'-coefficient ring, and $\mathcal{F}$ a smooth invertible R_0-sheaf on Y which is wild at all points $q_1, \ldots, q_m$. Let $g(\overline{Y})$ denote the genus of $\overline{Y}$. Set $s_i := \mathrm{sw}_{q_i}(\mathcal{F}) > 0$ for each i and put*

$$r := 2g(\overline{Y}) - 2 + \sum_i (1 + s_i).$$

Define an effective divisor $\mathfrak{m}_0$ on $\overline{Y}$ as

$$\mathfrak{m}_0 := \sum_i [q_i].$$

Let $J_{\mathfrak{m}_0}$ be the generalized Jacobian of $\overline{Y}$ with modulus $\mathfrak{m}_0$. Take a closed point P on Y and let $h : \mathrm{Sym}^r Y \to J_{\mathfrak{m}_0}$ be the morphism given by $D \mapsto D - r[P]$. Then $Rh_! \Gamma^r_{\mathrm{ext}} \mathcal{F}$ is supported on a closed point on $J_{\mathfrak{m}_0}$.

PROOF. We set

$$\mathfrak{m} := \sum_i (1 + s_i)[q_i].$$

Let $J_{\mathfrak{m}}$ (resp. $J_{\mathfrak{m}-\mathfrak{m}_0}$) be the generalized Jacobian of $\overline{Y}$ with modulus $\mathfrak{m}$ (resp. $\mathfrak{m}-\mathfrak{m}_0$). We write the morphism $h : \mathrm{Sym}^r Y \to J_{\mathfrak{m}_0}$ as the composition

$$\mathrm{Sym}^r Y \xrightarrow{g} J_{\mathfrak{m}} \xrightarrow{h'} J_{\mathfrak{m}-\mathfrak{m}_0} \xrightarrow{h''} J_{\mathfrak{m}_0}.$$

By the geometric class field theory, there exists a character sheaf $\mathcal{G}$ on $J_{\mathfrak{m}}$ such that $\mathcal{F} = h^* \mathcal{G}$. Furthermore, the restriction of $\mathcal{G}$ on the kernel of $J_{\mathfrak{m}} \to J_{\mathfrak{m}-[q_i]}$ is non-trivial for every i.

Define the reduced closed subschemes $K, K' \subset J_{\mathfrak{m}}$ with $K' \subset K$ as

$$K := \{\mathrm{div}(\omega) + \mathfrak{m} - r[P] \ ; \ \mathrm{Supp}(\mathrm{div}(\omega) + \mathfrak{m}) \subset Y\},$$

and by

$$K' := \{\mathrm{div}(\omega) + \mathfrak{m} - r[P] \in K \ ; \ \sum_i \mathrm{Res}_{q_i} \omega = 0\}.$$

By [Se3, V, §1, 2] and Riemann-Roch theorem [Se3, IV, §2, 6. Thm. 1], $g^{-1}(J_{\mathfrak{m}} - K) \to J_{\mathfrak{m}} - K$ is a vector bundle of rank $g(\overline{Y}) - 1$, $g^{-1}(K - K')$ is empty and $g^{-1}(K') \to K'$ is a vector bundle of rank $g(\overline{Y})$. We have $R(h'' \circ h')_! \mathcal{G} = 0$ by the orthogonality relation. Hence to prove the lemma, it suffices to show that

$$R(h'' \circ h'|_K)_! (\mathcal{G}|_K) = 0$$

and that

$$R(h'' \circ h'|_{K'})_! (\mathcal{G}|_{K'})$$

is supported on a closed point of $J_{\mathfrak{m}_0}$.

Since K is a translation of a sub k-group of $J_{\mathfrak{m}}$ which contains the kernel of $h'' \circ h'$, the first assertion follows from the orthogonality relation of character sheaves (§ 3.2). We prove the second assertion.

Take prime elements $\pi_{q_1}, \dots, \pi_{q_m}$ of $\mathcal{O}_{\overline{Y},q_1}, \dots, \mathcal{O}_{\overline{Y},q_m}$. Take a meromorphic differential ω_0 on $\overline{Y}$ such that

$$\frac{\omega_0}{d\pi_{q_i}} \in \frac{1}{\pi_{q_i}^{1+s_i}} + \mathcal{O}_{\overline{Y},q_i}$$

for all i.

Then K and K' are the translations of the sub k-group scheme

$$\begin{aligned} C_{\mathfrak{m}} &= (\prod_i (\mathcal{O}_{\overline{Y},q_i}/\mathfrak{m}_{\overline{Y},q_i}^{1+s_i})^\times)/\mathbb{G}_{m,k} \\ &\cong \{\operatorname{div}(f) \in J_{\mathfrak{m}} \; ; \; \operatorname{Supp}(\operatorname{div}(f)) \subset Y\} \end{aligned}$$

of $J_{\mathfrak{m}}$ and the sub k-scheme

$$\{(\sum_{j=0}^{s_i} b_{i,j}\pi_{q_i}^j)_i \in C_{\mathfrak{m}} \; ; \; \sum_i b_{i,s_i} = 0\}$$

of $C_{\mathfrak{m}}$ by the class of $\operatorname{div}(\omega_0) + \mathfrak{m} - r[P]$.

The image $h'(K)$ is the translation of the sub k-group scheme

$$\begin{aligned} C_{\mathfrak{m}-\mathfrak{m}_0} &= (\prod_i (\mathcal{O}_{\overline{Y},q_i}/\mathfrak{m}_{\overline{Y},q_i}^{s_i})^\times)/\mathbb{G}_{m,k} \\ &\cong \{\operatorname{div}(f) \in J_{\mathfrak{m}-\mathfrak{m}_0} \; ; \; \operatorname{Supp}(\operatorname{div}(f)) \subset Y\} \end{aligned}$$

of $J_{\mathfrak{m}-\mathfrak{m}_0}$ by the class of $\operatorname{div}(\omega_0) + \mathfrak{m} - r[P]$.

Let

$$Q = \operatorname{div}(\omega_0) + \mathfrak{m} - r[P] + (\sum_{j=0}^{s_i-1} b_{i,j}\pi_{q_i}^j)_i$$

be a closed point in $h'(K)$, $\widetilde{Q} \in K$ be the lift of Q defined as

$$\widetilde{Q} = \operatorname{div}(\omega_0) + \mathfrak{m} - r[P] + \sum_{j=0}^{s_i-1} b_{i,j}\pi_{q_i}^j + 0)_i.$$

Then the fiber $K' \cap (h')^{-1}(Q)$ of $h'|_{K'}$ at Q is equal to the translation of

$$\{(1 + b_{i,s_i}\pi_{q_i}^{s_i})_i \in C_{\mathfrak{m}} \; ; \; \sum_i b_{i,0}b_{i,s_i} = 0\}$$

by $\widetilde{Q}$.

It is easy to see that there is a unique closed point $(b'_{i,0})_i \in (\mathbb{G}_{m,k}^{m+1})/\mathbb{G}_{m,k}$ such that $R(h'|_{K'})_!(\mathcal{G}|_{K'})_Q = 0$ for $Q \in h'(K)$ as above with $(b_{i,0})_i \neq (b'_{i,0})_i$. This completes the proof. □

In view of the proof of the above lemma, we have:

COROLLARY 5.5. *In the notation of the above lemma, suppose that there exists a finite subfield k_0 of k such that Y is the base change of a smooth curve Y_0 over k_0, that q_1 comes from a k_0-rational point of the completion $\overline{Y}_0$ of Y_0, and that $\mathcal{F}$ is defined on Y_0. Let $\overline{Y}_0 - Y_0 = \{q'_1, \cdots, q'_{m'}\}$. Fix a non-trivial additive character $\psi : k_0 \to R_0^\times$. Take a non-zero meromorphic differential ω_0 on $\overline{Y}_0$. For each i, let $K_{q'_i}$ be the completion of the function field of Y_0 at q'_i. Take a non-zero meromorphic function f on $\overline{Y}_0$ such that for every i, the class of f in $K_{q'_i}/1 + \mathfrak{m}_{K_{q'_i}}$ is equal to the refined swan conductor $\mathrm{rsw}_{\psi_{\omega_0, q'_1}}(\mathcal{F}_{q'_1})$.*

Then $Rh_!\Gamma^r_{\mathrm{ext}}\mathcal{F}$ is supported on the closed point on $J_{\mathfrak{m}_0}$ corresponding to the divisor $\mathrm{div}(\omega_0) - \mathrm{div}(f) + \mathfrak{m} - r[P]$.

COROLLARY 5.6. *Proposition 5.1 (2) holds if V is of the form $V = \mathrm{Ind}_{W_L}^{W_K}\chi$ for a finite separable totally ramified extension L of K and a rank one wild object χ in $\mathrm{Rep}(W_L, R_0)$.*

PROOF. Let $f : Y \to \mathbb{G}_{m,k}$ be the finite etale extension corresponding to L/K by the theory of canonical extension. Let $Y \hookrightarrow \overline{Y}$ be its smooth compactification, let $\overline{f} : \overline{Y} \to \mathbb{P}^1_k$ be the morphism induced by f. Since L/K is totally ramified, Y is geometrically irreducible and $\overline{f}^{-1}(0)$ consists of one k-rational point q_1. Set $(\overline{Y} - Y)(\overline{k}) = \{q_1, q_2, \ldots, q_m\}$. There exists a smooth invertible R_0-sheaf $\mathcal{F}_1$ on Y which is wildly ramified at q_1 and tame at $q_2, \ldots q_m$ such that $\mathcal{F}_V \cong f_*\mathcal{F}_1$. Then $Y, \overline{Y}, q_1, \ldots, q_m$, and $\mathcal{F} = \mathcal{F}_1 \otimes f^*\mathcal{L}$ satisfy the conditions of the above lemma.

Let $J_{\mathfrak{m}_0}$ be the generalized Jacobian as in the above lemma. Take a point $P \in Y(\overline{k})$ such that $f(P) = 1 \in \mathbb{G}_{m,k}$. By using P, there is a canonical morphism $\mathrm{Sym}^r Y \to J_{\mathfrak{m}_0}$. We have a commutative diagram:

$$\begin{array}{ccc}
\mathrm{Sym}^r Y & \xrightarrow{\mathrm{Sym}^r f} & \mathrm{Sym}^r \mathbb{G}_{m,k} \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \tilde{s}_r} \\
J_{\mathfrak{m}_0} & \longrightarrow & \mathbb{G}_{m,k}.
\end{array}$$

Hence the assertion follows from the above lemma and $\Gamma^r_{\mathrm{ext}}\mathcal{F}'_V \cong (\mathrm{Sym}^r f)_* \Gamma^r_{\mathrm{ext}}\mathcal{F}$. $\square$

Therefore, to prove Proposition 5.1 (2), it suffices to prove the following lemma:

LEMMA 5.7.

(1) *Let*

$$0 \to V' \to V \to V'' \to 0$$

be an exact sequence of totally wild objects in $\mathrm{Rep}(W_K, R_0)$*. If Proposition 5.1 (2) holds for two of* V*,* V' *and* V''*, then it holds for all of* V*,* V' *and* V''*.*

(2) *Let* K'/K *be a finite unramified extension,* V *a totally wild object in* $\mathrm{Rep}(W'_K, R_0)$*. If Proposition 5.1 (2) holds for* V*, then it holds for all of* $\mathrm{Ind}^{W_K}_{W'_K} V$*.*

PROOF. We prove only the assertion 1. The assertion 2 is obtained by a similar method.

We set $r := \mathrm{sw}(V) + \mathrm{rank}\, V$, $r' := \mathrm{sw}(V') + \mathrm{rank}\, V'$, and $r'' := \mathrm{sw}(V'') + \mathrm{rank}\, V''$. The sheaf $\Gamma^r_{\mathrm{ext}}\mathcal{F}'_V$ has a natural filtration, whose graded pieces are equal to $\vee_{r_1,r_2,*}(\Gamma^{r_1}_{\mathrm{ext}}\mathcal{F}'_{V'} \boxtimes \Gamma^{r_2}_{\mathrm{ext}}\mathcal{F}_{V''})$ for some $r_1, r_2 \in \mathbb{Z}_{\geq 0}$ with $r_1 + r_2 = r$, where $\vee_{r_1,r_2} : \mathrm{Sym}^{r_1}\mathbb{G}_{m,k} \times \mathrm{Sym}^{r_2}\mathbb{G}_{m,k} \to \mathrm{Sym}^r\mathbb{G}_{m,k}$ is the canonical map.

We have the following commutative diagram :

$$\begin{array}{ccc}
\mathrm{Sym}^{r_1}\mathbb{G}_{m,k} \times \mathrm{Sym}^{r_2}\mathbb{G}_{m,k} & \longrightarrow & \mathrm{Sym}^r\mathbb{G}_{m,k} \\
{\scriptstyle \tilde{s}_{r_1} \times \tilde{s}_{r_2}}\downarrow & & \downarrow{\scriptstyle \tilde{s}_r} \\
\mathbb{G}_{m,k} \times \mathbb{G}_{m,k} & \xrightarrow{\mu} & \mathbb{G}_{m,k},
\end{array}$$

where $\mu : \mathbb{G}_{m,k}\times\mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$ is the multiplication map. By Proposition 5.1 1, we have

$$R\widetilde{s}_{r,!}\Gamma^{r}_{\mathrm{ext}}\mathcal{F}'_V \cong R\mu_!(R\widetilde{s}_{r'}\Gamma^{r'}_{\mathrm{ext}}\mathcal{F}'_{V'} \boxtimes R\widetilde{s}_{r''}\Gamma^{r''}_{\mathrm{ext}}\mathcal{F}'_{V''}).$$

Hence the assertion follows from simple computation. □

This completes the proof of Proposition 5.1. □

REMARK 5.8. In view of Corollary 5.5 and the proof of Proposition 5.1, we have the following refinement of Proposition 5.1 (2): in the notation of Proposition 5.1, fix a non-trivial additive character $\psi : k \to R_0^\times$. Take a non-zero meromorphic differential ω on $\mathbb{G}_{m,k}$. Let K_0 and K_∞, be the completions of the function field of $\mathbb{G}_{m,k}$ at 0 and ∞, respectively. We identify K_0 with K by the morphism i_0. Take a non-zero meromorphic function $f \in k(\pi_K)^\times$ on $\mathbb{G}_{m,k}$ such that the class of f in $K_0^\times/1+\mathfrak{m}_{K_0}$ and $K_\infty^\times/1+\mathfrak{m}_{K_\infty}$ are equal to the refined Swan conductors $\mathrm{rsw}_{\psi_{\omega,0}}(V)$ and $(\mathrm{rsw}_{\psi_{\omega,\infty}}(\mathcal{L}))^{\mathrm{rank}\,V}$, respectively. Let $g(\pi_K) \in k(\pi_K)^\times$ be a rational function such that

$$\mathrm{div}(g) = \mathrm{div}(\omega_0) - \mathrm{div}(f) + \mathfrak{m} - r[1]$$

Then $R\widetilde{s}_{r,!}\Gamma^{r}_{\mathrm{ext}}\mathcal{F}'_V$ is supported on the k-rational point $\frac{g(0)}{g(\infty)} \in k^\times = \mathbb{G}_{m,k}(k)$.

6. Determinant of $\mathcal{F}'_V$

Let the notation be the same as in the beginning of § 5. Fix a non-trivial additive character $\psi_0 : k \to R_0^\times$. Let $\psi : K \to R_0^\times$ be the continuous additive character given by $\psi(x) = \psi_0(\mathrm{Res}(x\frac{d\pi_K}{\pi_K}))$. Let $\beta \in k^\times$ be the unique element satisfying $\phi_0(\frac{x}{\pi_K}) = \psi_0(-\beta x)$ for all $x \in k$.

PROPOSITION 6.1. *Let V and (ρ, W) be two objects in* $\mathrm{Rep}(W_K, R_0)$. *Suppose that V is totally wild and that W is tamely ramified. Then we have*

$$\begin{aligned}&\det(-\mathrm{Fr}\,;\, R\Gamma_c(\mathcal{F}'_{V\otimes W}))^{-1}\\ &= \det(-\mathrm{Fr}\,;\, R\Gamma_c(\mathcal{F}'_V))^{-\mathrm{rank}\,W}\cdot \det W(\mathrm{rec}(\mathrm{rsw}_\psi(V)\cdot(\beta\pi_K)^{\mathrm{rank}\,V})).\end{aligned}$$

REMARK 6.2. Let $K_\infty = k((\frac{1}{\pi_K}))$ and let $\psi_\infty : K_\infty \to R_0^\times$ be the continuous additive character given by $\psi_\infty(x) = \psi_0(\mathrm{Res}(x\frac{d\pi_K}{\pi_K}))$. Then we have $\mathrm{rsw}_{\psi_\infty}(\mathcal{L}_\infty) = \beta^{-1}\pi_K^{-1}$. Hence $\mathrm{rsw}_{\psi_\infty}((\mathcal{F}'_V)_\infty) = \beta^{-\mathrm{rank}\,V}\pi_K^{-\mathrm{rank}\,V}$. Therefore, we have

$$\begin{aligned}\det(\mathcal{F}_W)_\infty(\mathrm{rec}_{K_\infty}(\mathrm{rsw}_{\psi_\infty}((\mathcal{F}'_V)_\infty))) &= \det(\mathcal{F}_W)_\infty(\mathrm{rec}_{K_\infty}(\beta\cdot\pi_K))^{-\mathrm{rank}\,V}\\ &= \det W(\mathrm{rec}(\beta\cdot\pi_K))^{\mathrm{rank}\,V}.\end{aligned}$$

PROOF. We set $G = \pi_1^{tm}(\mathbb{G}_{m,k})$, $I = \pi_1^{tm}(\mathbb{G}_{m,\bar{k}})$, and $I_m = I/I^m$. For every integer m which is prime to p, let $\pi_m : \mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$ be the m-th power map, $W_m = H_c^1(\mathbb{G}_{m,\bar{k}}, (\pi_{m,*}R_0)\otimes_{R_0}\mathcal{F}'_V)$. W_m is a free $R[I_m]$-module of rank r with a semi-linear action of G, where $r = \mathrm{rank}\,V + \mathrm{sw}(V)$. If n, m are two positive integers which are prime to p with $m|n$, the canonical map $W_n \to W_m$ induces an isomorphism

$$W_n \otimes_{R_0[I_n]} R_0[I_m] \cong W_m.$$

Let $\widehat{W} = \varprojlim_m W_m$ be the projective limit of W_m. $\widehat{W}$ is a free $R_0[[I]]$-module of rank r with a semi-linear action of G.

Consider the maximal exterior power $\det_{R_0[[I]]}\widehat{W} = \bigwedge^r_{R_0[[I]]}\widehat{W}$ of $\widehat{W}$. It is a free $R_0[[I]]$-module of rank one with a semi-linear action of G. We note that the action of $I \subset G$ on $\det_{R_0[[I]]}\widehat{W}$ does not necessarily coincide with the action of $I \subset R_0[[I]]^\times$.

Take a lift $\widetilde{\mathrm{Fr}}_q \in G$ of the geometric Frobenius Fr_q. An argument similar to that in [Y1, Remark 11.3] shows that the eigenvalue of $\widetilde{\mathrm{Fr}}_q$ on $\det_{R_0[[I]]}\widehat{W}$ gives a well-defined element $\widehat{u} \in (R_0[[I]]^\times)_G$ in the G-coinvariant. As in [Y1, § 11.2], we have

LEMMA 6.3. *The determinant of* $\det(\mathrm{Fr}_k; H^1_{c,et}(\mathbb{G}_{m,\bar{k}}, \mathcal{F}'_{V\otimes_{R_0}W}))$ *is equal to*

$$\det(\widetilde{\mathrm{Fr}}_q\,;\,W)^r\cdot\det\left(\int_I \rho(g)^{-1}d\widehat{u}(g)\right).$$

PROOF. Take a sufficiently divisible m which is prime to p such that the restriction $\pi_m^*\mathcal{F}_W$ is geometrically constant. By the same argument as in [Y1, § 11.1] we have $H^1_c(\mathbb{G}_{m,\bar{k}}, \mathcal{F}'_{V\otimes_{R_0}W}) \cong (W_m \otimes_{R_0} W)_{I_m}$. The lemma follows from an argument similar to that in [Y1, § 11.1]. □

Let us go back to the proof of Proposition 6.1. We consider W_K/W_K^{0+} as a subgroup of G in a canonical way. Then $\widetilde{\mathrm{Fr}}_q^r$ lies in W_K/W_K^{0+}. The element $\mathrm{rsw}_\psi(V) \cdot (\beta\pi_K)^{\mathrm{rank}\,V}$ in $K^\times/1+\mathfrak{m}_K$ and $\mathrm{rec}^{-1}(\widetilde{\mathrm{Fr}}_q^r) \in K^\times/1+\mathfrak{m}_K$ only differ by an element in $k^\times$ which we denote by $a_{\widetilde{\mathrm{Fr}}_q,\psi} \in k^\times$, that is

$$\mathrm{rsw}_\psi(V) \cdot (\beta\pi_K)^{\mathrm{rank}\,V} = a_{\widetilde{\mathrm{Fr}}_q,\psi} \cdot \mathrm{rec}^{-1}(\widetilde{\mathrm{Fr}}_q^r).$$

We consider $a_{\widetilde{\mathrm{Fr}}_q,\psi}$ as an element in $I/I^{q-1} \subset (R_0[[I]])_G^\times$ by the canonical isomorphism $I/I^{q-1} \cong k^\times$. Then it suffices to prove that

$$\widehat{u} = a_{\widetilde{\mathrm{Fr}}_q,\psi} \cdot \det(\mathrm{Fr}_k\,;\, H^1_c(\mathbb{G}_{m,\bar{k}}, \mathcal{F}'_V)) \text{ in } (R_0[[I]]^\times)_G.$$

By the same way as in [Y1, § 11.3], we have

$$\det{}_{R_0[[I]]}\widehat{W} \cong \varprojlim{}_m R\Gamma_c(\mathbb{G}_{m,\bar{k}}, (\pi_{m,*}R_0) \otimes_{R_0} R\widetilde{s}_{r,!}\Gamma^r_{\mathrm{ext}}\mathcal{F}'_V)[r].$$

By Proposition 5.1, $R\widetilde{s}_{r,!}\Gamma^r_{\mathrm{ext}}$ is supported on a closed point P in $\mathbb{G}_{m,k}$. By computation of rank, P is a k-rational point, $R^i\widetilde{s}_{r,!}\Gamma^r_{\mathrm{ext}}\mathcal{F}'_V = 0$ for $i \neq r$, and the geometric stalk $R^i\widetilde{s}_{r,!}\Gamma^r_{\mathrm{ext}}\mathcal{F}'_V$ at P is a free R_0-module of rank one. Hence there exists a prime element π'_K in K such that for every positive integer m with $(m,p)=1$, $\det{}_{R_0[I_m]}W_m$ is isomorphic to the free R_0-module

$$\left(\bigoplus_{x^m=\pi'_K} R_0\right) \otimes_{R_0} \det{}_{R_0} H^1_c(\mathbb{G}_{m,\bar{k}}, \mathcal{F}'_V),$$

endowed with the canonical action of G. Hence there exists an element $a' \in I/I^{q-1}$ such that $\widehat{u} = a' \cdot \det(\mathrm{Fr}_k\,;\, H^1_c(\mathbb{G}_{m,\bar{k}}, \mathcal{F}'_V))$ in $(R_0[[I]]^\times)_G$. It suffices to prove that $a' = a_{\widetilde{\mathrm{Fr}}_q,\psi}$. This assertion follows from Remark 5.8. □

Corollary 6.4. *Let R be a strict p'-coefficient ring which contains R_0, V a totally wild object in* $\mathrm{Rep}(W_K, R_0)$. *Then the product formula holds for $\mathcal{F}'_V \otimes_{R_0} R$.*

Proof. By Lemma 3 and [Y1, Prop. 9.14], we may assume that V is of the form $V = V_1 \otimes_{R_0} V_2$, where V_1 is a tamely ramified object in

$\mathrm{Rep}(W_K, R_0)$ and V_2 is the base change of an object in $\mathrm{Rep}(W_K, R_0')$ by a local ring homomorphisms $R_0' \to R_0$, where R_0' is a p'-coefficient ring which is a complete discrete valuation ring with a finite residue field whose field of fractions is of characteristic zero. By [Lau2, p. 187, Thm. 3.2.1.1], the product formula holds for V_2. Since V_2 is totally wild, the product formula also holds for V by [Y1, Prop. 8.3] and Proposition 6.1. □

7. Fourier Transforms

7.1. Deligne-Laumon's global Fourier transform $F^{(0,\infty')}$

Let k be a finite field of characteristic p, R_0 a finite p'-coefficient ring, U a non-empty open subscheme of the affine line $\mathbb{A}^1_k$.

Take a non-trivial additive character $\phi : k \to R_0^\times$. Let $\mathcal{L}_\phi$ be the Artin-Schreier sheaf on $\mathbb{A}^1_k$ associated to ϕ. Let $\langle\, ,\, \rangle : \mathbb{A}^1_k \times_k \mathbb{A}^1_k \to \mathbb{A}^1_k$ be the product map. According to [Lau2, p. 148, 1.4.1], let us denote the sheaf $\langle\, ,\, \rangle^* \mathcal{L}_\phi$ on $\mathbb{A}^1_k \times_k \mathbb{A}^1_k$ by $\mathcal{L}_\phi(xx')$. We also use the notation $\mathcal{L}_\psi(s.x')$, $\mathcal{L}_\psi(x.s')$ and $\mathcal{L}_\psi((x-s).x')$ in [Lau2, p. 148–149, 1.4.2, and p. 195]. Let $\overline{\mathcal{L}}_\phi(xx')$ be the extension by zero of $\mathcal{L}_\phi(xx')$ by the canonical inclusion $\mathbb{A}^1_k \times_k \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k \times_k \mathbb{P}^1_k$.

For a smooth R_0-flat R_0-sheaf $\mathcal{F}$ on U which is at most tamely ramified at $\infty \in \mathbb{P}^1_k - \mathbb{A}^1_k$, define the *global Fourier transform* $\mathrm{FT}_{\phi,U}(\mathcal{F})$ of $\mathcal{F}$ as

$$\mathrm{FT}_{\phi,U}(\mathcal{F}) := R^1\mathrm{pr}_{2,!}(\mathrm{pr}_1^*\mathcal{F} \otimes (\mathcal{L}_\phi(xx')|_{U\times\mathbb{G}_{m,k}})),$$

where $\mathrm{pr}_1 : U \times \mathbb{G}_{m,k} \to U$ and $\mathrm{pr}_2 : U \times \mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$ are projections.

LEMMA 7.1. *$\mathrm{FT}_{\phi,U}(\mathcal{F})$ is a smooth R_0-flat R_0-sheaf on $\mathbb{G}_{m,k}$.*

PROOF. It is easy to see that $\mathrm{FT}_{\phi,U}(\mathcal{F})$ is a R_0-flat R_0-sheaf on $\mathbb{G}_{m,k}$ whose all geometric fibers are of a same rank. To prove the smoothness of $\mathrm{FT}_{\phi,U}(\mathcal{F})$, we may assume that R_0 is a field. Let $\mathcal{G}$ be the extension by zero of the sheaf $\mathrm{pr}_1^*\mathcal{F} \otimes (\mathcal{L}_\phi(xx')|_{U\times\mathbb{G}_{m,k}})$ on $U \times \mathbb{G}_{m,k}$ to $\mathbb{P}^1_k \times \mathbb{G}_{m,k}$. Then $(\mathbb{P}^1_k \times \mathbb{G}_{m,k}, \mathcal{G}, \mathrm{pr}_2)$ is universally locally acyclic by [Lau1, p. 186, Thm. 2.1.1]. Hence $\mathrm{FT}_{\phi,U}(\mathcal{F})$ is smooth. □

Let $\alpha : U \hookrightarrow \mathbb{P}^1_k$ be the inclusion map, and $D_{0'}$ the henselizations of $\mathbb{P}^1_k$ at 0. Let $\overline{\mathrm{pr}}_1 : \mathbb{P}^1 \times D_{0'} \to \mathbb{P}^1$, $\overline{\mathrm{pr}}_2 : \mathbb{P}^1 \times D_{0'} \to D_{0'}$ be projections. Let $\eta_{0'}$ be the generic point of $D_{0'}$. Let us consider the vanishing cycle

$$R\Phi_{\eta_{\infty'}}(\overline{\mathrm{pr}}_1^*(\alpha_!\mathcal{F}) \otimes (\overline{\mathcal{L}}_\phi(xx')|_{\mathbb{P}^1\times D_{0'}})) \in D(\mathbb{P}^1 \times \eta_{0'}, R_0)$$

for the projection $\overline{\mathrm{pr}}_2$.

The following lemma is proved in a manner similar to that in [Lau2, p. 160, Prop. 2.3.2.1]:

LEMMA 7.2.

(1) *The restriction of* $R\Phi_{\eta_{0'}}(\overline{\mathrm{pr}}_1^*(\alpha_!\mathcal{F}) \otimes \mathcal{L})$ *to* $U \times \eta_{0'}$ *is zero.*

(2) *For any closed point* $s \in \mathbb{P}^1_k$, *let* $\overline{s}$ *denote a geometric point of* $\mathbb{P}^1_k$ *over* s. *Then for* $i \neq -1$,

$$R^i\Phi_{\overline{\eta}_{0'}}(\overline{\mathrm{pr}}_1^*(\alpha_!\mathcal{F}) \otimes \mathcal{L})_{(\overline{\infty},\overline{0}')}$$

is zero.

(3) *We have a canonical distinguished triangle*

$$R\Gamma_c(U \otimes_k \overline{k}, \mathcal{F}) \to \mathrm{FT}_{\phi,U}(\mathcal{F})_{\overline{\eta}_{0'}} \to R^{-1}\Phi_{\overline{\eta}_{0'}}(\overline{\mathrm{pr}}_1^*(\alpha_!\mathcal{F}) \otimes \mathcal{L})_{(\overline{\infty},\overline{0}')} \xrightarrow{+1} .$$

Let $D_{\infty'}$ be the henselizations of $\mathbb{P}^1_k$ at ∞. Let $\overline{\mathrm{pr}}_1 : \mathbb{P}^1 \times D_{\infty'} \to \mathbb{P}^1$, $\overline{\mathrm{pr}}_2 : \mathbb{P}^1 \times D_{\infty'} \to D_{\infty'}$ be projections. Let $\eta_{\infty'}$ be the generic point of $D_{\infty'}$. Let us consider the vanishing cycle

$$R\Phi_{\eta_{\infty'}}(\overline{\mathrm{pr}}_1^*(\alpha_!\mathcal{F}) \otimes (\overline{\mathcal{L}}_\phi(xx')|_{\mathbb{P}^1 \times D_{\infty'}})) \in D(\mathbb{P}^1 \times \eta_{\infty'}, R_0)$$

for the projection $\overline{\mathrm{pr}}_2$.

LEMMA 7.3.

(1) *The restriction of* $R\Phi_{\eta_{\infty'}}(\overline{\mathrm{pr}}_1^*(\alpha_!\mathcal{F}) \otimes \mathcal{L})$ *to* $U \times \eta_{\infty'}$ *is zero.*

(2) *For any closed point* $s \in \mathbb{P}^1_k$, *let* $\overline{s}$ *denote a geometric point of* $\mathbb{P}^1_k$ *over* s. *Then for* $s \in \mathbb{P}^1_k - U$ *and for* $i \neq -1$,

$$R^i\Phi_{\overline{\eta}_{\infty'}}(\overline{\mathrm{pr}}_1^*(\alpha_!\mathcal{F}) \otimes \mathcal{L})_{(\overline{s},\overline{\infty}')}$$

is zero.

(3) *For any closed point s on a curve C over k, let G_s denote the absolute Galois group of the fraction field of the henselization of C at s. Then we have*

$$\mathrm{FT}_{\phi,U}(\mathcal{F})_{\overline{\eta}_{\infty'}} = \bigoplus_{s\in\mathbb{P}^1_k-U} \mathrm{Ind}_{G_{s\times_k\infty'}}^{G_{\infty'}} R^{-1}\Phi_{\overline{\eta}_{\infty'}}(\overline{\mathrm{pr}}_1^*(\alpha_!\mathcal{F})\otimes\mathcal{L})_{(\overline{s},\overline{\infty}')}.$$

REMARK 7.4. We will see later that

$$R^{-1}\Phi_{\overline{\eta}_{\infty'}}(\overline{\mathrm{pr}}_1^*(\alpha_!\mathcal{F})\otimes\mathcal{L})_{(\overline{\infty},\overline{\infty}')}.$$

is zero.

PROOF. The assertion (1) follows from [KL, 2.4]. The assertions (2) and (3) are proved in a manner similar to that in [Lau2, p. 161, Prop. 2.3.3.1]. □

7.2. Laumon's local Fourier transform $F^{(0,\infty')}$

Let K be a p-local field of characteristic p with residue field k, π_K a prime element in K. Since the subring $\cap_n(K)^{p^n}$ of K is canonically isomorphic to k, the field K has a canonical structure of k-algebra. Let $\mathcal{O}_K^h$ be the henselization of $k[\pi_K]_{(\pi_K)}$, $K^h = \mathrm{Frac}\,(\mathcal{O}_K^h)$. Let R_0 be a finite local ring on which p is invertible. For an object V in $\mathrm{Rep}(W_K, R_0)$, let $\widetilde{V}$ denote the etale R_0-sheaf on $\mathrm{Spec}\,(K^h)$ corresponding to V. Let $j_K : \mathrm{Spec}\,(K^h) \to \mathrm{Spec}\,(\mathcal{O}_K^h)$ be the canonical inclusion. We consider the R_0-sheaf $j_{K,!}\widetilde{V}$ on $\mathrm{Spec}\,(\mathcal{O}_K^h)$.

Suppose further that there exists a non-trivial additive character $\phi : \mathbb{F}_p \to R_0^\times$. Let $A = \mathbb{A}_k^1 = \mathrm{Spec}\,(k[t])$ be the affine line over k, $\mathcal{L}_\phi$ be the smooth etale R_0-sheaf on A defined by the Artin-Schreier equation $s^p - s = t$ and ϕ. Let $\mathcal{L}_\phi^{(0,\infty')}$ be the pull-back of $\mathcal{L}_\phi$ by the morphism from $\mathrm{Spec}\,(\mathcal{O}_K^h)\times_k\mathrm{Spec}\,(K^h)$ to A whose associated homomorphism of coordinate rings is given by $t \mapsto \frac{\pi_K\otimes 1}{1\otimes\pi_K}$, and set $\overline{\mathcal{L}}_\phi^{(0,\infty')} = (id\times j_K)_!\mathcal{L}_\phi^{(0,\infty')}$. Let us consider the vanishing cycle

$$R^1\Phi_{\overline{\eta}'}(\mathrm{pr}_1^*(j_{K,!}\widetilde{V})\otimes_{R_0}\overline{\mathcal{L}}_\phi^{(0,\infty')})$$

relative to $\mathrm{pr}_2 : \mathrm{Spec}\,(\mathcal{O}_K^h)\times_k\mathrm{Spec}\,(\mathcal{O}_K^h) \to \mathrm{Spec}\,(\mathcal{O}_K^h)$.

Definition 7.5. For an object V in $\mathrm{Rep}(W_K, R_0)$, let

$$F^{(0,\infty')}(V) := R^1\Phi_{\overline{\eta}'}(\mathrm{pr}_1^*(j_{K,!}\widetilde{V}) \otimes_{R_0} \overline{\mathcal{L}}_\phi^{(0,\infty')})_{\overline{t},\overline{t}'},$$

where $\overline{t} = \overline{t}'$ is the spectrum of an algebraic closure of the residue field at the closed point of $\mathrm{Spec}\,(\mathcal{O}_K^h)$. Then $F^{(0,\infty')}(V)$ is also an object in $\mathrm{Rep}(W_K, R_0)$.

We also define objects $F^{(\infty,0')}(V)$ and $F^{(\infty,\infty')}(V)$ in $\mathrm{Rep}(W_K, R_0)$ in a manner similar to that in [Lau2, p. 163, Defn. 2.4.2.3]. The objects $F^{(0,\infty')}(V)$, $F^{(\infty,0')}(V)$ and $F^{(\infty,\infty')}(V)$ in $\mathrm{Rep}(W_K, R_0)$ are called the *local Fourier transforms* of V.

The following lemma is easily checked:

Lemma 7.6 (cf. [Lau2, 2.4.2.1]). *Let $\pi : \mathrm{Spec}\,(\mathcal{O}_K^h) \to A$ and $\pi' : \mathrm{Spec}\,(K^h) \to A$ denote the morphisms whose associated homomorphisms of coordinate rings are given by $t \mapsto \pi_K$, and $t \mapsto \frac{1}{\pi_K}$, respectively. Then for any smooth R_0-sheaf $\mathcal{F}$ on a nonempty open subscheme $U \subset A$ such that $\pi^*\alpha_!\mathcal{F} \cong j_{K,!}\widetilde{V}$, where $\alpha : U \hookrightarrow \mathbb{P}^1_k$ is the canonical inclusion, we have*

$$F^{(0,\infty')}(V) = (\pi \times \pi')^* R\Phi_{\eta_0'}(\overline{\mathrm{pr}}_1^*(\alpha)_!\mathcal{F} \otimes_{R_0} \overline{\mathcal{L}}_\phi)_{(\overline{t},\overline{t}')}.$$

Similar statements hold for $F^{(\infty,0')}$ and $F^{(\infty,\infty')}$.

As a corollary, we have the exactness of the three functors $F^{(0,\infty')}$, $F^{(\infty,0')}$, and $F^{(\infty,\infty')}$.

Proposition 7.7 (cf. [Lau2, p. 165, 2.5.3]). *Let $V \in \mathrm{Rep}(W_K, R_0)$ be a tamely ramified object. Then we have*

(1) *$F^{(0,\infty')}(V)$ is a tamely ramified object with the same rank as V.*

(2) *$F^{(\infty,0')}(V)$ is a tamely ramified object with the same rank as V. If V is unramified, then $F^{(\infty,0')}(V) \cong V(-1)$.*

(3) *$F^{(\infty,\infty')}(V) = 0$.*

Proof. There exists a smooth R_0-flat R_0-sheaf $\mathcal{F}$ on $\mathbb{G}_{m,k}$ which is tamely ramified both at 0 and at ∞ such that the geometric stalk of $\mathcal{F}$ at η_0

can be identified with V. Then global Fourier transform $\mathrm{FT}_{\phi,\mathbb{G}_{m,k}}(\mathcal{F})$ of $\mathcal{F}$ is a smooth R_0-sheaf on $\mathbb{G}_{m,k}$ whose geometric stalks are free R_0-modules of rank $\operatorname{rank} V$. Take any element $a \in k^\times$ and let $t_a : \mathbb{G}_{m,\overline{k}} \to \mathbb{G}_{m,\overline{k}}$ be the translation by a. It is easy to check that $t_a^*(\mathrm{FT}_{\phi,\mathbb{G}_{m,k}}(\mathcal{F})|_{\mathbb{G}_{m,\overline{k}}})$ is isomorphic to $\mathrm{FT}_{\phi,\mathbb{G}_{m,k}}(\mathcal{F})|_{\mathbb{G}_{m,\overline{k}}}$ for any $a \in \overline{k}$. By [V, p. 336, Prop. 1.1], $\mathrm{FT}_{\phi,\mathbb{G}_{m,k}}(\mathcal{F})$ is also tamely ramified both at 0 and at ∞. Hence (2) follows from Lemma 7.2 (3).

By Lemma 7.3 (3), $F^{(0,\infty')}(V) \oplus F^{(\infty,\infty')}(\mathcal{F}_{\overline{\eta}_\infty})$ is a tamely ramified object with the same rank as V. Hence to complete the proof of the proposition, it suffices to prove that $F^{(\infty,\infty')}(V) = 0$.

Let $1 \in \mathrm{Rep}(W_K, R_0)$ be the trivial representation. By considering the constant sheaf R_0 on $\mathbb{G}_{m,k} - \{1\}$, we have $2 \operatorname{rank} F^{(0,\infty')}(1) + \operatorname{rank} F^{(\infty,\infty')}(1) = 2$. Hence $F^{(\infty,\infty')}(1) = 0$. For general V, there exists a smooth R_0-sheaf $\mathcal{F}'$ on $\mathbb{G}_{m,k} - \{1\}$ such that $\mathcal{F}'_{\overline{\eta}_\infty}$ trivial and that $\mathcal{F}'_{\overline{\eta}_0}$ is isomorphic to V. Using this $\mathcal{F}'$, we have $\operatorname{rank} F^{(0,\infty')}(V) = \operatorname{rank} V$. Hence the assertion follows. □

Corollary 7.8. *Let V be an object in* $\mathrm{Rep}(W_K, R_0)$. *Then* $\operatorname{rank} F^{(0,\infty')}(V) = \operatorname{rank} V + \mathrm{sw}(V)$ *and* $\mathrm{sw}(F^{(0,\infty')}(V)) = \mathrm{sw}(V)$.

Proof. We have $F^{(0,\infty')} \cong (\mathrm{FT}_{\phi,\mathbb{G}_{m,k}}(\mathcal{F}_V))_{\overline{\eta}_{\infty'}}$. Applying the Grothendieck-Ogg-Shafarevich formula to $\mathcal{F}_V$, we have $\operatorname{rank} F^{(0,\infty')}(V) = \operatorname{rank} V + \mathrm{sw}(V)$. By Proposition 7.7, $\mathrm{FT}_{\phi,\mathbb{G}_{m,k}}(\mathcal{F}_V)$ is tamely ramified at $0'$. Since

$$
\begin{aligned}
&R\Gamma_c(\mathbb{G}_{m,\overline{k}}, \mathrm{FT}_{\phi,\mathbb{G}_{m,k}}(\mathcal{F}_V)) \\
&\cong R\Gamma_c(\mathbb{G}_{m,\overline{k}} \times \mathbb{G}_{m,\overline{k}}, \mathcal{F}_V \otimes (\mathcal{L}_\phi(xx')|_{\mathbb{G}_{m,k} \times \mathbb{G}_{m,k}}))[1] \\
&\cong R\Gamma_c(\mathbb{G}_{m,\overline{k}}, \mathcal{F}_V),
\end{aligned}
$$

the second assertion follows from the Grothendieck-Ogg-Shafarevich formula. □

Let U be a non-empty open subscheme of A, $\mathcal{F}$ a smooth R_0-flat R_0-sheaf on U which is unramified at ∞, $\alpha : U \hookrightarrow \mathbb{P}^1_k$ the canonical inclusion. Let $k\{t\}$ be the henselization of $k[t]_{(t)}$. For any closed point $s \in A$, let $P_s(t) \in k[t]$ be the unique monic irreducible polynomial which vanishes at s. Let $A_{(s)}$ be the henselization of A at s, η_s the generic point of $A_{(s)}$. Let

$\pi_s : A_{(s)} \to \mathrm{Spec}\,(k\{t\})$ be the morphism whose associated homomorphism of coordinate rings is given by $t \mapsto P_s(t)$.

PROPOSITION 7.9 (cf. [Lau2, p. 194, Thm. 3.4.2]). *Let $S = A - U$ be the complement of U. Then*

(1) *For all $s \in S$,*

$$\det(F^{(0,\infty')}(\pi_{s,*}(\mathcal{F}|_{\eta_s})))$$

is tamely ramified.

(2)

$$\bigotimes_{s\in S} \det(F^{(0,\infty')}(\pi_{s,*}(\mathcal{F}|_{\eta_s})))$$

is unramified and is isomorphic to

$$\det(R\Gamma_c(U \otimes_k \overline{k}, \mathcal{F}))^{-1} \otimes \det(\mathcal{F}_{\overline{\infty}}(-1)).$$

PROOF. (1) Consider the global Fourier transform

$$\mathcal{F}' = \mathrm{FT}_{\phi,U}(\mathcal{F})$$

of $\mathcal{F}$. The sheaf $\mathcal{F}'$ is a smooth R_0-flat R_0-sheaf on $\mathbb{G}_{m,k}$. There exists a distinguished triangle

$$R\Gamma_c(U \otimes_k \overline{k}, \mathcal{F})[1] \to \mathcal{F}'_{\overline{\eta}'_0} \to \mathcal{F}_{\overline{\infty}}(-1) \xrightarrow{+1} .$$

Hence $\det(\mathcal{F}')$ is unramified at 0 and

$$(j'_* \det(\mathcal{F}'))_{\overline{0}'} \cong \det(R\Gamma_c(U \otimes_k \overline{k}, \mathcal{F}))^{-1} \otimes \det(\mathcal{F}_\infty(-1)),$$

where $j' : \mathbb{G}_{m,k} \hookrightarrow A$.

(2) Since

$$\mathcal{F}'_{\overline{\eta}_{\overline{\infty}'}} \cong \bigoplus_{s\in S} \mathrm{Ind}^{G_s}_{G_s \times_k \infty'} (R^{-1}\Phi_{\overline{\eta}_{\overline{\infty}'}}(\mathcal{M}(\mathcal{F})))_{\overline{s},\overline{\infty'}},$$

where

$$\begin{aligned}&(R^{-1}\Phi_{\overline{\eta}_{\overline{\infty}'}}(\mathcal{M}(\mathcal{F})))_{\overline{s},\overline{\infty'}} \\ &\cong \overline{\mathcal{L}}_\phi(s.x')_{\overline{\eta}_{\overline{\infty}'}} \otimes_{R_0} (R^{-1}\Phi_{\overline{\eta}_{\overline{\infty}'}}(\overline{\mathrm{pr}}^*(\alpha_! \mathcal{F}) \otimes \overline{\mathcal{L}}_\phi((x-s).x')))_{\overline{s},\overline{\infty'}},\end{aligned}$$

we have

$$\det(\mathcal{F})_{\overline{\eta}_{\overline{\infty}'}} \cong \mathcal{L}_\phi(\delta_{\mathcal{F}}.x')_{\overline{\eta}_{\overline{\infty}'}} \otimes_{R_0} \bigotimes_{s\in S} \det(F^{0,\infty'}(\pi_{s,*}(\mathcal{F}|_{\eta_s}))),$$

where

$$\delta_{\mathcal{F}} = \sum_{s\in S}(\operatorname{rank}\mathcal{F} + \mathrm{sw}_s(\mathcal{F}))\cdot \mathrm{Tr}(s) \in k.$$

Hence, $\mathcal{L}(-\delta_{\mathcal{F}}.x')\otimes j'_*\det(\mathcal{F}')$ is a smooth invertible R_0-sheaf on A which is tamely ramified at ∞. By the global class field theory, $\mathcal{L}(-\delta_{\mathcal{F}}.x') \otimes j'_*\det(\mathcal{F}')$ must be geometrically constant. Therefore

$$\bigotimes_{s\in S} \det(F^{(0,\infty')}(\pi_s(\mathcal{F}|_{\eta_s})))$$

is an unramified object which is isomorphic to

$$\det(R\Gamma_c(U\otimes_k \overline{k},\mathcal{F}))^{-1}\otimes \det(\mathcal{F}_{\overline{\infty}}(-1)).$$

Hence the proposition follows. □

8. End of Proof

Definition 8.1. Let $\psi : K \to R_0^\times$ be a non-trivial continuous additive character. Let $a \in K^\times$ be the unique element satisfying $\psi(x) = \phi(\mathrm{Tr}_{k/\mathbb{F}_p}(\mathrm{Res}(axd\pi_K)))$ for every $x \in K$. Define the ε'_{0,R_0}-constant $\varepsilon'_{0,R_0}(V,\psi,\phi,\pi_K) \in R_0^\times$ as

$$\begin{aligned}\varepsilon'_{0,R_0}(V,\psi,\phi,\pi_K) := (-1)^{\operatorname{rank}V+\mathrm{sw}V} \\ \cdot \det V(\mathrm{rec}(a))q^{v_K(a)}\det(F^{(0,\infty')}(V))(\mathrm{rec}(\pi_K)),\end{aligned}$$

Proposition 8.2. *Let $A = \operatorname{Spec} k[t]$, U and $\mathcal{F}$ be as above. In the notation of Theorem 4.1, suppose that $X_0 = \mathbb{P}^1_k$, $U_0 = U$, $\omega = -dt$ and $\mathcal{F} = \mathcal{F}$. For $x \in A - U_0$, let $\pi_x = P_x(t)$ and let $\pi_\infty = \frac{1}{t}$. Then we have*

$$\varepsilon_{R_0}(U_0,\mathcal{F}) = q^{\frac{1}{2}\chi(X)\operatorname{rank}(\mathcal{F})}\prod_{x\in X_0-U_0}\varepsilon'_{0,R_0}(\mathcal{F}_x,\psi_{\omega,x},\phi,\pi_x).$$

PROOF. It follows from Proposition 7.9 (2). □

Hence to prove Theorem 4.1, it suffice to prove the following proposition:

PROPOSITION 8.3. *For any object V in $\mathrm{Rep}(W_K, R_0)$ and for any non-trivial continuous additive character $\psi : K \to R_0^\times$, we have*

$$\varepsilon_{0,R}(V \otimes_{R_0} R, \psi) = \varepsilon'_{0,R_0}(V, \psi, \phi, \pi_K).$$

LEMMA 8.4. *Proposition 8.3 holds for (V, ψ) if V is a tamely ramified object.*

PROOF. We may assume that the character ψ is of the form

$$\psi(x) = \phi(\mathrm{Tr}_{k/\mathbb{F}_p}(\mathrm{Res}(x\frac{d\pi_K}{\pi_K}))).$$

Let V be a tamely ramified object in $\mathrm{Rep}(W_K, R_0)$, and $\mathcal{F}_V$ the canonical extension sheaf on $\mathbb{G}_{m,k}$ corresponding to V. Consider the global Fourier transform $\mathrm{FT}_{\phi,\mathbb{G}_{m,k}}(\mathcal{F}_V)$.

Then for every closed point x in $\mathbb{G}_{m,k}$, we have, by [Y1, Thm. 10.5] and [Y1, Thm. 5.6]

$$\det(\mathrm{Fr}_x\,;\,\mathrm{FT}_{\phi,\mathbb{G}_{m,k}}(\mathcal{F}_V)) = \det(\mathrm{Fr}_x\,;\,\mathcal{F}_V) \cdot (q \cdot \varepsilon_{0,R}(V \otimes_{R_0} R, \psi))^{\deg(x)}.$$

Hence $\det(\mathrm{FT}_{\phi,\mathbb{G}_{m,k}}(\mathcal{F}_V)) \otimes_{R_0} \det(\mathcal{F}_V)^{\otimes -1}$ is geometrically constant. The lemma follows by comparing the traces of the geometric Frobenius at 1 and at ∞ on $\det(\mathrm{FT}_{\phi,\mathbb{G}_{m,k}}(\mathcal{F}_V)) \otimes_{R_0} \det(\mathcal{F}_V)^{\otimes -1}$. □

COROLLARY 8.5. *Let $\phi : k \to R_0^\times$ be a non-trivial character and $\chi : W_K^{ab} \to R_0^\times$ be the character given by*

$$\chi(\mathrm{rec}(x)) = \phi(\mathrm{Res}(\frac{dx}{\pi_K x})).$$

Then Proposition 8.3 holds for (V, ψ) if V is of the form $V = W \otimes \chi$ for a tamely ramified object W in $\mathrm{Rep}(W_K, R_0)$.

PROOF. This follows from the definition of $\varepsilon_{0,R}(W, \psi)$. □

Proof of Proposition 8.3. By Lemma 8.4, we may assume that V is totally wild. Let us consider the sheaf $\mathcal{F}'_V$. By Corollary 6.4, the product formula holds for $\mathcal{F}'_V \otimes_{R_0} R$. Hence the assertion follows from Proposition 8.2 and Corollary 8.5. □

This completes the proof of Theorem 4.1. □

9. Application to Saito's Formula

Let k be a finite field of q elements with characteristic p, X/k a smooth projective variety of pure dimension n, and $U \subset X$ an open subscheme such that $D = X - U$ is a divisor with simple normal crossing.

Let $c_{X,U}$ be the element in $\pi_1^{tm}(U)^{ab}$ which is introduced and is called the *relative canonical cycle* in [Sa1, p. 402]; the prime-to-p part of $c_{X,U}$ is equal to

$$\begin{aligned}&(-1)^n c_n(\Omega^1_{X/k}(\log D), \mathrm{Res}) \in H^{2n}(X \mod D, \widehat{\mathbb{Z}}'(n))\\ &:= \prod_{l \neq p} H^{2n}(U, j_! \mathbb{Z}_l(n)_U),\end{aligned}$$

where $j : U \hookrightarrow X$ is the canonical inclusion and $c_n(\Omega^1_{X/k}(\log D), \mathrm{Res})$ is the relative top chern class for the partially trivialized locally free sheaf $(\Omega^1_{X/k}(\log D), \mathrm{Res})$ which is introduced in [Sa1, p. 391], and the p-part of $c_{X,U}$ is the image of usual chern class $(-1)^n c_n(\Omega^1_{X/k}(\log D))$ by the reciprocity map $CH_0(X) \to \pi_1(X)^{ab}$.

Let R_0 be a finite p'-coefficient ring, $\psi_0 : k \to R_0^\times$ a non-trivial additive character. Let $\widehat{\mathbb{Z}}'(1)$ be the group

$$\widehat{\mathbb{Z}}'(1) := \varprojlim{}_{k'} (k')^\times,$$

where k' runs over all finite extensions of k in a fixed algebraic closure of k, and the projective limit is taken with respect to the norm maps. Let $\mathcal{C}_k$ be the exact category of object V in $\mathrm{Rep}(\widehat{\mathbb{Z}}'(1), R_0)$ which satisfies $q^*V \cong V$, where q^*V is the pull-back of V by the q-power map $\widehat{\mathbb{Z}}'(1) \xrightarrow{\cong} \widehat{\mathbb{Z}}'(1)$. For each object V in $\mathcal{C}_k$, we define an element $\tau(V, \psi_0) \in R_0^\times$ in the following way: take a triple $(R, \widetilde{V}, \psi)$, where R is a strict p'-coefficient ring containing R_0, $\widetilde{V}$ is a tamely ramified object in $\mathrm{Rep}(W_{k((t))}, R)$ such that

$\mathrm{Res}^{W_{k((t))}}_{W^0_{k((t))}}$ is identified with $V \otimes_{R_0} R$ via the canonical isomorphism $\widehat{\mathbb{Z}}'(1) \cong W^0_{k((t))}/W^{0+}_{k((t))}$, and $\psi : k((t)) \to R^\times$ is a continuous additive character of conductor -1 whose restriction to $k[[t]]$ induces ψ_0. We put $\tau(V, \psi_0) := (-q)^{\mathrm{rank}\, V} \varepsilon_{0,R}(\widetilde{V}, \psi)$. Then $\tau(V, \psi_0)$ belongs to $R_0^\times$ and is independent of the choice of $(R, \widetilde{V}, \psi)$.

LEMMA 9.1 (cf. [Sa1, p. 400–401, Lem. 1 (1)]). *Let k' be a finite extension of k and $r \in \mathbb{Z}_{>0}$ be a positive integer. Let $r = ms$ be the decomposition of r into the prime-to-p part m and the p-part s. For an object V in $\mathcal{C}_{k'}$, let* $\mathrm{Ind}\, V$ *be the object in $\mathcal{C}_k$ defined as*

$$\mathrm{Ind}\, V = \bigoplus_{i=0}^{[k':k]-1} (q^i)^* (\mathrm{Ind}_r V),$$

where $\mathrm{Ind}_r V$ *is the induced representation of V by the multiplication-by-r map $\widetilde{\mathbb{Z}}'(1) \to \widetilde{\mathbb{Z}}'(1)$. Then we have*

$$\tau(\mathrm{Ind}\, V, \psi_0) = (\det V)(m) \tau(V, \psi_0^{(s)} \circ \mathrm{Tr}_{k'/k}) \cdot (\tau_{k,m,\psi_0})^{[k':k] \cdot \mathrm{rank}\, V}.$$

Here $\det V$ *is considered as a character of $\widehat{\mathbb{Z}}'(1)/(q-1)\widehat{\mathbb{Z}}'(1) = k^\times$ and $\psi_0^{(s)}$ is the additive character of k given by $\psi_0^{(s)}(x) = \psi_0(x^s)$, and*

$$\tau_{k,m,\psi_0} = \begin{cases} \left(\frac{m}{k}\right) \left(\left(\frac{2}{k}\right) \tau_k \left(\left(\overline{k}\right), \psi_0\right)\right)^{m-1} & \text{if } p \neq 2, \\ \left(\frac{m}{k}\right)' q^{\frac{m-1}{2}} & \text{if } p = 2, \end{cases}$$

Here $\left(\frac{m}{k}\right)' = (-1)^{\frac{m^2-1}{8} \cdot [k:\mathbb{F}_2]}$.

PROOF. We may assume that $s = 1$. Let $L = k'((t))$ and $K = k((t^m)) \subset L$. Take a triple $(R, \widetilde{V}, \psi)$, where R is a strict p'-coefficient ring containing R_0, $\widetilde{V}$ is a tamely ramified object in $\mathrm{Rep}(W_L, R)$ such that $\mathrm{Res}^{W_L}_{W_L^0}$ is identified with $V \otimes_{R_0} R$, and $\psi : K \to R^\times$ is a continuous additive character of conductor -1 whose restriction to $\mathcal{O}_K$ induces ψ_0. Then we have $\tau(\mathrm{Ind}\, V, \psi_0) = (-q)^{\mathrm{rank}\, \mathrm{Ind}\, V} \varepsilon_{0,R}(\mathrm{Ind}^{W_K}_{W_L} \widetilde{V}, \psi)$ and $\tau(V, \psi_0 \circ \mathrm{Tr}_{k'/k}) = (-q)^{[k':k]\, \mathrm{rank}\, V} (\det V)(m)^{-1} \varepsilon_{0,R}(V, \psi \circ \mathrm{Tr}_{L/K})$. By [Y1, Thm. 5.6], it suffices to prove that

$$\tau^{[k':k]}_{k,m,\psi_0} = \frac{(-q)^{[k':k]m}}{(-q)^{[k':k]}} \cdot \lambda_R(L/K, \psi).$$

Let $L_0 = k'((t^m))$ be the maximal unramified subextension of L/K. By [Y1, Prop. 6.5 (5)] we have

$$\begin{aligned}\lambda_R(L/K,\psi) &= \lambda_R(L_0/K,\psi)^m \cdot \lambda_R(L/L_0,\psi\circ \mathrm{Tr}_{L_0/K}) \\ &= (-1)^{([k':k]-1)m} \cdot \lambda_R(L/L_0,\psi\circ \mathrm{Tr}_{L_0/K}).\end{aligned}$$

By [Y1, Lem. 6.7], we have

$$\lambda_R(L/L_0,\psi\circ \mathrm{Tr}_{L_0/K}) \\ = \begin{cases} q^{-\frac{m-1}{2}[k':k]}\left(\dfrac{(-1)^{\frac{m-1}{2}}m}{k'}\right) & \text{if } m \text{ is odd and } p\neq 2, \\ q^{-\frac{m-1}{2}[k':k]}(-1)^{\frac{m^2-1}{8}[k':\mathbb{F}_2]} & \text{if } m \text{ is odd and } p=2, \\ q^{-\frac{m}{2}[k':k]}\tau_R(L/L_0,\psi\circ_{L_0/K})\left(\dfrac{(-1)^{\frac{m}{2}-1}\frac{m}{2}}{k'}\right) & \text{if } m \text{ is even.}\end{cases}$$

Hence the assertion follows. □

Let V be an object in $\mathrm{Rep}(\pi_1^{tm}(U),R_0)$. We define $\tau_{D/k}(V,\psi_0)\in R_0^\times$ in the following way (cf. [Sa1, p. 403]). Let $(D_i)_{i\in I}$ denote the family of irreducible components of D. For $i\in I$, let k_i be the constant field of D_i, $c_i\in\mathbb{Z}$ the Euler number of $(D_i - \bigcup_{i'\neq i} D_{i'})\otimes_{k_i}\overline{k_i}$, and $V_i\in\mathcal{C}_{k_i}$ the restriction of V by the morphism $\widehat{\mathbb{Z}}'(1)_{\overline{k_i}}\to\pi_1(U)^{tm}$ which is canonically defined up to conjugacy. Then we put

$$\tau_{D/k}(V,\psi_0) := \prod_{i\in I}\tau_{k_i}(V_i,\psi_0\circ\mathrm{Tr}_{k_i/k})^{c_i}.$$

Proposition 9.2 (cf. [Sa1, p. 403, Thm. 1]). *Assume that $p\neq 2$. Let $\mathcal{F}$ be the smooth R_0-sheaf on U corresponding to an object V in $\mathrm{Rep}(\pi_1^{tm}(U),R_0)$. Then the global ε_0-constant*

$$\varepsilon_{0,R_0}(U/k,\mathcal{F}) = \det(-\mathrm{Fr}_k\,;\,R\Gamma_c(U\otimes_k\overline{k},\mathcal{F}))^{-1}\in R_0^\times$$

satisfies

$$\varepsilon_{0,R_0}(U/k,\mathcal{F}) = \det V(-c_{X,U})\cdot\tau_{D/k}(V,\psi_0)\cdot\varepsilon_{0,R_0}(U/k,R_0)^{\mathrm{rank}\,V}.$$

Let K be a p-local field with the residue field k, X a proper flat generically smooth $\mathcal{O}_K$-scheme which is regular of dimension n as a scheme, and $U \subset X$ an open subscheme contained in $X \otimes_{\mathcal{O}_K} K$. Assume that the complement $D = X - U$ is a divisor with simple normal crossing of X. Let $(D_i)_{i \in I}$ be the family of irreducible components of D and let $I_0 \subset I$ be the subset of $i \in I$ such that $D_i \subset X \otimes_{\mathcal{O}_K} k$. Let r_i be the multiplicity of of $X \otimes_{\mathcal{O}_K} k$ at D_i and m_i be the prime-to-p part of r_i. For a subset $J \subset I$, let $D_J = \bigcap_{i \in J} D_i$. We assume that $D_{J,K} = \bigcap_{i \in J} D_i \otimes_{\mathcal{O}_K} K$ is smooth over K for all $J \subset I$ and that (X, U) is tame over $\mathcal{O}_K$ in the sense of [Sa1, p. 404], that is, the $\mathcal{O}_X$-module $\Omega^1_{X/\mathcal{O}_K}(\log D / \log k)$ of differentials with logarithmic poles is locally free. We define (the tame part of) the relative canonical cycle

$$c_{X,U/\mathcal{O}_K} \in H^{2n}_{X_k}(X \bmod D, \widehat{\mathbb{Z}}'(n))$$

in the following way (cf. [Sa1, p. 404]). For $i \in I$, let $D_i^* = D_i - \bigcup_{i' \neq i} D_{i'}$. For $i \in I_0$, we also denote, by abuse of notation, by c_{D_i, D_i^*} the prime-to-p part in $H^{2n-2}(D_i \mod D, \widehat{\mathbb{Z}}'(n-1))$ of the relative canonical cycle $c_{D_i, D_i^*} \in \pi_1^{tm}(U)^{ab}$. Then we put

$$c_{X,U/\mathcal{O}_K} = i_*(-\sum_{i \in I_0} m_i \cup c_{D_i, D_i^*}) \in H^{2n}_{X_k}(X \mod D, \widehat{\mathbb{Z}}'(n)),$$

where $i_* : \bigoplus_{i \in I_0} H^{2n-1}(D_i \mod D, \widehat{\mathbb{Z}}'(n)) \to H^{2n}_{X_k}(X \mod D, \widehat{\mathbb{Z}}(n))$ is as in [Sa1, p403], and $m_i \in \mathbb{Z}$ is considered as an element in $H^1(D_i \mod D, \widehat{\mathbb{Z}}(1))$.

Let $h : R_0 \to R$ be a local ring homomorphism from R_0 to a strict p'-coefficient ring R, $\psi : K \to R^\times$ a non-trivial continuous additive character with $\operatorname{ord} \psi = -1$, and $\psi_0 : k \to R^\times$ the character given by the restriction $\psi|_{\mathcal{O}_K}$. For an object (ρ, V) in $\operatorname{Rep}(\pi_1^{tm}(U), R_0)$, we define $\tau_{D/k}(\rho, \psi_0) \in R_0^\times$ in a manner similar to that in [Sa1, p. 404], that is,

$$\tau_{D/k}(\rho, \psi_0) = \prod_{i \in I_0^t} \tau_{k_i}(\rho_i, \psi_0 \circ \operatorname{Tr}_{k_i/k})^{c_i},$$

where $I_0^t = \{i \in I_0 \,;\, p \nmid r_i\}$.

In view of the proof of [Sa1, p. 403, Thm. 1] given in [Sa1, p. 409–415], to prove Proposition 9.2, it suffices to prove the following proposition:

PROPOSITION 9.3 (cf. [Sa1, p. 405, Thm. 2]). *Assume that $p \neq 2$. Let $\mathcal{F}$ be the the smooth R_0-sheaf on U corresponding to (ρ, V). Then the local ε_0-constant*

$$\varepsilon_{0,R}(U/K, \mathcal{F}, \psi) = \varepsilon_{0,R}(R\Gamma_c(U \otimes_K \overline{K}, \mathcal{F}) \otimes_{R_0} R, \psi) \in R^\times$$

is equal to the image by h of

$$\det \rho(-c_{X,U/\mathcal{O}_K})\tau_{D/k}(\rho, \psi_0) \times \begin{cases} \left((-1)^{\chi_c(U_{\overline{K}})} \left(\frac{M}{k}\right) \left(\left(\frac{2}{k}\right) \tau_k\left(\left(\frac{}{k}\right), \psi_0\right)\right)^{\chi_c(U_{\overline{K}})-\chi_c^s(U_{\overline{K}})}\right)^{\operatorname{rank} V} & \text{if } p \neq 2, \\ \left((-1)^{\chi_c(U_{\overline{K}})} \left(\frac{M}{k}\right)' q^{\frac{1}{2}(\chi_c(U_{\overline{K}})-\chi_c^s(U_{\overline{K}}))}\right)^{\operatorname{rank} V} & \text{if } p = 2, \end{cases}$$

where $\chi_c(U_{\overline{K}})$ and $\chi_c^s(U_{\overline{K}})$ are as in [Sa1, p. 405, Thm. 2].

PROOF. We fix a subset $J \subset I$ and a connected component E of D_J and study the restriction $R\Psi\mathcal{F}|_{E^*}$ of the complex of the nearby cycles $R\Psi\mathcal{F}$ on $E^* = E - \bigcup_{i \in J} D_i$. For $i \in J$, let I_i and P_i ..... $\alpha_i : I_i/P_i \to \pi_1^{tm}(U)$. We may assume that the image of α_i is commutative to each other. Let $J_0 = J \cap I_0$ and M be the complex concentrated on degree 0 and -1:

$$\mathbb{Z}^{J_0} \to \mathbb{Z} \; ; \; e_i \mapsto r_i.$$

Then the canonical morphism

$$\widehat{\mu} : \prod_{i \in J_0} I_i/P_i \to I_K/P_K$$

is canonically identified with $\mu \otimes id_{\widehat{\mathbb{Z}}'(1)}$.

In view of the proof of [Sa1, p. 405, Thm. 2] given in [Sa1, p. 405–408], in order to prove the proposition, it suffices to prove the following analogue of [Sa1, p. 407, Cor. 1]. □

LEMMA 9.4 (cf. [Sa1, p. 407, Cor. 1]). *On each stratum E^*, $R\Psi\mathcal{F}|_{E^*}$ is a successive extension of bounded complexes of smooth R_0-flat R_0-sheaves which is tamely ramified along the boundary $E - E^*$. If $J = \{i\}$ for some $i \in I_0$, then $R^q\Psi\mathcal{F}|_{E^*} = 0$ for $q \neq 0$ and $R^0\Psi\mathcal{F}|_{E^*} = \operatorname{Ind}_{I_i/P_i}^{I_K/P_K} \rho_i$ as a representation of I_K/P_K. If J otherwise, we have*

$$[R\Psi\mathcal{F}|_{E^*}] = 0$$

in the Grothendieck group of the smooth R_0-flat R_0-sheaves on E^* which is tamely ramified along the boundary $E - E^*$.

PROOF. We use the notation in the proof of [Sa1, p. 405–406, Prop. 6]. Let $R_0[[G]] = \varprojlim_d R_0[G_d]$. Consider the $R_0[[G]]$-sheaf $(\varphi_{d,!}\varphi_d^*\mathcal{F})_d$. Since

$$\mathcal{F} \cong (\varphi_{d,!}\varphi_d^*\mathcal{F})_d \otimes^{\mathbb{L}}_{R_0[[G]]} R_0,$$

we have

$$R\Psi\mathcal{F}|_{E^*} \cong (\mathcal{F}_d|_{E^*} \otimes_{R_0} \mathrm{Ind}_{I_{K_d}}^{I_K} R\Psi_d R_{0,U_d})_d \otimes^{\mathbb{L}}_{R_0[[G]]} R_0.$$

Hence there exists a filtration on $R\Psi\mathcal{F}|_{E^*}$ such that each graded piece is isomorphic to

$$(\mathcal{F}_d|_{E^*} \otimes_{R_0} \mathrm{Ind}_{I_{K_d}}^{I_K} R_0)_d \otimes^{\mathbb{L}}_{R_0[[G]]} R_0 \otimes_{R_0} \bigwedge^q \mathrm{Hom}(H_1(M)(1), R_0).$$

Hence the lemma follows. □

This completes the proof of Proposition 9.2. □

References

[De1] Deligne, P., *Les constantes des équations fonctionnelles des fonctions L*, in Modular functions in one variable II, 501–597, Lecture Notes in Math. 349, Springer, Berlin (1973).

[De2] Deligne, P., *Cohomologie étale. Séminaire de Géométrie Algébrique du Bois-Marie SGA $4\frac{1}{2}$. Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier*, Lecture Notes in Math. 569, Springer, Berlin (1977).

[De3] Deligne, P., *Les constantes des équations fonctionnelles des fonctions L*, Seminaire a l'IHES (1980).

[DH] Deligne, and G. Henniart, Sur la variation, par torsion, des constantes locales d'équations fonctionnelles de fonctions L, Invent. Math. **64** (1981), no. 1, 89–118.

[Dr] Drinfeld, V. G., *Two-dimensional ℓ-adic representations of the fundamental group of a curve over a finite residue field and automorphic forms on $GL(2)$*, 85–114.

[He] Henniart, G., Galois ε-factors modulo roots of unity, Invent. Math. **78** (1984), 117–126.

[K1] Kato, K., *Swan conductors with differential values*, in Galois representations and arithmetic algebraic geometry (Kyoto, 1985/Tokyo, 1986), 315–342, Adv. Stud. Pure Math. **12**, North-Holland, Amsterdam-New York (1987).

[Kz1] Katz, N. M., Local-to-global extensions of representations fundamental groups, Ann. Inst. Fourier (Grenoble) **36** (1986), no. 4, 69–106.

[KL] Katz, N. M. and G. Laumon, Transformation de Fourier et majoration de sommes exponentielles, Inst. Hautes Études Sci. Publ. Math. **62** (1985), 361–418.

[Lau1] Laumon, G., *Semi-continuité du conducteur de Swan (d'après P. Deligne)*, in Caractéristique d'Euler-Poincaré, Astéisque, 82-83, Soc. Math. France, Paris.

[Lau2] Laumon, G., Transformation de Fourier, constantes d'équations fonctionnelles et conjecture de Weil, Inst. Hautes Études Sci. Publ. Math. **65** (1987), 131–210.

[R] Raynaud, M., *Caractéristique d'Euler-Poincaré d'un faisceau et cohomologie des variété abéliennes*, Séinaire Bourbaki, Exp. 286 (1965).

[Sa1] Saito, T., ε-factor of a tamely ramified sheaf on a variety, Invent. Math. **113** (1993), 389–417.

[Sa2] Saito, T., *Ramification groups and local constants*, UTMS preprint 96-19, University of Tokyo (1996).

[Se1] Serre, J.-P., Groupes proalgébriques, Inst. Hautes Études Sci. Publ. Math. **7** (1960), 1–67.

[Se2] Serre, J.-P., *Corps locaux*, Hermann, Paris (1968).

[Se3] Serre, J.-P., *Groupes algébriques et corps de classes*, Hermann, Paris (1975).

[SGA4] Artin, M., Grothendieck, A. and J. L. Verdier (ed., with the collaboration of P. Deligne and B. Saint-Donat) *Théorie des topos et cohomologie étale des schémas. Tome 3*, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Lecture Notes in Math. 305, Springer, Berlin-New York (1973).

[SGA5] Illusie, L. (ed.), *Cohomologie l-adique et fonctions L*, Séminaire de Géometrie Algébrique du Bois-Marie 1965-1966 (SGA 5), Lecture Notes in Math. 589, Springer, Berlin-New York (1977).

[T1] Tate, J., *Fourier analysis in number fields, and Hecke's zeta-functions*, in Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), 305–347.

[V] Verdier, J.-L., *Spécialisation de faisceaux et monodromie modérée*, in Analysis and topology on singular spaces, II, III (Luminy, 1981), 332–364, Astérisque, 101-102, Soc. Math. France, Paris (1983).

[Y1] Yasuda, S., *Local constants in torsion rings*, preprint (2001).

[Y2] Yasuda, S., Local ε_0-characters in torsion rings, J. Théor. Nombres Bordx. **19** (2007), 763–797.

(Received May 15, 2006)
(Revised May 11, 2009)

Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502, Japan
E-mail: yasuda@kurims.kyoto-u.ac.jp