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Quantization of Differential Systems with the Affine Weyl Group Symmetries of Type $C_N^{(1)}$

By Hajime NAGOYA

Abstract. We construct systems of nonlinear differential equations with affine Weyl group symmetries of type $C_N^{(1)}$, as compatibility conditions of linear differential equations. These systems are Hamiltonian systems and those Hamiltonians are mutually commutative. We also construct a quantization of these systems with affine Weyl group symmetries of type $C_N^{(1)}$.

1. Introduction

Classical integrable systems of finite dimension, such as the Painlevé equations or their higher order analogues, can be viewed as compatibility conditions of linear differential equations (Lax form) ([21], [22], [31], etc.). One of the systematic methods constructing Lax forms is by similarity reduction [24] of (generalized) Drinfel'd-Sokolov hierarchies [17]. The Drinfel'd-Sokolov hierarchies give soliton equations such as the Korteweg-de Vries (KdV) equation [11]. These hierarchies are constructed in terms of affine Lie algebras and their corresponding Lie groups by Gauss decomposition. In the construction of those integrable hierarchies, Heisenberg subalgebras of affine Lie algebras play a crucial role. It is well known that there exists a relationship between soliton equations and the Painlevé equations, which is explained clearly by similarity reduction from generalized Drinfel'd-Sokolov hierarchies.

Infinite-dimensional quantum integrable systems, such as quantum soliton equations, are widely studied (for example, [40], [9], [10], [4]-[8], [12]). Here, it is natural to quantize the Painlevé equations, their higher order analogues and general differential systems through similarity reduction from Drinfel'd-Sokolov hierarchies. This means, in particular, constructing mutually commuting quantum Hamiltonians using the appropriate algebra, whose symbols are the classical Hamiltonians, establishing group actions

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compatible with Hamiltonians and obtaining solutions of the corresponding Hamiltonian systems on representations of this algebra.

It has been known that the Painlevé equations P_{II}, P_{III}, P_{IV} P_{V} and P_{VI} admit the affine Weyl group actions of type $A_1^{(1)}$, $C_2^{(1)}$, $A_2^{(1)}$, $A_3^{(1)}$ and $D_4^{(1)}$, respectively, as groups of Bäcklund transformations [34]. A. P. Veselov and A. B. Shabat studied the dressing chains that can be considered as higher order analogues of the fourth Painlevé equation P_{IV} and the fifth Painlevé equation P_{V} [39]. By using the dressing chains, V. E. Adler introduced the symmetric form of P_{IV} [1]. M. Noumi and Y. Yamada constructed a class of discrete dynamical systems associated with affine Weyl group actions providing a general framework to describe the structure of Bäcklund transformations of differential systems of Painlevé type [30]. K. Hasegawa constructed quantization of the difference version of this action [19]. The Painlevé equations can be represented as non-autonomous Hamiltonian systems as well as their higher order analogues. Quantization of the autonomous Hamiltonians of the fourth Painlevé equation and its higher analogue was performed in [26]. Quantization of the fourth and fifth Painlevé equations and their higher analogues was proposed in [27]. Similar to the classical Painlevé equations, the quantum Painlevé equations also possess affine Weyl group symmetries as Bäcklund transformations of the quantum Painlevé equations. Quantization of the affine Weyl group actions on Poisson algebras of the Painlevé type differential systems in [32] was established in [25]. Takano's theory for the classical Painlevé equations [33] is also applicable to the quantum Painlevé equations [38]. Note that we have not obtained solutions of the quantum Painlevé equations yet.

About quantization of monodromy preserving deformation, the following is known. The Schlesinger equation is the first example of a nonlinear differential equation obtained as the compatibility condition for linear differential equations. It describes the monodromy preserving deformation of linear differential equations with regular singularities. In special case, the Schlesinger equation becomes the sixth Painlevé equation [14]. The Knizhnik-Zamolodchikov (KZ) equation is a set of constraints to be satisfied by the correlation functions in the conformal field theory and can be regarded as quantization of the Schlesinger equation [36], [18]. However, we have not yet derived the sixth quantum Painlevé equation from the KZ equation. In the cases of Poincaré rank 1 at the infinity, a quantization is constructed in [2], [13]. For any value of the Poincaré rank, confluent KZ equations were constructed for $\mathfrak{sl}_2$, and quantum Painlevé equations $\mathrm{QP_I}$, ... , $\mathrm{QP_V}$ are formally derived from these equations [20].

In a previous paper [28], we discussed the quantization of differential systems by similarity reduction from the Drinfel'd-Sokolov hierarchies of type $A^{(1)}_{n-1}$ with respect to the principal gradation. For this purpose, the Lax matrix is very important. Because the commuting quantum Hamiltonians were constructed using the trace of the power of the Lax matrix, in addition to the classical Hamiltonians. We would like to extend the results obtained in this study to other affine Lie algebras. In this paper, we quantize Hamiltonian systems with affine Weyl group symmetries of type $C^{(1)}_N$, which were obtained by similarity reduction from the Drinfel'd-Sokolov hierarchies of type $C^{(1)}_N$ with respect to the principal gradation. More precisely, the Lax matrix M is the same one obtained by similarity reduction from above condition and matrices $B_{s,k}$ in this paper are different from matrices B_l by similarity reduction in general. We have not yet know the connection between them.

The third Painlevé equation has the affine Weyl group action of type $C^{(1)}_2$. Although we deal with differential systems with the affine Weyl group symmetriy of type $C^{(1)}_2$ in this paper, $\mathrm{P_{III}}$ is not included in them. The easiest and most nontrivial case of type $C^{(1)}_2$ in this paper (the example 2.4) is a 2-parameter family of 2-coupled $\mathrm{P_{II}}$ systems in [29], [37]. Also the easiest and most nontrivial case of type $C^{(1)}_4$ in this paper is a 4-parameter family of 2-coupled $D^{(2)}_3$-systems in [37].

The structure of the rest of the paper is as follows. In the subsequent section, we introduce classical differential systems with affine Weyl group symmetries of type $C^{(1)}_N$. In Section 3, we quantize these classical differential systems in the following manner. First, we define a derivation using the Lax equation. Second, we present an action of the affine Weyl group of type $C^{(1)}_N$ compatible with the derivation using gauge transformations. Finally, we show that the differential systems defined by the derivation are Hamiltonian systems.

2. Classical Case

In this section, we present differential systems with affine Weyl group symmetries of type $C_N^{(1)}$, by using symplectic algebra $\mathfrak{sp}(2N,\mathbb{C})$. We use the following matrix representation:

$$\mathfrak{sp}(2N,\mathbb{C}) = \left\{X \in gl(n,\mathbb{C}) | XJ + JX^t = 0\right\}, \tag{2.1}$$

where $n = 2N$ for $N = 2, 3, \ldots$, X^t is the transpose of X and

$$J = \begin{pmatrix} 0 & E_N \\ -E_N & 0 \end{pmatrix}, \quad E_N = \sum_{i=1}^{N} E_{i,N+1-i}. \tag{2.2}$$

2.1. Lax equation

We introduce a fundamental rational function field $C_{m,n}$ $(m < n)$ for $m \in \mathbb{N}$ over $\mathbb{C}$. The generators of $C_{m,n}$ are ϵ_i, $f_{i,i+j}$ $(i = 1, \ldots, n, j = 1, \ldots, m)$ and the defining relations of $C_{m,n}$ are

$$\epsilon_i = s(i,i)\epsilon_{1-i}, \quad f_{i,j} = s(i,j)f_{1-i,1-j}, \quad f_{n+i,n+j} = f_{i,j} \tag{2.3}$$

where $s(i,j)$ is a function over $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ such that $s(i,j) = s(j,i)$ and

$$s(i,j) = \begin{cases} -1 & (1 \leq i,j \leq N \quad or \quad N+1 \leq i,j \leq n) \\ 1 & (1 \leq i \leq N, \quad N+1 \leq j \leq n) \end{cases}.$$

Note that $-s(i,k) = s(i,j)s(j,k)$.

DEFINITION 2.1. Let $C_{m,n}[z,z^{-1}]$ be the polynomial ring and we define a matrix element $M_{m,n}$ in $M_{n,n}(C_{m,n}[z,z^{-1}])$ by

$$(M_{m,n})_{ii} = \epsilon_i \quad (i = 1, \ldots, n), \tag{2.4}$$

$$(M_{m,n})_{i,j} = f_{i,j} \quad (i < j), \quad (M_{m,n})_{i,j} = zf_{i,j+n} \quad (i > j). \tag{2.5}$$

In what follows, we omit the index of $M_{m,n}$, namely we write simply M instead of $M_{m,n}$. We call M Lax operator.

Example 2.2. We give an example of M for $m = 3$ and $n = 4$:

$$M = \begin{pmatrix} \epsilon_1 & f_{12} & f_{13} & f_{14} \\ zf_{25} & \epsilon_2 & f_{23} & f_{24} \\ zf_{35} & zf_{36} & \epsilon_3 & f_{34} \\ zf_{45} & zf_{46} & zf_{47} & \epsilon_4 \end{pmatrix}.$$

If we write only independent elements, then

$$M = \begin{pmatrix} \epsilon_1 & f_{12} & f_{13} & f_{14} \\ zf_{25} & \epsilon_2 & f_{23} & f_{13} \\ zf_{35} & zf_{36} & -\epsilon_2 & -f_{12} \\ zf_{45} & zf_{35} & -zf_{25} & -\epsilon_1 \end{pmatrix}.$$

We define a Poisson bracket $\{,\}$ on $C_{m,n}$ by

$$\begin{aligned}\{f_{ij}, f_{kl}\} = \frac{1}{2}(&\delta_{j,k} f_{i,j+l-k} + \delta_{j,1-l} s(k,l) f_{i,j+l-k} \\ &- \delta_{l,i} f_{k,j+l-i} - \delta_{1-k,i} s(k,l) f_{1-l,1+j-i-k}),\end{aligned} \tag{2.6}$$

where $\delta_{j,k}$ is defined by

$$\delta_{j,k} = \begin{cases} 1 & (j \equiv k \pmod{n}) \\ 0 & (j \not\equiv k \pmod{n}) \end{cases}. \tag{2.7}$$

This Poisson bracket is induced from a certain subalgebra of the affine Lie algebra of type $C_N^{(1)}$.

Let $g = \mathfrak{sp}(2N, \mathbb{C}) \otimes C_{m,n}[z, z^{-1}]$ and let $g_{\geq 0}$, $g_{<0}$ be subalgebra of g defined by

$$\begin{aligned} g_{\geq 0} = \left\{ \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & * \end{pmatrix} + z \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix} \right. \\ \left. + z^2 \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix} + \cdots \right\}, \end{aligned} \tag{2.8}$$

$$g_{<0} = \left\{ \begin{pmatrix} 0 & 0 & \cdots & 0 \\ * & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ * & \cdots & * & 0 \end{pmatrix} + z^{-1} \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix} \right. \\ \left. + z^{-2} \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix} + \cdots \right\} \tag{2.9}$$

then it holds that $g = g_{\geq 0} \oplus g_{<0}$ and $M \in g_{\geq 0}$. We denote the decomposition of $X \in g$ by $X = X_+ + X_-$ for $X_+ \in g_{\geq 0}$ and $X_- \in g_{<0}$.

DEFINITION 2.3. For $s, k \in \mathbb{N}$ and that k is odd, we define a $\mathbb{C}$-derivation $\partial_{s,k}$ by the Lax equation

$$\partial_{s,k}(M) = [M, B_{s,k}] + \kappa z \partial_z(B_{s,k}), \tag{2.10}$$

where $B_{s,k} = (M^k z^{-s})_+$, $\kappa \in \mathbb{C}$ and ∂_z is the $C_{m,n}$-derivation which takes z to 1.

It is needed that k is odd, because if k is even, then $B_{s,k}$ and $[M, B_{s,k}]$ are not in g in general.

Example 2.4. We give an example of the system defined by $\partial_{s,k}$ for $m = 3$, $n = 4$, $s = 2$, $k = 3$.

The generators of $\mathcal{C}_{3,4}$ are

$$\epsilon_1, \epsilon_2, f_{12}, f_{23}, f_{45}, f_{13}, f_{35}, f_{14}, f_{25}, f_{36}. \tag{2.11}$$

We denote $f_1 = f_{12}$, $f_{23} = f_2$, $f_0 = f_{45}$, $g_1 = f_{13}$, $g_2 = f_{35}$, $f_{14} = 1$, $f_{25} = 1$, $f_{36} = 1$ for convenience. The Lax operator is the following:

$$M = \begin{pmatrix} \epsilon_1 & f_1 & g_1 & 1 \\ z & \epsilon_2 & f_2 & g_1 \\ zg_2 & z & -\epsilon_2 & -f_1 \\ zf_0 & zg_2 & -z & -\epsilon_1 \end{pmatrix}.$$

From the Lax equation

$$\partial_{2,3}(M) = [M, B_{2,3}] + \kappa z \partial_z(B_{2,3}) \tag{2.12}$$

and

$$B_{2,3} = \begin{pmatrix} g_1 & -1 & 0 & 0 \\ 0 & g_2 & -1 & 0 \\ 0 & 0 & -g_2 & 1 \\ -z & 0 & 0 & -g_1 \end{pmatrix}, \tag{2.13}$$

we write down $\partial_{2,3}$ as follows:

$$\partial_{2,3}(f_1) = f_1 g_2 - g_1 f_1 + \epsilon_2 - \epsilon_1, \tag{2.14}$$

$$\partial_{2,3}(f_2) = -f_2 g_2 - g_2 f_2 - 2\epsilon_2, \tag{2.15}$$

$$\partial_{2,3}(f_0) = f_0 g_1 + g_1 f_0 + 2\epsilon_1 - \kappa, \tag{2.16}$$

$$\partial_{2,3}(g_1) = -f_1 + f_2 - g_1 g_2 - g_1^2, \tag{2.17}$$

$$\partial_{2,3}(g_2) = f_1 - f_0 + g_2 g_1 + g_1^2. \tag{2.18}$$

This system has a simple solution. Let f_i and g_i be functions of t, and ϵ_i and κ parameters in $\mathbb{C}$, and let the derivation $\partial_{2,3}$ be d/dt. Then, the following is a solution of above system:

$$f_0 = f_1 = f_2 = t, \quad g_1 = g_2 = 0, \tag{2.19}$$

$$\epsilon_1 = -\frac{3}{2}, \quad \epsilon_2 = -\frac{1}{2}, \quad \kappa = -4. \tag{2.20}$$

2.2. Affine Weyl group symmetry

We construct affine Weyl group symmetry of type $C_N^{(1)}$ for the systems defined in the previous subsection.

We define matrices $G_i \in g$ $(i = 0, 1, \ldots, N)$ as follows:

$$G_i = \exp\left(\frac{\alpha_i^\vee}{f_{i,i+1}} \frac{E_{i+1,i} - E_{n+1-i,n-i}}{2}\right) \quad (i = 1, \ldots, N-1), \tag{2.21}$$

$$G_0 = \exp\left(\frac{\alpha_0^\vee}{f_{n,n+1}} E_{1,n} z^{-1}\right), \quad G_N = \exp\left(\frac{\alpha_N^\vee}{f_{N,N+1}} E_{N+1,N}\right), \tag{2.22}$$

where $\alpha_i^\vee = 2(\epsilon_i - \epsilon_{i+1})$ $(i = 1, \ldots, N-1)$ and $\alpha_0^\vee = -2\epsilon_1 + \kappa$, $\alpha_N^\vee = 2\epsilon_N$, and $E_{i,j}$ is the matrix with 1 at the (i, j)-entry and 0 for other entries.

Using the matrices G_i, we define an action of s_i $(i = 0, 1, \ldots, N)$ on the field $C_{m,n}$ by

$$\kappa z \partial_z + s_i(M) = G_i(\kappa \partial_z + M) G_i^{-1}. \tag{2.23}$$

These actions were obtained by M. Noumi and Y. Yamada [32] and s_i acts on $f_{k,l}$ as follows:

$$s_i(f_{k,l}) = \exp\left(\frac{\alpha_i^\vee}{f_{i,i+1}}\mathrm{ad}_{\{\}}(f_{i,i+1})\right)(f_{k,l}) \\ = f_{k,l} + \{f_{i,i+1}, f_{k,l}\}\frac{\alpha_i^\vee}{f_{i,i+1}} \\ + \{f_{i,i+1}, \{f_{i,i+1}, f_{k,l}\}\}\frac{1}{2}\left(\frac{\alpha_i^\vee}{f_{i,i+1}}\right)^2 + \cdots . \tag{2.24}$$

Moreover, we can regard $\alpha_i^\vee$ ($i = 0, 1, \dots, N$) as the simple coroots of the affine root system of type $C_N^{(1)}$. Since the Lax equation (2.10) is written as

$$[\kappa z\partial_z + M, \partial_{s,k} + B_{s,k}] = 0, \tag{2.25}$$

we have

PROPOSITION 2.5. *(1) The action of s_i ($i = 0, 1, \dots, N$) define a representation of the affine Weyl group $W = \langle s_0, s_1, \dots, s_N\rangle$ of type $C_N^{(1)}$, namely they satisfy the following relations*

$$s_i^2 = 1, \quad s_is_j = s_js_i \quad |i-j| \geq 2, \tag{2.26}$$

$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \quad i = 1, \dots, N-2, \tag{2.27}$$

$$s_0s_1s_0s_1 = s_1s_0s_1s_0, \quad s_{N-1}s_Ns_{N-1}s_N = s_Ns_{N-1}s_Ns_{N-1}. \tag{2.28}$$

(2) The derivation $\partial_{s,k}$ commutes with the action of the affine Weyl group $W = \langle s_0, s_1, \dots, s_N\rangle$ of type $C_N^{(1)}$.

2.3. Hamiltonian

For $F \in C_{m,n}[z]$, we denote by F_i the z^i's coefficient of the polynomial F. We define Hamiltonians by the trace of some power of M:

$$H_{s,k} = \frac{\mathrm{Tr}\left(M^{k+1}\right)_s}{k+1} \quad s, k \in \mathbb{N}. \tag{2.29}$$

The following proposition proves that $H_{s,k}$ is the Hamiltonian of the system defined by the derivation $\partial_{s,k}$.

PROPOSITION 2.6. *For positive numbers s, s' and positive odd numbers k, k', we have the following.*

$$\text{(1)} \quad \partial_{s,k}(f_{i,j}) = \{H_{s,k}, f_{i,j}\} + (\kappa z \partial_z(B_{s,k}))_{i,j}, \tag{2.30}$$

$$\text{(2)} \quad \{H_{s,k}, H_{s',k'}\} = 0. \tag{2.31}$$

We can prove this proposition by direct computation. To see examples, non trivial Hamiltonians are finite and if $\kappa = 0$, these systems are integrable systems.

3. Quantum Case

In this section, we quantize the differential systems with affine Weyl group symmetry of type $C_N^{(1)}$ defined in the previous section. We use a set $\mathfrak{sp}(2N, \mathcal{K})$, where $\mathcal{K}$ is a skew field. The set $\mathfrak{sp}(2N, \mathcal{K})$ is not a Lie algebra in general. However, for certain conditions, a quantization of the classical system exists.

3.1. Lax equation

Let $\mathcal{C}_{m,n}$ ($m < n$, $m \in \mathbb{N}$) be a skew field over $\mathbb{C}$ with generators

$$h, \epsilon_i, \hat{f}_{i,i+j} \quad (i = 1, \ldots, n, j = 1, \ldots, m), \tag{3.1}$$

and defining relations

$$\epsilon_i = s(i,i)\epsilon_{1-i}, \quad \hat{f}_{i,j} = s(i,j)\hat{f}_{1-i,1-j}, \quad \hat{f}_{n+i,n+j} = \hat{f}_{i,j}, \tag{3.2}$$

$$\begin{aligned}\left[\hat{f}_{ij}, \hat{f}_{kl}\right] = \frac{h}{2}\Big(&\delta_{j,k}\hat{f}_{i,j+l-k} + \delta_{j,1-l}s(k,l)\hat{f}_{i,j+l-k} \\ &-\delta_{l,i}\hat{f}_{k,j+l-i} - \delta_{1-k,i}s(k,l)\hat{f}_{1-l,1+j-i-k}\Big)\end{aligned} \tag{3.3}$$

and ϵ_i and h are central elements. We denote the commutator by $[,]$, namely $[a,b] = ab - ba$. The condition $m < n$ is needed in the definition of the affine Weyl group symmetry because of the fact that a matrix set $\mathfrak{sp}(2N, \mathcal{C}_{m,n})$ is not a Lie algebra.

DEFINITION 3.1. Let $\mathcal{C}_{m,n}[z, z^{-1}]$ be the polynomial ring and we define a matrix element $\widehat{M}$ in $M_{n,n}(\mathcal{C}_{m,n}[z, z^{-1}])$ by

$$(\widehat{M})_{ii} = \epsilon_i \quad (i = 1, \ldots, n), \tag{3.4}$$

$$(\widehat{M})_{i,j} = \hat{f}_{i,j} \quad (i<j), \quad (\widehat{M})_{i,j} = z\hat{f}_{i,j+n} \quad (i>j). \tag{3.5}$$

We introduce elements $e_{i,j} = E_{i,j'}z^s$ where $j = ns + j'$ and $0 \le j' < n$ in $M_{n,n}(\mathcal{C}_{m,n}[z,z^{-1}])$. Then, $\widehat{M} = \sum_{i=1}^n \epsilon_i e_{i,i} + \sum_{i=1}^n \sum_{j=1}^m \hat{f}_{i,i+j} e_{i,i+j}$.

We define $\hat{g}$, $\hat{g}_{\ge 0}$ and $\hat{g}_{<0}$ in the same way as the classical case. We also denote the decomposition of $\widehat{X} \in \hat{g}$ by $\widehat{X} = \widehat{X}_+ + \widehat{X}_-$ for $\widehat{X}_+ \in \hat{g}_{\ge 0}$ and $\widehat{X}_- \in \hat{g}_{<0}$.

PROPOSITION 3.2. *For a positive number s and an odd number k such that $mk \ge ns > m(k-1)$, we can define $\mathbb{C}[z,z^{-1}]$-derivation $\hat{\partial}_{s,k}$ on $\mathcal{C}_{m,n}[z,z^{-1}]$ by the Lax equation*

$$\partial_{s,k}(\widehat{M}) = [\widehat{M}, \widehat{B}_{s,k}] + \kappa z \partial_z(\widehat{B}_{s,k}), \tag{3.6}$$

where $\widehat{B}_{s,k} = (\widehat{M}^k z^{-s})_+$, $\kappa \in \mathbb{C}$.

REMARK 3.3. Note that if $mk < ns$, then $\widehat{B}_{s,k} = 0$, and if $mk = ns$, then it holds

$$\left[\widehat{M}, \widehat{B}_{s,k}\right] = \kappa z \partial_z(\widehat{B}_{s,k}) = 0. \tag{3.7}$$

To prove Proposition 3.2, we need the next lemma.

LEMMA 3.4. *Let s be a positive number and k an odd positive number, and i,j,p,q integers such that $1 \le i \le n$, $0 \le j \le mk - ns$. Under the assumption $ns > m(k-1)$, we have*

$$\begin{aligned} &\left[\left(\widehat{B}_{s,k}\right)_{i,i+j}, \hat{f}_{p,q}\right] \\ &\quad = \frac{h}{2}\left(\delta_{i+j,p}\left(\widehat{B}_{s,k}\right)_{i,i+j+q-p} + \delta_{i+j,1-q} s(p,q)\left(\widehat{B}_{s,k}\right)_{i,i+j+q-p}\right. \\ &\quad \left. -\delta_{q,i}\left(\widehat{B}_{s,k}\right)_{p,j+q} - \delta_{1-p,i} s(p,q)\left(\widehat{B}_{s,k}\right)_{1-q,1+j-p}\right), \end{aligned} \tag{3.8}$$

where $\left(\widehat{B}_{s,k}\right)_{i,i+j}$ is determined by $\widehat{B}_{s,k} = \sum_{i=1}^n \sum_{j=1}^{mk-ns} \left(\widehat{B}_{s,k}\right)_{i,i+j} e_{i,i+j}$.

Proof. It is convenient to introduce the following notation $I_t = i_1 + \cdots + i_t$ for integers $i_1, \ldots, i_t$ to calculate the commutator.

By the definition,

$$\left(\widehat{B}_{s,k}\right)_{i,i+j} = \sum_{I_k = ns+j} f_{i,i+I_1} f_{i+I_1,i+I_2} \cdots f_{i+I_{k-1},i+I_k}. \tag{3.9}$$

In turn it is straightforward to calculate the commutator of $\left(\widehat{B}_{s,k}\right)_{i,i+j}$ and $\hat{f}_{p,q}$ from the commutation rules and we obtain (3.8). □

Proof of Proposition 3.2. It is needed to check that matrices $\widehat{B}_{s,k}$ and $\left[\widehat{M}, \widehat{B}_{s,k}\right]$ are elements in $\mathfrak{sp}\left(n, \mathcal{C}_{m,n}[z, z^{-1}]\right)$, and the following two maps $\partial'_{s,k}$ and $\partial''_{s,k}$ defined by

$$\partial'_{s,k}(\widehat{M}) = \left[\widehat{M}, \widehat{B}_{s,k}\right], \quad \partial''_{s,k}(\widehat{M}) = \kappa z \partial_z \left(\widehat{B}_{s,k}\right) \tag{3.10}$$

are $\mathbb{C}[z, z^{-1}]$-derivations.

For the former, we compute $\left(\widehat{B}_{s,k}\right)_{1-i-j,1-i}$ as follows.

$$\begin{aligned}
\left(\widehat{B}_{s,k}\right)_{1-i-j,1-i} &= \sum_{I_k = ns+j} f_{1-i-j,1-i-j+I_1} f_{1-i-j+I_1,1-i-j+I_2} \\
&\qquad \cdots f_{1-i-j+I_{k-1},1-i-j+I_k} \\
&= \sum_{I_k = ns+j} s(i, i+i_{k-1}) s(i+i_{k-1}, i+i_{k-1}+i_{k-2}) \\
&\qquad \cdots s(i+i_{k-1}+\cdots+i_2, i+I_k) \\
&\quad \times f_{i,i+i_{k-1}} f_{i+i_{k-1},i+i_{k-1}+i_{k-2}} \cdots f_{i+i_{k-1}+\cdots+i_2,i+I_k} \\
&= \sum_{I_k = ns+j} s(i, i+j)(-1)^{k-1} f_{i,i+i_{k-1}} f_{i+i_{k-1},i+i_{k-1}+i_{k-2}} \\
&\qquad \cdots f_{i+i_{k-1}+\cdots+i_2,i+I_k}.
\end{aligned}$$

Since k is odd, $\left(\widehat{B}_{s,k}\right)_{1-i-j,1-i} = s(i, i+j)\left(\widehat{B}_{s,k}\right)_{i,j}$. Hence, $\widehat{B}_{s,k}$ is in $\mathfrak{sp}\left(n, \mathcal{C}_{m,n}[z, z^{-1}]\right)$. For $\left[\widehat{M}, \widehat{B}_{s,k}\right]$, the same computation proceeds and we obtain that the element $\left[\widehat{M}, \widehat{B}_{s,k}\right]$ is in $\mathfrak{sp}\left(n, \mathcal{C}_{m,n}[z, z^{-1}]\right)$.

For the latter, it is proved in Theorem 3.9 that the map $\partial'_{s,k}$ is $\mathbb{C}[z,z^{-1}]$-derivation, although we can check that by direct computation. Also we can check by direct computation using Lemma 3.4 that the map $\partial''_{s,k}$ is $\mathbb{C}[z,z^{-1}]$-derivation. □

3.2. Affine Weyl group symmetry

As the classical case, we can define the affine Weyl group action of type $C_N^{(1)}$ on $\mathcal{C}_{m,n}$ for $m<n$ compatible with the Lax equation, by matrices $\widehat{G}_i$.

Let matrices $\widehat{G}_i$ be defined by

$$\widehat{G}_i = \exp\left(\frac{\alpha_i^\vee}{\hat{f}_{i,i+1}}\frac{E_{i+1,i}-E_{n+1-i,n-i}}{2}\right) \quad (i=1,\dots,N-1), \tag{3.11}$$

$$\widehat{G}_0 = \exp\left(\frac{\alpha_0^\vee}{\hat{f}_{n,n+1}}E_{1,n}z^{-1}\right), \quad \widehat{G}_N = \exp\left(\frac{\alpha_N^\vee}{\hat{f}_{N,N+1}}E_{N+1,N}\right), \tag{3.12}$$

where $\alpha_i^\vee = 2(\epsilon_i - \epsilon_{i+1})$ $(i=1,\dots,N-1)$ and $\alpha_0^\vee = -2\epsilon_1+\kappa$, $\alpha_N^\vee = 2\epsilon_N$.

PROPOSITION 3.5. *Using the matrices $\widehat{G}_i$, we can define automorphisms s_i $(i=0,1,\dots,N)$ on the skew field $\mathcal{C}_{m,n}$ by*

$$s_i(\widehat{M}) = \widehat{G}_i\widehat{M}\widehat{G}_i^{-1} - \kappa z\partial_z\left(\widehat{G}_i\right)\widehat{G}_i^{-1}. \tag{3.13}$$

PROOF. From $m<n$, the right hand side of (3.13) is in $\mathfrak{sp}\big(n, \mathcal{C}_{m,n}[z,z^{-1}]\big)$. Hence s_i is a map of the free algebra generated by h, ϵ_i, $f_{i,j}$. Through a long computation, we can check the map s_i preserves the commutation relations. Thus s_i is an automorphism on the skew field $\mathcal{C}_{m,n}$. □

As the classical case, we can regard $\alpha_i^\vee$ $(i=0,1,\dots,N)$ as the simple coroots of the affine root system of type $C_N^{(1)}$. Note that from definition, the action of s_i on the generators $f_{k,l}$ is written down in the following.

$$\begin{aligned} s_i(f_{k,l}) =& f_{k,l} + \frac{1}{2}\left(\frac{\alpha_i^\vee}{h\hat{f}_{i,i+1}}\left[\hat{f}_{i,i+1},\hat{f}_{k,l}\right] + \left[\hat{f}_{i,i+1},f_{k,l}\right]\frac{\alpha_i^\vee}{h\hat{f}_{i,i+1}}\right) \\ &+ \frac{1}{2}\left(\frac{\alpha_i^\vee}{h\hat{f}_{i,i+1}}\right)^2\left[\hat{f}_{i,i+1},\left[\hat{f}_{i,i+1},\hat{f}_{k,l}\right]\right]. \end{aligned} \tag{3.14}$$

Example 3.6. We give an example of the action for $m = 3$, $n = 4$, $s = 2$, $k = 3$. We use the same notation in Section 2. Then,

$$
\begin{aligned}
&s_0(f_0) = f_0, \quad s_0(f_1) = f_1 + \frac{1}{2}\left(\frac{\alpha_0^\vee}{f_0}g_2 + g_2\frac{\alpha_0^\vee}{f_0}\right),\\
&s_0(f_2) = f_2, \quad s_0(g_1) = g_1 - \frac{\alpha_0^\vee}{f_0}, \quad s_0(g_2) = g_2,\\
&s_1(f_0) = f_0 - \frac{1}{2}\left(\frac{\alpha_1^\vee}{f_1}g_2 + g_2\frac{\alpha_1^\vee}{f_1}\right) + \frac{1}{4}\left(\frac{\alpha_1^\vee}{f_1}\right)^2, \quad s_1(f_1) = f_1,\\
&s_1(f_2) = f_2 + \frac{1}{2}\left(\frac{\alpha_1^\vee}{f_1}g_1 + g_1\frac{\alpha_1^\vee}{f_1}\right) + \frac{1}{4}\left(\frac{\alpha_1^\vee}{f_1}\right)^2,\\
&s_1(g_1) = g_1 + \frac{1}{2}\frac{\alpha_1^\vee}{f_1}, \quad s_1(g_2) = g_2 - \frac{1}{2}\frac{\alpha_1^\vee}{f_1},\\
&s_2(f_0) = f_0, \quad s_2(f_1) = f_1 - \frac{1}{2}\left(\frac{\alpha_2^\vee}{f_2}g_1 + g_1\frac{\alpha_2^\vee}{f_2}\right),\\
&s_2(f_2) = f_2, \quad s_0(g_1) = g_1, \quad s_0(g_2) = g_2 + \frac{\alpha_2^\vee}{f_2}.
\end{aligned}
$$

The (noncommutative) differential systems defined by $\partial_{s,k}$ using Lax equation have the affine Weyl group symmetry of type $C_N^{(1)}$.

THEOREM 3.7. *(1) The action of s_i $(i = 0, 1, \ldots, N)$ define a representation of the affine Weyl group $W = \langle s_0, s_1, \ldots, s_N\rangle$ of type $C_N^{(1)}$, namely they satisfy the following relations*

$$s_i^2 = 1, \quad s_i s_j = s_j s_i \quad |i-j| \geq 2, \tag{3.15}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad i = 1, \ldots, N-2, \tag{3.16}$$

$$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0, \quad s_{N-1} s_N s_{N-1} s_N = s_N s_{N-1} s_N s_{N-1}. \tag{3.17}$$

(2) The derivation $\partial_{s,k}$ commutes with the action of the affine Weyl group $W = \langle s_0, s_1, \ldots, s_N\rangle$ of type $C_N^{(1)}$.

PROOF. (1) We can check by direct computations that all fundamental relations hold.

(2) Since Lax equation is written as

$$[\kappa z \partial_z + \widehat{M}, \partial_{s,k} + \widehat{B}_{s,k}] = 0, \tag{3.18}$$

and from (3.13), we have

$$\kappa z\partial_z + s_i\left(\widehat{M}\right) = \widehat{G}_i\left(\kappa z\partial_z + \widehat{M}\right)\widehat{G}_i^{-1}, \tag{3.19}$$

it is sufficient to show

$$\partial_{s,k} + s_i\left(\widehat{B}_{s,k}\right) = \widehat{G}_i\left(\partial_{s,k} + \widehat{B}_{s,k}\right)\widehat{G}_i^{-1}. \tag{3.20}$$

From the definition of $\widehat{B}_{s,k}$ and (3.13), the left hand side of the above is

$$\begin{aligned}\partial_{s,k} + s_i\left(\widehat{B}_{s,k}\right) =& \partial_{s,k} + \left(s_i\left(\widehat{M}\right)^k z^{-s}\right)_+ \\ =& \partial_{s,k} + \left(\left(\widehat{G}_i\widehat{M}\widehat{G}_i^{-1} - \kappa\partial_z\left(\widehat{G}_i\right)\widehat{G}_i^{-1}\right)^k z^{-s}\right)_+ \\ =& \partial_{s,k} + \left(\left(\widehat{G}_i\widehat{M}\widehat{G}_i^{-1}\right)^k z^{-s}\right)_+ .\end{aligned}$$

We used the condition $m(k-1) < ns$ here. We compute the right hand side of (3.20) as follows:

$$\begin{aligned}\widehat{G}_i\left(\partial_{s,k} + \widehat{B}_{s,k}\right)\widehat{G}_i^{-1} =& \partial_{s,k} + \widehat{G}_i\partial_{s,k}\left(\widehat{G}_i^{-1}\right) + \widehat{G}_i\widehat{B}_{s,k}\widehat{G}_i^{-1} \\ =& \partial_{s,k} + \frac{\alpha_i^\vee}{\hat{f}_{i,i+1}}\partial_{s,k}\left(\hat{f}_{i,i+1}\right) \\ & \times \frac{1}{2\hat{f}_{i,i+1}}\left(E_{i+1,i} + s(i+1,i)E_{n+1-i,n-i}\right) \\ & \times \left(1 + \delta_{i,0}z^{-1}\right) + \left(\widehat{G}_i\widehat{M}^k\widehat{G}_i^{-1}z^{-s}\right)_+ \\ & + \frac{\alpha_i^\vee}{2\hat{f}_{i,i+1}}\Bigg(E_{i+1,i}\left(\widehat{B}_{s,k}\right)_{i,i} \\ & + s(i+1,i)E_{n+1-i,n-i}\left(\widehat{B}_{s,k}\right)_{n-i,n-i}\Bigg) \\ & \times \left(1 + \delta_{i,0}z^{-1}\right) \\ & - \Bigg(E_{i+1,i}\left(\widehat{B}_{s,k}\right)_{i+1,i+1} \\ & + s(i+1,i)E_{n+1-i,n-i}\left(\widehat{B}_{s,k}\right)_{n+1-i,n+1-i}\Bigg)\end{aligned}$$

$$
\begin{aligned}
&\times \left(1+\delta_{i,0}z^{-1}\right) \frac{\alpha_i^{\vee}}{2\hat{f}_{i,i+1}} \\
&- \frac{\alpha_i^{\vee}}{2\hat{f}_{i,i+1}} \left(E_{i+1,i} \left(\widehat{B}_{s,k}\right)_{i,i+1} \right. \\
&+ E_{n+1-i,n-i} \left(\widehat{B}_{s,k}\right)_{n-i,n-i+1} \\
&\left. +2E_{i+1,i} \left(\widehat{B}_{s,k}\right)_{i,i+1} \left(\delta_{i,0}+\delta_{i,N}\right) \right) \\
&\times \frac{\alpha_i^{\vee}}{2\hat{f}_{i,i+1}} \left(1+\delta_{i,0}z^{-1}\right) \\
=&\partial_{s,k} + \left(\left(\widehat{G}_i \widehat{M} \widehat{G}_i^{-1} \right)^k z^{-s} \right)_+ \\
&+ \frac{1}{2} \left(E_{i+1,i} + s(i+1,i) E_{n+1-i,n-i}\right) \frac{\alpha_i^{\vee}}{\hat{f}_{i,i+1}} \\
&\times \left(\partial_{s,k}\left(\hat{f}_{i,i+1}\right) + \left(\widehat{B}_{s,k}\right)_{i,i} \hat{f}_{i,i+1} \right. \\
&- \hat{f}_{i,i+1} \left(\widehat{B}_{s,k}\right)_{i+1,i+1} \\
&\left. - \frac{\alpha_i^{\vee}}{2} \left(1+\delta_{i,0}+\delta_{i,N}\right) \left(\widehat{B}_{s,k}\right)_{i,i+1} \right) \\
&\times \frac{1}{\hat{f}_{i,i+1}} \left(1+\delta_{i,0}z^{-1}\right).
\end{aligned}
$$

We used Lemma 3.4 here. From the definition of $\partial_{s,k}$, we have

$$
\begin{aligned}
\partial_{s,k}\left(\hat{f}_{i,i+1}\right) = &\hat{f}_{i,i+1} \left(\widehat{B}_{s,k}\right)_{i+1,i+1} - \left(\widehat{B}_{s,k}\right)_{i,i} \hat{f}_{i,i+1} \\
&+ \left(\epsilon_i - \epsilon_{i+1,i+1} + \delta_{i,n}\kappa\right) \left(\widehat{B}_{s,k}\right)_{i,i+1}.
\end{aligned} \tag{3.21}
$$

Therefore, we obtain the formula (3.20). □

3.3. Hamiltonian

In this section, we show that Hamiltonians of quantum differential systems defined in Section 3.1 are obtained from the trace of some power of Lax operator.

Definition 3.8. For $s, k \in \mathbb{N}$, we define Hamiltonian $\widehat{H}_{s,k}$ by

$$\widehat{H}_{s,k} = \frac{\mathrm{Tr}(\widehat{M}^{k+1})_s}{k+1}. \tag{3.22}$$

Theorem 3.9. *For $s, k \in \mathbb{N}$ such that $ns > m(k-1)$ and k is odd, it holds*

$$\frac{1}{h}\left[\widehat{H}_{s,k} I_n, \widehat{M}\right] = \left[\widehat{M}, \widehat{B}_{s,k}\right], \tag{3.23}$$

where I_n is the identity matrix $\sum_{i=1}^{n} E_{i,i}$.

We prove this theorem by direct computation of the commutator in the same way as the $A_{n-1}^{(1)}$ case [28]. We need the following lemmas.

For $1 \le i \le j$ and $1 \le j \le m-1$, let a_{ij} and b_{ij} in $\mathfrak{sp}\,(n, \mathcal{C}_{m,n})$ be

$$a_{ij} = \frac{1}{2}\sum_{l=1}^{m-j}\left(\hat{f}_{i-l,i+j}e_{i-l,i} + s(i,i+j)\hat{f}_{1-i-j-l,1-i}e_{1-i-j-l,1-i-j}\right), \tag{3.24}$$

$$b_{ij} = \frac{1}{2}\sum_{l=1}^{m-j}\left(\hat{f}_{i,i+j+l}e_{i+j,i+j+l} + s(i,i+j)\hat{f}_{1-i-j,1-i+l}e_{1-i,1-i+l}\right). \tag{3.25}$$

Lemma 3.10. *The commutator of $\widehat{M}$ and $\hat{f}_{i,i+j}$ $(1 \le i \le n, 1 \le j \le m-1)$ is given by*

$$\frac{1}{h}\left[\widehat{M}, \hat{f}_{i,i+j} I_n\right] = a_{ij} - b_{ij}. \tag{3.26}$$

Proof. We compute the left-hand side of (3.26) as follows:

$$\begin{aligned}\frac{1}{h}\left[\widehat{M}, \hat{f}_{i,i+j}\right] =& \frac{1}{h}\sum_{l=1}^{m-j}\sum_{p=1}^{n}[\hat{f}_{p,p+l}, \hat{f}_{i,i+j}]e_{p,p+l} \\ =& \frac{1}{2}\sum_{l=1}^{m-j}\sum_{p=1}^{n}\Big(\delta_{p+l,i}\hat{f}_{p,p+l+j} - \delta_{i+j,p}\hat{f}_{i,i+j+l}\end{aligned}$$

$$
\begin{aligned}
&+ s(i,i+j)\delta_{1-i-j,p+l}\hat{f}_{p,p+l+j} \\
&-s(i,i+j)\delta_{1-i,p}\hat{f}_{1-i-j,1-i+l}\Big)\, e_{p,p+l} \\
=&a_{ij} - b_{ij}. \ \square
\end{aligned}
$$

For $1 \leq i \leq j$, $1 \leq j \leq m-1$, $0 \leq p \leq k+1$ and $a, b \in \mathbb{Z}$, we define the following elements $A_{ij}^p(a,b)$, $B_{ij}^p(a,b)$ and $C_{ij}^p(a,b)$ in $\mathcal{C}_{m,n}$.

$$
\begin{aligned}
A_{ij}^p(a,b) =& \sum \hat{f}_{i,i+J_1} \cdots \hat{f}_{i+J_{a-1},i+J_a} \hat{f}_{i+J_b-J_k+J_a,i+J_b-J_k+J_{a+1}} \\
&\cdots \hat{f}_{i+J_b-J_k+J_{k-2},i+J_b-J_k+J_{k-1}} \\
&\cdot \hat{f}_{i+J_b-J_k+J_{k-1},i+J_b-J_k+J_{k+1}},
\end{aligned} \tag{3.27}
$$

where the summation is taken over all $j_1, \ldots, j_{k+1} \in \mathbb{N}$ such that $J_{k+1} = ns + j$, $j_p \geq j+1$ and $J_{k+1} - J_a \equiv 0 \pmod{n}$.

$$
\begin{aligned}
B_{ij}^p(a,b) =& \sum \hat{f}_{i,i+J_1} \cdots \hat{f}_{i+J_{b-1},i+J_{b+1}} \hat{f}_{i+J_{b+1},i+J_{b+2}} \cdots \hat{f}_{i+J_{a-1},i+J_a} \\
&\cdot \hat{f}_{i+J_b-J_{k+1}+J_a,i+J_b-J_{k+1}+J_{a+1}} \\
&\cdots \hat{f}_{i+J_b-J_{k+1}+J_k,i+J_b-J_{k+1}+J_{k+1}},
\end{aligned} \tag{3.28}
$$

where the summation is taken over all $j_1, \ldots, j_{k+1} \in \mathbb{N}$ such that $J_{k+1} = ns + j$, $j_p \geq j+1$ and $J_{k+1} - J_a \equiv 0 \pmod{n}$.

$$
\begin{aligned}
C_{ij}^p(a,b) =& \sum (-1)^b \hat{f}_{i,i+J_1} \\
&\cdots \hat{f}_{i+J_{a-2},i+J_{a-1}} \hat{f}_{i+J_{a-1},i+J_{a+1}} \hat{f}_{i+J_{q+1},i+J_{q+2}} \\
&\cdots \hat{f}_{i+J_k,i+J_{k+1}},
\end{aligned} \tag{3.29}
$$

where the summation is taken over all $j_1, \ldots, j_{k+1} \in \mathbb{N}$ such that $J_{k+1} = ns + j$, $j_p \geq j+1$ and $i + J_a \equiv 1 - (i + J_{a+b}) \pmod{n}$.

Lemma 3.11. *Suppose $1 \leq q \leq k$, $1 \leq i \leq n$, $1 \leq j \leq m-1$ and $ns > m(k-1)$. We have*

$$
2\,Tr\left(\widehat{M}^{q-1} \frac{1}{h}\left[a_{ij}, \widehat{M}\right] \widehat{M}^{k-q}\right)_s
$$

$$
\begin{aligned}
&= \sum_{t=1}^{q-1} \Big\{ A_{ij}^{k-t}(k-t,k-q) - B_{ij}^{k-t+2}(k-t+2,k-q+1) \Big\} \\
&\quad + A_{ij}^{k}(k-q,k+1) + B_{ij}^{k-q+2}(k-q+2,k-q+1) \\
&\quad + \sum_{t=q}^{k-1} \Big\{ -A_{ij}^{t}(t,t-q) + B_{ij}^{t+2}(t+2,t-q+1) \Big\} \\
&\quad + \sum_{t=2}^{q} \big\{ A_{ij}^{1}(k-t+1,q-t+1) - B_{ij}^{1}(k-t+3,q-t+2) \big\} \\
&\quad - B_{ij}^{1}(k-q+2,1) + A_{ij}^{k+1}(k-q,0) \\
&\quad + \sum_{t=q+1}^{k} \big\{ -A_{ij}^{1}(t-1,q) + B_{ij}^{1}(t+1,q+1) \big\} \\
&\quad + \sum_{t=1}^{q-1} \Big\{ C_{ij}^{k+1}(k-q+1,t) + C_{ij}^{k+1}(k-q+t,-t) \Big\} + C_{ij}^{k-q+2}(k-q+1,q) \\
&\quad + C_{ij}^{k+1}(k,-q) - \sum_{t=q}^{k-1} \Big\{ C_{ij}^{k+1}(t-q+1,k-t) + C_{ij}^{k+1}(k-q,-k+t) \Big\} \\
&\quad + \sum_{t=2}^{q} \big\{ C_{ij}^{1}(q,-t+1) + C_{ij}^{1}(q-t+2,t-1) \big\} + C_{ij}^{q}(q,-q) + C_{ij}^{1}(1,q) \\
&\quad - \sum_{t=q+1}^{k} \big\{ C_{ij}^{1}(k-t+q+1,-k+t-1) + C_{ij}^{1}(q+1,k-t+1) \big\},
\end{aligned}
$$

and

$$
\begin{aligned}
&2\,Tr\left(\widehat{M}^{q-1} \frac{1}{h} \left[\widehat{M}, b_{ij} \right] \widehat{M}^{k-q} \right)_s \\
&= \sum_{t=2}^{q} \big\{ B_{ij}^{1}(k-t+3,k-q+2) - A_{ij}^{1}(k+1-t,k-q+1) \big\} \\
&+ A_{ij}^{k+1}(q-1,0) - B_{ij}^{1}(q+1,1) \\
&+ \sum_{t=q+1}^{k} \big\{ -B_{ij}^{1}(t+1,t-q+1) + A_{ij}^{1}(t-1,t-q) \big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{t=1}^{q-1}\left\{-A_{ij}^{k-t}(k-t,q-1-t)+B_{ij}^{k+2-t}(k+2-t,q-t)\right\}-B_{ij}^{a}(q+1,q)\\
&+A_{ij}^{k}(q-1,k+1)+\sum_{t=q}^{k-1}\left\{A_{ij}^{t}(t,q-1)-B_{ij}^{t+2}(t+2,q)\right\}\\
&-\sum_{t=2}^{q}\left\{C_{ij}^{1}(k-q+2,t-1)+C_{ij}^{1}(k-q+t,-t+1)\right\}+C_{ij}^{1}(1,k-q+1)\\
&+C_{ij}^{k-q+1}(k-q+1,-k+q-1)\\
&+\sum_{t=q+1}^{k}\left\{C_{ij}^{1}(t-q+1,k-t+1)+C_{ij}^{1}(k-q+1,-k+t-1)\right\}\\
&-\sum_{t=1}^{q-1}\left\{C_{ij}^{k+1}(q-1,-t)+C_{ij}^{k+1}(q-t,t)\right\}+C_{ij}^{k+1}(k,-k+q-1)\\
&+C_{ij}^{q+1}(q,k-q+1)\\
&+\sum_{t=q}^{k-1}\left\{C_{ij}^{k+1}(k-t+q-1,-k+t)+C_{ij}^{k+1}(q,k-t)\right\}.
\end{aligned}
$$

Proof. We compute the left hand side of the first formula above as follows. First of all, we have

$$
\begin{aligned}
&\operatorname{Tr}\left(\widehat{M}^{q-1}\frac{1}{h}\left[\sum_{l=1}^{m-j}\hat{f}_{i-l,i+j}e_{i-l,i},\widehat{M}\right]\widehat{M}^{k-q}\right)_s\\
&=\sum_{l=1}^{m-j}\sum_{I_k+l=ns}\Big[\hat{f}_{i-l-I_{q-1},i-l-I_{q-1}+I_1}\cdots\hat{f}_{i-l-i_{q-1},i-l}\hat{f}_{i-l,i+j}\hat{f}_{i,i+i_q}\\
&\cdots\hat{f}_{i+I_{k-2}-I_{q-1},i+I_{k-1}-I_{q-1}},\hat{f}_{i+I_{k-1}-I_{q-1},i+I_k-I_{q-1}}\Big]\\
&=\mathrm{I}+\mathrm{II}+\mathrm{III},
\end{aligned}
$$

where

$$
\mathrm{I}=\sum_{l=1}^{m-j}\sum_{I_k+l=ns}\sum_{t=1}^{q-1}\Big(\hat{f}_{i-l-I_{q-1},i-l-I_{q-1}+I_1}\cdots\hat{f}_{i-l-I_{q-1}+I_{t-2},i-l-I_{q-1}+I_{t-1}}
$$

$$\cdot \left[\hat{f}_{i-l-I_{q-1}+I_{t-1},i-l-I_{q-1}+I_t}, \hat{f}_{i+I_{k-1}-I_{q-1},i+I_k-I_{q-1}}\right]$$
$$\cdot \hat{f}_{i-l-I_{q-1}+I_t,i-l-I_{q-1}+I_{t+1}} \cdots \hat{f}_{i-l-i_{q-1},i-l}$$
$$\left.\cdot\ \hat{f}_{i-l,i+j}\hat{f}_{i,i+i_q} \cdots \hat{f}_{i+I_{k-2}-I_{q-1},i+I_{k-1}-I_{q-1}}\right).$$

$$\mathrm{II} = \sum_{l=1}^{m-j} \sum_{I_k+l=ns} \Big(\hat{f}_{i-l-I_{q-1},i-l-I_{q-1}+I_1} \cdots \hat{f}_{i-l-i_{q-1},i-l}$$
$$\left.\cdot \left[\hat{f}_{i-l,i+j}, \hat{f}_{i+I_{k-1}-I_{q-1},i+I_k-I_{q-1}}\right] \hat{f}_{i,i+i_q} \cdots \hat{f}_{i+I_{k-2}-I_{q-1},i+I_{k-1}-I_{q-1}}\right).$$

$$\mathrm{III} = \sum_{l=1}^{m-j} \sum_{I_k+l=ns} \sum_{t=q}^{k-1} \Big(\hat{f}_{i-l-I_{q-1},i-l-I_{q-1}+I_1} \cdots \hat{f}_{i-l-i_{q-1},i-l}\hat{f}_{i-l,i+j}\hat{f}_{i,i+i_q}$$
$$\cdots \hat{f}_{i+I_{t-2}-I_{q-1},i+I_{t-1}-I_{q-1}}$$
$$\cdot \left[\hat{f}_{i+I_{t-1}-I_{q-1},i+I_t-I_{q-1}}, \hat{f}_{i+I_{k-1}-I_{q-1},i+I_k-I_{q-1}}\right] \hat{f}_{i+I_t-I_{q-1},i+I_{t+1}-I_{q-1}}$$
$$\left.\cdots \hat{f}_{i+I_{k-2}-I_{q-1},i+I_{k-1}-I_{q-1}}\right).$$

We compute I. Using the commutation relations, we see that

$$\left[\hat{f}_{i-l-I_{q-1}+I_{t-2},i-l-I_{q-1}+I_{t-1}}, \hat{f}_{i+I_{k-1}-I_{q-1},i+I_k-I_{q-1}}\right] \neq 0 \tag{3.30}$$

if and only if $i_t + i_k \leq m$, $1 \leq i_t, i_k$ and one of the next four conditions

$$\text{i)}\quad I_t \equiv -i_k \pmod{n}, \tag{3.31}$$
$$\text{ii)}\quad I_{t-1} \equiv 0 \pmod{n}, \tag{3.32}$$
$$\text{iii)}\quad i-l-I_{q-1}+I_t \equiv 1-i+l+I_{q-1} \pmod{n}, \tag{3.33}$$
$$\text{iv)}\quad 1-i+i_k+l+I_{q-1} \equiv i-l-I_{q-1}+I_{t-1} \pmod{n}. \tag{3.34}$$

When $I_t \equiv -i_k$,

$$\mathrm{I} = \sum_{l=1}^{m-j} \sum_{I_k+l=ns} \sum_{t=1}^{q-1} \Big(\hat{f}_{i-l-I_{q-1},i-l-I_{q-1}+I_1} \cdots \hat{f}_{i-l-I_{q-1}+I_{t-2},i-l-I_{q-1}+I_{t-1}}$$
$$\cdot \hat{f}_{i-l-I_{q-1}+I_{t-1},i-l-I_{q-1}+I_t+i_k}\hat{f}_{i-l-I_{q-1}+I_t,i-l-I_{q-1}+I_{t+1}} \cdots \hat{f}_{i-l-i_{q-1},i-l}$$
$$\left.\cdot\ \hat{f}_{i-l,i+j}\hat{f}_{i,i+i_q} \cdots \hat{f}_{i+I_{k-2}-I_{q-1},i+I_{k-1}-I_{q-1}}\right).$$

Replacing

$$(i_1, \ldots, i_t, i_{t+1}, \ldots, i_{q-1}, l+j, i_q, \ldots, i_{k-1}, i_k)$$

with

$$(j_{k-t+1}, \dots , j_k, j_{k-q+1}, \dots , j_{k-t-1}, j_{k-t}, j_1, \dots , j_{k-q}, j_{k+1}),$$

we have

$$\begin{aligned}
\mathrm{I} = \sum_{J_{k+1}=ns+j} \sum_{t=1}^{q-1} & \hat{f}_{i+J_{k-q}-J_k+J_{k-t}, i+J_{k-q}-J_k-J_{k-t+1}} \\
& \cdots \hat{f}_{i+J_{k-q}-J_k+J_{k-2}, i+J_{k-q}-J_k-J_{k-1}} \\
& \cdot \hat{f}_{i+J_{k-q}-J_k+J_{k-1}, i+J_{k-q}-J_k-J_{k+1}} \hat{f}_{i+J_{k-q}, i+J_{k-q+1}} \cdots \hat{f}_{i+J_{k-t-1}, i+J_{k-t}} \\
& \cdot \hat{f}_{i,i+J_1} \cdots \hat{f}_{i+J_{k-q-1}, i+J_{k-q}},
\end{aligned}$$

where $J_{k+1} - J_{k-t} \equiv 0 \pmod{n}$ and $j_{k-t} \geq j+1$. Since $m(k-1) < ns$, it holds that $j_1, \dots , j_{k+1} \in \mathbb{N}$ and all elements in the expression above are mutually commutative. Hence, we obtain

$$\mathrm{I} = \sum_{t=1}^{q-1} A_{ij}^{k-t}(k-t, k-q). \tag{3.35}$$

For the case ii) $I_{t-1} \equiv 0 \pmod{n}$, we obtain

$$\mathrm{I} = -\sum_{t=1}^{q-1} B_{ij}^{k-t+2}(k-t+2, k-q+1) \tag{3.36}$$

in the same way of i). For the case iii) $i - l - I_{q-1} + I_t \equiv 1 - i + l + I_{q-1} \pmod{n}$, we compute I as follows.

$$\begin{aligned}
I = \sum_{l=1}^{m-j} \sum_{I_k+l=ns} \sum_{t=1}^{q-1} \Big(& \hat{f}_{i-l-I_{q-1}, i-l-I_{q-1}+I_1} \cdots \hat{f}_{i-l-I_{q-1}+I_{t-2}, i-l-I_{q-1}+I_{t-1}} \\
& \cdot s(i + I_{k-1} - I_{q-1}, i + I_k - I_{q-1}) \hat{f}_{i-l-I_{q-1}+I_{t-1}, i-l-I_{q-1}+I_t+i_k} \\
& \cdot \hat{f}_{i-l-I_{q-1}+I_t, i-l-I_{q-1}+I_{t+1}} \cdots \hat{f}_{i-l-i_{q-1}, i-l} \\
& \cdot \hat{f}_{i-l,i+j} \hat{f}_{i,i+i_q} \cdots \hat{f}_{i+I_{k-2}-I_{q-1}, i+I_{k-1}-I_{q-1}} \Big) .
\end{aligned}$$

Replacing

$$(i_1, \dots , i_{t-1}, i_t, i_{t+1}, \dots , i_{q-1}, i_q, \dots , i_{k-1}, i_k, l+j)$$

with

$$(j_{k-q+t+1},\dots,j_{k-q+3},j_{k-q+2},j_{k-q+t+2},\dots,j_k,j_1,\dots,j_{k-q},j_{k-q+1},j_{k+1}),$$

we have

$$\begin{aligned} I = \sum_{J_{k+1}=ns+j}\sum_{t=1}^{q-1}(-1)^t \Big(&\hat{f}_{i+J_{k-q+t},i+J_{k-q+t+1}}\cdots\hat{f}_{i+J_{k-q+2},i+J_{k-q+3}} \\ &\cdot \hat{f}_{i+J_{k-q},i+J_{k-q+2}} \\ &\cdot \hat{f}_{i+J_{k-q+1+t},i+J_{k-q+2+t}}\cdots\hat{f}_{i+J_k,i+J_{k+1}} \\ &\cdot \hat{f}_{i,i+J_1}\cdots\hat{f}_{i+J_{k-q-1},i+J_{k-q}}\Big), \end{aligned}$$

where $i+J_{k-q+t+1}\equiv 1-(i-J_{k-q+1}) \pmod n$ and $j_{k+1}\geq j+1$. Since $m(k-1)<ns$, it holds that $j_1,\dots,j_{k+1}\in\mathbb{N}$ and all elements in the expression above are mutually commutative. Hence, we obtain

$$I=\sum_{t=1}^{q-1}C_{ij}^{k+1}(k-q+1,t). \tag{3.37}$$

For the case iv), we obtain

$$I=\sum_{t=1}^{q-1}C_{ij}^{k+1}(k-q+t,-t) \tag{3.38}$$

in the same way iii). For the each case of II and III, the computation is similar. □

LEMMA 3.12. *Suppose* $1\leq q\leq k$, $1\leq i\leq n$, $1\leq j\leq m-1$ *and* $ns>m(k-1)$. *We have*

$$Tr\left(\widehat{M}^{q-1}\left[a_{ij},\widehat{M}\right]\widehat{M}^{k-q}\right)_s - Tr\left(\widehat{M}^{k-q}\left[\widehat{M},b_{ij}\right]\widehat{M}^{q-1}\right)_s=0. \tag{3.39}$$

PROOF. This lemma immediately follows from Lemma 3.11. □

PROOF OF THEOREM 3.9. We compute the left-hand side of (3.23) as follows:

$$\frac{1}{h}\left[\widehat{H}_{s,k}I_n,\widehat{M}\right]=\frac{1}{h}\left[\frac{\mathrm{Tr}(\widehat{M}^{k+1})_s}{k+1}I_n,\widehat{M}\right]$$

$$= \sum_{j=1}^{m-1} \sum_{i=1}^{n} \sum_{q=0}^{k} \frac{1}{k+1} \mathrm{Tr} \left(\widehat{M}^q \frac{1}{h} \left[\widehat{M}, \hat{f}_{i,i+j} I_n \right] \widehat{M}^{k-q} \right)_s e_{i,i+j}.$$

Applying Lemma 3.10, we have

$$\begin{aligned}
&\frac{1}{h} \left[\widehat{H}_{s,k} I_n, \widehat{M} \right] \\
&\quad = \sum_{j=1}^{m-1} \sum_{i=1}^{n} \sum_{q=0}^{k} \frac{1}{k+1} \mathrm{Tr} \left(\widehat{M}^q \left(a_{ij} - b_{ij} \right) \widehat{M}^{k-q} \right)_s e_{i,i+j} \\
&\quad = \sum_{j=1}^{m-1} \sum_{i=1}^{n} \frac{1}{k+1} \mathrm{Tr} \Bigg((k+1) \left(\widehat{M}^k a_{ij} - b_{ij} \widehat{M}^k \right) \\
&\qquad + \sum_{q=1}^{k} q \left(\widehat{M}^{q-1} \left[a_{ij}, \widehat{M} \right] \widehat{M}^{k-q} - \widehat{M}^{k-q} \left[\widehat{M}, b_{ij} \right] \widehat{M}^{q-1} \right) \Bigg)_s e_{i,i+j}.
\end{aligned}$$

Using Lemma 3.12, we have

$$\frac{1}{h} \left[\widehat{H}_{s,k} I_n, \widehat{M} \right] = \sum_{j=1}^{m-1} \sum_{i=1}^{n} \mathrm{Tr} \left(\widehat{M}^k a_{ij} - b_{ij} \widehat{M}^k \right)_s e_{i,i+j}.$$

From the definition of a_{ij} and b_{ij},

$$\begin{aligned}
&\mathrm{Tr} \left(\widehat{M}^k a_{ij} - b_{ij} \widehat{M}^k \right)_s e_{i,i+j} \\
&\quad = \frac{1}{2} \sum_{l=1}^{m-j} \mathrm{Tr} \Big\{ \widehat{M}^k \left(\hat{f}_{i-l,i+j} e_{i-l,i} + s(i,i+j) \hat{f}_{1-i-j-l,1-i} e_{1-i-j-l,1-i-j} \right) \\
&\qquad - \left(\hat{f}_{i,i+j+l} e_{i+j,i+j+l} + s(i,i+j) \hat{f}_{1-i-j,1-i+l} e_{1-i,1-i+l} \right) \widehat{M}^k \Big\}_s e_{i,i+j} \\
&\quad = \frac{1}{2} \sum_{l=1}^{m-j} \sum_{I_k+l=ns} \Big(\hat{f}_{i-l-I_k,i-l-I_{k-1}} \cdots \hat{f}_{i-l-i_1,i-l} \hat{f}_{i-l,i+j} \\
&\qquad + s(i,i+j) \hat{f}_{1-i-j-l-I_k,1-i-j-l-I_{k-1}} \cdots \hat{f}_{1-i-j-l-I_1,1-i-j-l} \hat{f}_{1-i-j-l,1-i} \\
&\qquad - \hat{f}_{i,i+j+l} \hat{f}_{i+j+l,i+l+l+I_1} \cdots \hat{f}_{i+j+l+I_{k-1},i+l+l+I_k} \\
&\qquad - s(i,i+j) \hat{f}_{1-i-j,1-i+l} \hat{f}_{1-i+l,1-i+l+I_1} \cdots \hat{f}_{1-i+l+I_{k-1},1-i+l+I_k} \Big) e_{i,i+j}
\end{aligned}$$

$$= \frac{1}{2}\sum_{l=1}^{m-j}\left\{\left(\widehat{M}^k\right)_{i,i+ns-l}\hat{f}_{i-l,i+j}\right.$$
$$+ s(i,i+j)\left(\widehat{M}^k\right)_{1-i-j,1-i-j+ns-l}\hat{f}_{1-i-j-l,1-i}$$
$$- \hat{f}_{i,i+j+l}\left(\widehat{M}^k\right)_{i+j+l,i+l+ns}$$
$$\left.-s(i,i+j)\hat{f}_{1-i-j,1-i+l}\left(\widehat{M}^k\right)_{1-i+l,1-i+ns}\right\} e_{i,i+j},$$

where $\left(\widehat{M}^k\right)_{i,j}$ is determined by $\widehat{M}^k = \sum_{i,j}\left(\widehat{M}^k\right)_{i,j} e_{i,j}$. From Proposition 3.2, an element $[\left(\widehat{M}^k z^{-s}\right)_-, \widehat{M}^k]$ is in $\mathfrak{sp}(n, \mathcal{C}_{m,n}[z,z^{-1}])$. Hence,

$$\sum_{l=1}^{m-j} s(i,i+j)\left(\left(\widehat{M}^k\right)_{1-i-j,1-i-j+ns-l}\hat{f}_{1-i-j-l,1-i}\right.$$
$$\left.-\hat{f}_{1-i-j,1-i+l}\left(\widehat{M}^k\right)_{1-i+l,1-i+ns}\right)$$
$$= \left(\left[\left(\widehat{M}^k z^{-s}\right)_-, \widehat{M}\right]\right)_{i,i+j+ns},$$

and

$$\mathrm{Tr}\left(\widehat{M}^k a_{ij} - b_{ij}\widehat{M}^k\right)_s = \left(\left[\left(\widehat{M}^k z^{-s}\right)_-, \widehat{M}\right]\right)_{i,i+j}.$$

Therefore, we obtain

$$\frac{1}{h}\left[\widehat{H}_{s,k}I_n, \widehat{M}\right] = \left[\left(\widehat{M}^k z^{-s}\right)_-, \widehat{M}\right] = \left[\widehat{M}, \left(\widehat{B}_{s,k}\right)\right]. \ \square$$

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Graduate School of Mathematical Sciences
The University of Tokyo
Tokyo 153-8914, Japan
E-mail: nagoya@ms.u-tokyo.ac.jp