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2-Spheres of Square -1 and the Geography of Genus-2 Lefschetz Fibrations

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Abstract. Using the Gromov invariants and the Taubes' structure theorem, we investigate how spheres of square -1 are embedded against fibers in relatively minimal Lefschetz fibrations over S^2 and show the finiteness of the geography of relatively minimal genus-2 Lefschetz fibrations containing spheres of square -1. Furthermore, we present the list of possible pairs (n, s) of the numbers of irreducible singular fibers and reducible singular fibers in such a Lefschetz fibration.

1. Introduction

A Lefschetz fibration is a smooth map from a smooth 4-manifold onto a surface with finitely many critical points as complex analogs of Morse functions. The importance of Lefschetz fibrations from the viewpoint of topology was reverified by Matsumoto [22]. The remarkable works of Donaldson [6] and Gompf [10] show that Lefschetz fibrations provide a topological characterization of symplectic 4-manifolds, and the study of Lefschetz pencils and Lefschetz fibrations has played a major role in 4-dimensional symplectic topology since the works of Donaldson and Gompf. Moreover, they are also studied from various viewpoints of complex surfaces, mapping class groups and so on.

The geography problem in complex surfaces is the characterization of pairs of integers which are realized as (c_1^2, c_2) of complex surfaces, and it is well studied in algebraic geometry. By the classification of complex surfaces due to Kodaira, a simply connected complex surface is rational, elliptic or of general type. We know completely the range which rational surfaces and elliptic surfaces cover in the (c_1^2, c_2)-plane. Minimal surfaces of general type

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must satisfy that c_1^2, $c_2 > 0$ and $(c_2 - 36)/5 \le c_1^2 \le 3c_2$ (the Noether inequality and Bogomolov-Miyaoka-Yau inequality).

A simply connected complex surface is Kähler and so symplectic. Therefore, the geography problem for symplectic 4-manifolds comes into our mind. This problem was raised by McCarthy and Wolfson [23]. The region realized as complex surfaces are covered by some symplectic 4-manifolds. On the other hand, there are symplectic 4-manifolds which lie outside of this region. For example, there exist symplectic 4-manifolds which violate the Noether inequality [9], [33]. (See also [28].)

In the geography problem for symplectic 4-manifolds, the target manifold is usually required to be minimal. However, many important Lefschetz fibrations are not minimal, and so we will consider the geography problem for relatively minimal (possibly non-minimal) Lefschetz fibrations. Every lattice point (c_1^2, c_2) except finitely many lying in $(c_2 - 36)/5 \le c_1^2 \le 2c_2$ is realized as the total space of a Lefschetz fibration [29]. Fintushel and Stern showed that there exists a minimal Lefschetz fibration which does not satisfy the Noether inequality [8]. In [34], Stipsicz addressed the Bogomolov-Miyaoka-Yau inequality for Lefschetz fibrations.

Smith showed in [32] that only finitely many pairs (c_1^2, c_2) can be realized as genus-2 Lefschetz pencils. Since one can obtain examples of Lefschetz fibrations over S^2 with sections of square -1 from Lefschetz pencils by the blow-ups at the base locus points, this puts us in mind of the geography problem for genus-2 Lefschetz fibrations with spheres of square -1. In this paper, we show the finiteness of the geography of relatively minimal genus-2 Lefschetz fibrations over S^2 with spheres of square -1:

THEOREM 1.1. *Only finitely many pairs* (c_1^2, c_2) *can be realized as genus-2 Lefschetz fibrations over* S^2 *with 2-spheres of square* -1 *but without a fiber containing 2-spheres of square* -1.

Since a new example of a genus-2 Lefschetz fibration constructed by Auroux [2] is a non-minimal Lefschetz fibration which can not be obtained from any Lefschetz pencil by blow-ups, Theorem 1.1 does not follow from the Smith's finiteness result of genus-2 Lefschetz pencils.

We use the theory of pseudo-holomorphic curves and the Taubes' structure theorem on the Gromov invariants. Firstly we show that a genus-g Lefschetz fibration $f : X \to S^2$ without a fiber containing 2-spheres of square -1 can admit at most $2g-2$ 2-spheres of square -1 essentially and we investigate how such 2-spheres lie against fibers. Thus, we can know the topological placement of spheres with self-intersection number -1 in genus-2 Lefschetz fibrations and we will prove Theorem 1.1 by using such spheres and analyzing pseudo-holomorphic curves representing classes with non-trivial Gromov invariants. Moreover, we present examples of non-minimal genus-2 Lefschetz fibrations.

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2. Lefschetz Fibrations and Pencils

We recall the definitions on Lefschetz fibrations and Lefschetz pencils.

A smooth map $f : X \to \Sigma$ from a closed, connected, oriented smooth 4-manifold X onto a closed, connected, oriented smooth 2-manifold Σ is said to be a *Lefschetz fibration*, if f admits finitely many critical points $C = \{p_1, p_2, \ldots, p_k\}$ on which f is injective and around which there are orientation-preserving complex coordinate neighborhoods such that locally f can be expressed as $f(z_1, z_2) = z_1^2 + z_2^2$. It is a consequence of this definition that $f|_{X \setminus C} : X \setminus C \to \Sigma \setminus f(C)$ is a smooth fiber bundle. Generic fibers, which are fibers without critical points, are closed oriented 2-manifolds and are of the same diffeomorphism type. If the genus of a generic fiber is g, we refer to f as a *genus-g Lefschetz fibration*. Any fiber containing a critical point is called a *singular fiber*, which is obtained by collapsing a simple closed curve, called a *vanishing cycle*, on a nearby generic fiber to a point. A singular fiber is called *reducible* or *irreducible* according to whether the corresponding vanishing cycle separates or dose not separate in the generic fiber. The monodromy around a singular fiber of a Lefschetz fibration is given by a positive (or right-handed) Dehn twist along the corresponding vanishing cycle. A Lefschetz fibration $f : X \to \Sigma$ is said to be *relatively minimal* if there is no fiber containing a sphere of square -1. We assume that a Lefschetz fibration f is nontrivial, that is, f has at least one singular fiber through this paper. Moreover, we may construct a new Lefschetz fibration from two Lefschetz fibrations as follows: Let $f_i : X_i \to \Sigma_i$ $(i = 1, 2)$ be a genus-g Lefschetz fibration. Removing regular neighborhoods of generic fibers in each, we glue the boundaries of these remainders by using a fiber-preserving diffeomorphism which can be extended to f_i ($i = 1, 2$). Thus we obtain a genus-g Lefschetz fibration, which is denoted by $X_1 \sharp_f X_2 \to \Sigma_1 \sharp \Sigma_2$ and is called the *fiber sum* of f_1 and f_2.

A *Lefschetz pencil* is a nonempty finite set $B = \{b_1, b_2, \dots, b_\ell\}$ of X, called the *base locus*, together with a smooth map $f : X \setminus B \to \mathbb{C}P^1$ such that each b_i has an orientation-preserving complex coordinate neighborhood in which locally f can be expressed as $f(z_1, z_2) = z_1/z_2$, and each critical point of f has a local coordinate neighborhood as a Lefschetz fibration. Every smooth projective complex surface $S \subset \mathbb{C}P^N$ has a Lefschetz pencil: The intersections of S with hyperplane sections containing a generic linear subspace of complex codimension 2 are complex curves, which give S a Lefschetz pencil. By the definitions of Lefschetz fibrations and pencils, the blow-up at the base locus points of a Lefschetz pencil yields a Lefschetz fibration over S^2 with sections of square -1. For more details on Lefschetz fibrations and Lefschetz pencils, see [3], [5] and [10].

Combining the remarkable theorems of Donaldson and Gompf gives the following topological characterization of symplectic 4-manifolds.

THEOREM 2.1 ([6], [10]). *A 4-manifold X admits a symplectic structure if and only if it admits a Lefschetz pencil.*

We can estimate the number of points in the base locus of a Lefschetz pencil, which suggests an estimation of the essential number of spheres of square -1 in the Lefschetz fibration yielded from a Lefschetz pencil by the blow-up.

PROPOSITION 2.1 ([32]). *Let X be a smooth 4-manifold with $b_2^+(X) > 1$. If X admits a Lefschetz pencil whose fibers are of genus g, then the number of points in the base locus is bounded above by $2g - 2$.*

In §4, we will estimate the upper bound for the essential number of spheres of square -1 in a non-minimal Lefschetz fibration.

3. Gromov Invariants and the Taubes' Structure Theorem

In this section, we recall the Gromov invariants and the Taubes' structure theorem. The Seiberg-Witten invariants and the theory of pseudo-holomorphic curves make the topology of symplectic 4-manifolds rich. Let (M, ω) be a connected, closed symplectic 4-manifold with an ω-compatible almost complex structure J and with symplectic canonical class K_M. A smooth map $\varphi : \Sigma \to M$ from a possibly disconnected compact Riemann surface (Σ, j) to (M, J) is said to be *J-holomorphic* if the differential $d\varphi$ satisfies

$$d\varphi \circ j = J \circ d\varphi.$$

We call the image $\varphi(\Sigma)$ a *J-holomorphic curve* or a *pseudo-holomorphic curve* with respect to J. If C is a pseudo-holomorphic curve with respect to an ω-compatible almost complex structure, then C is also ω-symplectic.

The *Gromov invariant* Gr_T is defined by counting with signs the number of pseudo-holomorphic curves in a given homology class roughly as follows. For the more exact definition and details on the Gromov invariant, see [38, 39, 40]: Given a cohomology class $\alpha \in H^2(M; \mathbb{Z})$, we define $d(\alpha)$ by

$$d(\alpha) = -K_M \cdot \alpha + \alpha \cdot \alpha,$$

where $\cdot$ is the cup product pairing. If $d(\alpha) > 0$, let $\Omega \subset M$ be a set of $d(\alpha)/2$ distinct points. If $d(\alpha) \leq 0$, we set $\Omega = \emptyset$. Then, we consider the space $\mathcal{H}(\alpha, J)$ of J-holomorphic curves representing $PD(\alpha)$ and going through Ω. Here we denote $PD(\alpha)$ the Poincaré dual of α. One can show that $\mathcal{H}(\alpha, J)$ is an oriented 0-manifold for generic choices of J and Ω. See [38, 39] and [25]. In [39], Taubes proved the regularity theorem for $\mathcal{H}(\alpha, J)$ and introduced the Gromov invariant Gr_T. For a nonzero class α, $\mathrm{Gr}_T(\alpha)$ is defined to be the algebraic number of pseudo-holomorphic curves in $\mathcal{H}(\alpha, J)$;

$$\mathrm{Gr}_T(\alpha) = \sum_{C \in \mathcal{H}(\alpha, J)} \varepsilon(C),$$

where $\varepsilon(C) = \pm 1$. We set $\mathrm{Gr}_T(0) = 1$. The Gromov invariants Gr_T are independent of generic choices of J and Ω. Furthermore, Taubes showed that the Gromov invariants of symplectic 4-manifolds with $b_2^+ > 1$ are equivalent to the Seiberg-Witten invariants [38, 39], that is,

$$\mathrm{SW}(K_M^{-1} + 2\alpha) = \pm \mathrm{Gr}_T(\alpha),$$

and he obtained the following structure theorem.

THEOREM 3.1 (Taubes' structure theorem [38, 39]). *Let (M, ω) be a closed symplectic 4-manifold with $b_2^+(M) > 1$. Then the followings hold:*

(1) $\mathrm{Gr}_T(K_M) = \pm 1$. *In particular, the homology class $PD(K_M)$ has a (possibly disconnected) smooth pseudo-holomorphic representative. Hence, if M is minimal, then $c_1^2(M) = K_M^2 \geq 0$.*

(2) *(The duality formula) For any cohomology class $\alpha \in H^2(M; \mathbb{Z})$, $\mathrm{Gr}_T(\alpha) = \pm \mathrm{Gr}_T(K_M - \alpha)$.*

(3) *If $\mathrm{Gr}_T(\alpha) \neq 0$, then for a generic ω-compatible almost complex structure J, there exists a J-holomorphic representative $\{(C_i, m_i)\}_i$ of α, where $C_i \subset M$ is a J-holomorphic submanifold and m_i is an appropriate multiplicity (which is one, unless C_i is a torus of square 0).*

The Seiberg-Witten invariants can be also defined for manifolds with $b_2^+ = 1$. However, the invariants depend on a chamber structure. For details on the relation between the Seiberg-Witten theory and the Gromov-Taubes theory for symplectic 4-manifolds with $b_2^+ = 1$, see [19] and [20].

4. Spheres of Square -1 in the Total Spaces of Lefschetz Fibrations over S^2

We begin with some theorems on smoothly embedded spheres in a symplectic 4-manifold with self-intersection number -1.

THEOREM 4.1 ((-1)-curve theorem). *Let (M, ω) be a closed symplectic 4-manifold with symplectic canonical class K_M. Let $e \in H_2(M; \mathbb{Z})$ denote a homology class represented by a smoothly embedded sphere of square -1 and with $\omega(e) > 0$. Then the followings hold:*

(1) (Taubes [38, 39]) *If $b_2^+(M) > 1$, then we have $\mathrm{Gr}_T(e) = \pm 1$, in particular, for a generic ω-compatible almost complex structure J the class e is represented by a J-holomorphic (-1)-curve.*

(2) (Li and Liu [19]) *If* $b_2^+(M) = 1$ *and* $e \cdot K_M = -1$, *then for a generic* ω*-compatible almost complex structure* J *the class* e *is represented by a* J*-holomorphic* (-1)*-curve.*

(3) (Li [16]) *Furthermore, if* M *is the blow-up of a minimal symplectic 4-manifold which is not rational nor ruled, and* $e_1, e_2, \ldots, e_\ell$ *are the homology classes represented by exceptional curves, then these classes* e_i*'s are the only classes represented by a smoothly embedded sphere of square* -1. *Furthermore, if* C_1 *and* C_2 *are symplectic* (-1)*-curves in* M, *then they can not satisfy* $[C_1] \cdot [C_2] > 0$, *provided* $b_2^+(M) > 1$.

Moreover, rational or ruled surfaces can be characterized among all symplectic 4-manifolds in terms of smoothly embedded spheres of square -1.

THEOREM 4.2 ([16]). *If* $e \cdot K_M \neq -1$ *for a homology class* e *as in Theorem 4.1, then* M *is rational or ruled.*

Hence, if M is not rational or ruled, then every smoothly embedded sphere of square -1 is $\mathbb{Z}$-homologous to a J-holomorphic (-1)-curve for a generic ω-compatible almost complex structure J after the appropriate choice of an orientation of the sphere. On the other hand, if M is rational or ruled, and $e \cdot K_M \neq -1$, then via pullback by an orientation-preserving diffeomorphism, the class e is represented by a symplectic (-1)-curve, namely e is represented by a pseudo-holomorphic (-1)-curve [16].

Let $e_1, e_2, \ldots, e_n$ be homology classes represented by smooth spheres in M of square -1 with $\omega(e_i) > 0$ $(i = 1, 2, \ldots, n)$. By the (-1)-curve theorem, we see that if M is not rational nor ruled, then there exists an ω-compatible almost complex structure on M such that homology classes $e_1, e_2, \ldots, e_n$ can be represented by pseudo-holomorphic (-1)-curves simultaneously. On the other hand, if M is rational or ruled, then they can not always be represented by pseudo-holomorphic (-1)-curves simultaneously [17]. For example, we consider $M = \mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P^2}$. Let α be the homology class represented by the line in $\mathbb{C}P^2$. Let e_1 and e_2 be the homology classes represented by exceptional curves in $\mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P^2}$. Then, e_1 and $\alpha + e_1 + e_2$ are represented by smooth spheres of square -1. However, since distinct pseudo-holomorphic curves intersect each other positively [24] and the intersection number $e_1 \cdot$

$(\alpha+e_1+e_2)$ is negative, they can not be represented by pseudo-holomorphic (-1)-curves simultaneously.

As for spheres of square -1 in the total space of a Lefschetz fibration, the problems about the minimality or the fiber sum decomposability have been treated.

THEOREM 4.3 ([34]). *A Lefschetz fibration $f : X \to B$ with $g(B) > 0$ is relatively minimal if and only if X is minimal.*

The case of $B = S^2$ has many counterexamples for this theorem. In the rest of this paper, we consider Lefschetz fibrations over S^2 which are relatively minimal but not minimal. The followings are well known as theorems about the fiber sum decomposability.

THEOREM 4.4 ([35]). *For any Lefschetz fibration $f : X \to S^2$, the fiber sum $X \sharp_f X$ is minimal.*

THEOREM 4.5 ([36], [31]). *If a Lefschetz fibration $f : X \to S^2$ admits a section of square -1, then X cannot be decomposed as any nontrivial fiber sum $X = X_1 \sharp_f X_2$.*

These theorems naturally lead us to the problem of the minimality of nontrivial fiber sums. Usher proved the following conjecture proposed by Stipsicz [36].

THEOREM 4.6 (The Stipsicz's conjecture [41]). *Every nontrivial fiber sum $X_1 \sharp_f X_2$ is minimal.*

Therefore, non-minimal Lefschetz fibrations are "irreducible" building blocks in the fiber-sum construction.

Given a Lefschetz pencil on a smooth 4-manifold X, we obtain a genus-g Lefschetz fibration $X \sharp n \overline{\mathbb{C}P^2} \to S^2$ by the blow-up along the base locus. Then, each exceptional sphere is a section of square -1. Proposition 2.1 states that there are at most $2g-2$ sections of square -1 which are exceptional spheres. We will generalize this estimation about the numbers of spheres of square -1 to the case of Lefschetz fibrations.

Now let $E \in H^2(X;\mathbb{Z})$ be the Poincaré dual of the homology class which is represented by a smoothly embedded sphere of square -1 in a closed symplectic 4-manifold (X, ω). In the case where X is not rational nor ruled, by changing the orientation of a smooth sphere representing E if necessary, we may assume that $E \cdot [\omega] > 0$, because we have $E \cdot [\omega] \neq 0$ by the (-1)-curve theorem and the fact that $\omega_{|\Sigma}$ on a closed symplectic submanifold Σ is a volume form. We denote by $\mathcal{E}_X$ the set of all the Poincaré dual of the homology classes E which can be represented by smoothly embedded spheres of square -1 and satisfy $E \cdot [\omega] > 0$.

Let $f : X \to S^2$ be a relatively minimal genus-g Lefschetz fibration. The Gompf's result on symplectic structures [10] states that if the homology class of a generic fiber F of f is not torsion in $H_2(X; \mathbb{Z})$, in particular $g \geq 2$, then the total space X of a Lefschetz fibration admits a symplectic structure ω with symplectic fibers, and so X can be equipped with an ω-compatible almost complex structure J for which the fibers are pseudo-holomorphic.

Let F denote the Poincaré dual of the homology class represented by a generic fiber. Given a 2-dimensional submanifold Σ, we will avoid the complication by using the same symbol in order to denote the Poincaré dual of the homology class represented by Σ from now on.

Since a Lefschetz fibration f is relatively minimal, one can show the following key lemma:

Lemma 4.1 ([18]). *If X is not rational nor ruled, then $E \cdot F \geq 1$ for any $E \in \mathcal{E}_X$.*

Now we estimate spheres of square -1 in relatively minimal Lefschetz fibrations over S^2.

Theorem 4.7. *Let $f : X \to S^2$ be a relatively minimal genus-g (≥ 2) Lefschetz fibration. Suppose that $\mathcal{E}_X$ is not empty, and set $\mathcal{E}_X = \{E_1, E_2, \dots, E_n\}$. If X is not rational nor ruled, then the followings hold:*

(1) $n \leq 2g - 2$.

(2) $\left(\sum_{i=1}^{n} E_i \right) \cdot F \leq 2g - 2$.

(3) *For any i $(1 \leq i \leq n)$, $1 \leq E_i \cdot F \leq 2g - 2$.*

Proof. Assertions (1) and (3) follow from Lemma 4.1 and the assertion (2). We show the assertion (2). Since pseudo-holomorphic curves are symplectic, it follows from the (-1)-curve theorem that there are symplectic (-1)-curves $C_1, C_2, \dots, C_n$ representing the classes $E_1, E_2, \dots, E_n$, respectively. Then, X has a symplectic minimal model Y and is obtained by blowing up Y at n points [10]. Exceptional curves represent $E_1, E_2, \dots, E_n$. Let $\pi : X \to Y$ be the blow-down map. Then $K_X = \pi^* K_Y + \sum_{i=1}^n E_i$ in cohomology. Equip X with an almost complex structure J such that fibers are J-holomorphic curves. It follows from the adjunction formula that we have

$$\begin{aligned} 2g - 2 &= K_X \cdot F + F \cdot F = K_X \cdot F \\ &= \pi^* K_Y \cdot F + \left(\sum_{i=1}^n E_i \right) \cdot F. \end{aligned}$$

We need to show $\pi^* K_Y \cdot F \geq 0$. This inequality is true if K_Y is a torsion class. Suppose that K_Y is not a torsion class.

The case of $b_2^+(X) > 1$: Then, we have $b_2^+(Y) > 1$. By the Taubes' structure theorem and the blow-up formula of Gromov invariants, we have $Gr_T(\pi^* K_Y) \neq 0$. So, $\pi^* K_Y$ can be represented by a J-holomorphic curve Σ. Since F is represented by an irreducible J-holomorphic curve with $F \cdot F = 0$ and pseudo-holomorphic curves have locally positive intersections [24], the intersection number of any component of Σ with F is non-negative. Hence, we concludes that $\pi^* K_Y \cdot F \geq 0$ and so

$$2g - 2 \geq \left(\sum_{i=1}^n E_i \right) \cdot F.$$

The case of $b_2^+(X) = 1$: We can prove the assertion (2) in the same manner as the case of $b_2^+(X) > 1$. However, the question is whether $\pi^* K_Y$ can have a pseudo-holomorphic representative or not, that is to say, whether $\mathrm{Gr}_T(\pi^* K_Y) \neq 0$ or not. Since X is not the blow-up of a rational or ruled surface, it follows from a result, Proposition 5.2 in [20], of Li and Liu that $\mathrm{Gr}_T(m K_Y) \neq 0$ for $m \geq 2$. By the blow-up formula of Gromov invariants, we have $\mathrm{Gr}_T(m \pi^* K_Y) \neq 0$ for $m \geq 2$. Hence, $m \pi^* K_Y$ can be represented by a pseudo-holomorphic curve, and so we can show that $(m \pi^* K_Y) \cdot F \geq 0$. Thus $\pi^* K_Y \cdot F \geq 0$. Therefore, we can prove the inequality (2) in the same manner as the case of $b_2^+(X) > 1$. $\square$

REMARK 4.1. Theorem 4.5 is not true for the blow-up of a rational or ruled surface. For example, $\mathbb{C}P^2 \sharp 13\overline{\mathbb{C}P^2}$ and $S^2 \times T^2 \sharp 4\overline{\mathbb{C}P^2}$ admit a relatively minimal genus-2 Lefschetz fibration over S^2 (see §5), which contain more than two spheres of square -1.

5. The Geography of Genus-2 Lefschetz Fibrations over S^2 with Spheres of Square -1

In this section, we consider a relatively minimal genus-2 Lefschetz fibration $f : X \to S^2$ with spheres of square -1. If X is not rational nor ruled, then Theorem 4.7 states that $\mathcal{E}_X$ is one of the following three :

Type (1, 1) : $\mathcal{E}_X = \{E_1, E_2\},\ E_1 \cdot F = E_2 \cdot F = 1.$

Type (1) : $\mathcal{E}_X = \{E\},\ E \cdot F = 1.$

Type (2) : $\mathcal{E}_X = \{E\},\ E \cdot F = 2.$

In the first and the second cases, spheres representing $\mathcal{E}_X$ are sections of $f : X \to S^2$. Note that $E_1 \cdot E_2 = 0$ for E_1 and E_2 in the case of Type (1, 1), which follows from the proof of Corollary 3 in [16].

THEOREM 5.1. *Only finitely many pairs (c_1^2, c_2) can be realized as relatively minimal genus-2 Lefschetz fibrations $X \to S^2$ with 2-spheres of square -1. Here, $c_1^2 = c_1^2(X)([X])$ and $c_2 = c_2(X)([X])$.*

PROOF. Let $f : X \to S^2$ be a relatively minimal genus-2 Lefschetz fibration with spheres of square -1. Equip X with an almost complex structure J such that fibers are J-holomorphic curves. We begin with the proof in the case of $b_2^+(X) > 1$.

The case of $b_2^+(X) > 1$: We consider the case of Type (2), where $\mathcal{E}_X = \{E\}$ and $E \cdot F = 2$. Set $A = K_X - E$. By the adjunction formula, we have $K_X \cdot F = 2$, $K_X \cdot E = -1$ and so $A \cdot F = A \cdot E = 0$ and $A^2 = K_X \cdot A$. Since E is a basic class of the Gromov invariant, it follows from the duality formula that A is also a basic class, that is, $\mathrm{Gr}_T(A) \neq 0$. Hence, the class A has a J-holomorphic representative $\{(C_i, m_i)\}_i$ such that each C_i is a J-holomorphic curve and each $m_i (\geq 1)$ is a multiplicity. If we set $C = \bigcup_{i=1}^n C_i$, then we have $A = [C] = \sum_{i=1}^n m_i[C_i]$ in (co)homology. Noting that $A \cdot F = 0$, each component C_i of C is contained in a fiber because pseudo-holomorphic curves have locally positive intersections. Thus, we have $[C_i]^2 = 0$ or -1. Because of the relative minimality of f, fibers contain no sphere-component and each component C_i is not a sphere. Since $E \cdot F = 2$ and $A \cdot E = 0$, each component C_i is neither a generic fiber nor an irreducible singular fiber, and so it is a component of a reducible singular fiber. Furthermore, we have $C_i \cap C_j = \emptyset$ $(i \neq j)$, because fibers containing C_i, C_j are disjoint. Hence C consists of components $C_1, C_2, \ldots, C_n$ with $[C_i]^2 = -1$, $[C_i] \cdot [C_j] = 0$ $(i \neq j)$ and $genus(C_i) = 1$ $(i, j = 1, 2, \ldots, n)$. Then, by the adjunction formula, we have $K_X \cdot [C_i] = 1$ $(i = 1, 2, \ldots, n)$, and so we have

$$A^2 = \sum_{i=1}^{n} m_i^2 [C_i]^2 = -\sum_{i=1}^{n} m_i^2 \quad \text{and}$$
$$K_X \cdot A = \sum_{i=1}^{n} m_i K_X \cdot [C_i] = \sum_{i=1}^{n} m_i.$$

Since A satisfies that $A^2 = K_X \cdot A$, we obtain that $0 < \sum_{i=1}^{n} m_i = -\sum_{i=1}^{n} m_i^2 < 0$. Therefore, A has to satisfy that $A = 0$. Thus, we have that $K_X = E$ and $K_X^2 = -1$. Suppose that $f : X \to S^2$ has n irreducible singular fibers and s reducible singular fibers. By the Hirzebruch signature theorem and the Matsumoto's local signature formula [22], we have

$$\begin{cases} K_X^2 = 3\sigma(X) + 2e(X), \\ \sigma(X) = -\dfrac{3}{5}n - \dfrac{1}{5}s, \text{ and} \\ e(X) = n + s - 4. \end{cases}$$

Moreover, the structure on the abelianization $H_1(\Gamma_2; \mathbb{Z}) \cong \mathbb{Z}/10\mathbb{Z}$ of the mapping class group Γ_2 of genus 2 concludes that $n + 2s \equiv 0 \pmod{10}$ ([14], [22]). Therefore, the pair (n, s) satisfies

$$\begin{cases} n + 7s = 35, \\ n + 2s \equiv 0 \pmod{10}. \end{cases}$$

Since f in the case of $n = 0$ is trivial [35], the required pairs (n, s) are $(14, 3)$ and $(28, 1)$.

Next we consider the case of Type (1, 1), where $\mathcal{E}_X = \{E_1, E_2\}$ and $E_1 \cdot F = E_2 \cdot F = 1$. Set $A = K_X - E_1 - E_2$. Then it follows from the Taubes' structure theorem that A has a J-holomorphic representative. Furthermore, since $A \cdot F = A \cdot E_1 = A \cdot E_2 = 0$ and $A^2 = K_X \cdot A$, we can show that $A = 0$, i.e. $K_X = E_1 + E_2$ in the same manner as above. Hence, $K_X^2 = -2$, and so the pair (n, s) satisfies the equations

$$\begin{cases} n + 7s = 30, \\ n + 2s \equiv 0 \pmod{10}. \end{cases}$$

Since there are at least 7 singular fibers in any genus-2 Lefschetz fibration over S^2 ([14], [27]), the required pairs are $(n, s) = (16, 2)$ and $(30, 0)$.

Next we consider the case of Type (1), where $\mathcal{E}_X = \{E\}$ and $E \cdot F = 1$. Set $A = K_X - E$. Then, $A \cdot F = 1$, $A \cdot E = 0$ and $A^2 = K_X \cdot A$. Moreover, A has a J-holomorphic representative C. Because of $A \cdot F = 1$, C contains a section S. Then we can see that S is smooth. Suppose that S has a singular point x. The fiber $F_0 = f^{-1}(f(x))$ intersects S at the singular point x. This fact implies that $S \cdot F_0 \geq 2$, because pseudo-holomorphic curves have locally positive intersections. However, this is in contradiction to $A \cdot F_0 = A \cdot F = 1$. Hence, S is a smooth section. By Lemma 2.1 of [36], the self-intersection number of S is negative. Moreover, because of $A \cdot F = 1$, the multiplicity of S is one. Thus we have that $C = S \cup \bigcup_{i=1}^{n} C_i$ and $A = [C] = [S] + \sum_{i=1}^{n} m_i [C]_i$. Since $A \cdot F = [S] \cdot F + \sum_{i=1}^{n} m_i [C_i] \cdot F = 1 + \sum_{i=1}^{n} m_i [C_i] \cdot F$, the equation $A \cdot F = 1$ leads us to $\sum_{i=1}^{n} m_i [C_i] \cdot F = 0$, i.e. $[C_i] \cdot F = 0$ $(i = 1, 2, \ldots, n)$. Hence, the components $C_1, C_2, \ldots, C_n$ of C are components of fibers, because of the positivity of intersections. We divide these components into generic/irreducible fibers and reducible singular fibers, and so we can write $A = [S] + mF + \sum_{j=1}^{k} n_j [D_j]$ in (co)homology, where each $D_j (\subset C)$ is a component of a reducible singular fiber. Since the fibration f is relatively minimal, each D_j is a torus of square -1. Because of $\mathrm{Gr}_T(A) \neq 0$, the space $\mathcal{H}(A, J')$ of J'-holomorphic curves representing A is nonempty for a generic ω-compatible almost complex structure J' and the formal dimension of $\mathcal{H}(A, J')$ is non-negative, that is,

$$-K_X \cdot A + A^2 = \dim \mathcal{H}(A, J') \geq 0.$$

(We note that in fact $\dim \mathcal{H}(A, J') = -K_X \cdot A + A^2 = 0$ because of $A^2 = K_X \cdot A$.) Using the adjunction formula, we calculate $K_X \cdot A$ and A^2;

$$\begin{cases} K_X \cdot A = K_X \cdot [S] + mK_X \cdot F + \displaystyle\sum_{j=1}^{k} n_j K_X \cdot [D_j] \\ \qquad\qquad = (-2 - [S]^2) + 2m + \displaystyle\sum_{j=1}^{k} n_j, \\ A^2 = [S]^2 - \displaystyle\sum_{j=1}^{k} n_j^2 + 2m + 2\sum_{j=1}^{k} n_j [S] \cdot [D_j]. \end{cases}$$

Hence we have

$$\dim \mathcal{H}(A, J') = -K_X \cdot A + A^2 = 2([S]^2 + 1) + \sum_{j=1}^{k} n_j (2[S] \cdot [D_j] - n_j - 1).$$

Since $[S]^2 \leq -1$, $[S] \cdot [D_j] \leq 1$ and $n_j \geq 1$, we have that $2([S]^2 + 1) \leq 0$ and $\sum_{j=1}^{k} n_j (2[S] \cdot [D_j] - n_j - 1) \leq 0$. Thus, because of $\dim H(A, J') \geq 0$, we obtain $[S]^2 = -1$. Hence S is a smooth section of square -1, and so $[S] = E$, that is, $A = E + mF + \sum_{j=1}^{k} n_j [D_j]$. From the equation $A \cdot E = 0$, we get $1 = m + \sum_{j=1}^{k} n_j [D_j] \cdot E \geq m$, and so we have that $m = 0$ or 1.

When $m = 0$, we have $\sum_{j=1}^{k} n_j [D_j] \cdot E = 1$. Since $n_j \geq 1$ and $[D_j] \cdot E \geq 0$, we may assume that $n_1 = [D_1] \cdot E = 1$ and $\sum_{j=2}^{k} n_j [D_j] \cdot E = 0$ without loss of generality, that is, $A = E + [D_1] + \sum_{j=2}^{k} n_j [D_j]$. Then, we have

$$K_X \cdot A = K_X \cdot E + K_X \cdot [D_1] + \sum_{j=2}^{k} n_j K_X \cdot [D_j] = \sum_{j=2}^{k} n_j \quad \text{and}$$

$$\begin{aligned} A^2 &= E^2 + [D_1]^2 + \sum_{j=2}^{k} n_j^2 [D_j]^2 + 2E \cdot [D_1] + 2\sum_{j=2}^{k} n_j E \cdot [D_j] \\ &\quad + 2\sum_{j=2}^{k} n_j [D_1] \cdot [D_j] \\ &= -\sum_{j=2}^{k} n_j^2. \end{aligned}$$

By using the equation $A^2 = K_X \cdot A$, we can deduce that $A = E + [D_1]$, i.e. $K_X = 2E + [D_1]$. Thus we have $K_X^2 = -1$, and so the possible pair (n, s) satisfies $n + 7s = 35$ and $n + 2s \equiv 0 \pmod{10}$. On the other hand, we can show that a genus-2 Lefschetz fibration $X \to S^2$ with $K_X = 2E + [D_1]$ admits only one reducible singular fiber. Suppose that there is another reducible singular fiber R which is distinct from D_1. Since $E \cdot [R] = 0$ or 1, we obtain that $[D_1] \cdot [R] = \pm 1$ from the equation $K_X \cdot [R] = 1$. However, this is a contradiction because R is a pseudo-holomorphic curve which is disjoint from D_1. Hence, we have $s = 1$. Therefore, when $K_X = 2E + [D_1]$, the possible pair (n, s) is only $(28, 1)$.

When $m = 1$, we have $A = E + F + \sum_{j=1}^{k} n_j [D_j]$. The equation $A \cdot E = 0$ leads us to $\sum_{j=1}^{k} n_j [D_j] \cdot E = 0$. Hence, none of the D_j's meet the section E. Then, we have

$$K_X \cdot A = K_X \cdot E + K_X \cdot F + \sum_{j=1}^{k} n_j K_X \cdot [D_j] = 1 + \sum_{j=1}^{k} n_j \quad \text{and}$$

$$\begin{aligned} A^2 &= E^2 + F^2 + \sum_{j=1}^{k} n_j^2 [D_j]^2 + 2E \cdot F + 2 \sum_{j=1}^{k} n_j E \cdot [D_j] \\ &\quad + 2 \sum_{j=1}^{k} n_j F \cdot [D_j] \\ &= 1 - \sum_{j=1}^{k} n_j^2. \end{aligned}$$

Hence, by using the equation $A^2 = K_X \cdot A$, we can deduce that $A = E + F$, i.e. $K_X = 2E + F$. Then, we have $K_X^2 = 0$, and so the possible pair (n, s) satisfies $n + 7s = 40$ and $n + 2s \equiv 0 \pmod{10}$. On the other hand, we can show that a genus-2 Lefschetz fibration $X \to S^2$ with $K_X = 2E + F$ admits no reducible singular fiber. If there is a reducible singular fiber, then its irreducible component R is a pseudo-holomorphic torus of square -1. Though $K_X \cdot [R] = 2E \cdot [R] \equiv 0 \pmod 2$, the adjunction formula leads us to $K_X \cdot [R] = 1$. This is a contradiction. Hence, we have $s = 0$. Therefore, the possible pair (n, s) is only $(40, 0)$.

The case of $b_2^+(X) = 1$: Since X is a symplectic 4-manifold with $b_2^+(X) = 1$, X is either the blow-up of a ruled surface or $b_1(X) \in \{0, 2\}$

[37]. Moreover, we obtain that

$$1 - b_2^-(X) = -\frac{3}{5}n - \frac{1}{5}s \text{ and}$$
$$3 - 2b_1(X) + b_2^-(X) = n + s - 4$$

from the definitions of the signature and the Euler number. If X is the blow-up of a ruled surface over the surface Σ_h of genus h, then a genus-2 Lefschetz fibration $X \to S^2$ must satisfy that $0 \leq 2h \leq 2$ [37]. Moreover, because of $h = 0, 1$, we obtain that $b_1(X) = 0, 2$. Thus, we see that $b_1(X) \in \{0, 2\}$ holds for ruled surfaces with genus-2 Lefschetz fibrations as well. If $b_1(X) = 0$, then the above relations imply that $n + 2s = 20$. When $b_1(X) = 2$, $n + 2s = 10$. Hence, only finitely many pairs (n, s) satisfy either (i) $n + 2s = 20, n > 0, s \geq 0$ or (ii) $n + 2s = 10, n > 0, s \geq 0$.

Thus only finitely many pairs (n, s) arise. Since the pairs (n, s) of the numbers of singular fibers are equivalent to the pairs (c_1^2, c_2) by the Hirzebruch signature theorem and the Matsumoto's local signature formula $\sigma = -3n/5 - s/5$, and so only finitely many pairs (c_1^2, c_2) arise. □

REMARK 5.1. (1) Let $f : X \to S^2$ be a genus-2 Lefschetz fibration with $n+2s = 10$. The fiber sum $X \sharp_f X$ of two copies of f is a minimal symplectic 4-manifold with $b_2^+(X \sharp_f X) > 1$ by Theorem 4.4. Hence, it follows from Theorem 3.1 (1) that we have $2c_1^2(X)+8 = c_1^2(X \sharp_f X) \geq 0$, i.e. $c_1^2(X) \geq -4$. By the Matsumoto's local signature formula, we get $c_1^2(X) = 3\sigma(X) + 2e(X) = s - 6$, and so $s \geq 2$. Therefore, the pairs $(10, 0)$ and $(8, 1)$ do not occur as the pair of the numbers of singular fibers.

(2) By Theorem 5.1, we can prove that if a nontrivial fiber sum $X = X_1 \sharp_f X_2 \to S^2$ of genus-2 Lefschetz fibrations satisfies $b_2^+(X) > 1$, then X is minimal, which is the Stipsicz's conjecture in the case of genus 2 with $b_2^+ > 1$: Suppose that $X = X_1 \sharp_f X_2$ is not minimal. Let (n, s) be the pair of the numbers of singular fibers of $X \to S^2$. Noting Theorem 4.5, the pair (n, s) is $(14, 3)$ or $(28, 1)$ of Type (2) in TABLE 1. If we let (n_i, s_i) be the pair of the numbers of singular fibers of $X_i \to S^2$ $(i = 1, 2)$, then we have $n = n_1 + n_2$ and $s = s_1 + s_2$. Noting that $n_i + 2s_i \equiv 0 \pmod{10}$ and $n_i + s_i \geq 7$ $(i = 1, 2)$, the possible pairs (n_i, s_i) are the followings;

$$(n, s) = (14, 3) \ : \begin{cases} (n_1, s_1) = (4, 3) \\ (n_2, s_2) = (10, 0) \end{cases} , \quad \begin{cases} (n_1, s_1) = (6, 2) \\ (n_2, s_2) = (8, 1) \end{cases}$$

Table 1. Possible pairs (n, s) as geography.

b_2^+	Possible pairs (n, s)	$\mathcal{E}_X$
$b_2^+ > 1$	$(16, 2)$, $(30, 0)$	Type (1, 1)
	$(28, 1)$ $(40, 0)$	Type (1)
	$(14, 3)$, $(28, 1)$	Type (2)
$b_2^+ = 1$	$n + 2s = 20,\ n > 0,\ s \geq 0$	
	$n + 2s = 10,\ n > 0,\ s \geq 0$	

$$(n, s) = (28, 1)\ :\begin{cases}(n_1, s_1) = (8, 1)\\(n_2, s_2) = (20, 0)\end{cases},\quad \begin{cases}(n_1, s_1) = (18, 1)\\(n_2, s_2) = (10, 0).\end{cases}$$

However, there is no genus-2 Lefschetz fibration with $(n_i, s_i) = (10, 0)$ or $(8, 1)$ by the above remark (1). Hence, the above decompositions of (n, s) do not occur. Thus, $X = X_1 \sharp_f X_2$ is minimal.

(3) A genus-g Lefschetz fibration $f : X \to S^2$ is said to be *hyperelliptic* if the monodromy representation of f is equivalent to one taking isotopy classes commuting with the hyperelliptic involution $\iota : \Sigma_g \to \Sigma_g$ on a closed oriented surface Σ_g of genus g. Since the hyperelliptic mapping class group Γ_2^{hyp} of genus 2 agrees with Γ_2, every genus-2 Lefschetz fibration is hyperelliptic. When we restrict the signature cocycle τ_g to the hyperelliptic mapping class group Γ_g^{hyp} of genus g, its cohomology class $[\tau_g^H] \in H^2(\Gamma_g^{\mathrm{hyp}}; \mathbb{Z})$ is of finite order [7]. So we can calculate the terms of signature cocycles by the coboundary maps called Meyer's functions. Matsumoto [22] calculated Meyer's functions and obtained the local signature formula $\sigma(X) = -3n/5 - s/5$ for genus-2 Lefschetz fibrations. Endo [7] extended the signature formula for genus-2 Lefschetz fibrations to that for hyperelliptic genus-g Lefschetz fibrations. These formulae imply that the signature is determined by the number of singular fibers. So we can consider the geography of hyperelliptic genus-$g(\geq 3)$ Lefschetz fibrations in the same manner as Theorem 5.1. However, since all genus-g Lefschetz fibrations are not hyperelliptic, we will need another argument in order to consider the non-hyperelliptic case. In [30], the author considers the geography of

genus-3 Lefschetz fibrations.

6. Examples of Non-Minimal Genus-2 Lefschetz Fibrations

We consider whether the pairs (n, s) in TABLE 1 can be realized as non-minimal genus-2 Lefschetz fibrations. In particular, we will prove the following theorem:

THEOREM 6.1. *In the case of* $b_2^+ > 1$*, there exist non-minimal genus-2 Lefschetz fibrations realizing all pairs in* TABLE *1 except* $(14, 3)$ *of Type (2).*

It is open whether the pair $(14, 3)$ of Type (2) can be realized.

We can construct Lefschetz fibrations by using double branched coverings of surface bundles or by using *positive relations* in the mapping class groups. Many well-known Lefschetz fibrations are constructed from double branched coverings of surface bundles. We will give new examples by using positive relations, because the way of using positive relations has an advantage to judge whether Lefschetz fibrations corresponding to given positive relations admit sections of square -1. On the other hand, a non-minimal genus-2 Lefschetz fibration of Type (2) admits a sphere of square -1 intersecting with a generic fiber at two points, and so it is difficult to obtain an example of a non-minimal Lefschetz fibration of Type (2) from a positive relation. As an example of Type (2), we will present a Lefschetz fibration realizing the pair $(28, 1)$ of Type (2), which is constructed from a double branched covering of a Hirzebruch surface by Auroux.

Let Γ_2 be the mapping class group of genus 2 and $\Gamma_{2,k}$ the mapping class group of a genus-2 surface with k boundary components. To simplify the notation in the rest of the paper, we denote a positive Dehn twist along a simple closed curve α also by α. The inverse α^{-1} of α is denoted by $\overline{\alpha}$. For elements $\varphi, \psi \in \Gamma_2$, the product $\varphi \cdot \psi$ stands for applying φ first and then applying ψ, and sometimes the dot "$\cdot$" is dropped. Furthermore, we use the notation φ_ψ instead of $\overline{\psi} \cdot \varphi \cdot \psi$. If τ is a positive Dehn twist along a simple closed curve α on the closed orientable surface Σ_2 of genus 2, then τ_ψ is the positive Dehn twist along the curve $\psi(\alpha)$.

The monodromy around a singular fiber of a Lefschetz fibration is given by a positive Dehn twist along the corresponding vanishing cycle. Since the base spaces of Lefschetz fibrations in this paper are the sphere, the product $\tau_1\tau_2\cdots\tau_m$ of all monodromies is trivial in Γ_2. We call the relation $\tau_1\tau_2\cdots\tau_m = 1$ obtained from a factorization of the identity via positive Dehn twists $\tau_1, \tau_2, \ldots, \tau_m$ a *positive relation.* Then, by Kas [13] and Matsumoto [22], Lefschetz fibrations on 4-manifolds can be described by positive relations. Namely, for a given positive relation $\tau_1\tau_2\cdots\tau_m = 1$, there is a Lefschetz fibration with monodromies $\tau_1, \tau_2, \ldots, \tau_m$. If a lift of the relation $\tau_1\tau_2\cdots\tau_m = 1$ to $\Gamma_{2,k}$ is given by positive Dehn twists along circles δ_j's parallel to the boundary components;

$$\tau_1\tau_2\cdots\tau_m = \delta_1^{n_1}\delta_2^{n_2}\cdots\delta_k^{n_k},$$

then it shows the existence of k distinct sections of the corresponding Lefschetz fibration and the self-intersection number of the j-th section is $-n_j$ [15].

The mapping class group Γ_2 is generated by five Dehn twists ζ_i $(1 \le i \le 5)$ around curves indicated on Figure 1 [21] :

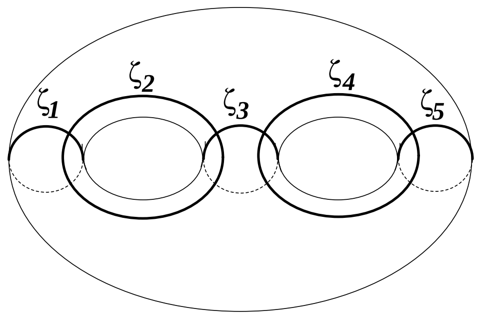

Fig. 1.

For the sake of brevity, we denote the Dehn twist ζ_j by j. It is well-known that Γ_2 has the following positive relations:

(6.1) $$(1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)^2 = 1$$

(6.2) $$(5\cdot 4\cdot 3\cdot 2\cdot 1)^6 = 1$$

(6.3) $$(4\cdot 3\cdot 2\cdot 1)^{10} = 1$$

Example 6.1. There exist genus-2 Lefschetz fibrations with the above positive relations, which are obtained from double branched covers of the Hirzebruch surfaces $\mathbb{F}_n$. See [10].

(1) $\mathbb{C}P^2 \sharp 13 \overline{\mathbb{C}P^2}$:

The positive relation (6.1) describes the genus-2 Lefschetz fibration on the rational surface $\mathbb{C}P^2 \sharp 13 \overline{\mathbb{C}P^2}$ obtained as a double covering of $\mathbb{F}_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$ branched along a smooth algebraic curve in the linear system $|6\Delta + 2F|$. Here Δ is a section of $\mathbb{F}_0$ and F is a fiber of $\mathbb{F}_0$. This fibration is obtained from the composition of the covering projection with the bundle projection $\mathbb{F}_0 \to S^2$ and has 20 irreducible singular fibers and sections of square -1, which represents $(n, s) = (20, 0)$.

(2) $K3 \sharp 2 \overline{\mathbb{C}P^2}$:

The positive relation (6.2) describes the genus-2 Lefschetz fibration on $K3 \sharp 2 \overline{\mathbb{C}P^2}$ obtained as a double covering of $\mathbb{F}_1 = \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ branched along a smooth algebraic curve in the linear system $|6L|$, where L is a line in $\mathbb{C}P^2$ avoiding the blown-up point. This fibration has 30 irreducible singular fibers and two sections of square -1. Hence, this fibration represents $(n, s) = (30, 0)$ of Type (1, 1) in TABLE 1.

(3) $H'(1)$ (Horikawa surface) :

The Hirzebruch surface $\mathbb{F}_2 = \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(2))$ has two disjoint holomorphic sections Δ_2 and Δ_{-2} of square ± 2. The Horikawa surface $H'(1)$ is obtained as a double branched cover of $\mathbb{F}_2$, branched along the disjoint union of a smooth curve in the linear system $|5\Delta_2|$ and Δ_{-2}. Then, the positive relation (6.3) describes the genus-2 Lefschetz fibration on the Horikawa surface $H'(1)$. This fibration has 40 irreducible singular fibers and a section of square -1. This section is a lift of the component of the branched set coming from Δ_{-2}. Hence, this fibration represents $(n, s) = (40, 0)$ of Type (1) in TABLE 1.

(4) $S^2 \times T^2 \sharp 4 \overline{\mathbb{C}P^2}$:

Matsumoto [22] showed that $S^2 \times T^2 \sharp 4 \overline{\mathbb{C}P^2}$ has a genus-2 Lefschetz fibration with 6 irreducible singular fibers and 2 reducible singular fibers, namely $(n, s) = (6, 2)$. This also has a section of square -1. The positive relation describing this fibration is $(\alpha_1 \cdot \sigma \cdot \alpha_2 \cdot \alpha_3)^2 = 1$, where $\alpha_1, \alpha_2, \alpha_3$ and σ are given by positive Dehn twists along curves indicated on Figure 2.

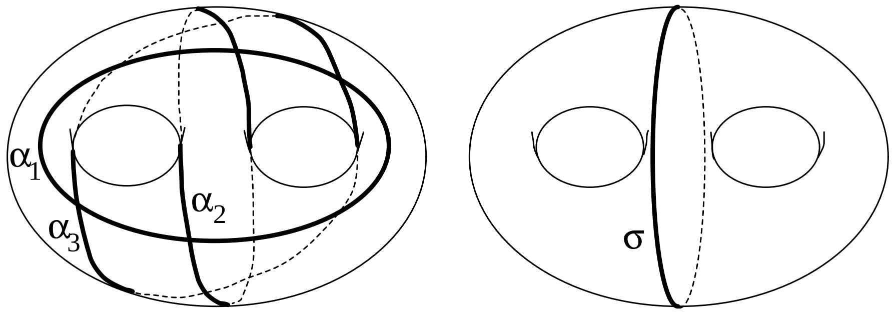

Fig. 2.

Hirose [11] constructed interesting positive relations as an application of his study of periodic homeomorphisms on Riemann surfaces. An orientation-preserving homeomorphism $\varphi : \Sigma_g \to \Sigma_g$ on a surface of genus g is said to be *periodic* if there is an integer n such that φ^n is isotopic to the identity map. We call the smallest positive such integer n the *period* of φ. By Nielsen's theorem, the conjugacy class $[\varphi]$ of a periodic homeomorphism φ is characterized by the period and the valencies of multiple points. For a periodic homeomorphism $\varphi : \Sigma_g \to \Sigma_g$, we denote the valency data of φ by $[g, n; \theta_1, \theta_2, \dots, \theta_b]$. Here n is the period of φ and each θ_i is the integer corresponding to the valency of a multiple point of φ. The sequence $\theta_1, \theta_2, \dots, \theta_b$ is corresponding to the total valency $n_1/m_1 + n_2/m_2 + \cdots + n_b/m_b$ in [1], namely $\theta_i/n = n_i/m_i$. For example, the valency data $[2, 6; 1, 1, 4]$ stands for the valency data $g = 2, n = 6, 1/6 + 1/6 + 2/3$. For more details on periodic homeomorphisms, see [26] and [1].

Hirose gave the list of periodic homeomorphisms of genus 2 by which any periodic homeomorphism is generated. See Table 2.

Hirose studies presentations of periodic homeomorphisms by positive Dehn twists. For examples, the conjugacy classes of periodic homeomorphisms with valency data $[2, 10; 1, 4, 5], [2, 8; 1, 3, 4], [2, 6; 2, 3, 3, 4], [2, 6; 1, 1, 4]$ are represented by the following products of positive Dehn twists :

$$[2, 10; 1, 4, 5] = [\zeta_4 \cdot \zeta_3 \cdot \zeta_2 \cdot \zeta_1], \qquad [2, 8; 1, 3, 4] = [\zeta_4 \cdot \zeta_4 \cdot \zeta_3 \cdot \zeta_2 \cdot \zeta_1]$$
$$[2, 6; 2, 3, 3, 4] = [\zeta_1 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_5 \cdot \zeta_{2\zeta_3\zeta_4} \cdot \zeta_{1\zeta_2\zeta_3} \cdot \sigma]$$
$$[2, 6; 1, 1, 4] = [\zeta_5 \cdot \zeta_4 \cdot \zeta_3 \cdot \zeta_2 \cdot \zeta_1].$$

Table 2. Valency data of periodic maps of genus 2.

period $= 10$	$[\varphi] = [2, 10; 1, 4, 5]$
φ^2	$[2, 5; 2, 2, 1]$
φ^3	$[2, 10; 8, 7, 5]$
φ^4	$[2, 5; 1, 1, 3]$
φ^5	$[2, 2; 1, 1, 1, 1, 1, 1]$
φ^6	$[2, 5; 4, 4, 2]$
φ^7	$[2, 10; 2, 3, 5]$
φ^8	$[2, 5; 3, 3, 4]$
φ^9	$[2, 10; 6, 9, 5]$
φ^{10}	id

period $= 8$	$[\varphi] = [2, 8; 1, 3, 4]$
φ^2	$[2, 4; 2, 2, 3, 1]$
φ^3	$[2, 8; 1, 3, 4]$
φ^4	$[2, 2; 1, 1, 1, 1, 1, 1]$
φ^5	$[2, 8; 7, 5, 4]$
φ^6	$[2, 4; 2, 2, 3, 1]$
φ^7	$[2, 8; 7, 5, 4]$
φ^8	id

period $= 6$	$[\varphi] = [2, 6; 2, 3, 3, 4]$
φ^2	$[2, 3; 1, 1, 2, 2]$
φ^3	$[2, 2; 1, 1, 1, 1, 1, 1]$
φ^4	$[2, 3; 1, 1, 2, 2]$
φ^5	$[2, 6; 2, 3, 3, 4]$
φ^6	id

period $= 6$	$[\varphi] = [2, 6; 1, 1, 4]$
φ^2	$[2, 3; 1, 1, 2, 2]$
φ^3	$[2, 2; 1, 1]$
φ^4	$[2, 3; 1, 1, 2, 2]$
φ^5	$[2, 6; 5, 5, 2]$
φ^6	id

For example, two products $(44321)^4$ and $(4321)^5$ have the same valency data $[2, 2; 1, 1, 1, 1, 1, 1]$ (see Table 2) and so there is a homeomorphism h such that $(44321)^4 = h^{-1}(4321)^5 h$. (We can show that $(44321)^4 = (4321)^5$ by using the braid relations only, and so we can take $h = id$ exactly.) Thus, from Table 2 and these presentations, we can obtain presentations of periodic homeomorphisms of genus 2 by positive Dehn twists and construct new examples of positive relations of genus 2. The followings are examples given by Hirose [12]:

Example 6.2. (1) By using the braid relations and the chain relation $(2 \cdot 1)^6 = \sigma$ on Γ_2, we can obtain a positive presentation of $(4 \cdot 3 \cdot 2 \cdot 1)^5$ by 9 Dehn twists as follows (we use the braid relations at the terms with underlines) :

$$\begin{aligned}(4 \cdot 3 \cdot 2 \cdot 1)^5 &= (4321)^3 432\underline{14}321 = (4321)^3 4324\underline{13}21 = (4321)^3 43\underline{24}3121 \\ &= (4321)^3 \underline{434}23121 = (4321)^3 34\underline{323}121 = (4321)^3 3423(21)^2 \\ &= \cdots = (4321)^2 234123(21)^3 = (4321)^2 234123(\bar{1}\,\bar{2})^3 (21)^6 \\ &= (4321)^2 234123(\bar{1}\,\bar{2})^3 \sigma = (4321)^2 23412\underline{3\bar{1}}\,\bar{2}\,\bar{1}\,\underline{\bar{2}\,\bar{1}\,\bar{2}}\sigma\end{aligned}$$

$$
\begin{aligned}
&= (4321)^2 234\underline{12\bar{1}}3\bar{2}\bar{1}\bar{1}\bar{2}\bar{1}\sigma = (4321)^2 234\bar{2}1\underline{23\bar{2}}\bar{1}\bar{1}\bar{2}\bar{1}\sigma \\
&= \cdots = 4 \cdot 3\underline{21\bar{2}}\bar{3} \cdot 4 \cdot \underline{32\bar{3}} \cdot 1 \cdot \underline{23\bar{2}} \cdot 4 \cdot 1\underline{23\bar{2}}\bar{1} \cdot \sigma \\
&= 4 \cdot \underline{3\bar{1}}2\underline{1\bar{3}} \cdot 4 \cdot \bar{2}32 \cdot \cdot \bar{3}23 \cdot 4 \cdot \underline{1\bar{3}}2\underline{3\bar{1}} \cdot \sigma \\
&= \cdots = 4 \cdot \bar{1}\bar{2}321 \cdot 4 \cdot \bar{2}32 \cdot 1 \cdot \bar{3}23 \cdot 4 \cdot \bar{3}\bar{2}123 \cdot \sigma \\
&= 4 \cdot 3_{21} \cdot 4 \cdot 3_2 \cdot 1 \cdot 2_3 \cdot 4 \cdot 1_{23} \cdot \sigma
\end{aligned}
$$

Since $((4 \cdot 3 \cdot 2 \cdot 1)^5)^2 = (4 \cdot 3 \cdot 2 \cdot 1)^{10} = 1$ in Γ_2, we have the positive relation

$$(4 \cdot 3_{21} \cdot 4 \cdot 3_2 \cdot 1 \cdot 2_3 \cdot 4 \cdot 1_{23} \cdot \sigma)^2 = 1 \tag{6.4}$$

and $(4 \cdot 3_{21} \cdot 4 \cdot 3_2 \cdot 1 \cdot 2_3 \cdot 4 \cdot 1_{23} \cdot \sigma)^2$ is the product of 16 positive Dehn twists along non-separating curves and two positive Dehn twists along separating curves. Furthermore, we see that the lift of the relation (6.4) to $\Gamma_{2,2}$ is

$$(4 \cdot 3_{21} \cdot 4 \cdot 3_2 \cdot 1 \cdot 2_3 \cdot 4 \cdot 1_{23} \cdot \sigma)^2 = \delta_1 \cdot \delta_2,$$

where δ_1 and δ_2 are Dehn twists along boundaries of disks indicated in Figure 3.

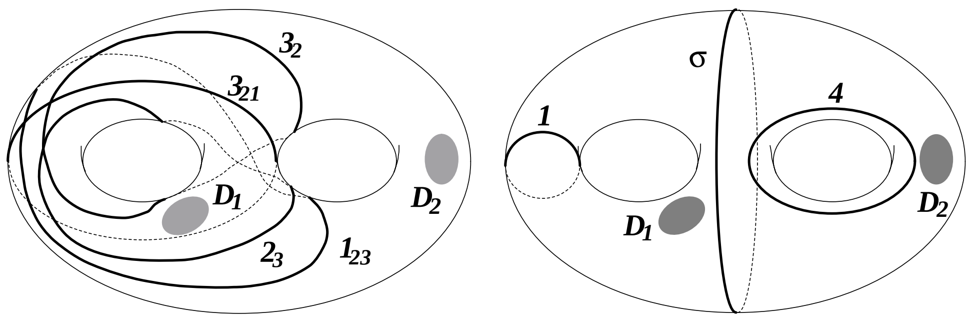

Fig. 3.

Therefore, the Lefschetz fibration induced from the positive relation (6.4) admits two sections with self-intersection number -1 and so it is a genus-2 Lefschetz fibration representing $(n, s) = (16, 2)$ of Type (1, 1) in Table 1.

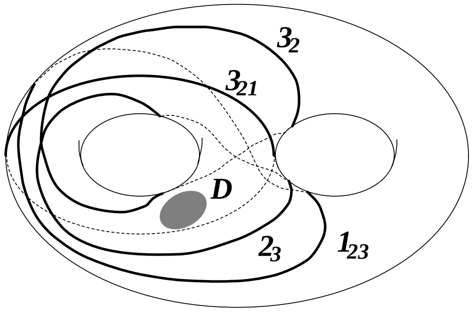

Fig. 4.

(2) Since $(4 \cdot 3 \cdot 2 \cdot 1)^{10} = 1$ and $(4 \cdot 3 \cdot 2 \cdot 1)^5 = 4 \cdot 3_{21} \cdot 4 \cdot 3_2 \cdot 1 \cdot 2_3 \cdot 4 \cdot 1_{23} \cdot \sigma$ in Γ_2, we have the positive relation

$$(4 \cdot 3 \cdot 2 \cdot 1)^5 \cdot 4 \cdot 3_{21} \cdot 4 \cdot 3_2 \cdot 1 \cdot 2_3 \cdot 4 \cdot 1_{23} \cdot \sigma = 1. \tag{6.5}$$

Hence, we can construct a genus-2 Lefschetz fibration with $(n, s) = (28, 1)$ from the positive relation (6.5). Since the lift of the relation (6.5) to $\Gamma_{2,1}$ is

$$(4 \cdot 3 \cdot 2 \cdot 1)^5 \cdot 4 \cdot 3_{21} \cdot 4 \cdot 3_2 \cdot 1 \cdot 2_3 \cdot 4 \cdot 1_{23} \cdot \sigma = \delta,$$

where δ is the Dehn twist along the boundary of a disk indicated in Figure 4.

Thus, this fibration has at least one section of square -1. Non-minimal genus-2 Lefschetz fibrations with $(n, s) = (28, 1)$ are of Type (1) or Type (2) in TABLE 1. Every non-minimal fibration of Type (2) admits no section of square -1. Hence, the fibration induced from the relation (6.5) has one section of square -1 and represents $(n, s) = (28, 1)$ of Type (1) in TABLE 1.

Since $(4 \cdot 4 \cdot 3 \cdot 2 \cdot 1)^8 = 1$ and $(4 \cdot 4 \cdot 3 \cdot 2 \cdot 1)^4 = (4 \cdot 3 \cdot 2 \cdot 1)^5$ in Γ_2, we can obtain the positive relation

$$\begin{aligned} 1 &= (4 \cdot 4 \cdot 3 \cdot 2 \cdot 1)^4 \cdot (4 \cdot 3 \cdot 2 \cdot 1)^5 \\ &= (4 \cdot 4 \cdot 3 \cdot 2 \cdot 1)^4 \cdot 4 \cdot 3_{21} \cdot 4 \cdot 3_2 \cdot 1 \cdot 2_3 \cdot 4 \cdot 1_{23} \cdot \sigma. \end{aligned} \tag{6.6}$$

From the relation (6.6), we can also construct a genus-2 Lefschetz fibration with $(n, s) = (28, 1)$. But two fibrations induced from the relations (6.5) and (6.6) are isomorphic, for $(4 \cdot 4 \cdot 3 \cdot 2 \cdot 1)^4$ and $(4 \cdot 3 \cdot 2 \cdot 1)^5$ are equivalent by

the braid relations and so the words $(4\cdot 3\cdot 2\cdot 1)^5\cdot 4\cdot 3_{21}\cdot 4\cdot 3_2\cdot 1\cdot 2_3\cdot 4\cdot 1_{23}\cdot\sigma$ and $(4\cdot 4\cdot 3\cdot 2\cdot 1)^4\cdot 4\cdot 3_{21}\cdot 4\cdot 3_2\cdot 1\cdot 2_3\cdot 4\cdot 1_{23}\cdot\sigma$ are Hurwitz equivalent.
(3) By using the braid relations and the chain relation, we obtain a positive presentation of $(5\cdot 4\cdot 3\cdot 2\cdot 1)^3$ by 4 Dehn twists as follows :

$$
\begin{aligned}
(5\cdot 4\cdot 3\cdot 2\cdot 1)^3 &= 543215432\underline{154 3}21 = 543215432\underline{54}3121\\
&= 54321543\underline{35}423121 = 54321\underline{545}3423121\\
&= 54321454\underline{34}23121 = 543214534\underline{323}121\\
&= 54321\underline{4534}232121 = \cdots\\
&= 345234123\cdot(21)^3 = 345234123(\bar{1}\bar{2})^3(21)^6\\
&= 34523412\underline{3\bar{1}}\bar{2}\bar{1}\bar{2}\bar{1}\bar{2}\sigma = 345234\underline{12\bar{1}}3\bar{2}\bar{1}\underline{\bar{2}\bar{1}\bar{2}}\sigma = \cdots\\
&= 3\underline{45\bar{4}}\bar{3}\cdot 2\underline{34\bar{3}}\bar{2}\cdot 1\underline{23\bar{2}}\bar{1}\cdot\sigma = \underline{3\bar{5}}4\underline{5\bar{3}}\cdot\underline{2\bar{4}}3\underline{4\bar{2}}\cdot\underline{1\bar{3}}2\underline{3\bar{1}}\cdot\sigma\\
&= \bar{5}\underline{34\bar{3}}5\cdot\bar{4}\underline{23\bar{2}}4\cdot\bar{3}\underline{12\bar{1}}3\cdot\sigma = \bar{5}\bar{4}345\cdot\bar{4}\bar{3}234\cdot\bar{3}\bar{2}123\cdot\sigma\\
&= 3_{45}\cdot 2_{34}\cdot 1_{23}\cdot\sigma.
\end{aligned}
$$

Since $(5\cdot 4\cdot 3\cdot 2\cdot 1)^6 = 1$ in Γ_2, we can obtain the positive relation

$$
\begin{aligned}
1 &= (5\cdot 4\cdot 3\cdot 2\cdot 1)^3\cdot(5\cdot 4\cdot 3\cdot 2\cdot 1)^3\\
&= (5\cdot 4\cdot 3\cdot 2\cdot 1)^3\cdot 3_{45}\cdot 2_{34}\cdot 1_{23}\cdot\sigma. \qquad (6.7)
\end{aligned}
$$

The genus-2 Lefschetz fibration induced from the positive relation (6.7) is a fibration representing $(n,s)=(18,1)$ and we can see that it has at least one section of square -1 .

Next we consider an interesting Lefschetz fibration constructed by Auroux :

Example 6.3. In [2], Auroux gave the interesting genus-2 Lefschetz fibration $f : X\to\mathbb{C}P^1$ with $(n,s)=(28,1)$, which is constructed as follows : Consider a curve C of degree 7 in $\mathbb{C}P^2$ with two triple points p_1 and p_2. Then, we can choose C such that the three branches of C through p_i intersect each other transversely. Let L_0 be the line through p_1 and p_2. Since $[C]\cdot[L_0]=7$, the line L_0 intersects C transversely in another point p. Next blow up $\mathbb{C}P^2$ at p and let B be the resulting curve obtained from C by the blow-up. The complex surface obtained by the blow-up is the Hirzebruch surface $\mathbb{F}_1=\mathbb{C}P^2\sharp\overline{\mathbb{C}P^2}$ and has a S^2-bundle over S^2. Let L be a line in

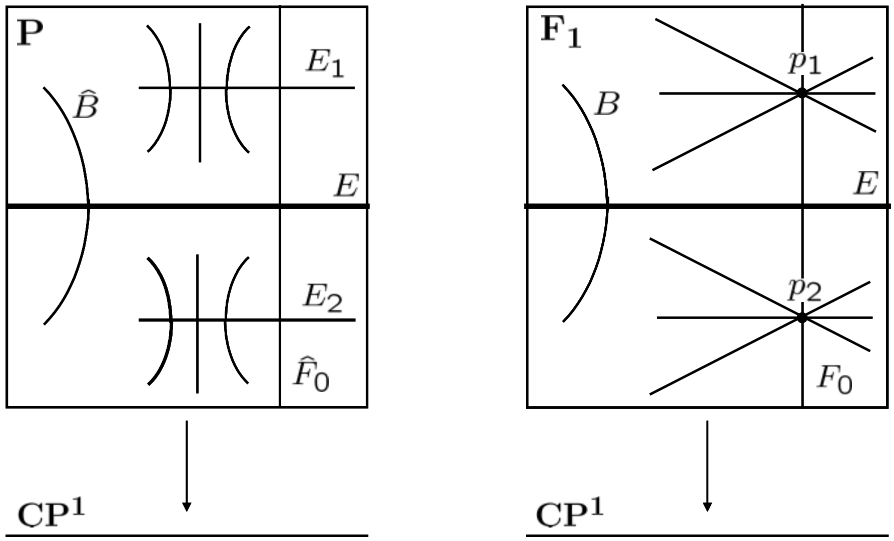

Fig. 5.

$\mathbb{C}P^2$, F a fiber of $\mathbb{F}_1$ and E the exceptional curve of the blow-up. Since B is the proper transform of C in $\mathbb{F}_1$, we have that $[E] = [L] - [F]$ and $[B] = [C] - [E] = 6[L] + [F]$. The proper transform F_0 of L_0 is the fiber of $\mathbb{F}_1$ through two triple points p_1 and p_2. The exceptional curve E intersects the curves B and F_0 in one point each transversely.

Next let P be the surface obtained by blowing up $\mathbb{F}_1$ at p_1 and p_2. We denote the proper transforms of B and F_0 in P by $\hat{B}$ and $\hat{F}_0$, respectively. See Figure 5.

If we let E_1 and E_2 be the exceptional curves of the two blow-ups, then we have that $[\hat{B}] = [B] - 3[E_1] - 3[E_2] = 6[L] + [F] - 3[E_1] - 3[E_2]$ and $[\hat{F}_0] = [F] - [E_1] - [E_2]$. Since $[\hat{B}] + [\hat{F}_0] = 2(3[L] + [F] - 2[E_1] - 2[E_2])$ is divisible by 2, we can consider the double cover $\pi : \hat{X} \to P$ branched along $\hat{B} \cup \hat{F}_0$. Because of $[F] \cdot [E_1] = [F] \cdot [E_2] = 0$, we have $[\hat{F}_0]^2 = -2$ and so $[\pi^{-1}(\hat{F}_0)]^2 = -2/2 = -1$. Hence, $\pi^{-1}(\hat{F}_0)$ is a rational curve of square -1. Next we want to blow down the (-1)-curve $\pi^{-1}(\hat{F}_0)$, but before that we consider the fibration $\hat{f} : \hat{X} \to \mathbb{C}P^1$ obtained by composing the double cover $\pi : \hat{X} \to P$ with the projection $P \to \mathbb{C}P^1$ induced from the bundle projection $\mathbb{F}_1 \to \mathbb{C}P^1$. Because of $([\hat{B}] + [\hat{F}_0]) \cdot [F] = 6$, a fiber of $\hat{f}$ is a

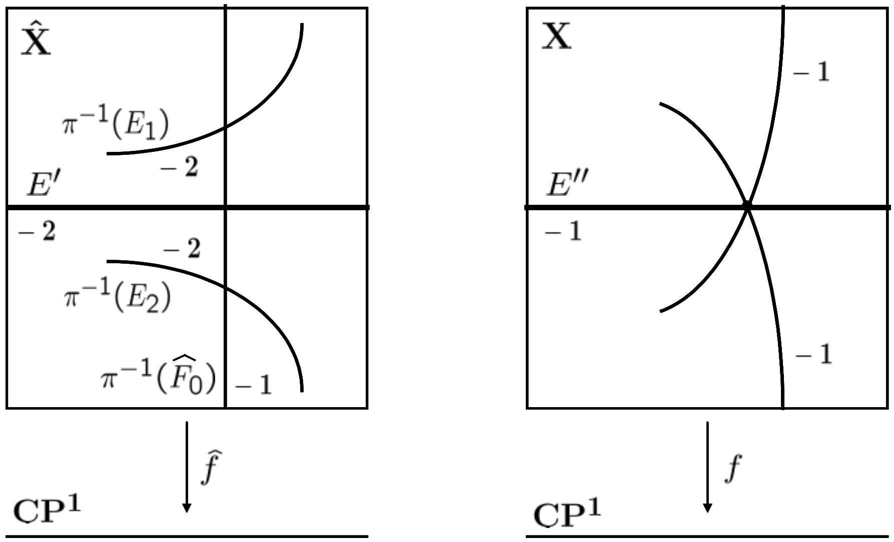

Fig. 6.

closed surface of genus 2 obtained as the double cover of $\mathbb{C}P^1$ branched at 6 points. Namely, $\hat{f}$ is a genus-2 fibration. Then, the fiber of $\hat{f}$ corresponding to F_0 is $\pi^{-1}(\hat{F_0} \cup E_1 \cup E_2) = \pi^{-1}(\hat{F_0}) \cup \pi^{-1}(E_1) \cup \pi^{-1}(E_2)$. The preimages $\pi^{-1}(E_1)$ and $\pi^{-1}(E_2)$ are elliptic curves of square -2, for these are obtained as the double covers of spheres E_1 and E_2 branched at 4 points each.

By blowing down $\pi^{-1}(\hat{F_0})$ in $\hat{X}$, we obtain a holomorphic genus-2 fibration $f : X \to \mathbb{C}P^1$ induced from the genus-2 fibration $\hat{f} : \hat{X} \to \mathbb{C}P^1$. This fibration f has one reducible singular fiber consisting of two elliptic curves of square -1. See Figure 6. Then, we have the following :

Proposition 6.1 ([2]). *The complex surface X obtained by blowing down $\pi^{-1}(\hat{F_0})$ admits a holomorphic genus-2 Lefschetz fibration $\overline{f} : X \to \mathbb{C}P^1$ with global monodromy $\sigma \cdot (\zeta_3 \cdot \zeta_4 \cdot \zeta_5 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_1 \cdot \zeta_2 \cdot \zeta_3)^2 \cdot (\zeta_1 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_5 \cdot \zeta_5 \cdot \zeta_4 \cdot \zeta_3 \cdot \zeta_2 \cdot \zeta_1)$.*

Hence, the Lefschetz fibration $\overline{f}$ has 28 irreducible singular fibers and one reducible singular fiber.

Next we chase the exceptional curve E in $\mathbb{F}_1$ intersecting the curves B and F_0 in one point each. Since E does not pass through two triple points p_1 and p_2, E gives an exceptional curve in P, which we call E again. The exceptional curve E in P intersects both $\hat{B}$ and $\hat{F_0}$ in one point each. Hence, the preimage $E' = \pi^{-1}(E)$ of E via the double cover $\pi : \hat{X} \to P$ is the double cover of the sphere branched at two points, and so E' is a sphere of square -2. Furthermore, since E' intersects $\pi^{-1}(\hat{F_0})$ in one point which is one of the two branch points of the double cover $E' \to E$. Therefore, after blowing down $\pi^{-1}(\hat{F_0})$, the sphere E' of square -2 becomes a sphere E'' in X of square -1.

The exceptional curve E'' passes through the singular point of the reducible singular fiber which is the intersection between two elliptic curves induced from $\pi^{-1}(E_1)$ and $\pi^{-1}(E_2)$, and E'' comes from a section of the Hirzebruch surface $\mathbb{F}_1$. Hence, the intersection number $[E'']\cdot[F]$ of E'' with a generic fiber F in X is 2. Therefore, we have the following:

PROPOSITION 6.2. *(1) The holomorphic genus-2 Lefschetz fibration* $\overline{f} : X \to \mathbb{C}P^1$ *constructed by Auroux in* [2] *represents* $(n, s) = (28, 1)$ *of Type (2) in* TABLE 1.

(2) There exists a non-minimal genus-2 Lefschetz fibration which can not be obtained from any Lefschetz pencil by blow-ups.

REMARK 6.1. (1) In [32], Smith showed that only finitely many pairs (c_1^2, c_2) are realized as the total spaces of genus-2 Lefschetz pencils. By Proposition 6.2, Theorem 5.1 does not follow from the Smith's finiteness result of genus-2 Lefschetz pencils.

(2) By taking the fiber sums, we can obtain infinitely many pairs (c_1^2, c_2) realized as minimal genus-2 Lefschetz fibrations. For example, for the genus-2 Lefschetz fibration $f : \mathbb{C}P^2 \sharp 13 \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ we consider the fiber sum $\sharp_m f$ of m copies of f. Then, $\sharp_m f$ has $(c_1^2, c_2) = (4m-8,\ 20m-4)$. It follows from Theorem 4.4 and TABLE 1 that the total space of $\sharp_m f$ is minimal, provided $m \geq 2$. Hence, Theorem 5.1 does not hold for *minimal* genus-2 Lefschetz fibrations.

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