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Asymptotic Behavior of the Sample Mean of a Function of the Wiener Process and the MacDonald Function

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Abstract. Explicit asymptotic formulas for large time of expectations or distribution functions of exponential functionals of Brownian motion in terms of MacDonald's function are proven.

1. Introduction. Main Results

Many problems in probability and physics concern the asymptotic behavior of the distribution function of the sample mean $T^{-1}\int_0^T g(\xi(t))dt$ for large T, where $\xi(t)$, $t \geq 0$, is a random almost surely (a. s.) continuous process taking values in $X \subset \mathbf{R}$ and g is a continuous function on X, see [29], [37], [30], [7], [38, page 208]. Distributions and means of exponential functionals of Brownian motion and Bessel processes are intensively treated in connection with mathematical problems of financial mathematics, [19], [43]. In many cases exact distributions are evaluated, [33]. Nevertheless, difficult problems remain open of evaluation of the asymptotic behavior of the sample means. As one of few examples of such the solutions, see [21], [22], where asymptotic behaviors of sample means of exponentials of a random walk, with applications to Brownian motion, where evaluated.

In order to work these problems out, a fruitful idea is to pass from the nonlinear functional of ξ to a linear functional of the sojourn time $L_t(\xi, \cdot)$ (for definition see (2.1)), by the relation (2.2). For a wide class of homogeneous Markov processes $\xi(t)$, main tools to evaluate such an asymptotic behavior are based on the Donsker-Varadhan *large deviation*

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principle, [8], which in turn actually is based on the Feynman-Kac formula. The large deviation principle allows one to evaluate the logarithmic (rough) asymptotic behavior of the sample mean, that is, the behavior of $E\exp\{-T\int_X g(x)L_T(\xi,dx)\}$ for large T, which, by (2.2), is equal to $E\exp\{-\int_0^T g(\xi(t))dt\}$. A further development is suggested in [28], where, basing on the theory of Markov semigroups and Dirichlet forms, the exact asymptotic behavior of the above expectations has been evaluated. See also the review [11]. In [12], starting from results of [28], a Laplace type method has been developed to derive the asymptotic behavior of the probability $P\{T^{-1}\int_0^T g(\xi(t))dt < d\}$ for large T. Using this method, the exact asymptotic behavior of the probability has been derived in [12], [13], [14] for the power function $g(x) = |x|^p$, $p > 0$ with $\xi(\cdot)$ the Wiener process, respectively the Brownian bridge, respectively the Ornstein-Uhlenbeck process, the Bessel process. In the present paper we consider another type of g, namely, $g(x) = e^{\theta|x|}$, $\theta > 0$ with $\xi(\cdot)$ the Wiener process $w(t)$, $t \geq 0$. It is necessary to note that d-dimensional case can be also considered. But explicit evaluations require the derivation of the minimal root of the corresponding Schrödinger operator (see below), which is much more complicated in d dimensions.

We use the notations

$$P_c\{w \in (\cdot)\} := P\{w \in (\cdot) \,|\, w(0) = c\},$$

$$P_{c,b}\{w \in (\cdot)\} := P\{w \in (\cdot) \,|\, w(0) = c,\, w(T) = b\},$$

and E_c, $E_{c,b}$ for the corresponding expectations.

In order to formulate our main results, we introduce $K_\nu(x)$, $x > 0$, the modified Bessel function of the second kind, also called the *MacDonald* function, see [9, 7.2.2], [1, 9.6], [44] and Lemma 1 below. It is well known that if ν is purely imaginary, $\nu = i\tau$, $i = \sqrt{-1}$, $\tau \in \mathbb{R}_+$, the function $K_{i\tau}(x)$ takes real values and can be represented as an integral,

$$K_{i\tau}(x) = \int_0^\infty e^{-x\cosh t}\cos(\tau t)dt, \quad x > 0, \quad \tau \in [0,\infty). \tag{1.1}$$

Arguing by analogy with [20, Chapter 7.3], one can show that for any fixed positive $x > 0$, the equation $K'_{i\tau}(x) = 0$ in τ has a countable number of strictly positive isolated roots, where $K'_{i\tau}(x) = \frac{d}{dx}K_{i\tau}(x)$. Further, for any $x \neq 0$, $K'_\nu(x)$ is an entire function of ν, [1, 9.6.1], so that it has no accumulating points of zeros. Using again a line of reasoning from [20, Chapter 7.3], one can see that for any positive θ there exists a minimal positive root of the equation

$$K'_{i\tau}\left(2\sqrt{2}/\theta\right) = 0. \tag{1.2}$$

Denote this root by $\tau_0 = \tau_0(\theta) > 0$. We need also the integral Bessel function,

$$Ki_\nu(x) := \int_x^\infty \frac{K_\nu(t)}{t} dt, \quad x > 0, \tag{1.3}$$

[35, Attachment II, II.15].

THEOREM 1 **(Asymptotic behavior of expectations)**. *For any positive θ and any real c, b, the following two asymptotic relations take place,*

$$E_c\left[\exp\left\{-\int_0^T e^{\theta|w(t)|} dt\right\}\right] \sim \tau_0 q_1 e^{-\frac{1}{8}T\theta^2\tau_0^2}, \tag{1.4}$$

as $T \to \infty$, with

$$q_1 = \frac{Ki_{\tau_0 i}(2\sqrt{2}/\theta)\ K_{i\tau_0}(2\sqrt{2}e^{\theta|c|/2}/\theta)}{K_{i\tau_0}(2\sqrt{2}/\theta)\ \left[\frac{\partial}{\partial\tau}K'_{i\tau}(2\sqrt{2}/\theta)\right]\big|_{\tau=\tau_0}} > 0; \tag{1.5}$$

and

$$E_{c,b}\left[\exp\left\{-\int_0^T e^{\theta|w(t)|} dt\right\}\right] \sim \sqrt{\frac{\pi}{8}}\theta\tau_0 q_2\sqrt{T} e^{-\frac{1}{8}T\theta^2\tau_0^2}, \tag{1.6}$$

as $T \to \infty$, with

$$q_2 = \frac{K_{i\tau_0}(2\sqrt{2}e^{\theta|c|/2}/\theta)\ K_{i\tau_0}(2\sqrt{2}e^{\theta|b|/2}/\theta)}{K_{i\tau_0}(2\sqrt{2}/\theta)\ \left[\frac{\partial}{\partial\tau}K'_{i\tau}(2\sqrt{2}/\theta)\right]\big|_{\tau=\tau_0}} > 0. \tag{1.7}$$

Now we turn to the asymptotic behavior of the respective probabilities, see (1.11,1.13) below. Though there is some analogy with the case of means, the study of the probabilities is more complicated. In the case of means we had the *one-parameter defining condition* (1.2), with respect to τ. In the case of probabilities we have to introduce a *two-parameter defining condition*, with respect to a and p, see below.

PROPOSITION 1. *Let $d > 1$ be a fixed number. Among all the possible solutions (a, p) to the equations set*

$$\begin{aligned} &K'_{ip}(2\sqrt{a}) = 0, \\ &\frac{\partial}{\partial p} K'_{ip}(2\sqrt{a}) = p\frac{p^2 - 4a}{8a^{3/2}d} K_{ip}(2\sqrt{a}), \qquad (1.8) \\ &a > 0, \quad p > 0, \end{aligned}$$

there exist a unique pair (a_0, p_0), which minimize $p^2 - 4ad$. Moreover,

$$p_0^2 - 4a_0 d > 0. \qquad (1.9)$$

Clear, a_0 and p_0 depend of d. Notice that for fixed $a > 0$, there exists a minimal positive root $p = p(a) > 0$ of the first equation of (1.8). We prove this proposition in Section 3.

Set

$$\begin{aligned} \sigma^2 := \frac{2a_0 d}{p_0^2 - 4a_0} \Bigg\{ &\frac{p_0^2}{4} + 1 - \frac{4a_0}{p_0^2} - 2a_0 d + \frac{a_0 d}{p_0}(4a_0 - p_0^2) \frac{\left[\frac{\partial}{\partial p} K_{ip}(2\sqrt{a_0})\right]|_{p=p_0}}{K_{ip_0}(2\sqrt{a_0})} \\ &- \frac{8da_0^{3/2}}{p_0^2} \frac{\left[\frac{\partial^2}{\partial p^2} K'_{ip}(2\sqrt{a_0})\right]|_{p=p_0}}{K_{ip_0}(2\sqrt{a_0})} \Bigg\}. \qquad (1.10) \end{aligned}$$

Notice that by (4.3) and (4.43), $\sigma^2 > 0$. Notice also that a_0, p_0 and σ^2 do not depend of θ.

THEOREM 2 **(Asymptotic behavior of probabilities)**. *For $d > 1$, $\theta > 0$, $b \in \mathbf{R}$ the following asymptotic relations, as $T \to \infty$, take place,*

$$P_0 \left\{ \frac{1}{T} \int_0^T e^{\theta|w(t)|} dt < d \right\} \sim \frac{q_3}{\sqrt{2\pi}\theta\sigma} e^{-\frac{1}{8}T\theta^2(p_0^2 - 4a_0 d)}, \qquad (1.11)$$

with

$$q_3 = \frac{8a_0 d}{p_0^2 - 4a_0 d} \frac{Ki_{ip_0}(2\sqrt{a_0})}{K_{ip_0}(2\sqrt{a_0})} > 0; \tag{1.12}$$

and

$$P_{0,b}\left\{\frac{1}{T}\int_0^T e^{\theta|w(t)|}dt < d\right\} \sim \frac{q_4}{\sigma} e^{-\frac{1}{8}T\theta^2(p_0^2 - 4a_0 d)}, \tag{1.13}$$

with

$$q_4 = \frac{2a_0 d}{p_0^2 - 4a_0 d} \frac{K_{ip_0}(2\sqrt{a_0}e^{\theta|b|/2})}{K_{ip_0}(2\sqrt{a_0})} > 0. \tag{1.14}$$

Notice that, using the approach described below, it is possible to prove an analogue of Theorem 2 for the probabilities conditioned on $w(0) = c$, for an arbitrary c. The corresponding expressions will be more complicated but similar to those of Theorem 2. In particular, an assertion corresponding to Proposition 1 must depend also on θ, compare with Lemma 4 below.

2. The Laplace Method for Sojourn Times. Proof of Theorem 1

2.1. Definitions, notations and necessary facts

Proofs of propositions like Theorems 1 and 2 are based on the substantial and effective theory of large deviations for sojourn times of homogenous Markov processes, see [8], [28], [11], [12]. We give here corresponding results as applied to the Wiener process.

Let $C[0,\infty)$ be the space of continuous real valued functions on $[0,\infty)$. For any $t > 0$, $z \in C[0,\infty)$ and Borel B from $\mathbf{R}$ denote by

$$L_t(z, B) := \frac{1}{t}\int_0^t I_B(z(s))\, ds, \tag{2.1}$$

the sojourn (occupation) time of a fixed trajectory $z = z(\cdot)$ to be in B in the time period $[0,t]$. Here $I_B(\cdot)$ is the indicator function of the set B. Shortly, $L_t(z, B)$ is the (normalized) sojourn time. The following important property takes place, [5, Chapter 1, (4.1)], which we formulate in a simplified form.

PROPOSITION 2. *Let a real valued function g be Borel measurable and integrable on any compact subset of* $\mathbf{R}$*. Then for any $t > 0$ and all $\omega \in \Omega$,*

$$\frac{1}{t}\int_0^t g(w(s))ds = \int_{-\infty}^{\infty} g(x)L_t(w,dx). \tag{2.2}$$

Notice that for $g(x) = I_B(x)$, (2.2) follows immediate from (2.1)

Remark that on the left-hand part of the above identity we see a *non-linear* functional of w whereas on the right we have a *linear* functional of the sojourn time of w.

Let $(L_2, \|\cdot\|)$ be the Hilbert space of real functions on $\mathbf{R}$ with the usual scalar product $\langle u, v\rangle := \int_{-\infty}^{\infty} u(x)v(x)dx$, and set $N(v) := \|v\|^2$. Remind that the generator $\mathcal{A}$ of the Wiener process is defined on the set

$$\mathcal{D}_{\mathcal{A}} := \{ \text{ the set of bounded functions } h \text{ such that } h, h' \text{ are absolutely continuous, and } h'' \text{ is bounded } \} \tag{2.3}$$

and acts as

$$(\mathcal{A}h)(x) = \frac{1}{2}h''(x), \quad h \in \mathcal{D}_{\mathcal{A}}, \tag{2.4}$$

see, for example, [23, Theorem 2.14]. The set $\widetilde{\mathcal{D}}_{\mathcal{A}} := \{h \in \mathcal{D}_{\mathcal{A}} \cap L^2 : h'' \in L^2\}$ is totally dense in L^2.

Properties of the generator $\mathcal{A}$ and the Dirichlet form.

The transition density of the Wiener process is given by

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}}\exp\{-\frac{(x-y)^2}{2t}\}, \quad t \in (0,\infty), \quad x, y \in \mathbf{R}, \tag{2.5}$$

[36], [24]. Evidently,

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}}(1 + o(1)) \quad x, y \in \mathbf{R} \quad t \to \infty. \tag{2.6}$$

We also see that the transition density is a symmetric function with respect to x, y, that is $p(t,x,y) = p(t,y,x)$, in fact, this is a common property of one-dimensional diffusions, [6, Chapter 2, 2.1.4]. Thus the Wiener process is an *m-symmetric Markov process*, [28], [16], [18], [17, page 311], with m the Lebesgue measure on $\mathbf{R}$, therefore the following defining relation takes place,

$$\langle P_t g, h\rangle = \langle g, P_t h\rangle\,, \quad g, h \in C_0(\mathbf{R}),$$

where $P_t h(x) = \int\limits_{-\infty}^{\infty} p(t,x,y)h(y)dy$ is the semigroup operator and $C_0(\mathbf{R})$ is the space of continuous compactly supported functions on $\mathbf{R}$. For m-symmetric Markov processes a fruitful theory of large deviations has been developed, see [28], [17], [11, sect. 5.1].

Proposition 3. *The Wiener generator $\mathcal{A} : \widetilde{\mathcal{D}}_{\mathcal{A}} \to L^2$ is symmetric, that is,*

$$\langle \mathcal{A}u, v\rangle = \langle \mathcal{A}v, u\rangle\,, \quad u, v \in \widetilde{\mathcal{D}}_{\mathcal{A}}. \tag{2.7}$$

It is also negative, that is,

$$\langle \mathcal{A}v, v\rangle \leq 0, \quad v \in \widetilde{\mathcal{D}}_{\mathcal{A}}. \tag{2.8}$$

Though the properties (2.7, 2.8) follow from general results about m-symmetric Markov processes, [16], [17, page 311], they are in fact immediate. Indeed, integrating by part, one has, $-\langle \mathcal{A}v, v\rangle = \frac{1}{2}\int\limits_{-\infty}^{\infty}[v'(x)]^2dx \geq 0.$

So, $-\mathcal{A}$ is symmetric and positive. Following arguments from [16] and taking into account (2.7) and (2.8), let us define the *bilinear Dirichlet form*,

$$\mathcal{E}(u,v) := \left\langle \sqrt{-\mathcal{A}}u, \sqrt{-\mathcal{A}}v \right\rangle, \quad u, v \in \mathcal{D}(\mathcal{E}) = \mathcal{D}(\sqrt{-\mathcal{A}}), \tag{2.9}$$

where $\sqrt{-\mathcal{A}}$ is the square root of $-\mathcal{A}$ and $\mathcal{D}(\sqrt{-\mathcal{A}})$ is its domain [2].

We have,

$$\mathcal{E}(u,v) = -\langle \mathcal{A}u, v\rangle\,, \quad u \in \widetilde{\mathcal{D}}_{\mathcal{A}}, \quad v \in \mathcal{D}(\mathcal{E}), \tag{2.10}$$

[16, Section 1.3].

The Large Deviation Principle and the logarithmic behavior.

We recall here some results from the theory of large deviations as applied to the case of Wiener process. Denote by $(\mathcal{M},\rho)$, the metric space of probability measures on $\mathbf{R}$ where ρ is the Prokhorov metric. Define on $(\mathcal{M},\rho)$ the *action functional* $I(\nu)$, by

$$I(\nu) := \begin{cases} \mathcal{E}(v,v), & \text{if } v \in \mathcal{D}(\mathcal{E}) \text{ and } \nu(dx) = v^2(x)dx, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.11}$$

The following theorem shows that the families of probability measures $[P_c\{L_T(w,\cdot) \in (\cdot)\}]_{T>0}$ and $[P_{c,b}\{L_T(w,\cdot) \in (\cdot)\}]_{T>0}$ satisfy the Large Deviation Principle in the space $(\mathcal{M},\rho)$ with the same action functional $I(\nu)$ and the speed T.

Theorem 3 **(the Large Deviation Principle)**.
(i) For any open $G \subset \mathcal{M}$, and $P = P_c$ or $P = P_{c,b}$, for any c, b,

$$\begin{aligned} -\inf_{\nu\in G} I(\nu) &\le \liminf_{T\to\infty} \frac{1}{T}\log P\{L_T(w,\cdot)\in G\} \\ &\le \limsup_{T\to\infty}\frac{1}{T}\log P\{L_T(w,\cdot)\in \overline{G}\} \le -\inf_{\nu\in\overline{G}} I(\nu), \end{aligned} \tag{2.12}$$

$\overline{G}$ is the closure of G.
(ii) For any weakly continuous functional F on $\mathcal{M}$,

$$\begin{aligned} &\lim_{T\to\infty}\frac{1}{T}\log E_c\{\exp[-TF(L_T(w,\cdot))]\} \\ &= \lim_{T\to\infty}\frac{1}{T}\log E_{c,b}\{\exp[-TF(L_T(w,\cdot))]\} = -\inf_{\nu\in\mathcal{M}}[F(\nu)+I(\nu)]. \end{aligned} \tag{2.13}$$

One can get the proof of the theorem using [8, III] and [42, III], see also [17]. It turns out that, unlike to logarithmic behaviors (2.13), the asymptotic behaviors of the expectations in (2.13) are different. They can be evaluated using results from [28], see Theorem 4 below.

According to the context it will be convenient to work either with functions, in L^2, or with measures, in $\mathcal{M}$. Thus, though the main theorem 3 deals with measures, it is easier to solve extremal problems like (2.13) in the space of functions, where one can use tools from the calculus of variations.

We recall that the action functional $I(\nu)$ is non-negative, semicontinuous from below and has compact level sets, [11, 3.2, Definition 2.1].

Set

$$\widetilde{\mathcal{H}} := \{v \in \widetilde{\mathcal{D}}_{\mathcal{A}} :\ N(v) = 1\}.$$

Notice that if $v \in \widetilde{\mathcal{H}}$, then v^2 is a probability density on $\mathbf{R}$, furthermore, $\lim\limits_{|x|\to\infty} v(x) = 0$.

The complete metric space $(\mathcal{M}, \rho)$ has no linear structure. Taking into account that the Fréchet derivative can be defined, as a rule, in normed linear spaces, [3, 2.2] and following [28, p.534], we introduce one more space of measures.

Let $C_b(\mathbb{R})$ be a Banach space of bounded continuous real functions on $\mathbb{R}$ with the supremum norm, $\|h\|_{C_b(\mathbb{R})} := \sup_{x\in\mathbb{R}} |h(x)|$. take a sequence $\{\phi_n\}_{n=1}^{\infty} \subset C_b(\mathbb{R})$ of squared integrable functions and a sequence of positive numbers $\{a_n\}_{n=1}^{\infty}$ such that the following three conditions be fulfilled:

1. $\phi_n \neq 0$ for all n and $\|\phi_n\|_{C_b(\mathbb{R})} \to 0$ as $n \to \infty$;
2. $\{\|\phi_n\|_{L^2}^{-1}\phi_n\}_{n=1}^{\infty}$ is a complete orthonormal basis in L_2;
3. $\sum_{n=1}^{\infty} a_n \|\phi_n\|_{L^2}^{-2} < \infty$.

Denote by $\mathcal{M}_A$ the normed linear space of signed measured μ on $\mathbb{R}$ of bounded variation and the norm

$$\|\mu\|_A := \left[\sum_{n=1}^{\infty} a_n ||\phi_n||_{L^2}^{-2} \left(\int_{-\infty}^{\infty} \phi_n d\mu\right)^2\right]^{1/2},$$

see [28, p.534] and [27, p.237].

The following result specifies the logarithmic asymptotic relation (2.13).

Theorem 4 **(Laplace asymptotics for a linear functional)**. *Suppose that for a bounded linear* $\Psi : \mathcal{M}_A \to \mathbf{R}$ *the extremal problem*

$$\Psi(u^2 dx) - \langle \mathcal{A}u, u\rangle \to \inf, \quad u \in \widetilde{\mathcal{H}},$$

has a unique solution $u_0 > 0$. *Then, as* $T \to \infty$,

$$E_c[\exp\{-T\Psi(L_T(w,\cdot))\}]$$

$$= \exp\bigl\{-T[\Psi(u_0^2 dx) - \langle \mathcal{A}u_0, u_0\rangle]\bigr\} u_0(c) \int_{-\infty}^{\infty} u_0(x)dx(1+o(1)); \tag{2.14}$$

$$E_{c,b}[\exp\{-T\Psi(L_T(w,\cdot))\}]$$

$$= \exp\bigl\{-T[\Psi(u_0^2 dx) - \langle \mathcal{A}u_0, u_0\rangle]\bigr\} u_0(c)u_0(b)\sqrt{2\pi T}\,(1+o(1)). \tag{2.15}$$

Asymptotic behaviors of the integrals from (2.14), (2.15) are actually evaluated in [28], Theorems 2.19 and 2.20. Remark only that in [28], Markov processes with transition probabilities having densities with respect to some *probability* reference measure on $\mathbf{R}$ (the speed measure). But careful inspection of the corresponding proofs from [28] shows that the results of Theorem 4 are also valid for the Wiener process, though its speed measure is the Lebesgue measure. Furthermore, following arguments from [28, page 543, Proposition (2.1)], [10, Lemma 1] and taking into account (2.10), one can prove that under the conditions of Theorem 4, the extremal problems $\Psi(u^2dx) + \mathcal{E}(u,v) \to \inf, \quad u \in \widetilde{\mathcal{H}}$, and $\Psi(u^2dx) - \langle \mathcal{A}u, u\rangle \to \inf, \quad u \in \widetilde{\mathcal{H}}$, are equivalent.

The relation (2.15) can be derived from an assertion similar to Theorem 2.19, [28], taking into account (2.6).

2.2. Proof of Theorem 1

Remind first that the first Fréchet derivative of a bounded linear functional Λ equals Λ and its second Fréchet derivative equals zero, [3, 32.2]. Thus evaluations with Tailor expansions used in [28] for a general nonlinear functional can be made easier when the functional is linear. This gives a possibility to apply the ideas and tools from [28]. Namely, we apply Theorem 4 to the following linear functional,

$$\Psi_\theta(\mu) := \int_{-\infty}^{\infty} e^{\theta|x|}\mu(dx), \quad \mu \in \mathcal{M}_A,\ \theta > 0. \tag{2.16}$$

Obviously, Ψ_θ is not bounded even on $\mathcal{M} \subset \mathcal{M}_A$. Nevertheless, the couple of objects, the functional Ψ_θ and the random probability measure $L_T(\omega,\cdot)$, has a number of good properties which allow us to use for this couple the assertion of Theorem 4.

For any $T > 0$ the random probability measure $L_T(\omega, \cdot)$ can be considered as a random element taking values in $(\mathcal{M},\rho)$. For fixed $c, b \in \mathbb{R}$, $T > 0$, we define

$$Q_c^T(G) := P_c\{L_T(\omega, \cdot) \in G\}, \quad Q_{c,b}^T(G) := P_{c,b}\{L_T(\omega, \cdot) \in G\},$$

where G is a Borel set in $(\mathcal{M},\rho)$. Define also

$$\mathcal{M}_0 := \{\mu \in \mathcal{M} : \Psi_\theta(\mu) < \infty\}.$$

The following two sentences take place.

Sentence 1. *For any $T > 0$, the random element $L_T(\omega, \cdot)$ takes values a.s. in $\mathcal{M}_0$. It means that for any $c, b \in \mathcal{R}$, $T > 0$,*

$$Q_c^T(\mathcal{M}_0) = Q_{c,b}^T(\mathcal{M}_0) = 1.$$

This sentence follows from Proposition 2. Indeed, setting in (2.2) $g(x) = e^{\theta|x|}$, $x \in \mathbb{R}$ and using a.s. continuity of $w(t)$, we get, that

$$\Psi_\theta(L_T(\omega, \cdot)) = \int_{-\infty}^{\infty} e^{\theta|x|} L_T(\omega, dx) = \frac{1}{T}\int_0^T e^{\theta|w(t)|}dt < \infty \quad \text{a.s.}$$

Using Sentence 1, we get the following,

Sentence 2. *(i) The integral Varadhan Theorem on logarithmic asymptotic behavior takes place for Ψ_θ and both probability measures $P = P_c$ and $P = P_{c,b}$. Namely, the limit relations*

$$\begin{aligned}&\lim_{T\to\infty} \frac{1}{T} \log E\{\exp[-T\Psi_\theta(L_T(w, \cdot))]\} \\ &= \lim_{T\to\infty} \frac{1}{T} \log \int_{\mathcal{M}} \exp[-T\Psi_\theta(\mu)]Q^T(d\mu) = -\inf_{\mu\in\mathcal{M}}[\Psi_\theta(\mu) + I(\mu)]\end{aligned}$$

take place for both $E = E_c$, $Q^T = Q_c^T$ and $E = E_{c,b}$, $Q^T = Q_{c,b}^T$, correspondingly. The action functional $I(\mu)$ is defined in (2.9).

(ii) The assertion of Theorem 4 is valid for Ψ_θ.

The proof of (i) follows the same scheme as the proof of Theorem 2.1 in [41, p.6] with taking into account Sentence 1 and positiveness of $\Psi_\theta(\mu)$, $\mu \in \mathcal{M}$. The proof of (ii) follows the methods of [28] with taking into account Sentence 1 and linearity of Ψ_θ

Now turn to the proof of Theorem 1. For any $\theta > 0$, the set

$$\mathcal{D}_\theta := \{ \text{ the set of functions } h \in L^2 \\ \text{such that } \; x \Rightarrow e^{\theta|x|}h(x) \text{ is in } L^2\} \tag{2.17}$$

is totally dense in L^2. Introduce a quadratic functional, associated with Ψ_θ,

$$U(v) := \int_{-\infty}^{\infty} e^{\theta|x|}v^2(x)dx \equiv \left\langle e^{\theta|x|}v, v\right\rangle, \quad v \in \mathcal{D}_\theta. \tag{2.18}$$

Clearly, we have,

$$\Psi_\theta(\mu) = U(v), \quad \text{w}ith \quad \mu(dx) = v^2(x)dx.$$

The set $\mathcal{D} := \mathcal{D}_\theta \cap \widetilde{\mathcal{D}}_\mathcal{A}$, is also totally dense in L^2; the set

$$\mathcal{H} := \{v \in \mathcal{D} : \, N(v) = 1\}, \tag{2.19}$$

is totally dense in $\widetilde{\mathcal{H}}$. By virtue of Theorem 4 and Sentence 2, in order to prove (1.4) and (1.6), one has to solve the following extremal problem for a non-negative quadratic functional,

$$U(v) - \langle \mathcal{A}v, v\rangle \to \inf, \quad v \in \mathcal{H}, \quad v > 0. \tag{2.20}$$

Auxiliary facts.

First we need some facts about modified Bessel functions of the first and third kinds. It is known that the modified Bessel equation with a complex ν,

$$x^2z''(x) + xz'(x) - (x^2 + \nu^2)z(x) = 0, \quad x \in \mathbf{R}, \tag{2.21}$$

has two linearly independent solutions, which are called modified Bessel functions of the first and third kinds, $I_\nu(x)$, $K_\nu(x)$, [9, 7.2.2]. We recall some properties of these functions, which are useful for our discussions.

LEMMA 1. *Let ν be a complex number. Then*

$$K_\nu(x) = e^{-x}\sqrt{\frac{\pi}{2x}}(1+o(1)), \qquad K'_\nu(x) = -e^{-x}\sqrt{\frac{\pi}{2x}}(1+o(1)),$$

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}}(1+o(1)), \quad x \to \infty; \tag{2.22}$$

$$\int x\, K_\nu^2(x)dx = \frac{1}{2}(x^2+\nu^2)K_\nu^2(x) - \frac{x^2}{2}\,[K'_\nu(x)]^2,$$

$$\int_x^\infty t\, K_\nu^2(t)dt = \frac{x^2}{2}\,[K'_\nu(x)]^2 - \frac{1}{2}(x^2+\nu^2)K_\nu^2(x), \quad x>0; \tag{2.23}$$

$$\int \frac{K_\nu^2(x)}{x}dx = \frac{x}{2\nu}\left[K_\nu(x)\frac{\partial}{\partial\nu}K'_\nu(x) - K'_\nu(x)\frac{\partial}{\partial\nu}K_\nu(x)\right],$$

$$\int_x^\infty \frac{K_\nu^2(t)}{t}dt = \frac{x}{2\nu}\left[K'_\nu(x)\frac{\partial}{\partial\nu}K_\nu(x) - K_\nu(x)\frac{\partial}{\partial\nu}K'_\nu(x)\right], \quad x>0. \tag{2.24}$$

PROOF. The relations (2.22) can be find in [1, 9.7]. The first equality from (2.23) is in [9, 7.14, Formula(13)], the second one follows from the first one and relation (2.22), see also [34, 1.12.3(2)].

The second equality in(2.24) is derived from the formula 1.12.3(3), [34], by letting $\mu \to \nu$. The first one follows from the second and (2.22), see also (12) in [9, 7.14]. So the Lemma follows. □

Now consider the Schrödinger operator $\mathcal{B} \equiv \mathcal{B}_\theta : L^2 \to L^2$, related with the extremal problem (2.20). It is defined on the set $\mathcal{D}$ ([31, 24.5]) by

$$\begin{cases} \mathcal{B}y(x) = -\frac{1}{2}y''(x) + e^{\theta|x|}y(x), \quad x \in \mathbf{R}, \\ y(-\infty) = y(\infty) = 0. \end{cases} \tag{2.25}$$

From (2.4) and (2.18) it follows that

$$U(v) - \langle \mathcal{A}v, v\rangle \equiv \langle \mathcal{B}v, v\rangle\,, \quad v \in \mathcal{D}. \tag{2.26}$$

LEMMA 2. *The Schrödinger operator $\mathcal{B}$ is self-adjoint positive, its spectrum is purely discrete and simple, that is, the equation $\mathcal{B}v = \beta v$ has a* *countable number of solutions* (v_n, β_n)*,* $n = 0, 1, 2, \ldots,$ *where*

$$0 < \beta_0 < \beta_1 < \cdots < \beta_n < \ldots$$

are eigenvalues and $\{v_n\}$ *are corresponding normalized eigenfunctions,* $\|v_n\| = 1$*.*

PROOF. The positivity is immediate, indeed, integrating by parts, we have

$$\langle \mathcal{B}y, y\rangle = \frac{1}{2}\int\limits_{-\infty}^{\infty}[y'(x)]^2 dx + \int\limits_{-\infty}^{\infty} e^{\theta|x|}y^2(x)dx > 0, \quad 0 \neq y \in \mathcal{D}.$$

The potential $e^{\theta|x|}$ is positive hence is in particular bounded from below and tends to infinity as $|x| \to \infty$, therefore the assertions about self-adjointness and the spectrum follows from well-known general facts, see for example [31, 24.6b], [27, IX.2a], taking into account the positiveness of $\mathcal{B}$. The Lemma is proven. □

Consider $\beta_0 > 0$, the minimal eigenvalue of $\mathcal{B}$ and $v_0(t) > 0$, the corresponding normalized eigenfunction. — In other words, v_0 *is the unique solution to the following boundary problem for the Schrödinger equation,*

$$\begin{cases} \frac{1}{2}v''(x) - e^{\theta|x|}v(x) = -\beta_0 v(x), \quad x \in \mathbf{R}, \\ v(-\infty) = v(\infty) = 0, \\ \int\limits_{-\infty}^{\infty} v^2(x)dx = 1. \end{cases} \tag{2.27}$$

Solution to the extremal problem (2.20).

For τ_0 defined in (1.2), let us set,

$$0 < C_0 := \frac{\sqrt{\theta}}{2}\Big(\int\limits_{2\sqrt{2}/\theta}^{\infty} \frac{K^2_{i\tau_0}(t)}{t}dt\Big)^{-1/2}$$

$$= \frac{\sqrt{\theta}}{2}\Big(\frac{K_{i\tau_0}(2\sqrt{2}/\theta)}{\tau_0}\Big[\frac{\partial}{\partial\tau}K'_{i\tau}(2\sqrt{2}/\theta)\Big]\Big|_{\tau=\tau_0}\Big)^{-1/2}, \tag{2.28}$$

the latter equality is taking place by virtue of (1.2) and (2.24) with $\nu = i\tau_0$.

LEMMA 3. *(i) We have*

$$\beta_0 = \frac{\theta^2 \tau_0^2}{8} \tag{2.29}$$

and

$$v_0(x) = C_0 K_{i\tau_0}\Big(\frac{2\sqrt{2}}{\theta}\exp\Big\{\frac{\theta|x|}{2}\Big\}\Big), \quad x \in \mathbf{R}. \tag{2.30}$$

(ii) The function v_0 is the unique solution to the extremal problem (2.20). Furthermore,

$$\inf[U(v) - \langle \mathcal{A}v, v\rangle : v \in \mathcal{H},\, v > 0] = U(v_0) - \langle \mathcal{A}v_0, v_0\rangle = \frac{\theta^2\tau_0^2}{8}. \tag{2.31}$$

PROOF. Applying Lagrange's method of multipliers [3, 3.2.1], found all local minimums correspond to the problem (2.20), and then select the global minimum. The Lagrange function is

$$\mathcal{L}(v) = \lambda_1[U(v) - \langle \mathcal{A}v, v\rangle] + \lambda_2[\langle v, v\rangle - 1], \quad v \in \mathcal{D}, \tag{2.32}$$

where

$$\lambda_1 \ge 0, \quad \lambda_2 \in \mathbf{R}, \quad \lambda_1^2 + \lambda_2^2 > 0, \tag{2.33}$$

are the Lagrange multipliers. Differentiating, we get that in order to have a point v as the minimum in (2.20), it is necessary to satisfy the stationary condition,

$$\mathcal{L}'(v)(x) \equiv 2\lambda_1[e^{\theta|x|}v(x) - \mathcal{A}v(x)] + 2\lambda_2 v(x) = 0, \quad v \in \mathcal{D},\ x \in \mathbf{R}. \tag{2.34}$$

and the complementary slackness condition,

$$\lambda_2[\langle v, v\rangle - 1] = 0. \tag{2.35}$$

It is easy to verify that for our problem $\lambda_1 > 0$. Indeed, suppose that $\lambda_1 = 0$. Then, in virtue of the last inequality in (2.33), $\lambda_2 \ne 0$, and relation (2.34) can be written as $2\lambda_2 v = 0$, which contradicts (2.35). Thus $\lambda_1 > 0$. Let us set $\beta = -\lambda_2/\lambda_1 \in \mathbf{R}$. Investigating relations (2.32)–(2.35), one can see that for a function v, yielding a local extremum in (2.20), there exists $\beta = \beta(v)$ such that the couple (v, β) satisfies the following boundary value problem,

$$\begin{cases} \frac{1}{2}v''(x) - e^{\theta|x|}v(x) = -\beta v(x), \quad x \in \mathbf{R}, \\ v(-\infty) = v(\infty) = 0, \\ \int\limits_{-\infty}^{\infty} v^2(x)dx = 1. \end{cases} \tag{2.36}$$

In accordance with the definition (2.25) and Lemma 2, the problem (2.36) has a countable number of solutions (v_n, β_n), $n = 0, 1, 2, \ldots$. Therefore, taking into account (2.26), we get

$$U(v_n) - \langle \mathcal{A}v_n, v_n\rangle = \langle \mathcal{B}v_n, v_n\rangle = \beta_n.$$

From here we conclude that the minimum in (2.20) is achieved in the only point v_0, and

$$\inf[U(v) - \langle \mathcal{A}v, v\rangle : v \in \mathcal{H},\, v > 0] = \langle \mathcal{B}v_0, v_0\rangle = \beta_0 > 0. \tag{2.37}$$

Notice that the above argument is completely similar to the variational principle in the problem of eigenvalues on an interval, [25, Part 2, 9.4].

Now we solve the boundary problem (2.36) with $\beta = \beta_0$ and prove both assertions of the lemma. Consider the differential equation

$$v''(x) + (2\beta_0 - 2e^{\theta|x|})v(x) = 0, \quad x \in \mathbf{R}, \tag{2.38}$$

with the normalization condition

$$\int_{-\infty}^{\infty} v^2(x)dx = 1. \tag{2.39}$$

It is easily seen that, since $v_0(x)$ is a solution to (2.38) for $x \geq 0$, the function $v_0(-x)$ satisfies (2.38) for $x \leq 0$. Therefore we solve first the differential equation

$$v''(x) + (2\beta_0 - 2e^{\theta x})v(x) = 0, \quad x \in [0, \infty). \tag{2.40}$$

Its solutions can be expressed in terms of Bessel functions, see for example 2.1.3 (2) in [33], or formula (23) with $a = 0$ in [25, Part 3, Chapter 2, 2.162].

Let us make in (2.40) the change of variables, $t = \frac{2\sqrt{2}}{\theta}e^{\theta x/2}$, $t \in [2\sqrt{2}/\theta, \infty)$, $z(t) = v(x)$. The equation (2.40) takes the form

$$t^2 z''(t) + tz'(t) - (t^2 + \nu^2)z(t) = 0, \quad t \in [2\sqrt{2}/\theta, \infty), \tag{2.41}$$

where $\nu := 2\sqrt{2\beta_0}\theta^{-1}i$. The equation (2.41) is the modified Bessel equation, (2.21). Taking into account (2.22), we conclude that the desired positive solution which, by (2.39), satisfies the condition $z(\infty) = 0$, is the Macdonald's function $z(t) = K_\nu(t)$, $t \in [2\sqrt{2}/\theta, \infty)$.

Thus, the differential equation (2.40) with the condition (2.39) has the unique positive solution

$$v_0(x) = C_0 K_{i\,2\sqrt{2\beta_0}/\theta}(2\sqrt{2}e^{\theta x/2}/\theta) \quad x \geq 0. \tag{2.42}$$

Let us show, that (2.28) and (2.29) hold. Let

$$v_0(x) = v_0(-x), \quad x \leq 0. \tag{2.43}$$

Obviously, $v_0'(x) = -v_0'(-x)$, $x \in \mathbf{R}$. Since the extremal v_0 is twice differentiable at zero, we have $v_0'(0+) = v_0'(0-) = 0$. From here, differentiating (2.42) and taking into account (1.2), one gets (2.29). (2.28) follows from the normalization condition (2.39). Thus, (2.30), (2.31) and therefore Lemma 3 follow.

Proofs of (1.4) and (1.6).

The relation (1.4) follows from (2.14) of Theorem 4, Lemma 3 and Definition (1.3). The relation (1.6) follows from (2.15) of Theorem 4 and Lemma 3. □

3. Beginning of Proof of Theorem 2: Logarithmic Behavior, Computation of an Extremal, Proof of Proposition 1

3.1. Scale properties and logarithmic behavior

The subject of Theorem 2 is the Wiener process starting at zero, therefore the functional under consideration has a scale property, so that one can dispense with the scale parameter θ^2. From the scale invariance (self-similarity) of the Wiener process, [23, 2.1], we get the following immediate Lemma.

LEMMA 4. *For any $\theta > 0$, $d > 1$, $T > 0$, $b \in \mathbf{R}$,*

$$P_0\Big\{\frac{1}{T}\int_0^T e^{\theta|w(t)|}dt < d\Big\} = P_0\Big\{\frac{1}{\theta^2 T}\int_0^{\theta^2 T} e^{|w(t)|}dt < d\Big\}, \tag{3.1}$$

$$P_{0,b}\Big\{\frac{1}{T}\int_0^T e^{\theta|w(t)|}dt < d\Big\} = P_0\Big\{\frac{1}{\theta^2 T}\int_0^{\theta^2 T} e^{|w(t)|}dt < d \,|\, w(\theta^2 T) = \theta b\Big\}. \tag{3.2}$$

REMARK 1. Lemma 4 shows that it is sufficient to prove Theorem 2 for $\theta = 1$ and the three parameters d, $T_1 = \theta^2 T \to \infty$ and $b_1 = \theta b$. Thus in what follows we take in (2.16–2.19) $\theta = 1$.

LEMMA 5 **(Logarithmic behavior of sojourn time)**. *For $d > 1$ and $b \in \mathbf{R}$,*

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\log P_0\{&\int_{-\infty}^{\infty} e^{|x|}L_T(w,dx) < d\} \\ &= \lim_{T\to\infty}\frac{1}{T}\log P_{0,b}\{\int_{-\infty}^{\infty} e^{|x|}L_T(w,dx) < d\} = \\ &= -\inf\{I(\nu):\, \Psi_1(\nu) < d\} = -\inf\{I(\nu):\, \Psi_1(\nu) \le d\}.\end{aligned} \tag{3.3}$$

PROOF. By Definition,

$$\begin{aligned}P_0\{\int_{-\infty}^{\infty} e^{|x|}L_T(w,dx) < d\} &= P_0\{\Psi_1(L_T(w)) < d\}, \\ P_{0,b}\{\int_{-\infty}^{\infty} e^{|x|}L_T(w,dx) < d\} &= P_{0,b}\{\Psi_1(L_T(w)) < d\}.\end{aligned} \tag{3.4}$$

Formulas (3.3) are derived using (3.4), the linear property of Ψ_1 and Theorem 3, by standard arguments of the Large Deviation Theory (compare with Remark 2.2, Theorem 2.2, Theorem 2.5 and Corollary 2.1 from [11]. Remark that the relations (3.3) correspond to the case $\inf_{\nu\in G} I(\nu) = \inf_{\nu\in\overline{G}} I(\nu)$, $G = \{\nu:\, \Psi_1(\nu) < d\}$ in (2.12). □

Thus, in accordance with (3.3), to apply the Laplace method to the probabilities under consideration, one has to solve the extremal problem:

$$I(\nu) \to \inf, \quad \Psi_1(\nu) \leq d. \tag{3.5}$$

3.2. Solution to the extremal problem (3.5)

LEMMA 6. *The extremal problem (3.5) is equivalent to the following extremal problem,*

$$-\langle \mathcal{A}v, v\rangle \to \inf, \quad v \in \mathcal{H}, \quad U_1(v) \leq d, \quad v \geq 0. \tag{3.6}$$

PROOF. From (2.16 – 2.19) with $\theta = 1$, one sees that the problem (3.5) is equivalent to

$$\mathcal{E}(v,v) \to \inf, \quad v \in \mathcal{D}(\mathcal{E}), \quad N(v) = 1, \quad U_1(v) \leq d. \tag{3.7}$$

For the Dirichlet form $\mathcal{E}(u,v)$, defined by(2.9) one has, [28, page 543],

$$v \in \mathcal{D}(\mathcal{E}), \quad |v| \in \mathcal{D}(\mathcal{E}) \quad \mathcal{E}(v,v) \geq \mathcal{E}(|v|,|v|). \tag{3.8}$$

Therefore, by (3.8), the extremal problem (3.7) is equivalent to

$$\mathcal{E}(v,v) \to \inf, \quad v \in \mathcal{D}(\mathcal{E}), \quad N(v) = 1, \quad U_1(v) \leq d, \quad v \geq 0. \tag{3.9}$$

By Weierstrass' First Theorem, [40, Theorem 9.2], the minimum in (3.9) can be achieved. Using arguments from [28, Proposition (2.1)] and [10, Lemma 1], it is possible to show that the points of minimum in (3.9) belong to the domain $\mathcal{D}_{\mathcal{A}}$ of the generator $\mathcal{A}$. From here, taking into account (2.10), it follows that the problems (3.6) and (3.9) are equivalent, therefore (3.5) and (3.6) are also equivalent. The lemma follows. □

A solution of the extremal problem (3.6) is tightly connected with Proposition 1. In the following proof of this proposition we need the following assertion of variational nature.

PROPOSITION 4. *For a fixed $d > 1$ the following assertions hold:*

(i) The extremal problem (3.6) has a unique solution $u_0(x) \equiv u_0(x,d)$, $x \in \mathbf{R}$, moreover, $u_0(x)$ is bounded and positive.

(ii) There is only a pair (λ_0, γ_0) *(the Lagrange multipliers) such that* $u_0(x)$ *is the unique positive solution to the Schrödinger boundary problem with symmetric potential* $e^{|x|}$,

$$\left\{\begin{array}{l}\frac{1}{2}u''(x) - \lambda_0 e^{|x|}u(x) + \gamma_0 u(x) = 0, \quad x \in \mathbf{R}, \\ u(-\infty) = u(\infty) = 0, \\ \int\limits_{-\infty}^{\infty} u^2(x)dx = 1, \\ \int\limits_{-\infty}^{\infty} e^{|x|}u^2(x)dx = d. \end{array}\right. \tag{3.10}$$

Moreover,

$$\begin{aligned}&\inf\{-\langle \mathcal{A}v, v\rangle : v \in \mathcal{H},\ U_1(v) \le d,\ v \ge 0\} \\ &\qquad = -\langle \mathcal{A}u_0, u_0\rangle = \gamma_0 - d\lambda_0 > 0.\end{aligned} \tag{3.11}$$

The boundedness of the integrals in (3.10) follows from the Liouville-Green approximation, [32, Chapter 6, Theorem 2.1]. Lemma 7 below gives an explicit solution to (3.10).

3.3. Proof of Proposition 4

We solve the problem (3.6) using Lagrange multipliers and Kuhn-Tucker's theorem, [3, 1.3.3, 3.2.1]. The Lagrange function is given by

$$\begin{aligned}\mathcal{L}(v) = -\lambda_1 \langle \mathcal{A}v, v\rangle + \lambda_2[U_1(v) - d] + \lambda_3[\langle v, v\rangle - 1], \\ v \in \mathcal{D}, \quad v \ge 0,\end{aligned} \tag{3.12}$$

where

$$\lambda_1 \ge 0, \quad \lambda_2 \ge 0, \quad \lambda_3 \in \mathbf{R}, \quad \lambda_1^2 + \lambda_2^2 + \lambda_3^2 > 0. \tag{3.13}$$

Differentiating the quadratic functionals we conclude that since v is a minimum in the problem (3.6) the stationary condition

$$\mathcal{L}'(v) \equiv -2\lambda_1 \mathcal{A}v + 2\lambda_2 e^{|x|}v + 2\lambda_3 v = 0, \quad v \in \mathcal{D}, \quad v \ge 0, \tag{3.14}$$

and the complementary slackness condition,

$$\lambda_2[U_1(v) - d] = 0, \quad \lambda_3[\langle v, v\rangle - 1] = 0 \tag{3.15}$$

are fulfilled. It is easy to verify that in our case $\lambda_1 > 0$. Let us set, $\lambda = \lambda_2/\lambda_1 > 0$, $\gamma = -\lambda_3/\lambda_1 \in \mathbf{R}$. On the other hand, from (3.12–3.15), it follows that a minimum point v for the problem (3.6) is a boundary point, moreover

$$\begin{cases} -\mathcal{A}v(x) + \lambda e^{|x|}v(x) - \gamma v(x) = 0, \quad x \in \mathbf{R}, \\ v(-\infty) = v(\infty) = 0, \\ \int\limits_{-\infty}^{\infty} v^2(x)dx = 1, \\ \int\limits_{-\infty}^{\infty} e^{|x|}v^2(x)dx = d. \end{cases} \tag{3.16}$$

Using this and following the proof of the Kuhn-Tucker theorem [3, 1.3.3], we conclude that there exists only a pair of positive numbers (λ_0, γ_0) such that

$$\begin{aligned} &\min\{-\langle \mathcal{A}z, z\rangle : z \in \mathcal{H},\ U_1(z) \le d,\ z \ge 0\} = \\ &\min\{-\langle \mathcal{A}z, z\rangle + \lambda_0[U_1(z) - d] + \gamma_0[\langle z, z\rangle - 1] :\ z \in \mathcal{D},\ z \ge 0\}. \end{aligned} \tag{3.17}$$

For a minimum point v on the right-hand past of (3.17) the relations (3.16) are fulfilled for $\lambda = \lambda_0$, $\gamma = \gamma_0$, that is, relations (3.10).

To conclude the proof of Proposition 4 the following Lemma is going to be proven.

Lemma 7. *The function u_0 from the formulation of Proposition 4 is the only solution to the boundary problem (3.10).*

First we prove an auxiliary assertion. Consider a linear differential Schrödinger operator $S : L^2 \to L^2$, acting by the rule,

$$(Su)(x) = -\frac{1}{2}u''(x) + \lambda_0 e^{|x|}u(x), \quad x \in \mathbf{R}, \tag{3.18}$$

with domain $\mathcal{D}_S := \{u \in L^2 :\ Su \in L^2\}$ which is totally dense in L^2

The operator S is close to $\mathcal{B}_1$, defined by (2.25) with $\theta = 1$. Since $\lambda_0 > 0$, the following analogue of Lemma 2 is valid for S.

Lemma 8. *The Schrödinger operator S is self-adjoint, positive and has purely discrete spectrum, that is the equation $Su = \mu u$ has a countable* *number of solutions* (u_n, μ_n)*,* $n = 0, 1, 2, \dots,$ *where*

$$0 < \mu_0 < \mu_1 < \cdots < \mu_n < \dots$$

are eigenvalues and $\{u_n\}$ *are corresponding normalized eigenfunctions,* $\|u_n\| = 1$*.*

PROOF OF LEMMA 7. The differential equation of the problem (3.10) can be now written in terms of the operator S as

$$Su = \gamma_0 u, \quad u \in \mathcal{D}_S. \tag{3.19}$$

By Lemma 8, for fixed $\lambda_0 > 0$ and $\gamma_0 \in \mathbf{R}$ the equation (3.19) has at most one solution and the solution exists only if $\gamma_0 > 0$. It has already proven that the extremal problem (3.6) does have a solution, therefore equation (3.19) has exactly one solution, namely, the function $u_0 \in L^2$ defined in Lemma 8. At that $\gamma_0 = \mu_0$ and u_0 satisfies (3.10).

From the above it follows that u_0 is positive (see (3.8)) and it is the only solution of the boundary problem (3.10). Thus Lemma 7 follows. □

Now we can conclude the proof of Proposition 4. We first prove the second equality in (3.11). By the already proved part of Proposition 4 we have for the extremal $u_0(x)$,

$$-\frac{1}{2}u_0''(x) = \gamma_0 u_0(x) - \lambda_0 e^{|x|} u_0(x), \quad x \in \mathbf{R}, \tag{3.20}$$

$$\int_{-\infty}^{\infty} u_0^2(x)dx = 1, \quad \int_{-\infty}^{\infty} e^{|x|} u_0^2(x)dx = d. \tag{3.21}$$

Multiplying (3.20) on $u_0(x)$ and integrating both sides from $-\infty$ to ∞ we get, taking into account (2.4) and (3.21):

$$-\langle \mathcal{A}u_0, u_0 \rangle = \gamma_0 \langle u_0, u_0 \rangle - \lambda_0 \left\langle e^{|x|} u_0, u_0 \right\rangle = \gamma_0 - d\lambda_0 \geq 0. \tag{3.22}$$

The last equality follows from the negativeness of $\mathcal{A}$, see (2.8). Thus we know that $\gamma_0 \geq d\lambda_0 > 0$.

Now we shall prove the strict inequality $-\langle \mathcal{A}u_0, u_0\rangle > 0$. It is easy to see that the generator $\mathcal{A}$ is injective in the domain $\mathcal{D}_\mathcal{A} \cap L^2$, it means that for any v from the domain, we have $\mathcal{A}v = 0$ if and only if $v = 0$. Thus, by virtue of Proposition 3, the operator $-\mathcal{A}$ with the domain $\mathcal{D}$ is symmetric positive and injective. From here, by Proposition 1.3 from [25, Chapter 3], it follows that the quadratic form $\langle \mathcal{A}u, u\rangle$ equals zero then and only then, when $u = 0$. Thus $-\langle \mathcal{A}u_0, u_0\rangle > 0$ and (3.11) and therefore Proposition 4 follow. □

3.4. Proof of Proposition 1

The boundary problem (3.10) similarly as for the problem (2.27), allows an explicit solution represented by the Macdonald's function $K_{i\tau}(x)$ of purely imaginary order.

Proof of Proposition 1. We shall find the solution of the boundary problem (3.10). The differential equation from (3.10)

$$u''(x) + (2\gamma - 2\lambda e^{|x|})u(x) = 0, \quad x \in \mathbf{R}, \tag{3.23}$$

is very similar to the already solved equation (2.38). By similar arguments we conclude that the desired positive solution to (3.23) has the form

$$\begin{cases} u_0(x) = QK_{2\sqrt{2\gamma}\,i}(2\sqrt{2\lambda}e^{x/2}), & x \geq 0, \\ u_0(x) = u_0(-x), & x \leq 0, \end{cases} \tag{3.24}$$

where $Q > 0$ is the normalizing constant and the following connection between parameters γ and λ takes place. For any fixed $\lambda > 0$, $\gamma = \gamma(\lambda) > 0$ is defined uniquely as the minimal root of the equation

$$K'_{2\sqrt{2\gamma}\,i}(2\sqrt{2\lambda}) = 0. \tag{3.25}$$

with respect to γ. Using the two last equalities in (3.10) and relations (2.23), (2.24), (3.24), (3.25), we get two extra equations for (Q, γ, λ),

$$\begin{aligned} 1 &= 4Q^2 \int\limits_{2\sqrt{2\lambda}}^{\infty} \frac{K^2_{2\sqrt{2\gamma}\,i}(t)}{t}\,dt \\ &\equiv K_{2\sqrt{2\gamma}\,i}(2\sqrt{2\lambda})\frac{1}{2\sqrt{2\gamma}}\Big[\frac{\partial}{\partial p}K'_{ip}(2\sqrt{2\lambda})\Big]\Big|_{p=2\sqrt{2\gamma}}, \end{aligned} \tag{3.26}$$

(3.27) $$d = \frac{Q^2}{2\lambda} \int\limits_{2\sqrt{2\lambda}}^{\infty} t K^2_{2\sqrt{2\gamma}\,i}(t)dt \equiv \frac{Q^2}{\lambda}(\gamma - \lambda) K^2_{2\sqrt{2\gamma}\,i}(2\sqrt{2\lambda}).$$

From Proposition 4 it follows that the equation set (3.25 – 3.27) with respect to (Q, γ, λ) has a unique solution, say $(Q_0, \gamma_0, \lambda_0)$.

Passing from the parameters (Q, γ, λ) to (Q, p, a), setting $a = 2\lambda$, $p = 2\sqrt{2\gamma}$ and

(3.28) $$a_0 = 2\lambda_0, \quad p_0 = 2\sqrt{2\gamma_0},$$

(3.25) takes the form of the first equation from (1.8). Eliminating Q from (3.26, 3.27) we get the second equation for a, p in (1.8). Using (3.26) and the second equation of (1.8), we find that

(3.29) $$0 < Q_0 = \frac{1}{2}\Big(\int\limits_{2\sqrt{a_0}}^{\infty} \frac{K^2_{ip_0}(t)}{t} dt \Big)^{-1/2} = \Big[\frac{2a_0 d}{(p_0^2 - 4a_0) K^2_{ip_0}(2\sqrt{a_0})} \Big]^{1/2}.$$

Taking into account the derivation of (3.22), we make sure that from the (3.28), (3.29) and Proposition 4, using $p^2 - 4ad = 8(\gamma - d\lambda)$, Proposition 1 follows. Inequality (1.9) follows from (3.28) and (3.11). □

Notice that from the proof of Proposition 1 we get also the following.

PROPOSITION 5. *The extremal introduced in Proposition 4 is*

(3.30) $$u_0(x) = Q_0 K_{ip_0}\big(2\sqrt{a_0}\exp\{\frac{|x|}{2}\}\big), \quad x \in \mathbf{R},$$

where Q_0 is given by (3.29) and the numbers a_0, p_0 are defined in Proposition 1.

4. Conclusion of Proof of Theorem 2

First we formulate a general result which is in some sense a generalization of Theorem 4 and which is the base of our proof of Theorem 2. This result is already given in [12] for the case of a power potential, in a more detailed form, see also [13], [14].

Define a continuous function:

$$V(x) := \gamma_0 - \lambda_0 e^{|x|}, \quad x \in \mathbf{R}. \tag{4.1}$$

The operator $\mathcal{A} + V : L^2 \to L^2$ is defined on a totally dense set $\mathcal{D}$ in L^2. Similarly to [12, (3.11)], it can be shown that the kernel E_0 of $\mathcal{A} + V$ is a one-dimensional subspace of L^2,

$$E_0 := Ker(\mathcal{A} + V) = \{c_0\, u_0,\, c_0 \in \mathbf{R}\}.$$

Consider the decomposition of L^2 in a direct sum,

$$L^2 = E_0 \oplus E_1,$$

where

$$E_1 = E_0^{\perp} = \{v \in L^2 : \int_{-\infty}^{\infty} v(x) u_0(x) dx = 0\}$$

is the orthogonal supplement of E_0, see [26, Chapter 4, 5.4]. Since $\mathcal{A} + V$ is self-adjoint, its restriction to E_1 (denoted again by $\mathcal{A} + V$) acts from E_1 to E_1in one-to-one way, (compare with [26, Chapter 9, sect. 4.5]). Therefore one can define its inverse $(\mathcal{A} + V)^{-1} y$, $y \in E_1$. Let

$$\psi(x) := \frac{e^{|x|}}{d} - 1, \quad x \in \mathbf{R}. \tag{4.2}$$

Then, by two last equalities of (3.10), we have $u_0 \psi \in E_1$. Set

$$\sigma_0^2 := -2\lambda_0^2 d \int_{-\infty}^{\infty} (\mathcal{A} + V)^{-1}[u_0\psi](x) e^{|x|} u_0(x) dx > 0. \tag{4.3}$$

By the above argument, the element $(\mathcal{A} + V)^{-1}[u_0\psi](x)$ satisfies

$$\int_{-\infty}^{\infty} (\mathcal{A} + V)^{-1}[u_0\psi](x) u_0(x) dx = 0. \tag{4.4}$$

The assertion $\sigma_0^2 > 0$ follows from (4.2), (4.4) and the positivity of $-(\mathcal{A} + V)^{-1}$. In turn, the positivity of $-(\mathcal{A} + V)^{-1}$ follows from (2.4), (4.1) and Lemma 8 with $\mu_0 = \gamma_0$.

THEOREM 5 **(Exact asymptotic behavior for sojourn times)**. *For the Wiener process w, any $d > 1$ and $b \in \mathbf{R}$, as $T \to \infty$,*

$$(4.5)\quad P_0\{\int_{-\infty}^{\infty} e^{|x|} L_T(w,dx) < d\} \sim \frac{\exp\{-T(\gamma_0 - d\lambda_0)\}u_0(0)}{\sqrt{2\pi T}\sigma_0} \int_{-\infty}^{\infty} u_0(x)dx;$$

$$(4.6)\quad P_{0,b}\{\int_{-\infty}^{\infty} e^{|x|} L_T(w,dx) < d\} \sim \frac{1}{\sigma_0} \exp\{-T(\gamma_0 - d\lambda_0)\}u_0(0)u_0(b).$$

The proof of Theorem 5 is similar to the proof of Proposition 1 from [12]. The main reason why Theorem 5 holds is that the extremal in Proposition 4 is unique and that the spectrum of the Schrodinger operator in Lemma 8 is discrete.

4.1. Proof of Theorem 2

We derive the assertion of Theorem 2 from Theorem 5. We let $\theta = 1$ and compute σ_0^2 defined by (4.3) and (4.4). We need several auxiliary assertions.

LEMMA 9. *The Wronskian of the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$ equals*

$$\mathcal{W}(I_\nu, K_\nu) = I_\nu(x)K'_\nu(x) - I'_\nu(x)K_\nu(x) = -\frac{1}{x}, \quad x \in \mathbf{R}.$$

For any complex ν, the following antiderivatives hold,

$$(4.7)\quad \int^y \frac{dx}{xK_\nu^2(x)} = \frac{I_\nu(y)}{K_\nu(y)},$$

$$(4.8)\quad \int^y \frac{I_\nu(x)K_\nu(x)}{x}dx = \frac{y}{2\nu}\Big[I_\nu(y)\frac{\partial}{\partial\nu}K'_\nu(y) - I'_\nu(y)\frac{\partial}{\partial\nu}K_\nu(y)\Big].$$

PROOF. The expression for the Wronskian is well known, see [9, 7.11]. The equality (4.7) can be proven by differentiating and using the expression for the Wronskian, one can also find it in [34, 1.12.5(2)]. The antiderivative (4.8) is derived from 1.12.4(4) [34] by letting there $\mu \to \nu$. Notice that this relation is an analogue of the first equality in (2.24)). The Lemma follows. □

For short we set $\nu_0 = ip_0$. Let us compute several antiderivatives.

LEMMA 10. *For all $y \in [0, \infty)$ and any complex ν,*

$$\int^y u_0^2(x) e^x dx$$

$$= \frac{Q_0^2}{4a_0}\Big[(4a_0 e^y + \nu_0^2) K_{\nu_0}^2(2\sqrt{a_0}e^{y/2}) - 4a_0 e^y [K'_{\nu_0}(2\sqrt{a_0}e^{y/2})]^2\Big]; \tag{4.9}$$

$$\int^y x\Big[\frac{K'_\nu(x)}{K_\nu(x)}\Big]^2 dx = \frac{y^2}{2} + \nu^2 \log y - y\frac{K'_\nu(y)}{K_\nu(y)}; \tag{4.10}$$

$$\Omega_1(y) := \frac{1}{d}\int^y \frac{1}{u_0^2(x)}\Big(\int u_0^2(x)e^x dx\Big)dx = \frac{e^{y/2}}{d\sqrt{a_0}} \frac{K'_{\nu_0}(2\sqrt{a_0}e^{y/2})}{K_{\nu_0}(2\sqrt{a_0}e^{y/2})}; \tag{4.11}$$

$$\int^y \frac{dx}{u_0^2(x)} = \frac{2}{Q_0^2}\frac{I_{\nu_0}(2\sqrt{a_0}e^{y/2})}{K_{\nu_0}(2\sqrt{a_0}e^{y/2})}; \tag{4.12}$$

$$\Omega_2(y) := \int^y \frac{1}{u_0^2(x)}\Big(\int u_0^2(x)dx\Big)dx = \frac{2}{\nu_0}\frac{\big[\frac{\partial}{\partial\nu}K_\nu(2\sqrt{a_0}e^{y/2})\big]\big|_{\nu=\nu_0}}{K_{\nu_0}(2\sqrt{a_0}e^{y/2})}. \tag{4.13}$$

PROOF. Taking into account (3.30) and changing variable

$$t = 2\sqrt{a_0}e^{x/2}, \tag{4.14}$$

we get

$$\int u_0^2(x)e^x dx = Q_0^2 \int e^x K_{\nu_0}^2(2\sqrt{a_0}e^{x/2})dx = \frac{Q_0^2}{2a_0}\int^{2\sqrt{a_0}e^{x/2}} tK_{\nu_0}^2(t)dt.$$

From here and (2.23) (4.9) follows.

Let us prove (4.10). Since $K_\nu(x)$ satisfies the equation (2.21), we get

$$\Big[\frac{K'_\nu(x)}{K_\nu(x)}\Big]^2 = \frac{K''_\nu(x)}{K_\nu(x)} - \Big[\frac{K'_\nu(x)}{K_\nu(x)}\Big]' = 1 + \frac{\nu^2}{x^2} - \frac{1}{x}\frac{K'_\nu(x)}{K_\nu(x)} - \Big[\frac{K'_\nu(x)}{K_\nu(x)}\Big]'. \tag{4.15}$$

Integrating by parts, we get

$$\int^y x\Big[\frac{K'_\nu(x)}{K_\nu(x)}\Big]' dx = y\frac{K'_\nu(y)}{K_\nu(y)} - \int^y \frac{K'_\nu(x)}{K_\nu(x)}dx. \tag{4.16}$$

Inserting (4.15) into (4.10), computing the obtaining integrals, simplifying by exploiting cancelations and using (4.16), we deduce (4.10).

Let us prove (4.11). Inserting (4.9) into $\Omega_1(y)$, we get,

$$\Omega_1(y) = \frac{1}{4da_0}\Big\{\int^y (4a_0e^x + \nu_0^2)dx - 4a_0\int^y e^x \frac{[K'_{\nu_0}(2\sqrt{a_0}e^{x/2})]^2}{K^2_{\nu_0}(2\sqrt{a_0}e^{x/2})}dx\Big\}. \tag{4.17}$$

Changing variable (4.14) and using (4.10), for the second integral here we obtain,

$$4a_0\int^y e^x \frac{[K'_{\nu_0}(2\sqrt{a_0}e^{x/2})]^2}{K^2_{\nu_0}(2\sqrt{a_0}e^{x/2})}dx = 4a_0e^y + \nu_0^2 y - 4\sqrt{a_0}e^{y/2}\frac{K'_{\nu_0}(2\sqrt{a_0}e^{y/2})}{K_{\nu_0}(2\sqrt{a_0}e^{y/2})}. \tag{4.18}$$

Inserting (4.18) into (4.17) and taking into account that $\int^y (4a_0e^x + \nu_0^2)dx = 4a_0e^y + \nu_0^2 y$, after simplifying by exploiting cancelations we get (4.11).

Now let us prove (4.12). Using (3.30) and changing variables according to (4.14), we obtain

$$\int^y \frac{dx}{u_0^2(x)} = \frac{2}{Q_0^2}\int^{2\sqrt{a_0}e^{y/2}} \frac{dt}{tK^2_\nu(t)}.$$

From here, applying (4.7), we get (4.12).

Finally let us prove (4.13). Taking into account (4.12), the definition of Ω_2 and integrating by parts, we have

$$\begin{aligned}\Omega_2(z) &= \int^z \Big[\int^x u_0^2(t)dt\Big] d_x\Big[\int^x \frac{ds}{u_0^2(s)}\Big] \\ &= \int^z u_0^2(t)dt\int^z \frac{ds}{u_0^2(s)} - \int^z u_0^2(x)\int^x \frac{ds}{u_0^2(s)}dx \\ &= \frac{2}{Q_0^2}\frac{I_{\nu_0}(2\sqrt{a_0}e^{z/2})}{K_{\nu_0}(2\sqrt{a_0}e^{z/2})}\int^z u_0^2(x)dx - \frac{2}{Q_0^2}\int^z u_0^2(x)\frac{I_{\nu_0}(2\sqrt{a_0}e^{x/2})}{K_{\nu_0}(2\sqrt{a_0}e^{x/2})}dx.\end{aligned} \tag{4.19}$$

Again, changing variables (4.14) in the two last integrals of (4.19), we obtain for $z \in [0, \infty)$,

$$\Omega_2(z) = 4\Big[\frac{I_{\nu_0}(2\sqrt{a_0}e^{z/2})}{K_{\nu_0}(2\sqrt{a_0}e^{z/2})}\int^{2\sqrt{a_0}e^{z/2}}\frac{K^2_{\nu_0}(t)}{t}dt$$

$$- \int^{2\sqrt{a_0}e^{z/2}}\frac{K_{\nu_0}(t)I_{\nu_0}(t)}{t}dt\Big]. \tag{4.20}$$

Denoting for short $y := 2\sqrt{a_0}e^{z/2}$ and using (2.24) and (4.8), from (4.20), abbreviating by cancelation, we obtain, that

$$\Omega_2(z) = \frac{2y}{\nu_0}\frac{\big[\frac{\partial}{\partial\nu}K_\nu(y)\big]|_{\nu=\nu_0}}{K_{\nu_0}(y)}[K_{\nu_0}(y)I'_{\nu_0}(y) - I_{\nu_0}(y)K'_{\nu_0}(y)].$$

From here, using the obtained expression for the Wronskian, we get (4.13). The Lemma follows. □

For the function $\psi(x)$ defined by (4.2) let us find a function

$$g_0(x) := (\mathcal{A} + V)^{-1}[u_0\psi](x) \in L^2 \tag{4.21}$$

such that

$$\int_{-\infty}^{\infty} g_0(x)u_0(x)dx = 0. \tag{4.22}$$

This is achieved in the following:

LEMMA 11.

$$\begin{aligned} g_0(x) &= C_1u_0(x) + 2u_0(x)[\Omega_1(x) - \Omega_2(x)] \\ &\equiv C_1Q_0K_{\nu_0}(2\sqrt{a_0}e^{x/2}) + \frac{2Q_0}{d\sqrt{a_0}}e^{x/2}K'_{\nu_0}(2\sqrt{a_0}e^{x/2}) \\ &-\frac{4Q_0}{\nu_0}\big[\frac{\partial}{\partial\nu}K_\nu(2\sqrt{a_0}e^{x/2})\big]|_{\nu=\nu_0}, \quad x \ge 0, \;\; g_0(x) = g_0(-x), \;\; x \le 0, \end{aligned} \tag{4.23}$$

where

$$C_1 = \frac{2Q_0^2}{da_0}K_{\nu_0}(2\sqrt{a_0})\Big[\big(1 - \frac{4a_0+\nu_0^2}{a_0\nu_0^2 d}\big)K_{\nu_0}(2\sqrt{a_0})$$

$$+ \frac{8\sqrt{a_0}}{\nu_0^2}\big[\frac{\partial^2}{\partial\nu^2}K'_\nu(2\sqrt{a_0})\big]|_{\nu=\nu_0}\Big], \tag{4.24}$$

$C_1 > 0$.

PROOF. By definition, g_0 is a solution to $(\mathcal{A}+V)g = u_0\psi$, or

$$g''(x) + (2\gamma_0 - 2\lambda_0 e^{|x|})g(x) = 2u_0(x)\Big[\frac{e^{|x|}}{d} - 1\Big], \quad x \in \mathbf{R}, \tag{4.25}$$

where the numbers γ_0 and λ_0 satisfy (3.28).

Since u_0 is even, the equation (4.25) has an even solution, $g(x) = g(-x)$, $x \in \mathbf{R}$. This solution is obtained by solving

$$g''(x) + (2\gamma_0 - 2\lambda_0 e^{x})g(x) = 2u_0(x)\Big[\frac{e^{x}}{d} - 1\Big], \quad x \in [0,\infty). \tag{4.26}$$

This is a non-homogeneous equation, its homogeneous counterpart is the equation (3.23). Having two linearly independent solutions to (3.23), by a standard method, [25, Part 1, 24.2b], we find the desired solution to (4.26), which satisfies (4.22).

Using arguments from the proof of Lemma 3 and Proposition 1, one can see, that the equation (3.23) has two linearly independent solutions, $u_0 \in L^2$, given by (3.30), and $u_1(x) = I_{\nu_0}(2\sqrt{a_0}e^{x/2})$, $x \in [0,\infty)$, given by (2.22). Observe that the second solution lies in L^2. Therefore, by the above mentioned standard method and the Wronskian definition, the general solution to (4.26) has the form,

$$\begin{aligned} g(y) &= C_1u_0(y) + C_2u_1(y) \\ &\quad + 2u_0(y)\int^y \frac{1}{u_0^2(x)}\Big(\int^y u_0^2(x)\left[\frac{e^{|x|}}{d} - 1\right] dx\Big)dx, \end{aligned} \tag{4.27}$$

where $y \geq 0$, and C_1, C_2 are constants. Since we are interested in the real solution $g(\cdot) \in L^2$, C_2 must be zero and C_1 must be real. Thus, using functions defined in (4.11) and (4.13), we get

$$g_0(x) = u_0(x)[C_1 + 2\Omega_1(x) - 2\Omega_2(x)], \quad x \geq 0. \tag{4.28}$$

Set

$$g_0(x) = g_0(-x), \quad x \leq 0. \tag{4.29}$$

Since $v_0'(0+) = v_0'(0-) = 0$, it is easily to verify that $g_0'(0+) = g_0'(0-) = 0$ as well. Thus, taking into account the condition (4.22), we conclude that the solution to (4.25) defined by (4.28), (4.29), is the only possible solution satisfying our assumptions.

Formula (4.23) will follow if we show that the constant C_1 is given by (4.24). Inserting (4.28) into (4.22) and using that $g_0(x)$ and $v_0(x)$ are even, we obtain, that

$$\int_0^\infty u_0^2(x)[C_1 + 2\Omega_1(x) - 2\Omega_2(x)]dx = 0. \tag{4.30}$$

Let us now compute the integrals. Since $\|u_0\| = 1$, we have

$$\int_0^\infty C_1 u_0^2(x) = \frac{C_1}{2}. \tag{4.31}$$

Using (3.30), (4.11) and changing variables according to (4.14), we find, that

$$\begin{aligned} &2\int_0^\infty u_0^2(x)\Omega_1(x)dx \\ &= \frac{2Q_0^2}{d\sqrt{a_0}} \int_0^\infty e^{x/2} K_{\nu_0}(2\sqrt{a_0}e^{x/2}) K'_{\nu_0}(2\sqrt{a_0}e^{x/2})dx \\ &= \frac{2Q_0^2}{da_0} \int_{2\sqrt{a_0}}^\infty K_{\nu_0}(t)K'_{\nu_0}(t)dt = -\frac{Q_0^2}{da_0} K_{\nu_0}^2(2\sqrt{a_0}), \end{aligned} \tag{4.32}$$

where the first relation from (2.22) was used. Taking into account (3.30), (4.13) and again changing variables according to (4.14), we get the equalities

$$\begin{aligned} &2\int_0^\infty u_0^2(x)\Omega_2(x)dx \\ &= \frac{4Q_0^2}{\nu_0} \int_0^\infty K_{\nu_0}(2\sqrt{a_0}e^{x/2})\big[\frac{\partial}{\partial\nu} K_\nu(2\sqrt{a_0}e^{x/2})\big]\big|_{\nu=\nu_0} dx \\ &= \frac{8Q_0^2}{\nu_0} \int_{2\sqrt{a_0}}^\infty K_{\nu_0}(t)\big[\frac{\partial}{\partial\nu}K_\nu(t)\big]\big|_{\nu=\nu_0}\frac{dt}{t}. \end{aligned} \tag{4.33}$$

Letting $\nu = ip$, $p \geq 0$ and applying the equality (4.36) below, we conclude, that one can change the order of differentiating in ν and integrating in t in the last integral of (4.33). By this, using also (2.24), (1.8), we get

$$\begin{aligned}
&\int_{2\sqrt{a_0}}^{\infty} K_{\nu_0}(t)\big[\frac{\partial}{\partial\nu}K_\nu(t)\big]|_{\nu=\nu_0}\frac{dt}{t} = \frac{1}{2}\frac{\partial}{\partial\nu}\Big[\int_{2\sqrt{a_0}}^{\infty} K_\nu^2(t)\frac{dt}{t}\Big]|_{\nu=\nu_0} \\
&= \frac{\sqrt{a_0}}{2\nu_0^2}K_{\nu_0}(2\sqrt{a_0})\Big\{\big[\frac{\partial}{\partial\nu}K'_\nu(2\sqrt{a_0})\big]|_{\nu=\nu_0} \\
&-\nu_0\big[\frac{\partial^2}{\partial\nu^2}K'_\nu(2\sqrt{a_0})\big]|_{\nu=\nu_0}\Big\} \\
&= K_{\nu_0}(2\sqrt{a_0})\Big\{\frac{4a_0+\nu_0^2}{16a_0\nu_0 d}K_{\nu_0}(2\sqrt{a_0}) \\
&-\frac{\sqrt{a_0}}{2\nu_0}\big[\frac{\partial^2}{\partial\nu^2}K'_\nu(2\sqrt{a_0})\big]|_{\nu=\nu_0}\Big\}.
\end{aligned} \tag{4.34}$$

The equality (4.24) follows now from (4.30 – 4.34). The Lemma 11 will follow provided the following Lemma will be proved.

We recall that $\nu_0 = ip_0$.

LEMMA 12. *(i) For any fixed $p_1 > 0$,*

$$\frac{\partial}{\partial p}K_{ip}(t)|_{p=p_1} = -\int_0^\infty e^{-t\cosh s} s\sin(p_1 s)ds, \quad t > 0. \tag{4.35}$$

(ii)

$$\frac{\partial}{\partial p}\Big[\int_{2\sqrt{a_0}}^{\infty} K_{ip}^2(t)\frac{dt}{t}\Big]|_{p=p_0} = \int_{2\sqrt{a_0}}^{\infty}\Big[\frac{\partial}{\partial p}K_{ip}^2(t)\Big]|_{p=p_0}\frac{dt}{t}, \tag{4.36}$$

$$\frac{\partial}{\partial p}\Big[\int_{2\sqrt{a_0}}^{\infty} K_{ip}^2(t)t\,dt\Big]|_{p=p_0} = \int_{2\sqrt{a_0}}^{\infty}\Big[\frac{\partial}{\partial p}K_{ip}^2(t)\Big]|_{p=p_0}t\,dt. \tag{4.37}$$

PROOF. To prove (4.35) we shall use the integral representation,

$$K_{ip}(t) = \int_0^\infty e^{-t\cosh s}\cos(ps)ds, \quad t > 0, \quad p \in [0,\infty) \tag{4.38}$$

(see (1.1)) and apply well known conditions for changing orders of differentiating and integrating. Take $\delta \in (0, p_1)$. For a fixed $t > 0$ we have

$$\Big|\int_0^\infty e^{-t\cosh s} s \sin(ps)ds\Big| \le \int_0^\infty e^{-t\cosh s} s ds < \infty \tag{4.39}$$

uniformly for $[p \in [p_1 - \delta, p_1 + \delta]$. From here it follows (see [15, 515]) convergence of the integral

$$\int_0^\infty \frac{\partial}{\partial p}[e^{-t\cosh s}\cos(ps)]ds \equiv -\int_0^\infty e^{-t\cosh s} s \sin(ps)ds.$$

which is uniform for $[p \in [p_1 - \delta, p_1 + \delta]$. Consequently, we may differentiate,

$$\begin{aligned}\frac{\partial}{\partial p}K_{ip}(t)|_{p=p_1} &= \int_0^\infty \frac{\partial}{\partial p}\Big[e^{-t\cosh s}\cos(ps)\Big]|_{p=p_1}ds \\ &= -\int_0^\infty e^{-t\cosh s} s\sin(p_1 s)ds,\end{aligned}$$

so (4.35) follows.

The relations (4.36) and (4.37) can be proved similarly. We give here the necessary uniform estimations. Choose $\delta_0 \in (0, p_0)$. In virtue of (4.38) and (4.35),

$$|K_{ip}(t)| \le K_0(t) \tag{4.40}$$

for all $t > 2\sqrt{a_0}$ and $p \in [p_0 - \delta_0, p_0 + \delta_0]$, and

$$\Big|\frac{\partial}{\partial p}K_{ip}(t)\Big| \le \int_0^\infty e^{-2\sqrt{a_0}\cosh s} s ds =: r_0 < \infty \tag{4.41}$$

for all $t > 2\sqrt{a_0}$ and $p \in [p_0 - \delta_0, p_0 + \delta_0]$. From (4.40), (4.41) and (1.3) we get

$$\int_{2\sqrt{a_0}}^\infty K_{ip}(t)\frac{\partial}{\partial p}K_{ip}(t)\frac{dt}{t} \le r_0 \int_{2\sqrt{a_0}}^\infty K_0(t)\frac{dt}{t} < \infty$$

for all $p \in [p_0 - \delta_0, p_0 + \delta_0]$. From here, since $\frac{\partial}{\partial p} K_{ip}^2(t) = 2K_{ip}(t)\frac{\partial}{\partial p} K_{ip}(t)$, (4.36) follows. Thus Lemma 12 and therefore Lemma 11 also follow. □

LEMMA 13. *The σ^2 defined by (1.10) is equal to σ_0^2 defined by (4.3), (4.4), (4.23),*

$$\sigma_0^2 = \sigma^2. \tag{4.42}$$

PROOF. Taking into account (4.3), (4.4), (4.21), (4.23), we obtain, that

$$\sigma_0^2 = -2\lambda_0^2 d \int_0^\infty e^x u_0^2(x)[C_1 + 2\Omega_1(x) - 2\Omega_2(x)]dx. \tag{4.43}$$

Let us compute the integrals. The even function u_0 given by (3.30), is the solution to the boundary problem (3.10), therefore, in accordance with the last equality in (3.10) we find that

$$\int_0^\infty C_1 e^x u_0^2(x)dx = \frac{C_1 d}{2}. \tag{4.44}$$

Taking into account (3.30), (4.11) and changing variables according to (4.14), we get

$$\begin{aligned} &2\int_0^\infty e^x u_0^2(x)\Omega_1(x)dx \\ &= \frac{2Q_0^2}{d\sqrt{a_0}} \int_0^\infty e^{3x/2} K_{\nu_0}(2\sqrt{a_0}e^{x/2})K'_{\nu_0}(2\sqrt{a_0}e^{x/2})dx \\ &= \frac{Q_0^2}{2da_0^2} \int_{2\sqrt{a_0}}^\infty t^2 K_{\nu_0}(t)K'_{\nu_0}(t)dt. \end{aligned} \tag{4.45}$$

Integrating by parts and applying (2.23), (1.8),

$$\begin{aligned} &\int_{2\sqrt{a_0}}^\infty t^2 K_{\nu_0}(t)K'_{\nu_0}(t)dt = \tfrac{1}{2}\int_{2\sqrt{a_0}}^\infty t^2 dK_{\nu_0}^2(t) \\ &= -2a_0 K_{\nu_0}(2\sqrt{a_0}) - \int_{2\sqrt{a_0}}^\infty tK_{\nu_0}^2(t)dt = \tfrac{\nu_0^2}{2}K_{\nu_0}^2(2\sqrt{a_0}). \end{aligned} \tag{4.46}$$

From (4.45), (4.46),

$$2\int_0^\infty e^x u_0^2(x)\Omega_1(x)dx = \frac{\nu_0^2 Q_0^2}{4da_0^2} K_{\nu_0}^2(2\sqrt{a_0}). \tag{4.47}$$

Using (3.30), (4.13) and changing variables according to (4.14), we obtain that

$$2\int_0^\infty e^x u_0^2(x)\Omega_2(x)dx = \frac{2Q_0^2}{\nu_0 a_0}\int_{2\sqrt{a_0}}^\infty tK_{\nu_0}(t)[\frac{\partial}{\partial\nu}K_\nu(t)]|_{\nu=\nu_0}dt. \tag{4.48}$$

Using (4.37), we change orders of differentiating in ν and integrating in t and apply (2.23), (1.8) to get

$$\begin{aligned}\int_{2\sqrt{a_0}}^\infty tK_{\nu_0}(t)[\tfrac{\partial}{\partial\nu}K_\nu(t)]|_{\nu=\nu_0}dt &= \tfrac{1}{2}\tfrac{\partial}{\partial\nu}\Big[\int_{2\sqrt{a_0}}^\infty tK_\nu^2(t)dt\Big]|_{\nu=\nu_0}\\ &= -\tfrac{1}{2}K_{\nu_0}(2\sqrt{a_0})\Big\{\nu_0 K_{\nu_0}(2\sqrt{a_0}) + (4a_0+\nu_0^2)[\tfrac{\partial}{\partial\nu}K_\nu(2\sqrt{a_0})]|_{\nu=\nu_0}\Big\}.\end{aligned} \tag{4.49}$$

From (4.48) and (4.49) it follows that

$$\begin{aligned}&2\int_0^\infty e^x u_0^2(x)\Omega_2(x)dx\\ &= -\frac{Q_0^2}{\nu_0 a_0}K_{\nu_0}(2\sqrt{a_0})\Big\{\nu_0 K_{\nu_0}(2\sqrt{a_0})\\ &+ (4a_0+\nu_0^2)[\frac{\partial}{\partial\nu}K_\nu(2\sqrt{a_0})]|_{\nu=\nu_0}\Big\}.\end{aligned} \tag{4.50}$$

Now put (4.24), (4.44), (4.47), (4.50) into (4.43) and, using (3.28, 3.29), group members. We get (4.42) and thus Lemma 13.

Proof of Theorem 2. For $\theta = 1$ the Theorem follows from Theorem 5, Lemma 13 and (1.3). For an arbitrary $\theta > 0$ we use Remark 1. □

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