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Higher Direct Images of Local Systems in Log Betti Cohomology

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Abstract. We prove that the higher direct images of a locally constant sheaf by the associated continuous map of a proper separated log smooth morphism of fs log analytic spaces are locally constant.

Contents

Introduction

In this paper we prove the following theorem.

Theorem 0.1. *Let $f\colon X \longrightarrow Y$ be a proper separated log smooth morphism of fs log analytic spaces and F a locally constant sheaf of* $\mathbf{Z}$*-modules on* $X^{\log}$*. Then* $\mathrm{R}^q f^{\log}_* F$ *is locally constant for any* $q \geq 0$.

This is a generalization to log geometry in the sense of Fontaine-Illusie of the following well-known, classical result:

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THEOREM 0.1.1. *Let $f\colon X \longrightarrow Y$ be a proper separated smooth morphism of analytic spaces and F a locally constant sheaf of $\mathbf{Z}$-modules on X. Then $\mathrm{R}^q f_* F$ is locally constant for any $q \geq 0$.*

Sampei Usui proved the above theorem 0.1 in the case where f is a multi-generalized semistable family over a polydisk ([U1], [U2]).

As corollaries to 0.1, we also have the following.

COROLLARY 0.2. *In* 0.1, *if F is a local system on $X^{\log}$, that is, a locally constant sheaf of finite dimensional $\mathbf{C}$-vector spaces on $X^{\log}$, then $\mathrm{R}^q f_*^{\log} F$ is also a local system for any $q \geq 0$.*

COROLLARY 0.3. *In* 0.1, *$\mathrm{R}^q f_*^{\log}\mathbf{Z}$ is a locally constant sheaf of finitely generated abelian groups for any $q \geq 0$.*

See [KN], [IKN] for basic concepts on log geometry in the analytic situation in detail, including the definition of the topological space $X^{\log}$ for an fs log analytic space X. For readers' convenience, we briefly review the construction of $X^{\log}$. As a set,

$$X^{\log} = \{(x,h) | x \in X, h \in \mathrm{Hom}\,(M^{\mathrm{gp}}_{X,x}, \mathbf{S}^1), \quad h(f) = \frac{f(x)}{|f(x)|} \text{ for any } f \in \mathcal{O}^{\times}_{X,x}\}.$$

For example, if $X = \mathrm{Spec}\,(\mathbf{C}[P])_{\mathrm{an}}$ for an fs monoid P, endowed with the canonical log structure, we have $X^{\log} = \mathrm{Hom}\,(P, \mathbf{R}^{\mathrm{mult}}_{\geq 0} \times \mathbf{S}^1)$. Here $\mathbf{R}^{\mathrm{mult}}_{\geq 0}$ is $\{r \in \mathbf{R} \,|\, r \geq 0\}$ regarded as a monoid with multiplication, and $\mathbf{S}^1 = \{z \in \mathbf{C} \,|\, |z| = 1\}$. In this case the topology of $X^{\log}$ is the one induced by $\mathbf{R}^{\mathrm{mult}}_{\geq 0}$ and $\mathbf{S}^1$. In general case, any fs log analytic space X is locally isomorphic to a (locally) closed analytic subspace of such a space. The topology of $X^{\log}$ is the glued one.

These corollaries are important in the log Hodge theory ([IKN], [KMN]). For instance, our 0.2 plays an essential role for showing the functoriality of log Riemann-Hilbert correspondence with an arbitrary base, which is one of the main results (6.2) of [IKN].

REMARK 0.4. An l-adic analogue of 0.2 (proper log smooth base change theorem) was already proved by Kazuya Kato (unpublished), which is: Let $f\colon X \longrightarrow Y$ be a proper log smooth morphism of fs log schemes. Let $n \geq 1$ be an integer invertible on Y, and F a locally constant and constructible sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on $X^{\log}_{\text{ét}}$ (Kummer log étale site; see [N]). Then $\mathrm{R}^q f_* F$ is also locally constant and constructible for any $q \geq 0$.

This also suggested our main theorems.

In Section 1, we explain a key idea which will be used in Section 4. Section 2 is a preliminary from algebraic topology. In Sections 3–5, we prove main theorems. Section 3 treats the case where the base has a constant log structure. This case is reduced to a theorem on the topological local triviality. Section 4 treats the case where the base is ideally log smooth. This case is reduced to the previous case by the key idea explained in Section 1 together with some facts on convex bodies which are proved in Appendix A. In Section 5, we reduce the general case to the ideally log smooth base case by resolutions of singularities. In Appendixes B and C, we discuss some related problems.

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Convention: A *topological manifold* (resp. a *topological manifold with boundary*) is a topological space which is locally isomorphic to an open subspace of the Euclidean space (resp. the half Euclidean space). We do not assume that it is Hausdorff. We do not assume that it is paracompact, either.

A topological space is *locally connected* if each point admits a fundamental system of connected neighborhoods. A topological space is *compact* if any open covering has a finite subcovering. Here we do not assume that it is Hausdorff. A topological space is *locally compact* (resp. *locally Hausdorff*) if each point has at least one compact (resp. Hausdorff) neighborhood.

A *ring* (resp. a *monoid*) means a commutative ring (resp. monoid) having a unit element. A homomorphism of monoids (resp. rings) is required to preserve the unit elements. For a monoid P, the monoid algebra over $\mathbf{C}$ is denoted by $\mathbf{C}P$, the set of all prime ideals of P (cf. [K2] (5.1)) is denoted by Spec P, and the tensor product $P \otimes_{\mathbf{N}} \mathbf{R}_{\geq 0}$ (cf. 3.3.2) by $P_{\mathbf{R}_{\geq 0}}$. For a polyhedral cone or a polyhedron P, $\tau < P$ means that τ is a face of P.

For readers' convenience, we briefly review here some terms in log geometry: a homomorphism $h\colon P \longrightarrow Q$ of integral monoids is said to be *exact* (resp. *integral*, resp. *dominating*) if $P = (h^{\mathrm{gp}})^{-1}(Q)$ in P^{gp} (resp. if for any homomorphism $P \longrightarrow P'$ of integral monoids, the push out of $P' \longleftarrow P \longrightarrow Q$ in the category of monoids is integral, resp. if for any element $q \in Q$, there are elements $p \in P$ and $q' \in Q$ such that $qq' = h(p)$). For a morphism $f\colon X \longrightarrow Y$ of fs log analytic spaces and a point x of X, we say that f is *strict* (resp. *exact*, resp. *integral*, resp. *vertical*) *at* x if $(M_Y/\mathcal{O}_Y^\times)_{f(x)} \longrightarrow (M_X/\mathcal{O}_X^\times)_x$ is an isomorphism (resp. exact, resp. integral, resp. dominating). For an fs log analytic space X, the open subspace $\{x \in X \mid M_{X,x} = \mathcal{O}_{X,x}^\times\}$ of X consisting of the points where the log structure of X is trivial is denoted by X_{triv}.

1. A Test Case (Log Smooth Case)

To explain a key idea of the proof of 0.1, we prove the following test case in this section. (This will not be used in the sequel.)

THEOREM IN CASE 1.1. *Let $f\colon X \longrightarrow Y$ be as in Theorem* 0.1. *Assume moreover that Y is log smooth over $\mathbf{C}$ and that f is vertical. Then the conclusion of* 0.1 *holds for f.*

LEMMA 1.2. *(Cf.* [KN] *Remark* (1.5.1)*.) Let X be an fs log analytic space which is log smooth over $\mathbf{C}$. Then $X^{\log}$ is a topological manifold with the boundary $X - X_{\mathrm{triv}}$.*

PROOF. Taking a chart, we may assume that $X = (\operatorname{Spec} \mathbf{C}P)_{\mathrm{an}}$ with P a sharp fs monoid ("sharp" means it has no invertible element other than the unit element). Then, by Proposition A.1, there is a homeomorphism from $X^{\log} = \operatorname{Hom}(P, \mathbf{R}_{\geq 0}^{\mathrm{mult}}) \times \operatorname{Hom}(P, \mathbf{S}^1)$ onto $P_{\mathbf{R}_{\geq 0}} \times \operatorname{Hom}(P, \mathbf{S}^1)$ which maps $X_{\mathrm{triv}} = \operatorname{Hom}(P, \mathbf{R}_{>0}) \times \operatorname{Hom}(P, \mathbf{S}^1)$ onto $\operatorname{Int}(P_{\mathbf{R}_{\geq 0}}) \times \operatorname{Hom}(P, \mathbf{S}^1)$. Since the polyhedral cone $P_{\mathbf{R}_{\geq 0}}$ is a topological manifold with boundary $P_{\mathbf{R}_{\geq 0}} - \operatorname{Int}(P_{\mathbf{R}_{\geq 0}})$, this implies the required fact. □

REMARK 1.3. As a corollary, we have that for a locally constant sheaf F of $\mathbf{Z}$-modules on $X^{\log}$, $F \longrightarrow \mathrm{R}j_*j^{-1}F$ is an isomorphism. Here j is the canonical inclusion map $X_{\rm triv} \longrightarrow X^{\log}$. This was proved by Arthur Ogus [Og] by a different method. See the paragraph next to Claim 4.2.

1.4. Proof of 1.1. By the verticality, $X_{\rm triv} = f^{-1}(Y_{\rm triv})$. Hence the commutative diagram of topological spaces

$$\begin{array}{ccc} X^{\log} & \xleftarrow{\ j\ } & X_{\rm triv} \\ {\scriptstyle f^{\log}}\downarrow & & \downarrow{\scriptstyle f_{\rm triv}} \\ Y^{\log} & \xleftarrow[\ j\ ]{} & Y_{\rm triv} \end{array}$$

is cartesian and $f_{\rm triv}$ is proper. Since Y is log smooth over $\mathbf{C}$, X is also log smooth. Then, by Lemma 1.2, $F \xrightarrow{\cong} \mathrm{R}j_*j^{-1}F$. Hence $\mathrm{R}f^{\log}_*F \xrightarrow{\cong} \mathrm{R}f^{\log}_*\mathrm{R}j_*j^{-1}F = \mathrm{R}j_*\mathrm{R}f_{{\rm triv}*}j^{-1}F$ by the commutativity of the diagram. By Theorem 0.1.1, $\mathrm{R}^qf_{{\rm triv}*}j^{-1}F$ is locally constant for any q. Then, again by Lemma 1.2, $\mathrm{R}^qf^{\log}_*F = j_*\mathrm{R}^qf_{{\rm triv}*}j^{-1}F$ which is locally constant. □

Remark 1.5. (This is a remark due to one of the referees.) The proof in 1.4 shows that the assumption in 1.1 that f is log smooth can be weakened into the assumption that $f_{\rm triv}$ is smooth and that X is log smooth over $\mathbf{C}$.

2. Sheaves on Topological Spaces

In this section we review some basic facts on sheaves on topological spaces.

First we review the proper base change theorem for topological spaces. Recall that a map of topological spaces is called *proper* if it is universally closed, and called *separated* if the associated diagonal map is closed ([Gr] Remarque 2.8).

Proposition 2.1. *(Cf.* [V]*) Let*

$$\begin{array}{ccc} X' & \xrightarrow{\ f'\ } & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[\ f\ ]{} & Y \end{array}$$

be a cartesian diagram of topological spaces with f proper and separated. Then, for any sheaf F of $\mathbf{Z}$*-modules on X and for any $q \in \mathbf{Z}$, the natural homomorphism $g^{-1}\mathrm{R}^q f_* F \longrightarrow \mathrm{R}^q f'_* g'^{-1} F$ is an isomorphism.*

This is well-known (cf. [SGA4$\frac{1}{2}$] p. 39), and this proposition reduces to the lemma 2.1.1 below as [Go] II, Théorème 4.11.1 reduces to loc. cit. Corollaire 1 to II, Théorème 3.3.1. See [KU] Appendix for another, detailed explanation of 2.1.

LEMMA 2.1.1. *Let X be a topological space, K a compact subset of X, and F a sheaf on X. Assume that any two points of K can be separated by open sets of X. Then we have*

$$\varinjlim_{U} \Gamma(U, F) \xrightarrow{\cong} \Gamma(K, F|_K),$$

where U ranges over all open neighborhoods of K.

PROOF. The injectivity is clear. To prove the surjectivity, let s be an element of $\Gamma(K, F|_K)$. Take open subsets $U_1, \dots, U_n$ of X which cover K, and sections $s_i \in F(U_i)$ $(i = 1, \dots, n)$ such that $s_i|_{U_i \cap K} = s|_{U_i \cap K}$. It is enough to show that we can take $n = 1$. We may assume $n = 2$. Let U_{12} be the open subset of $U_1 \cap U_2$ where s_1 and s_2 coincide. Take open subsets U'_1, U'_2 such that $U'_i \supset K - U_{12}$ $(i = 1, 2)$ and such that $U'_1 \cap U'_2 = \varnothing$. Shrinking them if necessary, we may assume that $U_i \supset U'_i$ $(i = 1, 2)$. Then $s_i|_{U'_i}$ $(i = 1, 2)$ and $s_1|_{U_{12}} = s_2|_{U_{12}}$ glue into the desired section. □

Note that this lemma also implies that, in the notation there, the pull-back of any flasque sheaf ([Go] II, 3.1) on X to K is soft ([KS] p.132).

In the rest of section, we discuss local constantness of a sheaf.

LEMMA 2.2. *Let X be a Hausdorff topological space, K_1 and K_2 compact subsets of X, and F a sheaf on X. Then the sequence*

$$F(K_1 \cup K_2) \to F(K_1) \times F(K_2) \rightrightarrows F(K_1 \cap K_2)$$

is exact.

PROOF. By Lemma 2.1.1, this lemma reduces to the following lemma 2.2.1. □

LEMMA 2.2.1. *Let X, K_1, K_2 be as in Lemma 2.2. For any open subsets U_1, U_2, U'_{12} of X such that $U_i \supset K_i$ $(i = 1, 2)$ and such that $U_1 \cap U_2 \supset U'_{12} \supset K_1 \cap K_2$, there exist open subsets U'_1, U'_2 such that $U_i \supset U'_i \supset K_i$ $(i = 1, 2)$ and such that $U'_1 \cap U'_2 \subset U'_{12}$.*

PROOF. Take open subsets U''_1, U''_2 such that $U''_i \supset K_i - U'_{12}$ $(i = 1, 2)$ and such that $U''_1 \cap U''_2 = \varnothing$. Shrinking them if necessary, we may assume that $U_i \supset U''_i$ $(i = 1, 2)$. Then $U'_i := U''_i \cup U'_{12}$ $(i = 1, 2)$ satisfy the desired condition. □

PROPOSITION 2.3. *Let $f\colon X \longrightarrow Y$ be a proper separated surjective map of topological spaces. Assume that X is locally compact and locally connected, that Y is locally Hausdorff, and that each fiber of f is connected and locally connected. Let F be a sheaf on Y. If $f^{-1}F$ is locally constant, then F is locally constant.*

PROOF. Since $F \longrightarrow f_*f^{-1}F$ is an isomorphism by a set-valued variant of Proposition 2.1, it is enough to show the following: for each fiber K, the sheaf $G := f^{-1}F$ is constant on a neighborhood of K. To prove this, fix an isomorphism $j\colon G|_K \xrightarrow{\cong} M_K$, where M is the local value of G. By shrinking Y if necessary, we may and do assume that Y is Hausdorff. Then X is Hausdorff. By the local connectedness of X and K, we can take a triple (U_x, K_x, j_x) for each $x \in K$, where U_x (resp. K_x) is a compact neighborhood of x in X (resp. in K) such that $K_x \subset \mathrm{Int}\,(U_x)$, and j_x is an isomorphism $G|_{U_x} \xrightarrow{\cong} M_{U_x}$ such that $j_x|_{K_x}$ coincides with $j|_{K_x}$. By the compactness of K, a finite number of $\mathrm{Int}\,(K_x)$ cover K. It is enough to show that we can decrease this number. Let (U_i, K_i, j_i) $(i = 1, 2)$ be a pair of such triples as above. We show that there is another triple of the form $(U, K_1 \cup K_2, j)$. Since j_1 and j_2 coincide on $K_1 \cap K_2$, they coincide on a connected neighborhood of each point of $K_1 \cap K_2$, and hence they coincide also near $K_1 \cap K_2$. By Lemma 2.2.1, shrinking U_i $(i = 1, 2)$ if necessary, we may assume that j_1 and j_2 coincide on $U_1 \cap U_2$. The two triples glue by Lemma 2.2. □

PROPOSITION 2.4. *Let X be a Hausdorff space, X_1 and X_2 closed subsets of X such that $X = X_1 \cup X_2$. Assume that X_1 is locally compact.*

Assume further that for each $x \in X_1 \cap X_2$ and each neighborhood U of x in X_2, there is a compact connected subneighborhood $K_2 \subset U$ such that $X_1 \cap K_2$ is connected. Let F be a sheaf on X. If $F|_{X_i}$ is locally constant for each $i = 1, 2$, then F is locally constant.

PROOF. We show that F is locally constant around $x \in X_1 \cap X_2$. Take a compact neighborhood K_1 of x in X_1 such that $F|_{K_1}$ is constant. Fix an isomorphism $F|_{K_1} \cong M_{K_1}$, where M is the local value of F. Since $(K_1 \cap X_2) \cup (X_2 - X_1)$ is a neighborhood of x in X_2, by the assumption, there is a compact connected neighborhood K_2 of x in X_2 such that $X_1 \cap K_2$ is connected and contained in K_1 and such that $F|_{K_2}$ is also constant. Since $K_1 \cap K_2 = X_1 \cap K_2$ and K_2 are connected, the canonical map $F(K_2) \longrightarrow F(K_1 \cap K_2)$ is an isomorphism. Hence we have a homomorphism $M \longrightarrow F(K_1) \times F(K_2)$ by composing $M \longrightarrow F(K_1) \longrightarrow F(K_1 \cap K_2)$ and $F(K_1 \cap K_2) \xleftarrow{\cong} F(K_2)$. By Lemma 2.2, this induces $M_{K_1 \cup K_2} \xrightarrow{\cong} F|_{K_1 \cup K_2}$. $\square$

REMARK 2.5. When we are concerned only with locally constant sheaves having local values with some finiteness structure, Propositions 2.3 and 2.4 could be combined. For instance, the following is easy to see: let A be a ring, $f\colon X \longrightarrow Y$ a closed surjective map of topological spaces, F a sheaf of A-modules on Y. Assume that $f^{-1}F$ is locally constant and with local values of finite presentation. Then F is also locally constant. Note that this does not hold without the assumption of finite presentedness because in general for a locally constant sheaf of modules, the constantness on a compact set K does not necessarily implies that around K (for example, consider the product space $X := L \times \mathbf{S}^1$, where L is the subspace $\{0\} \cup \{\frac{1}{n} \mid n \geq 1\}$ of $\mathbf{R}$, and the locally constant sheaf over X whose local value is the free module with a countable base $e_1, e_2, \dots$, whose monodromy around the circle $K := \{0\} \times \mathbf{S}^1$ is the identity, and, for each $n \geq 1$, whose monodromy around the circle $\{\frac{1}{n}\} \times \mathbf{S}^1$ sends e_i to e_i (resp. $-e_i$) if $i \leq n$ (resp. $i > n$)).

PROPOSITION 2.6. *Let $f\colon X \longrightarrow Y$ be a proper separated and locally trivial map of topological spaces. Assume that X is locally connected. Let F be a locally constant sheaf of $\mathbf{Z}$-modules on X. Then $\mathrm{R}^q f_* F$ is locally constant for any $q \in \mathbf{Z}$.*

Proof. Let $y \in Y$ and we work around y. We may assume that $X = Y \times K$, where $K = f^{-1}(y)$. Consider the map $p\colon X = Y \times K \xrightarrow{\text{proj}} \{y\} \times K \hookrightarrow X$. By Proposition 2.1, we have $\mathrm{R}^q f_* p^{-1} F = p_Y^{-1} \mathrm{R}^q f_* F$, where p_Y is the composite $Y \to \{y\} \hookrightarrow Y$, so that it is enough to show that $p^{-1}F$ is isomorphic to F. To construct such an isomorphism, we take a triple $(Y_x \times U_x, K_x, j_x)$ for each $x \in K$, where K_x is a compact neighborhood of x, U_x (resp. Y_x) is an open neighborhood of K_x (resp. y) such that $F|_{Y_x \times U_x}$ is constant, and $j_x\colon F(Y_x \times U_x) \longrightarrow (p^{-1}F)|_{Y_x \times U_x}$ is an isomorphism such that $j_x|_{\{y\} \times U_x}$ is the natural one. The rest is similar to the proof of Proposition 2.3 and omitted. We use Lemma 2.2.1 to glue such triples. □

Remark 2.6.1. There is a remark for Proposition 2.6 which is similar to Remark 2.5. In fact, when F is a sheaf of A-modules (A is a ring), we can replace the assumption of the local connectedness of X by the finite generatedness of the local values of F without changing the conclusion of Proposition 2.6. But in general the direct image of a locally constant sheaf of modules by a proper separated product map is not necessarily locally constant (for example, consider the first projection $X \to L$, where X and L are in the example in the end of 2.5).

Note that if we only have to prove 0.1 under some finiteness assumption on F like the one in Remark 2.5 or 2.6.1, the use of 2.3, 2.4, and 2.6 in the following sections could be replaced with that of easier ones mentioned in Remarks 2.5 and 2.6.1.

3. Constant Log Case

In this section we prove the main theorem 0.1 under the assumption that Y has a constant log structure. More precisely, we prove the following.

Theorem in Case 3.1. *Let $f\colon X \longrightarrow Y$ be as in Theorem* 0.1. *Assume that Y* has a constant log structure, *that is, that there is a chart $P \longrightarrow \Gamma(Y, M_Y)$ of Y with P fs such that the induced homomorphism $P_Y \longrightarrow M_Y/\mathcal{O}_Y^{\times}$ is an isomorphism. Assume further that f is exact. Then the conclusion of* 0.1 *holds for f.*

3.2. Actually we show the local triviality of $f^{\log}$ for such an f. (Cf. Appendix B.) For this, we start with some lemmas.

LEMMA 3.2.1. *Let $f\colon X \longrightarrow Y$ be a morphism of fs log analytic spaces. Then $f^{\log}$ is proper (resp. separated) if f is proper (resp. separated).*

PROOF. First note that, locally on Y, $Y^{\log}$ is locally compact Hausdorff.

In the following, we show the proper case. The other case is proved similarly.

Consider the commutative diagram

$$\begin{array}{ccc} X^{\log} & \longrightarrow & Y^{\log} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

of topological spaces. By the assumption, the bottom arrow is proper. Since the vertical maps are also proper, the top arrow $f^{\log}$ is proper. □

LEMMA 3.2.2. *Let $f\colon X \longrightarrow Y$ be a log smooth morphism of fs log analytic spaces. Assume that Y has a constant log structure* (3.1). *Then $f^{\log}$ is submersive.*

Recall that a map $f\colon X \longrightarrow Y$ of topological spaces is called *submersive* (T_0-submersive in [S] Definition 6.8) if locally on X and on Y, f factors as the composite $X \overset{i}{\hookrightarrow} Y \times Z \overset{\mathrm{pr}_1}{\longrightarrow} Y$ of an open immersion i and the first projection pr_1, where Z is a topological space.

PROOF. Take a constant chart $Q \longrightarrow \Gamma(Y, M_Y)$. Let $f'\colon X' \longrightarrow Y'$ be the base change of f by $\mathrm{Spec}\,(\mathbf{C}Q/\langle Q - \{1\}\rangle)_{\mathrm{an}} \longrightarrow \mathrm{Spec}\,(\mathbf{C}Q)_{\mathrm{an}}$. Since the strict closed immersion $Y' \longrightarrow Y$ is surjective by the assumption, it is, in particular, a strict homeomorphism. Hence $X' \longrightarrow X$ is also a strict homeomorphism so that we can replace f by f' and we may assume that there is a strict morphism $Y \longrightarrow \mathrm{Spec}\,(\mathbf{C}Q/\langle Q - \{1\}\rangle)_{\mathrm{an}}$. By the log smoothness, we may assume further that there is a chart $Q \hookrightarrow P$ of f with P fs such that $X \longrightarrow Y \times_{\mathrm{Spec}\,(\mathbf{C}Q)} \mathrm{Spec}\,(\mathbf{C}P)$ is a strict open immersion (cf. [K1] Theorem (3.5)), and it is enough to show that the map $\mathrm{Spec}\,(\mathbf{C}P/\langle Q - \{1\}\rangle)^{\log}_{\mathrm{an}} \longrightarrow \mathrm{Spec}\,(\mathbf{C}Q/\langle Q - \{1\}\rangle)^{\log}_{\mathrm{an}}$ is submersive. But this map is the product of the map $\{h\colon P \longrightarrow \mathbf{R}^{\mathrm{mult}}_{\geq 0} \mid h(Q - \{1\}) = \{0\}\} \overset{a}{\longrightarrow} \{h\colon Q \longrightarrow \mathbf{R}^{\mathrm{mult}}_{\geq 0} \mid h(Q - \{1\}) = \{0\}\}$ and the map $\mathrm{Hom}\,(Q \hookrightarrow P, \mathbf{S}^1)$. The last map is clearly submersive. Since the target of a is the trivial space, the product map is submersive. □

3.3. Next we review "**Q**-integral homomorphisms" briefly for the proof of Proposition 3.4 below.

DEFINITION 3.3.1. A local homomorphism $h\colon P \longrightarrow Q$ of sharp fs monoids ("local" means that the inverse image of the maximal ideal is the maximal ideal) is called **Q**-*integral* if one of the following equivalent conditions is satisfied.
(1) $h \otimes_{\mathbf{N}} \mathbf{Q}_{\geq 0}$ is integral.
(2) $\operatorname{Spec} \mathbf{Z}[h]$ is exact.
(3) h satisfies the condition GD (it means "going-down"), that is, if for any $\mathfrak{q} \in \operatorname{Spec} Q$ and $\mathfrak{p} \in \operatorname{Spec} P$ such that $\mathfrak{p} \subset h^{-1}(\mathfrak{q})$, there exists $\mathfrak{q}' \in \operatorname{Spec} Q$ such that $h^{-1}(\mathfrak{q}') = \mathfrak{p}$ and $\mathfrak{q}' \subset \mathfrak{q}$.

For the above equivalence, see [IKN] Appendix, Proposition (A.3.2).

REMARK 3.3.2. For a monoid P and M, $P \otimes_{\mathbf{N}} M$ means the tensor product of monoids over the semiring $(\mathbf{N}, +, \times)$. This is defined by the universality as in the definition of the tensor product of modules over the ring $\mathbf{Z}$. In this paper, only the cases $M = \mathbf{Q}_{\geq 0}$ and $M = \mathbf{R}_{\geq 0}$ are necessary. Note that it is easy to see that the tensor product $P \otimes_{\mathbf{N}} \mathbf{Q}_{\geq 0}$ equals to $(\mathbf{Q}_{\geq 0} \times P)/\sim$, where $\sim$ denotes the equivalence relation defined as that $(q_1, p_1) \sim (q_2, p_2)$, $q_1, q_2 \in \mathbf{Q}_{\geq 0}$, $p_1, p_2 \in P$, if and only if there exists an integer $n \geq 1$ such that $nq_1, nq_2 \in \mathbf{N}$ and $p_1^{nq_1} = p_2^{nq_2}$ in P. Further, it can be proved that, for a fine P, the tensor product $P \otimes_{\mathbf{N}} \mathbf{R}_{\geq 0}$ equals to the convex hull of P in $P^{\mathrm{gp}}_{\mathbf{R}}$. Note also that $\operatorname{Spec} P = \operatorname{Spec}(P \otimes_{\mathbf{N}} \mathbf{Q}_{\geq 0}) = \operatorname{Spec}(P \otimes_{\mathbf{N}} \mathbf{R}_{\geq 0})$.

PROPOSITION 3.3.3. *Let $f\colon X \longrightarrow Y$ be an exact log smooth morphism of fs log analytic spaces. Let x be a point of X and $y = f(x)$. Then the induced homomorphism $h\colon (M_Y/\mathcal{O}_Y^{\times})_y \longrightarrow (M_X/\mathcal{O}_X^{\times})_x$ is* **Q**-*integral.*

PROOF. This can be proved by the equivalence of (1) and (2) in Definition 3.3.1. Log smoothness and exactness imply that $\operatorname{Spec} \mathbf{C}[h]$ is exact. Cf. [IKN] Appendix, Proposition (A.3.3). □

DEFINITION 3.3.4. For a morphism $f\colon X \longrightarrow Y$ of fs log analytic spaces and a point x of X, we say f is *vertical at* x if $(M_Y/\mathcal{O}_Y^{\times})_{f(x)} \longrightarrow (M_X/\mathcal{O}_X^{\times})_x$ is dominating ([N] (7.3), see also Convention). Note that a homomorphism $h\colon P \longrightarrow Q$ of fs monoids is dominating if and only if $1 \overset{\mathrm{int}}{\oplus}_P Q = Qh(P^{\mathrm{gp}})/h(P^{\mathrm{gp}})$ is a group. (Here $1 \overset{\mathrm{int}}{\oplus}_P Q = \varinjlim(\{1\} \longleftarrow P \longrightarrow Q)$ in the category of integral monoids.)

Now we prove the following.

PROPOSITION 3.4. *Let $f\colon X \longrightarrow Y$ be an exact log smooth morphism of fs log analytic spaces. Then each fiber of $f^{\log}$ is a topological manifold with boundary consisting of the points lying over the points of X where f is not vertical.*

PROOF. Let x be a point of $X^{\log}$ and y its image in $Y^{\log}$. We prove that $f^{\log\,-1}(y)$ is a topological manifold with boundary near x and that x is an interior of $f^{\log\,-1}(y)$ if and only if f is vertical at $\tau(x)$. By [IKN] Lemma (5.3), locally on X and on Y, there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ j\ } & X' = \mathrm{Spec}\,(\mathbf{C}Q)_{\mathrm{an}} \times \mathbf{C}^A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{\ i\ } & Y' = \mathrm{Spec}\,(\mathbf{C}P)_{\mathrm{an}} \times \mathbf{C}^{A'} \end{array}$$

of fs log analytic spaces, where i is a strict immersion, P and Q are sharp fs monoids, A a finite set and A' a subset of A, and g is the natural morphism induced by a local homomorphism $h\colon P \longrightarrow Q$ such that the induced morphism $X \longrightarrow X' \times_{Y'} Y$ is a strict open immersion and such that $j(\tau(x))$ (resp. $i(\tau(y))$) is the origin of X' (resp. Y'). Thus the problem is reduced to the one for the fiber of $\mathrm{Hom}\,(h, \mathbf{R}_{\geq 0}^{\mathrm{mult}})$ at the origin. By Proposition 3.3.3, h is $\mathbf{Q}$-integral. Take a homeomorphism from $\mathrm{Hom}\,(Q, \mathbf{R}_{\geq 0}^{\mathrm{mult}})$ to $Q_{\mathbf{R}_{\geq 0}}$ as in Proposition A.1, where $(-)_{\mathbf{R}_{\geq 0}}$ denotes $(-) \otimes_{\mathbf{N}} \mathbf{R}_{\geq 0}$ (cf. 3.3.2). Then the concerned fiber $\{\varphi\colon Q \longrightarrow \mathbf{R}_{\geq 0}^{\mathrm{mult}} \,|\, (\varphi \circ h)^{-1}(\{0\}) = P - \{1\}\} = \bigcup_{\sigma < Q_{\mathbf{R}_{\geq 0}}, h_{\mathbf{R}_{\geq 0}}^{-1}(\sigma) = \{1\}} \{\varphi\colon Q \longrightarrow \mathbf{R}_{\geq 0}^{\mathrm{mult}} \,|\, \varphi^{-1}(\{0\}) = Q - \sigma\} \subset \mathrm{Hom}\,(Q, \mathbf{R}_{\geq 0}^{\mathrm{mult}})$ is mapped onto the subspace $U := \bigcup_{\sigma < Q_{\mathbf{R}_{\geq 0}}, h_{\mathbf{R}_{\geq 0}}^{-1}(\sigma) = \{1\}} \mathrm{Int}\,(\sigma) = \bigcup_{\sigma < Q_{\mathbf{R}_{\geq 0}}, h_{\mathbf{R}_{\geq 0}}^{-1}(\sigma) = \{1\}} \sigma$ of $Q_{\mathbf{R}_{\geq 0}}$, where $\sigma < Q_{\mathbf{R}_{\geq 0}}$ means that

σ is a face of $Q_{\mathbf{R}_{\geq 0}}$ (cf. Convention). Hence it is enough to show that U is a topological manifold with boundary and that the origin 0 of U is an interior if and only if h is dominating. Consider the continuous map $\iota\colon U \hookrightarrow Q_{\mathbf{R}_{\geq 0}} \longrightarrow QP^{\mathrm{gp}}_{\mathbf{R}}/P^{\mathrm{gp}}_{\mathbf{R}}$. Since $h_{\mathbf{R}_{\geq 0}}$ satisfies GD, we can apply to it the $\mathbf{R}_{\geq 0}$-version of [IKN] Appendix, Proposition (A.3.2.2). (The proof of it is obtained by replacing $\mathbf{Q}_{\geq 0}$ there with $\mathbf{R}_{\geq 0}$.) Then we see that for any $b \in Q_{\mathbf{R}_{\geq 0}}$, there exists a unique element $b' \in Q_{\mathbf{R}_{\geq 0}}$ such that $P^{\mathrm{gp}}_{\mathbf{R}} b \cap Q_{\mathbf{R}_{\geq 0}} = P_{\mathbf{R}_{\geq 0}} b'$ and that the image of the map $Q_{\mathbf{R}_{\geq 0}} \longrightarrow Q_{\mathbf{R}_{\geq 0}}\,;\ b \mapsto b'$ is the union of the faces σ of $Q_{\mathbf{R}_{\geq 0}}$ such that $\sigma \cap P_{\mathbf{R}_{\geq 0}} = \{1\}$. This means that ι is bijective. On the other hand, for any $\sigma < Q_{\mathbf{R}_{\geq 0}}$, the restriction $\iota|_{\sigma}$ is an injective affine map so that ι is a closed map. Thus ι is a homeomorphism. It is easy to see that 0 of U maps to an interior by this map if and only if $QP^{\mathrm{gp}}_{\mathbf{R}} = Q^{\mathrm{gp}}_{\mathbf{R}}$, which is equivalent to that h is dominating (cf. 3.3.4). $\square$

The next is a special case of [S] Corollary 6.14 together with loc. cit. Theorem 2.3. See also [CK].

Proposition 3.5. *Let $f\colon X \longrightarrow Y$ be a proper separated submersive map of topological spaces. Assume that each fiber of f is a topological manifold with boundary. Then f is locally trivial.*

Remark 3.5.1. There is a trivial remark on [S] Corollary 6.14. We need to impose the separatedness of p in loc. cit. 6.9, 6.15, and 6.14. In fact, $X =$ the real line and $E =$ the real line with the origin doubled do not satisfy the conclusion of 6.14. The same for loc. cit. Union lemma 6.9 and Corollary 6.15, from which 6.14 is deduced. In the proof in loc. cit. p. 151 of 6.9, the set of double points of $h|_{W_1 \times N}$ is not necessarily closed.

Remark 3.5.2. In fact, L. C. Siebenmann [S] proved far more generalized results. For example, “a topological manifold with boundary” in 3.5 can be replaced by “a WCS set.”

Lemma 3.6. *Let X be an fs log analytic space. Then $X^{\log}$ is locally connected.*

Proof. We prove here the case where X has a constant log structure, which is all that is needed in this section. The general case will be proved later in 5.4. In the constant log case, by the argument in the proof of Lemma 3.2.2, we may assume that there is a strict morphism $X \longrightarrow X_0 := \mathrm{Spec}\,(\mathbf{C}Q/\langle Q-\{1\}\rangle)_{\mathrm{an}}$. Then $X^{\log} = X \times_{X_0} X_0^{\log}$ as topological spaces. Here the analytic space X is locally connected, X_0 is trivial, and $X_0^{\log}$ is a product of finite number of copies of $\mathbf{S}^1$. Hence $X^{\log}$ is locally connected. □

3.7. We prove Case 3.1. By Lemma 3.2.1 and Lemma 3.2.2, $f^{\log}$ is proper, separated and submersive. By Proposition 3.4, each fiber of $f^{\log}$ is a topological manifold with boundary. Hence $f^{\log}$ is locally trivial by Proposition 3.5. This implies that $\mathrm{R}^q f_*^{\log}$ preserves the local constancy by Proposition 2.6 together with Lemma 3.6. This completes the proof of Theorem in Case 3.1.

4. Ideally Log Smooth Case

In this section we prove the main theorem 0.1 under the assumption that Y is ideally log smooth. More precisely, we prove the following. (We do not explain the term "ideally log smooth." See [Og].)

THEOREM IN CASE 4.1. *Let Y be a polydisk Δ^n endowed with the log structure associated to $\mathbf{N}^{m+r} \longrightarrow \Gamma(Y, \mathcal{O}_Y)$; $e_j \mapsto z_j$ $(1 \le j \le m)$, $e_j \mapsto 0$ $(j > m)$, where $(e_j)_j$ is the canonical base of $\mathbf{N}^{m+r}$, $(z_j)_j$ are coordinate functions on Δ^n, and m is an integer such that $0 \le m \le n$. Let $f\colon X \longrightarrow Y$ be as in Theorem 0.1. Assume that f is exact. Then the conclusion of 0.1 holds for f.*

In the rest of this section we prove this case. Let $Y_0 \hookrightarrow Y$ be the strict open immersion from the biggest open subspace that has a constant log structure by $\mathbf{N}^r$, that is, the open subspace defined by $z_1 \ne 0, \ldots, z_m \ne 0$. Let $X_0 = f^{-1}(Y_0)$. Then we have a cartesian diagram of topological spaces

$$\begin{array}{ccc} X^{\log} & \xleftarrow{\ j\ } & X_0^{\log} \\ {\scriptstyle f^{\log}}\downarrow & & \downarrow{\scriptstyle f_0^{\log}} \\ Y^{\log} & \xleftarrow[\ j\ ]{} & Y_0^{\log} \end{array}$$

and $\mathrm{R}f_*^{\log}\mathrm{R}j_*j^{-1}F = \mathrm{R}j_*\mathrm{R}f_{0*}^{\log}j^{-1}F$. We use the same argument in Section 1. Since Y_0 has a constant log structure, Case 3.1 implies that $\mathcal{H}^q$ of the right hand side of the above last equality is a locally constant sheaf $j_*\mathrm{R}^q f_{0*}^{\log} j^{-1}F$. Thus the problem is reduced to the following.

Claim 4.2. $j\colon X_0^{\log} \hookrightarrow X^{\log}$ has the following property: Let U be an open set of $X^{\log}$ and F a locally constant sheaf of $\mathbf{Z}$-modules on U. Let j_U denotes the open immersion $X_0^{\log} \cap U \hookrightarrow U$. Then $F \longrightarrow \mathrm{R}j_{U*}j_U^{-1}F$ is an isomorphism.

We call this property of open immersions of topological spaces (A) for short. As remarked in 1.3, Arthur Ogus showed that the open immersion $X_{\mathrm{triv}} \hookrightarrow X^{\log}$ has (A) for any log smooth fs log analytic space X ([Og] Theorem 5.12 and Corollary 5.14).

Proposition 4.2.1. (1) *Let $U \hookrightarrow X$ be an open immersion having* (A). *Let X' be an open subspace of X. Then $U \cap X' \hookrightarrow X'$ also has* (A).
(2) *Let $U \overset{j_0}{\hookrightarrow} X$ be an open immersion having* (A). *Let Z be a topological manifold. Then $U \times Z \overset{j}{\hookrightarrow} X \times Z$ also has* (A).
(3) *For any locally contractible space Z, $\mathbf{R}_{>0} \times Z \overset{j}{\hookrightarrow} \mathbf{R}_{\geq 0} \times Z$ has* (A).

Proof. (1) is clear.

For (2), we may assume that $Z = \mathbf{R}^n$, $n \geq 1$. Further we may assume that $Z = \mathbf{R}$ and it is enough to show that for any locally constant F on $X \times Z$, the homomorphism $F \longrightarrow \mathrm{R}j_*j^{-1}F$ is an isomorphism. By [KS] Proposition 2.7.8, F is the inverse image of a locally constant F_0 on X, and the smooth base change theorem for topological spaces (which means that the higher direct images commute with the pull-backs by the projection $X \times \mathbf{R} \longrightarrow X$) reduces the problem to that $F_0 \longrightarrow \mathrm{R}j_{0*}j_0^{-1}F_0$ is an isomorphism.

(3) We may assume that Z is contractible. It is enough to show that for any constant F on $\mathbf{R}_{\geq 0} \times Z$, the homomorphism $F \longrightarrow \mathrm{R}j_*j^{-1}F$ is an isomorphism. This is proved similarly as (2). □

To prove Claim 4.2, take a point x of $X^{\log}$, and let y be its image in $Y^{\log}$. It is enough to show (A) near x. We may assume that $\tau(y)$ is the origin of $\overset{\circ}{Y}$. Consider the same diagram as in the proof of Proposition 3.4. We use the same notation there. Then, $P = \mathbf{N}^{m+r}$. By 4.2.1 (1), it is enough to show that $W_0^{\log} \hookrightarrow W^{\log}$ has (A), where W is the strict closed subspace of X' defined by the ideal of Q generated by the image of $\{e_{m+1}, \dots, e_{m+r}\}$, and $W_0 = \{x \in W \mid z_1 \neq 0, \cdots, z_m \neq 0\}$. By 4.2.1 (2), we may assume further that the set A in that diagram is empty and can replace $(\operatorname{Spec} \mathbf{C}Q)^{\log}_{\mathrm{an}}$ with $\operatorname{Hom}(Q, \mathbf{R}^{\mathrm{mult}}_{\geq 0})$. Thus Claim 4.2 reduces to the next claim (by taking $P = \mathbf{N}^{m+r}$, $\mathfrak{p} = (e_1, \dots, e_r)$, and $e = e_{r+1} \cdots e_{r+m}$).

Claim 4.3. Let $h\colon P \hookrightarrow Q$ be a local $\mathbf{Q}$-integral homomorphism of sharp fs monoids. Let $f := \operatorname{Hom}(h, \mathbf{R}^{\mathrm{mult}}_{\geq 0})\colon \operatorname{Hom}(Q, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \longrightarrow \operatorname{Hom}(P, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) =: V$. Let $\mathfrak{p}$ be a prime ideal of P, $e \in P - \mathfrak{p}$, and U and U_0 the subspaces of V defined by $\{g\colon P \longrightarrow \mathbf{R}^{\mathrm{mult}}_{\geq 0} \mid g(\mathfrak{p}) = 0\}$ and $\{g \mid g(\mathfrak{p}) = 0, g(e) \neq 0\}$ respectively. Then $j\colon f^{-1}(U_0) \hookrightarrow f^{-1}(U)$ has the property (A).

By Proposition A.1, j is homeomorphic to $X_0 \hookrightarrow X$ of subspaces of $Q_{\mathbf{R}_{\geq 0}}$, where X and X_0 are $\bigcup_{\sigma < Q,\, \sigma \cap \mathfrak{p} = \varnothing} \sigma_{\mathbf{R}_{\geq 0}}$ and $\bigcup_{\sigma < Q,\, \sigma \cap \mathfrak{p} = \varnothing,\, e \in \sigma} \operatorname{Int}(\sigma_{\mathbf{R}_{\geq 0}})$ respectively. Here we regard P as a subset of Q for simplicity and $\sigma < Q$ means that σ is a face of Q. Hence we reduce the problem to the next claim.

Claim 4.4. Let the notation be as in 4.3. Let $X := \bigcup_{\sigma < Q,\, \sigma \cap \mathfrak{p} = \varnothing} \sigma_{\mathbf{R}_{\geq 0}}$ and $Z := \bigcup_{\tau < Q,\, e \notin \tau} \tau_{\mathbf{R}_{\geq 0}}$. Then $X - Z \hookrightarrow X$ has (A).

A key step is to reduce this to the next.

Claim 4.5. Let Q be a sharp fs monoid, $e \in Q$, $\sigma_1, \dots, \sigma_k$ faces of Q containing e, $X := \bigcup (\sigma_i)_{\mathbf{R}_{\geq 0}}$ and $Z := \bigcup_{e \notin \tau < Q} \tau_{\mathbf{R}_{\geq 0}}$. Then $X - Z \hookrightarrow X$ has (A).

This implies the previous one as follows. In the situation of Claim 4.4, let $\sigma_1, \dots, \sigma_k$ be the faces that do not intersect with $\mathfrak{p}$ but contain e. Then we have $\bigcup_{\sigma < Q,\, \sigma \cap \mathfrak{p} = \varnothing} \sigma_{\mathbf{R}_{\geq 0}} = \bigcup (\sigma_i)_{\mathbf{R}_{\geq 0}}$ by $\mathbf{Q}$-integrality as follows. Let σ be a face that does not intersect with $\mathfrak{p}$. Then $h^{-1}(\sigma) = \sigma \cap P \subset P - \mathfrak{p}$. Since h satisfies the condition GD, there is a face σ' such that $\sigma' \supset \sigma$ and $h^{-1}(\sigma') = P - \mathfrak{p}$, the latter of which means that σ' is one of the σ_i. Thus Claim 4.4 reduces to Claim 4.5.

Since $Q_{\mathbf{R}_{\geq 0}}$ is the union of the convex polytopes $Q_{\mathbf{R}_{\geq 0}} \cap H^+$ obtained by cutting $Q_{\mathbf{R}_{\geq 0}}$ with hyperplanes H not passing through the origin (H^+

is the half space including the origin and the boundary H), we see that the rest is to show the following.

CLAIM 4.6. Let C be a convex polytope, $e \in C$, $\sigma_1, \dots, \sigma_k$ faces containing e, $X = \bigcup \sigma_i$, and $Z := \bigcup_{e \notin \tau < C} \tau$. Then $X - Z \hookrightarrow X$ has (A).

PROOF. The case where e is an interior is well-known (and unnecessary for the argument in this section). If e is a vertex of C, by Proposition A.2, there is a homeomorphism $i \colon C \longrightarrow C'$ of convex polytopes such that C' is a cone over one of its face Z' with the vertex $i(e)$, such that $i(Z) = Z'$, and such that each $i(\sigma_j)$ is a cone over a face of Z'. By 4.2.1 (3), the conclusion follows.

If $e \in \partial C$ is not a vertex, by Proposition A.3, there is a homeomorphism $i \colon C \longrightarrow C'$ of convex polytopes such that $i(e)$ is a vertex of C', such that the image of a face of C (resp. a face of C containing e) is the union of faces of C' (resp. faces of C' containing $i(e)$), and such that the image of Z coincides with the union of the faces that do not contain $i(e)$. Then this case is reduced to the case where e is a vertex. □

5. General Case

In this section we prove the main theorem 0.1 and its corollaries.

First we prove a log version of the proper base change theorem.

PROPOSITION 5.1. *Let*

$$\begin{array}{ccc} X_4 & \xrightarrow{p'} & X_3 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X_2 & \xrightarrow[p]{} & X_1 \end{array}$$

be a cartesian diagram D of fs log analytic spaces.

(1) *Assume that the following condition is satisfied:*

(∗) *For any two points $x_2 \in X_2$ and $x_3 \in X_3$ lying over the same point x_1 of X_1, if P_i denotes $(M_{X_i}/\mathcal{O}_{X_i}^{\times})_{x_i}$ $(1 \leqq i \leqq 3)$, the inverse image of $P_2 \oplus P_3$ under*

$$P_1^{\mathrm{gp}} \longrightarrow P_2^{\mathrm{gp}} \oplus P_3^{\mathrm{gp}}\,;\ a \mapsto (a, a^{-1})$$

is $\{1\}$.

Then the diagram $D^{\log}$ *of topological spaces is also cartesian.*
(2) *Assume further that* p *is proper and separated. Then, for any sheaf* F *of* $\mathbf{Z}$*-modules on* $X_2^{\log}$ *and for any* $q \in \mathbf{Z}$*, the natural homomorphism* $f^{\log -1}\mathrm{R}p_*^{\log}F \longrightarrow \mathrm{R}p'^{\log}_* f'^{\log -1}F$ *is an isomorphism.*

REMARK 5.1.1. The condition $(*)$ above is satisfied if either p or f is exact. In particular, $(*)$ is satisfied if M_{X_1} is trivial. See [KN] Lemma (1.3) (3) for another special case of 5.1 (1) where p or f is strict.

REMARK 5.1.2. In $(*)$, "the inverse image of ... is $\{1\}$" can be restated as that $P_1 \overset{a\mapsto(a,a)}{\longrightarrow} P_2 \oplus P_3$ is injective and there exists a point of X_4 that maps to x_2 and also to x_3. The proof of this is essentially explained in [N] Remark (5.1.2). More remarks on this condition $(*)$ are in the subsection [N] (5.1), which includes the l-adic analogue of 5.1 (2).

PROOF. Since (2) is reduced to 2.1 by (1) and 3.2.1, we prove (1).

Since $X_4 \longrightarrow X_2 \times X_3$ is finite, $X_4^{\log} \longrightarrow (X_2 \times X_3)^{\log}$ is closed by 3.2.1. Further, it is easily checked that $(X_2 \times X_3)^{\log} \longrightarrow X_2^{\log} \times X_3^{\log}$ is closed. Hence $X_4^{\log} \longrightarrow X_2^{\log} \times_{X_1^{\log}} X_3^{\log}$ is closed so that it is enough to show that $D^{\log}$ is cartesian as a diagram of sets. Hence we may assume that $\overset{\circ}{X}_i = \operatorname{Spec}\mathbf{C}$ $(1 \le i \le 3)$. By [N] Lemma (2.2.3), if we write $P_i := \Gamma(X_i, M_{X_i}/\mathcal{O}_{X_i}^{\times})$ $(1 \le i \le 3)$, $X_2 \longrightarrow X_1 \longleftarrow X_3$ is isomorphic to $(\operatorname{Spec}\mathbf{C}, \mathbf{C}^{\times} \oplus (P_2 \longleftarrow P_1 \longrightarrow P_3))$. Thus $X_4^{\log} = S \times \operatorname{Hom}(P_4, \mathbf{S}^1)$, where P_4 is the pushout of $P_2 \longleftarrow P_1 \longrightarrow P_3$ in the category of fs monoids and S is the set of homomorphisms $P_4 \longrightarrow \mathbf{R}_{\ge 0}^{\mathrm{mult}}$ that send the images of $P_2 - \{1\}$ and $P_3 - \{1\}$ to $\{0\}$. By the assumption $(*)$, S is not empty (cf. Remark 5.1.2) so that S is a singleton. On the other hand, $\operatorname{Hom}(P_4, \mathbf{S}^1) = \operatorname{Hom}(P_4^{\mathrm{gp}}, \mathbf{S}^1) = \varprojlim \operatorname{Hom}(P_2^{\mathrm{gp}} \longleftarrow P_1^{\mathrm{gp}} \longrightarrow P_3^{\mathrm{gp}}, \mathbf{S}^1)$. Hence $X_4^{\log} = \varprojlim(X_2^{\log} \longrightarrow X_1^{\log} \longleftarrow X_3^{\log})$. $\square$

Using the above, we prove the following case.

THEOREM IN CASE 5.2. *Let* $(z_j)_j$ *be the coordinate functions of* Δ^n*, and* $g_1, \dots, g_r$ *are monomials of* $z_1, \dots, z_n$ *or* 0*. Let* Y *be the union of closed subspaces* $\{z_j = 0\}$ *of* Δ^n *for some* j*'s, endowed with the log structure* *associated to* $\mathbf{N}^r \longrightarrow \Gamma(Y, \mathcal{O}_Y)$; $e_i \mapsto g_i$ $(1 \le i \le r)$. *Let* $f\colon X \longrightarrow Y$ *be as in Theorem* 0.1 *and assume that* f *is exact. Then the conclusion of Theorem* 0.1 *holds.*

Proof. First we reduce this to the case where each g_i is either one of z_j's or 0: If $\deg g_r > 1$, decompose $g_r = g'_r g'_{r+1}$, where g'_r and g'_{r+1} are monomials whose degrees $\geqq 1$, consider $Y' := \overset{\circ}{Y}$ endowed with the log structure associated to $\mathbf{N}^{r+1} \longrightarrow \Gamma(Y, \mathcal{O}_Y)$; $e_i \mapsto g_i$ $(1 \le i < r)$, $e_r \mapsto g'_r$, $e_{r+1} \mapsto g'_{r+1}$, and consider the morphism $Y' \longrightarrow Y$ defined by $\mathrm{id}_{\mathbf{N}^{r-1}} \oplus$ diagonal: $\mathbf{N}^r \longrightarrow \mathbf{N}^{r+1}; (a_1, \dots, a_{r-1}, a_r) \mapsto (a_1, \dots, a_{r-1}, a_r, a_r)$. If we can apply Proposition 2.3 to $Y'^{\log} \longrightarrow Y^{\log}$, by Proposition 5.1 (2) with Remark 5.1.1, the problem is reduced to the case of Y'. We check the assumption of Proposition 2.3 for $Y'^{\log} \longrightarrow Y^{\log}$. By 3.2.1, $Y'^{\log} \longrightarrow Y^{\log}$ is proper and separated, and each fiber of it is $\mathbf{S}^1$ or the one point. The rest is the local connectedness of $Y'^{\log}$, a part of Lemma 3.6 which has not yet proved. What is to be seen here is that $Y^{\log}$ is locally connected for such a Y as in Case 5.2:

Claim. Let Y be as in 5.2. Then $Y^{\log}$ is locally connected.

Proof. This claim reduces to the case where each g_i is some z_j or 0 by the above method, because for a morphism $Y' \to Y$ as above, if $Y'^{\log}$ is locally connected, then $Y^{\log}$ is also locally connected. Then we use the inductions on $\dim Y$ and the number of the irreducible components of Y. First, in the case where $\dim Y = 0$, $Y^{\log}$ is the product of copies of $\mathbf{S}^1$. Next let $\overset{\circ}{Y}_2$ be an irreducible component of $\overset{\circ}{Y}$. Let Y'_2 be the biggest open subspace of $(\overset{\circ}{Y}_2, M_Y|_{\overset{\circ}{Y}_2})$ that has a constant log structure, $Y''_2 := \overset{\circ}{Y}_2$ endowed with the log structure given by the divisor $\overset{\circ}{Y}_2 - Y'_2$, $Y'''_2 := \overset{\circ}{Y}_2$ endowed with the constant log structure that coincides with M_Y on Y'_2, $Y_2 := Y''_2 \times_{\overset{\circ}{Y}_2} Y'''_2$, and $Y_1 := Y - Y'_2$. Then we see that $Y_1^{\log} \cup Y_2^{\log}$ is a closed covering of $Y^{\log}$ as follows. Since $Y_1 \longrightarrow Y$ is a strict closed immersion, $Y_1^{\log} \longrightarrow Y^{\log}$ is a closed immersion. Since $Y_2 \longrightarrow Y$ is proper, $Y_2^{\log} \longrightarrow Y^{\log}$ is closed by 3.2.1. It is easy to see that it is also injective. Finally, $Y^{\log} = Y_1^{\log} \amalg Y_2'^{\log}$ as sets so that $Y_1^{\log}$ and $Y_2^{\log}$ can cover $Y^{\log}$. By the inductive hypothesis applied to Y_1, we may assume that $Y = Y_2$. But

$Y_2^{\log}$ is the product of copies of the interval $[0,1)$, copies of $\mathbf{S}^1$, and copies of Δ. Thus it is locally connected, and the claim is proved. □

We return to the proof of 5.2. As mentioned before, by the claim, we can apply Proposition 2.3 and we may and will assume that each g_i is some z_j or 0. To complete the proof of 5.2, we use the same induction as in the proof of the claim. We use the same notation as in the proof of the claim. When $\dim Y = 0$, the log structure of Y is constant so that we can use Case 3.1. When $Y = Y_2$, we use Case 4.1. Thus the rest is to see the assumption of Proposition 2.4 for $Y^{\log} = Y_1^{\log} \cup Y_2^{\log}$ to apply Proposition 2.4. But, since $Y_2^{\log}$ is a topological manifold with the boundary $Y_2^{\log} - Y_2'^{\log} = Y_2^{\log} - Y_1^{\log}$, the assumption there is certainly satisfied. □

Next we prove the invariance of cohomology under log blowing-ups.

Proposition 5.3. *Let Y be an fs log analytic space, and let $p\colon X \longrightarrow Y$ be a blowing-up along log structure. Then*
(1) *For any sheaf F of $\mathbf{Z}$-modules on $Y^{\log}$, $F \longrightarrow \mathrm{R}p_*^{\log} p^{\log\,-1} F$ is an isomorphism.*
(2) *For any locally constant sheaf F of $\mathbf{Z}$-modules on $X^{\log}$, $p^{\log\,-1}\mathrm{R}p_*^{\log} F \longrightarrow F$ is an isomorphism and $p_*^{\log} F$ is locally constant.*

Remark 5.3.1. The l-adic analogue of Proposition 5.3 (1) was proved by Kazuhiro Fujiwara and Kazuya Kato ([FK], see also [I]).

Since 5.3 is local on the base, it is reduced to the next variant.

Proposition 5.3.2. *Let $p\colon X \longrightarrow Y$ be the monoidal transformation of fs log analytic spaces associated to a proper subdivision Σ' of a fan Σ satisfying* ($\mathrm{S}_{\mathrm{fan}}$) *in the sense of* [K2] *Section* 9*. Then the same conclusions in the previous proposition hold.*

Remark 5.3.3. The assumption in 5.3.2 means that $X = Y \times_\Sigma \Sigma'$ for a "strict morphism" $Y \longrightarrow \Sigma$; we are working with the analytic variant of [K2] Section 9. More precisely, let $\mathcal{C}_1$ be the category of fs log analytic spaces and $\mathcal{C}_2$ the category of fans satisfying ($\mathrm{S}_{\mathrm{fan}}$). Let $\mathcal{C}$ be the unique category satisfying $\mathrm{Ob}\,\mathcal{C} = \mathrm{Ob}\,\mathcal{C}_1 \amalg \mathrm{Ob}\,\mathcal{C}_2$, $\mathcal{C}_i \longrightarrow \mathcal{C}$ is fully faithful ($i = 1,2$), and for any $X \in \mathcal{C}_1$, $F \in \mathcal{C}_2$, $\mathrm{Hom}_{\mathcal{C}}(X,F) = \mathrm{Hom}\,((X, M_X/\mathcal{O}_X^\times), F)$ and

$\mathrm{Hom}\,(F, X) = \varnothing$. For X, F as above, $X \longrightarrow F$ is called strict if it is so as a morphism of monoidal spaces. As in Proposition (9.9) in [K2], we see the following: Let $Y \xrightarrow{a} F \longleftarrow F'$ be a diagram in $\mathcal{C}$, $Y \in \mathcal{C}_1$, $F, F' \in \mathcal{C}_2$, such that a is strict and F' is a subdivision of F. Then there is a fiber product $X := Y \times_F F'$ in $\mathcal{C}$, $X \in \mathcal{C}_1$, and $X \longrightarrow F'$ is strict. We call such an X the monoidal transformation associated to F'. A blowing-up along log structure is, locally on the base, the monoidal transformation associated to a proper subdivision. (In fact, (9.9) in [K2] should be modified slightly. But the statement is easily corrected by imposing the integrality. Since, in the above, we work under the fs condition, this does not concern our statements.)

We prove 5.3.2. We may assume that Σ is associated to a fan in the classical sense (that is denoted by the same symbol) and that $\dim \Sigma = \dim N_{\mathbf{R}}$ ($N_{\mathbf{R}}$ is the notation in [Od]). We first remark that, for the monoidal transformation $p\colon X \longrightarrow Y$ (resp. $q\colon W \longrightarrow X$) associated to a proper subdivision Σ' of Σ (resp. Σ'' of Σ'), if Proposition 5.3.2 holds for q and pq, then so does it for p. Likewise, if 5.3.2 holds for p and q, then so does it for pq. Since the hyperplane arrangement defined by all facets of Σ' induces a proper subdivision of Σ, we may assume that Σ' is obtained from Σ by "adding" one hyperplane. By 3.2.1, $X^{\log} \longrightarrow Y^{\log}$ is proper and separated. Further, its fibers are contractible by the following lemma so that we have 5.3 (1) by the proper base change theorem, Proposition 2.1.

Lemma 5.3.4. *Let Σ be an affine fan and Σ' a proper subdivision obtained from Σ by adding one hyperplane. Then each fiber of* $\mathrm{Mc}\,(\Sigma') \longrightarrow \mathrm{Mc}\,(\Sigma)$ *is a point or homeomorphic to the closed interval* $[0, 1]$. *Here* Mc *means the manifold with corners (cf.* [Od] p. 13*).*

Proof. Let x be a point of $\mathrm{Mc}\,(\Sigma)$. We show that the fiber X of x is a point or homeomorphic to the closed interval $[0, 1]$. First we may assume that $\{x\}$ is a closed orbit by the $\mathbf{R}_{\geq 0}$-action, that is, $\{x\}$ is associated to a cone C of $\dim N_{\mathbf{R}}$. Then X is the union of the orbits that are associated to the cones which have nonempty intersection with the interior of C. By the assumption the cardinality of this set of cones is 1 (when the hyperplane does not cut C) or 3 (otherwise). In the first case X is a point. In the second case there is a unique 1-dimensional orbit in X and the other two are contained in its closure. Since the closure of an orbit of a toric variety is again a toric variety (cf. [F] p. 51), and since a proper 1-dimensional toric variety is $\mathbf{P}^1(\mathbf{C})$, the fiber X is homeomorphic to the manifold with corners associated to the toric variety $\mathbf{P}^1(\mathbf{C})$, i.e., the closed interval $[0,1]$. □

Thus we have 5.3 (1). In particular, for a blowing-up f along log structure, $f^{\log}$ has connected fibers. Further, we have 5.3 (2) also by this lemma if we can apply 2.3. To see that we can apply 2.3, it is enough to show that $X^{\log}$ is locally connected, that is, to prove the general case of 3.6, which we will see.

5.4. Proof of 3.6. We already proved a special case of 3.6 as a claim in the proof of 5.2. We will reduce the general case to this special case. By [KKMS] I, Theorem 11, locally on X, there is a blowing-up along log structure $f\colon X' \longrightarrow X$ such that X' admits local charts by free monoids. Since $f^{\log}$ has connected fibers as noted above, we may assume that X is a closed subspace of Δ^n endowed with the log structure associated to $\mathbf{N}^m \longrightarrow \Gamma(X, \mathcal{O}_X)$; $e_j \mapsto z_j$ $(1 \leq j \leq m \leq n)$, where z_j's are the coordinate functions of Δ^n. By Hironaka's resolution of singularities ([H] Main Theorem II$'$), shrinking Δ^n if necessary, we may assume that there is a composite of strict blowing-ups with smooth centers $p\colon M \longrightarrow \Delta^n$ with M being a complex manifold such that $p^{-1}(X)$ locally satisfies the condition in Case 5.2. By the claim in the proof of 5.2, $p^{-1}(X)^{\log}$ is locally connected. Since each fiber of $p^{-1}(X)^{\log} \longrightarrow X^{\log}$ is a connected analytic space, $X^{\log}$ is also locally connected. This completes the proof of 3.6. Hence the proof of 5.3 is also completed.

5.5. Now we prove Theorem 0.1. We use a similar reduction as in the previous subsection 5.4. By the exactification lemma ([IKN]), locally on Y, there is a blowing-up along log structure $Y' \longrightarrow Y$ such that $X \times_Y Y' \longrightarrow Y'$ is exact. Hence we may assume that f is exact by Proposition 5.3. Further, by [KKMS] I, Theorem 11 as in 5.4 and again by Proposition 5.3, we may assume further that Y is a closed subspace of Δ^n endowed with the log structure associated to $\mathbf{N}^m \longrightarrow \Gamma(Y, \mathcal{O}_Y)$; $e_j \mapsto z_j$ $(1 \leq j \leq m \leq n)$, as in 5.4, where z_j's are the coordinate functions of Δ^n. Using [H] as in 5.4, we reduce the problem to Case 5.2 by Proposition 2.3.

5.6. We prove Corollaries 0.2 and 0.3. As in the previous subsection

5.5, we may assume that f is exact by the exactification and 5.3. Then 0.2 and 0.3 are deduced from Theorem 0.1 since all the fibers of f are compact topological manifolds with the boundaries by Proposition 3.4.

Appendix A. Convex Bodies

In this appendix, we collect the facts on topologies of convex bodies which are used in the text. See, for example, [E] for the terminology on convex bodies.

PROPOSITION A.1. *Let Q be a sharp fs monoid. Then there is a homeomorphism from* $\mathrm{Hom}\,(Q, \mathbf{R}^{\mathrm{mult}}_{\geq 0})$ *to* $Q_{\mathbf{R}_{\geq 0}}$ *(see Convention or* 3.4 *for the notation) such that for each $\tau < Q$ the subspace* $\mathrm{Hom}\,(\tau, \mathbf{R}_{>0}) := \{h\colon Q \longrightarrow \mathbf{R}^{\mathrm{mult}}_{\geq 0} \mid h^{-1}(\{0\}) = Q - \tau\}$ *of* $\mathrm{Hom}\,(Q, \mathbf{R}^{\mathrm{mult}}_{\geq 0})$ *is mapped onto* $\mathrm{Int}\,(\tau_{\mathbf{R}_{\geq 0}})$.

PROOF. Taking a supporting hyperplane $H \subset Q^{\mathrm{gp}}_{\mathbf{Q}}$ of $\{1\}$ for Q and cut Q by a hyperplane H' which is parallel to H such that the vertices of $H' \cap Q_{\mathbf{R}_{\geq 0}}$ belong to the lattice Q^{gp}. We denote by $P \subset Q^{\mathrm{gp}}_{\mathbf{R}}$ the convex hull of the cut part of Q which includes the origin. Let $X = X(\Delta_P)$ be the toric variety associated to P ([F]). Then, by Proposition in p. 81 of [F] with its proof, a moment map defines a homeomorphism μ from $X_{\geq}$ onto P which induces $(O_{\tau'})_{\geq} \xrightarrow{\cong} \mathrm{Int}\,(\tau)$ for each face τ of P (cf. [F] 4.1 for the symbol $(\ \ )_{\geq}$), where τ' is the cone of the fan Δ_P corresponding to τ. On the other hand, there is a natural one-to-one correspondence between the set of faces of Q and the set of faces of P which is not contained in H'. For $\tau < Q$, we denote by τ_P the corresponding face of P. Restricting μ on the open subspace $(U_{\{1\}'_P})_{\geq} = \mathrm{Hom}\,(Q, \mathbf{R}_{\geq 0})$ of $X_{\geq}$, we have a homeomorphism $\mathrm{Hom}\,(Q, \mathbf{R}_{\geq 0}) \xrightarrow{\cong} P - H'$ which induces $\mathrm{Hom}\,(\tau^{\mathrm{gp}}, \mathbf{R}_{>0}) = (O_{\tau'_P})_{\geq} \xrightarrow{\cong} \mathrm{Int}\,(\tau_P)$ for each $\tau < Q$. By composing this with a homeomorphism $P - H' \longrightarrow Q_{\mathbf{R}_{\geq 0}}$ which induces $\mathrm{Int}\,(\tau_P) \xrightarrow{\cong} \mathrm{Int}\,(\tau_{\mathbf{R}_{\geq 0}})$ for each $\tau < Q$, the desired homeomorphism is obtained. □

REMARK A.1.1. Note that we use only the statement of A.1 in our text. On the other hand, one of the referees suggested the following alternative construction of the homeomorphism as in A.1. Take a finite subset S which generates $Q_{\mathbf{R}_{\geq 0}}$ as an $\mathbf{R}_{\geq 0}$-monoid. Then the map $\mu\colon \mathrm{Hom}\,(Q, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \longrightarrow$

$Q_{\mathbf{R}_{\geq 0}}$ defined by

$$\mu(x) = \sum_{s \in S} x(s)s \qquad (x \in \mathrm{Hom}\,(Q, \mathbf{R}^{\mathrm{mult}}_{\geq 0}))$$

is the desired homeomorphism. We sketch the proof. First, we can show that μ is bijective by using the statement of (A_n) in p.83 of [F]. Hence, for any set $I = (I_s)_{s \in S}$ of compact intervals in $\mathbf{R}_{\geq 0}$, the restriction of μ to the compact subspace $C_I := \{h \colon Q \longrightarrow \mathbf{R}^{\mathrm{mult}}_{\geq 0} \mid h(s) \in I_s \text{ for all } s \in S\}$ of $\mathrm{Hom}\,(Q, \mathbf{R}^{\mathrm{mult}}_{\geq 0})$ is a homeomorphism onto its image $\mu(C_I)$. Since we can take a locally finite closed covering of $Q_{\mathbf{R}_{\geq 0}}$ consisting of subsets of the form $\mu(C_I)$, we can conclude that μ itself is a homeomorphism.

PROPOSITION A.2. *Let C be a convex polytope and e its vertex. Let Z be the union of all faces of C which do not contain e. Then there is a homeomorphism $i \colon C \longrightarrow C'$ of convex polytopes such that C' is a cone over one of its face Z' with the vertex $i(e)$, such that $i(Z) = Z'$, and such that for each $\sigma < C$ that includes e, $i(\sigma)$ is a cone over a face of Z'.*

PROOF. Take a supporting hyperplane H of $\{e\}$ for C, and cut C by a hyperplane which is parallel and sufficiently near to H. Then the part C' of C including e is a cone. Let $i \colon C \longrightarrow C'$ be the map sending x to $e + \frac{|l \cap C'|}{|l \cap C|}(x - e)$, where l is the line through e and x and $| - |$ is the length of the segment. Then i is a homeomorphism and satisfies the required conditions. □

PROPOSITION A.3. *Let C be a convex polytope and $e \in \partial C$. Then there is a homeomorphism $i \colon C \longrightarrow C'$ of convex polytopes such that the following conditions* (1)—(4) *are satisfied*:

(1) *$i(e)$ is a vertex of C'.*

(2) *For each face of C, its image by i is the union of some faces of C'.*

(3) *For each face of C containing e, its image by i is the union of some faces of C' containing $i(e)$.*

(4) *The image of the union of all the faces that do not contain e coincides with the union of all the faces that do not contain $i(e)$.*

PROOF. If e is a vertex of C, then we may define $C' = C$, and i as the identity. So, we assume that e is not a vertex of C. Let F_0 be the minimal face of C that contains e. Take a supporting hyperplane H of C satisfying that $H \cap C = F_0$ and that, for a normal vector u_0 of H, the point $e + u_0$ is in the interior of C. We show that, for a sufficiently small $\epsilon > 0$, the convex hull C' of $C \cup \{e' := e - \epsilon u_0\}$ and the projection $i \colon C \to C'$ from the point $e + \epsilon u_0$ satisfy (1)—(4) in this proposition.

We first introduce notation for hyperplanes. For a non-zero vector $u \in \mathbf{R}^n$ and a point $x_0 \in \mathbf{R}^n$, we denote by $H_{u,x_0} := \{x \in \mathbf{R}^n \mid \langle u, x \rangle = \langle u, x_0 \rangle\}$ the hyperplane through x_0 with normal vector u. Here, $\langle\ ,\ \rangle$ is the standard inner product of $\mathbf{R}^n$. We also denote, by $H^+_{u,x_0} := \{x \in \mathbf{R}^n \mid \langle u, x \rangle \geq \langle u, x_0 \rangle\}$ the half space with boundary H_{u,x_0}, and by H^-_{u,x_0} the other half space with boundary H_{u,x_0}. For a hyperplane H, we denote, simply by H^+ and H^- the two half spaces with boundary H.

We first show that the set of vertices of C' is the union of $\{e'\}$ and the set of vertices of C. Let H be a supporting hyperplane of C such that $\{v\} = H \cap C$ for a vertex v of C. Suppose that the half space H^+ with boundary H contains C. Since $e \in H^+ - H$, we have $e' \in H^+ - H$ for a sufficiently small ϵ. So, every vertex $v \in C$ is also a vertex of C'. On the other hand, $H^+_{u_0,e'} - H_{u_0,e'}$ contains all the vertices of C, because, for a vertex v of C, $\langle u_0, v \rangle - \langle u_0, e' \rangle \geq \epsilon \langle u_0, u_0 \rangle > 0$. Hence e' is a vertex of C'. We verified (1).

If a face F of C' does not contain e', then it is easily seen that F is a face of C. So, the projection i is the identity on such an F, and the statement (4) holds. If we verify (3), then (2) follows immediately. Hence, to complete the proof, we have only to show (3). Let F be a face of C which contains e, and F' a face of F which does not contain e. It is enough to verify that the convex hull $\tilde{F}$ of F' and e' is a face of C'. Suppose that $H_{u,e}$ (resp. $H_{u',x'}$) is a supporting hyperplane of C with $C \cap H_{u,e} = F$ (resp. $C \cap H_{u',x'} = F'$ and $x' \in F'$). Let H be a hyperplane defined by

$$\langle u + \delta u', x \rangle - \langle u, e \rangle - \delta \langle u', x' \rangle = 0.$$

Here $\delta := \epsilon \langle u, u_0 \rangle / (\langle u', e' \rangle - \langle u', x' \rangle)$. We can easily verify that, for a sufficiently small $\epsilon > 0$, H is a supporting hyperplane of C' with $C' \cap H = \tilde{F}$. □

Appendix B. On Local Triviality

In this section, we discuss related problems on the local triviality and C^∞-structures.

First we explain the motivation. Recall that Sampei Usui ([U1], [U2]) proved Theorem 0.1 in the case mentioned in Introduction by proving that $f^{\log}$ is locally trivial in that case and his local trivialization respects C^∞-structures piecewisely. Further, the classical result also mentioned in Introduction is deduced from the C^∞-local triviality, too, that is, 0.1.1 is a corollary to the following.

THEOREM B.1. *Let f be as in* 0.1.1. *Then f is locally trivial in the category of C^∞-spaces.*

Here a C^∞-space means a ringed space over $\operatorname{Spec}\mathbf{R}$ which is locally isomorphic to (A, C_A^∞) for a subset A of $\mathbf{R}^n$ for some $n \geq 0$, where C_A^∞ is the sheaf of $\mathbf{R}$-valued maps which can be locally extended to C^∞-functions on open subsets of $\mathbf{R}^n$.

In general, for a proper separated log smooth morphism f, $f^{\log}$ is not necessarily locally trivial because a blowing-up f along log structure is proper, separated, and log smooth, though mostly $f^{\log}$ is not locally trivial.

But we can expect the following has an affirmative answer.

Problem B.2. Let $f\colon X \longrightarrow Y$ be a proper separated log smooth exact morphism of fs log analytic spaces. Is $f^{\log}$ locally trivial?

Note that the case was solved in Section 3 where Y has a constant log structure, and that the case was solved by Sampei Usui ([U1], [U2]) where f is a multi-generalized semistable family over a polydisk, as said in the beginning in this section.

REMARK B.2.1. To solve this problem would give an alternative proof of 0.1. In fact, to prove 0.1, by the argument in the beginning of 5.5, we may assume that f is exact. Then 2.6 together with 3.2.1 and 3.6 reduces 0.1 to the above problem.

REMARK B.2.2. By Proposition 4.2 in [KMN], one can show that for a log smooth exact morphism f of fs log analytic spaces, $f^{\log}$ is an open map. This supports the affirmation of the above problem.

Problem B.2 is reduced to the following local problem.

Problem B.3. (topological relative rounding problem) For any local $\mathbf{Q}$-integral homomorphism $h\colon Q \hookrightarrow P$ of sharp fs monoids, is $\mathrm{Hom}\,(h, \mathbf{R}_{\geq 0}^{\mathrm{mult}})$ submersive?

In fact, taking a $\mathbf{Q}$-integral chart as in the proof of 3.4, we see that, if B.3 is valid, $\mathrm{Spec}\,\mathbf{C}[h]_{\mathrm{an}}^{\log} = \mathrm{Hom}\,(h, \mathbf{R}_{\geq 0}^{\mathrm{mult}}) \times \mathrm{Hom}\,(h, \mathbf{S}^1)$ is submersive and hence $f^{\log}$ is submersive for such an f as in B.2. Then we can use 3.5 together with 3.2.1 and 3.4 to see B.2.

The authors have not yet solved even the case of B.3 where $P = \mathbf{N}$. But a partial solution of B.3 gives a partial solution of B.2. For example,

PROPOSITION B.3.1. *Let $\mathcal{C}$ be the minimal class of homomorphisms of fs monoids which contains the diagonal $\mathbf{N} \longrightarrow \mathbf{N}^2$ and which is closed under cobase changes and composition. (Explicitly, a homomorphism belongs to this class if and only if it is the composite of finite number of the cobase changes of the diagonal $\mathbf{N} \longrightarrow \mathbf{N}^r$ (r varies).)*

For an f as in B.2, *if for any point x of X, the homomorphism $(M_Y/\mathcal{O}_Y^\times)_{f(x)} \longrightarrow (M_X/\mathcal{O}_X^\times)_x$ belongs to $\mathcal{C}$, $f^{\log}$ is locally trivial.*

PROOF. As before, the problem lies in that $f^{\log}$ is submersive. This is seen in this case from the fact that the map $\mathbf{R}_{\geq 0}^2 \longrightarrow \mathbf{R}_{\geq 0}$; $(x, y) \mapsto xy$ is submersive. □

We discuss another approach to B.2. The following is a C^∞-analogue of 3.5.

PROPOSITION B.4. *Let $f\colon X \longrightarrow Y$ be a proper separated C^∞-submersive map of C^∞-spaces. Assume that each fiber of f is a differentiable manifold. Then f is locally trivial in the category of C^∞-spaces.*

This is a direct consequence of the inverse mapping theorem. Note that B.1 is a corollary to this proposition. Further, it can be applied to some log cases. For example, consider a proper separated semistable curve $f\colon X \longrightarrow \Delta$ over the log pointed disk (= the unit disk endowed with the log structure associated to the origin), and endow the natural C^∞-space structures on $X^{\log} \longrightarrow \Delta^{\log}$. Then, since the set of points of X on which $f^{\log}$ is not C^∞-submersive is discrete, by modifying the C^∞-structure of $X^{\log}$ (straightening $\mathbf{R}_{\geq 0}^2$ into $\mathbf{R} \times \mathbf{R}_{\geq 0}$; $(x, y) \mapsto (x^2 - y^2, 2xy)$), we can still apply B.4 to this situation so that $f^{\log}$ is C^{∞}-locally trivial in a sense. We do not know whether or not such a modification is possible generally. We do not know, either, whether or not such a consideration is natural. But we still hope that the following problem is a reasonable one.

Problem B.5. For an fs log analytic space X, endow $X^{\log}$ with some smooth structure functorially, and solve B.2 by a sort of the inverse mapping theorem.

Finally we remark that the conclusion of B.2 can be valid for some non-exact f:

PROPOSITION B.6. *Let* $f\colon X \longrightarrow Y$ *be as in* 0.1. *Assume that* Y *has a constant log structure* (3.1). *Then* $f^{\log}$ *is locally trivial.*

PROOF. By lemmas 3.2.1 and 3.2.2 as in 3.7, it is enough to show that each fiber of $f^{\log}$ satisfies the condition in Corollary 6.14 of [S] (cf. Remark 3.5.2). It can be checked locally on that fiber (cf. loc. cit. Remark 2.2). By Proposition A.1 as in the proof of 3.4, locally, the fiber is an open set of the product of a topological manifold and the subspace $\bigcup_{\sigma\in\Phi}\operatorname{Int}(\sigma)$ of a real cone, where Φ is a subcomplex of the set of faces of that cone. Since Φ is a subcomplex, the latter subspace is equal to the union of some faces of that cone so that it is a CS set in the sense of [S] Definition 1.2. Thus Deformation Theorem 2.3 in [S] implies the required condition in loc. cit. 6.14. □

Appendix C. Comparison between Log Betti and Log Étale Cohomologies

In this section we prove the following comparison theorem. This explains analogies between log Betti cohomology and log étale cohomology (cf. Remarks 0.4, 5.1.2, and 5.3.1), and even gives partially alternative proofs for results in the former by reducing them to those in the latter.

THEOREM C.1. *Let* $f\colon X \longrightarrow Y$ *be a proper log injective morphism of fs log schemes over* $\mathbf{C}$ *(see* [N] *Definition* (5.5.1) *for the log injectivity).*

Assume that $\overset{\circ}{Y}$ *is locally of finite type over* $\mathbf{C}$*. Consider the diagram of topoi*

$$\begin{array}{ccc} (X_{\mathrm{an}}^{\log})^{\sim} & \xrightarrow{f=f_{\mathrm{an}}^{\log}} & (Y_{\mathrm{an}}^{\log})^{\sim} \\ \varepsilon \downarrow & & \downarrow \varepsilon \\ (X_{\text{ét}}^{\log})^{\sim} & \xrightarrow[f]{} & (Y_{\text{ét}}^{\log})^{\sim}, \end{array}$$

where $(\ )_{\text{ét}}^{\log}$ *is the Kummer log étale topos (*[N] (2.2)*) and* ε *is the morphism of topoi defined in* [KN] (2.1)*. Let* F *be a torsion sheaf of modules on* $X_{\text{ét}}^{\log}$ *("torsion" means that each stalk is a torsion module). Then the natural homomorphism*

$$\varepsilon^* \mathrm{R} f_* F \longrightarrow \mathrm{R} f_* \varepsilon^* F$$

is an isomorphism.

Remark C.1.1. When $Y = \mathrm{Spec}\,(\mathbf{C})$ with the trivial log structure, the properness of f can be replaced with the constructibility of F. See [KN] Theorem (0.2) (1).

Remark C.1.2. In the above theorem, the properness is necessary. In fact, the strict open immersion $Y := \mathrm{Spec}\,\mathbf{C}[\mathbf{N}^2]/\langle e_1 - e_2\rangle \hookleftarrow X := Y[\frac{1}{e_1}]$, where $(e_i)_i$ is the canonical base of $\mathbf{N}^2$, does not satisfy the conclusion of the theorem for $F = \mathbf{Z}/n\mathbf{Z}$, $n > 1$. See [KN] Remark (2.7) for another example that tells the necessity of the log injectivity.

Proof. Since f is quasi-compact and quasi-separated and since $f_{\mathrm{an}}^{\log}$ is proper and separated (GAGA in [SGA1] Proposition 3.2 (v) and 3.2.1), both the $\mathrm{R}^q f_*$ commute with filtered inductive limits ($q \geq 0$). On the other hand, since F is torsion, the natural map $\varinjlim_n \mathrm{Ker}\,(n\colon F \longrightarrow F) \longrightarrow F$ is an isomorphism. Hence we may assume that there is an $N \geq 1$ such that F is a $\mathbf{Z}/N\mathbf{Z}$-Module. Denote $A := \mathbf{Z}/N\mathbf{Z}$.

We may assume either

(A) f is strict, or

(B) $\overset{\circ}{f}$ is an isomorphism.

Further, the latter case is reduced to the case where $\overset{\circ}{X} = \overset{\circ}{Y} = \mathrm{Spec}\,\mathbf{C}$ by the proper base change theorems (Theorem (5.1) in [N] and Proposition

5.1 (2) together with Remark 5.1.1), and to the case of $f = (\operatorname{Spec}\mathbf{C}, \mathbf{C}^\times \oplus (h\colon P \hookrightarrow Q))$, where h is an injective homomorphism of sharp fs monoids satisfying either

(B1) $\operatorname{cok}(h^{\mathrm{gp}})$ is finite, or

(B2) h is isomorphic to the natural inclusion $P \longrightarrow P \oplus \mathbf{N}$,

by the argument in [N] (5.6).

We will treat the above three cases (A), (B1), and (B2).

In any case, since X is quasi-separated, we may assume that F is a constructible A-Module by [N] Proposition (3.3) 9.

The case (A) is reduced to the non-log case as follows. Since the problem is strict étale local on Y, we may assume that $\overset{\circ}{Y}$ is affine and Y is charted by a sharp fs monoid P. Then, by [N] Lemma (5.6.2), we may assume that $F = \iota_! G$, where $\iota\colon X' \longrightarrow X$ is the strict immersion from a constructible reduced subscheme of X and G is an A-Module on $(X')^{\log}_{\text{ét}}$ such that for some $n \geq 1$, $G|_{X'_n}$ is the inverse image of a constructible A-Module on $\overset{\circ}{X}{}'_n$ with the trivial log structure (there is a typographical error in [N] Lemma (5.6.2); $\mathcal{S}^A_{X'^{\mathrm{cl}}_n}$ there should be $\mathcal{S}^A_{X'^{\mathrm{cl}}_n,\mathrm{ctr}}$). Here and hereafter $(\ )_n = (\ ) \times_{\operatorname{Spec}\mathbf{Z}P} \operatorname{Spec}\mathbf{Z}P^{\frac{1}{n}}$. Consider the commutative diagram

$$\begin{array}{ccccc} X'_n & \xrightarrow{\iota} & X_n & \longrightarrow & Y_n \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ X' & \xrightarrow[\iota]{} & X & \longrightarrow & Y \end{array}$$

of fs log schemes. Since $\pi^{\log}_{\mathrm{an}}$ is a surjective local homeomorphism, and since $\pi^* \iota_! = \iota_! \pi^*$, we may assume that the above n is 1. Then F is the inverse image of a torsion Module H on the classical étale site of $\overset{\circ}{X}$. Again by the proper base change theorems, the pull backs with respect to $X \longrightarrow \overset{\circ}{X}$ and $Y \longrightarrow \overset{\circ}{Y}$ commute with both the $\mathrm{R}f_*$. Hence the problem is reduced to [SGA4] XVI Théorème 4.1 (i).

Next we treat the case (B1). Let n be the order of $\operatorname{cok}(h^{\mathrm{gp}})$. Since the problem is Kummer étale local on Y, considering the base change of f by $Y_n \longrightarrow Y$ in the category of fs log schemes, we may assume that $n = 1$ so that $P^{\mathrm{gp}} = Q^{\mathrm{gp}}$. Then $(X^{\log}_{\text{ét}})^\sim = (\text{continuous}\,\pi_1\text{-set}) = (Y^{\log}_{\text{ét}})^\sim$, where $\pi_1 = \operatorname{Hom}(P^{\mathrm{gp}}, \widehat{\mathbf{Z}}'(1)) = \operatorname{Hom}(Q^{\mathrm{gp}}, \widehat{\mathbf{Z}}'(1))$ (cf. [N] Proposition (4.6)) and $X^{\log}_{\mathrm{an}} = Y^{\log}_{\mathrm{an}}$, and hence the conclusion is trivial.

Finally we prove the case (B2). Since F is locally constant and constructible, there is an $n \geq 1$ such that $F|_{X_n}$ is constant, where $X_n = X \times_{\mathrm{Spec}\,\mathbf{Z}Q} \mathrm{Spec}\,\mathbf{Z}Q^{\frac{1}{n}}$. Denote by p the projection $X_n \longrightarrow X$ and by F' the cokernel of the injection $F \longrightarrow p_* p^* F$. Then $p^* F'$ is also constant and constructible. Thus we can take a resolution

$$0 \longrightarrow F \longrightarrow p_* G^0 \longrightarrow p_* G^1 \longrightarrow \cdots$$

of F, where each G^i is constant and constructible. Since the cohomological dimension of p is zero by [N] Proposition (5.3), we have $F = p_* G^\bullet = \mathrm{R}p_* G^\bullet$. Thus $\varepsilon^* \mathrm{R}f_* F = \varepsilon^* \mathrm{R}(f \circ p)_* G^\bullet$. On the other hand, we have $\mathrm{R}f^{\log}_{\mathrm{an}*} \varepsilon^* \mathrm{R}p_* G^\bullet = \mathrm{R}(f \circ p)^{\log}_{\mathrm{an}*} \varepsilon^* G^\bullet$ by the cases (A) and (B1), which are already proved. Thus it is enough to show the original statement for $f \circ p$ and a constant G^i. Decompose $f \circ p$ into $X_n \longrightarrow (X_n)_{\mathrm{red}} \longrightarrow X' \longrightarrow Y$, where $X' = (\mathrm{Spec}\,\mathbf{C}, \mathbf{C}^\times \oplus P^{\frac{1}{n}})$. Again by the cases (A) and (B1), the problem is reduced to that for $(X_n)_{\mathrm{red}} \longrightarrow X'$. But this morphism is a base change of the morphism $f_0 = (\mathrm{Spec}\,\mathbf{C}, \mathbf{C}^\times \oplus (\{1\} \longrightarrow \mathbf{N}^{\frac{1}{n}}))$. Hence, again by the proper base change theorems, the problem is reduced to that for f_0, and to [KN] Theorem (0.2) (1). □

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