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Time Periodic Navier-Stokes Flow with Nonhomogeneous Boundary Condition

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Abstract. It is known that the Navier-Stokes initial boundary value problem for non-homogeneous boundary condition has a unique local solution (e.g., O. A. Ladyzhenskaya[5]). Nevertheless, it seems to the author that there is no results for the periodic problem with non-homogeneous boundary condition satisfying the general outflow condition. We consider the periodic problem for the Navier-Stokes equations in a two dimensional bounded domain. In case of a symmetric domain, we obtain a periodic weak solution for symmetric boundary values satisfying only the general outflow condition.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^2$. The boudary $\partial\Omega$ consists of $N+1$ smooth connected components $\Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_N$, that is, simple closed curves, where $N \geq 1$, Ω being inside of Γ_0. We suppose that Ω is symmetric with respect to the x_2-axis and every Γ_i $(0 \leq i \leq N)$ intersects the x_2-axis. We call this assumption (SYM). Let $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$.

We consider the periodic problem for the Navier-Stokes equations.

$$(1.1) \qquad \begin{cases} u_t &= \nu\Delta u - (u\cdot\nabla)u - \nabla p + f \quad \text{in} \quad Q \\ \operatorname{div} u &= 0 \quad \text{in} \quad Q \\ u &= \beta \quad \text{on} \quad \Sigma \\ u(x,0) &= u(x,T) \quad \text{for} \quad x \in \Omega \end{cases}$$

where the fluid velosity $u = u(x,t)$ and the pressure $p = p(x,t)$ are unknown, the external force $f = f(x,t)$ and the boundary value $\beta = \beta(x,t)$ are given. The function β should satisfy the outflow condition:

$$(1.2) \qquad \int_{\partial\Omega} \beta\cdot n d\sigma = 0$$

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which we call the general outflow condition (GOC). Here n is an outward unit normal vector to $\partial\Omega$. The following condition, which is stronger than (GOC), is called the stringent outflow condition (SOC).

$$\int_{\Gamma_k} \beta \cdot n d\sigma = 0 \quad (\forall k = 0, 1, 2, \cdots, N). \tag{1.3}$$

(GOC) and (SOC) are equivalent if the boundary $\partial\Omega$ has only one connected component.

We suppose that β depends only on x and not on t. Let $b = b(x)$ be a divergence free extension of $\beta = \beta(x)$.

$$\begin{cases} \operatorname{div} b &= 0 \quad \text{in} \quad \Omega \\ \quad b &= \beta \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.4}$$

A result for the β depending on t and x will be in the forthcoming paper T-P. Kobayashi[4].

Notation. Before stating our result, we introduce some function spaces.

$C_0^\infty(\Omega)$ and $L^2(\Omega)$ are as usual. The inner product and the norm of $L^2(\Omega)$ are denoted by $(\cdot, \cdot)$ and $\|\cdot\|$. $H^1(\Omega)$ is a usual Sobolev space.

$C_{0,\sigma}^\infty(\Omega) = \{u \in C_0^\infty(\Omega) \times C_0^\infty(\Omega); \operatorname{div} u = 0 \text{ in } \Omega\}$

$H = H(\Omega)$ is the closure of $C_{0,\sigma}^\infty(\Omega)$ in $L^2(\Omega) \times L^2(\Omega)$ and

$H_\sigma^1(\Omega) = \{u \in H^1(\Omega) \times H^1(\Omega); \operatorname{div} u = 0 \text{ in } \Omega\}$

$V = V(\Omega)$ is the closure of $C_{0,\sigma}^\infty(\Omega)$ in $H^1(\Omega) \times H^1(\Omega)$. Since Ω is bounded, we use the Dirichlet norm $\|\nabla u\|$ for $u \in V$, which is equivalent to the H^1 norm.

V' is the dual space of V.

We use the notation

$$B(u, v, w) = ((u \cdot \nabla)v, w) = \int_\Omega \sum_{i,j} u_i \frac{\partial v_j}{\partial x_i} w_j dx.$$

For a vector function defined in Ω, $\varphi(x) = \varphi(x_1, x_2)$, we put

$$\varphi^s(x_1, x_2) = \frac{1}{2}(\varphi_1(x_1, x_2) - \varphi_1(-x_1, x_2),\ \varphi_2(x_1, x_2) + \varphi_2(-x_1, x_2))$$

$$\varphi^a(x_1, x_2) = \frac{1}{2}(\varphi_1(x_1, x_2) + \varphi_1(-x_1, x_2),\ \varphi_2(x_1, x_2) - \varphi_2(-x_1, x_2)).$$

φ^s is called the symmetric part of φ and φ^a antisymmetric part of φ. It holds

$$\varphi = \varphi^s + \varphi^a.$$

Definition 1.1. A vector valued function $u(x_1, x_2) = (u_1(x_1, x_2), u_2(x_1, x_2))$ defined in Ω is called symmetric with respect to the x_2-axis if $u = u^s$, that is,

$$u_1(-x_1, x_2) = -u_1(x_1, x_2),\ u_2(-x_1, x_2) = u_2(x_1, x_2).$$

holds true. u is called antisymmetric with respect to the x_2-axis if $u = u^a$, that is,

$$u_1(-x_1, x_2) = u_1(x_1, x_2),\ u_2(-x_1, x_2) = -u_2(x_1, x_2).$$

holds true.

$$H^s = H^s(\Omega) = \{u \in H(\Omega); u = u^s\}$$
$$V^s = V^s(\Omega) = \{u \in V(\Omega); u = u^s\}$$

It is to be remarked that the trace to the axis of symmetry of the second component of $u \in V^s(\Omega)$ vanishes, that is, $u(x) = (0, u_2(0, x_2))$ for $x = (0, x_2) \in \Omega$. See Fujita[2] for details.

Our result is as follows.

Theorem 1.1. *Let Ω satisfy the assumption* (SYM), $f \in L^2(0, T; (V^s)')$ *and $\beta = \beta(x)$ be smooth, symmetric and satisfy* (GOC). *Then, there exists u such that $u - b \in L^2(0, T; V^s) \cap L^\infty(0, T; H^s)$ and*

$$(1.5) \quad \begin{cases} < u', \varphi > + \nu(\nabla u, \nabla \varphi) + B(u, u, \varphi) = < f, \varphi > \quad (\forall \varphi \in V^s) \\ u(0) = u(T) \end{cases}$$

hold true. Here b is a solenoidal symmetric extension of β, and $< \cdot, \cdot >$ means the duality between $(V^s)'$ and V^s.

Remark 1.1. For the Navier-Stokes initial-boundary value problem, the solvability is well known. It is due to the possibility to use Gronwall's lemma. See, e.g., O. A. Ladyzhenskaya [5].

However, only partial results are known for the existence of solution to the stationary problem under (GOC). In 1984, Ch.Amick[1] showed the existence of symmetric solution for 2-dimensinal case assuming the symmetry for the domain and the data. In 1997, H.Fujita[2] obtained a Leray type inequality for 2-dimesional symmetric functions and proved the existence of symmetric solutions for the stationary problem.

It is not known that there exists a periodic Navier-Stokes flow for a general domain with the boundary value satisfying only (GOC). If the boundary value satisfies (SOC) or the integrals $|\int_{\Gamma_k} \beta\cdot n d\sigma| (k = 0, 1, \cdots, N)$ are small, the theorem holds. Our result admits the large $|\int_{\Gamma_k} \beta\cdot n d\sigma| (k = 0, 1, \cdots, N)$ with (GOC).

For the case $\beta = 0$ there are many results. See Prodi[9] ($n = 2$), Kaniel-Shinbrot[3] ($n = 3$), Takeshita[11] ($n = 2$). For $n = 2, 3$, Yudovic[12] treated $\beta \neq 0$ with (SOC). Serrin[10] treated the case for $n = 3$ with small Reynolds number. See also Morimoto[8].

2. Symmetric Bases

Let Ω be a 2-dimensional bounded domain, symmetric with respect to the x_2-axis. We consider the weak formulation of the Stokes boundary value problem in Ω. Let $f \in H^s(\Omega)$. Then, by Riesz' theorem, we can show that there exists one and only one $u \in V^s(\Omega)$ satisfying

$$(\nabla u, \nabla v) = (f, v) \quad (\forall v \in V^s(\Omega)).$$

Define the operator $T : H^s(\Omega) \to H^s(\Omega)$ as $Tf = u$. Then T is a bounded linear operator from $H^s(\Omega)$ into $H^s(\Omega)$. T is symmetric, therefore it is selfadjoint. T is also injective. Using Rellich's theorem, we find T is a compact operator defined on $H^s(\Omega)$. By the general theory for compact operator, the non-zero spectrum of T is eigenvalues μ_j and corresponding eigenfunctions f_j are complete in $H^s(\Omega)$. Furthermore, all the eigenvalues are positive: $\mu_j > 0$.

Put $\lambda_j = \mu_j^{-1}$, $w_j = Tf_j$. After normalizing $\{w_j\}_j$ and using the same symbol, we find $\{w_j\}_j$ is a complete ortho-normal system in $H^s(\Omega)$ and $\{w_j/\sqrt{\lambda_j}\}_j$ is a complete ortho-normal system in $V^s(\Omega)$.

3. Preliminaries

Let $\Omega \subset \mathbb{R}^2$.

LEMMA 3.1. *Let $u, v, w \in H^1(\Omega) \times H^1(\Omega)$, $\operatorname{div} u = 0$ and one of the trace of u, v, w to $\partial\Omega$ vanishes. Then*

$$B(u, v, w) = -B(u, w, v).$$

LEMMA 3.2. *The trilinear form B satisfies*

(i) $|B(u, v, u)| \leq \|u\|_4^2 \|\nabla v\| \quad (u \in L^4(\Omega), v \in V)$

(ii) $|B(u, v, w)| \leq C_1 \|\nabla u\| \|\nabla v\| \|\nabla w\| \quad (u, v, w \in V)$

(iii) $|B(u, v, u)| \leq C_2 \|\nabla u\|^2 \|v\|_4 \quad (u \in V, v \in H^1)$

where the constants C_1, C_2 depend on Ω.

LEMMA 3.3. *(Poincaré's inequality)*

$$\|u\| \leq C_3 \|\nabla u\| \ (u \in V)$$

where C_3 is a constant depending on Ω.

These three Lemmas hold true even for $\Omega \subset \mathbb{R}^3$.

LEMMA 3.4. *Let Ω be a bounded domain of $\mathbb{R}^2$. Then there exists an absolute constant c_0 such that*

$$\|v\|_4 \leq c_0 \|\nabla v\|_2^{1/2} \|v\|_2^{1/2} \quad (\forall v \in H_0^1(\Omega)).$$

LEMMA 3.5. *If $v \in L^2(0, T : V) \cap L^\infty(0, T : H)$, then,*

$$(v \cdot \nabla) v \in L^2(0, T : V').$$

Lemma 3.6. *Suppose* $f \in L^2(0,T:V')$ *and* $v \in L^2(0,T:V) \cap L^\infty(0,T:H)$ *and* $u = v + b$ *satisfies* (1.5). *Then* $v' \in L^2(0,T:V')$. *Furthermore,* v *is continuous a.e. in* $[0,T]$ *taking value in* V'.

The next lemma is essential for the proof of our result.

Lemma 3.7 ([2], [7]). *Let* Ω *satisfy* (SYM) *and* β *be a symmetric smooth function defined on* $\partial\Omega$ *satisfying* (GOC). *Then, for every* $\varepsilon > 0$, *there exists a solenoidal symmetric extension* b *of* β *such that*

$$|B(v,v,b)| \le \varepsilon \|\nabla v\|^2 \quad (\forall v \in V^s).$$

Remark 3.1. It is well known that for the general bounded domain in $\mathbb{R}^n(n = 2,3)$, the similar inequality holds for $\forall v \in V$ if β satisfy (SOC).

Remark 3.2. If $u = v + b$ satisfies (1.5), then v satisfies the following.

$$(3.1) \qquad < v', \varphi > + \nu(\nabla v, \nabla\varphi) + B(v,v,\varphi) + B(b,v,\varphi) + B(v,b,\varphi)$$

$$= < f, \varphi > - \nu(\nabla b, \nabla\varphi) - B(b,b,\varphi) \ (\forall \varphi \in V^s)$$

4. Proof of Theorem

Let $\{w_j\}_j$ be as in Section 2, $b = b(x)$ a symmetric solenoidal extension to Ω of the boundary value β obtained in Lemma 3.7. First, we consider the following finite dimensional problem:

Find a solution

$$v_m(t) = \sum_{k=1}^{m} g_{km}(t) w_k$$

to the initial value problem of ordinary differential equation:

$$(4.1) \qquad (v'_m, w_j) + \nu(\nabla v_m, \nabla w_j) + B(v_m, v_m, w_j) + B(v_m, b, w_j)$$

$$+ B(b, v_m, w_j) = < f, w_j > - \nu(\nabla b, \nabla w_j) - B(b,b,w_j) \quad (1 \le j \le m)$$

$$v_m(0) = v_0 \in [w_1, w_2, \cdots, w_m].$$

It is immediate to see that there exists a positive t_m such that a solution $v_m(t)$ exists for $t \in [0, t_m]$. Let us show $t_m = T$. Multiply (4.1) by $g_{jm}(t)$ and sum up with respect to j. Using Lemma 3.1, we find

$$\frac{1}{2}\frac{d}{dt}\|v_m(t)\|^2 + \nu\|\nabla v_m(t)\|^2 + B(v_m, b, v_m) \tag{4.2}$$

$$= < f, v_m > -\nu(\nabla b, \nabla v_m) - B(b, b, v_m).$$

Let $\varepsilon > 0$ arbitrary. By Lemma 3.7, we have

$$|B(v_m, b, v_m)| = | - B(v_m, v_m, b)| \le \varepsilon\|\nabla v_m\|^2.$$

Estimate the right side of (4.2) using Lemma 3.2 and Hölder's inequality and we obtain

$$| < f, v_m > -\nu(\nabla b, \nabla v_m) - B(b, b, v_m)| \le (\|f\|_{V'} + \nu\|\nabla b\|_2 + \|b\|_4^2)\|\nabla v_m\|$$

$$\le \varepsilon\|\nabla v_m\|^2 + C_\varepsilon(\|f\|_{V'}^2 + \nu^2\|\nabla b\|_2^2 + \|b\|_4^4)$$

where the constant C_ε depends only on ε. Choosing $\varepsilon = \nu/2$, we obtain

$$\frac{d}{dt}\|v_m(t)\|^2 + \nu\|\nabla v_m(t)\|^2 \le F(t) \tag{4.3}$$

where

$$F(t) = 2C_\varepsilon(\|f(t)\|_{V'}^2 + \nu^2\|\nabla b\|_2^2 + \|b\|_4^4).$$

$F(t)$ is an integrable function independent of m. Integrating the both sides, we have

$$\|v_m(t)\|^2 + \nu\int_0^t \|\nabla v_m(s)\|^2 ds \tag{4.4}$$

$$\le \|v_0\|^2 + \int_0^t F(s)ds \le \|v_0\|^2 + \int_0^T F(s)ds.$$

The right hand side is a constant independing of m. Therefore, we can take $t_m = T$.

Using Lemma 3.3 for (4.3), we obtain the following inequality with some constant $c_1 > 0$ independent of m:

$$\frac{d}{dt}\|v_m(t)\|^2 + c_1\|v_m(t)\|^2 \le F(t). \tag{4.5}$$

Integration of this inequality yields:

$$\|v_m(t)\|^2 \le \|v_0\|^2 e^{-c_1 t} + e^{-c_1 t}\int_0^t e^{c_1 s}F(s)ds. \tag{4.6}$$

Now, we consider the finite dimensional periodic problem:

$$\begin{gathered}(v_m', w_j) + \nu(\nabla v_m, \nabla w_j) + B(v_m, v_m, w_j) + B(v_m, b, w_j) \\ +B(b, v_m, w_j) = < f, w_j > -\nu(\nabla b, \nabla w_j) - B(b, b, w_j) \quad (1 \le j \le m) \\ v_m(0) = v_m(T).\end{gathered} \tag{4.7}$$

According to the previous investigation, there exists a unique solution $v_m(t)$ for the initial value problem with the initial condition

$$v_m(0) = v_0 \in [w_1, w_2, \cdots, w_m].$$

Define the mapping $\mathcal{T}_m$ as

$$\mathcal{T}_m : [w_1, w_2, \cdots, w_m] \to [w_1, w_2, \cdots, w_m], \quad \mathcal{T}_m v_0 = v_m(T).$$

Then $\mathcal{T}_m$ is a continuous mapping from $[w_1, w_2, \cdots, w_m]$ to $[w_1, w_2, \cdots, w_m]$. Put $B_m(R) = \{u \in [w_1, w_2, \cdots, w_m] : \|u\| \le R\}$.

Now let us show that there exists a positive number R independent of m such that $\mathcal{T}_m(B_m(R)) \subset B_m(R)$. Choose R as

$$R^2 = \frac{e^{-c_1 T}\int_0^T e^{c_1 s}F(s)ds}{1 - e^{-c_1 T}}.$$

Then R is independent of m, and if $\|v_0\| \le R$, we have

$$\|v_0\|^2 + \int_0^T e^{c_1 s}F(s)ds \le R^2 + R^2 e^{c_1 T}(1 - e^{-c_1 T}) = R^2 e^{c_1 T}.$$

Therefore, by (4.6), we obtain

$$\|\mathcal{T}_m v_0\|^2 = \|v_m(T)\|^2 \le e^{-c_1 T}(\|v_0\|^2 + \int_0^T e^{c_1 s}F(s)ds) \le R^2$$

and $\mathcal{T}_m(B_m(R)) \subset B_m(R)$ holds. By Brouwer's fixed point theorem, there exists $v_0 \in [w_1, \cdots, w_m]$ such that $\mathcal{T}_m(v_0) = v_0$. Let v_m be the solution with the initial condition $v_m(0) = v_0$. Then v_m is a periodic solution for (4.7). Note that $\|v_m(0)\| \le R$ for all m. From the estimate (4.4), it follows

$$\{v_m\}_m \text{ is a bounded sequence in } L^\infty(0,T:H^s). \tag{4.8}$$

Let $t = T$ in (4.4). Then we assure

$$\{v_m\}_m \text{ is a bounded sequence in } L^2(0,T:V^s). \tag{4.9}$$

Since $\{w_j\}_j$ are chosen as the eigenfuctions of the Stokes operator, we find, using Lemma 3.4, Lemma 3.5, Lemma 3.6, that

$$\{v'_m\}_m \text{ is a bounded sequence in } L^2(0,T:(V^s)'). \tag{4.10}$$

See J. L. Lions[6] for details. We can choose a subsequence which converges to a suitable solution to the periodic problem (1.5).

5. Uniqueness

Let u_i $(i = 1, 2)$ be solutions to the periodic problem (1.5) for the boundary condition $u = \beta$ and the external force f, that is,

$$u_i - b_i \in L^2(0,T;V^s) \cap L^\infty(0,T;H^s)$$
$$\begin{cases} < u'_i, \varphi > +\nu(\nabla u_i, \nabla\varphi) + B(u_i, u_i, \varphi) =< f, \varphi > \quad (\forall\varphi \in V^s) \\ u_i(0) = u_i(T) \end{cases}$$

where b_i is a solenoidal symmetric extension of β. Put $u = u_1 - u_2$. Then $u \in V^s$ and

$$< u', \varphi > +\nu(\nabla u, \nabla\varphi) + B(u, u_1, \varphi) + B(u_2, u, \varphi) = 0 \ (\varphi \in V^s).$$

Taking $\varphi = u$, we have

$$< u', u > +\nu(\nabla u, \nabla u) + B(u, u_1, u) = 0.$$

By Lemma 3.2 (iii), it holds

$$|B(u, u_1, u)| \le C_2\|\nabla u\|^2\|u_1\|_4,$$

therefore, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + (\nu - C_2\|u_1\|_4)\|\nabla u\|^2 \le 0.$$

Put $\mathcal{U}(t) := \nu - C_2\|u_1\|_4$. If u_1 is so small that $\mathcal{U}(t) > 0$ holds *a.e.* $t \in [0, T]$, then, using Poincaré's inequality, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + C_3^{-2}\mathcal{U}(t)\|u\|^2 \leq 0$$

Integrating this inequality, we obtain the estimate

$$\|u(t)\|^2 \exp\{2C_3^{-2}\int_0^t \mathcal{U}(s)ds\} \leq \|u(0)\|^2 \quad (\forall t \in [0, T]). \tag{5.1}$$

Put $t = T$. Since $u(0) = u(T)$ and $\exp\{2C_3^{-2}\int_0^T \mathcal{U}(s)ds\} > 1$, we have $\|u(0)\| = 0$. Therefore, using again (5.1), we have $u(t) = 0$ for $0 \leq t \leq T$.

Theorem 5.1. *If the periodic solution is small, then it is unique.*

Remark 5.1. We do not know if the small periodic solution exists or not.

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