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Gap Conjecture for 3-Dimensional Canonical Thresholds

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Abstract. We prove that the interval (5/6, 1) contains no 3-dimensional canonical thresholds.

1. Introduction

We work over the complex number field $\mathbb{C}$.

Let $(X \ni P)$ be a three-dimensional canonical singularity and let $S \subset X$ be a $\mathbb{Q}$-Cartier divisor. The *canonical threshold* of the pair (X, S) is

$$\mathrm{ct}(X, S) := \sup\{c \mid \text{the pair } (X, cS) \text{ is canonical}\}.$$

It is easy to see that $\mathrm{ct}(X, S)$ is rational and non-negative. Moreover, if S is effective and integral, then $\mathrm{ct}(X, S) \in [0, 1]$. Define the subset $\mathcal{T}_n^{\mathrm{can}} \subset [0, 1]$ as follows

$$\mathcal{T}_n^{\mathrm{can}} := \{\mathrm{ct}(X, S) \mid \dim X = n,\ S \text{ is integral and effective}\}.$$

The following conjecture is an analog of corresponding conjectures for log canonical thresholds and minimal discrepancies, see [Sho88], [Kol92], [Kol97], [MP04], [Kol08].

CONJECTURE 1.1. *The set $\mathcal{T}_n^{\mathrm{can}}$ satisfies the ascending chain condition.*

The conjecture is interesting for applications to birational geometry, see, e.g., [Cor95]. It was shown in [BS06] that much more general form of 1.1 follows from ACC for minimal log discrepancies and weak Borisov-Alexeev conjecture. The important particular case of 1.1 is the following

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Conjecture 1.2 (cf. [Kol08]). $\epsilon_n^{\mathrm{can}} := 1 - \sup(\mathcal{T}_n^{\mathrm{can}} \setminus \{1\}) > 0.$

The aim of this note is to prove Conjecture 1.2 for $n = 3$ in a precise form:

Theorem 1.3. $\epsilon_3^{\mathrm{can}} = 1/6.$

An analog of this theorem for log canonical thresholds was proved by J. Kollár [Kol94]: $\epsilon_3^{\mathrm{lc}} = 1/42$.

Note that replacing $(X \ni P)$ with its terminal $\mathbb{Q}$-factorial modification we may assume that $(X \ni P)$ is terminal. Thus the following is a stronger form of Theorem 1.3:

Theorem 1.4. *Let $(X \ni P)$ be a three-dimensional terminal singularity and let $S \subset X$ be an (integral) effective Weil $\mathbb{Q}$-Cartier divisor such that the pair (X, S) is not canonical. Then $\mathrm{ct}(X, S) \le 5/6$ and this bound is sharp. Moreover, if $(X \ni P)$ is singular, then $\mathrm{ct}(X, S) \le 4/5$.*

In Section 3 we give examples where the values 5/6 and 4/5 in the above theorem are achieved (see Examples 3.10 and 3.11).

The proof is rather standard. We use the classification of terminal singularities and weighted blowups techniques, cf. [Kaw92], [Kol94], [Mar96].

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2. Preliminaries

2.1. Notation. For a polynomial ϕ, $\mathrm{ord}_0 \phi$ denotes the order of vanishing of ϕ at 0 and ϕ_d is the homogeneous component of degree d.

Throughout this paper we let $(X \ni P)$ be the germ of a three-dimensional terminal singularity and let $S \subset X$ be an effective Weil $\mathbb{Q}$-Cartier divisor such that the pair (X, S) is not canonical. Put $c := \mathrm{ct}(X, S) > 0$. Since (X, S) is not canonical, $c < 1$. We assume that $c > 1/2$.

Lemma 2.2. *In the above notation the singularity $(S \ni P)$ is not Du Val.*

Proof. Assume that $(S \ni P)$ is Du Val. Since $X \ni P$ is an isolated singularity, by the inversion of adjunction [Sho93, §3] we see that the pair (X, S) is PLT. Further, since K_S is Cartier lifting its nonwhere vanishing section to X we can show that $K_X + S$ is also Cartier. Hence, the pair (X, S) is canonical. □

Lemma 2.3. *In the above notation S is reduced, irreducible and normal.*

Proof. Indeed, otherwise by blowing up a curve in the singular locus of S we get $c \le 1/2$. □

2.4. We use the techniques of weighted blowups. For definitions and basic properties we refer, for example, to [Mar96], [Rei87]. By fixing coordinates $x_1, \dots, x_n$ we regard the affine space $\mathbb{C}^n$ as a toric variety. Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ be a weight (a primitive lattice vector in the positive octant) and let $\sigma_\alpha \colon \mathbb{C}^n_{\boldsymbol{\alpha}} \to \mathbb{C}^n$ be the weighted blowup with weight $\boldsymbol{\alpha}$ ($\boldsymbol{\alpha}$-blowup). The exceptional divisor $E_{\boldsymbol{\alpha}}$ is irreducible and determines a discrete valuation $v_{\boldsymbol{\alpha}}$ of the function field $\mathbb{C}(\mathbb{C}^n)$ such that $v_{\boldsymbol{\alpha}}(x_i) = \alpha_i$.

2.5. Now let $X \subset \mathbb{C}^n$ be a hypersurface given by the equation $\phi = 0$ and let $X_{\boldsymbol{\alpha}} \subset \mathbb{C}^n_{\boldsymbol{\alpha}}$ be its proper transform. Fix an irreducible component G of $E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ such that $X_{\boldsymbol{\alpha}}$ is smooth at the generic point of G. Let v_G be the corresponding discrete valuation of $\mathbb{C}(X)$. Write

$$E_{\boldsymbol{\alpha}} \mid_{X_{\boldsymbol{\alpha}}} = m_G G + (\text{other components}).$$

Assume that $m_G = 1$ and G is not a toric subvariety in $\mathbb{C}^n_{\boldsymbol{\alpha}}$. Then the discrepancy of G with respect to K_X is computed by the formula

$$a(G, K_X) = |\boldsymbol{\alpha}| - 1 - v_{\boldsymbol{\alpha}}(\phi), \quad |\boldsymbol{\alpha}| = \sum \alpha_i,$$

see [Mar96]. Let $S \subset X$ be a Cartier divisor and let ψ be a local defining equation of S in $\mathcal{O}_{0,X}$. Then $v_G(\psi) = v_{\boldsymbol{\alpha}}(\psi)$ and the discrepancy of G with respect to $K_X + cS$ is computed by the formula

$$a(G, K_X + cS) = a(G, K_X) - c v_G(\psi) = |\boldsymbol{\alpha}| - 1 - v_{\boldsymbol{\alpha}}(\phi) - c v_{\boldsymbol{\alpha}}(\psi).$$

Therefore,

$$c \le a(G, K_X)/v_{\boldsymbol{\alpha}}(\psi) = (|\boldsymbol{\alpha}| - 1 - v_{\boldsymbol{\alpha}}(\phi))/v_{\boldsymbol{\alpha}}(\psi).$$

Definition 2.6 (cf. [Mar96]). A weight $\boldsymbol{\alpha}$ is said to be *admissible* if $E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ contains at least one reduced non-toric component.

3. Gorenstein Case

In this section we consider the case where $(X \ni P)$ is either smooth or an index one singularity.

Lemma 3.1. *If $(X \ni P)$ is smooth, then $c \leq 5/6$.*

Proof. Let $c > 5/6$. We may assume that $X = \mathbb{C}^3$. Let $\psi(x, y, z) = 0$ be an equation of S. Consider a weighted blowup $\sigma_{\boldsymbol{\alpha}} \colon \mathbb{C}^3_{\boldsymbol{\alpha}} \to \mathbb{C}^3$ with a suitable weight $\boldsymbol{\alpha}$. Let $E_{\boldsymbol{\alpha}}$ be the exceptional divisor. Recall that $(S \ni P)$ is not Du Val. Since S is normal, up to analytic coordinate change there are the following cases (cf. [KM98, 4.25]):

3.2. Case $\operatorname{ord}_0 \psi \geq 3$**.** Take $\boldsymbol{\alpha} = (1, 1, 1)$ (usual blowup of 0). Then $a(E_{\boldsymbol{\alpha}}, K_X) = 2$, $v_{\boldsymbol{\alpha}}(\psi) = \operatorname{ord}_0 \psi \geq 3$. Hence $c \leq a(E_{\boldsymbol{\alpha}}, K_X)/v_{\boldsymbol{\alpha}}(\psi) \leq 2/3$, a contradiction.

3.3. Case $\psi = x^2 + \eta(y, z)$**, where** $\operatorname{ord}_0 \eta \geq 4$**.** Take $\boldsymbol{\alpha} = (2, 1, 1)$. Then $a(E_{\boldsymbol{\alpha}}, K_X) = 3$, $v_{\boldsymbol{\alpha}}(\psi) = 4$. Hence $c \leq a(E_{\boldsymbol{\alpha}}, K_X)/v_{\boldsymbol{\alpha}}(\psi) \leq 3/4$, a contradiction.

3.4. Case $\psi = x^2 + y^3 + \eta(y, z)$**, where** $\operatorname{ord}_0 \eta \geq 4$**.** We may assume that $\eta(y, z) = u_a y z^a + u_b z^b$ (see, e.g., [KM98, 4.25]). Since he singularity $(S \ni P)$ is not Du Val, we have $a \geq 4$, $b \geq 6$ and u_a, u_b are either units or zero. Take $\boldsymbol{\alpha} = (3, 2, 1)$. Then $a(E_{\boldsymbol{\alpha}}, K_X) = 5$, $v_{\boldsymbol{\alpha}}(\psi) = 6$. Hence $c \leq a(E_{\boldsymbol{\alpha}}, K_X)/v_{\boldsymbol{\alpha}}(\psi) = 5/6$, a contradiction. □

Lemma 3.5. *Assume that $(X \ni P)$ is a Gorenstein terminal singularity and $(X \ni P)$ is not smooth. Then $c \leq 4/5$.*

Proof. Let $c > 4/5$. We may assume that X is a hypersurface in $\mathbb{C}^4$ (it is an isolated cDV-singularity [Rei80]). Let $\phi(x, y, z, t) = 0$ be the equation of X. Since $(X \ni P)$ is a cDV-singularity, $\operatorname{ord}_0 \phi = 2$. According to [Mar96], in a suitable coordinate system (x, y, z, t), there is an admissible weighted blowup $\sigma_{\boldsymbol{\alpha}} \colon \mathbb{C}^4_{\boldsymbol{\alpha}} \to \mathbb{C}^4$ such that at least for one component G of

$E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ we have $a(G, K_X) = 1$. Then $c \le 1/v_{\boldsymbol{\alpha}}(\psi)$, so $v_{\boldsymbol{\alpha}}(\psi) = 1$. This means, in particular, that $\operatorname{ord}_0 \psi = 1$. Up to coordinate change we may assume that $\psi = t$. Write

$$\phi = \eta(x, y, z) + t\zeta(x, y, z, t).$$

Then S is a hypersurface in $\mathbb{C}^3_{x,y,z}$ given by $\eta(x, y, z) = 0$. As in the proof of Lemma 3.1, using Morse Lemma we get the following cases:

3.6. Case $\operatorname{ord}_0 \eta \ge 3$. Since $\operatorname{ord}_0 \phi = 2$, ζ contains a linear term. Take $\boldsymbol{\alpha} = (1, 1, 1, 2)$. By the terminality condition [Rei87, Th. 4.6], we have $4 = v_{\boldsymbol{\alpha}}(xyzt) - 1 > v_{\boldsymbol{\alpha}}(\phi)$.

First we assume that ζ contains at least one of the terms x, y, or z. By symmetry we may assume that ζ contains x. After the analytic coordinate change $x \longleftarrow \zeta(x, y, z, t)$ we obtain

$$\phi = \eta(x, y, z) + tx.$$

In the affine chart $U_x := \{x \ne 0\}$ the map $\sigma_{\boldsymbol{\alpha}}^{-1}$ is given by

$$x \mapsto x', \quad y \mapsto y'x', \quad z \mapsto z'x', \quad t \mapsto t'x'^2. \tag{3.7}$$

$E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ is given in $\sigma_{\boldsymbol{\alpha}}^{-1}(U_x) \simeq \mathbb{C}^4$ by

$$x' = \eta_3(1, y', z') + t' = 0.$$

Hence $\boldsymbol{\alpha}$ is admissible, i.e., $E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ has a reduced non-toric component G. Then $a(G, K_X) = 1$, $v_G(\psi) = 2$ and $c \le a(G, K_X)/v_G(\psi) = 1/2$, a contradiction.

Now we assume that ζ does not contain any of the terms x, y, z. Then ζ contains t. So,

$$\phi = \eta(x, y, z) + t^2 + t\xi(x, y, z, t), \quad \operatorname{ord}_0 \xi \ge 2.$$

Further, $v_{\boldsymbol{\alpha}}(\eta) \le 3$ and $\eta_3 \ne 0$. We claim that $\boldsymbol{\alpha}$ is admissible. Using (3.7) we see that $E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ is given in $\sigma_{\boldsymbol{\alpha}}^{-1} \simeq \mathbb{C}^4$ by the equations $x' = \eta_3(1, y', z') = 0$. If η_3 is not a cube of a linear form, then $E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ has a reduced non-toric component G. Then, as above, $c \le 1/2$, a contradiction.

Finally assume that ζ does not contain any of the terms x, y, z and η_3 is a cube of a linear form. Then, as above, $\eta_3 \neq 0$ and up to linear coordinate change we have $\eta_3(x,y,z) = y^3$. So,

$$\phi = y^3 + \eta^\bullet(x,y,z) + t^2 + t\xi(x,y,z,t), \quad \operatorname{ord}_0 \xi \geq 2, \quad \operatorname{ord}_0 \eta^\bullet \geq 4.$$

Put $\boldsymbol{\alpha}' = (2,2,2,3)$. Again, in the affine chart $U_x := \{x \neq 0\}$ the map $\sigma_{\boldsymbol{\alpha}'}^{-1}$ is given by $x \mapsto x'^2$, $y \mapsto y'x'^2$, $z \mapsto z'x'^2$, $t \mapsto t'x'^3$, where $\sigma_{\boldsymbol{\alpha}'}^{-1}(U_x) \simeq \mathbb{C}^4/\boldsymbol{\mu}_2(1,0,0,1)$ and

$$E_{\boldsymbol{\alpha}'} \cap X_{\boldsymbol{\alpha}'} \cap \sigma_{\boldsymbol{\alpha}'}^{-1}(U_x) = \{x' = 0,\ y'^3 + t'^2 = 0\}.$$

Thus $\boldsymbol{\alpha}'$ is admissible and for some component G' of $X_{\boldsymbol{\alpha}'} \cap E_{\boldsymbol{\alpha}'}$ we have $a(G', K_X) = 2$, $v_{G'}(\psi) = 3$, $c \leq 2/3$, a contradiction.

3.8. Case $\eta = x^2 + \xi(y,z)$, where $\operatorname{ord}_0 \xi \geq 4$. By Morse Lemma we may assume that ζ does not depend on x. Write the linear part of ζ in the form $\zeta_1 = \delta_1 y + \delta_2 z + \delta_3 t$, $\delta_i \in \mathbb{C}$. Take $\boldsymbol{\alpha} = (2,1,1,3)$. In the affine chart $U_y := \{y \neq 0\}$ the map $\sigma_{\boldsymbol{\alpha}}^{-1}$ is given by $x \mapsto x'y'^2$, $y \mapsto y'$, $z \mapsto z'y'$, $t \mapsto t'y'^3$ and

$$E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}} \cap \sigma_{\boldsymbol{\alpha}}^{-1}(U_y) = \{y' = 0,\ x'^2 + \xi_4(1,z') + \delta_1 t' + \delta_2 t'z' = 0\}.$$

If either $\delta_1 \neq 0$ or $\delta_2 \neq 0$ or $\xi_4 \neq 0$, then $E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ is reduced (at least over U_y). Hence, $\boldsymbol{\alpha}$ is admissible and for some component G of $E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ we have $c \leq a(G, K_X)/v_G(\psi) = 2/3$, a contradiction. Thus $\delta_1 = \delta_2 = 0$ and $\xi_4 = 0$. Then we can write

$$\phi = x^2 + \xi(y,z) + \delta_3 t^2 + t\zeta^\bullet(y,z,t), \quad \operatorname{ord}_0 \xi \geq 5, \quad \operatorname{ord}_0 \zeta^\bullet \geq 2.$$

Take $\boldsymbol{\alpha}' = (2,1,1,2)$. In the affine chart $U_y := \{y \neq 0\}$ the map $\sigma_{\boldsymbol{\alpha}'}^{-1}$ is given by $x \mapsto x'y'^2$, $y \mapsto y'$, $z \mapsto z'y'$, $t \mapsto t'y'^2$ and

$$E_{\boldsymbol{\alpha}'} \cap X_{\boldsymbol{\alpha}'} \cap \sigma_{\boldsymbol{\alpha}'}^{-1}(U_y) = \{y' = 0,\ x'^2 + \delta_3 t'^2 + t'\zeta^\bullet_{(2)}(1,z',0) = 0\},$$

where $\zeta^\bullet_{(2)}(y,z,t) = \zeta^\bullet_{(2)}(y,z,0)$ is the degree 2 weighted homogeneous part of $\zeta^\bullet$. If $\delta_3 \neq 0$ or $\zeta^\bullet_{(2)} \neq 0$, as above, $\boldsymbol{\alpha}'$ is admissible and $c \leq 1/2$, a contradiction. Thus $\delta_3 = 0$, $\zeta^\bullet_{(2)} = 0$, and

$$\phi = x^2 + \xi(y,z) + \delta t^3 + t\zeta^\circ(y,z,t), \quad \delta \in \mathbb{C}, \quad \operatorname{ord}_0 \xi \geq 5, \quad \operatorname{ord}_0 \zeta^\circ \geq 3.$$

Applying the terminality condition [Rei87, Th. 4.6] with weight $(2,1,1,1)$ we get $\delta \neq 0$.

Take $\boldsymbol{\alpha}'' = (3,1,1,2)$. Again by the terminality condition $\xi_5 \neq 0$ or $\zeta^{\circ}_{(3)} \neq 0$, where $\zeta^{\circ}_{(3)}$ is the degree 3 weighted homogeneous part of ζ°. As above we get

$$E_{\boldsymbol{\alpha}''} \cap X_{\boldsymbol{\alpha}''} \cap \sigma_{\boldsymbol{\alpha}''}^{-1}(U_x) = \{x' = 0,\ \xi_5(y', z') + t'\zeta^{\circ}_{(3)}(y', z', t') = 0\}.$$

If either $\zeta^{\circ}_{(3)} \neq 0$ or ξ_5 has a factor of multiplicity 1, then $\boldsymbol{\alpha}''$ is admissible and $c \leq 1/2$, a contradiction.

Therefore, we may assume that $\zeta^{\circ}_{(3)} = 0$ and ξ_5 has only multiple factors. Up to linear coordinate change of y and z we can write $\xi_5 = y^5$ or $\xi_5 = y^2z^3$. Take $\boldsymbol{\alpha}''' = (3,2,1,2)$. Then $\boldsymbol{\alpha}'''$ is admissible and $c \leq 1/2$, a contradiction.

3.9. Case $\eta = x^2 + y^3 + \xi(y,z)$, where $\operatorname{ord}_0 \xi \geq 4$. As in [KM98, 4.25] we may assume that $\xi(y,z) = u_a y z^a + u_b z^b$. Since the singularity $(S \ni P)$ is not Du Val, we have $a \geq 4$, $b \geq 6$ and u_a, u_b are either units or zero. Write the linear part of ζ in the form $\zeta_1 = cz + \ell(x,y,t)$. Let $\xi_{(6)}$ is the degree 6 weighted homogeneous part of ξ with respect to $\operatorname{wt}(y,z) = (2,1)$. Clearly, $\xi_{(6)}$ is a linear combination of z^6 and yz^4. Take $\boldsymbol{\alpha} = (3,2,1,5)$. In the affine chart $U_z := \{z \neq 0\}$ the map $\sigma_{\boldsymbol{\alpha}}^{-1}$ is given by $x \mapsto x'z'^3$, $y \mapsto y'z'^2$, $z \mapsto z'$, $t \mapsto t'z'^5$ and

$$E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}} \cap \sigma_{\boldsymbol{\alpha}}^{-1}(U_z) = \{z' = 0,\ x'^2 + y'^3 + \xi_{(6)}(y', 1) + \delta t' = 0\},$$

where δ is a constant and $\xi_{(6)}(y',1)$ contains no y'^3. Hence $\boldsymbol{\alpha}$ is admissible, i.e., $E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ has a reduced non-toric component G. Then $a(G, K_X) = 4$, $v_G(\psi) = 5$, and $c \leq a(G,K_X)/v_G(\psi) \leq 4/5$, a contradiction. □

The following examples show that bounds $\operatorname{ct}(X,S) \leq 5/6$ and $\leq 4/5$ in Theorem 1.4 are sharp.

Example 3.10. Let $X = \mathbb{C}^3$ and let $S = S^d$ is given by $x^2 + y^3 + z^d$, $d \geq 6$. Then $\operatorname{ct}(\mathbb{C}^3, S^d) = 5/6$. We prove this by descending induction on $\lfloor d/6 \rfloor$. Take $\boldsymbol{\alpha} = (3,2,1)$ and consider the $\boldsymbol{\alpha}$-blowup $\sigma_{\boldsymbol{\alpha}} \colon \mathbb{C}^3_{\boldsymbol{\alpha}} \to \mathbb{C}^3$. Let $S_{\boldsymbol{\alpha}} \subset X_{\boldsymbol{\alpha}}$ be the proper transform of S. We have $a(E_{\boldsymbol{\alpha}}, K_X) = 5$ and $v_{\boldsymbol{\alpha}}(\psi) = 6$. Hence, $\operatorname{ct}(\mathbb{C}^3, S^d) \leq 5/6$. Further,

$$K_{\mathbb{C}^3_{\boldsymbol{\alpha}}} + \tfrac{5}{6} S_{\boldsymbol{\alpha}} = \sigma_{\boldsymbol{\alpha}}^*(K_{\mathbb{C}^3} + \tfrac{5}{6} S).$$

Thus it is sufficient to show that $\mathrm{ct}(X_{\boldsymbol{\alpha}}, \frac{5}{6}S_{\boldsymbol{\alpha}})$ is canonical. We have three affine charts:

- $U_x := \{x \neq 0\}$. Here $\sigma_{\boldsymbol{\alpha}}^{-1}\colon\ x \mapsto x'^3$, $y \mapsto y'x'^2$, $z \mapsto z'x'$, $S_{\boldsymbol{\alpha}}$ is given in $\sigma_{\boldsymbol{\alpha}}^{-1}(U_x) \simeq \mathbb{C}^3/\boldsymbol{\mu}_3(-1,2,1)$ by the equation $1 + y'^3 + z'^d x'^{d-6} = 0$. Hence, in this chart, $S_{\boldsymbol{\alpha}}$ is smooth and does not pass through a (unique) singular point of $\sigma_{\boldsymbol{\alpha}}^{-1}(U_x)$.

- $U_y := \{y \neq 0\}$. Here $\sigma_{\boldsymbol{\alpha}}^{-1}\colon\ x \mapsto x'y'^3$, $y \mapsto y'^2$, $z \mapsto z'y'$, $S_{\boldsymbol{\alpha}}$ is given in $\sigma_{\boldsymbol{\alpha}}^{-1}(U_y) \simeq \mathbb{C}^3/\boldsymbol{\mu}_2(3,-1,1)$ by the equation $x'^2 + 1 + z'^d y'^{d-6} = 0$. Again, in this chart, $S_{\boldsymbol{\alpha}}$ is smooth and does not pass through a (unique) singular point of $\sigma_{\boldsymbol{\alpha}}^{-1}(U_y)$.

- $U_z := \{z \neq 0\}$. Here $\sigma_{\boldsymbol{\alpha}}^{-1}\colon\ x \mapsto x'z'^3$, $y \mapsto y'z'^2$, $z \mapsto z'$, $S_{\boldsymbol{\alpha}}$ is given in $\sigma_{\boldsymbol{\alpha}}^{-1}(U_z) \simeq \mathbb{C}^3$ by the equation $x'^2 + y'^3 + z'^{d-6} = 0$. In this chart, $(X_{\boldsymbol{\alpha}}, S_{\boldsymbol{\alpha}}) \simeq (\mathbb{C}^3, S^{d-6})$.

Thus $X_{\boldsymbol{\alpha}}$ has only terminal singularities, $S_{\boldsymbol{\alpha}}$ does not pass through any singular point of $X_{\boldsymbol{\alpha}}$, and the pair $(X_{\boldsymbol{\alpha}}, S_{\boldsymbol{\alpha}})$ is terminal in charts U_x and U_y. In the chart U_z the pair by induction $(X_{\boldsymbol{\alpha}}, \frac{5}{6}S_{\boldsymbol{\alpha}})$ is canonical (moreover, $(X_{\boldsymbol{\alpha}}, S_{\boldsymbol{\alpha}})$ is canonical if $d \leq 11$). Therefore, $\mathrm{ct}(X, S) = 5/6$.

Example 3.11. Let $X \subset \mathbb{C}^4$ is given by $x^2 + y^3 + z^d + tz = 0$, $d \geq 7$ and let S cut out by $t = 0$. Take $\boldsymbol{\alpha} = (3,2,1,5)$ and consider the $\boldsymbol{\alpha}$-blowup $\sigma_{\boldsymbol{\alpha}}\colon X_{\boldsymbol{\alpha}} \to X$. Let $S_{\boldsymbol{\alpha}} \subset X_{\boldsymbol{\alpha}}$ be the proper transform of S. We see below that $\boldsymbol{\alpha}$ is admissible. Moreover, the exceptional divisor $G := E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ is reduced and irreducible. We have four charts:

- $U_x := \{x \neq 0\}$. Here $\sigma_{\boldsymbol{\alpha}}^{-1}\colon x \mapsto x^3$, $y \mapsto yx^2$, $z \mapsto zx$, $t \mapsto tx^5$, $X_{\boldsymbol{\alpha}}$ is given in $\sigma_{\boldsymbol{\alpha}}^{-1}(U_x) \simeq \mathbb{C}^4/\boldsymbol{\mu}_3(-1,2,1,5)$ by the equation $1 + y^3 + z^d x^{d-6} + tz = 0$ and $S_{\boldsymbol{\alpha}}$ by two equations $x = 1 + y^3 + tz = 0$. Hence, in this chart, both $X_{\boldsymbol{\alpha}}$ and $S_{\boldsymbol{\alpha}}$ are smooth.

- $U_y := \{y \neq 0\}$. Here $\sigma_{\boldsymbol{\alpha}}^{-1}\colon x \mapsto xy^3$, $y \mapsto y^2$, $z \mapsto zy$, $t \mapsto ty^5$, $\sigma_{\boldsymbol{\alpha}}^{-1}(U_y) \simeq \mathbb{C}^4/\boldsymbol{\mu}_2(3,-1,1,5)$, $X_{\boldsymbol{\alpha}} = \{x^2 + 1 + z^d y^{d-6} + tz = 0\}$, and $S_{\boldsymbol{\alpha}} = \{y = x^2 + 1 + tz = 0\}$. As above, both $X_{\boldsymbol{\alpha}}$ and $S_{\boldsymbol{\alpha}}$ are smooth in this chart.

- $U_z := \{z \neq 0\}$. Here $\sigma_{\boldsymbol{\alpha}}^{-1}\colon x \mapsto xz^3$, $y \mapsto yz^2$, $z \mapsto z$, $t \mapsto tz^5$, $\sigma_{\boldsymbol{\alpha}}^{-1}(U_z) \simeq \mathbb{C}^4$, $X_{\boldsymbol{\alpha}} = \{x^2 + y^3 + z^{d-6} + t = 0\}$, and $S_{\boldsymbol{\alpha}} = \{z = x^2 + y^3 + t = 0\}$. As above, both $X_{\boldsymbol{\alpha}}$ and $S_{\boldsymbol{\alpha}}$ are smooth in this chart.

- $U_t := \{t \neq 0\}$. Here $\sigma_{\boldsymbol{\alpha}}^{-1}\colon x \mapsto xt^3$, $y \mapsto yt^2$, $z \mapsto zt$, $t \mapsto t^5$, $\sigma_{\boldsymbol{\alpha}}^{-1}(U_t) \simeq \mathbb{C}^4/\boldsymbol{\mu}_5(3,2,1,-1)$, $X_{\boldsymbol{\alpha}} = \{x^2 + y^3 + z^d t^{d-6} + z = 0\}$, and $S_{\boldsymbol{\alpha}} = \{t = x^2 + y^3 + z = 0\}$. The variety $X_{\boldsymbol{\alpha}}$ has a unique singular point Q at the origin and this point is terminal of type $\frac{1}{5}(3,2,-1)$. In this case, $S_{\boldsymbol{\alpha}} \in |-K_{U_t}|$ and the pair $(U_t, S_{\boldsymbol{\alpha}})$ is canonical.

Thus we have $a(G, K_X) = 4$, $v_{\boldsymbol{\alpha}}(\psi) = 5$, and $a(G, K_X + \frac{4}{5}S) = 0$. Therefore,

$$K_{X_{\boldsymbol{\alpha}}} + \tfrac{4}{5}S_{\boldsymbol{\alpha}} = \sigma_{\boldsymbol{\alpha}}^*(K_X + \tfrac{4}{5}S).$$

Since the pair $K_{X_{\boldsymbol{\alpha}}} + \frac{4}{5}S_{\boldsymbol{\alpha}}$ is canonical, $\mathrm{ct}(X, S) = 4/5$.

4. Non-Gorenstein Case

Now we assume that $(X \ni P)$ is a (terminal) point of index $r > 1$. Let $\pi\colon (X^\sharp \ni P^\sharp) \to (X \ni P)$ be the index-one cover and let $S^\sharp := \pi^{-1}(S)$.

Lemma 4.1. *If $(X \ni P)$ is a cyclic quotient singularity, then $\mathrm{ct}(X, S) \leq 1/2$.*

Proof. By our assumption we have $X \simeq \mathbb{C}^3/\boldsymbol{\mu}_r(a, -a, 1)$ for some $r \geq 2$, $1 \leq a < r$, $\gcd(a, r) = 1$. Assume that $c = \mathrm{ct}(X, S) > 1/2$. Let $\psi = 0$ be a defining equation of $S^\sharp$. Consider the weighted blowup $\sigma_{\boldsymbol{\alpha}}\colon X_{\boldsymbol{\alpha}} \to X$ with weights $\boldsymbol{\alpha} = \frac{1}{r}(a, r-a, 1)$. Then $a(E_{\boldsymbol{\alpha}}, K_X) = 1/r$. Since $a(E_{\boldsymbol{\alpha}}, K_X) - c v_{\boldsymbol{\alpha}}(\psi) \geq 0$, we have $v_{\boldsymbol{\alpha}}(\psi) \leq a(E_{\boldsymbol{\alpha}}, K_X)/c < 2a(E_{\boldsymbol{\alpha}}, K_X) = 2/r$ and so $v_{\boldsymbol{\alpha}}(\psi) = 1/r$. Thus we may assume that ψ contains x_3 (if $a \equiv \pm 1$ we possibly have to permute coordinates). Then $S^\sharp \simeq \mathbb{C}^2$ is smooth and $S \simeq \mathbb{C}^2/\boldsymbol{\mu}_r(a, -a)$, i.e., S is Du Val of type A_{r-1}. □

Lemma 4.2. *If $(X \ni P)$ is a terminal singularity of index $r > 1$ and $\mathrm{ct}(X, S) > 1/2$, then $K_X + S \sim 0$.*

Proof. By Lemma 4.1 $(X \ni P)$ is not a cyclic quotient singularity. There is an analytic $\boldsymbol{\mu}_r$-equivariant embedding $(X^\sharp, P^\sharp) \subset (\mathbb{C}^4, 0)$. Let (x_1, x_2, x_3, x_4) be coordinates in $\mathbb{C}^4$, let $\phi = 0$ be an equation of $X^\sharp$, and let $\psi = 0$ be an equation of $S^\sharp$. We can take (x_1, x_2, x_3, x_4) and ϕ to be semi-invariants such that one of the following holds [Mor85] (see also [Rei87]):

- **Main series.** $\mathrm{wt}(x_1, x_2, x_3, x_4; \phi) \equiv (a, -a, 1, 0; 0) \mod r$, where $\gcd(a, r) = 1$.

- **Case** $cAx/4$**.** $r = 4$, $\mathrm{wt}(x_1, x_2, x_3, x_4; \phi) \equiv (1, 3, 1, 2; 2) \mod 4$.

In both cases $\mathrm{wt}(x_1x_2x_3x_4) - \mathrm{wt}\,\phi \equiv \mathrm{wt}\,x_3 \mod r$. According to [Kaw92] there is a weight $\boldsymbol{\alpha}$ such that for the corresponding $\boldsymbol{\alpha}$-blowup $\sigma_{\boldsymbol{\alpha}}\colon X_{\boldsymbol{\alpha}} \subset W \to X \subset \mathbb{C}^4/\boldsymbol{\mu}_r$ the exceptional divisor $E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ has a reduced component G of discrepancy $a(G, K_X) = 1/r$. Moreover, $r\alpha_i \equiv \mathrm{wt}\,x_i \mod r$, $i = 1, 2, 3, 4$. Since $c > 1/2$, we have $1/r - cv_{\boldsymbol{\alpha}}(\psi) \geq 0$, i.e., $rv_{\boldsymbol{\alpha}}(\psi) < 2$, so $rv_{\boldsymbol{\alpha}}(\psi) = 1$. In particular, $\mathrm{wt}\,\psi \equiv 1 \mod r$.

Let ω be a section of $\mathcal{O}_X(-K_X)$. Then ω can be written as

$$\omega = \lambda(\partial\phi/\partial x_4)(dx_1 \wedge dx_2 \wedge dx_3)^{-1},$$

where λ is a semi-invariant function with

$$\mathrm{wt}\,\lambda - \mathrm{wt}(x_1x_2x_3x_4) + \mathrm{wt}\,\phi \equiv \mathrm{wt}\,\omega \equiv 0 \mod r.$$

Thus, $\mathrm{wt}\,\psi \equiv \mathrm{wt}\,\lambda \mod r$. Hence, $S \sim -K_X$. $\square$

LEMMA 4.3. *If $(X \ni P)$ is a terminal singularity of index $r > 1$, then $c \leq 4/5$.*

PROOF. Since π is étale in codimension one, we have $K_{X^\sharp} + cS^\sharp = \pi^*(K_X + cS)$. Hence the pair $(X^\sharp, cS^\sharp)$ is canonical (see, e.g., [Kol97, 3.16.1]). Assume that $c > 4/5$. By Lemma 4.1 the point $(X^\sharp \ni P^\sharp)$ is singular. Then by Lemma 3.5 the pair $(X^\sharp, S^\sharp)$ is canonical. Therefore, $(S^\sharp \ni P^\sharp)$ is a Du Val singularity. Then the singularity $(S \ni P) = (S^\sharp \ni P^\sharp)/\boldsymbol{\mu}_r$ is log terminal. On the other hand, by Lemma 4.2 the divisor K_S is Cartier. Hence, $(S \ni P)$ is Du Val, a contradiction. $\square$

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