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Local Constants in Torsion Rings

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Abstract. Let p be a rational prime and K a local field of residue characteristic p. In this paper, generalizing the theory of Deligne [De1], we construct a theory of local ε_0-constants for representations, over a complete local ring with an algebraically closed residue field of characteristic $\neq p$, of the Weil group W_K of K.

1. Introduction

Let K be a complete discrete valuation field whose residue field k is finite of characteristic p. In this paper, such a field is called a *p-local field*. Let q denote the cardinality of k. Let W_K denote the Weil group of K. In [De1], Deligne defined the local constants $\varepsilon(V, \psi, dx)$ and $\varepsilon_0(V, \psi, dx)$ for triples (V, ψ, dx) where V is a complex or an ℓ-adic representation of W_K of finite rank, ψ an additive character of K, and dx a Haar measure of K. These local constants play an important role in the theory of L-functions for representations of global Weil groups.

For a topological ring R, let $\mathrm{Rep}(W_K, R)$ denote the category of continuous representations of W_K on finitely generated free R-modules. A *strict p'-coefficient ring* is a noetherian commutative local ring with an algebraically closed residue field of characteristic $\neq p$ such that $(R^\times)^p = R^\times$. In this paper, we generalize the theory of Deligne to the representation of W_K over strict p'-coefficient rings. We consider a triple $(R, (\rho, V), \psi)$ where R is a strict p'-coefficient ring, (ρ, V) is an object in $\mathrm{Rep}(W_K, R)$, and $\psi : K \to R^\times$ is a non-trivial continuous additive character. The main theorem of this paper is the following:

THEOREM 1.1 (See Theorem 5.1 for the precise statements). *Let K be a p-local field. Then for each such triple $(R, (\rho, V), \psi)$ we can attach, in a canonical way, an element*

$$\varepsilon_{0,R}((\rho, V), \psi) \in R^\times$$

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which satisfy several properties including the following:

(1) *For fixed R and ψ, the element $\varepsilon_{0,R}((\rho,V),\psi) \in R^\times$ depends only on the isomorphism class of (ρ,V).*

(2) *Let $(R,(\rho,V),\psi)$ be such a triple, R' a strict p'-coefficient ring, and $h : R \to R'$ a local ring homomorphism. Then we have*

$$h(\varepsilon_{0,R}(V,\psi)) = \varepsilon_{0,R'}(V \otimes_R R', h \circ \psi).$$

(3) *Let $(R,(\rho,V),\psi)$ be such a triple. Suppose that R is a field. Then*

$$\varepsilon_{0,R}(V,\psi) = \varepsilon_0(V,\psi,dx),$$

where dx is the R-valued Haar measure of K in the sense of Deligne [De1, p. 554, 6.1] *satisfying $\int_{\mathcal{O}_K} dx = 1$.*

We call the element $\varepsilon_{0,R}(V,\psi)$ the *local ε_0-constant* of the triple $(R,(\rho,V),\psi)$.

For a fixed K, our local ε_0-constants satisfy many properties analogous to those of Deligne's ε_0-constants; for example additivity, formula for rank one objects, formula for changes of ψ, and formula for unramified twists (see § 5, Theorem 5.1 for details). We also prove that the well-known formula for local ε_0-constants for induced representations also holds for our case:

THEOREM 1.2 (Theorem 5.6). *Let L be a finite separable extension of K, let R be a strict p'-coefficient ring, and let $\psi : K \to R^\times$ be a non-trivial continuous additive character. Then there exists an element*

$$\lambda_R(L/K,\psi) \in R^\times$$

such that for every object V in $\mathrm{Rep}(W_L,R)$, we have

$$\varepsilon_{0,R}(\mathrm{Ind}_{W_L}^{W_K} V,\psi) = \varepsilon_{0,R}(V,\psi \circ \mathrm{Tr}_{L/K}) \cdot \lambda_R(L/K,\psi)^{\mathrm{rank}\, V}.$$

Furthermore, $\lambda_R(L/K,\psi)$ is compatible with the base change by $h : R \to R'$.

Let k be a finite field. When R_0 is the ring of integers of a finite extension of $\mathbb{Q}_\ell$ for a prime $\ell \neq p$, the product formula of Deligne-Laumon describes the determinant of Frobenius on the etale cohomologies of a smooth R_0-sheaf on a curve over k as a product of local ε_0-constants. In the forthcoming paper [Y], we generalize the product formula to the case where R_0 is a profinite p'-coefficient ring, giving evidence that our construction provides a good theory of local ε_0-constants.

1.1. The local ε conjecture

In [K2, p. 5, 1.8], Kato gives a conjecture concerning local ε-constants, which he named as "local ε conjecture". While Kato deals only with $K = \mathbb{Q}_p$ case, the formulation of the "$\ell \neq p$"-part of his conjecture can be generalized without any difficulty to the case where K is an arbitrary p-local field. Let us briefly explain his conjecture. (We do not recall the exact form of his conjecture in this introduction because it is rather lengthy. In § 5, we recall his conjecture in a form slightly different from his original one.)

Let ℓ be a rational prime different from p. We consider a triple $(\Lambda, (\rho, V), \psi)$, where $\Lambda = (\Lambda, \mathfrak{m}_\Lambda)$ is a complete noetherian commutative local ring whose residue field is finite of characteristic ℓ, (ρ, V) is an object in $\mathrm{Rep}(W_K, \Lambda)$ and $\psi : K \to W(\overline{\mathbb{F}}_\ell)^\times$ is a non-trivial continuous additive character.

Let (ρ, V) be an object in $\mathrm{Rep}(W_K, \Lambda)$. Let r denote the Λ-rank of V. Then the r-th exterior power of (ρ, V) defines a continuous homomorphism $\det(\rho) : W_K^{\mathrm{ab}} \to \Lambda^\times$.

We set

$$a_V = a_{(\rho,V)} = \det(\rho)(\mathrm{rec}(\ell)) \in \Lambda^\times.$$

The ring Λ has a canonical structure of a $\mathbb{Z}_\ell$-algebra. Define $\Lambda_V = \Lambda_{(\rho,V)}$ by

$$\Lambda_{(\rho,V)} = \{x \in \Lambda \widehat{\otimes}_{\mathbb{Z}_\ell} W(\overline{\mathbb{F}}_\ell)\,;\, (1 \otimes \varphi)(x) = (a_{(V,\rho)} \otimes 1)x\}.$$

Λ_V is a Λ-submodule of $\Lambda \otimes_{\mathbb{Z}_\ell} W(\overline{\mathbb{F}}_\ell)$ which is free of rank one.

The "$\ell \neq p$ part" of his conjecture ([K2, p. 5, Conj. 1.8]) predicts the existence of a canonical basis $\varepsilon_{\Lambda,\psi}(V)$ of the invertible Λ-module

$$\Delta_\Lambda(V) = \det{}_\Lambda R\Gamma(\mathbb{Q}_\ell, V) \otimes_\Lambda \Lambda_V,$$

which satisfies certain conditions and has a connection with Deligne's local constants.

As a corollary of Theorem 1.1, we have

Corollary 1.3. *The $\ell \neq p$ part of Kato's local ε conjecture is true.*

1.2. Other results in this paper

In viewing the proof of "independence of ϕ_0" which we have briefly described above, we get a formula expressing tame ε_0-constants as an integral on the tame inertia group of K. By taking a prime element of K, we identify X_0 with $\mathbb{G}_{m,k}$. We set $G = W_K/(W_K)^{0+}$ and $I = (W_K)^0/(W_K)^{0+}$. For every positive integer n prime to p, let $[n] : \mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$ denote the n-th power map. By taking the projective limit of $H^1_c(\mathbb{G}_{m,\overline{k}}, [n]^*\widetilde{\mathcal{L}}'_{\phi_0})$, we get a free $R[[I]]$-module $\widehat{W}$ of rank one with a semi-linear action of G. Take a lift $\widetilde{\mathrm{Fr}} \in G$ of the geometric Frobenius. The eigenvalue of the action of $\widetilde{\mathrm{Fr}}$ gives a well-defined element u in the G-coinvariant $(R[[I]]^\times)_G$. Then $\varepsilon_{0,R}(V,\psi)$ has the following description:

PROPOSITION 1.4 (Proposition 11.4). *Take an arbitrary representative $\widehat{u} \in R[[I]]$ of u. We consider $\widehat{u}$ as a measure on I. Let $\psi : K \to R^\times$ be an additive character with conductor -1 satisfying*

$$\psi(x) = \phi_0(\mathrm{rec}^{-1}(\widetilde{\mathrm{Fr}}^{-1})x)$$

for all $x \in \mathcal{O}_K$. Then for any tamely ramified object (ρ, V) in $\mathrm{Rep}(W_K, R)$, we have

$$\varepsilon_{0,R}(V,\psi) = \det\left(\frac{1}{q}\int_{g\in I} \rho(g)^{-1} d\widehat{u}(g)\right).$$

This paper also deals with results (Proposition 10 and Proposition 8.3) analogous to that in Deligne-Henniart [DH, p. 108, Thm. 4.2 and p. 110, Thm. 4.6].

Let us explain the outline of our proof of Theorem 5.1. Let K be a p-local field. Let be R be a strict p'-coefficient ring. For an object (ρ, V) in $\mathrm{Rep}(W_K, R)$, let $V = V^0 \oplus V^{>0}$ be the decomposition of V into the tamely ramified part V^0 and the totally wild part $V^{>0}$. We construct the epsilon constants $\varepsilon_{0,R}(V^{>0},\psi)$, $\varepsilon_{0,R}(V^0,\psi)$ for $V^{>0}$, V^0 separately and then define $\varepsilon_{0,R}(V,\psi)$ as the product $\varepsilon_{0,R}(V^{>0},\psi \cdot \varepsilon_{0,R}(V^0,\psi)$. Let $\mathfrak{m}_R \subset R$ denote the maximal ideal of R. We construct $\varepsilon_{0,R}(V^{>0},\psi)$ by lifting $\varepsilon_{0,R/\mathfrak{m}_R}(V^{>0}\otimes_R R/\mathfrak{m}_R,\psi)$constructed by Deligne ([De1, p. 555-556, Thm. 6.5]) in a unique way such that $\varepsilon_{0,R}(V^{>0},\psi)$ satisfies a version of Henniart's formula (cf. Theorem 5.3). The original Henniart's formula in [He, Theorem] is a formula for complex representations of W_K, however, it can be stated as a formula for $\varepsilon_{0,R/\mathfrak{m}_R}(V^{>0} \otimes_R R/\mathfrak{m}_R, \psi)$. We identify the tame quotient of W_K with that of the Weil group $W_{K'}$ of the completion of $\mathbb{A}^1_k$ at 0 and then construct $\varepsilon_{0,R}(V^0, \psi)$ in the spirit of Laumon's definition ([Lau1]) of ε_0-constants for ℓ-adic representations of $W_{K'}$ (cf. Theorem 5.4).

Let us briefly review the contents of this paper. After recalling in § 3 some basic facts necessary in this paper, we recall, in § 4, basic properties of Langlands-Deligne's local ε-constants. Main results of this paper will be given in § 5. After the preparation of λ-constants in § 6 and of Henniart and Saito's results on the description of local ε-constants in § 7, we give, in § 8, the definition of the local ε_0-constant $\varepsilon_{0,R}(V, \psi)$ for totally wild V. In § 9, we give a proof of a formula of $\varepsilon_{0,R}$ for induced representations. In § 10, we define $\varepsilon_{0,R}(V, \psi)$ for tamely ramified representations. In § 11, we prove that the constant $\varepsilon_{0,R}(V, \psi)$ defined in § 10 does not depend on the choice of an auxiliary parameter and completes the proof of the main results in § 5.

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2. Notation

Let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers respectively.

Let $\mathbb{Z}_{>0}$ (resp. $\mathbb{Z}_{\geq 0}$) be the ordered set of positive (resp. non-negative) integers. We also define $\mathbb{Q}_{\geq 0}$, $\mathbb{Q}_{>0}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ in the same way. For $\alpha \in \mathbb{R}$, let $\lfloor \alpha \rfloor$ (resp. $\lceil \alpha \rceil$) denote the maximum integer not larger than α (resp. the minimum integer not smaller than α).

For a prime number ℓ, we denote by $\mathbb{F}_\ell$ the finite field of ℓ elements. For $n \in \mathbb{Z}_{>0}$, we let $\mathbb{F}_{\ell^n}$ denote the unique extension of $\mathbb{F}_\ell$ of degree n. We denote by $\overline{\mathbb{F}}_\ell$ a fixed algebraic closure of $\mathbb{F}_\ell$, by $\mathbb{Z}_\ell = W(\mathbb{F}_\ell)$ (resp. by $W(\overline{\mathbb{F}}_\ell)$) the ring of Witt vectors of $\mathbb{F}_\ell$ (resp. $\overline{\mathbb{F}}_\ell$), and by $\mathbb{Q}_\ell$ the field of fractions

$\mathrm{Frac}(\mathbb{Z}_\ell))$ of $\mathbb{Z}_\ell$. Let $\varphi : W(\overline{\mathbb{F}}_\ell) \to W(\overline{\mathbb{F}}_\ell)$ be the Frobenius automorphism of $W(\overline{\mathbb{F}}_\ell)$.

For a ring R, we denote by $R^\times$ the group of units in R. For a positive integer $n \in \mathbb{Z}_{>0}$, we denote by $\boldsymbol{\mu}_n(R)$ the group of n-th roots of unity in R, and by $\boldsymbol{\mu}_{n^\infty}$ the union $\cup_i \boldsymbol{\mu}_{n^i}(R)$.

For a finite extension L/K of fields, we let $[L : K]$ denote the degree of L over K. For a subgroup H of a group G of finite index, we denote its index by $[G : H]$.

For a finite field k of characteristic $\neq 2$, we let $\left(\frac{}{k}\right) : k^\times \to \{\pm 1\}$ denote the unique surjective homomorphism.

Throughout this paper, we fix once for all a prime number p. We consider a complete discrete valuation field K whose residue field is finite of characteristic p. Such a field K is called a *p-local field*.

For a p-local field K, we denote by $\mathcal{O}_K$ its ring of integers, by $\mathfrak{m}_K$ the maximal ideal of $\mathcal{O}_K$, by k_K the residue field $\mathcal{O}_K/\mathfrak{m}_K$ of $\mathcal{O}_K$, and by v_K the normalized valuation $K^\times \to \mathbb{Z}$. We also denote by $q_K = \sharp k_K$ the cardinality of k_K, by W_K the Weil group of K, by $\mathrm{rec} = \mathrm{rec}_K : K^\times \xrightarrow{\cong} W_K^{\mathrm{ab}}$ the reciprocity map given by the local class field theory, which sends a prime element of K to a lift of the geometric Frobenius of k. We denote by $(\ ,\ )_K : K^\times \times K^\times \to \{\pm 1\}$ the Hilbert symbol (resp. the trivial biadditive map) if $\mathrm{char}\, K \neq 2$ (resp. $\mathrm{char}\, K = 2$). We often abbreviate k_K and q_K by k and q respectively if there is no risk of confusion.

If L/K is a finite separable extension of p-local fields, we let $e_{L/K} \in \mathbb{Z}$, $f_{L/K} \in \mathbb{Z}$, $D_{L/K} \in \mathcal{O}_L/\mathcal{O}_L^\times$, and $d_{L/K} \in \mathcal{O}_K/\mathcal{O}_K^{\times 2}$ denote the ramification index of L/K, the residual degree of L/K, the different of L/K, and the discriminant of L/K respectively.

For a topological group (or more generally for a topological monoid) G and a commutative topological ring R, let $\mathrm{Rep}(G, R)$ denote the category whose object is a pair (ρ, V) of a finitely generated free R-module V and a continuous group homomorphism $\rho : G \to GL_R(V)$ (we endow $GL_R(V)$ with the topology induced from the direct product topology of $\mathrm{End}_R(V)$), and whose morphisms are R-linear maps compatible with actions of G.

A sequence

$$0 \to (\rho', V') \to (\rho, V) \to (\rho'', V'') \to 0$$

of morphisms in $\mathrm{Rep}(G, R)$ is called a *short exact sequence* in $\mathrm{Rep}(G, R)$ if $0 \to V' \to V \to V'' \to 0$ is the short exact sequence of R-modules.

In this paper, a noetherian commutative local ring with residue field of characteristic $\neq p$ is called a *p'-coefficient ring*. Any p'-coefficient ring $(R, \mathfrak{m}_R)$ is considered as a topological ring with the $\mathfrak{m}_R$-preadic topology. A *strict p'-coefficient ring* is a p'-coefficient ring R with an algebraically closed residue field such that $(R^\times)^p = R^\times$.

3. Review of Basic Facts

3.1. Ramification subgroups

Let K be a p-local field with residue field k. Take a separable closure $\overline{K}$ (resp. $\overline{k}$) of K (resp. k) and let $G = W_K$ denote the Weil group of K. Let $G^v = G \cap \mathrm{Gal}(\overline{K}/K)^v$ and $G^{v+} = G \cap \mathrm{Gal}(\overline{K}/K)^{v+}$ be the *upper numbering ramification subgroups* of G. They have the following properties:

- G^v and G^{v+} are closed normal subgroups of G.
- $G^v \supset G^{v+} \supset G^w$ for every $v, w \in \mathbb{Q}_{\geq 0}$ with $w > v$.
- G^{v+} is equal to the closure of $\bigcup_{w>v} G^w$.
- $G^0 = I_K$, the inertia subgroup of W_K. $G^{0+} = P_K$, the wild inertia subgroup of W_K. In particular, the group G^w for $w > 0$ and the group G^{w+} for $w \geq 0$ are pro p-groups.
- For $w \in \mathbb{Q}$ with $w > 0$, the group G^w/G^{w+} is an abelian group which is killed by p.

3.2. Herbrand's function $\psi_{L/K}$

For a finite separable extension L/K of p-local fields, let $\psi_{L/K} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ denote the Herbrand function (cf. [Se1, IV, §3], [Lan2], [De5] and [FV, Chap. III, 3]). The function $\psi_{L/K}$ has the following properties:

- $\psi_{L/K}$ is continuous, strictly increasing, piecewise linear, and convex function on $\mathbb{R}_{\geq 0}$.
- For sufficiently large w, the function $\psi_{L/K}(w)$ is linear with slope $e_{L/K}$.
- We have $\psi_{L/K}(0) = 0$.
- We have $\psi_{L/K}(\mathbb{Z}_{\geq 0}) \subset \mathbb{Z}_{\geq 0}$ and $\psi_{L/K}(\mathbb{Q}_{\geq 0}) = \mathbb{Q}_{\geq 0}$.

PROPOSITION 3.1. *We set $G = W_K$ and $H = W_L$. Then for $w \in \mathbb{Q}_{\geq 0}$, we have $G^w \cap H = H^{\psi_{L/K}(w)}$ and $G^{w+} \cap H = H^{\psi_{L/K}(w)+}$. Furthermore, the slope of $\psi_{L/K}$ at w is equal to $\frac{e_{L/K}}{[G^w : H^{\psi_{L/K}(w)}]}$.*

PROOF. If L/K is Galois, the first assertion is essentially in [Se1], The first assertion in general case follows from Galois case by [Se1, IV, §3, Prop. 15]. The second assertion is found in [DH, p.103, (3.2.1)]. □

COROLLARY 3.2. *Let $v_0 \in \mathbb{Q}_{\geq 0}$ be a non-negative rational number. Then the function $\psi_{L/K}(v)$ is linear for $v \geq v_0$ if and only if W_L contains W_K^{v+}.*

Let $m \in \mathbb{Z}_{>0}$ be a positive integer. Put $n = \psi_{L/K}(m)$. We have $\mathrm{N}_{L/K}(1 + \mathfrak{m}_L^n) \subset 1 + \mathfrak{m}_K^m$ and $\mathrm{N}_{L/K}(1 + \mathfrak{m}_L^{n+1}) \subset 1 + \mathfrak{m}_K^{m+1}$. Let $\alpha_{L/K,m} : \mathfrak{m}_L^n/\mathfrak{m}_L^{n+1} \to \mathfrak{m}_K^m/\mathfrak{m}_K^{m+1}$ be the homomorphism given by $1 + \alpha_{L/K,m}(x) = \mathrm{N}_{L/K}(1 + x) \mod 1 + \mathfrak{m}_K^{m+1}$ for all $x \in \mathfrak{m}_L^n$.

LEMMA 3.3. *Suppose that $\psi_{L/K}(v)$ is linear for $v \geq v_0$. Then for any integer $m > v_0$, the map $\alpha_{L/K,m}$ is surjective and is equal to the trace map $\mathrm{Tr}_{L/K} : \mathfrak{m}_L^{\psi_{L/K}(m)}/\mathfrak{m}_L^{\psi_{L/K}(m)+1} \to \mathfrak{m}_K^m/\mathfrak{m}_K^{m+1}$.*

PROOF. Let $\widetilde{L}$ be the Galois closure of L/K. Let $v_0 \in \mathbb{Q}_{\geq 0}$ be the minimal rational number such that $\psi_{L/K}(v)$ is linear for $v \geq v_0$. Then W_L contains W_K^{v+}. Since W_K^{v+} is a normal subgroup of W_K, the group $W_{\tilde{L}}$ also contains W_K^{v+}. Hence $\psi_{\tilde{L}/K}(v)$ is linear for $v \geq v_0$ and $\psi_{\tilde{L}/L}(v)$ is linear for $v \geq \psi_{L/K}(v_0)$. Hence we may assume that L/K is Galois. Since the lemma for L/K and that for M/L imply that of M/L, we may assume that L/K is cyclic of prime degree. Then the lemma follows from the discussion in [Se1, V, §3]. □

LEMMA 3.4. *Let L and K' be two finite separable extensions of K (in a fixed separable closure $\overline{K}$ of K). Suppose that there exist $v_1, v_2 \in \mathbb{Q}_{\geq 0}$ with $v_1 < v_2$ such that $\psi_{L/K}(v) = v$ for $0 < v < v_2$ and that $\psi_{K'/K}(v)$ is linear for $v > v_1$. Let $L' = L \cdot K'$ be the composite field. Then*

(1) *$\psi_{L'/K'}(v) = v$ for $0 < v < \psi_{K'/K}(v_2)$.*

(2) *$\psi_{L'/L}(v)$ is linear for $v > v_1$.*

PROOF. We use Proposition 3.1.

(1) Let $w \in \mathbb{Q}_{>0}$ be a rational number satisfying $v_1 < w < v_2$. Let $v = \psi_{K'/K}(w)$. Since $W_{K'} \supset (W_K)^w$, we have

$$\begin{aligned}[L' : K'] &\geq [W_{K'}^v : W_{L'}^{\psi_{L'/K'}(v)}] = [W_{K'} \cap W_K^w : W_{L'} \cap (W_K)^w] \\ &= [W_K^w : W_L \cap W_{K'} \cap W_K^w] \\ &= [W_K^w : W_L \cap W_K^w] = [W_K^w : W_L^{\psi_{L/K}(w)}] \\ &= [L : K].\end{aligned}$$

Hence the assertion follows.

(2) Let $v \in \mathbb{Q}_{>0}$ be a rational number satisfying $v_1 < v < v_2$. Since $W_{K'} \supset W_K^v$, we have

$$W_{L'}^{\psi_{L'/L}(v)} = W_{L'} \cap W_K^v = W_L \cap W_{K'} \cap W_K^v = W_L \cap W_K^v = W_L^v.$$

Hence the assertion follows. □

3.3. Refined different (See [K1, p. 321, §2] and [Sa2, p. 2])

Let L/K be a finite separable extension of p-local fields. The *refined different* $\widetilde{D}_{L/K}$ is the unique element in $L^\times/1 + \mathfrak{m}_L$ satisfying $\mathrm{Tr}_{L/K}(\widetilde{D}_{L/K}^{-1}\mathcal{O}_L) \subset \mathcal{O}_K$ and $\mathrm{Tr}_{L/K}(\widetilde{D}_{L/K}^{-1}\mathfrak{m}_L) \subset \mathfrak{m}_K$ which makes the following diagram commutative:

$$\begin{array}{ccc} \widetilde{D}_{L/K}^{-1}\mathcal{O}_L & \xrightarrow{\mathrm{Tr}_{L/K}} & \mathcal{O}_K \\ {\scriptstyle \widetilde{D}_{L/K}\times}\downarrow & & \downarrow{\scriptstyle \bmod \mathfrak{m}_K} \\ \mathcal{O}_L & \xrightarrow{\mathrm{Tr}_{k_L/k_K}} & k_K. \end{array}$$

If M is a finite separable extension of L, we have $\widetilde{D}_{M/K} = \widetilde{D}_{M/L}\widetilde{D}_{L/K}$. If L/K is at most tamely ramified, then $\widetilde{D}_{L/K} = e_{L/K}$.

LEMMA 3.5. *Suppose that $\psi_{L/K}(v)$ is linear for $v \geq v_0$. Then we have $\psi_{L/K}(v) = e_{L/K}v - v_L(\widetilde{D}_{L/K})$. In particular $v_L(\widetilde{D}_{L/K}) = v_L(D_{L/K}) + 1 - e_{L/K}$. Furthermore for any integer $m > v_0$, $\alpha_{L/K,m}$ is equal to the composite*

$$\mathfrak{m}_L^{\psi_{L/K}(m)}/\mathfrak{m}_L^{\psi_{L/K}(m)+1} \xrightarrow{\widetilde{D}_{L/K}} \mathfrak{m}_L^{e_{L/K}m}/\mathfrak{m}_L^{e_{L/K}m+1}$$

$$\cong (\mathfrak{m}_K^m/\mathfrak{m}_K^{m+1}) \otimes_{k_K} k_L \xrightarrow{1\otimes \mathrm{Tr}_{k_L/k_K}} \mathfrak{m}_K^m/\mathfrak{m}_K^{m+1}.$$

Proof. This follows from Lemma 3.3. □

Proposition 3.6. *Let L and K' be two finite separable extensions of K (in a fixed separable closure $\overline{K}$ of K). Suppose that there exist $v_1, v_2 \in \mathbb{Q}_{\geq 0}$ with $v_1 < v_2$ such that $\psi_{L/K}(v) = v$ for $0 < v < v_2$ and that $\psi_{K'/K}(v)$ is linear for $v > v_1$. Let $L' = L \cdot K'$ be the composite field. Then we have*

$$\widetilde{D}_{K'/K} = \mathrm{N}_{L'/K'}(\widetilde{D}_{L'/L}) = \widetilde{D}_{L'/L}^{[L:K]}.$$

Proof. The assertion is clear if K'/K is at most tamely ramified. We may assume that K'/K is totally wildly ramified. Take a sufficiently large integer N with $p \nmid N$, so that there exist an integer $m \in \mathbb{Z}$ satisfying $Nv_1 < mNv_2$. Let K_1/K a totally ramified extension whose ramification index $e_{K_1/K}$ is equal to N. Put $L_1 = K_1 \cdot L$, $K_1' = K_1 \cdot K'$ and $L_1' = K_1 \cdot L'$.

For $n \in \mathbb{Z}_{>0}$, let $K_{1,n}$ denote the unique unramified extension of K_1 of degree n. Define $L_{1,n}$, $K_{1,n}'$ and $L_{1,n}'$ in similar ways. Then we have $\alpha_{K_{1,n}'/K_{1,n},m} \circ \alpha_{L_{1,n}'/K_{1,n}',\psi_{K_1'/K_1}(m)} = \alpha_{L_{1,n}/K_{1,n},m} \circ \alpha_{L_{1,n}'/L_{1,n},m}$, By taking direct limit, we get the following commutative diagram:

$$\begin{CD}
\mathfrak{m}_{L_1'}^{\psi_{L'/K}(m)}/\mathfrak{m}_{L_1'}^{\psi_{L'/K}(m)+1} \otimes_k \overline{k} @>{\tilde{\alpha}_{L_1'/K_1',\psi_{K_1'/K_1}(m)}}>> \mathfrak{m}_{K_1'}^{\psi_{K_1'/K_1}(m)}/\mathfrak{m}_{K_1'}^{\psi_{K_1'/K_1}(m)+1} \otimes_k \overline{k} \\
@V{\tilde{\alpha}_{L_1'/L_1,m}}VV @VV{\tilde{\alpha}_{K_1'/K_1,m}}V \\
\mathfrak{m}_{L_1}^m/\mathfrak{m}_{L_1}^{m+1} \otimes_k \overline{k} @>{\tilde{\alpha}_{L_1/K_1,m}}>> \mathfrak{m}_{K_1}^m/\mathfrak{m}_{K_1}^{m+1} \otimes_k \overline{k}.
\end{CD}$$

If we take k-bases for $\mathfrak{m}_{K_1}^m/\mathfrak{m}_{K_1}^{m+1} \otimes_k \overline{k}$, $\mathfrak{m}_{L_1}^m/\mathfrak{m}_{L_1}^{m+1} \otimes_k \overline{k}$, $\mathfrak{m}_{K_1'}^{\psi_{K_1'/K_1}(m)}/\mathfrak{m}_{K_1}^{\psi_{K_1'/K_1}(m)+1} \otimes_k \overline{k}$, and $\mathfrak{m}_{K_1}^m/\mathfrak{m}_{K_1}^{m+1} \otimes_k \overline{k}$, all the morphisms in the above diagram are represented by additive polynomials with coefficients in $\overline{k}$. The above diagram remains commutative if we replace all the morphisms by the highest degree parts of them. In particular we have the following commutative diagram

$$\begin{array}{ccc}
\mathfrak{m}_{L_1'}^{\psi_{L'/K}(m)}/\mathfrak{m}_{L_1'}^{\psi_{L'/K(m)+1}} & \xrightarrow{\mathrm{N}_{L_1'/K_1'}} & \mathfrak{m}_{K_1'}^{\psi_{K_1'/K_1}(m)}/\mathfrak{m}_{K_1'}^{\psi_{K_1'/K_1}(m)+1} \\
{\scriptstyle \times \tilde{D}_{L_1'/L_1}}\downarrow & & {\scriptstyle \times \tilde{D}_{K_1'/K_1}}\downarrow \\
\mathfrak{m}_{L_1}^{m}/\mathfrak{m}_{L_1}^{m+1} & \xrightarrow{\mathrm{N}_{L_1/K_1}} & \mathfrak{m}_{K_1}^{m}/\mathfrak{m}_{K_1}^{m+1} \otimes_k \overline{k}.
\end{array}$$

Hence the proposition follows. □

3.4. Break decomposition and refined break decomposition

Let K be a p-local field and let $G = W_K$ denote the Weil group of K. Let $(R, \mathfrak{m}_R)$ be a p'-coefficient ring.

Let V be an $R[G]$-module. We say that V is *tamely ramified* or *pure of break* 0 if G^{0+} acts trivially on V. V is called *totally wild* if $V^{G^{0+}} = \{0\}$. For $v \in \mathbb{Q}_{>0}$, we say that V is *pure* of break v if the G^v-fixed part V^{G^v} of V is 0 and if G^{v+} acts trivially on V.

Let (ρ, V) be an object in $\mathrm{Rep}(G, R)$. Then for any $v \in \mathbb{Q}_{\geq 0}$, there exists a unique maximal sub $R[G]$-module V^v of V which is pure of break v. We have $V^v = \{0\}$ except for a finite number of v and we have a decomposition

$$V = \bigoplus_{v \in \mathbb{Q}_{\geq 0}} V^v$$

in $\mathrm{Rep}(G, R)$. For $v \in \mathbb{Q}_{\geq 0}$, the object V^v in $\mathrm{Rep}(G, R)$ is called the *break-v-part* of (ρ, V). The assignment $V \mapsto V^v$ gives a functor from $\mathrm{Rep}(G, R)$ to itself which preserves short exact sequences. When we consider such functors for various R's, they are compatible with the base changes of the representations by a local ring homomorphism $R \to R'$.

Definition 3.7. Let (ρ, V) be an object in $\mathrm{Rep}(G, R)$, and let $V = \bigoplus_{v \in \mathbb{Q}_{\geq 0}} V^v$ be its break decomposition. We define the *Swan conductor* *sw*(V) of V as

$$\mathrm{sw}(V) = \sum_{v \in \mathbb{Q}_{\geq 0}} v \cdot \mathrm{rank}\, V^v.$$

Since $\mathrm{sw}(V) = \mathrm{sw}(V \otimes_R R/\mathfrak{m}_R)$, we have $\mathrm{sw}(V) \in \mathbb{Z}_{\geq 0}$.

Assume further that the ring R contains a primitive p-th root of unity. Let $v \in \mathbb{Q}_{>0}$, and let (ρ, V) be an object in $\mathrm{Rep}(G, R)$. Let (ρ^v, V^v) denote the break-v-part of (ρ, V). We have a decomposition

$$V^v = \bigoplus_{1 \neq \chi \in \mathrm{Hom}(G^v/G^{v+}, R^\times)} V_\chi$$

of V^v by the sub $R[G^v/G^{v+}]$-modules V_χ on which G^v/G^{v+} acts by χ. The group G acts on the set $\mathrm{Hom}(G^v/G^{v+}, R^\times)$ by conjugation : $(g.\chi)(h) = \chi(g^{-1}hg)$. The action of $g \in G$ on V^v induces an R-linear isomorphism $V_\chi \xrightarrow{\cong} V_{g.\chi}$. Let X^v denote the set of G-orbits in the G-set of the non-trivial homomorphisms from G^v/G^{v+} to $R^\times$. For any $\Sigma \in X^v$, the direct sum $V^\Sigma = \bigoplus_{\chi \in \Sigma} V_\chi$ is a sub $R[G]$-module of V^v and thus we have the decomposition

$$V = V^0 \oplus \bigoplus_{v \in \mathbb{Q}_{>0}} \bigoplus_{\Sigma \in X^v} V^\Sigma$$

in $\mathrm{Rep}(G, R)$, which we call the *refined break decomposition* of V. The object V^Σ in $\mathrm{Rep}(G, R)$ is called the *refined-break-Σ-part* of (ρ, V). We say that (ρ, V) is *pure of refined break* Σ if $V = V^\Sigma$. The assignment $V \mapsto V^\Sigma$ gives a functor from $\mathrm{Rep}(G, R)$ to itself which preserves short exact sequences. When we consider such functors for various R's, they are compatible with the base changes of the representations by a local ring homomorphism $R \to R'$.

LEMMA 3.8. *Let (ρ, V) be a non-zero object in $\mathrm{Rep}(G, R)$ which is pure of refined break $\Sigma \in X^v$. Choose $\chi \in \Sigma$ and let $V_\chi \subset \mathrm{Res}^G_{G^v} V$ denote the χ-part of $\mathrm{Res}^G_{G^v} V$. Let $H_\chi \subset G$ denote the stabilizing subgroup of χ.*

(1) *H_χ is a subgroup of G of finite index.*

(2) *V_χ is stable under the action of H_χ on V.*

(3) *V is, as an object in $\mathrm{Rep}(G, R)$, isomorphic to $\mathrm{Ind}^G_{H_\chi} V_\chi$.*

PROOF. Obvious. □

REMARK 3.9. Finiteness of $[G : H_\chi]$ also follows from the explicit description of the homomorphism $\mathrm{Hom}(G^v/G^{v+}, R^\times)$ given in [Sa2, p. 3, Thm. 1] (See also § 7).

4. Deligne's Local Constant $\varepsilon_0(V, \psi, dx)$

Let K be a p-local field with residue field k. In this section we recall the basic properties of $\varepsilon_0(V, \psi, dx)$. Let R be a discrete commutative ring on which p is invertible. Assume that there exists a non-trivial continuous additive character $\psi : K \to R^\times$. Take such a character ψ and an R-valued Haar measure dx of K. (We use the terminology "R-valued Haar measure" to indicate an R-valued Haar measure in the sense of Deligne [De1, p. 554, 6.1].) The *conductor* of ψ, denoted by $\mathrm{ord}\,\psi$, is the unique integer $n \in \mathbb{Z}$ satisfying $\psi|_{\mathfrak{m}^{-n}} = 1$ and $\psi|_{\mathfrak{m}^{-n-1}} \neq 1$. For $a \in K^\times$, let ψ_a be the additive character of K defined by $\psi_a(x) = \psi(ax)$. Then we have $\mathrm{ord}\,\psi_a = \mathrm{ord}\,\psi + v_K(a)$. If L is a finite separable extension of K, then we have $\mathrm{ord}\,(\psi \circ \mathrm{Tr}_{L/K}) = e_{L/K}\mathrm{ord}\,\psi + v_L(D_{L/K})$.

For a continuous multiplicative quasi-character $\chi : K^\times \to R^\times$ of $K^\times$ (we endow R with discrete topology), the ε-constant $\varepsilon(\chi, \psi, dx) \in R$ of χ is defined by the following integral:

$$\varepsilon(\chi, \psi, dx) = \begin{cases} q^{\mathrm{ord}\,\psi}\chi(\pi^{\mathrm{ord}\,\psi}) \int_{\mathcal{O}_K} dx, & \text{if } \chi\text{: unramified,} \\ \int_{K^\times} \chi^{-1}(x)\psi(x)dx, & \text{if } \chi\text{: ramified.} \end{cases}$$

For an object (ρ, V) in $\mathrm{Rep}(W_K, R)$ with $\mathrm{rank}_R V = 1$, we define the ε-constant $\varepsilon(V, \psi, dx) = \varepsilon((\rho, V), \psi, dx)$ of (ρ, V) by

$$\varepsilon(\rho, \psi, dx) = \varepsilon(\rho \circ \mathrm{rec}, \psi, dx).$$

When $R = \mathbb{C}$ with discrete topology, Langlands [Lan2] defines, after the pioneering work of Dwork [Dw], the local ε-constant $\varepsilon(\rho, \psi)$ for any object (ρ, V) in $\mathrm{Rep}(W_K, \mathbb{C})$, generalizing $\varepsilon(V \otimes \omega_{1/2}, \psi, dx_K)$ discussed above for (ρ, V) with $\mathrm{rank}_R V = 1$, where $\omega_{1/2} : W_K^{ab} \to \mathbb{C}^\times$ is an unramified quasi-character defined by $\omega_{1/2}(x) = q_K^{-v_K(\mathrm{rec}^{-1}(x))/2}$, and dx_K is the self-dual Haar measure of K (see [W, Chap. VII, §2] for the definition of self-dual Haar measure). It is not difficult to construct a candidate of $\varepsilon(\rho, \psi)$ by using Brauer's theorem, however the proof of the well-definedness of $\varepsilon(\rho, \psi)$ given in [Lan2] is much complicated.

In [De1], Deligne discusses Langlands' result and gives a simpler proof of the well-definedness of ε-constants. Deligne uses the terminology "$\varepsilon(V, \psi, dx)$". For any (ρ, V) in $\mathrm{Rep}(W_K, \mathbb{C})$, Langlands' $\varepsilon(\rho, \psi)$ is equal to Deligne's $\varepsilon(V \otimes \omega_{1/2}, \psi, dx_K)$. In this paper, we use Deligne's terminology for local constants, since it has the advantage that we can generalize the theory of Deligne's $\varepsilon(V, \psi, dx)$ to the case where $R \neq \mathbb{C}$. For example, the proof of [De1, p. 555, Théoréme 6.5] shows that we can define $\varepsilon(V, \psi, dx)$ without much effort for (ρ, V) in $\mathrm{Rep}(W_K, R)$ when

(4.1) R is an arbitrary discrete field of characteristic zero

in such a way that most properties of $\varepsilon(V, \psi, dx)$ for (ρ, V) in $\mathrm{Rep}(W_K, \mathbb{C})$ (for example, the properties (1), (3), (6), (7), (8), (9) in Theorem 5.1 below) are automatically satisfied by $\varepsilon(V, \psi, dx)$ for (ρ, V) in $\mathrm{Rep}(W_K, R)$. As we can see from [De1, p. 572, 8.12], we can define $\varepsilon(V, \psi, dx)$ for (ρ, V) in $\mathrm{Rep}(W_K, R)$ even when

(4.2) R is the topological field $\overline{\mathbb{Q}}_\ell$ for $\ell \neq p$ and and V is defined over a finite extension of $\mathbb{Q}_\ell$.

Under the assumption (4.1) or (4.2), Deligne [De1, p. 548, 5.1] also defines ε_0-constants $\varepsilon_0(V, \psi, dx)$ which satisfies

$$\varepsilon_0(V, \psi, dx) = \varepsilon(V, \psi, dx) \det(-\mathrm{Fr}_k | V^{I_K}).$$

There are several properties that the ε-constants and the ε_0-constants satisfy (cf. [De1, p. 535, thm 4.1. and p. 548, 5.1.] and [Lau1, p. 187]). In [De1, p. 555–556, Thm. 6.5.], Deligne also considers ε_0 of representations of W_K over fields of characteristic $\neq p$, which satisfies additivity, a formula for a change of dx, an induction formula, an explicit formula in rank one case, the compatibility with inclusions of coefficient fields, and the compatibility with reduction of the coefficients from a complete discrete valuation ring to its residue field.

5. Statements of the Main Results

Theorem 5.1. *Let K be a p-local field. Then for each triple $(R, (\rho, V), \psi)$ where R is a strict p'-coefficient ring, (ρ, V) is an object in $\mathrm{Rep}(W_K, V)$, and $\psi : K \to R^\times$ is a non-trivial continuous additive character, we can attach, in a canonical way, an element*

$$\varepsilon_{0,R}((\rho, V), \psi) \in R^\times$$

which satisfy the following properties:

(1) *For fixed R and ψ, the element $\varepsilon_{0,R}((\rho, V), \psi) \in R^\times$ depends only on the isomorphism class of (ρ, V).*

(2) *Let $(R, (\rho, V), \psi)$ be a triple as above, R' a strict p'-coefficient ring, and $h : R \to R'$ a local ring homomorphism. Then we have*

$$h(\varepsilon_{0,R}(V, \psi)) = \varepsilon_{0,R'}(V \otimes_R R', h \circ \psi).$$

(3) *Let $(R, (\rho, V), \psi)$, $(R, (\rho', V'), \psi)$ and $(R, (\rho'', V''), \psi)$ be three triples as above with common R and ψ. Suppose that there exists an exact sequence*

$$0 \to V' \to V \to V'' \to 0$$

in $\mathrm{Rep}(W_K, R)$*. Then we have*

$$\varepsilon_{0,R}(V, \psi) = \varepsilon_{0,R}(V', \psi) \cdot \varepsilon_{0,R}(V'', \psi).$$

(4) *Let $(R, (\rho, V), \psi)$ be a triple as above. Suppose that R is a field. Then*

$$\varepsilon_{0,R}(V, \psi) = \varepsilon_0(V, \psi, dx),$$

where dx is the R-valued Haar measure of K satisfying $\int_{\mathcal{O}_K} dx = 1$.

(5) *Let R_0 be a complete discrete valuation ring with a finite residue field of characteristic $\neq p$. We denote by F_0 the field of fractions* $\mathrm{Frac}\,(R_0)$ *of R_0, by F the completion of the maximal unramified extension of F_0, and by R the ring of integers in F. Let $(R, (\rho, V), \psi)$ be a triple as above. Suppose that (ρ, V) is isomorphic to the base change $(\rho_0, V_0) \otimes_{R_0} R$ of an object (ρ_0, V_0) in* $\mathrm{Rep}(W_K, R_0)$*. Then*

$$\varepsilon_{0,R}(V, \psi) = \varepsilon_0(V_0 \otimes_{R_0} \overline{F_0}, \psi, dx),$$

where dx is the R_0-valued Haar measure of K satisfying $\int_{\mathcal{O}_K} dx = 1$.

(6) *Let $(R, (\rho, V), \psi)$ be a triple as above with* $\mathrm{rank}\, V = 1$*, then $\varepsilon_{0,R}(V, \psi)$ coincides with $\varepsilon_0(\rho \circ \mathrm{rec}, \psi, dx)$ defined in* [De1, p. 555, 6.4]*, where dx is the R-valued Haar measure of K satisfying $\int_{\mathcal{O}_K} dx = 1$.*

(7) *Let $(R, (\rho, V), \psi)$ be a triple as above. Let $a \in K^\times$ and let $\psi_a : K \to R^\times$ be the additive character defined by $\psi_a(x) = \psi(ax)$. Then we have*

$$\varepsilon_{0,R}(V, \psi_a) = \det(V)(\mathrm{rec}(a)) q_K^{v_K(a) \cdot \mathrm{rank}\, V} \varepsilon_{0,R}(V, \psi).$$

(8) *Let* $(R, (\rho, V), \psi)$ *be a triple as above. Let* W *be an object in* $\mathrm{Rep}(W_K, R)$ *on which* W_K *acts via* $W_K/W_K^0 \cong \mathbb{Z}$*. Let* $\mathrm{Fr} \in W_K/W_K^0$ *be the geometric Frobenius. Then we have*

$$\varepsilon_{0,R}(V \otimes W, \psi) = \det W(\mathrm{Fr}^{\mathrm{sw}(V)+\mathrm{rank}\, V\cdot(\mathrm{ord}\,\psi+1)})\varepsilon_{0,R}(V, \psi)^{\mathrm{rank}\, W}.$$

(9) *Let* $(R, (\rho, V), \psi)$ *be a triple as above. Suppose that the coinvariant* $(V)_{W_K^0}$ *is zero. Let* V^* *be the* R*-linear dual of* V*. Then we have*

$$\varepsilon_{0,R}(V, \psi) \cdot \varepsilon_{0,R}(V^*, \psi) = \det V(\mathrm{rec}(-1)) \cdot q^{\mathrm{sw}(V)+\mathrm{rank}\, V\cdot(2\mathrm{ord}\,\psi+1)}.$$

REMARK 5.2. A partial result for the uniqueness of ε_0-constants is given in Corollary 9.17.

Here we give an outline of the proof of Theorem 5.1. Let K be a p-local field.

Let be R be a strict p'-coefficient ring. For an object (ρ, V) in $\mathrm{Rep}(W_K, R)$, let $V = V^0 \oplus V^{>0}$ be the decomposition of V into the tamely ramified part V^0 and the totally wild part $V^{>0}$. By § 3.4, for a short exact exact sequence $0 \to V' \to V \to V'' \to 0$ in $\mathrm{Rep}(W_K, R)$, $0 \to (V')^0 \to V^0 \to (V'')^0 \to 0$ and $0 \to (V')^{>0} \to V^{>0} \to (V'')^{>0} \to 0$ are also exact sequences.

We divide Theorem 5.1 into the following two theorems:

THEOREM 5.3. *For each triple* $(R, (\rho, V), \psi)$ *where* R *is a strict* p'*-coefficient ring,* (ρ, V) *is a totally wild object in* $\mathrm{Rep}(W_K, V)$*, and* $\psi : K \to R^\times$ *is a non-trivial continuous additive character, we can attach, in a canonical way, an element*

$$\varepsilon_{0,R}((\rho, V), \psi) \in R^\times$$

which satisfies the properties (1)–(9) in Theorem 5.1.

THEOREM 5.4. *For each triple* $(R, (\rho, V), \psi)$ *where* R *is a strict* p'*-coefficient ring,* (ρ, V) *is a tamely ramified object in* $\mathrm{Rep}(W_K, V)$*, and* $\psi : K \to R^\times$ *is a non-trivial continuous additive character, we can attach, in a canonical way, an element*

$$\varepsilon_{0,R}((\rho, V), \psi) \in R$$

which satisfies

(0) $\varepsilon_{0,R}((\rho, V), \psi) \in R^{\times}$

and the properties (1)–(9) in Theorem 5.1.

In § 7, Definition 7.5, we define, following Henniart [He] and Saito [Sa2], an element

$$\overline{\varepsilon}_{0,R}(V, \psi) \in R^{\times}/\boldsymbol{\mu}_{p^{\infty}}(R).$$

in a canonical way, for each triple $(R, (\rho, V), \psi)$ as above such that V is totally wild. In Theorem 7.8, we prove that which satisfies the nine properties corresponding to the properties (1)–(9) in Theorem 5.1.

Using this element, we define, in § 8, Definition 8.1, an element $\varepsilon_{0,R}(V, \psi) \in R^{\times}$ for each triple $(R, (\rho, V), \psi)$ as above such that V is totally wild. In § 8, we prove Theorem 5.3.

In § 10, Definition 10.7, we define, in a canonical way, an element $\varepsilon_{0,R}(V, \psi) \in R^{\times}$ for each triple $(R, (\rho, V), \psi)$ as above such that V is tamely ramified. In § 10.3 and § 10.5, we prove Theorem 5.4.

Application to Kato's local ε conjecture Let the notation be as in § 1.1. In view of [K2, 3.2], we see that the "$\ell \neq p$ part" of his conjecture ([K2, Conj. 1.8]) is equivalent to the following conjecture modulo ± 1 in the case where $K = \mathbb{Q}_p$ and Λ is a pro-ℓ commutative ring:

Conjecture 5.5 (Local ε conjecture). *Let K be as above. Then for each triple $(\Lambda, (\rho, V), \psi)$ as above, we can define an element $\varepsilon_{0,\Lambda}(V, \psi) = \varepsilon_{0,\Lambda}((\rho, V), \psi)$ in $\Lambda_{(\rho,V)}$ satisfying the following conditions:*

(1) *Assume that we are given two triples $(\Lambda, (\rho, V), \psi)$ and $(\Lambda', (\rho', V'), \psi)$ as above with common ψ, a local ring homomorphism $h : \Lambda \to \Lambda'$, and an isomorphism $(\rho, V) \otimes_{\Lambda} \Lambda' \xrightarrow{\cong} (\rho', V')$ in* $\mathrm{Rep}(W_K, \Lambda')$. *Then the isomorphism $\Lambda_{(\rho,V)} \otimes_{\Lambda} \Lambda' \xrightarrow{\cong} \Lambda'_{(\rho,V)}$ induced by h sends $\varepsilon_{0,\Lambda}(V, \psi) \otimes 1$ to $\varepsilon_{0,\Lambda'}(V', \psi)$.*

(2) *Let $(\Lambda, (\rho, V), \psi)$, $(\Lambda, (\rho', V'), \psi)$ and $(\Lambda, (\rho'', V''), \psi)$ be three triples as above with common Λ and ψ. Assume that there is a short exact sequence*

$$0 \to (\rho', V') \to (\rho, V) \to (\rho'', V'') \to 0$$

in $\mathrm{Rep}(W_K, \Lambda)$. *There is a canonical isomorphism*

$$\Lambda_{(\rho,V)} \xrightarrow{\cong} \Lambda_{(\rho',V')} \otimes_{\Lambda} \Lambda_{(\rho'',V'')}.$$

Then this isomorphism sends $\varepsilon_{0,\Lambda}(V,\psi)$ *to* $\varepsilon_{0,\Lambda}(V',\psi)\otimes\varepsilon_{0,\Lambda}(V'',\psi)$.

(3) *Let* $(\Lambda,(\rho,V),\psi)$ *be a triple as above and* $a\in K^\times$. *Let* $\psi_a : K\to W(\overline{\mathbb{F}}_\ell^\times)$ *denote the additive character defined by* $\psi_a(x)=\psi(ax)$ *for* $x\in K$. *Then we have*

$$\varepsilon_{0,\Lambda}(V,\psi_a)=\det(\rho)(\mathrm{rec}(a))q_K^{v_K(a)\cdot\mathrm{rank}\,(V)}\varepsilon_{0,\Lambda}(V,\psi).$$

(4) *Let* $(\Lambda,(\rho,V),\psi)$ *be a triple as above. Assume that* Λ *is a finite flat reduced local* $\mathbb{Z}_\ell$*-algebra.* $\Lambda\otimes_{\mathbb{Z}_\ell}\mathrm{Frac}\,W(\overline{\mathbb{F}}_\ell)$ *is isomorphic to a direct product* $\prod_i K_i$ *of finite extensions* K_i *of* $\mathrm{Frac}\,W(\overline{\mathbb{F}}_\ell)$. *For each* i *the base change* $(\rho_i,V_i)=(\rho,V)\otimes_\Lambda K_i$ *is a continuous representation of* W_K *on a finite dimensional* K_i*-vector space which is defined over a finite extension of* $\mathbb{Q}_\ell$ *in* K_i. *Then the image of* $\varepsilon_{0,\Lambda}(V,\psi)$ *in* K_i *is equal to the local* ε_0*-constant* $\varepsilon_0(V_i,\psi_i,dx)$ *in Deligne (* [De1, p. 535, Thm. 4.1], *on which we have reviewed in § 4), where* dx *is the* K_i*-valued Haar measure of the additive group* K *with* $\int_{\mathcal{O}_K}dx=1$.

PROOF OF CONJECTURE 5.5 (cf. [K2, p. 14, 3.2]). Let $(\Lambda,(\rho,V),\psi)$ be a triple as above.

Then $\Lambda\widehat{\otimes}_{\mathbb{Z}_\ell}W(\overline{\mathbb{F}}_\ell)$, is a finite product $\Lambda\widehat{\otimes}_{W(\mathbb{F}_\ell)}W(\overline{\mathbb{F}}_\ell)=\prod_i R_i$ of p'-coefficient rings R_i.

Define $\varepsilon_{0,\Lambda}(V,\psi)\in\Lambda\widehat{\otimes}_{W(\mathbb{F}_\ell)}W(\overline{\mathbb{F}}_\ell)$ by

$$\varepsilon_{0,\Lambda}(V,\psi)=(\varepsilon_{0,R}(V,\psi))_i.$$

Then, by Theorem 5.1 (4), we have $\varepsilon_{0,\Lambda}(V,\psi)\in\Lambda_{(\rho,V)}$.

It is easy to check that this element $\varepsilon_{0,\Lambda}(V,\psi)$ satisfies the desired properties. $\square$

THEOREM 5.6. *Let* L/K *be a finite separable extension of* p*-local fields, let* R *be a strict* p'*-coefficient ring, and let* $\psi : K\to R^\times$ *be a non-trivial continuous additive character. Then there exists an element* $\lambda_R(L/K,\psi)\in R^\times$ *such that for every object* (ρ,V) *in* $\mathrm{Rep}(W_L,R)$*., we have*

$$\varepsilon_{0,R}(\mathrm{Ind}_{W_L}^{W_K}V,\psi)=\lambda_R(L/K,\psi)^{\mathrm{rank}\,V}\varepsilon_{0,R}(V,\psi\circ\mathrm{Tr}_{L/K}).$$

Here we give an outline of the proof of Theorem 5.6.

For L/K, R and ψ as above, we define, in § 6, Definition 6.3, in a canonical way an element $\lambda_R(L/K, \psi) \in R^\times$.

We divide Theorem 5.6 into four parts in the following way:

THEOREM 5.7. *Let $\lambda_R(L/K, \psi) \in R^\times$ be as in Definition 6.3. Let (ρ, V) be an object in* $\mathrm{Rep}(W_L, R)$*. Then Theorem 5.6 holds for (ρ, V) and for this $\lambda_R(L/K, \psi) \in R^\times$ in the following four cases:*

(1) *V is totally wild.*

(2) *V is tamely ramified and L/K is unramified.*

(3) *V is tamely ramified and L/K is totally tamely ramified.*

(4) *V is tamely ramified and L/K is totally ramified and $[L : K]$ is a power of p.*

The proof of (1) is given in § 9. (2) is proved in § 10.5, Lemma 10.14. The proofs of (3) and (4) are given in § 11.3.

REMARK 5.8. In § 8.2, we prove a result analogous to Deligne-Henniart's result [DH, p. 108, Thm. 4.2 and p. 110, Thm. 4.6].

6. λ-Constants

In this section, we consider a triple $(L/K, R, \psi)$, where L/K is a finite separable extension of a p-local field K with residue field k of q elements, R is a p'-coefficient ring, and $\psi : K \to R^\times$ is a non-trivial continuous additive character.

The aim of this section is to define, for a triple $(L/K, R, \psi)$ as above, an element $\lambda_R(L/K, \psi) \in R^\times$ and to prove some basic properties of $\lambda_R(L/K, \psi)$.

6.1. Review on lambda constants

Let L be a finite separable extension of K, and let dx and dy be Haar measures of K and L respectively. When $R = \mathbb{C}$, Deligne [De1, p. 549, (5.6)] shows that there exists

$$\lambda(L/K, \psi, dx, dy) \in \mathbb{C}^\times$$

such that for any representation V of W_L over $\mathbb{C}$, we have

$$\varepsilon(\mathrm{Ind}_L^K V, \psi, dx) = \lambda(L/K, \psi, dx, dy)^{\mathrm{rank}\, V} \cdot \varepsilon(V, \psi \circ \mathrm{Tr}_{L/K}, dy)$$

and

$$\varepsilon_0(\mathrm{Ind}_L^K V, \psi, dx) = \lambda(L/K, \psi, dx, dy)^{\mathrm{rank}\, V} \cdot \varepsilon_0(V, \psi \circ \mathrm{Tr}_{L/K}, dy).$$

6.2. Universal λ-constant $\lambda_{\widetilde{\mathbb{Z}}_K}(L/K, \psi)$

For a complete discrete valuation field K whose residue field k is finite of characteristic p, let $\widetilde{\mathbb{Z}}_K$ be the following commutative ring

$$\widetilde{\mathbb{Z}}_K = \begin{cases} \mathbb{Z}[\frac{1}{2}][X]/(1+X^4), & \text{if } p = 2 \text{ and } \mathrm{char}\, K = 0, \\ \mathbb{Z}[\frac{1}{p}][X]/(1+X+\cdots+X^{p-1}), & \text{otherwise.} \end{cases}$$

The ring $\widetilde{\mathbb{Z}}_K$ depends only on the pair $(\mathrm{char}\, K, \mathrm{char}\, k)$. In particular, for a finite separable extension L of K, we have $\widetilde{\mathbb{Z}}_L = \widetilde{\mathbb{Z}}_K$.

DEFINITION 6.1.

(1) Assume that $\mathrm{char}\, K = 0$. A *universal partial character* of K is an additive character $\psi' : I \to \widetilde{\mathbb{Z}}_K^\times$ defined on a fractional ideal $I \subset K$ of K such that ψ' is either trivial on $4\mathfrak{m}_K I$ and is non-trivial on $4I$.

(2) Assume that $\mathrm{char}\, K = p$. A *universal partial character* of K is a non-trivial continuous additive character $\psi' : I = K \to \widetilde{\mathbb{Z}}_K^\times$ of K.

Let L be a finite separable extension of K. Take an embedding $\iota : \widetilde{\mathbb{Z}}_K \hookrightarrow \mathbb{C}$. For every universal partial character $\psi' : I \to \widetilde{\mathbb{Z}}_K^\times$, take a continuous additive character $\psi : K \to \mathbb{C}^\times$ whose restriction to I is equal to $\iota\psi'$.

LEMMA 6.2. *Let dx (resp. dy) be the Haar measure on K (resp. L) satisfying $\int_{\mathcal{O}_K} dx = 1$ (resp. $\int_{\mathcal{O}_L} dy = 1$). Then the λ-constant $\lambda(L/K, \psi) = \lambda(L/K, \psi, dx, dy) \in \mathbb{C}^\times$ belongs to $\iota(\widetilde{\mathbb{Z}}_K^\times)$.*

PROOF. Let $V = \mathrm{Ind}_{W_L}^{W_K} 1$. We have

$$\lambda(L/K, \psi, dx, dy) = \frac{\varepsilon(V \oplus \det V, \psi, dx)}{\varepsilon(1, \psi \circ \mathrm{Tr}_{L/K}, dy)\varepsilon(\det V, \psi, dx)}.$$

Since $V \oplus \det V$ is self-dual, we have

$$\varepsilon(V \oplus \det V, \psi, dx)^2 = q^{a(V)+a(\det V)+2(\operatorname{rank} V+1)\cdot \operatorname{ord} \psi}.$$

Here $a(V)$ and $a(\det V)$ denote the Artin conductors of V and $\det V$, respectively. By Serre [Se2], $a(V) + a(\det V)$ is an even integer. Hence $\varepsilon(V \oplus \det V, \psi, dx)$ lies in the image of $\widetilde{\mathbb{Z}}_K^\times$ by ι. It is easily checked that $\varepsilon(1, \psi \circ \operatorname{Tr}_{L/K}, dy)$ and $\varepsilon(\det V, \psi, dx)$ belong to $\iota(\widetilde{\mathbb{Z}}_K^\times)$. $\square$

We define $\lambda_{\widetilde{\mathbb{Z}}_K}(L/K, I, \psi') \in \widetilde{\mathbb{Z}}_K^\times$ to be the inverse image $\iota^{-1}(\lambda(L/K, \psi))$ by ι. The element $\lambda_{\widetilde{\mathbb{Z}}_K}(L/K, I, \psi')$ does not depend on the choice of ι or ψ.

Let $a \in (\mathbb{Z}_p/4p\mathbb{Z}_p)^\times$, $h_a : \widetilde{\mathbb{Z}}_K \to \widetilde{\mathbb{Z}}_K$ be the automorphism of the ring $\widetilde{\mathbb{Z}}_K$ given by $h_a(X) = X^a$. Then for any universal partial character $\psi' : I \to \widetilde{\mathbb{Z}}_K^\times$ of K, we have $h_a(\lambda_{\widetilde{\mathbb{Z}}_K}(L/K, I, \psi')) = \lambda_{\widetilde{\mathbb{Z}}_K}(L/K, I, h_a \circ \psi')$.

6.3. Definition of $\lambda_R(L/K, \psi)$

Let R be a p'-coefficient ring, $\psi : K \to R^\times$ a non-trivial continuous additive character. There exists a universal partial character $\psi' : I \to \widetilde{\mathbb{Z}}_K^\times$ of K and a homomorphism $h : \widetilde{\mathbb{Z}}_K \to R$ of rings such that $\psi|_I = h \circ \psi'$.

Definition 6.3. Take I, ψ' and h as above. We define the λ-*constant* $\lambda_R(L/K, \psi) \in R^\times$ of $(L/K, R, \psi)$ to be

$$\lambda_R(L/K, \psi) := h(\lambda(L/K, I, \psi')).$$

This $\lambda_R(L/K, \psi)$ does not depend on the choice of I, ψ' and h.

Proposition 6.4.

(1) *Let $(L/K, R, \psi)$ and $(L/K, R', \psi')$ be two such triples with common L/K, and $h : R \to R'$ a local ring homomorphism satisfying $\psi = \psi' \circ h$. Then we have*
$$h(\lambda_R(L/K, \psi)) = \lambda_{R'}(L/K, \psi').$$

(2) *Let $q = q_K$. Then*

$$\lambda_R(L/K, \psi)^2 = (d_{L/K}, -1)_K \cdot q^{-v_K(d_{L/K})}.$$

(3) *If $R = \mathbb{C}$, then $\lambda_R(L/K,\psi)$ coincides with Deligne's $\lambda(L/K,\psi, dx, dy)$, where dx and dy are Haar measures with $\int_{\mathcal{O}_K} dx = 1$ and $\int_{\mathcal{O}_L} dy = 1$.*

(4) *Let $q = q_K$. Let $a \in K^\times$ and ψ_a be the additive character defined as $\psi_a(x) = \psi(ax)$. Then we have*

$$\lambda_R(L/K,\psi_a) = \lambda_R(L/K,\psi)\cdot(d_{L/K}, a)_K.$$

(5) *If M is a finite separable extension of L, then we have*

$$\lambda_R(M/K,\psi) = \lambda_R(L/K,\psi)^{[M:L]}\cdot\lambda(M/L,\psi\circ\mathrm{Tr}_{L/K}).$$

PROOF. (1) and (3) are Obvious. (4) and (5) are immediate consequences of (3). We prove (2).

Let $a(V)$ be the Artin conductor of $V = \mathrm{Ind}_{W_L}^{W_K} 1$. Then,

$$\lambda_R(L/K,\psi)^2 = \det(V)(\mathrm{rec}(-1))\cdot\frac{q^{a(V)+2[L:K]\mathrm{ord}\,\psi}}{q_L^{2\mathrm{ord}\,(\psi\circ\mathrm{Tr}_{L/K})}}.$$

By [Se1, VI, Prop. 4], we have $a(V) = v_K(d_{L/K})$. Since $\mathrm{ord}\,(\psi\circ\mathrm{Tr}_{L/K}) = e_{L/K}\mathrm{ord}\,\psi + v_L(D_{L/K})$, we have

$$q_L^{2\mathrm{ord}\,(\psi\circ\mathrm{Tr}_{L/K})} = q^{2f_{L/K}(e_{L/K}\mathrm{ord}\,\psi+v_L(D_{L/K}))} = q^{2[L:K]\mathrm{ord}\,\psi+2v_K(d_{L/K})}.$$

Hence the lemma follows from $\det(V)(\mathrm{rec}(-1)) = (d_{L/K},-1)_K$. $\square$

6.4. Description of $\lambda_R(L/K,\psi)$ in some special cases

Let $q = q_K$. Let $n = [L:K]$ be the degree of L/K.

When $p \neq 2$ and $v_K(d_{L/K})$ is odd, we denote by $\tau_R(L/K,\psi)$ the quadratic Gauss sum

$$\tau_R(L/K,\psi) = \sum_{x\in k^\times}(d_{L/K},\pi_K^{-\mathrm{ord}\,\psi-1}x)_K\,\psi(\pi_K^{-\mathrm{ord}\,\psi-1}x)$$

where $\pi_K \in K$ is an arbitrary prime element in K. The Gauss sum $\tau_R(L/K,\psi)$ does not depend on the choice of π_K. We have $\tau_R(L/K,\psi)^2 = \left(\frac{-1}{k}\right)q$. In particular $\tau_R(L/K,\psi)$ is a unit in R.

LEMMA 6.5. *Suppose that L/K is unramified. Then*

$$\lambda_R(L/K, \psi) = (-1)^{([L:K]-1)\operatorname{ord}\psi}.$$

PROOF. It follows from direct computation of $\lambda_R(L/K, \psi)$ (cf. [M, p. 879, (2.5.3)]). $\square$

LEMMA 6.6. *Suppose that L/K is totally tamely ramified and let $n = [L : K]$. Then*

$$\lambda_R(L/K, \psi) = \begin{cases} q^{-\frac{n-1}{2}} \left(\dfrac{(-1)^{\frac{n-1}{2}} n}{k} \right)^{\operatorname{ord}\psi} & \text{if } n \text{ is odd and } p \neq 2, \\ q^{-\frac{n-1}{2}} (-1)^{\frac{n^2-1}{8}[k:\mathbb{F}_2]\operatorname{ord}\psi}. & \text{if } n \text{ is odd and } p = 2, \\ q^{-\frac{n}{2}} \tau_R(L/K, \psi) \left(\dfrac{(-1)^{\frac{n}{2}-1} \frac{n}{2}}{k} \right) & \text{if } n \text{ is even.} \end{cases}$$

PROOF. There exists a prime element $\pi_L \in L$ such that $\pi_K = \pi_L^n$ is a prime element in K. Since $\{1, \ldots, \pi_L^{n-1}\}$ is a $\mathcal{O}_K$-basis of $\mathcal{O}_L$, we have $d_{L/K} = (-1)^{\lfloor \frac{n-1}{2} \rfloor} n^n \pi_K^{n-1}$. If n is odd, $v_K(d_{L/K}) = n - 1$ is even. Hence if $p \neq 2$, by [He, p. 124, Prop. 2], we have

$$\lambda_R(L/K, \psi) = q^{-\frac{n-1}{2}} ((-1)^{\frac{n-1}{2}} n, 2\pi_K^{\operatorname{ord}\psi})_K = q^{-\frac{n-1}{2}} \left(\frac{(-1)^{\frac{n-1}{2}} n}{k} \right)^{\operatorname{ord}\psi}.$$

If $p = 2$, let dx be the Haar measure of K such that $\int_{\mathcal{O}_K} dx = 1$. Since $\chi := \det(\operatorname{Ind}_{W_L}^{W_K} 1_{\mathbb{C}})$ is unramified, by [He, p. 124, Prop. 2] (cf. [M, p. 881, Prop. 2.5.11]), we have

$$\begin{aligned} \lambda_{\mathbb{C}}(L/K, \psi) &= q^{-\operatorname{ord}(\psi\circ\operatorname{Tr}_{L/K})} \varepsilon(\operatorname{Ind}_{W_L}^{W_K} 1_{\mathbb{C}}, \psi, dx) \\ &= q^{-\operatorname{ord}(\psi\circ\operatorname{Tr}_{L/K}) + \frac{n-1}{2}} \varepsilon(\chi, \psi, dx)^n \\ &= q^{-\frac{n-1}{2}} \chi(\operatorname{rec}(\pi_K))^{n \operatorname{ord}\psi}. \end{aligned}$$

If furthermore $\operatorname{char} K = 0$, then

$$\chi(\operatorname{rec}(\pi_K)) = (d_{L/K}, \pi_K)_K = ((-1)^{\frac{n-1}{2}} n, \pi_K)_K = (-1)^{\frac{n^2-1}{8}[k:\mathbb{F}_2]}.$$

The formula $\chi(\mathrm{rec}(\pi_K)) = (-1)^{\frac{n^2-1}{8}[k:\mathbb{F}_2]}$ holds even when $\operatorname{char} K = 2$. Hence

$$\lambda_R(L/K,\psi) = q^{-\frac{n-1}{2}}(-1)^{\frac{n^2-1}{8}[k:\mathbb{F}_2]\operatorname{ord}\psi}.$$

If n is even, by [Sa1, p. 508, Thm.], we have

$$\begin{aligned}\lambda_R(L/K,\psi) &= q^{-\frac{n}{2}}\tau_R(L/K,\psi)\left(\frac{(-1)^{\frac{n}{2}-1}n}{k}\right)(d_{L/K},2)_K \\ &= q^{-\frac{n}{2}}\tau_R(L/K,\psi)\left(\frac{(-1)^{\frac{n}{2}-1}\frac{n}{2}}{k}\right). \ \square\end{aligned}$$

7. Local ε_0-Constant for Totally Wild Representations Modulo p-th Power Roots of Unity

Let K be a p-local field with residue field k. Let $q = q_K$. Let R be a strict p'-coefficient ring. In this section, inspired by the result of Henniart in [He], we define the local ε_0-constants for pairs $((\rho,V),\psi)$ up to p-th power roots of unity, where (ρ,V) is an object in $\mathrm{Rep}(W_K,R)$ and $\psi : K \to R^\times$ is a non-trivial continuous additive character of K.

Let $G = W_K$ denote the Weil group of K, and let G^v and G^{v+} denote its ramification subgroups.

7.1. The isomorphism σ_ψ

Let $\overline{K}$ be a separable closure of K. The valuation v_K of K canonically extends to a valuation $v_K : \overline{K} \to \mathbb{Q}\cup\{\infty\}$ of $\overline{K}$. For $w\in\mathbb{Q}$, let $N^w = N_K^w$ be the k-vector space

$$N^w := \{x\in\overline{K}\,;\, v_K(x)\geq w\}/\{x\in\overline{K}|v_K(x)>w\}$$

endowed with a canonical W_K-action. Furthermore, $N^\bullet = \bigoplus_{w\in\mathbb{Q}} N^w$ has a structure of a graded k-algebra.

Let $\overline{k}$ denote the residue field of the valuation field $\overline{K}$. There is a canonical isomorphism

$$\mathrm{Hom}(G^v/G^{v+},\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\cong} \mathrm{Hom}_{\overline{k}}(N^v,\overline{k}) \cong N^{-v}$$

of G-modules (cf. [Hi] and [Sa2, p. 3, Thm. 1]). Let us recall this in the notation of [Sa2]:

Let $\chi \in \mathrm{Hom}(G^v/G^{v+}, \mathbb{Z}/p\mathbb{Z})$ be a non-trivial character of G^v/G^{v+}. Take a finite Galois extension L of K such that $\mathrm{Gal}(L/K)^{v+} = \{1\}$ and that χ is factored by a homomorphism $\chi_L : \mathrm{Gal}(L/K)^v \to \mathbb{Z}/p\mathbb{Z}$. Let K' be the subextension of L/K corresponding to $\mathrm{Gal}(L/K)^v$. By [Se1], $\psi_{L/K}(v)$ and $\psi_{K'/K}(v)$ are integers, and the group $\mathrm{Gal}(L/K)^v$ is canonically isomorphic to the kernel of the homomorphism

$$\alpha_{L/K',\psi_{K'/K}(v)} : \mathfrak{m}_L^{\psi_{L/K}(v)}/\mathfrak{m}_L^{\psi_{L/K}(v)+1} \to \mathfrak{m}_{K'}^{\psi_{K'/K}(v)}/\mathfrak{m}_{K'}^{\psi_{K'/K}(v)+1}.$$

Let $\widetilde{D}_{K'/K} \in K'^{\times}/1 + \mathfrak{m}_K$ be the refined different of K'/K. Multiplication by $\widetilde{D}_{K'/K}$ defines an isomorphism $\mathfrak{m}_{K'}^{\psi_{K'/K}(v)}/\mathfrak{m}_{K'}^{\psi_{K'/K}(v)+1} \otimes_{k'} \overline{k} \cong N^v$ (where k' is the residue field of K').

The map $\mathfrak{m}_L^{\psi_{L/K}(v)}/\mathfrak{m}_L^{\psi_{L/K}(v)+1} \to N^v$ defines a finite Galois covering of an affine algebraic group N^v over $\overline{k}$ with Galois group $\mathrm{Gal}(L/K)^v$. This covering and χ induces a finite Galois covering $N_\chi \to N^v$ with Galois group $\mathbb{Z}/p\mathbb{Z}$. Then there exists a unique morphism $N^v \to \mathbb{A}^1_{\overline{k}} = \mathrm{Spec}\,(\overline{k}[t])$ of line bundles over $\overline{k}$ such that N_χ is isomorphic to the pull-back of the Artin-Schreier covering $\mathrm{Spec}\,(\overline{k}[t][s]/(s - s^p - t))$ of $\mathbb{A}^1_{\overline{k}}$. This defines an element in $\mathrm{Hom}_{\overline{k}}(N^v, \overline{k})$.

Fix a non-trivial continuous additive character $\psi : K \to R^\times$. For $v \in \mathbb{Q}_{>0}$, we set $w = -v - \mathrm{ord}\,\psi - 1$ and define an isomorphism

$$\sigma_\psi = \sigma_{\psi,v} : \mathrm{Hom}(G^v/G^{v+}, R^\times) \xrightarrow{\cong} N^w$$

to be the composite

$$\begin{aligned}
&\mathrm{Hom}(G^v/G^{v+}, R^\times)\\
\cong\ & N^{-v} \otimes_{\mathbb{Z}/p\mathbb{Z}} \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, R^\times)\\
\cong\ & N^w \otimes_k \mathrm{Hom}_k(\mathfrak{m}_K^{-\mathrm{ord}\,\psi-1}/\mathfrak{m}_K^{-\mathrm{ord}\,\psi}, k) \otimes_{\mathbb{Z}/p\mathbb{Z}} \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, R^\times)\\
\xrightarrow{\mathrm{Tr}_{k/\mathbb{F}_p}}\ & N^w \otimes_k \mathrm{Hom}_{\mathbb{F}_p}(\mathfrak{m}_K^{-\mathrm{ord}\,\psi-1}/\mathfrak{m}_K^{-\mathrm{ord}\,\psi}, \mathbb{F}_p) \otimes_{\mathbb{Z}/p\mathbb{Z}} \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, R^\times)\\
\cong\ & N^w \otimes_k \mathrm{Hom}(\mathfrak{m}_K^{-\mathrm{ord}\,\psi-1}/\mathfrak{m}_K^{-\mathrm{ord}\,\psi}, R^\times)\\
\xrightarrow{\psi}\ & N^w.
\end{aligned}$$

As in § 3.4, let X^v denote the set of G-orbits in the G-set $\mathrm{Hom}(G^v/G^{v+}, R^\times)$.

Let Σ be an element in X^v. Take a $\chi \in \Sigma$ and let K_χ be the extension of K corresponding to the stabilizing subgroup of χ. The field K_χ is an at most tamely ramified finite extension of K. It is easily checked that $\sigma_\psi(\chi) \in (N^w)^{W_{K_\chi}} \subset \amalg_{v' \in \mathbb{Q}} N^{v'}$ belongs to the image of the injective group homomorphism $(K_\chi^\times/1+\mathfrak{m}_{K_\chi}) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}] \hookrightarrow \amalg_{v' \in \mathbb{Q}} N^{v'}$.

Let us abbreviate k_{K_χ} and q_{K_χ} by k_χ and q_χ, respectively. Let $H = W_{K_\chi}$ be the Weil group of K_χ, and H^v,H^{v+} the upper numbering ramification subgroups of H. Since K_χ/K is at most tamely ramified, the inclusion map $H \hookrightarrow G$ induces a canonical isomorphism $H^{e_{K_\chi/K}v}/H^{e_{K_\chi/K}v+} \cong G^v/G^{v+}$. Then by direct computation the diagram

$$\begin{array}{ccc}
\mathrm{Hom}(G^v/G^{v+}, R^\times) & \xrightarrow[\cong]{\sigma_\psi} & N_K^w \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{Hom}(H^{e_{K_\chi/K}v}/H^{e_{K_\chi/K}v+}, R^\times) & \xrightarrow[\cong]{\sigma_{\psi \circ \mathrm{Tr}_{K_\chi/K}}} & N_{K_\chi}^{-e_{K_\chi/K}v - \mathrm{ord}\,(\psi \circ \mathrm{Tr}_{K_\chi/K}) - 1}.
\end{array}$$

is commutative (a more general results in this direction will be discussed in § 9.6).

7.2. Refined swan conductor

DEFINITION 7.1. Let V be an object in $\mathrm{Rep}(G, R)$ which is pure of refined break Σ. Choose a character $\chi \in \Sigma$. We define *refined ψ-Swan conductor* $\mathrm{rsw}_\psi(V)$ to be the element

$$\mathrm{rsw}_\psi(V) = \mathrm{N}_{K_\chi/K}(\sigma_\psi(\chi))^{-\frac{\mathrm{rank}\,V}{[K_\chi:K]}}$$

in $K^\times/1+\mathfrak{m}_K$, which is independent of the choice of χ.

For an arbitrary object W in $\mathrm{Rep}(G, R)$, define $\mathrm{rsw}_\psi(W) \in K^\times/1+\mathfrak{m}_K$ by

$$\mathrm{rsw}_\psi(W) = \prod_{\Sigma'} \mathrm{rsw}_\psi(W^{\Sigma'}),$$

where $W = W^0 \oplus \bigoplus_{\Sigma'} W^{\Sigma'}$ is the refined break decomposition of W.

REMARK 7.2. When R is a field of characteristic zero, this element $\mathrm{rsw}_\psi(V)$ is related to Kato's refined swan conductor defined in [K1, p. 324, (3.1)]. cf. [Sa2, p. 6, Thm. 2].

7.3. A quadratic Gauss sum

Assume that $p \neq 2$. For $x \in K^\times$ with $v_K(x) + \operatorname{ord}\psi = 2b+1$ is odd, let $\tau_{K,\psi}(x)$ be the quadratic Gauss sum defined as

$$\tau_{K,\psi}(x) = \sum_{y \in \mathfrak{m}_K^{-b-1}/\mathfrak{m}_K^{-b}} \psi(x\frac{y^2}{2}).$$

We have $\tau_{K,\psi}(x)^2 = \left(\frac{-1}{k}\right) q$. In particular $\tau_{K,\psi}(x)$ is a unit in R.

The Gauss sum $\tau_{K,\psi}(x)$, for fixed K and ψ, depends only on the class of $x \in \{x \in K^\times \,;\, v_K(x) + \operatorname{ord}\psi \equiv 1 \mod 2\}$ in $(K^\times/1+\mathfrak{m}_K) \otimes \mathbb{Z}/2\mathbb{Z}$. Thus we can define $\tau_{K,\psi}(x)$ for $x \in \{x \in (K^\times/1+\mathfrak{m}_K) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}] \,;\, v_K(x) + \operatorname{ord}\psi \in 1 + 2\mathbb{Z}[\frac{1}{p}]\}$.

7.4. Definition of local $\bar{\varepsilon}_0$-constants for totally wild representations

Let $v \in \mathbb{Q}_{>0}$ and $\Sigma \in X^v$. Choose a character $\chi \in \Sigma$. We define the *Gauss sum* $g_R(\Sigma, \psi)$ *associated with* Σ and ψ to be the element

$$\begin{aligned} &g_R(\Sigma, \psi) \\ &= q_\chi^{\operatorname{ord}(\psi\circ\operatorname{Tr}_{K_\chi/K})} \cdot \lambda_R(K_\chi/K, \psi) \\ &\quad \times \begin{cases} q_\chi^{(1+w)/2} & \text{if } p = 2 \text{ or } p \neq 2 \text{ and } \operatorname{ord}_2(v) \leq 0, \\ q_\chi^{w/2} \cdot \tau_{K_\chi, \psi\circ\operatorname{Tr}_{K_\chi/K}}(\sigma_\psi(\chi)) & \text{if } p \neq 2 \text{ and } \operatorname{ord}_2(v) > 0, \end{cases} \end{aligned}$$

in $R^\times$, where $w = e_{K_\chi/K} v$. The following two lemmas are easily checked:

LEMMA 7.3. *$g_R(\Sigma, \psi)$ depends only on Σ and ψ, and does not depend on the choice of χ.*

LEMMA 7.4. *Let d_v be the p-primary part of the denominator of v. Let $\widetilde{\chi} : G^v/G^{v+} \to \widetilde{\mathbb{Z}}_K^\times$ be a non-trivial homomorphism, $\widetilde{\Sigma}$ be the set of G-conjugates of $\widetilde{\chi}$. Then for a universal partial character $\psi' : I \to \widetilde{\mathbb{Z}}_K^\times$, there exists a canonical element $\widetilde{g}_R(\widetilde{\Sigma}, I, \psi')^{d_v} \in \widetilde{\mathbb{Z}}_K^\times$ satisfying the following property: for any strict p'-coefficient ring R, for any homomorphism $h : \widetilde{\mathbb{Z}} \to R$ of rings, for any continuous additive character $\psi : K \to R^\times$ whose restriction to I is equal to $h \circ \psi'$, and for any object (ρ, V) in* $\operatorname{Rep}(G, R)$ *which is pure of refined break $h(\widetilde{\Sigma}) = \{h \circ \widetilde{\chi} \mid \widetilde{\chi} \in \widetilde{\Sigma}\}$, we have*

$$g_R(h(\widetilde{\Sigma}), \psi) = (h(\widetilde{g}_R(\widetilde{\chi}, I, \psi')^{d_v}))^{1/d_v}.$$

Definition 7.5. Let $v \in \mathbb{Q}_{>0}$ and let (ρ, V) be an object in $\mathrm{Rep}(G, R)$ which is pure of refined break $\Sigma \in X^v$. Choose a character $\chi \in \Sigma$ and let $V_\chi = (\mathrm{Res}^G_{G^v} V)_\chi$ be the χ-part of $\mathrm{Res}^G_{G^v} V$. Using Lemma 3.8, we regard V_χ as an object in $\mathrm{Rep}(W_{K_\chi}, R)$. We define the *local* $\overline{\varepsilon}_0$*-constant* $\overline{\varepsilon}_{0,R}(V, \psi)$ *for* V *and* ψ to be the element

$$\overline{\varepsilon}_{0,R}(V, \psi) = \det(V_\chi)(\mathrm{rec}(\sigma_\psi(\chi)))^{-1} \cdot g_R(\Sigma, \psi)^{\mathrm{rank}\, V_\chi}$$

in $R^\times/\boldsymbol{\mu}_{p^\infty}(R)$. We call $g_R(\Sigma, \psi)^{\mathrm{rank}\, V_\chi}$ the *Gauss sum part* of $\overline{\varepsilon}_{0,R}(V, \psi)$.

Lemma 7.6. *The element* $\overline{\varepsilon}_{0,R}(V, \psi)$ *does not depend on the choice of a character* $\chi \in \Sigma$.

Proof. It suffices to prove that $\det(V_\chi)(\mathrm{rec}(\sigma_\psi(\chi)))$ is independent of the choice of χ. Let $\chi' \in \Sigma$ be another character and take an element $g \in G$ such that $\chi' = g.\chi$. We then have $W_{K_{\chi'}} = g W_{K_\chi} g^{-1}$ and $V_{\chi'}$ is isomorphic to the $R[W_{K_{\chi'}}]$-module with underlying R-module V_χ on which the group $W_{K_{\chi'}}$ acts via the isomorphism $W_{K_{\chi'}} \cong W_{K_\chi}$ which sends $h \in W_{K_{\chi'}}$ to $g^{-1} h g$. Since the homomorphism σ_ψ is equivariant under the action of G, we have

$$\mathrm{rec}(\sigma_\psi(\chi')) = \mathrm{rec}(g(\sigma_\psi(\chi))) = g \mathrm{rec}(\sigma_\psi(\chi)) g^{-1},$$

which proves the claim. □

Definition 7.7. Let (ρ, V) be an object in $\mathrm{Rep}(G, R)$ which is totally wild. Let

$$V = \bigoplus_{v \in \mathbb{Q}_{>0}} \bigoplus_{\Sigma \in X^v} V^\Sigma$$

be the refined break decomposition of V. We define the element $\overline{\varepsilon}_{0,R}(V, \psi)$ in $R^\times/\boldsymbol{\mu}$ to be

$$\overline{\varepsilon}_{0,R}(V, \psi) = \prod_{v \in \mathbb{Q}_{>0}} \prod_{\Sigma \in X^v} \overline{\varepsilon}_{0,R}(V^\Sigma, \psi).$$

7.5. Properties of local $\overline{\varepsilon}_0$-constants

Theorem 7.8. *The local* $\overline{\varepsilon}_0$*-constants* $\overline{\varepsilon}_{0,R}(V, \psi)$ *satisfy the following properties:*

(1) *For fixed* R *and* ψ*, the element* $\overline{\varepsilon}_{0,R}((\rho, V), \psi) \in R^\times/\boldsymbol{\mu}$ *depends only on the isomorphism class of* (ρ, V)*.*

(2) *Let R' be another strict p'-coefficient ring, and $h : R \to R'$ a local ring homomorphism. Then we have*

$$h(\overline{\varepsilon}_{0,R}(V, \psi)) = \overline{\varepsilon}_{0,R'}(V \otimes_R R', h \circ \psi).$$

(3) *Let V, V', and V'' be three totally wild objects in* $\mathrm{Rep}(W_K, R)$. *Suppose that there exists an exact sequence*

$$0 \to V' \to V \to V'' \to 0$$

in $\mathrm{Rep}(W_K, R)$. *Then we have*

$$\overline{\varepsilon}_{0,R}(V, \psi) = \overline{\varepsilon}_{0,R}(V', \psi) \cdot \overline{\varepsilon}_{0,R}(V'', \psi).$$

(4) *Suppose that R is a field. Then*

$$\overline{\varepsilon}_{0,R}(V, \psi) = \varepsilon_0(V, \psi, dx) \mod \boldsymbol{\mu}(R),$$

where dx is the R-valued Haar measure of K satisfying $\int_{\mathcal{O}_K} dx = 1$.

(5) *Let R_0 be a complete discrete valuation ring with a finite residue field of characteristic $\neq p$. Let F_0 denote the field of fractions* $\mathrm{Frac}\,(R_0)$ *of R_0, and let F denote the completion of the maximal unramified extension of F_0. Suppose that R the ring of integers in F. and that (ρ, V) is isomorphic to the base change $(\rho_0, V_0) \otimes_{R_0} R$ of an object (ρ_0, V_0) in* $\mathrm{Rep}(W_K, R_0)$. *Then*

$$\overline{\varepsilon}_{0,R}(V, \psi) = \varepsilon_0(V_0 \otimes_{R_0} \overline{F_0}, \psi, dx) \mod \boldsymbol{\mu}(F),$$

where dx is the R_0-valued Haar measure of K satisfying $\int_{\mathcal{O}_K} dx = 1$.

(6) *Suppose that* $\mathrm{rank}\, V = 1$, *then $\overline{\varepsilon}_{0,R}(V, \psi)$ coincides with $\varepsilon_0(\rho \circ \mathrm{rec}, \psi, dx) \mod \boldsymbol{\mu}(R)$ defined in* [De1, p. 555, 6.4], *where dx is the R-valued Haar measure of K satisfying $\int_{\mathcal{O}_K} dx = 1$.*

(7) *Let $a \in K^\times$ and let $\psi_a : K \to R^\times$ be the additive character defined by $\psi_a(x) = \psi(ax)$. Then we have*

$$\overline{\varepsilon}_{0,R}(V, \psi_a) = \det(V)(\mathrm{rec}(a)) q_K^{v_K(a) \cdot \mathrm{rank}\, V} \overline{\varepsilon}_{0,R}(V, \psi).$$

(8) *Let W be an object in* $\mathrm{Rep}(W_K, R)$ *on which* W_K *acts via* $W_K/W_K^0 \cong \mathbb{Z}$. *Let* $\mathrm{Fr} \in W_K/W_K^0$ *be the geometric Frobenius. Then we have*

$$\overline{\varepsilon}_{0,R}(V \otimes W, \psi) = \det W(\mathrm{Fr}^{\mathrm{sw}(V)+\mathrm{rank}\, V\cdot(\mathrm{ord}\,\psi+1)})\overline{\varepsilon}_{0,R}(V, \psi)^{\mathrm{rank}\, W}.$$

(9) *Let V^* be the R-linear dual of V. Then we have*

$$\overline{\varepsilon}_{0,R}(V, \psi) \cdot \overline{\varepsilon}_{0,R}(V^*, \psi) = \det V(\mathrm{rec}(-1)) \cdot q^{\mathrm{sw}(V)+\mathrm{rank}\, V\cdot(2\mathrm{ord}\,\psi+1)}.$$

(10) *(cf.* [DH, p. 108, Thm. 4.2]*) Let $V \neq \{0\}$ be a totally wild object in* $\mathrm{Rep}(G, R)$. *Take the smallest $v \in \mathbb{Q}_{>0}$ such that $V^v \neq \{0\}$. Then for every object W in* $\mathrm{Rep}(G, R)$ *satisfying $W^w = \{0\}$ for all $w \in \mathbb{Q}_{\geq 0}$ with $w \geq v$, we have*

$$\overline{\varepsilon}_{0,R}(V \otimes_R W, \psi) = \det W(\mathrm{rec}(\mathrm{rsw}_\psi(V))) \cdot \overline{\varepsilon}_{0,R}(V, \psi)^{\mathrm{rank}\, W}.$$

Proof. (1), (2) and (3) are obvious.

(4) and (5) follows from the main theorem of Henniart [He, p. 122, Thm. and Remark 4] and the proof of Saito [Sa2, p. 10, Thm. 3].

(6) Let $a(V)$ denotes the Artin conductor of V. The representation V is pure of refined break $\{\chi\}$, where $\chi = \rho|_{W_K^{a(V)-1}}$. Then $\sigma_\psi(\chi)$ is the unique element in $K^\times/1 + \mathfrak{m}_K$ such that

$$\rho(\mathrm{rec}(1 + x)) = \psi(\sigma_\psi(\chi)x)$$

holds for all $x \in \mathfrak{m}_K^{a(V)-1}$. Then we have

$$\overline{\varepsilon}_{0,R}(V, \psi) = \varepsilon_0(\rho \circ \mathrm{rec}, \psi, dx) \mod \boldsymbol{\mu}_{p^\infty},$$

by the standard computation of the local constant for character (see [T3, p.95, prop.1 and p.97, proof of Cor. 1]).

For (7) (8) (9) and (10), we may assume that V is pure of refined break $\Sigma \in X^v$. Then V^* is pure of refined break $\Sigma^{-1} = \{\chi^{-1} \mid \chi \in \Sigma\}$. Choose a character $\chi \in \Sigma$ and let K_χ/K be the extension corresponding to the stabilizing subgroup of χ and $q_\chi = q_{K_\chi}$. Let $V_\chi \in \mathrm{Rep}(W_{K_\chi}, R)$ be the χ-part of $\mathrm{Res}^G_{G^v} V$.

(7) We have $\sigma_{\psi_a} = a^{-1}\sigma_\psi$. Hence by Proposition 6.4 (4),

$$\overline{\varepsilon}_{0,R}(V, \psi_a) = \overline{\varepsilon}_{0,R}(V, \psi) \cdot \det(V_\chi)(\mathrm{rec}(a))$$

$$\cdot\,(q^{e_{K_\chi/K}v_K(a)+v_K(a)([K_\chi:K]-e_{K_\chi/K})}(d_{K_\chi/K},a)_K)^{\operatorname{rank}V_\chi}$$
$$=\overline{\varepsilon}_{0,R}(V,\psi)\det(V)(\operatorname{rec}(a))q^{v_K(a)\cdot\operatorname{rank}V}.$$

(9) We have

$$\begin{aligned}
&\overline{\varepsilon}_{0,R}(V,\psi)\cdot\overline{\varepsilon}_{0,R}(V^*,\psi)\\
=\ &\det(V_\chi)(\operatorname{rec}_{K_\chi}(\sigma_\psi(\chi)))^{-1}\cdot g_R(\Sigma,\psi)^{\operatorname{rank}V_\chi}\\
&\cdot\det(V_\chi)(\operatorname{rec}_{K_\chi}(-\sigma_\psi(\chi)))\cdot g_R(\Sigma^{-1},\psi)^{\operatorname{rank}V_\chi}\\
=\ &\det(V_\chi)(\operatorname{rec}_{K_\chi}(-1))\cdot(q_\chi^{\operatorname{ord}(\psi\circ\operatorname{Tr}_{K_\chi/K})}\cdot\lambda_R(K_\chi/K,\psi))^{2\operatorname{rank}V}\\
&\cdot q_\chi^{\operatorname{rank}V_\chi\cdot(1+e_{K_\chi/K}v)}\\
=\ &\det(V)(\operatorname{rec}_K(-1))\cdot\det(\operatorname{Ind}_{W_{K_\chi}}^{W_K}1)(\operatorname{rec}_K(-1))^{\operatorname{rank}V_\chi}\\
&\cdot((d_{K_\chi/K},-1)_K\cdot q^{v_K(d_{K_\chi/K})+2[K_\chi:K]\operatorname{ord}\psi})^{\operatorname{rank}V_\chi}\\
&\cdot q^{\operatorname{rank}V_\chi\cdot(f_{K_\chi/K}+[K_\chi:K]v)}\\
=\ &\det(V)(\operatorname{rec}_K(-1))\cdot q^{\operatorname{rank}V_\chi\cdot(v_K(d_{K_\chi/K})+2[K_\chi:K]\operatorname{ord}\psi)+f_{K_\chi/K}+[K_\chi:K]v)}\\
=\ &\det(V)(\operatorname{rec}_K(-1))\cdot q^{\operatorname{sw}(V)+\operatorname{rank}V\cdot(2\operatorname{ord}\psi+1)}.
\end{aligned}$$

(10) $V\otimes W$ is pure of refined break Σ and $V_\chi\otimes W$ is the χ-part of $\operatorname{Res}^G_{G^v}V\otimes W$. Hence

$$\begin{aligned}
\overline{\varepsilon}_{0,R}(V\otimes W,\psi)&=\det(V_\chi\otimes W)(\operatorname{rec}(\sigma_\psi(\chi)))^{-1}\cdot g_R(\Sigma,\psi)^{\operatorname{rank}V_\chi\otimes W}\\
&=\det W(\operatorname{rec}(\sigma_\psi(\chi)))^{-\operatorname{rank}V_\chi}\cdot\overline{\varepsilon}_{0,R}(V,\psi)^{\operatorname{rank}W}\\
&=\det W(\operatorname{rec}(\operatorname{rsw}_\psi(V)))\cdot\overline{\varepsilon}_{0,R}(V,\psi)^{\operatorname{rank}W}.
\end{aligned}$$

(8) By (10), we have

$$\overline{\varepsilon}_{0,R}(V\otimes W,\psi)=\det(W)(\operatorname{Fr}_q^{f_{K_\chi/K}\,v_{K_\chi}(\sigma_\psi(\chi))})^{-\operatorname{rank}V_\chi}\overline{\varepsilon}_{0,R}(V,\psi)^{\operatorname{rank}W}.$$

The assertion follows from $v_{K_\chi}(\sigma_\psi(\chi))=-e_{K_\chi/K}(v+\operatorname{ord}\psi+1)$. □

8. Local ε_0-Constants for Totally Wild Representations

Let K be a p-local field with residue field k and let $G=W_K$ denote the Weil group of K. Let $(R,\mathfrak{m}_R)$ be a strict p'-coefficient ring. Let $\boldsymbol{\mu}=\boldsymbol{\mu}_{p^\infty}(R)\subset R^\times$ denote the group of p power roots of unity in R.

8.1. Definition of local ε_0-constants for totally wild representations

DEFINITION 8.1. Let $\psi : K \to R^\times$ be a non-trivial continuous additive character. For a totally wild object (ρ, V) in $\mathrm{Rep}(G, R)$, we define the *local* ε_0*-constant* $\varepsilon_{0,R}(V, \psi)$ to be the unique element of $R^\times$ satisfying

$$\varepsilon_{0,R}(V,\psi) \mod \boldsymbol{\mu} = \overline{\varepsilon}_{0,R}(V,\psi)$$

and

$$\varepsilon_{0,R}(V,\psi) \mod \mathfrak{m}_R = \varepsilon_0(V \otimes_R R/\mathfrak{m}_R, \psi, dx).$$

REMARK 8.2. Existence of $\varepsilon_{0,R}(V, \psi)$ follows from Theorem 7.8 (4). Uniqueness of $\varepsilon_{0,R}(V, \psi)$ follows from the bijectivity of the canonical map $\boldsymbol{\mu}_{p^\infty}(R) \to \boldsymbol{\mu}_{p^\infty}(R/\mathfrak{m}_R)$.

PROOF OF THEOREM 5.3. It suffices to check that the element $\varepsilon_{0,R}(V, \psi)$ in Definition 8.1 satisfies the properties (1)–(9) in Theorem 5.3. All these properties follows immediately from Theorem 7.8 and the properties of $\varepsilon_0(V \otimes_R R/\mathfrak{m}_R, \psi, dx)$ reviewed in the last part of § 4. □

8.2. Result of Deligne-Henniart type

PROPOSITION 8.3 (cf. [DH, p. 110, Thm. 4.6]). *Let $V \neq \{0\}$ be a totally wild object in* $\mathrm{Rep}(G, R)$. *Take the smallest $v \in \mathbb{Q}_{>0}$ such that $V^v \neq \{0\}$. Then there exists an element $\gamma = \gamma_{V,\psi} \in K^\times$, unique modulo $1 + \mathfrak{m}_K^{\lceil \frac{v}{2} \rceil}$, which satisfies the following property: for every object W in* $\mathrm{Rep}(G, R)$ *satisfying $W^w = \{0\}$ for all $w \in \mathbb{Q}_{\geq 0}$ with $w > \frac{v}{2}$, we have*

$$\varepsilon_{0,R}(V \otimes_R W, \psi) = \det W(\mathrm{rec}(\gamma)) \cdot \varepsilon_{0,R}(V, \psi)^{\mathrm{rank}\, W}.$$

Furthermore, we have $\gamma \equiv \mathrm{rsw}_\psi(V) \mod 1 + \mathfrak{m}_K$, in particular $v_K(\gamma) = \mathrm{sw}(V) + \mathrm{ord}\, \psi \cdot \mathrm{rank}\, V$.

PROOF. We may assume that V is pure of refined break $\Sigma \in X^v$. If R is a field of characteristic zero, then the assertion follows from [DH, p. 110, Thm. 4.6] and [Sa2, p. 10, Cor. of Thm. 3].

Assume that R is a field of characteristic $\neq 0, p$. Since any irreducible object (ρ, V) in $\mathrm{Rep}(G, R)$ is a twist by an unramified character of a representation of G whose image is finite, (ρ, V) can be lifted to characteristic zero as a virtual representation $(\widetilde{\rho}, \widetilde{V})$. Further we can take $\widetilde{V}$ such that $\widetilde{V}$ has a pure refined break. The continuous additive character ψ is also lifted to characteristic zero, which we denote by $\widetilde{\psi}$. Thus we can take $\gamma_{V,\psi} = \gamma_{\widetilde{V},\widetilde{\psi}}$.

For general R, let $\gamma \in K^{\times}$ be the element which satisfies the assertion of the proposition for $V \otimes_R R/\mathfrak{m}_R$. Then, by Theorem 7.8 (10), γ satisfies the assertion of the proposition also for V. $\square$

9. Proof of Theorem 5.7 (1)

9.1. Statement

Let L be a finite separable extension of K. Let $\mathcal{O}_L$ denote the ring of integers in L, and let $\mathfrak{m}_L$ denote the maximal ideal of $\mathcal{O}_L$. Let $\psi : K \to R^{\times}$ be a non-trivial continuous additive character.

The aim of this section is to give a proof of Theorem 5.7 (1), that is, to prove the following theorem:

THEOREM 9.1. *Let (ρ, V) be a totally wild object in* $\mathrm{Rep}(W_L, R)$. *Let $W = \mathrm{Ind}_{W_L}^{W_K} V$. Then*

$$\varepsilon_{0,R}(W, \psi) = \varepsilon_{0,R}(V, \psi \circ \mathrm{Tr}_{L/K}) \cdot \lambda_R(L/K, \psi)^{\mathrm{rank}\, V}.$$

We denote W_K and W_L by G and H respectively. Let $\psi_{L/K}$ is the Herbrand function of L/K. Then for $w \in \mathbb{Q}_{\geq 0}$ and for $v = \psi_{L/K}(w)$, we have $G^w \cap H = H^v$ (resp. $G^{w+} \cap H = H^{v+}$), where G^w, H^v, G^{w+} and H^{v+} are the upper numbering ramification subgroups. The inclusion $H \hookrightarrow G$ induces canonical inclusions $H/H^v \hookrightarrow G/G^w$, $H/H^{v+} \hookrightarrow G/G^{w+}$ and $H^v/H^{v+} \hookrightarrow G^w/G^{w+}$.

9.2. Break decomposition of $\mathrm{Ind}_H^G V$

Let (ρ, V) be an object in $\mathrm{Rep}(H, R)$ which is pure of break $v_0 \in \mathbb{Q}_{\geq 0}$. We put $W = \mathrm{Ind}_H^G V$. There exists a unique $w_0 \in \mathbb{Q}_{\geq 0}$ satisfying $v_0 = \psi_{L/K}(w_0)$.

The following lemma is easily checked:

LEMMA 9.2.

(1) *If $w_0 > 0$, then $W^{G^{w_0}} = \{0\}$.*

(2) *If* $w \geq w_0$ *and* $v = \psi_{L/K}(w)$, *then* $W^{G^{w+}}$ *is canonically isomorphic to* $\mathrm{Ind}_{H/H^{v+}}^{G/G^{w+}} V$ *as an object of* $\mathrm{Rep}(G, R)$.

(3) *If* $w > w_0$ *and* $v = \psi_{L/K}(w)$, *then* W^{G^w} *is canonically isomorphic to* $\mathrm{Ind}_{H/H^v}^{G/G^w} V$ *as an object of* $\mathrm{Rep}(G, R)$.

COROLLARY 9.3. *For* $w \in \mathbb{Q}_{\geq 0}$, *let* W^w *denote the break-*w*-part of* W.

(1) $W^w = \{0\}$ *for* $w < w_0$

(2) $W^{w_0} \cong \mathrm{Ind}_{H/H^{v_0+}}^{G/G^{w_0+}} V$

(3) *For* $w > w_0$ *and for* $v = \psi_{L/K}(w)$, *there exists an exact sequence*

$$0 \to \mathrm{Ind}_{H/H^v}^{G/G^w} V \to \mathrm{Ind}_{H/H^{v+}}^{G/G^{w+}} V \to W^w \to 0$$

in $\mathrm{Rep}(G, R)$.

9.3. Reduction to $\overline{\varepsilon}_{0,R}$

LEMMA 9.4. *If* R *is a field, then for any object* (ρ, V) *in* $\mathrm{Rep}(H, R)$, *we have*

$$\varepsilon_{0,R}(\mathrm{Ind}_H^G V, \psi) = \varepsilon_{0,R}(V, \psi \circ \mathrm{Tr}_{L/K}) \cdot \lambda_R(L/K, \psi)^{\mathrm{rank}\, V}.$$

PROOF. If $\mathrm{char}\, R = 0$, this is due to Proposition 6.4 (3) and Deligne [De1, 4.1]. If $\mathrm{char}\, R \neq 0$, this is an immediate consequence of Deligne [De1, 6.5]. $\square$

If (ρ, V) is a totally wild object in $\mathrm{Rep}(H, R)$ then $\mathrm{Ind}_H^G V$ is also a totally wild object. Therefore, to prove the Theorem 9.1, it suffices to prove the following proposition:

PROPOSITION 9.5. *If* (ρ, V) *is a totally wild object in* $\mathrm{Rep}(W_L, R)$ *then*

$$\overline{\varepsilon}_{0,R}(\mathrm{Ind}_H^G V, \psi) = \overline{\varepsilon}_{0,R}(V, \psi \circ \mathrm{Tr}_{L/K}) \cdot \lambda_R(L/K, \psi)^{\mathrm{rank}\, V}.$$

Before proving this proposition, we investigate the refined break decomposition of $\mathrm{Ind}_H^G V$.

9.4. Refined break decomposition of $(\mathrm{Ind}_H^G V)^{>w_0}$

As in § 9.2, let (ρ, V) be an object in $\mathrm{Rep}(H, R)$ which is pure of break $v_0 \in \mathbb{Q}_{\geq 0}$. Set $W = \mathrm{Ind}_H^G V$. There exists a unique $w_0 \in \mathbb{Q}_{\geq 0}$ satisfying $v_0 = \psi_{L/K}(w_0)$. In this subsection, we consider the refined break decomposition of W^w for $w > w_0$.

Let $w > w_0$ and set $v = \psi_{L/K}(w)$. Let C_w be the set of all R-valued characters of the abelian p-group G^w/G^{w+} which is trivial on H^v/H^{v+}. The group H/H^{v+} acts on C_w by conjugation. Let B_w denote the set of H/H^{v+}-orbits of C_w. For $\chi' \in C_w$, let $H_{\chi'} \subset H/H^{v+}$ be the stabilizing subgroup of χ'. Then the representation $\mathrm{Res}_{H_{\chi'}}^{H/H^{v+}} V$ can be uniquely lifted to a representation $V_{\chi'}$ of $G^w H_{\chi'}/G^{w+}$ on which G^w/G^{w+} acts by χ'. For $\Sigma' \in B_w$, take an element $\chi' \in \Sigma'$ and set $V_{\Sigma'} = \mathrm{Ind}_{G^w H_{\chi'}/G^{w+}}^{G^w H/G^{w+}} V_{\chi'}$. Then $V_{\Sigma'}$ does not depend on the choice of χ'.

The following lemma is easily checked:

Lemma 9.6.

(1) *As an object in* $\mathrm{Rep}(G^w H, R)$, *the induced representation* $\mathrm{Ind}_{H/H^{v+}}^{G^w H/G^{w+}} V$ *is canonically isomorphic to the direct sum* $\bigoplus_{\Sigma' \in B_w} V_{\Sigma'}$.

(2) *For* $\Sigma' \in B_w$ *with* $\Sigma' \neq \{1\}$, *let* $\widetilde{\Sigma}'$ *denote the unique* G/G^{w+}*-orbit of characters of* G^w/G^{w+} *which contains* Σ'. *Then, as an object in* $\mathrm{Rep}(G, R)$, *the induced representation* $\mathrm{Ind}_{G^w H/G^{w+}}^{G/G^{w+}} V_{\Sigma'}$ *is pure of refined break* $\widetilde{\Sigma}'$.

Let $G_{\chi'} \subset G/G^{w+}$ be the stabilizing subgroup of χ'. Then $G_{\chi'} \supset G^w/G^{w+}$ and $G_{\chi'} \cap H/H^{v+} = H_{\chi'}$. Hence

$$\mathrm{Ind}_{G^w H/G^{w+}}^{G/G^{w+}} V_{\Sigma'} = \mathrm{Ind}_{G^w H_{\chi'}/G^{w+}}^{G/G^{w+}} V_{\chi'} = \mathrm{Ind}_{G_{\chi'}/G^{w+}}^{G/G^{w+}} \mathrm{Ind}_{G^w H_{\chi'}/G^{w+}}^{G_{\chi'}/G^{w+}} V_{\chi'}.$$

Using this description, we shall compute the $\overline{\varepsilon}_0$-constant of the break-w-part W^w of W.

9.5. Refined break decomposition of $(\mathrm{Ind}_H^G V)^{w_0}$

Let (ρ, V) be an object in $\mathrm{Rep}(H, R)$ which is pure of break $v_0 \in \mathbb{Q}_{>0}$. Set $W = \mathrm{Ind}_H^G V$. Assume further that V is pure of refined break Σ. There exists a unique $w_0 \in \mathbb{Q}_{>0}$ such that $\psi_{L/K}(v_0) = w_0$. In this subsection, we consider the refined break decomposition of W^{w_0}.

Take an element $\chi \in \Sigma$. Let $H_\chi \subset H/H^{v_0+}$ be the stabilizing subgroup of χ. Then $H_\chi \supset H^{v_0}/H^{v_0+}$. There exists a representation V' of H_χ such that H^{v_0}/H^{v_0+} acts on V' by χ and that V is isomorphic to $\mathrm{Ind}_{H_\chi}^{H/H^{v_0+}} V'$. Let C_{w_0} be the set of all characters of the abelian p-group G^{w_0}/G^{w_0+} whose restriction on H^{v_0}/H^{v_0+} is isomorphic to χ. The group H_χ acts on C_{w_0} by conjugation. et B_{w_0} denote the set of H_χ-orbits of C_{w_0}. For $\chi' \in C_{w_0}$, let $H_{\chi'} \subset H_\chi$ be the stabilizing subgroup of χ'. Then the representation $\mathrm{Res}_{H_{\chi'}}^{H_\chi} V'$ can be uniquely lifted to a representation $V'_{\chi'}$ of $G^{w_0}H_{\chi'}/G^{w_0+}$ on which G^{w_0}/G^{w_0+} acts by χ'. For $\Sigma' \in B_{w_0}$, take an element $\chi' \in \Sigma'$ and set $V'_{\Sigma'} = \mathrm{Ind}_{G^{w_0}H_{\chi'}/G^{w_0+}}^{G^{w_0}H_\chi/G^{w_0+}} V'_{\chi'}$. $V'_{\Sigma'}$ does not depend on the choice of χ'.

The following lemma is easily checked:

Lemma 9.7.

(1) *The object* $\mathrm{Ind}_{H_\chi/H^{v_0+}}^{G^{w_0}H_\chi/G^{w_0+}} V'$ *in* $\mathrm{Rep}(G^{w_0}H_\chi/G^{w_0+}, R)$ *is canonically isomorphic to the direct sum* $\bigoplus_{\Sigma' \in B_{w_0}} V'_{\Sigma'}$.

(2) *For* $\Sigma' \in B_{w_0}$, *let* $\widetilde{\Sigma}'$ *denote the unique* G/G^{w_0+}*-orbit of characters of* G^{w_0}/G^{w_0+} *which contains* Σ'. *Then as an object in* $\mathrm{Rep}(G, R)$, *the induced representation* $\mathrm{Ind}_{G^{w_0}H_\chi/G^{w_0+}}^{G/G^{w_0+}} V'_{\Sigma'}$ *is pure of refined break* $\widetilde{\Sigma}'$.

Let $G_{\chi'} \subset G/G^{w_0+}$ be the stabilizing subgroup of χ'. Then $G_{\chi'} \supset G^{w_0}/G^{w_0+}$ and $G_{\chi'} \cap H/H^{v_0+} = H_{\chi'}$. Hence

$$\mathrm{Ind}_{G^{w_0}H_\chi/G^{w_0+}}^{G/G^{w_0+}} V'_{\Sigma'} = \mathrm{Ind}_{G^{w_0}H_{\chi'}/G^{w_0+}}^{G/G^{w_0+}} V'_{\chi'} = \mathrm{Ind}_{G_{\chi'}/G^{w_0+}}^{G/G^{w_0+}} \mathrm{Ind}_{G^{w_0}H_{\chi'}/G^{w_0+}}^{G_{\chi'}/G^{w_0+}} V'_{\chi'}.$$

Using this description, we shall compute the $\overline{\varepsilon}_0$-constant of the break-w_0-part W^{w_0} of W.

9.6. The restriction map $\mathrm{Hom}(G^w/G^{w+}, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Hom}(H^v/H^{v+}, \mathbb{Z}/p\mathbb{Z})$

Let L/K be an finite separable extension of p-fields such that $L \neq K$ and that L/K has no non-trivial intermediate extension. When L/K is ramified, there exists a unique $w_1 \in \mathbb{Q}_{\geq 0}$ such that $\psi_{L/K}(w) = w$ for $0 \leq w \leq w_1$

and that $\psi_{L/K}(w)$ is linear with slope $[L : K]$ for $w > w_1$. When L/K is unramified, we put $w_1 = 0$.

Let $w \in \mathbb{Q}_{\geq 0}$ and set $v = \psi_{L/K}(w)$. In this subsection, we will investigate the restriction map

$$\mathrm{Hom}(G^w/G^{w+}, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\mathrm{Res}} \mathrm{Hom}(H^v/H^{v+}, \mathbb{Z}/p\mathbb{Z}),$$

where $G = W_K$ and $H = W_L$ as before. For every finite separable extension M of K (resp. of L), we set $w_M = \psi_{M/K}(w)$ (resp. $v_M = \psi_{M/L}(v)$).

Lemma 9.8. *Let $\widetilde{K}$ be a finite Galois extension of K satisfying $\mathrm{Gal}(\widetilde{K}/K)^{w+} = \{1\}$ and $\mathrm{Gal}(\widetilde{K}/K)^w \neq \{1\}$. Let $\widetilde{L} = L \cdot \widetilde{K}$. Assume that $[\widetilde{L} : \widetilde{K}] = [G^{w+} : H^{v+}]$. Let K' (resp. L') be the subextension of $\widetilde{K}/K$ (resp. $\widetilde{L}/L$) corresponding to the subgroup $\mathrm{Gal}(\widetilde{K}/K)^w$ (resp. $\mathrm{Gal}(\widetilde{L}/L)^v$) of $\mathrm{Gal}(\widetilde{K}/K)$ (resp. $\mathrm{Gal}(\widetilde{L}/L)$). Then $w_{\widetilde{K}}$, $v_{\widetilde{L}}$, $w_{K'}$, and $v_{L'}$ are integers and the natural map*

$$\phi_{L/K,w} : N_K^{-w} \cong \mathrm{Hom}(G^w/G^{w+}, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\mathrm{Res}} \mathrm{Hom}(H^v/H^{v+}, \mathbb{Z}/p\mathbb{Z}) \cong N_L^{-v}$$

is dual to the map

$$N_L^v \xrightarrow{\times \widetilde{D}_{L'/L}^{-1}} \mathfrak{m}_{L'}^{v_{L'}}/\mathfrak{m}_{L'}^{v_{L'}} \otimes_{k_{L'}} \overline{k} \xrightarrow{\alpha_{L'/K',w_{K'}}} \mathfrak{m}_{K'}^{w_{K'}}/\mathfrak{m}_{K'}^{w_{K'}+1} \otimes_{k_{K'}} \overline{k} \xrightarrow{\times \widetilde{D}_{K'/K}} N_K^w,$$

where $\alpha_{L'/K',w_{K'}}$ is the homomorphism defined in § 3.2, that is, $\alpha_{L'/K',w_{K'}}$ is the homomorphism induced by the norm map

$$\mathrm{N}_{L'/K'} : (1 + \mathfrak{m}_{L'}^{v_{L'}})/(1 + \mathfrak{m}_{L'}^{v_{L'}}) \to (1 + \mathfrak{m}_{K'}^{w_{K'}})/\mathfrak{m}_{K'}^{w_{K'}+1}.$$

Proof. Let $\widetilde{K}_1$ be another finite Galois extension of K satisfying $\widetilde{K}_1 \supset \widetilde{K}$ and $\mathrm{Gal}(\widetilde{K}_1/K)^{w+} = \{1\}$. We have $\mathrm{Gal}(\widetilde{K}_1/K)^w \neq \{1\}$. Let $\widetilde{L}_1 = L \cdot \widetilde{K}_1$. Let K_1' (resp. L_1') be the subextension of $\widetilde{K}_1/K$ (resp. $\widetilde{L}_1/L$) corresponding to the subgroup $\mathrm{Gal}(\widetilde{K}_1/K)^w$ (resp. $\mathrm{Gal}(\widetilde{L}_1/L)^v$) of $\mathrm{Gal}(\widetilde{K}_1/K)$ (resp. $\mathrm{Gal}(\widetilde{L}_1/L)$).

Then $w_{\widetilde{K}_1}$, $v_{\widetilde{L}_1}$, $w_{K_1'}$ and $v_{L_1'}$ are integers. We have the following commutative diagram:

$$\begin{tikzcd}[column sep=small]
 & \mathrm{Gal}(\widetilde{L}_1/L)^v \arrow[rr, hook] \arrow[dl, hook] \arrow[dd, two heads] & & \mathfrak{m}_{\widetilde{L}_1}^{v_{\widetilde{L}_1}}/\mathfrak{m}_{\widetilde{L}_1}^{v_{\widetilde{L}_1}+1} \arrow[rr, "\alpha_{\widetilde{L}_1/L_1',v_{L_1'}}"] \arrow[dl, "\alpha_{\widetilde{L}_1/\widetilde{K}_1,w_{\widetilde{K}_1}}"'] \arrow[dd, "\alpha_{\widetilde{L}_1/\widetilde{L},v_{\widetilde{L}}}" near end] & & \mathfrak{m}_{L_1'}^{v_{L_1'}}/\mathfrak{m}_{L_1'}^{v_{L_1'}+1} \arrow[rr, hook, "\times\widetilde{D}_{L_1'/L}"'] \arrow[dl, "\alpha_{L_1'/K_1',w_{K_1'}}"'] \arrow[dd, "\alpha_{L_1'/L',v_{L'}}" near start] & & N_L^v \\
\mathrm{Gal}(\widetilde{K}_1/K)^w \arrow[rr, hook] \arrow[dd, two heads] & & \mathfrak{m}_{\widetilde{K}_1}^{w_{\widetilde{K}_1}}/\mathfrak{m}_{\widetilde{K}_1}^{w_{\widetilde{K}_1}+1} \arrow[rr, "\alpha_{\widetilde{K}_1/K_1',w_{K_1'}}"] \arrow[dd, "\alpha_{\widetilde{K}_1/\widetilde{K},w_{\widetilde{K}}}" near end] & & \mathfrak{m}_{K_1'}^{w_{K_1'}}/\mathfrak{m}_{K_1'}^{w_{K_1'}+1} \arrow[rr, hook, "\times\widetilde{D}_{K_1'/K}"] \arrow[dd, "\alpha_{K_1'/K',w_{K'}}" near end] & & N_K^w & \\
 & \mathrm{Gal}(\widetilde{L}/L)^v \arrow[rr, hook] \arrow[dl, hook] & & \mathfrak{m}_{\widetilde{L}}^{v_{\widetilde{L}}}/\mathfrak{m}_{\widetilde{L}}^{v_{\widetilde{L}}+1} \arrow[rr, "\alpha_{\widetilde{L}/L',v_{L'}}"] \arrow[dl, "\alpha_{\widetilde{L}/\widetilde{K},w_{\widetilde{K}}}"] & & \mathfrak{m}_{L'}^{v_{L'}}/\mathfrak{m}_{L'}^{v_{L'}+1} \arrow[rr, hook, "\times\widetilde{D}_{L'/L}"'] \arrow[dl, "\alpha_{L'/K'.w_{K'}}"] & & N_L^v \\
\mathrm{Gal}(\widetilde{K}/K)^w \arrow[rr, hook] & & \mathfrak{m}_{\widetilde{K}}^{w_{\widetilde{K}}}/\mathfrak{m}_{\widetilde{K}}^{w_{\widetilde{K}}+1} \arrow[rr, "\alpha_{\widetilde{K}/K',w_{K'}}"'] & & \mathfrak{m}_{K'}^{w_{K'}}/\mathfrak{m}_{K'}^{w_{K'}+1} \arrow[rr, hook, "\times\widetilde{D}_{K'/K}"'] & & N_K^w. &
\end{tikzcd}$$

Since $W_{K_1'} \supset G^w$, we have $W_{K_1'} \supset (W_{K'})^{w_{K'}}$. There exists a rational number $\epsilon \in \mathbb{Q}_{>0}$ such that $\psi_{K_1'/K'}(x)$ is linear for $x > w_{K'} - \epsilon$. By Lemma 3.5, $\alpha_{K_1'/K',w_{K'}}$ is equal to the multiplication by $\widetilde{D}_{K_1'/K'}$. For the same reason $\alpha_{L_1'/L',v_{L'}}$ is equal to the multiplication by $\widetilde{D}_{L_1'/L'}$. Hence the lemma follows. $\square$

PROPOSITION 9.9. *Let us consider the canonical map*

$$\sigma_{L/K,\psi,w} = \sigma_{\psi\circ\mathrm{Tr}_{L/K}} \circ \mathrm{Res} \circ \sigma_\psi^{-1} : N_K^{-w-\mathrm{ord}\,\psi-1} \to N_L^{-v-\mathrm{ord}\,(\psi\circ\mathrm{Tr}_{L/K})-1}.$$

(1) *If* $w > w_1$, *then* $\sigma_{L/K,\psi,w}$ *is equal to the identity map* : $N_K^{-w-\mathrm{ord}\,\psi-1} \xrightarrow{=} N_L^{-v-\mathrm{ord}\,(\psi\circ\mathrm{Tr}_{L/K})-1}$.

(2) *Suppose that* $w < w_1$. *Take a finite Galois extension* $\widetilde{K}$ *of* K *such that* $\mathrm{Gal}(\widetilde{K}/K)^{w+} = \{1\}$ *and that* $\mathrm{Gal}(\widetilde{K}/K)^w \neq \{1\}$. *Then the field* $\widetilde{L} = L\cdot\widetilde{K}$ *is a finite Galois extension of* L *such that* $\mathrm{Gal}(\widetilde{L}/L)^{v+} = \{1\}$ *and that* $\mathrm{Gal}(\widetilde{L}/L)^v \neq \{1\}$. *Let* K' *(resp.* L'*) be the subextension of* $\widetilde{K}/K$ *(resp.* $\widetilde{L}/L$*) corresponding to* $\mathrm{Gal}(\widetilde{K}/K)^w$ *(resp.* $\mathrm{Gal}(\widetilde{L}/L)^v$*). Take prime elements* $\pi_{L'} \in L'$ *and* $\pi_{K'} \in K'$ *satisfying* $\mathrm{N}_{K'/L'}(\pi_{K'}) = \pi_{L'}$. *Then* $\sigma_{L/K,\psi,w}$ *sends* $a \cdot a_{\psi,\zeta}^{-1} \cdot \widetilde{D}_{K'/K}^{-1}\pi_{K'}^{-w_{K'}}$ *to*

$$a^{\frac{1}{[L:K]}} \cdot a_{\psi,\zeta}^{-1}\widetilde{D}_{L'/K}^{-1}\pi_{L'}^{-v_{L'}}.$$

(3) *Let* $\widetilde{L}$ *be the Galois closure of* L/K*. Then* $\mathrm{Gal}(\widetilde{L}/K)^{w_1+} = \{1\}$*. Let* $\widetilde{K}$ *(resp.* L'*) be the subextension of* $\widetilde{L}/K$ *(resp.* $\widetilde{L}/L$*) corresponding to* $\mathrm{Gal}(\widetilde{L}/K)^{w_1}$ *(resp.* $\mathrm{Gal}(\widetilde{L}/L)^{w_1}$*). Take prime elements* $\pi_{L'} \in L'$ *and* $\pi_{\widetilde{K}} \in \widetilde{K}$ *satisfying* $\mathrm{N}_{\widetilde{K}/L'}(\pi_{\widetilde{K}}) = \pi_{L'}$*. Then there exists an additive polynomial*
$$P(t) = a_0 \cdot t^{[L:K]} + \cdots + 1 \in \overline{k}[t]$$
of degree $[L:K]$ *with* $a_0 = \widetilde{D}_{L'/\widetilde{K}} \cdot \frac{\pi_{L'}^{w_{1,L'}}}{\pi_{\widetilde{K}}^{w_{1,K}}}$ *and with the constant term* 1 *such that the homomorphism* $\sigma_{L/K,\psi,w}$ *sends* $a \cdot a_{\psi,\zeta}^{-1} \cdot \widetilde{D}_{\widetilde{K}/K}^{-1} \pi_{\widetilde{K}}^{-w_{1,K}}$ *to*
$$P(a^{\frac{1}{[L:K]}}) \cdot a_{\psi,\zeta}^{-1} \widetilde{D}_{L'/K}^{-1} \pi_{L'}^{-w_{1,L'}}.$$

Remark 9.10. We need only (1) to prove Theorem 5.7 (1). In § 11, (3) is used to prove Theorem 5.7 (4).

To prove the proposition, we use the following lemma which is easily checked.

Lemma 9.11. *Let* $V = V' = \mathrm{Spec}\,\overline{k}[t]$*, let* $P(t) = a_0 t + a_1 t^p + \cdots + a_n t^{p^n} \in \overline{k}[t]$ *be an additive polynomial, and let* $P : V' \to V$ *denote the morphism given by* $t \mapsto P(t)$*. Then the map*
$$\begin{aligned}\overline{k} &\cong \mathrm{Hom}_{\overline{k}}(V, \overline{k}) \cong \mathrm{Hom}(\pi_1(V), \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Hom}(\pi_1(V'), \mathbb{Z}/p\mathbb{Z}) \\ &\cong \mathrm{Hom}_{\overline{k}}(V', \overline{k}) \cong \overline{k}\end{aligned}$$
induced by P *is described as*
$$a \mapsto a_0 a + a_1^{\frac{1}{p}} a^{\frac{1}{p}} + \cdots + a_n^{\frac{1}{p^n}} a^{\frac{1}{p^n}}.$$

Proof. We prove only (3). (1) and (2) are easier and their proofs are left to the reader. By Lemma 9.8, the natural map
$$N_L^{w_1} \to N_K^{w_1}$$
of $\overline{k}$-group schemes is the composite
$$N_L^{w_1} \xrightarrow{\widetilde{D}_{L'/L}^{-1}} \mathfrak{m}_{L'}^{w_{1,L'}}/\mathfrak{m}_{L'}^{w_{1,L'}+1} \xrightarrow{\alpha_{L'/\widetilde{K},w_{1,K}}} \mathfrak{m}_{\widetilde{K}}^{w_{1,K}}/\mathfrak{m}_{\widetilde{K}}^{w_{1,K}+1} \xrightarrow{\widetilde{D}_{\widetilde{K}/K}} N_K^{w_1}.$$

By taking $\widetilde{D}_{L'/L}\pi_{L'}^{w_{1,L'}}$ (resp. $\widetilde{D}_{\widetilde{K}/K}\pi_{\widetilde{K}}^{w_{1,K}}$) as a $\overline{k}$-basis of $N_L^{w_1}$ (resp. $N_K^{w_1}$), we identify $N_L^{w_1}$ (resp. $N_K^{w_1}$) as the affine line over $\overline{k}$.

Apply Lemma 9.11 to $V = N_K^{w_1}$ and $V' = N_L^{w_1}$. By simple calculation, $P(t)$ is of the form

$$P(t) = a_0 t + \cdots + t^{[L':\widetilde{K}]},$$

where $a_0 = \frac{\mathrm{Tr}_{L'/K}(\pi_{L'}^{w_{1,L'}})}{\pi_K^{w_{1,K}}} = \widetilde{D}_{L'/\widetilde{K}} \cdot \frac{\pi_{L'}^{w_{1,L'}}}{\pi_K^{w_{1,K}}}$. Hence $\phi_{L/K,w}$ sends $a \cdot \widetilde{D}_{\widetilde{K}/K}^{-1}\pi_{\widetilde{K}}^{-w_{1,K}}$ to

$$(a_0 \cdot a + \cdots + a^{\frac{1}{[L:K]}}) \cdot \widetilde{D}_{L'/L}^{-1}\pi_{L'}^{-w_{1,L'}}.$$

Let $\psi : K \to R^\times$ be a non-trivial continuous additive character. Take a primitive p-th root of unity $\zeta \in R$. There exists a unique element $a_{\psi,\zeta} \in \mathfrak{m}_K^{-\mathrm{ord}\,\psi-1}/\mathfrak{m}_K^{\mathrm{ord}\,\psi}$ such that $\psi(x) = \zeta^{\mathrm{Tr}_{k/\mathbb{F}_p}(a_{\psi,\zeta}x)}$ for all $x \in \mathfrak{m}_K^{-\mathrm{ord}\,\psi-1}/\mathfrak{m}_K^{-\mathrm{ord}\,\psi}$. Then, for all $y \in \mathfrak{m}_L^{-\mathrm{ord}\,(\psi\circ\mathrm{Tr}_{L/K})-1}/\mathfrak{m}_L^{-\mathrm{ord}\,(\psi\circ\mathrm{Tr}_{L/K})}$, we have by Lemma 3.5,

$$\begin{aligned}\psi(\mathrm{Tr}_{L/K}(y)) = \psi(\mathrm{Tr}_{k_L/k}(\widetilde{D}_{L/K}y)) &= \zeta^{\mathrm{Tr}_{k/\mathbb{F}_p}(a_{\psi,\zeta}\mathrm{Tr}_{k_L/k}(\widetilde{D}_{L/K}y))}\\ &= \zeta^{\mathrm{Tr}_{k_L/\mathbb{F}_p}(a_{\psi,\zeta}\widetilde{D}_{L/K}y)}.\end{aligned}$$

We have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}(G^w/G^{w+}, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{1\mapsto\zeta} & \mathrm{Hom}(G^w/G^{w+}, R^\times) \\ {\scriptstyle\cong}\big\downarrow & & \big\downarrow{\scriptstyle\sigma_\psi} \\ N_K^{-w} & \xrightarrow{a_{\psi,\zeta}^{-1}} & N_K^{-w-\mathrm{ord}\,\psi-1}. \end{array}$$

Hence the proposition follows. □

9.7. Representation of p-groups over p'-coefficient rings

Let G be a finite p-group, R a strict p'-coefficient ring which contains a primitive p-th root of unity.

We call an object V in $\mathrm{Rep}(G, R)$ *indecomposable* if it cannot be written as a direct sum of two non-trivial objects in $\mathrm{Rep}(G, R)$.

It is well known that any irreducible complex representation of a finite p-group is monomial. (see [I, Chap. 6, Cor. (6.14)] for example). In the same way of its proof, we have

LEMMA 9.12. *If (ρ, V) is an indecomposable objects in $\mathrm{Rep}(G,R)$, then there exists a subgroup H of G and an object W in $\mathrm{Rep}(H,R)$ of rank one such that V is isomorphic to $\mathrm{Ind}_H^G W$.*

COROLLARY 9.13. *Let R' be another strict p'-coefficient ring, $h : R \to R'$ a local ring homomorphism. Then the functor $V \mapsto V \otimes_R R'$ gives a categorical equivalence $\mathrm{Rep}(G,R) \otimes_R R' \cong \mathrm{Rep}(G,R')$, where $\mathrm{Rep}(G,R) \otimes_R R'$ denotes the category with the same objects as as $\mathrm{Rep}(G,R)$ whose morphisms are defined as $\mathrm{Hom}_{\mathrm{Rep}(G,R)\otimes_R R'}(X,Y) := \mathrm{Hom}_{\mathrm{Rep}(G,R)}(X,Y) \otimes_R R'$.*

9.8. A key proposition

Let K be a p-local field with residue field k of q elements, R a strict p'-coefficient ring which contains a primitive p-th root of unity.

The aim of this subsection is to prove the following result.

PROPOSITION 9.14. *Let (ρ, V) be a totally wild object in $\mathrm{Rep}(W_K, R)$ which is defined over a finite subring $R_0 \subset R$. Assume that V is indecomposable and that V is not of the form $\mathrm{Ind}_{W_L}^{W_K} V'$ for a non-trivial finite separable at most tamely ramified extension L of K and for an object $V' \in \mathrm{Rep}(W_L, R)$. Then there exist a strict p'-coefficient ring R', p'-coefficient ring R'' which is a complete discrete valuation ring with a finite residue field whose field of fractions is of characteristic zero, local ring homomorphisms*

$$R \xrightarrow{h} R' \xleftarrow{h'} R''$$

such that h is injective, a tamely ramified object V' in $\mathrm{Rep}(W_K, R')$ and an object V'' in $\mathrm{Rep}(W_K, R'')$ such that

$$V \otimes_R R' \cong V' \otimes_{R'} (V'' \otimes_{R''} R').$$

PROOF. Let G, I, and P denote the image of W_K, $(W_K)^0$, and $(W_K)^{0+}$ under ρ, respectively. We have $G \rhd I \rhd P$ and $G \rhd P$. By assumption, I is a finite group. I/P is a cyclic group of finite order m which is prime to

p. Take a lift $\widetilde{\zeta} \in I$ of a generator $\zeta \in I/P$ such that the order of $\widetilde{\zeta}$ in I is also m. Then we have $I \cong \langle \widetilde{\zeta} \rangle \ltimes P$. Also take a lift $\widetilde{\sigma} \in G$ of the geometric Frobenius in G/I.

The restriction $\mathrm{Res}^G_P V$ is a direct sum of indecomposable objects $V = \bigoplus_{i=0}^n V_i$. Since P is a p-group, for $0 \le i, j \le n$ we have $V_i \cong V_j$ or $\mathrm{Hom}_{\mathrm{Rep}(P,R)}(V_i, V_j) = \{0\}$. By assumption on V, all V_i are isomorphic and for any $g \in G$, the conjugation of V_0 by g is isomorphic to V_0. Replacing R_0 by a larger subring of R if necessary, we may assume that V_0 is defined over R_0. Let ℓ denote the residue characteristic of R. Then there exists a ring R_1 which is the integer ring of a finite unramified extension of $\mathbb{Q}_\ell$, a local ring homomorphism $R_1 \to R_0$, and an object V_0' in $\mathrm{Rep}(P, R_1)$ such that $V_0 \cong V_0' \otimes_{R_1} R$.

There is an automorphism $\alpha, \beta \in GL_{R_1}(V_0')$ such that $\alpha \circ g = (\widetilde{\zeta} g \widetilde{\zeta}^{-1}) \circ \alpha$ and $\beta \circ g = (\widetilde{\sigma} g \widetilde{\sigma}^{-1}) \circ \beta$ on V_0' for any $g \in P$. Let g_0 be the element in P defined by $\widetilde{\sigma}^{-1}\widetilde{\zeta}\widetilde{\sigma} = \widetilde{\zeta}^q g_0$. Then there exist two elements $a, b \in R_1^\times$ such that $\alpha^m = a$ and $\beta^{-1}\alpha\beta = b\alpha^q g_0$. Let $\overline{a}$ and $\overline{b}$ denote the image of a and b in R_0, respectively. Adjusting α by an element in $R_1^\times$, we may assume that the order m' of $\overline{a}$ in $R_0^\times$ is prime to p. Take a power $q' > 1$ of q which is congruent to 1 modulo mm'. Then we have

$$\begin{aligned}
\beta^{-1}\alpha^{q'}\beta &= (b\alpha^q g_0)^{q'} \\
&= b^{q'}\alpha^{qq'}(\alpha^{-q(q'-1)} g_0 \alpha^{q(q'-1)}) \cdots (\alpha^{-q} g_0 \alpha^q) g_0 \\
&= b^{q'}\alpha^{qq'}(\widetilde{\zeta}^{-q(q'-1)} g_0 \widetilde{\zeta}^{q(q'-1)}) \cdots (\widetilde{\zeta}^{-q} g_0 \widetilde{\zeta}^q) g_0 \\
&= b^{q'}\alpha^{qq'}(\widetilde{\zeta}^{-q}(\widetilde{\zeta}^q g_0)^{q'}) \\
&= b^{q'}\alpha^{qq'}(\widetilde{\zeta}^{-q}(\widetilde{\sigma}^{-1}\widetilde{\zeta}\widetilde{\sigma})^{q'}) \\
&= b^{q'}\alpha^{qq'} g_0.
\end{aligned}$$

Hence $\overline{b} = \overline{b}^{q'}$. In particular the order of $\overline{b}$ in $R_0^\times$ is prime to p.

Let R_1' be the ring of integers in the field adjoining a $q-1$-th power root c of b to $\mathrm{Frac}\,(R_1)$.

There exists a strict p'-coefficient ring R', and local R_1-algebra homomorphisms $h : R \hookrightarrow R'$ and $h' : R_1' \to R'$. Define $\alpha' \in GL_{R_1'}(V_0' \otimes_{R_1} R_1')$ as $\alpha' = c\alpha$. Then we have $\alpha'^m = ac^m$, $\quad \beta^{-1}\alpha'\beta = \alpha'^q g_0$. We note that the order of the image of ac^m in R' is finite and prime to p.

Take a lift $\widetilde{\zeta}', \widetilde{\sigma}' \in G' := W_K/\mathrm{Ker}\,(W_K^{0+} \twoheadrightarrow P)$ of $\widetilde{\zeta}, \widetilde{\sigma} \in G$. Then the action of P on $V_0' \otimes_{R_1} R_1'$ is uniquely extended to a continuous action of G'

by $\widetilde{\zeta}' \mapsto \alpha'$ and $\widetilde{\sigma}' \mapsto \beta$; in fact G' is the projective limit $G' = \varprojlim_{(M,p)=1} G'_M$ of discrete groups G'_M, where G'_M is the quotient of G' by the inverse image of $(W_K^0/W_K^{0+})^M$ by $G' \twoheadrightarrow W_K/W_K^{0+}$. The group G'_M is isomorphic to the group with a set of generators

$$\{x\} \amalg \{y\} \amalg \{z_h\,;\, h \in P\},$$

and with fundamental relations

$$\begin{aligned} z_h z_{h'} = z_{hh'},\ y^M = 1,\ y z_h y^{-1} = z_{\widetilde{\zeta}' h \widetilde{\zeta}'^{-1}},\ x z_h x^{-1} \\ = z_{\widetilde{\sigma}' h \widetilde{\sigma}'^{-1}},\ x^{-1} y x = y^q z_{g_0}. \end{aligned}$$

Let $\widetilde{V}'_0$ denote $V'_0 \otimes_{R_1} R'_1$ regarded as an object in $\mathrm{Rep}(G', R'_1)$ in the above way.

It is easily checked that there exists a tamely ramified object W in $\mathrm{Rep}(W_K, R')$ such that $V \otimes_R R' \cong (\widetilde{V}'_0 \otimes_{R'_1} R') \otimes_{R'} W$. □

9.9. Proof of Theorem 5.7 (1)

LEMMA 9.15. *Let L/K be a finite separable totally ramified extension of p-local fields, R an algebraically field of characteristic zero, $\psi : K \to R^\times$ a non-trivial continuous additive character, and V a totally wild object in* $\mathrm{Rep}(W_L, R)$. *Then we have*

$$\mathrm{rsw}_\psi(\mathrm{Ind}_{W_L}^{W_K} V) = N_{L/K}(\mathrm{rsw}_{\psi\circ \mathrm{Tr}_{L/K}}(V)).$$

PROOF. By [Sa2, p. 6, Thm. 2], rsw_ψ is related to the refined Swan conductor defined in [K1, p. 324, (3.1)]. The lemma follows from [K1, p. 325, Prop. 3.3 (2)]. □

LEMMA 9.16. *Let L/K be a finite separable at most tamely ramified extension of p-local fields, R a strict p'-coefficient ring, and $\psi : K \to R^\times$ a non-trivial continuous additive character. Let V be a totally wild object in* $\mathrm{Rep}(W_L, R)$ *which is pure of break v and of refined break Σ. Suppose that Theorem 5.7 (1) holds for L/K, ψ and V. Then for any tamely ramified object V_1 in* $\mathrm{Rep}(W_L, R)$, *Theorem 5.7 (1) also holds for L/K, ψ and $V \otimes_R V_1$.*

PROOF. We set $W = \mathrm{Ind}_{W_L}^{W_K} V$ and $W_1 = \mathrm{Ind}_{W_L}^{W_K}(V \otimes_R V_1)$.

By Corollary 9.3, W and W_1 are also totally wild. It suffices to prove that

$$\bar{\varepsilon}_{0,R}(W_1, \psi) = \bar{\varepsilon}_{0,R}(V \otimes_R V_1, \psi \circ \mathrm{Tr}_{L/K}) \cdot \lambda_R(L/K, \psi)^{\mathrm{rank}\, V \cdot \mathrm{rank}\, V_1}.$$

Since V is also totally wild, we have, by Proposition 8.3,

$$\begin{aligned} &\varepsilon_{0,R}(V \otimes V_1, \psi \circ \mathrm{Tr}_{L/K}) \\ &= \det V_1(\mathrm{rec}(\mathrm{rsw}_{\psi \circ \mathrm{Tr}_{L/K}}(V))) \cdot \varepsilon_{0,R}(V, \psi \circ \mathrm{Tr}_{L/K})^{\mathrm{rank}\, V_1}. \end{aligned}$$

Hence it suffices to prove that

$$\bar{\varepsilon}_{0,R}(W_1, \psi) = \det V_1(\mathrm{rec}(\mathrm{rsw}_{\psi \circ \mathrm{Tr}_{L/K}}(V))) \cdot \varepsilon_{0,R}(W, \psi)^{\mathrm{rank}\, V_1}.$$

Let $w = \frac{v}{e_{L/K}}$. Then the canonical map $W_L^v / W_L^{v+} \to W_K^w / W_K^{w+}$ is bijective. Let

$$r : \mathrm{Hom}(W_K^w / W_K^{w+}, R^\times) \xrightarrow{\cong} \mathrm{Hom}(W_L^v / W_L^{v+}, R^\times)$$

be the canonical bijection. Let Σ' be the unique W_K-orbit containing $r^{-1}(\Sigma)$. By § 9.5, W and W_1 are pure of refined break Σ'.

Take an element $\chi \in \Sigma$ and let $H_\chi \subset W_L$ be the stabilizing subgroup of χ. Let $V' \subset V$ be the χ-part of V. Let $\chi' = r^{-1}(\chi) \in \Sigma'$ and $G_{\chi'} \subset W_K$ the stabilizing subgroup of W_K. Then by § 9.5, the χ'-part W' of W is isomorphic to $\mathrm{Ind}_{H_\chi}^{G_{\chi'}} V'$.

The object $V \otimes_R V_1$ is pure of refined break Σ, and the χ-part of $V \otimes_R (\mathrm{Res}_{H_\chi}^{W_L} V_1)$ is equal to $V' \otimes_R V_1$. Hence the χ'-part W_1' of W_1 is isomorphic to $\mathrm{Ind}_{H_\chi}^{G_{\chi'}}(V' \otimes (\mathrm{Res}_{H_\chi}^{W_L} V_1))$ Hence

$$\begin{aligned} \frac{\bar{\varepsilon}_{0,R}(W_1, \psi)}{\varepsilon_{0,R}(W, \psi)^{\mathrm{rank}\, V_1}} &= \frac{\det W'(\mathrm{rec}(\sigma_\psi(\chi')))^{\mathrm{rank}\, V}}{\det W_1'(\mathrm{rec}(\sigma_\psi(\chi')))} \\ &= \frac{(\mathrm{Ind}_{H_\chi}^{G_{\chi'}} 1)(\mathrm{rec}(\sigma_\psi(\chi')))^{\mathrm{rank}\, V - \mathrm{rank}\, V \cdot \mathrm{rank}\, V_1}}{\det V_1(\mathrm{Ver}_{H_\chi}^{G_{\chi'}}(\mathrm{rec}(\sigma_\psi(\chi'))))^{\mathrm{rank}\, V'}}. \end{aligned}$$

By Proposition 9.9 (1), we have $\sigma_{\psi \mathrm{Tr}_{L/K}}(\chi) = \sigma_\psi(\chi')$. Hence

$$\det V_1(\mathrm{Ver}_{H_\chi}^{G_{\chi'}}(\mathrm{rec}(\sigma_\psi(\chi'))))^{\mathrm{rank}\, V'} = \det(\mathrm{Res}_{H_\chi}^{W_L} V_1)(\sigma_\psi(\chi'))^{\mathrm{rank}\, V'}$$

$$= \det V_1(\mathrm{rec}(\mathrm{rsw}_{\psi\circ \mathrm{Tr}_{L/K}}(V)))^{-1}.$$

Hence the assertion follows. □

PROOF OF THEOREM 5.7 (1). Let L/K be a finite separable extension of p-local fields, and let $\psi : K \to R^\times$ be a non-trivial continuous additive character. Let V be a totally wild object in $\mathrm{Rep}(W_L, R)$.

We prove the theorem by induction on $r = \mathrm{rank}\, V$. We may assume that V is indecomposable. Suppose that V is of the form $V = \mathrm{Ind}_{W'_L}^{W_L} V'$ for a non-trivial finite separable at most tamely ramified extension L' of L and for an object $V' \in \mathrm{Rep}(W_{L'}, R)$. Then V' is also totally wild and the theorem holds for V by induction and by Proposition 6.4 (5). Hence we may assume that V is not of the form $V = \mathrm{Ind}_{W'_L}^{W_L} V'$ as above.

We apply Proposition 9.14. Replacing R by a larger strict p'-coefficient ring if necessary, we may assume that V is of the form $V = V_1 \otimes_R V_2$, where V_1 is a tamely ramified object in $\mathrm{Rep}(W_K, R)$ and V_2 is the base change of an object in $\mathrm{Rep}(W_K, R')$ by a local ring homomorphisms $R' \to R$, where R' is a p'-coefficient ring which is a complete discrete valuation ring with a finite residue field whose field of fractions is of characteristic zero.

Let L_1 be the maximal at most tamely ramified subextension of L/K. Let $V'_1 = \mathrm{Ind}_{W_L/W_L^{0+}}^{W_{L_1}/W_{L_1}^{0+}} V_1$ be the tamely ramified object in $\mathrm{Rep}(W_{L_1}, R)$ whose restriction to W_L is isomorphic to V_1. Then we have a canonical isomorphism

$$\mathrm{Ind}_{W_L}^{W_{L_1}} V \cong V'_1 \otimes_R (\mathrm{Ind}_{W_L}^{W_{L_1}} V_2).$$

Since the theorem holds for L_1/K, ψ, and $\mathrm{Ind}_{W_L}^{W_{L_1}} V_2$, it also holds for L_1/K, ψ and $\mathrm{Ind}_{W_L}^{W_{L_1}} V$ by Lemma 9.16. Hence

$$\varepsilon_{0,R}(\mathrm{Ind}_{W_L}^{W_K} V, \psi) = \varepsilon_{0,R}(\mathrm{Ind}_{W_L}^{W_{L_1}} V, \psi \circ \mathrm{Tr}_{L_1/K}) \cdot \lambda_R(L_1/K, \psi)^{\mathrm{rank}\, V \cdot [L:L_1]}.$$

Since $\mathrm{Ind}_{W_L}^{W_{L_1}} V_2$ is also totally wild, we have, by Proposition 8.3,

$$\begin{aligned}&\varepsilon_{0,R}(\mathrm{Ind}_{W_L}^{W_{L_1}} V, \psi \circ \mathrm{Tr}_{L_1/K})\\ &= \det V'_1(\mathrm{rec}(\mathrm{rsw}_{\psi\circ \mathrm{Tr}_{L_1/K}}(\mathrm{Ind}_{W_L}^{W_{L_1}} V_2))) \cdot \varepsilon_{0,R}(\mathrm{Ind}_{W_L}^{W_{L_1}} V_2, \psi \circ \mathrm{Tr}_{L_1/K})^{\mathrm{rank}\, V_1}.\end{aligned}$$

Since the theorem holds for L/L_1, $\psi \circ \mathrm{Tr}_{L_1/K}$, and V_2, we have

$$\varepsilon_{0,R}(\mathrm{Ind}_{W_L}^{W_{L_1}} V_2, \psi\circ\mathrm{Tr}_{L_1/K}) = \varepsilon_{0,R}(V_2, \psi\circ\mathrm{Tr}_{L/K})\cdot\lambda_R(L/L_1, \psi\circ\mathrm{Tr}_{L_1/K})^{\mathrm{rank}\, V_2}.$$

By Proposition 6.4 (5), it suffices to prove that

$$\det V_1'(\mathrm{rec}_{L_1}(\mathrm{rsw}_{\psi\circ\mathrm{Tr}_{L_1/K}}(\mathrm{Ind}_{W_L}^{W_{L_1}} V_2))) = \det V_1(\mathrm{rec}_L(\mathrm{rsw}_{\psi\circ\mathrm{Tr}_{L/K}} V_2)).$$

By Lemma 9.15, we have

$$\mathrm{rsw}_{\psi\circ\mathrm{Tr}_{L_1/K}}(\mathrm{Ind}_{W_L}^{W_{L_1}} V_2) = N_{L/L_1}(\mathrm{rsw}_{\psi\circ\mathrm{Tr}_{L/K}}(V_2)).$$

Hence the assertion follows. □

From Proposition 9.14, we have the following corollary:

COROLLARY 9.17 (Characterization of ε_0-constants for totally wild objects). *The attachment*

$$(L, R, (\rho, V), \psi) \mapsto \varepsilon_{0,R}(V, \psi) \in R^\times$$

for each quadruple $(L, R, (\rho, V), \psi)$ where L is a finite separable at most tamely ramified extension of K, R is a strict p'-coefficient ring, (ρ, V) is a totally wild object in $\mathrm{Rep}(W_L, V)$, *and $\psi : L \to R^\times$ is a non-trivial continuous additive character, is characterized by the following properties.*

(1) *For fixed L, R and ψ, the element $\varepsilon_{0,R}((\rho, V), \psi) \in R^\times$ depends only on the isomorphism class of (ρ, V).*

(2) *Let $(L, R, (\rho, V), \psi)$ be a quadruple as above, R' a strict p'-coefficient ring, and $h : R \to R'$ a local ring homomorphism. Then we have*

$$h(\varepsilon_{0,R}(V, \psi)) = \varepsilon_{0,R'}(V \otimes_R R', h \circ \psi).$$

(3) *Let $(L, R, (\rho, V), \psi)$, $(L, R, (\rho', V'), \psi)$, and $(L, R, (\rho'', V''), \psi)$ be three quadruples as above with common L, R and ψ. Suppose that there exists an exact sequence*

$$0 \to V' \to V \to V'' \to 0$$

in $\mathrm{Rep}(W_L, R)$. *Then we have*

$$\varepsilon_{0,R}(V, \psi) = \varepsilon_{0,R}(V', \psi) \cdot \varepsilon_{0,R}(V'', \psi).$$

(4) *Let R_0 be a complete discrete valuation ring with a finite residue field of characteristic $\neq p$. We denote by F_0 the field of fractions* $\mathrm{Frac}\,(R_0)$ *of R_0, by F the completion of the maximal unramified extension of F_0, and by R the ring of integers in F. Let $(L, R, (\rho, V), \psi)$ be a quadruple as above. Suppose that (ρ, V) is isomorphic to the base change $(\rho_0, V_0) \otimes_{R_0} R$ of an object (ρ_0, V_0) in* $\mathrm{Rep}(W_K, R_0)$*. Then*

$$\varepsilon_{0,R}(V, \psi) = \varepsilon_0(V_0 \otimes_{R_0} \overline{F_0}\, \psi, dx),$$

where dx is the R-valued Haar measure of K satisfying $\int_{\mathcal{O}_K} dx = 1$.

(5) *Let L_1 and L_2 be two finite separable at most tamely ramified extensions of K with $L_1 \subset L_2$, let R be a strict p'-coefficient ring, and let $\psi : L_1 \to R^\times$ be a non-trivial continuous additive character. Then there exists an element $\lambda_R(L_2/L_1, \psi) \in R^\times$ such that for every totally wild object (ρ, V) in* $\mathrm{Rep}(W_{L_2}, R)$*, we have*

$$\varepsilon_{0,R}(V, \psi \circ \mathrm{Tr}_{L_2/L_2}) = \varepsilon_{0,R}(W, \psi) \times \lambda_R(L_2/L_1, \psi)^{\mathrm{rank}\, V}.$$

(6) *Let $(L, R, (\rho, V), \psi)$ be a quadruple as above. Then for every tamely ramified object W in* $\mathrm{Rep}(W_L, R)$*, we have*

$$\varepsilon_{0,R}(V \otimes_R W, \psi) = \det W(\mathrm{rec}(\mathrm{rsw}_\psi(V)) \cdot \varepsilon_{0,R}(V, \psi)^{\mathrm{rank}\, W}.$$

10. Local ε_0-Constants for Tamely Ramified Representations

Let K be a p-local field with residue field k of q elements and R a strict p'-coefficient ring. The aim of this section is to define $\varepsilon_{0,R}(V, \psi)$ for a tamely ramified object (ρ, V) in $\mathrm{Rep}(W_K, R)$ and a non-trivial continuous additive character $\psi : K \to R^\times$.

10.1. Global tame ε-constants

For a finite separable extension L of K, with $(\mathcal{O}_L, \mathfrak{m}_L)$ its ring of integers, let $\mathrm{Gr}^\bullet L$ and $\mathrm{Gr}^{\geq 0} L$ denote the graded $\mathcal{O}_L/\mathcal{O}_K$-algebras given by

$$\mathrm{Gr}^\bullet L = \oplus_{n \in \mathbb{Z}} \mathfrak{m}_L^n/\mathfrak{m}_L^{n+1}, \quad \mathrm{Gr}^{\geq 0} L = \oplus_{n \geq 0} \mathfrak{m}_L^n/\mathfrak{m}_L^{n+1},$$

respectively and let $\widehat{\mathrm{Gr}}^\bullet L$ denote the complete discrete valuation field given by

$$\widehat{\mathrm{Gr}}^\bullet L = \mathrm{Frac}\,(\prod_{i=0}^{\infty} \mathfrak{m}_L^{-i}/\mathfrak{m}_L^{-i+1}).$$

If L/K is a finite at most tamely ramified Galois extension of K, then $\mathrm{Gr}^\bullet L$ (resp. $\widehat{\mathrm{Gr}}^\bullet L$) is a finite etale Galois $\mathrm{Gr}^\bullet K$-algebra (resp. a finite, at most tamely ramified Galois extension of $\widehat{\mathrm{Gr}}^\bullet K$) whose Galois group is canonically isomorphic to $\mathrm{Gal}(L/K)$. We note that $X_0 := \mathrm{Spec}\,(\mathrm{Gr}^\bullet K)$ is (non-canonically) isomorphic to $\mathbb{G}_{m,k}$. Let $W_{\mathrm{Gr}^\bullet K}$ (resp. $W_{\mathrm{Gr}^{\geq 0} K}$) denote the subgroup of $\pi_1^{et}(X_0)$ (resp. of $\pi_1^{et}(\mathrm{Spec}\,(\mathrm{Gr}^{\geq 0} K)))$ consisting of the elements whose image in $\pi_1^{et}(\mathrm{Spec}\,(k))$ are integral powers of Frobenius. For any tamely ramified object (ρ, V) in $\mathrm{Rep}(W_K, R)$, we associate an object $(\rho, V)_{\mathrm{Gr}^\bullet}$ in $\mathrm{Rep}(W_{\mathrm{Gr}^\bullet}, R)$ (resp. a tamely ramified object $(\rho, V)_{\widehat{\mathrm{Gr}}}$ in $\mathrm{Rep}(W_{\widehat{\mathrm{Gr}}^\bullet K}, R))$ in a canonical way.

Fix a non-trivial additive character $\phi_0 : \mathfrak{m}_K^{-1}/\mathcal{O}_K \to R^\times$. Let $K' = \mathrm{Gr}^{\geq 0} K$ (resp. $K' = \widehat{\mathrm{Gr}}^\bullet K$). Take a non-zero element $x \in \mathfrak{m}_K/\mathfrak{m}_K^2$ and consider the K'-algebra $L' = K'[t]/(t - t^q - x)$. L' is a finite etale Galois K'-algebra with its Galois group canonically isomorphic to k. Define a homomorphism $\mathrm{Gal}(L'/K') \cong k \to R^\times$ by $k \ni a \mapsto \phi_0(\frac{a}{x})$. This defines a rank one object $\mathcal{L}_{\phi_0}$ (resp. $\widehat{\mathcal{L}}_{\phi_0}$) in $\mathrm{Rep}(W_{\mathrm{Gr}^{\geq 0} K}, R)$ (resp. in $\mathrm{Rep}(W_{\widehat{\mathrm{Gr}}^\bullet K}, R))$ which does not depend on the choice of x. Let $\mathcal{L}'_{\phi_0}$ be the restriction of $\mathcal{L}_{\phi_0}$ to $W_{\mathrm{Gr}^\bullet K}$.

For a moment let us assume

(*) there exists a finite subring R_0 of R such that (ρ, V) comes from an object (ρ_0, V_0) in $\mathrm{Rep}(W_K, R_0)$ by the base change, and that the image of ϕ_0 is contained in $R_0^\times$.

Then $(\rho, V)_{\mathrm{Gr}^\bullet}$ and $\mathcal{L}'_{\phi_0}$ define smooth etale R_0-sheaves $\widetilde{V}$ and $\widetilde{\mathcal{L}}'_{\phi_0}$ respectively on the algebraic curve X_0 over k. By the perfect complex argument (see [De3, Rapport]), $H_c^1(X_0 \otimes_k \overline{k}, \widetilde{V} \otimes_{R_0} \widetilde{\mathcal{L}}'_{\phi_0})$ (where $\overline{k}$ is an algebraic closure of k) is a free R_0-module of the same rank as V, endowed with an action of the geometric Frobenius Fr_q. We define the global ε-constant $\varepsilon_R(\widetilde{V} \otimes_{R_0} \widetilde{\mathcal{L}}'_{\phi_0})$ to be

$$\varepsilon_R(\widetilde{V} \otimes_{R_0} \widetilde{\mathcal{L}}'_{\phi_0}) = \det(-\mathrm{Fr}_q; H_c^1(X_0 \otimes_k \overline{k}, \widetilde{V} \otimes_{R_0} \otimes \widetilde{\mathcal{L}}'_{\phi_0})).$$

Let us go back to the situation where the condition (*) is not necessarily satisfied. For an effective divisor $D = \sum_{i=1}^n m_i[P_i]$ on $\mathrm{Spec}\,(\mathrm{Gr}^\bullet K)$, where m_i are positive integers and $P_1, \dots, P_n$ are mutually distinct closed points on Spec $(\mathrm{Gr}^\bullet K)$, we define the symmetric trace $T(D; V \otimes \mathcal{L}'_{\phi_0})$ by

$$T(D; V \otimes \mathcal{L}'_{\phi_0}) = \prod_{i=1}^{n} \mathrm{Tr}(\mathrm{Fr}_{P_i}; \mathrm{TS}^{m_i}(V \otimes_R \mathcal{L}'_{\phi_0})),$$

where $\mathrm{Fr}_{P_i} \in W_{\mathrm{Gr}^\bullet K}$ is any element in the conjugacy class of the geometric Frobenius at P_i, and $\mathrm{TS}^{m_i}(\ )$ denotes the sheaf of m_i-th symmetric tensors.

Definition 10.1. We define the global ε-constant $\varepsilon_R(V \otimes_R \mathcal{L}'_{\phi_0})$ to be

$$\varepsilon_R(V \otimes_R \mathcal{L}'_{\phi_0}) = \sum_{D} T(D; V \otimes \mathcal{L}'_{\phi_0}),$$

where D runs over all effective divisors on Spec $(\mathrm{Gr}^\bullet K)$ of degree $r = \mathrm{rank}\, V$.

Proposition 10.2 (Trace formula). *Under the condition (*), we have*

$$\varepsilon_R(\widetilde{V} \otimes_{R_0} \widetilde{\mathcal{L}}'_{\phi_0}) = \varepsilon_R(V \otimes_R \mathcal{L}'_{\phi_0}).$$

In particular, $\varepsilon_R(V \otimes_R \mathcal{L}'_{\phi_0})$ is a unit in R.

Proof. This follows immediately from [De3, bFonction $L \mod \ell^n$]. □

10.2. Definition of tame local ε_0-constants

Definition 10.3. Let (ρ, V) be a tamely ramified object in $\mathrm{Rep}(W_K, R)$. For a non-trivial additive character $\psi_0 : k \to R^\times$, we define the ε_0-*constant* $\varepsilon_{0,R}(V, \psi_0, \phi_0) \in R$ *with an additional parameter* ϕ_0 as

$$\varepsilon_{0,R}(V, \psi_0, \phi_0) := q^{-\mathrm{rank}\, V} \cdot \frac{\varepsilon_R(V \otimes_R \mathcal{L}'_{\phi_0})}{\varepsilon_{0,R}((\rho, V)_{\widehat{\mathrm{Gr}}} \otimes_R \widehat{\mathcal{L}}_{\phi_0}, \psi')},$$

where ψ' is an additive character of $\widehat{\mathrm{Gr}}^\bullet K$ induced from the additive character of $\mathrm{Gr}^\bullet(K) = \bigoplus_{n \in \mathbb{Z}} \mathfrak{m}_K^n / \mathfrak{m}_K^{n+1}$ which is 1 on $\bigoplus_{n \neq 0} \mathfrak{m}_K^n / \mathfrak{m}_K^{n+1}$ and $x \mapsto \psi_0(-x)$ for $x \in \mathfrak{m}_K^0 / \mathfrak{m}_K^1$.

Remark 10.4. Let $y \in \mathfrak{m}_K^{-1} / \mathcal{O}_K$ be the unique element satisfying

$$\phi_0(xy) = \psi_0(-x)$$

for all $x \in k$. Then, by Proposition 8.3, we have

$$\varepsilon_{0,R}((\rho, V)_{\widehat{Gr}} \otimes_R \widehat{\mathcal{L}}_{\phi_0}, \psi') = \det V(y).$$

The following theorem will be proved in § 11.

THEOREM 10.5. *$\varepsilon_{0,R}(V, \psi_0, \phi_0)$ does not depend on the choice of ϕ_0. We denote it by $\varepsilon_{0,R}(V, \psi_0)$.*

LEMMA 10.6. *For $a \in k^\times$, let $\psi_{0,a} : k \to R^\times$ be the homomorphism defined as $\psi_{0,a}(x) = \psi(ax)$. Take a lift $\widetilde{a} \in \mathcal{O}_K^\times$ of a. Then we have*

$$\varepsilon_{0,R}(V, \psi_{0,a}) = \det(V)(\mathrm{rec}(\widetilde{a}))\varepsilon_{0,R}(V, \psi_0)$$

PROOF. We will show that

$$\varepsilon_{0,R}(V, \psi_{0,a}, \phi_0) = \det(V)(\mathrm{rec}(\widetilde{a}))\varepsilon_{0,R}(V, \psi_0, \phi_0)$$

We show that

$$\varepsilon_{0,R}((\rho, V)_{\widehat{\mathrm{Gr}}} \otimes_R \widehat{\mathcal{L}}_{\phi_0}, (\psi_a)') = \det(V)(\mathrm{rec}(\widetilde{a}))^{-1}\varepsilon_{0,R}((\rho, V)_{\widehat{\mathrm{Gr}}} \otimes_R \widehat{\mathcal{L}}_{\phi_0}, \psi').$$

Since $(\psi_a)' = (\psi')_a$, it suffices to show that

$$\det((\rho, V)_{\widehat{\mathrm{Gr}}} \otimes_R \widehat{\mathcal{L}}_{\phi_0})(\mathrm{rec}_{\widehat{\mathrm{Gr}}^\bullet K}(a)) = \det(V)(\mathrm{rec}(\widetilde{a}))^{-1}.$$

By the reciprocity law, we have

$$\widehat{\mathcal{L}}_{\phi_0}(\mathrm{rec}_{\widehat{\mathrm{Gr}}^\bullet K}(a)) = 1$$

and

$$\det((\rho, V)_{\widehat{\mathrm{Gr}}})(\mathrm{rec}_{\widehat{\mathrm{Gr}}^\bullet K}(a)) = \det(V)(\mathrm{rec}(\widetilde{a}))^{-1}.$$

Hence the assertion follows. □

DEFINITION 10.7. Let $\psi : K \to R^\times$ be a non-trivial continuous additive character of K. Take an element $a \in K^\times$ such that $v_K(a) + \mathrm{ord}\,\psi = -1$. Let $\psi_a : K \to R^\times$ be the additive character of K defined as $\psi_a(x) = \psi(ax)$. We define the ε_0-*constant* $\varepsilon_{0,R}(V, \psi, a)$ to be

$$\varepsilon_{0,R}(V, \psi) = \det(V)(\mathrm{rec}(a))^{-1} q^{-v_K(a)\cdot \mathrm{rank}\, V} \varepsilon_{0,R}(V, \psi_a).$$

By Lemma 10.6, $\varepsilon_{0,R}(V, \psi)$ does not depend on the choice of a.

10.3. Properties of tame local ε_0-constants (I)

In this subsection and in § 10.5, we prove that the ε_0-constants $\varepsilon_{0,R}(V,\psi)$ defined in Definition 10.7 satisfy the properties (0)–(9) in Theorem 5.4.

Proof of Theorem 5.4 (1), (2), (6), and (7). (1) and (2) are obvious. (7) is clear from the definition of $\varepsilon_{0,R}(V,\psi)$.

(6). By (7), we may assume that $\operatorname{ord}\psi = -1$. Let $\psi_0 : k \to R^\times$ be the character induced by $\psi|_{\mathcal{O}_K}$. Then the assertion follows from the definition of the global ε-constant $\varepsilon_R(V \otimes \mathcal{L}'_{\phi_0})$ and Remark 10.4. □

Lemma 10.8 (Stability for totally wild extensions). *Let K be a p-local field, and (ρ, V) be a tamely ramified object in $\operatorname{Rep}(W_K, R)$. Let L/K be a totally ramified finite separable extension whose ramification index is a power of p. We have a canonical isomorphism $W_L/W_L^{0+} \cong W_K/W_K^{0+}$. Let (ρ_L, V_L) be the tamely ramified object in $\operatorname{Rep}(W_L, R)$ corresponding to (ρ, V) via this isomorphism.*

Then we have

$$\varepsilon_{0,R}(V_L, \psi_0) = \varepsilon_{0,R}(V, \psi_0^{([L:K])}),$$

where $\psi_0^{([L:K])}$ is the composition of the $[L:K]$-th power map $k \to k$ with ψ_0.

Proof. For every $n \in \mathbb{Z}$, the norm map $\mathrm{N}_{L/K} : L^\times \to K^\times$ induces an group isomorphism $\mathfrak{m}_L^n/\mathfrak{m}_L^{n+1} \xrightarrow{\cong} \mathfrak{m}_K^n/\mathfrak{m}_K^{n+1}$. This induces isomorphisms

$$\mathrm{Gr}^\bullet L \xrightarrow{\cong} \mathrm{Gr}^\bullet K, \quad \mathrm{Gr}^{\geq 0} L \xrightarrow{\cong} \mathrm{Gr}^{\geq 0} K, \quad \widehat{\mathrm{Gr}}^\bullet L \xrightarrow{\cong} \widehat{\mathrm{Gr}}^\bullet K$$

of rings. Then $(\rho_L, V_L)_{\mathrm{Gr}^\bullet}$, $(\rho_L, V_L)_{\widehat{\mathrm{Gr}}}$, and $\mathcal{L}_{\phi_0 \circ \mathrm{N}_{L/K}}$ corresponds respectively to $(\rho, V)_{\mathrm{Gr}^\bullet}$, $(\rho, V)_{\widehat{\mathrm{Gr}}}$, and $\mathcal{L}_{\phi_0}$ via these isomorphisms.

Hence we have

$$\varepsilon_R(V_L \otimes_R \mathcal{L}'_{\phi_0 \circ \mathrm{N}_{L/K}}) = \varepsilon_R(V \otimes_R \mathcal{L}'_{\phi_0})$$

and

$$\varepsilon_{0,R}((\rho_L, V_L)_{\widehat{\mathrm{Gr}}} \otimes \widehat{\mathcal{L}}_{\phi_0 \circ \mathrm{N}_{L/K}}, \psi' \circ \mathrm{N}_{L/K}) = \varepsilon_{0,R}((\rho, V)_{\widehat{\mathrm{Gr}}} \otimes \widehat{\mathcal{L}}_{\phi_0}, \psi').$$

Hence the lemma follows. □

Take a primitive p-th root of unity $\zeta \in R$. Let $a_{\psi,\zeta} \in \mathfrak{m}_K^{-\operatorname{ord}\psi-1}/\mathfrak{m}_K^{\operatorname{ord}\psi}$ be the element defined in §9.6, that is, $a_{\psi,\zeta}$ is the unique element satisfying

$$\psi(x) = \zeta^{\operatorname{Tr}_{k/\mathbb{F}_p}(a_{\psi,\zeta}x)}$$

for all $x \in \mathfrak{m}_K^{-\operatorname{ord}\psi-1}/\mathfrak{m}_K^{-\operatorname{ord}\psi}$. Then by the above lemma, we have

COROLLARY 10.9.

$$\begin{aligned}
&\varepsilon_0(V_L, \psi \circ \operatorname{Tr}_{L/K}) \\
= \; &\det V(\operatorname{rec}(a_{\psi,\zeta}^{[L:K]-1} \cdot N_{L/K}(\widetilde{D}_{L/K})))q^{(([L:K]-1)(\operatorname{ord}\psi+1)+v_L(\tilde{D}_{L/K}))\cdot\operatorname{rank}V} \\
&\cdot \varepsilon_{0,R}(V, \psi).
\end{aligned}$$

10.4. Reduction to finite rings

10.4.1 A preliminary from commutative ring theory

The aim of this subsubsection is to prove the following proposition:

PROPOSITION 10.10. *Let A be a finitely generated commutative $\mathbb{Z}$-algebra. Then for every non-zero element $f \in A$, there exists a finite commutative ring R and a homomorphism $\varphi : A \to R$ of rings such that $\varphi(f) \neq 0$.*

This proposition follows immediately from the following lemma:

LEMMA 10.11. *Let A be a noetherian commutative ring. Then for any non-zero element $f \in A$, there exists a maximal ideal $\mathfrak{m} \subset A$ and a positive integer $n \in \mathbb{Z}_{>0}$ such that $f \notin \mathfrak{m}^n$.*

PROOF. Let $I = \{x \in A\,;\, xf = 0\}$. Since $f \neq 0$, we have $I \neq A$. Take a maximal ideal $\mathfrak{m}$ of A containing I, and put $N = \bigcap_n \mathfrak{m}^n$. Assume that $f \in N$. By Krull intersection theorem, there exists an element $m \in \mathfrak{m}$ such that $(1-m)f = 0$. We then have $1 = (1-m) + m \in I + \mathfrak{m} = \mathfrak{m}$, which is a contradiction. Thus $f \notin \mathfrak{m}^n$ for some n. $\square$

10.4.2 A universal ring $\mathcal{R}_{q,r}$

Let q and r be two positive integers. Let us consider the functor from the category of commutative rings to the category of sets, which associates a commutative ring R to the set

$$\{(\sigma, A) \in GL_r(R)^2\ ;\ \sigma^{-1}A\sigma = A^q\}.$$

We easily see that this functor is representable by a finitely generated $\mathbb{Z}$-algebra, which we denote by $\mathcal{R}_{q,r}$.

Let K be a p-local field with residue field k of q elements. Fix a lift $F \in W_K/P_K$ of the geometric Frobenius and fix a topological generator ζ of I_K/P_K. Let R be a p'-coefficient ring. If we take an R-basis of V for any tamely ramified object (ρ, V) in $\mathrm{Rep}(W_K, R)$ of rank r, the pair $(\rho(F),\ \rho(\zeta))$ of two elements in $GL_r(R)$ satisfies $\rho(F)^{-1}\rho(\zeta)\rho(F) = \rho(\zeta)^q$. Let $\varphi_V : \mathcal{R}_{q,r} \to R$ be the ring homomorphism corresponding to the pair $(\rho(F), \rho(\zeta))$.

Lemma 10.12. *If R_0 is a finite local ring of order prime to p, then $V \mapsto \varphi_V$ gives a bijection from the set of isomorphism classes of tamely ramified objects (ρ, V) in* $\mathrm{Rep}(W_K, R_0)$ *of rank r with R_0-bases to the set of ring homomorphisms $\varphi : \mathcal{R}_{q,r} \to R_0$.*

Proof. Let (σ, A) be the pair of elements in $GL_r(R_0)$ corresponding to φ. Then the relation $\sigma^{-1}A\sigma = A^q$ implies that the order of A in $GL_r(R_0)$ is prime to p. Hence φ defines an object in $\mathrm{Rep}(W_K, R_0)$. $\square$

To study tame ε_0-constants, the ring $\mathcal{R}_{q,r}$ is often useful to reduce the assertion to the case where the condition (*) is satisfied. We will explain this by proving the following lemma as an example:

Lemma 10.13. *Let (ρ, V) be a tamely ramified object in* $\mathrm{Rep}(W_K, R)$ *of rank r. For a positive integer s, set*

$$\Delta(V, \phi_0, s) = \sum_D T(D; V \otimes \mathcal{L}'_{\phi_0}),$$

where D runs over all effective divisors on $\mathrm{Spec}\,(\mathrm{Gr}^\bullet K)$ *of degree s. Then $\Delta(V, \phi_0, s) = 0$ for all $s > r$.*

Proof. Let
$$\mathcal{R}'_{q,r} := \mathcal{R}_{q,r}[\frac{1}{p}].$$
and let $\widetilde{\phi}_0 : \mathfrak{m}_K^{-1}/\mathcal{O}_K \to (\mathbb{Z}[X]/(1+X+\cdots+X^{p-1}))^\times$ be a non-trivial additive character. Then for any non-trivial character $\phi_0 : \mathfrak{m}_K^{-1}/\mathcal{O}_K \to R^\times$ whose kernel is equal to the kernel of $\widetilde{\phi}_0$, there exists a unique ring homomorphism $h_{\phi_0} : \mathbb{Z}[X]/(1+X+\cdots+X^{p-1}) \to R$ such that $\phi_0 = h_{\phi_0} \circ \widetilde{\phi}_0$.

There exists an element $\widetilde{\Delta}(\widetilde{\phi}_0, s)$ in $\mathcal{R}'_{q,r} \otimes_{\mathbb{Z}} \mathbb{Z}[X]/(1+X+\cdots+X^{p-1})$ such that for any p'-coefficient ring R and for any tamely ramified object (ρ, V) in $\mathrm{Rep}(W_K, R)$ of rank r with an R-basis of V, and for any non-trivial character $\phi_0 : \mathfrak{m}_K^{-1}/\mathcal{O}_K \to R^\times$ whose kernel is equal to the kernel of $\widetilde{\phi}_0$, the element $\Delta(V, \phi_0, s)$ is equal to the image of $\widetilde{\Delta}(\widetilde{\phi}_0, s)$ by the ring homomorphism
$$\varphi_V \otimes h_{\phi_0} : \mathcal{R}'_{q,r} \otimes_{\mathbb{Z}} \mathbb{Z}[X]/(1+X+\cdots+X^{p-1}) \to R.$$

To prove the lemma, it suffices to prove that $\widetilde{\Delta}(\widetilde{\phi}_0, s) = 0$. By Proposition 10.10, it suffices to prove that $\varphi(\widetilde{\Delta}(\widetilde{\phi}_0, s)) = 0$ for any homomorphisms $\varphi : \mathcal{R}'_{q,r} \otimes_{\mathbb{Z}} \mathbb{Z}[X]/(1+X+\cdots+X^{p-1}) \to R_0$ from $\mathcal{R}'_{q,r}$ to a finite local ring R_0.

Hence it suffices to show the lemma for every R and (ρ, V) which satisfy the condition $(*)$. In this case, the assertion of the lemma is obvious since $\bigwedge^s H^1_c(\mathrm{Spec}\,(\mathrm{Gr}^\bullet K) \otimes_k \overline{k}, \widetilde{V} \otimes_{R_0} \widetilde{\mathcal{L}}'_{\phi_0}) = 0$. □

10.5. Properties of tame local ε_0-constants (II)

Lemma 10.14 (=Theorem 5.7 (2)). *Let L be an unramified extension of K. We denote by $\mathcal{O}_L$ its ring of integers, by $\mathfrak{m}_L$ the maximal ideal of $\mathcal{O}_L$, and by k_L the residue field of $\mathcal{O}_L$. Let (ρ, V) be a tamely ramified object in* $\mathrm{Rep}(W_L, R)$. *Then we have*
$$\varepsilon_{0,R}(\mathrm{Ind}_{W_L}^{W_K} V, \psi) = \varepsilon_{0,R}(V, \psi \circ \mathrm{Tr}_{L/K}) \cdot \lambda_R(L/K, \psi)^{\mathrm{rank}\, V}.$$

Remark 10.15. By Lemma 6.5, we have
$$\lambda_R(L/K, \psi) = (-1)^{([L:K]-1)\mathrm{ord}\,\psi}.$$

PROOF. By reduction to finite rings, we may assume that (ρ, V) satisfies the condition (*).

Let $f : \mathrm{Spec}\,(\mathrm{Gr}^\bullet L) \to \mathrm{Spec}\,(\mathrm{Gr}^\bullet K)$ be the canonical etale covering induced from L/K. Since

$$\begin{aligned}
&\mathrm{Ind}_{W_{k_L}}^{W_k} H_c^1(\mathrm{Spec}\,(\mathrm{Gr}^\bullet L) \otimes_{k_L} \overline{k}_L, \widetilde{V} \otimes_{R_0} \widetilde{\mathcal{L}}'_{\phi_0 \circ \mathrm{Tr}_{k_L/k}}) \\
&= H_c^1(\mathrm{Spec}\,(\mathrm{Gr}^\bullet K) \otimes_k \overline{k}, f_*(\widetilde{V} \otimes_{R_0} \widetilde{\mathcal{L}}'_{\phi_0 \circ \mathrm{Tr}_{k_L/k}})) \\
&= H_c^1(\mathrm{Spec}\,(\mathrm{Gr}^\bullet K) \otimes_k \overline{k}, f_*(\widetilde{V}) \otimes_{R_0} \widetilde{\mathcal{L}}'_{\phi_0}),
\end{aligned}$$

we have

$$\varepsilon_R(\mathrm{Ind}_{W_L}^{W_K} V \otimes \mathcal{L}'_{\phi_0}) = \varepsilon_R(V \otimes \mathcal{L}'_{\phi_0 \circ \mathrm{Tr}}) \cdot (-1)^{[L:K]-1}.$$

Hence the lemma follows by Lemma 6.5. □

PROOF OF THEOREM 5.4. We check the properties (0)-(9) in the statement of the theorem. The properties (1) and (2) are clear.

Let V, V', and V'' be as in the statement of property (3). By the definition of $\varepsilon_{0,R}(V, \psi, \phi_0)$, we see that

$$\varepsilon_{0,R}(V, \psi, \phi_0) = \varepsilon_{0,R}(V' \oplus V'', \psi, \phi_0).$$

We may assume that $V = V' \oplus V''$. We set $r' = \mathrm{rank}\, V'$ and $r'' = \mathrm{rank}\, V''$.

For the property (3), it suffices to show that a certain element in the ring

$$(\mathcal{R}'_{q,r'} \times \mathcal{R}'_{q,r''}) \otimes_{\mathbb{Z}} \mathbb{Z}[X]/(1 + X + \cdots X^{p-1})$$

is zero. We reduce, by reduction to finite rings, the problem to the case where both (ρ', V') and (ρ'', V'') satisfy the condition (*). In this case the assertion is immediate from the cohomological interpretation of the global ε-constants.

(4). We may assume that R is of characteristic zero and that $\mathrm{ord}\,\psi = -1$. If K is of characteristic p, the assertion follows from the product formula. If K is of characteristic zero, let

$$K' = \mathrm{Frac}\,(\varprojlim_n \oplus_{i=0}^n \mathfrak{m}_K^i / \mathfrak{m}_K^{i+1})$$

and let V' denote the object in $\mathrm{Rep}(W_{K'}, R)$ which canonically corresponds to V. The representation V is the direct sum of the representation of the form $\mathrm{Ind}_{W_L}^{W_K}\chi$, where L is an unramified extension of K and χ is a rank one tamely ramified object in $\mathrm{Rep}(W_L, R)$. Hence we can check, by direct computation, that

$$\varepsilon_0(V, \psi, dx) = \varepsilon_0(V', \psi', dx'),$$

where ψ' is an additive character of K' with $\operatorname{ord}\psi' = -1$ whose restriction to $\mathcal{O}_K/\mathfrak{m}_K$ is equal to that of ψ, and dx' is the R-valued Haar measure of K' satisfying $\int_{\mathcal{O}_{K'}} dx' = 1$. Hence the proposition follows.

(5) follows from (3) and (4).

(0) follows from (4).

(8) By reduction to finite rings, we may assume that (ρ, V) satisfies the condition (*). By (7), we may assume that $\operatorname{ord}\psi = -1$. Let ψ_0 be the additive character of k induced by $\psi|_{\mathcal{O}_K}$. It suffices to prove that

$$\varepsilon_{0,R}(V \otimes W, \psi_0, \phi_0) = \varepsilon_{0,R}(V, \psi_0, \phi_0)^{\operatorname{rank} W}.$$

By Theorem 5.3 (8), we have

$$\begin{aligned} &\varepsilon_{0,R}((\rho, V \otimes W)_{\widehat{\mathrm{Gr}}} \otimes_R \widehat{\mathcal{L}}_{\phi_0}, \psi') \\ = {}&\det W(\mathrm{Fr}^{\operatorname{rank} V}) \cdot \varepsilon_{0,R}((\rho, V)_{\widehat{\mathrm{Gr}}} \otimes_R \widehat{\mathcal{L}}_{\phi_0}, \psi')^{\operatorname{rank} W}. \end{aligned}$$

On the other hand, by the cohomological interpretation of the global ε-constant, we have

$$\varepsilon_R(V \otimes W \otimes_R \mathcal{L}'_{\phi_0}) = \det W(\mathrm{Fr}^{\operatorname{rank} V}) \cdot \varepsilon_R(V \otimes_R \mathcal{L}'_{\phi_0})^{\operatorname{rank} W}$$

Hence the assertion follows.

(9) Let (ρ, V) be a tamely ramified object in $\mathrm{Rep}(W_K, R)$ such that the coinvariant $(V)_{W_K^0}$ is zero. Let ζ be a topological generator of W_K^0/W_K^{0+}. Then $\rho(\zeta) - 1 : V \to V$ is invertible. By reduction to finite rings, we may assume that (ρ, V) satisfies the condition (*). By (7), we may assume that $\operatorname{ord}\psi = -1$. Let ψ_0 be the additive character of k induced by $\psi|_{\mathcal{O}_K}$. Let $\phi_0 : \mathfrak{m}_K^{-1}/\mathcal{O}_K \to R_0^\times$ be a non-trivial character and set $\phi_{0,1}(x) = \phi_0(-x)$. It suffices to prove that

$$\varepsilon_{0,R}(V, \psi_0, \phi_0) \cdot \varepsilon_{0,R}(V^*, \psi_0, \phi_{0,-1}) = \det(V)(\mathrm{rec}(-1))q^{-\operatorname{rank} V}.$$

To prove this, it suffices to prove that

$$\varepsilon_R(\widetilde{V} \otimes_{R_0} \widetilde{\mathcal{L}}'_{\phi_0}) \cdot \varepsilon_R(\widetilde{V^*} \otimes_{R_0} \widetilde{\mathcal{L}}'_{\phi_{0,-1}}) = q^{\operatorname{rank}(V)},$$

which follows from Poincare duality. This completes the proof of Theorem 5.4. □

11. Proofs of Theorem 10.5 and Theorem 5.7 (3) (4)

In the first part of this section, we prove Theorem 10.5, that is, independence of ϕ_0 of tame ε_0-constants stated in the previous section. As a corollary, we get a formula describing tame ε_0-constants as integrals, on which we will discuss in § 11.2. § 11.3 is devoted to the proof of Theorem 5.7 (3). The proof consists of the following reduction steps:

$$\text{Theorem 5.7 (3)} \Leftarrow \text{Prop. 11.5} \Leftarrow \text{Prop. 11.6}.$$

In § 11.4, we remark that, if $K = \mathbb{Q}_p$, Gross-Koblitz formula [GK] yields an integration formula analogous to that in § 11.2. The last two subsections in this section are devoted to the proof of Theorem 5.7 (4).

11.1. Proof of Theorem 10.5

By reduction to finite rings, it suffices to prove the theorem under the assumption (*) in § 10.1.

Let K be a p-local field, R_0 a finite local ring on which p is invertible, $\phi_0 : \mathfrak{m}_K^{-1}/\mathcal{O}_K \to R_0^\times$ a non-trivial additive character. Let V be a tamely ramified object in $\mathrm{Rep}(W_K, R_0)$. We use the notation $\mathrm{Gr}^\bullet K$, $\widetilde{V}$, and $\widetilde{\mathcal{L}}'_{\phi_0}$ in § 10.1. We set $X_0 = \mathrm{Spec}\,(\mathrm{Gr}^\bullet K)$ and $X = X_0 \otimes_k \overline{k}$. Take an element $a \in k^\times$ and let $\phi'_0 : \mathfrak{m}_K^{-1}/\mathcal{O}_K \to R_0^\times$ be the non-trivial additive character defined by $\phi'_0(x) = \phi_0(ax)$. Define the smooth invertible sheaf $\widetilde{\mathcal{L}}'_{\phi'_0}$ on X_0 in a similar way as we have defined $\widetilde{\mathcal{L}}'_{\phi_0}$.

By Remark 10.4, Theorem 10.5 is implied by the following proposition:

Proposition 11.1. *We have*

$$\det(\mathrm{Fr}_q\,; H^1_c(X, \mathcal{F} \otimes_{R_0} \widetilde{\mathcal{L}}'_{\phi_0})) = \det(V)(\mathrm{rec}_K(a)) \cdot \det(\mathrm{Fr}_q\,; H^1_c(X, \mathcal{F} \otimes_{R_0} \widetilde{\mathcal{L}}'_{\phi'_0})).$$

Proof. For a positive integer $m \in \mathbb{Z}_{>0}$, let $\pi_m : X_m \to X$ be the unique connected etale covering of X of degree m which is tamely ramified at boundaries. Take a sufficiently divisible $m \in \mathbb{Z}_{>0}$ such that the restriction of $\mathcal{F}$ to X_m is constant.

We define an object W_m in $\mathrm{Rep}(\mathrm{Gal}(X_m/X_0), R_0)$ as

$$W_m := H^1_c(X, (\pi_{m*} R_0) \otimes_{R_0} \widetilde{\mathcal{L}}'_{\phi_0}).$$

We put $I_m := \mathrm{Gal}(X_m/X)(\cong \mathbb{Z}/m\mathbb{Z})$. By duality and Hochschild-Serre spectral sequence, we have a canonical isomorphism

$$H^1_c(X, \widetilde{V} \otimes_{R_0} \widetilde{\mathcal{L}}'_{\phi_0}) \cong (V \otimes_{R_0} W_m)_{I_m},$$

where $(\quad)_{I_m}$ denotes the I_m-coinvariant.

By the perfect complex argument, as an $R_0[I_m]$-module, W_m is free of rank one. Take an $R_0[I_m]$-basis b of W_m. Then the map

$$\varphi : V \to (V \otimes_{R_0} W_m)_{I_m}$$

defined as $\varphi(v) = v \otimes b$ is an isomorphism of R_0-modules. Take a lift $\widetilde{\mathrm{Fr}}_q \in \mathrm{Gal}(X_m/X_0)$ of the geometric Frobenius and let us write $\widetilde{\mathrm{Fr}}_q(b) = ub$ with $u = \sum_{g \in I_m} r_g[g] \in R_0[I_m]$. Then we have

$$\begin{aligned} \mathrm{Fr}_q(v \otimes b) &= \widetilde{\mathrm{Fr}}_q(v) \otimes \sum_{g \in I_m} r_g[g]b \\ &= (\sum_{g \in I_m} r_g[g^{-1}]\widetilde{\mathrm{Fr}}_q)v \otimes b \end{aligned}$$

in $(V \otimes_{R_0} W_m)_{I_m}$. Therefore

$$\det(\mathrm{Fr}_q; H^1_c(X, \mathcal{F} \otimes_{R_0} \widetilde{\mathcal{L}}'_{\phi_0})) = \det(\sum_{g \in I_m} r_g[g^{-1}]\widetilde{\mathrm{Fr}}_q; V). \tag{11.1}$$

Define the object W'_m in $\mathrm{Rep}(\mathrm{Gal}(X_m/X_0), R_0)$ by

$$W'_m := H^1_c(X, (\pi_{m*}R_0) \otimes_{R_0} \widetilde{\mathcal{L}}'_{\phi'_0}).$$

Then we have a canonical isomorphism

$$H^1_c(X, \widetilde{V} \otimes_{R_0} \widetilde{\mathcal{L}}'_{\phi'_0}) \cong (V \otimes_{R_0} W'_m)_{I_m}.$$

Take an element $\alpha \in \overline{k}$ satisfying $\alpha^m = a$. Then the map $X_m \to X_m$ induced by the multiplication-by-α map $\mathfrak{m}_K/\mathfrak{m}_K^2 \to \mathfrak{m}_K/\mathfrak{m}_K^2$ induces an isomorphism $\varphi : W_m \cong W'_m$ of $R_0[I_m]$-modules. Let $[\alpha^{q-1}] \in I_m$ be the element corresponding to $\alpha^{q-1} \in \boldsymbol{\mu}_m(\overline{k})$ by the canonical isomorphism $I_m \cong \boldsymbol{\mu}_m(\overline{k})$. It is easily checked that the action of $\widetilde{\mathrm{Fr}}_q$ on W_m is identified with the action of $\widetilde{\mathrm{Fr}}_q \cdot [\alpha^{q-1}]$ in W'_m by φ. Hence the proposition follows. $\square$

This completes the proof of Theorem 10.5. □

Corollary 11.2. *For fixed K, R, and ψ_0 the local ε_0-constant $\varepsilon_{0,R}(V,\psi_0)$ for a tamely ramified object (ρ, V) in* $\mathrm{Rep}(W_K, R)$ *depends only on the restriction of V to W_K^0.*

11.2. A measure defined by $(W_m)_m$

Let K, X_0, X, X_m and I_m be as in the proof of Proposition 11.1. Let $G := \pi_1^{tm}(X_0) = \varprojlim_m \mathrm{Gal}(X_m/X_0)$ denote the tame fundamental group of X, and let $I := \pi_1^{tm}(X) = \varprojlim_m I_m \subset G$ denote the inertia subgroup of G. We use the canonical identifications $G \cong W_K/(W_K)^{0+}$ and $I \cong (W_K)^0/(W_K)^{0+}$. Take a prime number ℓ different from p. Set $R = W(\mathbb{F}_\ell(\boldsymbol{\mu}_p))$. Let $\phi_0 : \mathfrak{m}_K^{-1}/\mathcal{O}_K \to R^\times$ be a non-trivial additive character.

Since R is isomorphic to the projective limit $\varprojlim_n R/\ell^n R$ of finite local rings, we can define, for each positive integer m, the cohomology group $W_m := H_c^1(X, (\pi_{m*}R) \otimes_R \widetilde{\mathcal{L}}_{\phi_0})$ as the projective limit of the cohomology groups for ϕ_0 modulo ℓ^n $(n = 0, 1, 2 \cdots)$ which appear in the proof of Proposition 11.1. Then W_m is an object in $\mathrm{Rep}(\mathrm{Gal}(X_m/X_0), R)$ and as an $R[I_m]$-module, W_m is free of rank one. Let us consider W_m as an object in $\mathrm{Rep}(G, R)$ on which G act via the quotient $\mathrm{Gal}(X_m/X_0)$. For two integers m, n with $m|n$, the canonical morphism $W_m \to W_n$ is compatible the action of G. Let $\widehat{W}$ be the $R[[G]]$-module $\widehat{W} = \varprojlim_m W_m$. As an $R[[I]]$-module, $\widehat{W}$ is free of rank one. Take an $R[[I]]$-basis $\widehat{b}$ of $\widehat{W}$. Take a lift $\widetilde{\mathrm{Fr}}_q \in G$ of the geometric Frobenius and define an element $u_{\widehat{b}}$ in $R[[I]]$ by $\widetilde{\mathrm{Fr}}_q \widehat{b} = u_{\widehat{b}} \widehat{b}$. It is simple to see that $u_{\widehat{b}}$ lies in $R[[I]]^\times$. We note that $u_{\widehat{b}}$ depends on the choice of ϕ_0 and $\widetilde{\mathrm{Fr}}_q$, not only on that of $\widehat{b}$.

Remark 11.3. Let us define the action of G on $R[[I]]$ by the conjugation $g.[i] = [gig^{-1}]$. The $R[[I]]$-action $R[[I]] \times \widehat{W} \to \widehat{W}$ on $\widehat{W}$ is compatible with the actions of G. The class $\widehat{u}$ of $u_{\widehat{b}}$ in the G-coinvariant $(R[[I]]^\times)_G$ does not depend on the choice of $\widehat{b}$. In fact, if $\widehat{b}' = a\widehat{b}$, with $a \in R[[I]]^\times$, is another basis of $\widehat{W}$, then we have $u_{\widehat{b}'} = (\widetilde{\mathrm{Fr}}.a) u_{\widehat{b}} a^{-1}$.

By (11.1) and by Remark 10.4, we have:

Proposition 11.4. *We canonically regard the element $u_{\widehat{b}} \in R[[I]]$ as an R-valued measure on I.*

Let R' be a strict p'-coefficient ring whose residue field is of characteristic ℓ. Then R' has a canonical structure of an R-algebra. Let $\psi : K \to R'^{\times}$ an additive character with $\operatorname{ord}\psi = -1$ *satisfying*

$$\psi(x) = \phi_0(\operatorname{rec}^{-1}(\widetilde{\operatorname{Fr}}_q^{-1})x)$$

for all $x \in \mathcal{O}_K$. Then for any tamely ramified object $(\rho, V) \in \operatorname{Rep}(W_K, R')$, we have

$$\varepsilon_{0,R'}(V, \psi) = \det\left(\frac{1}{q}\int_{g\in I} \rho(g)^{-1} du_{\widehat{b}}(g)\right).$$

11.3. Proof of Theorem 5.7 (3)

Let L be a finite separable totally tamely ramified extension of K degree n. Let R be a strict p'-coefficient ring, and let (ρ, V) be a tamely ramified object $\in \operatorname{Rep}(W_K, R)$ which satisfies the condition (*) in § 10.1.

We set $Y_0 = \operatorname{Spec}(\operatorname{Gr}^{\bullet} L)$. Let $f : Y_0 \to X_0$ denote the morphism associated with the extension L/K and put $Y = Y_0 \otimes_k \overline{k} \cong X_n$. Let $\psi_L : L \to R^{\times}$ be a non-trivial continuous additive character. To prove Theorem 5.7 (3), it suffices to prove that

$$(**) \qquad \varepsilon_{0,R}(V, \psi_L) = q^{-\operatorname{rank} V} \cdot \frac{\det(-\operatorname{Fr}_q \,;\, H^1_c(Y, \widetilde{V} \otimes f^* \widetilde{\mathcal{L}}'_{\phi_0}))}{\varepsilon_{0,R}(\widehat{V} \otimes \operatorname{Res}^{W_{\mathbb{G}\mathrm{r}_K}}_{W_{\mathbb{G}\mathrm{r}_L}} \widehat{\mathcal{L}}_{\phi_0}, \psi'_L)}.$$

Let $g_R \in R^{\times}$ denotes the Gauss sum part of $\overline{\varepsilon}_{0,R}(\widehat{V} \otimes \operatorname{Res}^{W_{\mathbb{G}\mathrm{r}_K}}_{W_{\mathbb{G}\mathrm{r}_L}} \widehat{\mathcal{L}}_{\phi_0}, \psi'_L)$ (Definition 7.5).

Let ℓ be a prime number different from p. Let R, ϕ_0, W_m, $\widehat{W}$, $\widehat{b}$ and $u_{\widehat{b}}$ be as in the previous subsection.

Consider the n-th power map $I \to I$. To avoid confusion, we denote it by $I_L \to I_K$. We regard $R[[I_K]]$ as a representation of I_K over a free $R[[I_L]]$-module of rank n. Then $\det_{R[[I_L]]} R[[I_K]]$ defines a representation $\widehat{\rho}_n$ of $I = I_K$ over a free $R[[I]] = R[[I_L]]$-module of rank one.

In the same way as in the proof of Proposition 11.4, the right hand side of (**) is expressed using

$$\det\left(\frac{1}{q}\int_{g\in I} \rho(g)^{-1} d(\widehat{\rho}_n(u_{\widehat{b}}))(g)\right).$$

Thus to prove Theorem 5.7 (3), it suffices to prove the following proposition:

Proposition 11.5. *The two elements $g_R u_{\widehat{b}}$ and $\widehat{\rho}_n(u_{\widehat{b}})$ in $R[[I]]^\times$ coincide in $(R[[I]]^\times)_G$.*

Let $\mathrm{Rep}^s(G, R[[I]])$ denote the category of finitely generated projective $R[[I]]$-modules endowed with a continuous semi-linear action of G. The G-module $\widehat{W}$ is an object in $\mathrm{Rep}^s(G, R[[I]])$ of $R[[I]]$-rank one. Furthermore, the action of $I \subset G$ on $\widehat{W}$ and that of $I \subset R[[I]]^\times$ on $\widehat{W}$ coincide. We note that these two actions of I do not necessarily coincide on a general object V in $\mathrm{Rep}^s(G, R[[I]])$.

If V, V' are two objects in $\mathrm{Rep}^s(G, R[[I]])$, then the tensor product $V \otimes_{R[[I]]} V'$ is canonically viewed as an object in $\mathrm{Rep}^s(G, R[[I]])$. For an integer $n \in \mathbb{Z}_{>0}$ which is prime to p, let $I = I_L \to I_K = I$ be the n-th power map of I. Let V be an object in $\mathrm{Rep}^s(G, R[[I_K]])$. We regard V as an object in $\mathrm{Rep}^s(G, R[[I_L]])$ via the map $I_L \to I_K$ as above. Set $V_{(n)} = \wedge^n_{R[[I_L]]} V$. The assignment $V \mapsto V_{(n)}$ gives a functor from $\mathrm{Rep}^s(G, R[[I]])$ to itself. If the two actions of I mentioned above coincide on V, then so does on $V_{(n)}$.

Proposition 11.5 is equivalent to the following:

Proposition 11.6. *The object $\widehat{W}_{(n)}$ in $\mathrm{Rep}^s(G, R[[I]])$ is isomorphic to $\widehat{W}_{g_R}$, where $\widehat{W}_{g_R}$ is the unramified twist of $\widehat{W}$ by the unramified character defined by $\widetilde{\mathrm{Fr}}_q \mapsto g_R$.*

Proof. Let us recall that our extension L/K is a totally tamely ramified extension of degree n. Taking a prime element π_L in L such that $\pi_K = \pi_L^n$ is a prime element in K, we identify $X_0 = Y_0 = \mathbb{G}_{m,k}$. Then $Y_0 \to X_0$ is the n-th power map : $\mathbb{G}_{m,k} \xrightarrow{n} \mathbb{G}_{m,k}$.

For a positive integer m, let $Y_{0,m} = \mathbb{G}_{m,k}$ endowed with the structure of $Y_0 = \mathbb{G}_{m,k}$-scheme by the m-th power map

$$\pi'_m : Y_{0,m} = \mathbb{G}_{m,k} \xrightarrow{m} \mathbb{G}_{m,k} = Y_0.$$

Set $Y_m = Y_{0,m} \otimes_k \overline{k} = \mathbb{G}_{m,\overline{k}}$. We set $J_m = \mathrm{Gal}(Y_m/Y) \cong \mathbb{Z}/m\mathbb{Z}$. If we identify Y with X_n as X-schemes, then J_m is identified with a subgroup of I_{mn}.

We consider the sheaf of $R[J_m]$-module $\pi'_{m,*}R$ on Y. Let $(\pi'_{m,*}R)^{\boxtimes n}$ is the external tensor product of n copies of $\pi_{n,nm,*}R$ over $R[J_m]$; it is an invertible $R[J_m]$-sheaf on the n-fold product $Y^n = Y \times \cdots \times Y$ of Y.

LEMMA 11.7. *Let* $s_n : Y^n = \mathbb{G}^n_{m,\overline{k}} \to \mathbb{G}_{m,\overline{k}} = Y$ *be the product map. Then we have a canonical isomorphism*

$$(\pi'_{m,*}R)^{\boxtimes_{R[J_m]} n} \cong s_n^*(\pi'_{m,*}R)$$

of $R[J_m]$*-sheaves on* Y^m.

PROOF. Since s_n comes from the group law of $\mathbb{G}_{m,\overline{k}}$, the map $\pi_1^{tm}(Y^n) \to \pi_1^{tm}(Y)$ induced by s_n comes from the corresponding group law. Hence the lemma follows. □

Since $W_{mn} = H^1_c(Y, \pi'_{m,*}R \otimes_R \widetilde{\mathcal{L}}_{\phi_0}|_Y)$, the above lemma yields a canonical isomorphism

$$\begin{aligned} W_{mn}^{\otimes_{R[J_m]},n} &\cong H^n_c(Y^n, (\pi'_{m,*}R \otimes_R \widetilde{\mathcal{L}}_{\phi_0}|_Y)^{\boxtimes_{R[J_m]},n}) \\ &\cong H^n_c(Y^n, (\pi'_{m,*}R)^{\boxtimes_{R[J_m]},n} \otimes_R (\widetilde{\mathcal{L}}_{\phi_0}|_Y)^{\boxtimes_R,n}) \\ &\cong H^1_c(Y, \pi'_{m,*}R \otimes_R Rs_{n,!}(\widetilde{\mathcal{L}}_{\phi_0}|_Y)^{\boxtimes_R,n})[n-1]). \end{aligned}$$

The n-th symmetric group $\mathfrak{S}_n$ acts on Y^n and the morphism s_n factors through the quotient $\mathrm{Sym}^n Y = Y^n/\mathfrak{S}_n$ of Y^n. Following [De4] we denote by $\Gamma^n_{\mathrm{ext}}(\widetilde{\mathcal{L}}_{\phi_0}|_Y)$ the $\mathfrak{S}_n$-invariant part of the direct image of $(\widetilde{\mathcal{L}}_{\phi_0}|_Y)^{\boxtimes_R,n}$ under the quotient morphism $Y^n \to \mathrm{Sym}^n Y$. Taking actions of the n-th symmetric group $\mathfrak{S}_n$ into account, we have

$$\det{}_{R[J_m]} W_{mn} \cong H^1_c(Y, \pi'_{m,*}R \otimes_R R\widetilde{s}_{n,!}(\Gamma^n_{\mathrm{ext}}(\widetilde{\mathcal{L}}_{\phi_0}|_Y))[n-1]),$$

where $\widetilde{s}_n : \mathrm{Sym}^n Y \to Y$ is the morphism induced by s_n.

Next we will compute $R\widetilde{s}_{n,!}(\Gamma^n_{\mathrm{ext}}(\widetilde{\mathcal{L}}_{\phi_0}|_Y))[n-1])$. The scheme $\mathrm{Sym}^n Y$ is identified with the moduli scheme of monic polynomials of degree n with invertible constant terms. Hence $\mathrm{Sym}^n \mathbb{G}_{m,\overline{k}}$ is identified with $\mathbb{A}^{n-1}_{\overline{k}} \times \mathbb{G}_{m,\overline{k}}$ by associating a polynomial $P(X) = X^n + \sum_i (-1)^i a_i X^{n-i}$ to the point $((a_1, \ldots, a_{n-1}), a_n)$. The morphism $\widetilde{s}_n$ is identified with the second projection $\mathrm{pr}_2 : \mathbb{A}^{n-1}_{\overline{k}} \times \mathbb{G}_{m,\overline{k}} \to \mathbb{G}_{m,\overline{k}}$.

Let $N_n(X_1, \cdots, X_n)$ be the n-th Newton polynomial, that is, the polynomial with $\mathbb{Z}$-coefficients characterized by

$$N_n(a_1, \cdots, a_n) = \alpha_1^n + \cdots + \alpha_n^n \text{ if } \prod_i (X - \alpha_i) = X^n + \sum_i (-1)^i a_i X^{n-i}.$$

Then N_n is of the form

$$N_n(X_1, \cdots, X_n) = (-1)^{n-1} n X_n + Q(X_1, \cdots, X_{n-1}).$$

Let $Q : \mathbb{A}^{n-1}_{\bar{k}} \to \mathbb{A}^1_{\bar{k}}$ be the morphism defined by $Q(X_1, \cdots, X_{n-1})$. We have a canonical isomorphism

$$\Gamma^n_{\text{ext}}(\widetilde{\mathcal{L}}'_{\phi_0}|_Y) \cong Q^* \widetilde{\mathcal{L}}_{\phi_0} \boxtimes_R \widetilde{\mathcal{L}}'_{\phi_{0,(-1)^{n-1}n}},$$

where $\phi_{0,(-1)^{n-1}n} : \mathfrak{m}_K^{-1}/\mathcal{O}_K \to R^\times$ is the composition of ϕ_0 with the multiplication by $(-1)^{n-1}n$. Hence,

$$R\widetilde{s}_{n,!}(\Gamma^n_{\text{ext}}(\widetilde{\mathcal{L}}'_{\phi_0}|_Y)) \cong R\Gamma_c(\mathbb{A}^{n-1}_{\bar{k}}, Q^* \widetilde{\mathcal{L}}_{\phi_0}) \otimes_R \widetilde{\mathcal{L}}'_{\phi_{0,(-1)^{n-1}n}}.$$

We compute the cohomology group
$R\Gamma_c(\mathbb{A}^{n-1}_{\bar{k}}, Q^* \widetilde{\mathcal{L}}_{\phi_0})$. Since $Q(X_1, \cdots, X_n)$ is characterized by

$$\begin{aligned} Q(a_1, \cdots, a_{n-1}) = \alpha_1^n + \cdots + \alpha_{n-1}^n \text{ if } & \prod_{i=1}^{n-1} (X - \alpha_i) \\ &= X^{n-1} + \sum_{i=1}^{n-1} (-1)^i a_i X^{n-1-i}, \end{aligned}$$

we have

$$\begin{aligned} R\Gamma_c(\mathbb{A}^{n-1}_{\bar{k}}, Q^* \widetilde{\mathcal{L}}_{\phi_0}) &= R\Gamma_c(\mathrm{Sym}^{n-1}(\mathbb{A}^1_{\bar{k}}), \Gamma^{n-1}_{\text{ext}} \widetilde{\mathcal{L}}_{\phi_0(x^n)}) \\ &= L\Gamma^{n-1}_{\text{ext}} R\Gamma_c(\mathbb{A}^1_{\bar{k}}, \widetilde{\mathcal{L}}_{\phi_0(x^n)}). \end{aligned}$$

Here $\widetilde{\mathcal{L}}_{\phi_0(x^n)}$ is the pull-back of $\widetilde{\mathcal{L}}_{\phi_0}$ by the morphism $\mathbb{A}^1_{\bar{k}} \to \mathbb{A}^1_{\bar{k}} = \mathrm{Spec}\,(\mathrm{Gr}^{\geq 0} K)$, $x \mapsto x^n$.

By the wildness of $\widetilde{\mathcal{L}}_{\phi_0(x^n)}$ at infinity and the Grothendieck-Ogg-Shafarevich formula, $H^i_c(\mathbb{A}^1_{\bar{k}}, \widetilde{\mathcal{L}}_{\phi_0(x^n)})$ is zero except $i = 1$ and $H^1_c(\mathbb{A}^1_{\bar{k}}, \widetilde{\mathcal{L}}_{\phi_0(x^n)})$ is a free R-module of rank $n - 1$.

LEMMA 11.8. *We have*

$$\det(\mathrm{Fr}_q\ ;\ H^1_c(\mathbb{A}^1_{\overline{k}}, \widetilde{\mathcal{L}}_{\phi_0(x^n)})) = -g_R.$$

PROOF. This follows from the product formula for the global ε-constant of $\widetilde{\mathcal{L}}_{\phi_0(x^n)}$. □

Summing up, we have

$$\begin{aligned}\det{}_{R[J_m]} W_{mn} &\cong H^1_c(Y, \pi'_{m,*}R \otimes_R R\widetilde{s}_{n,!}(\Gamma^n_{\mathrm{ext}}(\widetilde{\mathcal{L}}_{\phi_0}|_Y))[n-1]) \\ &\cong R\Gamma_c(\mathbb{A}^{n-1}_{\overline{k}}, Q^*\widetilde{\mathcal{L}}_{\phi_0})[n-1] \otimes_R H^1_c(Y, \pi'_{m,*}R \otimes_R \widetilde{\mathcal{L}}'_{\phi_{0,(-1)^{n-1}n}}) \\ &\cong H^{n-1}_c(\mathbb{A}^{n-1}_{\overline{k}}, Q^*\widetilde{\mathcal{L}}_{\phi_0}) \otimes_R H^1_c(Y, \pi'_{m,*}R \otimes_R \widetilde{\mathcal{L}}'_{\phi_{0,(-1)^{n-1}n}}) \\ &\cong \det{}_R H^1_c(\mathbb{A}^1_{\overline{k}}, \widetilde{\mathcal{L}}_{\phi_0(x^n)}) \otimes_R H^1_c(Y, \pi'_{m,*}R \otimes_R \widetilde{\mathcal{L}}'_{\phi_{0,(-1)^{n-1}n}}).\end{aligned}$$

Hence the proposition follows. □

This completes the proof of Theorem 5.7 (3).

11.4. A question on an integration formula for ε-constants

In this subsection, we assume that $K = \mathbb{Q}_p$. As a coefficient ring, we take $R = \overline{\mathbb{Q}}_p$ the algebraic closure of $\mathbb{Q}_p$. Here we endow R with discrete topology.

Let $\mathbb{Z}_{(p)} = \mathbb{Z}_p \cap \mathbb{Q}$ be the ring of rational numbers whose denominator is prime to p. Choose a group homomorphism $\varpi : \mathbb{Z}_{(p)} \to R^\times$ such that $\varpi(1) = -p$. Let $\psi_0 : \mathbb{F}_p \to R^\times$ be the non-trivial homomorphism characterized by the following property:

$$\frac{\psi(1)-1}{\varpi(\frac{1}{p-1})} \in 1 + \mathfrak{m}_{\mathbb{Q}_p(\boldsymbol{\mu}_p(R))}.$$

Let $e_{tm,\varpi}$ be the formal sum defined by

$$e_{tm,\varpi} = \sum_{x \in \mathbb{Z}_{(p)}, 0 \le x < 1} \Gamma_p(x)\varpi(x).$$

where $\Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p$ is Morita's p-adic Gamma function;

$$\Gamma_p(x) = \lim_{m\to x,\, m\in\mathbb{Z}_{>0}} (-1)^m \prod_{0<j<m,\, (p,j)=1} j.$$

Let $d\sigma$ be the Haar measure of $W_{\mathbb{Q}_p}$ such that $\int_{W^0_{\mathbb{Q}_p}} d\sigma = 1$. For a locally constant compactly supported R-valued function f on $W_{\mathbb{Q}_p}/W^{0+}_{\mathbb{Q}_p}$, define the integral

$$\int_{W_{\mathbb{Q}_p}} f(\sigma)\sigma(e_{tm,\varpi})d\sigma \in R$$

by the sum

$$\sum_{x\in\mathbb{Z}_{(p)},0\leq x<1} \int_{W_{\mathbb{Q}_p}} f(x)\Gamma_p(x)\varpi(x)d\sigma.$$

Since these summands vanish except for finitely many x, this sum has a well-defined meaning.

PROPOSITION 11.9. *Let $\psi : \mathbb{Q}_p \to R^\times$ be an continuous additive character of $\mathbb{Q}_p$ with* $\operatorname{ord}\psi = -1$ *whose restriction to $\mathbb{Z}_p$ is equal to ψ_0. Then for any tamely ramified object (ρ, V) in* $\operatorname{Rep}(W_{\mathbb{Q}_p}, R)$*, we have*

$$\varepsilon_{0,R}(V,\psi) = \det\left(\int_{W^0_{\mathbb{Q}_p}/(W_{\mathbb{Q}_p})^{0+}} \rho(\sigma)^{-1}\sigma(e_{tm,\varpi})d\sigma\right).$$

PROOF. Because of the additivity, it suffices to prove the proposition when V is of the form $V = \operatorname{Ind}_{W_{K_n}}^{W_{\mathbb{Q}_p}} \chi$, where K_n is the unique unramified extension of $\mathbb{Q}_p$ of degree n, and $\chi \in \operatorname{Rep}(W_{K_n}, R)$ is a rank one tamely ramified object. Then the restriction of $\chi \circ \operatorname{rec}$ on $\mathcal{O}_{K_n}$ defines a multiplicative character $\chi_0 : \mathbb{F}_{p^n} \to R^\times$. We have

$$\varepsilon_0(\rho,\psi) = (-1)^{n-1}\frac{1}{p^n}\sum_{x\in\mathbb{F}^\times_{p^n}} \chi_0(x)^{-1}\psi_0(\operatorname{Tr}_{\mathbb{F}_{p^n}/\mathbb{F}_p}(x)).$$

Let N denote the order of χ_0. Set $\mathbb{F}_{p^d} := \mathbb{F}_p(\boldsymbol{\mu}_N(\mathbb{F}_{p^n}))$. Let $a \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$ be the unique element which makes the following diagram commutative:

$$\begin{tikzcd} \mathbb{F}_{p^n}^\times \arrow[r, "\chi_0"] \arrow[d, "N_{\mathbb{F}_{p^n}/\mathbb{F}_{p^d}}"'] & \boldsymbol{\mu}_N(R) \arrow[d, "\cong"', "can"] \\ \mathbb{F}_{p^d}^\times \arrow[r, "(p^d-1)a"] & \boldsymbol{\mu}_N(\mathbb{F}_{p^d}). \end{tikzcd}$$

Then by Gross-Koblitz formula [GK, p. 571, Thm. 1.7], and Davenport-Hasse formula, we have

$$\begin{aligned} \varepsilon_0(\rho,\psi) &= (-1)^{n-1}\cdot(-1)^{\frac{n}{d}-1}\left(-\frac{1}{p^d}\prod_{j=0}^{d-1}\varpi(\langle p^j a\rangle)\Gamma_p(\langle p^j a\rangle)\right)^{\frac{n}{d}} \\ &= \frac{(-1)^n}{p^n}\prod_{j=0}^{n-1}\varpi(\langle p^j a\rangle)\Gamma_p(\langle p^j a\rangle), \end{aligned}$$

where $\langle\ \rangle$ denote the fractional part. Then the proposition follows by simple calculation. □

Question. Assume that $p \neq 2$. For general $v \in \mathbb{Q}_{\geq 0}$, does there exists a explicitly defined measure $e_{v,\psi}$ on $W_{\mathbb{Q}_p}/(W_{\mathbb{Q}_p})^{v+}$ such that the formula

$$\varepsilon_{0,R}(V,\psi) = \det\left(\int_{W_{\mathbb{Q}_p}/(W_{\mathbb{Q}_p})^{v+}} \rho(\sigma)^{-1}\sigma(e_{v,\psi})d\sigma\right)$$

holds for any object $(\rho, V) \in \mathrm{Rep}(W_{\mathbb{Q}_p}, R)$ which is pure of break v ?

11.5. An auxiliary lemma

The contents of this subsection are preliminary to the proof of Theorem 5.7 (4) given in § 11.6. Let K be a p-local field. Take a prime element π_K of K. For every integer $n \geq 1$, let L_n be the finite separable extension of K given by

$$L_n = K[X]/(X^{p^n} + \pi_K X - \pi_K).$$

Then it is easily checked that the Herbrand function $\psi_{L_n/K} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ of L_n/K is given by

$$\psi_{L/K}(w) = \begin{cases} w, & \text{for } 0 \leq w \leq \frac{1}{p^n-1}, \\ p^n w - 1, & \text{for } w \geq \frac{1}{p^n-1}. \end{cases}$$

LEMMA 11.10. *Let C be a separably closed field of characteristic $\neq p$. Let $G = W_K$ denote the Weil group of K. Let $w \in W_{>0}$. Then, for any non-trivial character $\sigma \in (\mathrm{Hom}(G^w/G^{w+}, C^\times))^G$, there exists an object (ρ, V) in $\mathrm{Rep}(G, C)$ which is pure of refined break $\{\sigma\}$ and that $\mathrm{rank}\, V$ is a power of p.*

PROOF. Take a sufficiently large $n \in \mathbb{Z}_{>0}$ so that $w > \frac{1}{p^n-1}$ and that $p^n w$ is an integer. Then $v = \psi_{L_n/K}(w)$ is an integer.

Let $H = W_{L_n}$ denote the Weil group of L_n. We have a canonical isomorphism $H^v/H^{v+} \cong G^w/G^{w+}$. Let $\sigma' : H^v/H^{v+} \to C^\times$ be the character corresponding to σ.

By the local class field theory, there exists a character $\chi : H \to C^\times$ of H which is pure of refined break $\{\sigma'\}$. We set $V = \mathrm{Ind}_H^G \chi$. It follows from the argument in § 9.5 that V is pure of refined break $\{\sigma\}$. □

COROLLARY 11.11. *Let C be a separably closed field of characteristic $\neq p$. Let $G = W_K$ denote the Weil group of K. Let $w \in W_{>0}$. Then, for any G-orbit Σ in the set of non-trivial characters in $\mathrm{Hom}(G^w/G^{w+}, C^\times)$, there exists an object (ρ, V) in $\mathrm{Rep}(G, C)$ which is pure of refined break Σ and that $\frac{\mathrm{rank}\, V}{\sharp \Sigma}$ is a power of p.*

11.6. Proof of Theorem 5.7 (4)

PROOF. Let L/K be a totally wild finite separable extension. We set $G = W_K$ and $H = W_L$. Let (ρ, V) be a tamely ramified object in $\mathrm{Rep}(H, R)$. Let $W = \mathrm{Ind}_H^G V$. Let W^0 (resp. $W^{>0}$) denote the tamely ramified part (resp. wild part) of W. We prove that

$$\varepsilon_{0,R}(W^{>0}, \psi) \cdot \varepsilon_{0,R}(W^0, \psi) = \varepsilon_{0,R}(V, \psi \circ \mathrm{Tr}_{L/K}) \cdot \lambda_R(L/K, \psi)^{\mathrm{rank}\, V}.$$

We may assume that L/K has no non-trivial intermediate extension. There exists a unique $w_1 \in \mathbb{Q}_{\geq 0}$ such that $\psi_{L/K}(w) = w$ for $0 \leq w \leq w_1$ and that $\psi_{L/K}(w)$ is linear of slope $[L : K]$ for $w > w_1$. By corollary 9.3, W is a direct sum $W = W^0 \oplus W^{w_1}$ of the tamely ramified part W^0 and the break-w_1-part W^{w_1}.

Let us consider W^{w_1}. We use the notation in § 9.4. We have a canonical element $\sigma_\psi(\chi')$ for $\chi' \in C_{w_1}$. We have

$$\det(\mathrm{Ind}_{G^{w_1} H_{\chi'}/G^{w_1+}}^{G_{\chi'}/G^{w_1+}} V_{\chi'})(\mathrm{rec}(\sigma_\psi(\chi')))$$

$$= \det(V_{\chi'})(\mathrm{rec}(\sigma_\psi(\chi'))) \cdot (\mathrm{Ind}_{G^{w_1}H_{\chi'}/G^{w_1+}}^{G_{\chi'}/G^{w_1+}} 1)(\mathrm{rec}(\sigma_\psi(\chi')))^{\mathrm{rank}\, V_{\chi'}}.$$

For each $\Sigma' \in B_{w_0}$, take an element $\chi'_{\Sigma'} \in \Sigma'$. We abbreviate the functor $\mathrm{Ind}_{G^{w_1}H_{\chi'_{\Sigma'}}/G^{w_1+}}^{G_{\chi'_{\Sigma'}}/G^{w_1+}}$ by $\mathrm{Ind}_{\chi'_{\Sigma'}}$ for simplicity. Then we have

$$\begin{aligned}
&\overline{\varepsilon}_{0,R}(W^{w_1}, \psi) \\
&= \prod_{\Sigma' \in B_{w_1} - \{1\}} \det(\mathrm{Ind}_{\chi'_{\Sigma'}} V_{\chi'_{\Sigma'}})(\mathrm{rec}(\sigma_\psi(\chi'_{\Sigma'})))^{-1} \cdot g_R(\chi'_{\Sigma'}, \psi)^{\mathrm{rank}\ \mathrm{Ind}_{\chi'_{\Sigma'}} V_{\chi'_{\Sigma'}}} \\
&= \prod_{\Sigma' \in B_{w_1} - \{1\}} \det(V_{\chi'_{\Sigma'}})(\mathrm{rec}(\sigma_\psi(\chi'_{\Sigma'})))^{-1} \cdot (\mathrm{Ind}_{\chi'_{\Sigma'}} 1)(\mathrm{rec}(\sigma_\psi(\chi'_{\Sigma'})))^{-\mathrm{rank}\, V_{\chi'_{\Sigma'}}} \\
&\qquad \cdot g_R(\chi'_{\Sigma'}, \psi)^{\mathrm{rank}\ \mathrm{Ind}_{\chi'_{\Sigma'}} V_{\chi'_{\Sigma'}}}
\end{aligned}$$

Let $\widetilde{L}$ be the Galois closure of L/K. Then $\mathrm{Gal}(\widetilde{L}/K)^{w_1+} = \{1\}$. Let $\widetilde{K}$ (resp. L') be the subextension of $\widetilde{L}/K$ (resp. $\widetilde{L}/L$) corresponding to $\mathrm{Gal}(\widetilde{L}/K)^{w_1}$ (resp. $\mathrm{Gal}(\widetilde{L}/L)^{w_1}$). Take prime elements $\pi_{L'} \in L'$ and $\pi_{\widetilde{K}} \in \widetilde{K}$ satisfying $\mathrm{N}_{\widetilde{K}/L'}(\pi_{\widetilde{K}}) = \pi_{L'}$. By Proposition 9.9 (3), the map

$$\sigma_{L/K,\psi,w_1} : N_K^{-w_1 - \mathrm{ord}\,\psi - 1} \to N_L^{-w_1 - \mathrm{ord}\,(\psi \circ \mathrm{Tr}_{L/K}) - 1}$$

is of the form

$$a \cdot a_{\psi,\zeta}^{-1} \cdot \widetilde{D}_{\widetilde{K}/K}^{-1} \pi_{\widetilde{K}}^{-w_{1,K}} \mapsto (a_0 \cdot a + \cdots + a^{\frac{1}{[L:K]}}) \cdot a_{\psi,\zeta}^{-1} \cdot \widetilde{D}_{L'/K}^{-1} \pi_{L'}^{-w_{1,L'}},$$

where $a_0 = \widetilde{D}_{L'/\widetilde{K}} \cdot \frac{\pi_{L'}^{w_{1,L'}}}{\pi_K^{w_{1,K}}}$.

Hence by Proposition 3.6,

$$\begin{aligned}
\prod_{x \in N_K^{-w_1 - \mathrm{ord}\,\psi - 1}, x \neq 0, \sigma_{L/K,\psi,w_1}(x) = 0} x &= (a_{\psi,\zeta}^{-1} \cdot \widetilde{D}_{\widetilde{K}/K}^{-1} \pi_{\widetilde{K}}^{-w_{1,K}})^{[L:K]-1} \cdot \frac{1}{a_0^{[L:K]}} \\
&= \frac{(a_{\psi,\zeta}^{-1} \cdot \widetilde{D}_{L'/K}^{-1} \pi_{L'}^{-w_{1,L'}})^{[L:K]}}{a_{\psi,\zeta}^{-1} \cdot \widetilde{D}_{\widetilde{K}/K}^{-1} \pi_{\widetilde{K}}^{-w_{1,K}}} \\
&= a_{\psi,\zeta}^{1-[L:K]} \cdot \widetilde{D}_{L/K}^{-[L:K]}.
\end{aligned}$$

Let K^{w_1} (resp. L^{v_1}) be the Galois extension of K (resp. L) corresponding to H^{v_1} (resp. K^{w_1}). For $\chi' \in C_{w_1}$, let $M_{\chi'}$ be the finite subextension of L^{v_1}/K^{w_1} corresponding to $\operatorname{Ker}\chi'$. Let $L_{\chi'}$ be the finite extension of L corresponding to $H_{\chi'}$, and set $K_{\chi'} = K^{w_1} \cap L_{\chi'}$ and $M'_{\chi'} = M_{\chi'} \cap L_{\chi'}$. Then there is a canonical isomorphism $\mathrm{Gal}(L^{v_1}/L_{\chi'}) \cong \mathrm{Gal}(K^{w_1}/K_{\chi'})$ and $\mathrm{Gal}(M_{\chi'}/K^{w_1}) \cong \mathrm{Gal}(M_{\chi'}/K_{\chi'})$. Let $V'_{\chi'}$ (resp. $\overline{\chi'}$) be the representation of $\mathrm{Gal}(K^{w_1}/K_{\chi'})$ (resp. $\mathrm{Gal}(M'_{\chi'}/K_{\chi'})$) over R corresponding to $\mathrm{Res}^{H}_{H_{\chi'}} V$ (resp. χ') via the above isomorphism. Then $V_{\chi'}$ is canonically isomorphic to $V'_{\chi'} \otimes \overline{\chi'}$.

Consider the following commutative diagram

$$\begin{array}{ccc} L^{\times}_{\chi'}/1+\mathfrak{m}_{L_{\chi'}} & \longrightarrow & K^{\times}_{\chi'}/1+\mathfrak{m}_{K_{\chi'}} \\ \downarrow & & \downarrow \\ L^{\times}/1+\mathfrak{m}_L & \longrightarrow & K^{\times}/1+\mathfrak{m}_K \end{array}$$

where all the arrows are homomorphisms induced by norms. Since L/K and $L_{\chi'}/K_{\chi'}$ are totally wildly ramified extensions, the horizontal maps are isomorphisms. Let $\sigma'_{\psi}(\chi') \in (L^{\times}/1+\mathfrak{m}_L) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}]$ be the unique element satisfying $\mathrm{N}_{L/K}(\sigma'_{\psi}(\chi')) = \mathrm{N}_{K_{\chi'}/K}(\sigma_{\psi}(\chi'))$. Then we have

$$\begin{aligned} \det V_{\chi'}(\mathrm{rec}(\sigma_{\psi}(\chi'))) &= \det(V)(\mathrm{rec}(\sigma'_{\psi}(\chi'))) \cdot \overline{\chi'}(\mathrm{rec}(\sigma_{\psi}(\chi')))^{\mathrm{rank}\, V} \\ &= \det(V)(\mathrm{rec}(\sigma'_{\psi}(\chi'))). \end{aligned}$$

in $R^{\times}/\boldsymbol{\mu}$.

Since

$$\begin{aligned} \prod_{\Sigma' \in B_{w_1}-\{1\}} \sigma'_{\psi}(\chi'_{\Sigma'}) &= \mathrm{N}^{-1}_{L/K}\Big(\prod_{\Sigma' \in B_{w_1}-\{1\}} \mathrm{N}_{K_{\chi'_{\Sigma'}}/K}(\sigma_{\psi}(\chi'_{\Sigma'}))\Big) \\ &= \mathrm{N}^{-1}_{L/K}\Big(\prod_{\chi' \in C_{w_1}-\{1\}} \sigma_{\psi}(\chi')\Big) \\ &= \mathrm{N}^{-1}_{L/K}\big(a_{\psi,\zeta}^{1-[L:K]} \cdot \widetilde{D}_{L/K}^{-[L:K]}\big) \\ &= a_{\psi,\zeta}^{-1+\frac{1}{[L:K]}} \cdot \widetilde{D}_{L/K}^{-1}. \end{aligned}$$

we have

$$\prod_{\Sigma' \in B_{w_1}-\{1\}} \det(V_{\chi'})(\mathrm{rec}(\sigma_{\psi}(\chi'_{\Sigma'})))^{-1} = \det(V)\left(\mathrm{rec}\left(a_{\psi,\zeta}^{-1+\frac{1}{[L:K]}} \cdot \widetilde{D}_{L/K}^{-1}\right)\right)^{-1}.$$

Take an element $\chi \in \Sigma$ and let L_χ be the extension of L corresponding to the stabilizing subgroup H_χ of χ. Let V' be the χ-part of V. Since V is isomorphic to $\mathrm{Ind}_{H_\chi}^H V'$, we have

$$\begin{aligned}
&\prod_{\Sigma' \in B_{w_1} - \{1\}} \det(V_{\chi'})(\mathrm{rec}(\sigma_\psi(\chi'_{\Sigma'})))^{-1} \\
&= \det(V')(\mathrm{rec}_{L_\chi}\left(a_{\psi,\zeta}^{-1+\frac{1}{[L:K]}} \cdot \widetilde{D}_{L/K}^{-1}\right))^{-1} \\
&\quad \cdot (\mathrm{Ind}_{H_\chi}^H 1)(\mathrm{rec}_L\left(a_{\psi,\zeta}^{-1+\frac{1}{[L:K]}} \cdot \widetilde{D}_{L/K}^{-1}\right))^{-1}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\overline{\varepsilon}_{0,R}(W^{w_1}, \psi) \\
&= \det V(\mathrm{rec}(a_{\psi,\zeta}^{-1+\frac{1}{[L:K]}} \cdot \widetilde{D}_{L/K}^{-1}))^{-1} \\
&\quad \cdot \prod_{\Sigma' \in B_{w_1} - \{1\}} g_R(\chi'_{\Sigma'}, \psi)^{\mathrm{rank}\, \mathrm{Ind}_{\chi'_{\Sigma'}} V_{\chi'_{\Sigma'}}}.
\end{aligned}$$

On the other hand, by corollary 10.9, we have

$$\begin{aligned}
&\varepsilon_{0,R}(W^0, \psi, \phi_0) \\
&= \det V(\mathrm{rec}(a_{\psi,\zeta}^{-1+\frac{1}{[L:K]}} \cdot \widetilde{D}_{L/K}^{-1})) q^{(-([L:K]-1)(\mathrm{ord}\, \psi+1) - v_L(\tilde{D}_{L/K}))\cdot \mathrm{rank}\, V} \\
&\quad \cdot \varepsilon_0(V, \psi \circ \mathrm{Tr}_{L/K}, \phi_0 \circ \mathrm{N}_{L/K}).
\end{aligned}$$

Therefore, it suffices to prove that

$$\begin{aligned}
&\prod_{\Sigma' \in B_{w_1} - \{1\}} g_R(\chi'_{\Sigma'}, \psi)^{\mathrm{rank}\, \mathrm{Ind}_{\chi'_{\Sigma'}} V_{\chi'_{\Sigma'}}} \\
&= q^{(([L:K]-1)(\mathrm{ord}\, \psi+1) + v_L(\tilde{D}_{L/K}))\cdot \mathrm{rank}\, V} \cdot \lambda_R(L/K, \psi)^{\mathrm{rank}\, V}.
\end{aligned}$$

By Corollary 11.11, it follows from the similar computation for $R = \mathbb{C}$ case. $\square$

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