J. Math. Sci. Univ. Tokyo
15 (2008), 521–543.

Exact Power Series in the Asymptotic Expansion of the Matrix Coefficients with the Corner K-type of P_J-Principal Series Representations of $Sp(2,\mathbb{R})$

By Masatoshi Iida and Takayuki Oda

Abstract. Let G be a symplectic Lie group of rank 2, $Sp(2,\mathbb{R})$ and P_J be its maximal parabolic subgroup called the Jacobi parabolic subgroup with non-abelian unipotent radical. The radial parts of matrix coefficients of the P_J-principal series representations of G were studied in relation to the Appell's hypergeometric function. The leading terms of the expansion of the functions around the infinity were well investigated in general cases (semisimple Lie groups and representations). In this paper, we determine the power series expansion other than leading terms for the above special case.

1. Introduction

In a previous paper [5], one of the authors gave a new explicit integral formula for the radial part of the spherical function with the corner K type of a P_J-principal series, in terms of the Appell's hypergeometric function F_2. This integral expression seems to be much easier to handle than the Eisenstein integral of matrix coefficients, because we can utilize the classical library of special functions for further analysis.

In this paper, we consider the power series expression of this integral at the infinity in the double coset decomposition $G = KAK$. For the matrix element with the trivial K-type of the class one principal series representations of a general semisimple group G, Harish-Chandra obtained an expression as the sum over the Weyl group of certain hypergeometric series (Harish-Chandra's hypergeometric series)([2]). This kind of result of Harish-Chandra was generalized for more general representations, say, by Casselman-Miličić ([1]), as the theory of 'asymptotic expansion'. From the viewpoint of algebraic analysis, Oshima investigated asymptotic behavior of spherical functions and their boundary values ([9]). Among others the

2000 *Mathematics Subject Classification.* 43A90.

leading terms of 'functions' appearing in the 'expansion' were precisely investigated.

However the whole functions with these leading terms seem to be very difficult to grasp in general. For example the higher degree terms of the power series expansions of these 'functions' depend on the choice of the local coordinates at the infinity of A, contrary to the fact that the leading terms are independent of choice of the local coordinates at the infinity.

The main result of this paper is the following.

We consider the matrix coefficient with the corner K-type of the P_J-principal series representation of $G = Sp(2, \mathbb{R})$.

Since its radial part satisfies the holonomic system of two variables of rank 4, which is called a modified F_2 system in the sense of Takayama [10], we have the other integral expression different from the Eisenstein integral, which is valid on a neighborhood of G in $G_{\mathbb{C}}$, the complexification of G.

Then we can apply the monodromy argument of the hypergeometric function to have a decomposition formula(Theorem 6.1). And this is found to be the sum of the asymptotic expansions(Theorem 7.1).

Among others we have precise formula for coefficients of the expansion in terms of Pochhammer symbols, which should be a special case of an analogous formula of the coefficients in terms of the values at 1 of generalized hypergeometric series of one variable(see, another example in [4]).

In this paper, we treat the matrix coefficient corresponding to 1-dimensional K-types, which we called "even case" in [5]. The system of differential operators which annihilate the matrix coefficient is understood in the framework of Heckman and Opdam. Almost same result will be obtained for 2-dimensional K-types(or "odd case" in [5]) just changing parameters of hypergeometric functions, despite that corresponding (difference-)differential operators are beyond the framework of Heckman and Opdam.

Our method of the proof is done by a very down-to-earth or 'elementary' manner. We take the advantage to start from an (Eulerian) integral expression of our matrix coefficient in terms of the Gaussian hypergeometric function, obtained in a previous paper [5]. What we need is the classical connection formula of Kummer and some general framework of the asymptotic behavior of the ideally analytic solution of holonomic systems in our setting (§4 and §5). We can find the theory of differential equations with regular singularities and ideally analytic solutions of them in [9] and its references.

2. P_J-Principal Series Representations

In this section, we recall some facts about representations of $Sp(2,\mathbb{R})$ and their K-type. Notations are same as those of [5].

Let $G = Sp(2,\mathbb{R})$ be a split real semisimple Lie group of real rank 2 with a maximal compact subgroup K which is isomorphic to the unitary group $U(2)$. The group G has two standard maximal parabolic subgroups. One is associated with the short simple root $e_1 - e_2$ and called the Siegel parabolic subgroup. The other is associated with the long simple root $2e_2$ and called the Jacobi parabolic subgroup P_J.

We set the Langlands decomposition of P_J as $P_J = M_J A_J N_J$, then M_J is isomorphic to $SL(2,\mathbb{R}) \times \{\pm 1\}$.

Let D_l^+ is the anti-holomorphic discrete series representation of $SL(2,\mathbb{R})$ with the Blattner parameter l ($l \in \mathbb{N}, l \geq 2$) and D_l^- its contragredient representation.

We denote the character of $\{\pm 1\}$ by ε and the complex valued linear form on $\mathfrak{a}_J = \mathrm{Lie}(A_J) \otimes \mathbb{C}$ by ν. The generalized principal series representation of $Sp(2,\mathbb{R})$ which we call the P_J-principal series representation is the induced representation $\pi_{(D_l^{\pm},\varepsilon),\nu} = \mathrm{Ind}_{P_J}^{G}((D_l^{\pm},\varepsilon) \boxtimes e_1^{\nu+\rho_J} \boxtimes \mathrm{id}_{N_J})$.

Here, id_{N_J} is the trivial representation of N_J and ρ_J is the half sum of positive roots corresponding to N_J.

The P_J-principal series representation has a special K-type of multiplicity free. We call the K-type as "the corner K-type".

If the character ε of $\{\pm 1\}$ satisfies $\varepsilon(-1) = (-1)^l$, then the corner K-type of $\pi_{(D_l^{\pm},\varepsilon),\nu}$ is the one dimensional representation $\tau_{(l,l)}$ whose highest weight is (l,l) and if $\varepsilon = (-1)^{l+1}$ holds, then the corner K-type is the two dimensional representation $\tau_{(l,l-1)}$ whose highest weight is $(l, l-1)$.

Let $(\eta, V_\eta), (\tau, V_\tau)$ be in $\hat{K}$. We denote the contragredient representation of τ by τ^*. We define the space of spherical functions

$$\begin{aligned} C^\infty_{\eta,\tau}(K\backslash G/K) = \{ f : G \to V_\eta \otimes V_{\tau^*} \mid & \ f \text{ is a } C^\infty \text{ function}, \\ & f(k_1 g k_2) = \eta(k_1) \otimes \tau^*(k_2)^{-1} f(g), \\ & \forall g \in G, \forall k_1, \forall k_2 \in K \}. \end{aligned}$$

In this paper, we consider the matrix coefficient $\phi \in C^{\infty}_{\tau_{(k,k)},\tau_{(l,l)}}(K\backslash G/K)$ of $\pi_{(D_l^{\pm},\varepsilon),\nu}$ for $k \geq l$, $k \equiv l \mod 2$ and $\varepsilon(-1) = (-1)^l$ (It goes almost same way for the case of $\varepsilon(-1) = (-1)^{l+1}$.). If $k < l$ or $k \not\equiv l \mod 2$ holds, then $\phi = 0$ (Proposition 3.4 and Lemma 4.2 in [5]). That is why we call $\tau_{(l,l)}$ the corner K-type.

We denote the standard split Cartan subgroup of G by

$$A = \{\mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_1, a_2 \in \mathbb{R}_{>0}\}.$$

The system of partial differential equations satisfied by the A-radial part of ϕ is the holonomic system. We choose the coordinates of A as (x_1, x_2) determined by $a_1 = \exp x_1, a_2 = \exp x_2$.

We recall the system of differential equations satisfied by ϕ (Theorem 7.5 in [5]).

THEOREM 2.1. *ϕ satisfies the following system of differential equations :*

$$\begin{aligned}(2.1)\qquad & \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2}\phi + \sum_{i=1}^{2}\{2\coth 2x_i + \coth(x_1+x_2)\}\frac{\partial}{\partial x_i}\phi \\ &\quad + \coth(x_1-x_2)\frac{\partial}{\partial x_1}\phi - \coth(x_1-x_2)\frac{\partial}{\partial x_2}\phi \\ &\quad - (k^2+l^2)(\mathrm{sh}^{-2} x_1 + \mathrm{sh}^{-2} x_2)\phi \\ &\quad + 2kl(\mathrm{ch}\, 2x_1 \cdot \mathrm{sh}^{-2} 2x_1 + \mathrm{ch}\, 2x_2 \cdot \mathrm{sh}^{-2} 2x_2)\phi \\ &= \{\nu^2 + (l-1)^2 - 5\}\phi,\end{aligned}$$

$$\begin{aligned}(2.2)\qquad & 2\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\phi + \{2l\coth 2x_2 - 2k\,\mathrm{sh}^{-1} 2x_2 \\ &\quad + \coth(x_1+x_2) - \coth(x_1-x_2)\}\frac{\partial}{\partial x_1}\phi \\ &\quad + \{2l\coth 2x_1 - 2k\,\mathrm{sh}^{-1} 2x_1 \\ &\quad + \coth(x_1+x_2) + \coth(x_1-x_2)\}\frac{\partial}{\partial x_2}\phi \\ &\quad + 2(l\coth 2x_1 - k\,\mathrm{sh}^{-1} 2x_1)(l\coth 2x_2 - k\,\mathrm{sh}^{-1} 2x_2)\phi \\ &\quad + (l\coth 2x_2 - k\,\mathrm{sh}^{-1} 2x_2) \\ &\quad \times (\coth(x_1+x_2) + \coth(x_1+x_2))\phi\end{aligned}$$

$$+ (l \coth 2x_1 - k \operatorname{sh}^{-1} 2x_1)$$
$$\times (\coth(x_1 + x_2) - \coth(x_1 + x_2))\phi = 0.$$

In the following section, we will determine characteristic roots of the system around the infinity, $a_1/a_2 = 0, a_2 = 0$ (Note that a_1/a_2 and a_2^2 correspond simple roots $e_1 - e_2$ and $2e_2$ respectively). There are 4 characteristic roots, so the system has 4 independent solutions. A solution ϕ is a linear combination of these solutions and its coefficients are analogues of c-functions.

3. The Holonomic System

We put $\delta(x_1, x_2) = (\operatorname{ch} x_1 \operatorname{ch} x_2)^{(l+k)/2} (\operatorname{sh} x_1 \operatorname{sh} x_2)^{(l-k)/2}$ and

$$\psi(x_1, x_2) = \delta(x_1, x_2)\phi(x_1, x_2).$$

Proposition 3.1. *ψ satisfies the following system of partial differential equations:*

$$(3.1) \quad \sum_{i=1}^{2} \frac{\partial}{\partial x_i^2}\psi + \sum_{i=1}^{2} \{2k \operatorname{sh}^{-1} 2x_i - 2(l-1)\coth 2x_i\} \frac{\partial}{\partial x_i}\psi$$
$$+ \frac{\operatorname{sh} 2x_1}{\operatorname{sh}^2 x_1 - \operatorname{sh}^2 x_2} \frac{\partial}{\partial x_1}\psi - \frac{\operatorname{sh} 2x_2}{\operatorname{sh}^2 x_1 - \operatorname{sh}^2 x_2} \frac{\partial}{\partial x_2}\psi$$
$$= \{\nu^2 - (l-2)^2\}\psi,$$

$$(3.2) \quad \frac{\partial^2}{\partial x_1 \partial x_2}\psi - \frac{1}{2} \frac{\operatorname{sh} 2x_2}{\operatorname{sh}^2 x_1 - \operatorname{sh}^2 x_2} \frac{\partial}{\partial x_1}\psi + \frac{1}{2} \frac{\operatorname{sh} 2x_1}{\operatorname{sh}^2 x_1 - \operatorname{sh}^2 x_2} \frac{\partial}{\partial x_2}\psi = 0.$$

Proof. This system is easily obtained from Theorem 2.1. □

We will transform this system into the system with variables

$$y_1 = (a_1/a_2)^2 = \exp 2(x_1 - x_2), \quad y_2 = a_2^2 = \exp 2x_2.$$

Since $y_1 y_2 = \exp 2x_1$, we have

$$\operatorname{sh} 2x_1 = \frac{y_1 y_2 - y_1^{-1} y_2^{-1}}{2} = \frac{y_1^2 y_2^2 - 1}{2y_1 y_2}, \quad \operatorname{sh} 2x_2 = \frac{y_2 - y_2^{-1}}{2} = \frac{y_2^2 - 1}{2y_2},$$

$$\operatorname{sh}^2 x_1 - \operatorname{sh}^2 x_2 = \frac{1}{2}(\operatorname{ch} 2x_1 - \operatorname{ch} 2x_2) = \frac{y_1 y_2 + y_1^{-1} y_2^{-1}}{4} - \frac{y_2 + y_2^{-1}}{4}$$
$$= \frac{(y_1 - 1)(y_1 y_2^2 - 1)}{4 y_1 y_2},$$
$$\coth 2x_1 = \frac{y_1 y_2 + y_1^{-1} y_2^{-1}}{y_1 y_2 - y_1^{-1} y_2^{-1}} = \frac{y_1^2 y_2^2 + 1}{y_1^2 y_2^2 - 1}, \quad \coth 2x_2 = \frac{y_2 + y_2^{-1}}{y_2 - y_2^{-1}} = \frac{y_2^2 + 1}{y_2^2 - 1},$$

and

$$\frac{\partial}{\partial x_1} = 2y_1 \frac{\partial}{\partial y_1}, \quad \frac{\partial}{\partial x_2} = -2y_1 \frac{\partial}{\partial y_1} + 2y_2 \frac{\partial}{\partial y_2}.$$

We regard ϕ and ψ as functions in variables y_1, y_2. Then, we have the system of differential equations in y_1, y_2 as follows.

PROPOSITION 3.2. *ψ satisfies the system differential equations:*

$$\begin{aligned}
(3.3) \qquad & 4\left\{2\left(y_1 \frac{\partial}{\partial y_1}\right)^2 - 2\left(y_1 \frac{\partial}{\partial y_1}\right)\left(y_2 \frac{\partial}{\partial y_2}\right) + \left(y_2 \frac{\partial}{\partial y_2}\right)^2\right\}\psi \\
& + \left\{\frac{4k y_1 y_2}{y_1^2 y_2^2 - 1} - \frac{4k y_2}{y_2^2 - 1} - 2(l-1)\frac{y_1^2 y_2^2 + 1}{y_1^2 y_2^2 - 1}\right. \\
& \left. + 2(l-1)\frac{y_2^2 + 1}{y_2^2 - 1}\right\}\left(2y_1 \frac{\partial}{\partial y_1}\right)\psi \\
& + \left\{\frac{4k y_2}{y_2^2 - 1} - 2(l-1)\frac{y_2^2 + 1}{y_2^2 - 1}\right\}\left(2y_2 \frac{\partial}{\partial y_2}\right)\psi \\
& + 4\frac{y_1^2 y_2^2 - 1 + y_1(y_2^2 - 1)}{(y_1 - 1)(y_1 y_2^2 - 1)}\left(y_1 \frac{\partial}{\partial y_1}\right)\psi \\
& - 4\frac{y_1(y_2^2 - 1)}{(y_1 - 1)(y_1 y_2^2 - 1)}\left(y_2 \frac{\partial}{\partial y_2}\right)\psi \\
& = \{\nu^2 - (l-2)^2\}\psi, \\
(3.4) \qquad & \left(y_1 \frac{\partial}{\partial y_1}\right)\left(-y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}\right)\psi \\
& - \frac{1}{2}\frac{y_1(y_2^2 - 1) + (y_1^2 y_2^2 - 1)}{(y_1 - 1)(y_1 y_2^2 - 1)}\left(y_1 \frac{\partial}{\partial y_1}\right)\psi \\
& + \frac{1}{2}\frac{y_1^2 y_2^2 - 1}{(y_1 - 1)(y_1 y_2^2 - 1)}\left(y_2 \frac{\partial}{\partial y_2}\right)\psi = 0.
\end{aligned}$$

4. Characteristic Indices

Let $\xi(y_1, y_2)$ be a solution of the system of differential equations (3.3) and (3.4) which has the expansion around $(y_1, y_2) = (0, 0)$ as

$$\xi(y_1, y_2) = y_1^\alpha y_2^\beta \sum_{p,q \geq 0} c_{p,q} y_1^p y_2^q, \qquad c_{0,0} = 1,$$

that is, (α, β) is the leading exponent of ξ at $(y_1, y_2) = (0, 0)$.

PROPOSITION 4.1. *The leading exponent (α, β) of ξ is one of the followings :*

$$(\frac{1}{2}, \mu_\pm), \quad (\mu_\pm, \mu_\pm).$$

Here, $\mu_\pm = \pm\frac{\nu}{2} - \frac{l-2}{2}$.

PROOF. If we substitute above $\xi(y_1, y_2)$ for $\psi(y_1, y_2)$ in the equation (3.4), then we obtain the indicial equation

$$\alpha(-\alpha + \beta) - \frac{1}{2}\frac{(-1)}{(-1)^2}\alpha + \frac{1}{2}\frac{(-1)}{(-1)^2}\beta = 0.$$

The solutions of this equation is

$$\alpha = \frac{1}{2} \text{ or } \alpha = \beta.$$

We obtain the other indicial equation from the equation (3.3) :

$$4(2\alpha^2 - 2\alpha\beta + \beta^2) + \left\{-2(l-1)\frac{1}{(-1)} + 2(l-1)\frac{1}{(-1)}\right\}(2\alpha) + \left\{-2(l-1)\frac{1}{(-1)}\right\}(2\beta) + 4\frac{(-1)}{(-1)^2}\alpha = \nu^2 - (l-2)^2. \tag{4.1}$$

In both cases $\alpha = 1/2$ and $\alpha = \beta$, the equation (4.1) is equivalent to

$$4\beta^2 + 4(l-2)\beta = \nu^2 - (l-2)^2.$$

Hence $\beta = \pm\nu/2 - (l-2)/2$. □

We denote ξ with the leading term $y_1^{\alpha}y_2^{\beta}$ by $\psi_{\alpha,\beta}$.

Since the multiplier $\delta(x_1,x_2)^{-1}$ is expanded as

$$\begin{aligned}
\delta(x_1,x_2)^{-1} &= (\mathrm{ch}\,x_1\,\mathrm{ch}\,x_2)^{-\frac{l+k}{2}}(\mathrm{sh}\,x_1\,\mathrm{sh}\,x_2)^{-\frac{l-k}{2}} \\
&= \left(\frac{a_1+a_1^{-1}}{2}\cdot\frac{a_2+a_2^{-1}}{2}\right)^{-\frac{l+k}{2}}\left(\frac{a_1-a_1^{-1}}{2}\cdot\frac{a_2-a_2^{-1}}{2}\right)^{-\frac{l-k}{2}} \\
&= 2^{2l}a_1^l a_2^l(1+\text{higher order}) \\
&= 2^{2l}(y_1y_2)^{\frac{l}{2}}y_2^{\frac{l}{2}}(1+\text{higher order}) \\
&= 2^{2l}y_1^{\frac{l}{2}}y_2^l(1+\text{higher order}),
\end{aligned}$$

we have

$$\delta(x_1,x_2)^{-1}\psi_{\alpha,\beta}(y_1,y_2) = 2^{2l}y_1^{\frac{l}{2}}y_2^l(1+\text{higher order})\cdot(y_1^{\alpha}y_2^{\beta}+\text{higher order}).$$

So we set

$$\phi_{\alpha+l/2,\beta+l}(y_1,y_2) = 2^{-2l}\delta(x_1,x_2)^{-1}\psi_{\alpha,\beta}(y_1,y_2). \tag{4.2}$$

The leading term of this function $\phi_{\alpha+l/2,\beta+l}$ is

$$y_1^{\frac{l}{2}}y_2^l\cdot y_1^{\alpha}y_2^{\beta} = \begin{cases} y_1^{\frac{l+1}{2}}y_2^{\mu_\pm+l} & \text{if } \alpha=\frac{1}{2},\ \beta=\mu_\pm \\ y_1^{\mu_\pm+\frac{l}{2}}y_2^{\mu_\pm+l} & \text{if } \alpha=\beta=\mu_\pm. \end{cases}$$

These exponents are same as the Siegel-Whittaker function and the Whittaker function([3]).

5. The Singular Boundary Value Problem

We would like to represent ψ as the linear combination of $\psi_{\alpha,\beta}$.

To do that, we will obtain analytic continuation of ψ from $(y_1,y_2)=(1,1)$, which corresponds to the identity of G, to $(y_1,y_2)=(1,0)$ at first, then $(y_1,y_2)=(1,0)$ to $(y_1,y_2)=(0,0)$. The first part was almost done in §9 of [5]. So we discuss the latter part in this section. A general reference for the singular boundary value problem is [10].

5.1. Justification of the singular boundary problem

We obtain the following equation by $1/4 \times$ the equation (3.3) $+ 2 \times$ the equation (3.4).

$$\left(y_2\frac{\partial}{\partial y_2}\right)^2\psi + \left\{\frac{2k(1-y_1)y_2(1+y_1y_2^2) - 2(l-1)(1-y_1^2)y_2^2}{(y_2^2-1)(y_1^2y_2^2-1)}\right\}\left(y_1\frac{\partial}{\partial y_1}\right)\psi$$
$$+\left\{\frac{2ky_2-(l-1)(y_2^2+1)}{y_2^2-1} + \frac{y_1y_2^2+1}{y_1y_2^2-1}\right\}\left(y_2\frac{\partial}{\partial y_2}\right)\psi = \frac{1}{4}\{\nu^2-(l-2)^2\}\psi.$$

This differential equation has regular singularities along $y_2 = 0$ and its indicial equation is

$$\beta^2 + \left\{\frac{-(l-1)}{(-1)} + \frac{1}{(-1)}\right\}\beta = \frac{1}{4}\{\nu^2-(l-2)^2\}.$$

Hence, we have $\beta = \pm\nu/2 - (l-2)/2 = \mu_\pm$, which is independent from y_1. Therefore, when the difference $\mu_+ - \mu_- = \nu$ is not an integer, the above differential equation has a solution

$$\psi(y_1, y_2) = a_+(y_1, y_2)y_2^{\mu_+} + a_-(y_1, y_2)y_2^{\mu_-}.$$

Functions $a_\pm(y_1, y_2)$ are real analytic function around $0 < y_1 \le 1, y_2 = 0$. This solution is called the ideally analytic solution.

So we assume that ν is not an integer hereafter.

5.2. The equation of the singular boundary value

In the beginning, we will find the equation which is satisfied by $f_\pm(y_1) = a_\pm(y_1, 0)$.

Lemma 5.1. *The function $f_\pm(y_1)$ satisfies the following ordinary differential equation.*

$$\left\{\left(y_1\frac{d}{dy_1}\right)^2 - \mu_\pm\left(y_1\frac{d}{dy_1}\right) + \frac{1}{2}\frac{y_1+1}{y_1-1}\left(y_1\frac{d}{dy_1}\right) - \frac{1}{2}\frac{\mu_\pm}{y_1-1}\right\}f_\pm(y_1) = 0.$$

Proof. Inserting $\psi(y_1, y_2) = a_\pm(y_1, y_2)y_2^{\mu_\pm}$ into the equation (3.4), we have

$$y_2^{\mu_\pm}\left(-\left(y_1\frac{\partial}{\partial y_1}\right)^2 a_\pm(y_1, y_2) + \mu_\pm\left(y_1\frac{\partial}{\partial y_1}\right)a_\pm(y_1, y_2)\right.$$

$$
\begin{aligned}
&+ y_2\left(y_1\frac{\partial}{\partial y_1}\right)\frac{\partial a_\pm(y_1,y_2)}{\partial y_2}\\
&-\frac{1}{2}\frac{y_1(y_2^2-1)+(y_1^2y_2^2-1)}{(y_1-1)(y_1y_2^2-1)}\left(y_1\frac{\partial}{\partial y_1}\right)a_\pm(y_1,y_2)\\
&+\frac{1}{2}\frac{y_1^2y_2^2-1}{(y_1-1)(y_1y_2^2-1)}\left(\mu_\pm a_\pm(y_1,y_2)+y_2\frac{\partial a_\pm(y_1,y_2)}{\partial y_2}\right)\bigg)=0.
\end{aligned}
$$

Dividing both sides of this equation by $y_2^{\mu_\pm}$ and taking limit $y_2 \to 0$, then we obtain

$$
\begin{aligned}
-\left(y_1\frac{d}{dy_1}\right)^2 f_\pm(y_1)+\mu_\pm\left(y_1\frac{d}{dy_1}\right)f_\pm(y_1)&\\
-\frac{1}{2}\frac{y_1(-1)+(-1)}{(y_1-1)(-1)}\left(y_1\frac{d}{dy_1}\right)f_\pm(y_1)&\\
+\frac{1}{2}\frac{(-1)\mu_\pm}{(y_1-1)(-1)}f_\pm(y_1)=0.\ \square&
\end{aligned}
$$

Now changing variables as $y_1 = 1/\zeta$, the equation in the previous lemma changes into the Gaussian hypergeometric equation of $\tilde{f}_\pm(\zeta) = f_\pm(y_1)$ with parameters $a = 1/2, b = \mu_\pm, c = \mu_\pm + 1/2 = a + b$:

$$
\left[\zeta(1-\zeta)\frac{d^2}{d\zeta^2}+\left\{(\mu_\pm+\frac{1}{2})-(\mu_\pm+\frac{1}{2}+1)\zeta\right\}\frac{d}{d\zeta}-\frac{1}{2}\mu_\pm\right]\tilde{f}_\pm(\zeta)=0.
$$

The solution of this equation is

$$
\tilde{f}_\pm(\zeta)=c_\pm\mathcal{P}\left\{\begin{matrix}0 & \infty & 1 & \\ 0 & \frac{1}{2} & 0 & :\zeta\\ \frac{1}{2}-\mu_\pm & \mu_\pm & 0 & \end{matrix}\right\}=c_\pm\mathcal{P}\left\{\begin{matrix}0 & \infty & 1 & \\ 0 & \frac{1}{2} & 0 & :1-\zeta\\ 0 & \mu_\pm & \frac{1}{2}-\mu_\pm & \end{matrix}\right\}.
$$

Here, $c_\pm$ are some constants. Therefore, the regular solution around $\zeta = 1$(this means $y_1 = 1$) is

$$
{}_2F_1\left(\frac{1}{2},\mu_\pm;1;1-\zeta\right)={}_2F_1\left(\frac{1}{2},\mu_\pm;1;1-\frac{1}{y_1}\right)
$$

up to a constant multiple.

Using the connection formula of ${}_2F_1$ ([8] equation (9.5.8)):

$$\begin{aligned} {}_2F_1(a,b;c;z) &= (1-z)^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}\,{}_2F_1\left(a, c-b; 1+a-b; \frac{1}{1-z}\right) \\ &+ (1-z)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}\,{}_2F_1\left(c-a, b; 1-a+b; \frac{1}{1-z}\right), \end{aligned}$$

and $1/\{1-(1-1/y_1)\} = y_1$, we obtain

$$\begin{aligned} f_\pm(y_1) &= c_\pm\,{}_2F_1\left(\frac{1}{2}, \mu_\pm; 1; 1-\frac{1}{y_1}\right) \\ &= c_\pm\left\{\left(\frac{1}{y_1}\right)^{-\frac{1}{2}}\frac{\Gamma(1)\Gamma(\mu_\pm-\frac{1}{2})}{\Gamma(1-\frac{1}{2})\Gamma(\mu_\pm)}\,{}_2F_1\left(\frac{1}{2}, 1-\mu_\pm; \frac{3}{2}-\mu_\pm; y_1\right)\right. \\ &+ \left.\left(\frac{1}{y_1}\right)^{-\mu_\pm}\frac{\Gamma(1)\Gamma(\frac{1}{2}-\mu_\pm)}{\Gamma(1-\mu_\pm)\Gamma(\frac{1}{2})}\,{}_2F_1\left(\frac{1}{2}, \mu_\pm; \frac{1}{2}+\mu_\pm; y_1\right)\right\} \\ &= c_\pm\left\{\frac{\Gamma(\mu_\pm-\frac{1}{2})}{\sqrt{\pi}\Gamma(\mu_\pm)}y_1^{\frac{1}{2}}\,{}_2F_1\left(\frac{1}{2}, 1-\mu_\pm; \frac{3}{2}-\mu_\pm; y_1\right)\right. \\ &+ \left.\frac{\Gamma(\frac{1}{2}-\mu_\pm)}{\sqrt{\pi}\Gamma(1-\mu_\pm)}y_1^{\mu_\pm}\,{}_2F_1\left(\frac{1}{2}, \mu_\pm; \frac{1}{2}+\mu_\pm; y_1\right)\right\}. \end{aligned}$$

Note that our hypothesis $\nu \notin \mathbb{Z}$ guarantees that the Gamma functions in numerators have no poles.

The function $a_\pm(y_1,y_2)y_2^{\mu_\pm} = f_\pm(y_1)y_2^{\mu_\pm}(1+O(y_2))$ is a linear combination of $\psi_{\alpha,\beta}$. Comparing the leading term, we have

$$\psi_{\frac{1}{2},\mu_\pm}(y_1,y_2) = y_1^{\frac{1}{2}}y_2^{\mu_\pm}\,{}_2F_1\left(\frac{1}{2}, 1-\mu_\pm; \frac{3}{2}-\mu_\pm; y_1\right) + \text{ (higher order term)},$$

$$\psi_{\mu_\pm,\mu_\pm}(y_1,y_2) = y_1^{\mu_\pm}y_2^{\mu_\pm}\,{}_2F_1\left(\frac{1}{2}, \mu_\pm; \frac{1}{2}+\mu_\pm; y_1\right) + \text{ (higher order term)}$$

and

$$\begin{aligned} &a_\pm(y_1,y_2)y_2^{\mu_\pm} \\ &\quad = c_\pm\left\{\frac{\Gamma(\mu_\pm-\frac{1}{2})}{\sqrt{\pi}\Gamma(\mu_\pm)}\psi_{\frac{1}{2},\mu_\pm}(y_1,y_2) + \frac{\Gamma(\frac{1}{2}-\mu_\pm)}{\sqrt{\pi}\Gamma(1-\mu_\pm)}\psi_{\mu_\pm,\mu_\pm}(y_1,y_2)\right\}. \end{aligned}$$

6. The Exact Asymptotic Expansion

In this section, we determine the value of constants $c_{\pm}$.

The matrix coefficient ϕ corresponding to the corner K-type of $\pi_{(D_l^{\pm},\varepsilon),\nu}$ was proved to be represented as

$$\phi(x_1, x_2) = \delta(x_1, x_2)^{-1} F_{10}\left(\begin{matrix} \mu_+ & \mu_- & \frac{1}{2} & \frac{1}{2} \\ 1 & C & & \end{matrix} ; -\operatorname{sh}^2 x_1, -\operatorname{sh}^2 x_2\right),$$

where F_{10} is the hypergeometric function

$$F_{10}\left(\begin{matrix} a & b & c_1 & c_2 \\ d & e & & \end{matrix} ; x_1, x_2\right) = \sum_{m_i \geq 0} \frac{(a)_{m_1+m_2}(b)_{m_1+m_2}(c_1)_{m_1}(c_2)_{m_2}}{m_1! m_2! (d)_{m_1+m_2}(e)_{m_1+m_2}} x_1^{m_1} x_2^{m_2}$$

and $(\lambda)_n = \dfrac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$ (Theorem 8.1 in [5]).

Theorem 6.1. *Assume $\nu \notin \mathbb{Z}$. We set $C = \dfrac{3+k-l}{2} \in \dfrac{1}{2}\mathbb{Z} - \mathbb{Z}$ and $\mu_{\pm}$ as above. The A-radial part of the matrix coefficient of the P_J-principal series representation with respect to the corner K-type $\tau_{(l,l)}$*

$$\delta(x_1, x_2)^{-1} F_{10}\left(\begin{matrix} \mu_+ & \mu_- & \frac{1}{2} & \frac{1}{2} \\ 1 & C & & \end{matrix} ; -\operatorname{sh}^2 x_1, -\operatorname{sh}^2 x_2\right)$$

has the following expansion around $y_1 = y_2 = 0$:

$$\begin{aligned} &\frac{4^{\mu_+ + l}\Gamma(-\nu)\Gamma(C)}{\sqrt{\pi}\Gamma(\mu_-)\Gamma(C-\mu_+)}\left\{\frac{\Gamma(\mu_+ - \frac{1}{2})}{\Gamma(\mu_+)}\phi_{(l+1)/2, \mu_+ + l}\right. \\ &\left. + \frac{\Gamma(\frac{1}{2}-\mu_+)}{\Gamma(1-\mu_+)}\phi_{\mu_+ + l/2, \mu_+ + l}\right\} \\ &+ \frac{4^{\mu_- + l}\Gamma(\nu)\Gamma(C)}{\sqrt{\pi}\Gamma(\mu_+)\Gamma(C-\mu_-)}\left\{\frac{\Gamma(\mu_- - \frac{1}{2})}{\Gamma(\mu_-)}\phi_{(l+1)/2, \mu_- + l}\right. \\ &\left. + \frac{\Gamma(\frac{1}{2}-\mu_-)}{\Gamma(1-\mu_-)}\phi_{\mu_- + l/2, \mu_- + l}\right\}. \end{aligned}$$

Proof. By setting $B_1 = B_2 = \dfrac{1}{2}, B = B_1 + B_2 = 1$ and $\mu_\pm = \pm\frac{\nu}{2} - \frac{l-2}{2}$, we have

$$F_{10}\left(\begin{matrix} \mu_+ & \mu_- & \frac{1}{2} & \frac{1}{2} \\ 1 & C & & \end{matrix}; \eta_1, \eta_2\right) \tag{6.1}$$
$$= \frac{\Gamma(-\nu)\Gamma(C)}{\Gamma(\mu_-)\Gamma(C-\mu_+)}(-\eta_2)^{-\mu_+}$$
$$\times F_2(\mu_+; \frac{1}{2}, \mu_+ - C + 1; 1, \nu + 1; 1 - \eta_1/\eta_2, 1/\eta_2)$$
$$+ \frac{\Gamma(\nu)\Gamma(C)}{\Gamma(\mu_+)\Gamma(C-\mu_-)}(-\eta_2)^{-\mu_-}$$
$$\times F_2(\mu_-; \frac{1}{2}, \mu_- - C + 1; 1, -\nu + 1; 1 - \eta_1/\eta_2, 1/\eta_2)$$

from the equation (9.8) in [5]. Though we required the condition $B \notin \mathbb{Z}$ in Theorem 9.2 in [5], that condition should be corrected as $B \notin \{0, -1, -2, ...\}$ (see [6]). So we can apply the theorem in the current problem.

We would like to know the asymptotic behavior of the matrix coefficient as $y_1, y_2 \to 0$. Since we put $y_1 = (a_1/a_2)^2, y_2 = (a_2)^2$ and $a_i = \exp x_i$ $(i = 1, 2)$ in §3, the limit $y_1, y_2 \to 0$ corresponds to $x_1, x_2 \to -\infty$.

As $x_i \to -\infty$ (that is, $a_i \to 0$),

$$\begin{aligned} \eta_i &= -\operatorname{sh}^2 x_i = -\frac{\exp(-2x_i)}{4}(1 + O(\exp(2x_i))) \\ &= -\frac{1}{4a_i^2}(1 + O(a_i^2)) \quad (i = 1, 2). \end{aligned}$$

Then we have

$$1 - \frac{\eta_1}{\eta_2} = 1 - \frac{a_2^2}{a_1^2}(1 + O(a_1^2))(1 + O(a_2^2)) = 1 - \frac{1}{y_1}(1 + O(y_1 y_2))(1 + O(y_2)),$$
$$\frac{1}{\eta_2} = -4a_2^2(1 + O(a_2^2)) = -4y_2(1 + O(y_2)).$$

Using results of §5, the equation (6.1) is asymptotically written as

$$F_{10}\left(\begin{matrix} \mu_+ & \mu_- & \frac{1}{2} & \frac{1}{2} \\ 1 & C & & \end{matrix}; -\operatorname{sh}^2 x_1, -\operatorname{sh}^2 x_2\right)$$
$$\sim \frac{\Gamma(-\nu)\Gamma(C)}{\Gamma(\mu_-)\Gamma(C-\mu_+)}(4y_2)^{\mu_+} F_2(\mu_+; \frac{1}{2}, \mu_+ - C + 1; 1, \nu + 1; 1 - y_1^{-1}, -4y_2)$$

$$
\begin{aligned}
&\quad + \frac{\Gamma(\nu)\Gamma(C)}{\Gamma(\mu_+)\Gamma(C-\mu_-)}(4y_2)^{\mu_-} \\
&\quad \times F_2(\mu_-; \frac{1}{2}, \mu_- - C + 1; 1, -\nu+1; 1-y_1^{-1}, -4y_2) \\
&\sim \frac{4^{\mu_+}\Gamma(-\nu)\Gamma(C)}{\Gamma(\mu_-)\Gamma(C-\mu_+)} y_2^{\mu_+} {}_2F_1(\mu_+, \frac{1}{2}; 1; 1-y_1^{-1}) \\
&\quad + \frac{4^{\mu_-}\Gamma(\nu)\Gamma(C)}{\Gamma(\mu_+)\Gamma(C-\mu_-)} y_2^{\mu_-} {}_2F_1(\mu_-, \frac{1}{2}; 1; 1-y_1^{-1}) \\
&= \frac{4^{\mu_+}\Gamma(-\nu)\Gamma(C)}{\Gamma(\mu_-)\Gamma(C-\mu_+)} y_2^{\mu_+} \cdot \frac{1}{c_+} f_+(y_1) \\
&\quad + \frac{4^{\mu_-}\Gamma(\nu)\Gamma(C)}{\Gamma(\mu_+)\Gamma(C-\mu_-)} y_2^{\mu_-} \cdot \frac{1}{c_-} f_-(y_1) \\
&\sim \frac{4^{\mu_+}\Gamma(-\nu)\Gamma(C)}{\Gamma(\mu_-)\Gamma(C-\mu_+)} \cdot \frac{1}{c_+} a_+(y_1,y_2) y_2^{\mu_+} \\
&\quad + \frac{4^{\mu_-}\Gamma(\nu)\Gamma(C)}{\Gamma(\mu_+)\Gamma(C-\mu_-)} \cdot \frac{1}{c_-} a_-(y_1,y_2) y_2^{\mu_-} \\
&= \frac{4^{\mu_+}\Gamma(-\nu)\Gamma(C)}{\Gamma(\mu_-)\Gamma(C-\mu_+)} \left(\frac{\Gamma(\mu_+ - \frac{1}{2})}{\sqrt{\pi}\Gamma(\mu_+)} \psi_{\frac{1}{2},\mu_+}(y_1,y_2) \right. \\
&\qquad \left. + \frac{\Gamma(\frac{1}{2}-\mu_+)}{\sqrt{\pi}\Gamma(1-\mu_+)} \psi_{\mu_+,\mu_+}(y_1,y_2) \right) \\
&\quad + \frac{4^{\mu_-}\Gamma(\nu)\Gamma(C)}{\Gamma(\mu_+)\Gamma(C-\mu_-)} \left(\frac{\Gamma(\mu_- - \frac{1}{2})}{\sqrt{\pi}\Gamma(\mu_-)} \psi_{\frac{1}{2},\mu_-}(y_1,y_2) \right. \\
&\qquad \left. + \frac{\Gamma(\frac{1}{2}-\mu_-)}{\sqrt{\pi}\Gamma(1-\mu_-)} \psi_{\mu_-,\mu_-}(y_1,y_2) \right).
\end{aligned}
$$

Since $y_2 = a_2^2 > 0$, the blanch of the complex power $y_2^{\mu_\pm}$ is determined.

Multiplying $\delta(x_1,x_2)^{-1}$ on both sides of the above equation, we have the result by the equation (4.2). □

From the above theorem, we find that analogues of c-functions are

$$
\begin{aligned}
c_1(k,l,\nu) &= \frac{4^{\mu_+ + l}\Gamma(-\nu)\Gamma(C)\Gamma(\mu_+ - \frac{1}{2})}{\sqrt{\pi}\Gamma(\mu_-)\Gamma(C-\mu_+)\Gamma(\mu_+)} \\
&= \frac{2^{\nu+l+2}\Gamma(-\nu)\Gamma(\frac{k-l+3}{2})\Gamma(\frac{\nu-l+1}{2})}{\sqrt{\pi}\Gamma(\frac{-\nu-l+2}{2})\Gamma(\frac{-\nu+k+1}{2})\Gamma(\frac{\nu-l+2}{2})},
\end{aligned}
$$

$$\begin{aligned}
c_2(k,l,\nu) &= \frac{4^{\mu_-+l}\Gamma(\nu)\Gamma(C)\Gamma(\mu_- - \frac{1}{2})}{\sqrt{\pi}\Gamma(\mu_+)\Gamma(C-\mu_-)\Gamma(\mu_-)} \\
&= \frac{2^{-\nu+l+2}\Gamma(\nu)\Gamma(\frac{k-l+3}{2})\Gamma(\frac{-\nu-l+1}{2})}{\sqrt{\pi}\Gamma(\frac{\nu-l+2}{2})\Gamma(\frac{\nu+k+1}{2})\Gamma(\frac{-\nu-l+2}{2})} = c_1(k,l,-\nu), \\
c_3(k,l,\nu) &= \frac{4^{\mu_++l}\Gamma(-\nu)\Gamma(C)\Gamma(\frac{1}{2}-\mu_+)}{\sqrt{\pi}\Gamma(\mu_-)\Gamma(C-\mu_+)\Gamma(1-\mu_+)} \\
&= \frac{2^{\nu+l+2}\Gamma(-\nu)\Gamma(\frac{k-l+3}{2})\Gamma(\frac{-\nu+l-1}{2})}{\sqrt{\pi}\Gamma(\frac{-\nu-l+2}{2})\Gamma(\frac{-\nu+k+1}{2})\Gamma(\frac{-\nu+l}{2})}, \\
c_4(k,l,\nu) &= \frac{4^{\mu_-+l}\Gamma(\nu)\Gamma(C)\Gamma(\frac{1}{2}-\mu_-)}{\sqrt{\pi}\Gamma(\mu_+)\Gamma(C-\mu_-)\Gamma(1-\mu_-)} \\
&= \frac{2^{-\nu+l+2}\Gamma(\nu)\Gamma(\frac{k-l+3}{2})\Gamma(\frac{\nu+l-1}{2})}{\sqrt{\pi}\Gamma(\frac{\nu-l+2}{2})\Gamma(\frac{\nu+k+1}{2})\Gamma(\frac{\nu+l}{2})} = c_3(k,l,-\nu).
\end{aligned}$$

7. Power Series Expansion of the Fundamental System of Solutions

In this section, we have explicit power series expression of $\psi_{\alpha,\beta}$.

THEOREM 7.1. *Let u_1, u_2 be*

$$u_1 = \frac{\operatorname{sh}^2 x_2}{\operatorname{sh}^2 x_1}, \qquad u_2 = -\frac{1}{\operatorname{sh}^2 x_2}. \tag{7.1}$$

Then, $\psi_{\alpha,\beta}(y_1,y_2) = (-4)^{-\beta}\Psi_{\alpha,\beta}(u_1,u_2)$, where

$$\begin{aligned}
&\Psi_{\frac{1}{2},\mu_\pm}(u_1,u_2) \\
&= u_1^{\frac{1}{2}} u_2^{\mu_\pm} \sum_{m,n\geq 0} \frac{(-\mu_\pm - n + 1)_m \left(\frac{1}{2}\right)_m}{\left(-\mu_\pm - n + \frac{3}{2}\right)_m m!} \\
&\quad \cdot \frac{(\mu_\pm - \frac{1}{2})_n(\mu_\pm - \frac{1}{2} + \frac{l-k}{2})_n}{(2\mu_\pm + l - 1)_n n!} u_1^m u_2^n \\
&= u_1^{\frac{1}{2}} u_2^{\mu_\pm} \sum_{n=0}^{\infty} \frac{(\mu_\pm - \frac{1}{2})_n(\mu_\pm - \frac{1}{2} + \frac{l-k}{2})_n}{(2\mu_\pm + l - 1)_n n!}
\end{aligned} \tag{7.2}$$

$$\times {}_2F_1\left(\frac{1}{2}, -\mu_\pm - n + 1; -\mu_\pm - n + \frac{3}{2}; u_1\right) u_2^n$$

and

(7.3)
$$\begin{aligned}
&\Psi_{\mu_\pm,\mu_\pm}(u_1,u_2)\\
&= u_1^{\mu_\pm} u_2^{\mu_\pm} \sum_{n,j\geq 0} \frac{(\frac{1}{2})_j(\mu_\pm+n)_j}{j!(\mu_\pm+n+\frac{1}{2})_j} \cdot \frac{(\mu_\pm)_n(\mu_\pm - 1 + \frac{l-k}{2})_n}{(2\mu_\pm + l - 1)_n n!} u_1^{n+j} u_2^n\\
&= u_1^{\mu_\pm} u_2^{\mu_\pm} \sum_{n=0}^{\infty} \frac{(\mu_\pm)_n(\mu_\pm - 1 + \frac{l-k}{2})_n}{(2\mu_\pm + l - 1)_n n!}\\
&\quad \times {}_2F_1\left(\frac{1}{2}, \mu_\pm + n; \mu_\pm + n + \frac{1}{2}; u_1\right) u_1^n u_2^n.
\end{aligned}$$

Here

$$(a)_n = a(a+1)\cdots(a+n-1) \quad (n \neq 0), \qquad (a)_0 = 1.$$

As is seen in the proof of Theorem 6.1, we have $u_1 \sim y_1$ and $u_2 \sim -4y_2$ as $y_1, y_2 \to 0$.

At first, we will express differential operators in Proposition 3.2 with new variables u_1, u_2.

PROPOSITION 7.2. *Differential operators in equations* (3.3) *and* (3.4) *are written as following operators* P *and* Q*(up to function multiple) respectively.*

(7.4)
$$\begin{aligned}
P \ =\ & 4(2-u_2-u_1u_2)\vartheta_1^2 + 4(1-u_2)\vartheta_2^2 - 8(1-u_2)\vartheta_1\vartheta_2\\
&+\frac{4}{u_1-1}\left\{1+u_1-\frac{l-k}{2}u_2+(l-k-2)u_1u_2-\frac{l-k}{2}u_1^2u_2\right\}\vartheta_1\\
&-\frac{4}{u_1-1}\left\{l-1-(l-2)u_1-\frac{l-k}{2}u_2+\frac{l-k-2}{2}u_1u_2\right\}\vartheta_2\\
&+(l-2)^2-\nu^2,
\end{aligned}$$

(7.5)
$$Q \ =\ \vartheta_1^2 - \vartheta_1\vartheta_2 + \frac{1}{2}\frac{u_1+1}{u_1-1}\vartheta_1 - \frac{1}{2}\frac{1}{u_1-1}\vartheta_2.$$

Here, we set $\vartheta_i = u_i\dfrac{\partial}{\partial u_i}$ $(i = 1, 2)$.

Indicial equations of P and Q are

$$8\alpha^2 + 4\beta^2 - 8\alpha\beta - 4\alpha + 4(l-1)\beta + (l-2)^2 - \nu_1^2 = 0,$$
$$\alpha^2 - \alpha\beta - \frac{1}{2}\alpha + \frac{1}{2}\beta = 0$$

respectively and the solutions of these equations are

$$(\alpha, \beta) = (\frac{1}{2}, \mu_\pm), (\mu_\pm, \mu_\pm),$$

where $\mu_\pm = \pm\nu/2 - (l-2)/2$.

This is a kind of the modified Appell's F_2 system.

Now we put the analytic kernel of P, Q as

$$\Psi_{\alpha,\beta}(u_1, u_2) = u_1^\alpha u_2^\beta \sum_{m,n\geq 0} a_{m,n} u_1^m u_2^n.$$

We normalize this solution as $a_{0,0} = 1$.

Comparing the leading term of $\Psi_{\alpha,\beta}$ with that of $\psi_{\alpha,\beta}$ as $y_1, y_2 \to 0$, we can see that

$$\Psi_{\alpha,\beta}(u_1, u_2) = (-4)^\beta \psi_{\alpha,\beta}(y_1, y_2).$$

Note that $\mu_\pm \notin \frac{1}{2}\mathbb{Z}$, since we assume that $\nu \notin \mathbb{Z}$ in §5.

It is easy to prove the following lemma.

Lemma 7.3. *If $Q\Psi_{\alpha,\beta}(u_1, u_2) = 0$ holds, then $a_{m,n}$ satisfy the following recurrence equations.*

$$(7.6)\quad (\alpha + m - 1)(\alpha - \beta + m - n - \frac{1}{2})a_{m-1,n} + (-\alpha + \beta - m + n)(\alpha + m - \frac{1}{2})a_{m,n} = 0 \qquad (m \neq 0)$$

$$(7.7)\quad (-\alpha + \beta + n)(\alpha - \frac{1}{2})a_{0,n} = 0$$

7.1. Case 1 : $\alpha = \frac{1}{2}$

In this subsection we determine $a_{m,n}$ for the case $(\alpha, \beta) = (1/2, \mu_\pm)$.

In this case, any $a_{0,n}$ satisfies (7.7). On the other hand, we have

$$(7.8) \qquad a_{m,n}$$
$$= \frac{(\beta - m + n)(\beta - m + n + 1)\cdots(\beta + n - 1)\cdot(m - \frac{1}{2})(m - \frac{3}{2})\cdots\frac{1}{2}}{(\beta - m + n - \frac{1}{2})(\beta - m + n + \frac{1}{2})\cdots(\beta + n - \frac{3}{2})\cdot m(m-1)\cdots 1} a_{0,n}$$
$$= \frac{(-\beta - n + 1)_m \left(\frac{1}{2}\right)_m}{\left(-\beta - n + \frac{3}{2}\right)_m m!} a_{0,n}$$
$$= \frac{(-\mu_\pm - n + 1)_m \left(\frac{1}{2}\right)_m}{\left(-\mu_\pm - n + \frac{3}{2}\right)_m m!} a_{0,n}$$

from (7.6).

Therefore we have only to determine $a_{0,n}$ $(n \geq 0)$ from $P\Psi_{\alpha,\beta}(u_1, u_2) = 0$.

Setting $\phi_m(u_2) = \sum_{n=0}^{\infty} a_{m,n} u_2^n$, we have

$$\Psi_{\alpha,\beta}(u_1, u_2) = u_1^\alpha u_2^\beta \sum_{m,n \geq 0} a_{m,n} u_1^m u_2^n = u_1^\alpha u_2^\beta \sum_{m=0}^{\infty} \phi_m(u_2) u_1^m.$$

So we have only to determine $\phi_0(u_2)$.

LEMMA 7.4. *$\phi_0(u_2)$ is a solution of the Gaussian hypergeometric equation :*

$$u_2(1 - u_2)\phi_0''(u_2) + \{r - (p + q + 1)u_2\}\, \phi_0'(u_2) - pq\phi_0(u_2) = 0.$$

In particular, we have $\phi_0(u_2) = {}_2F_1(p, q; r; u_2)$ and

$$(7.9) \qquad a_{0,n} = \frac{(p)_n (q)_n}{(r)_n n!},$$

with parameters

$$p = \mu_\pm - \frac{1}{2}, \qquad q = \mu_\pm - \frac{1}{2} + \frac{l - k}{2}, \qquad r = 2\mu_\pm + l - 1.$$

Proof. We obtain the Gaussian hypergeometric equation from

$$\frac{1}{4}u_1^{-\alpha}u_2^{-\beta-1}P\Psi_{\alpha,\beta}(u_1,u_2)|_{u_1=0}=0$$

using

$$\begin{aligned}
\vartheta_1\Psi_{\alpha,\beta} &= u_1^{\alpha}u_2^{\beta}\sum_{m=0}^{\infty}(\alpha+m)\phi_m(u_2)u_1^m,\\
\vartheta_1^2\Psi_{\alpha,\beta} &= u_1^{\alpha}u_2^{\beta}\sum_{m=0}^{\infty}(\alpha+m)^2\phi_m(u_2)u_1^m,\\
\vartheta_2\Psi_{\alpha,\beta} &= \beta\Psi_{\alpha,\beta}+u_1^{\alpha}u_2^{\beta+1}\sum_{m=0}^{\infty}\phi'_m(u_2)u_1^m\\
&= u_1^{\alpha}u_2^{\beta}\sum_{m=0}^{\infty}\left\{\beta\phi_m(u_2)+u_2\phi'_m(u_2)\right\}u_1^m,\\
\vartheta_2^2\Psi_{\alpha,\beta} &= u_1^{\alpha}u_2^{\beta}\sum_{m=0}^{\infty}\left\{\beta^2\phi_m(u_2)+(2\beta+1)u_2\phi'_m(u_2)+u_2^2\phi''_m(u_2)\right\}u_1^m,\\
\vartheta_1\vartheta_2\Psi_{\alpha,\beta} &= u_1^{\alpha}u_2^{\beta}\sum_{m=0}^{\infty}(\alpha+m)\left\{\beta\phi_m(u_2)+u_2\phi'_m(u_2)\right\}u_1^m
\end{aligned}$$

and indicial equations above.

The latter part is obvious from the assumption that $a_{0,0}=1$. □

Now, we have

$$a_{m,n}=\frac{(-\mu_\pm-n+1)_m\left(\frac{1}{2}\right)_m}{\left(-\mu_\pm-n+\frac{3}{2}\right)_m m!}\cdot\frac{(\mu_\pm-\frac{1}{2})_n(\mu_\pm-\frac{1}{2}+\frac{l-k}{2})_n}{(2\mu_\pm+l-1)_n n!}$$

from equations (7.8), (7.9) and we have shown the former half of Theorem 7.1.

7.2. Case 2 : $\alpha\neq\frac{1}{2}$

In this subsection we assume that $\alpha\neq 1/2$, that is, $\alpha=\beta=\mu_\pm$. From this condition and the equation (7.7), we have $a_{0,n}=0$ $(n\geq 1)$. At the same time we have

$$(\alpha+m-1)(m-n-\frac{1}{2})a_{m-1,n}+(-m+n)(\alpha+m-\frac{1}{2})a_{m,n}=0$$

from (7.6).

Note that factors except $(-m+n)$ are not zero by the assumption $\alpha = \mu_\pm \notin \frac{1}{2}\mathbb{Z}$.

If $m=n$, then we have $a_{m-1,m}=0$.

If $m<n$, then we have $a_{m,n} = Ca_{0,n} = 0$ (C is a constant which depends on m and n).

If $m>n$, then we have

(7.10)
$$\begin{aligned} a_{m,n} &= \frac{(m-n-\frac{1}{2})(m-n-\frac{3}{2})\cdots\frac{1}{2}\cdot(\alpha+m-1)(\alpha+m-2)\cdots(\alpha+n)}{(m-n)(m-n-1)\cdots 1\cdot(\alpha+m-\frac{1}{2})(\alpha+m-\frac{3}{2})\cdots(\alpha+n+\frac{1}{2})} a_{n,n} \\ &= \frac{(\frac{1}{2})_{m-n}(\alpha+n)_{m-n}}{(m-n)!(\alpha+n+\frac{1}{2})_{m-n}} a_{n,n} \\ &= \frac{(\frac{1}{2})_{m-n}(\mu_\pm+n)_{m-n}}{(m-n)!(\mu_\pm+n+\frac{1}{2})_{m-n}} a_{n,n} \end{aligned}$$

from (7.6).

Therefore it remains to determine $a_{n,n}$ $(n \geq 0)$ from $P\Psi_{\alpha,\beta}(u_1,u_2)=0$.

Since

$$\begin{aligned} \Psi_{\mu_\pm,\mu_\pm}(u_1,u_2) &= u_1^{\mu_\pm} u_2^{\mu_\pm} \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} a_{n+j,n} u_1^{n+j} u_2^{n} \\ &= (u_1u_2)^{\mu_\pm} \sum_{j=0}^{\infty} u_1^j \sum_{n=0}^{\infty} a_{n+j,n}(u_1u_2)^n \end{aligned}$$

holds, we have

$$\Psi_{\mu_\pm,\mu_\pm}(u_1,u_2) = t^{\mu_\pm}\sum_{j=0}^{\infty}\varphi_j(t)s^j$$

where $s=u_1, t=u_1u_2, \varphi_j(t) = \sum_{n=0}^{\infty} a_{n+j,n}t^n$. We have only to determine $\varphi_0(t)$.

Lemma 7.5. *$\varphi_0(t)$ is a solution of the Gaussian hypergeometric equation :*

$$t(1-t)\varphi_0''(t) + \{c-(a+b+1)t\}\varphi_0'(t) - ab\varphi_0(t) = 0.$$

Hence we have $\varphi_0(t) = {}_2F_1(a, b; c; t)$ *and*

$$a_{n,n} = \frac{(a)_n(b)_n}{(c)_n n!}, \tag{7.11}$$

with parameters

$$a = \mu_\pm, \qquad b = \mu_\pm - 1 + \frac{l-k}{2}, \qquad c = 2\mu_\pm + l - 1.$$

Proof. We denote $\vartheta_s = s\frac{\partial}{\partial s}, \vartheta_t = t\frac{\partial}{\partial t}$. Then we have

$$\begin{aligned} \vartheta_1 &= u_1\left(\frac{\partial s}{\partial u_1}\frac{\partial}{\partial s} + \frac{\partial t}{\partial u_1}\frac{\partial}{\partial t}\right) = u_1\left(\frac{\partial}{\partial s} + u_2\frac{\partial}{\partial t}\right) = \vartheta_s + \vartheta_t, \\ \vartheta_2 &= u_2\left(\frac{\partial s}{\partial u_2}\frac{\partial}{\partial s} + \frac{\partial t}{\partial u_2}\frac{\partial}{\partial t}\right) = \vartheta_t. \end{aligned}$$

Using above equations, we obtain the Gaussian hypergeometric equation from

$$P\Psi_{\mu_\pm,\mu_\pm}(s,t) = 0$$

in the similar way as Proposition 7.4. □

Now, we have

$$a_{m,n} = \begin{cases} 0 & (m < n) \\ \frac{(\frac{1}{2})_{m-n}(\mu_\pm + n)_{m-n}}{(m-n)!(\mu_\pm + n + \frac{1}{2})_{m-n}} \cdot \frac{(\mu_\pm)_n(\mu_\pm - 1 + \frac{l-k}{2})_n}{(2\mu_\pm + l - 1)_n n!} & (m \geq n) \end{cases}$$

from equations (7.10), (7.11) and we have shown the latter half of Theorem 7.1.

Remark 7.6. It is an interesting problem to compare our power series solutions with the confluenced ones which were discussed in [3].

References

[1] Casselman, W. and D. Miličić, Asymptotic behavior of matrix coefficients of admissible representations, Duke Math. J. **49** (1982), no. 4, 869–930.

[2] Harish-Chandra, Spherical functions on a semi-simple Lie group I, Amer. J. Math. **80** (1958), 241–310.
[3] Hirano, M., Ishii, T. and T. Oda, Confluence Siegel-Whittaker functions to Whittaker functions on $Sp(2, \mathbb{R})$, Math. Proc. Camb. Phil. Soc. **141** (2006), 15–31.
[4] Hirano, M. and T. Oda, Secondary Whittaker functions for P_J-principal series representations of $Sp(3, \mathbb{R})$, Proc. Japan Acad. Ser. A Math. Sci. **81** (2005), no. 6, 105–109.
[5] Iida, M., Spherical functions of the principal series representations of $Sp(2, \mathbb{R})$ as hypergeometric functions of C_2-type, Publ. RIMS, Kyoto Univ. **32** (1996), 689–727.
[6] Iida, M., Errata: Spherical functions of the principal series representations of $Sp(2, \mathbb{R})$ as hypergeometric functions of C_2-type (Publ. RIMS, Kyoto Univ. **32** (1996), 689–727), Publ. RIMS, Kyoto Univ. **43** (2007), no. 2, 521–524.
[7] Koseki, H. and T. Oda, Matrix coefficients of representations of $SU(2, 2)$: the case of P_J-principal series, Internat. J. Math. **15** (2004), no. 10, 1033–1064.
[8] Levedev, N. N., Special functions and their applications, translated by Richard A. Silverman, Dover Publ. Inc., 1972.
[9] Oshima, T., Asymptotic behavior of spherical functions on semisimple symmetric spaces, Adv. Stud. Pure Math. **14**, Academic Press, Boston, 561–601, 1988.
[10] Takayama, N., Propagation of singularities of solutions of the Euler-Darboux equations and a global structure of holomorphic solutions II, Funkcial. Ekvac. **36** (1993), 187–234.
[11] Wallach, N. R., Asymptotic expansions of generalized matrix entries of representations of real reductive groups, Lie group representations I(College park, Md. 1982/1983), 287–369, Lecture Notes in Math. 1024, Springer, Berlin, 1983.

(Received April 16, 2007)
(Revised November 17, 2008)

Masatoshi IIDA
Department of Mathematics
Josai University
Keyakidai, Sakado-shi,
Saitama, Japan
E-mail: iida@math.josai.ac.jp

Takayuki ODA
Graduate school of Mathematical Sciences
University of Tokyo
Komaba, Meguro
Tokyo, Japan
E-mail: takayuki@ms.u-tokyo.ac.jp