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A Cohen Type Inequality for Polynomial Expansions Associated with the Measure $(1-x)^{\alpha}(1+x)^{\beta}dx + M\delta_{-1} + N\delta_1$

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Abstract. The purpose of this paper is to establish a Cohen type inequality for Fourier expansion with respect to polynomials associated with the measure $(1-x)^{\alpha}(1+x)^{\beta}dx + M\delta_{-1} + N\delta_1$, where δ_t is the delta function at a point t and $M, N \geq 0$.

1. Introduction and Main Result

In a well known paper [1] Cohen proved that for any trigonometric polynomial $P_N(x) = \sum_{k=1}^{N} a_k e^{in_k x}$, where $0 < n_1 < ... < n_N$, $N \geq 2$, and $|a_k| \geq 1$ for $1 \leq k \leq N$, the following inequality holds:

$$\int_0^{2\pi} |P_N(x)|dx \geq c \left(\frac{\log N}{\log\log N} \right)^{1/8}.$$

Motivated by the work of Cohen, inequalities of this type have been established in various other contexts, e.g., for classical orthogonal expansions or on compact groups (see [1], [2], [3], [6], [10], [12]). The aim of this article is to prove a Cohen type inequality for Fourier expansions in terms of orthonormal polynomials relative to the Jacobi measure with two masses at points $x = \pm 1$.

Let $\omega_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $(\alpha, \beta > -1)$, be the Jacobi weight on the interval $[-1, 1]$. In [11] T. H. Koornwinder introduced the sequence of polynomials $\{P_n^{(\alpha,\beta,M,N)}(x)\}_{n=0}^{\infty}$ which are orthogonal on the interval $[-1, 1]$ with respect to the measure

$$d\mu(x) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\omega_{\alpha,\beta}(x)dx + M\delta_{-1} + N\delta_1,$$

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where $\alpha > -1$, $\beta > -1$, and $M, N \geq 0$. They are called Koornwinder's Jacobi-type polynomials. We denote the orthonormal Koornwinder's Jacobi-type polynomial by $p_n^{(\alpha,\beta,M,N)}$, which differs from $P_n^{(\alpha,\beta,M,N)}$ by normalization constant (see [15, p. 81]). For $M = N = 0$, denoted by $\{p_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$, we have the classical Jacobi orthonormal polynomials (see [14, Chapter IV]). It is known that, unlike the Jacobi orthonormal polynomials, the polynomials $p_n^{(\alpha,\beta,M,N)}$ for $M > 0$, $N > 0$ decay at the rate of $n^{-\alpha-3/2}$ and $n^{-\beta-3/2}$ at the end points 1 and -1.

We shall say that $f(x) \in L^p(d\mu)$ if $f(x)$ is μ-measurable on the $[-1,1]$ and $\|f\|_{L^p(d\mu)} < \infty$, where

$$\|f\|_{L^p(d\mu)} = \begin{cases} \left(\int_{-1}^{1} |f(x)|^p d\mu(x)\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \underset{-1<x<1}{ess\ sup}\ |f(x)| & \text{if } p = \infty. \end{cases}$$

Throughout this paper we denote by $[L^p(d\mu)]$ the space of all bounded, linear operators $T : L^p(d\mu) \to L^p(d\mu)$, furnished with the usual operator norm

$$||T||_{[L^p(d\mu)]} = \sup_{0 \neq f \in L^p(d\mu)} \frac{||T(f)||_{L^p(d\mu)}}{||f||_{L^p(d\mu)}}.$$

For $f \in L^1(d\mu)$, the Fourier expansion in Koornwinder's Jacobi-type polynomials is

$$\sum_{k=0}^{\infty} \hat{f}(k) p_k^{(\alpha,\beta,M,N)}(x), \tag{1.1}$$

where the Fourier coefficients are

$$\begin{aligned} \hat{f}(k) &= \int_{-1}^{1} f(x) p_k^{(\alpha,\beta,M,N)}(x) d\mu(x) \\ &= \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^{1} f(x) p_k^{(\alpha,\beta,M,N)}(x) \omega_{\alpha,\beta}(x) dx \\ &\quad + M f(-1) p_k^{(\alpha,\beta,M,N)}(-1) + N f(1) p_k^{(\alpha,\beta,M,N)}(1). \end{aligned}$$

The Cesàro means of order ρ of the expansion (1.1) are defined by (see [16, p. 76-77])

$$\sigma_n^{\rho} f(x) = \sum_{k=0}^{n} \frac{A_{n-k}^{\rho}}{A_n^{\rho}} \hat{f}(k) p_k^{(\alpha,\beta,M,N)}(x),$$

where $A_k^\rho = \binom{k+\rho}{k}$.

For a given sequence $\{c_{k,n}\}_{k=0}^n$, $n \in \mathbf{N} \cup \{0\}$, of complex numbers with $|c_{n,n}| > 0$, we define the operators $T_n^{\alpha,\beta,M,N} \in [L^p(d\mu)]$ by

$$T_n^{\alpha,\beta,M,N}(f) = \sum_{k=0}^{n} c_{k,n}\hat{f}(k)p_k^{(\alpha,\beta,M,N)}.$$

The main result of the present paper is the following theorem:

THEOREM. *Let $\alpha \geq \beta \geq -1/2$, $\alpha > -1/2$ and $1 \leq p \leq \infty$. There exists a positive constant c, independent of n, such that*

$$\|T_n^{\alpha,\beta,M,N}\|_{[L^p(d\mu)]} \geq c|c_{n,n}| \begin{cases} n^{\frac{2\alpha+2}{p}-\frac{2\alpha+3}{2}} & \text{if } a \leq p < p_0 \\ (\log\ n)^{\frac{2\alpha+1}{4\alpha+4}} & \text{if } p = p_0,\ p = q_0 \\ n^{\frac{2\alpha+1}{2}-\frac{2\alpha+2}{p}} & \text{if } q_0 < p \leq b \end{cases}$$

where $p_0 = (4\alpha+4)/(2\alpha+3)$, $q_0 = (4\alpha+4)/(2\alpha+1)$, and

i) *if $M > 0$, $N = 0$, then $a = 1$ and $b = \infty$,*

ii) *if $M > 0$, $N > 0$, then $1 < a$, $b < \infty$ and $1/a + 1/b = 1$.*

COROLLARY 1.1. *Let α, β, and p be as in Theorem. For $c_{k,n} = 1$, $k = 0, ..., n$, and for p outside the Pollard interval (p_0, q_0)*

$$\|S_n\|_{[L^p(d\mu)]} \to \infty, \qquad n \to \infty$$

where S_n denotes the nth partial sum of expansion (1.1).

For $c_{k,n} = \frac{A_{n-k}^\rho}{A_n^\rho}$, $0 \leq k \leq n$, the Theorem 1 yields:

COROLLARY 1.2. *Let α, β, p, and ρ be given numbers such that $\alpha > -1/2$,*

$$\begin{cases} -\frac{1}{2} \leq \beta \leq \alpha, \\ a \leq p \leq b, \\ 0 \leq \rho < \frac{2\alpha+2}{p} - \frac{2\alpha+3}{2} & \text{if } 1 \leq p < p_0, \\ 0 \leq \rho < \frac{2\alpha+1}{2} - \frac{2\alpha+2}{p} & \text{if } q_0 < p \leq \infty. \end{cases}$$

Then, for $p \notin [p_0, q_0]$

$$\|\sigma_n^{\rho}\|_{[L^p(d\mu)]} \to \infty, \qquad n \to \infty.$$

Remark 1.1. Using the symmetry formula in [11], for the case $M = 0$ and $N > 0$ we get the same results as above but exchanging α and β.

Notice that the study of the convergence of Fourier expansions (1.1) has been discussed in [7], [9].

2. Estimates for Koornwinder's Jacobi-type Polynomials

In order to prove our main result, we need some estimates for Koornwinder's Jacobi-type orthonormal polynomials. The representation of the $p_n^{(\alpha,\beta,M,N)}$ in terms of $p_n^{(\alpha,\beta)}$, a strong asymptotic on $(-1,1)$, a Mehler-Heine type formula, Lebesgue norms of $p_n^{(\alpha,\beta,M,N)}$ are derived. Throughout this paper, the positive constants are denoted by $c,\ c', ...$ and they may be different at different occurrence. The notation $u_n \cong v_n$ means that the sequence u_n/v_n converges to 1 and notation $u_n \sim v_n$ means $cu_n \leq v_n \leq c'u_n$ for sufficiently large n.

Proposition 2.1. *The representation of the* $p_n^{(\alpha,\beta,M,N)}$ *in terms of* $p_n^{(\alpha,\beta,M,0)}$ *is*

$$p_n^{(\alpha,\beta,M,N)}(x) = A_n p_n^{(\alpha,\beta,M,0)}(x) + B_n(x-1)p_{n-1}^{(\alpha+2,\beta,4M,0)}(x) \tag{2.1}$$

where

$$A_n \cong cn^{-2\alpha-2}, \qquad B_n \cong 1. \tag{2.2}$$

Proof. Let $\{P_n^1\}_{n=0}^{\infty}$ be the orthonormal polynomials with respect to the measure (see proof of the Proposition 6 in [8])

$$(x-1)^2[\omega_{\alpha,\beta}(x)dx + M\delta_{-1}] = \omega_{\alpha+2,\beta}(x)dx + 4M\delta_{-1}.$$

Therefore $P_n^1 = p_n^{(\alpha+2,\beta,4M,0)}$. From [8, Proposition 4] it follows

$$p_n^{(\alpha,\beta,M,N)}(x) = A_n p_n^{(\alpha,\beta,M,0)}(x) + B_n(x-1)p_{n-1}^{(\alpha+2,\beta,4M,0)}(x)$$

where

$$\lim_{n\to\infty} A_n L_{n-1}(1,1) = \frac{1}{\lambda(1)+N},$$

$$\lim_{n\to\infty} B_n = \frac{N}{\lambda(1)+N},$$

$$\lambda(1) = \lim_{n\to\infty} \frac{1}{L_n(1,1)}.$$

Since (see [5, (3)] and [14, (4.5.8)])

$$L_n(1,1) = \sum_{i=0}^{n} p_i^{(\alpha,\beta,M,0)}(1) p_i^{(\alpha,\beta,M,0)}(1) \cong cn^{2\alpha+2}$$

we get (2.2). □

Combining the above proposition with [5, (7)] we obtain:

COROLLARY 2.1. *The representation of the $p_n^{(\alpha,\beta,M,N)}$ in terms of $p_n^{(\alpha,\beta)}$ is*

$$p_n^{(\alpha,\beta,M,N)}(x) = a_n p_n^{(\alpha,\beta)}(x) + b_n(x+1)p_{n-1}^{(\alpha,\beta+2)}(x) + c_n(x-1)p_{n-1}^{(\alpha+2,\beta)}(x) + d_n(x^2-1)p_{n-2}^{(\alpha+2,\beta+2)}(x)$$

where

i) *if $M>0$, $N=0$, then*

$$a_n \cong cn^{-2\beta-2}, \qquad b_n \cong 1, \qquad c_n = 0, \qquad d_n = 0,$$

ii) *if $M>0$, $N>0$, then*

$$a_n \cong cn^{-2\alpha-2\beta-4}, \qquad b_n \cong cn^{-2\alpha-2}, \qquad c_n \cong cn^{-2\beta-2}, \qquad d_n \cong 1.$$

The following proposition establishes a strong asymptotic on $(-1,1)$ for $p_n^{(\alpha,\beta,M,N)}$.

PROPOSITION 2.2. *For $\theta \in [\epsilon, \pi-\epsilon]$ and $\epsilon > 0$*

$$p_n^{(\alpha,\beta,M,N)}(x) = l_n^{\alpha,\beta}(1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4}\cos(k\theta+\gamma) + O(n^{-1}),$$

where $x = \cos\theta$, $k = n + (\alpha+\beta+1)/2$, $\gamma = -(\alpha+1/2)\pi/2$, *and* $\lim_{n\to\infty} l_n^{\alpha,\beta} = \sqrt{2/\pi}$.

Proof. From (2.1) and [5, Lemma 1]

$$p_n^{(\alpha,\beta,M,N)}(x) = [A_n s_n^{\alpha,\beta} + B_n s_{n-1}^{\alpha+2,\beta}](1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4} \times \cos(k\theta+\gamma) + [A_n + B_n(x-1)]O(n^{-1}),$$

$\lim_{n\to\infty} s_n^{\alpha,\beta} = \sqrt{2/\pi}$. Now taking into account (2.2), the result follows. □

Next we give a Mehler-Heine type formula of the polynomials $p_n^{(\alpha,\beta,M,N)}$ for $M > 0$ and $N > 0$.

Proposition 2.3. *Let* $|z| \le R$, *and* R *a given positive real number. Then*

$$\lim_{n\to\infty} n^{-\alpha-1/2} p_n^{(\alpha,\beta,M,N)}(\cos\frac{z}{n}) = -2^{-\frac{\alpha+\beta}{2}} z^{-\alpha} J_{\alpha+2}(z)$$

where $J_\alpha(z)$ *is the Bessel function of order* α.

Proof. By (2.1) we have

$$n^{-\alpha-1/2} p_n^{(\alpha,\beta,M,N)}(\cos\frac{z}{n}) = A_n n^{-\alpha-1/2} p_n^{(\alpha,\beta,M,0)}(\cos\frac{z}{n}) - 2B_n \sin^2(\frac{z}{2n})\, n^{-\alpha-1/2} p_n^{(\alpha+2,\beta,4M,0)}(\cos\frac{z}{n}).$$

Using the estimates for the coefficients A_n, B_n, and the Mehler-Heine type formula for orthonormal polynomials $p_n^{(\alpha,\beta,M,0)}(\cos\frac{z}{n})$ (see [5, Lemma 2])

$$\lim_{n\to\infty} n^{-\alpha-1/2} p_n^{(\alpha,\beta,M,0)}(\cos\frac{z}{n}) = 2^{-\frac{\alpha+\beta}{2}} (z/2)^{-\alpha} J_\alpha(z),$$

the result follows. □

The proof of main result is based on following proposition.

Proposition 2.4. *Let* $\alpha \ge -1/2$ *and* $M, N \ge 0$. *For* $1 \le q < \infty$

$$\int_0^1 (1-x)^\alpha |p_n^{(\alpha,\beta,M,N)}(x)|^q dx \sim \begin{cases} c & \text{if } 2\alpha > q\alpha - 2 + q/2, \\ \log\ n & \text{if } 2\alpha = q\alpha - 2 + q/2, \\ n^{q\alpha+q/2-2\alpha-2} & \text{if } 2\alpha < q\alpha - 2 + q/2. \end{cases}$$

Proof. The upper estimate has been proved in [5, Theorem 1]. In order to prove the lower estimate, we follow the same line as in [14, Theorem 7.34] (see also [5, Theorem 2]).

Le $\alpha \geq -1/2$, $M > 0$ and $N > 0$. By using the Proposition 2.3 we get

$$\int_0^{\pi/2} \theta^{2\alpha+1} |p_n^{(\alpha,\beta,M,N)}(\cos\theta)|^q d\theta > \int_0^{n^{-1}} \theta^{2\alpha+1} |p_n^{(\alpha,\beta,M,N)}(\cos\theta)|^q d\theta$$
$$\cong c \int_0^1 (z/n)^{2\alpha+1} n^{q\alpha+q/2} |z^{-\alpha} J_{\alpha+2}(z)|^q \; n^{-1} dz \sim n^{q\alpha+q/2-2\alpha-2}.$$

In the same way , for $2\alpha = q\alpha - 2 + q/2$, from [13, Lemma 2.1] we obtain

$$\int_0^{\pi/2} \theta^{2\alpha+1} |p_n^{(\alpha,\beta,M,N)}(\cos\theta)|^q d\theta > \int_0^{n^{-1/2}} \theta^{2\alpha+1} |p_n^{(\alpha,\beta,M,N)}(\cos\theta)|^q d\theta$$
$$\cong c \int_0^{n^{1/2}} z^{2\alpha+1} |z^{-\alpha} J_{\alpha+2}(z)|^q \; dz \geq c \log n.$$

Finally, we use Proposition 2.2 to obtain

$$\int_0^{\pi/2} \theta^{2\alpha+1} |p_n^{(\alpha,\beta,M,N)}(\cos\theta)|^q d\theta > \int_{\pi/4}^{\pi/2} \theta^{2\alpha+1} |p_n^{(\alpha,\beta,M,N)}(\cos\theta)|^q d\theta \sim c.$$

The proof of case when $M > 0$ and $N = 0$ can be done in a similar way (see [5, Theorem 2]). □

3. Proof of Theorem

Let us consider the test functions

$$g_n^{\alpha,\beta,j}(x) = (1-x^2)^j p_n^{(\alpha+j,\beta+j)}(x)$$

where $\alpha \geq \beta \geq -1/2$, $\alpha > -1/2$, and $j \in \mathbf{N}$. The following estimates of the Fourier coefficients for the functions $g_n^{\alpha,\beta,j}(x)$ as well as the upper bounds of their Lebesgue norms plays the essential role in proof of theorem. By applying the operators $T_n^{\alpha,\beta,M,N}$ to the test functions $g_n^{\alpha,\beta,j}$, for some $j > \alpha + 1/2 - (2\alpha+2)/p$, we get

$$T_n^{\alpha,\beta,M,N}(g_n^{\alpha,\beta,j}) = \sum_{k=0}^{n} c_{k,n} (g_n^{\alpha,\beta,j})\hat{\ }(k) p_k^{(\alpha,\beta,M,N)} \tag{3.1}$$

where

$$(g_n^{\alpha,\beta,j})\hat{}\,(k)=\int_{-1}^{1}g_n^{\alpha,\beta,j}(x)p_k^{(\alpha,\beta,M,N)}(x)d\mu(x),\qquad k=0,\ 1,\ \dots,n.$$

From Corollary 2.1, the Fourier coefficients of the function $g_n^{\alpha,\beta,j}(x)$ are

$$\begin{aligned}
&\frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\,(g_n^{\alpha,\beta,j})\hat{}\,(k)\\
&=\int_{-1}^{1}(1-x^2)^j p_n^{(\alpha+j,\beta+j)}(x)p_k^{(\alpha,\beta,M,N)}(x)\omega_{\alpha,\beta}(x)dx\\
&=a_k\int_{-1}^{1}(1-x^2)^j p_n^{(\alpha+j,\beta+j)}(x)p_k^{(\alpha,\beta)}(x)\omega_{\alpha,\beta}(x)dx\\
&+b_k\int_{-1}^{1}(1-x^2)^j p_n^{(\alpha+j,\beta+j)}(x)(x+1)p_{k-1}^{(\alpha,\beta+2)}(x)\omega_{\alpha,\beta}(x)dx\\
&+c_k\int_{-1}^{1}(1-x^2)^j p_n^{(\alpha+j,\beta+j)}(x)(x-1)p_{k-1}^{(\alpha+2,\beta)}(x)\omega_{\alpha,\beta}(x)dx\\
&-d_k\int_{-1}^{1}(1-x^2)^j p_n^{(\alpha+j,\beta+j)}(x)(1-x^2)p_{k-2}^{(\alpha+2,\beta+2)}(x)\omega_{\alpha,\beta}(x)dx\\
&=I_1^{k,n}+I_2^{k,n}+I_3^{k,n}-I_4^{k,n}
\end{aligned}$$

where $k=0,\ 1,\ \dots,n$, and it is assumed $p_i^{(\cdot,\cdot)}(x)=0$ for $i=-1,-2$.

According to the [12, (2.8)] and [14, (4.3.4)] we get

$$(3.2)\quad (1-x^2)^j p_n^{(\alpha+j,\beta+j)}(x)=\{h_n^{(\alpha+j,\beta+j)}\}^{-1/2}\sum_{m=0}^{2j}b_{m,j}(\alpha,\beta,n)\{h_{n+m}^{(\alpha,\beta)}\}^{1/2}p_{n+m}^{(\alpha,\beta)}(x),$$

where $h_n^{(\alpha,\beta)}\cong 2^{\alpha+\beta}n^{-1}$, $b_{0,j}(\alpha,\beta,n)\cong 4^j$ and $b_{2j,j}(\alpha,\beta,n)\cong(-4)^j$.

Taking into account (3.2)

$$I_1^{k,n}=A_k\{h_n^{(\alpha+j,\beta+j)}\}^{-1/2}\sum_{m=0}^{2j}b_{m,j}(\alpha,\beta,n)\{h_{n+m}^{(\alpha,\beta)}\}^{1/2}\times\int_{-1}^{1}p_{n+m}^{(\alpha,\beta)}(x)p_k^{(\alpha,\beta)}(x)\omega_{\alpha,\beta}(x)dx.$$

Thus

$$\text{(3.3)}\qquad \begin{cases} I_1^{k,n} = 0, & 0 \le k \le n-1, \\ I_1^{n,n} \cong 2^j a_n, & m = 0, \\ I_1^{n,n} = 0, & 0 < m \le 2j. \end{cases}$$

Again, for $k \ge 1$, according to the [12, (2.8)] and [14, (4.3.4)]

$$\begin{aligned} I_2^{k,n} &= b_k \{h_n^{(\alpha+j,\beta+j)}\}^{-1/2} \{h_{k-1}^{(\alpha,\beta+2)}\}^{-1/2} \\ &\quad \times \int_{-1}^{1} (1-x^2)^j P_n^{(\alpha+j,\beta+j)}(x)(x+1) P_{k-1}^{(\alpha,\beta+2)}(x) \omega_{\alpha,\beta}(x) dx \\ &= b_k \{h_n^{(\alpha+j,\beta+j)}\}^{-1/2} \{h_{k-1}^{(\alpha,\beta+2)}\}^{-1/2} \sum_{m=0}^{2j} b_{m,j}(\alpha,\beta,n) \\ &\qquad \times \int_{-1}^{1} P_{n+m}^{(\alpha,\beta)}(x)(x+1) P_{k-1}^{(\alpha,\beta+2)}(x) \omega_{\alpha,\beta}(x) dx. \end{aligned}$$

Since (see [14, (4.5.4)])

$$\begin{aligned} (x+1) P_{k-1}^{(\alpha,\beta+2)}(x) &= \frac{2k}{2k+\alpha+\beta+1} P_k^{(\alpha,\beta+1)}(x) \\ &\quad + \frac{2(k+\beta+1)}{2k+\alpha+\beta+1} P_{k-1}^{(\alpha,\beta+1)}(x) \end{aligned}$$

and $\deg P_{k-1}^{(\alpha,\beta+1)} \le n-1$, we have

$$\begin{aligned} I_2^{k,n} &= \frac{2kb_k}{2k+\alpha+\beta+1} \{h_n^{(\alpha+j,\beta+j)}\}^{-1/2} \{h_{k-1}^{(\alpha,\beta+2)}\}^{-1/2} \\ &\quad \times \sum_{m=0}^{2j} b_{m,j}(\alpha,\beta,n) \int_{-1}^{1} P_{n+m}^{(\alpha,\beta)}(x)\ P_k^{(\alpha,\beta+1)}(x) \omega_{\alpha,\beta}(x) dx. \end{aligned}$$

Formula 16.4 (11) in [4, p. 285] shows that

$$\begin{aligned} \text{(3.4)}\quad & \int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\ P_n^{(\alpha,\beta+1)}(x)\ \omega_{\alpha,\beta}(x) dx \\ &= \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(n+1)\Gamma(\alpha+\beta+n+2)} \cong 2^{\alpha+\beta+1} n^{-1}. \end{aligned}$$

This formula can also be proved by using identity (9.4.1) in [14, p. 256]. Thus

$$\begin{cases} I_2^{k,n} = 0, & 0 \le k \le n-1, \\ I_2^{n,n} \cong 2^j b_n, & m = 0, \\ I_2^{n,n} = 0, & 0 < m \le 2j. \end{cases} \tag{3.5}$$

In a similar way, for $k \ge 1$, using symmetry $p_n^{(\alpha,\beta)}(-x) = (-1)^n p_n^{(\beta,\alpha)}(x)$ we find that

$$\begin{cases} I_3^{k,n} = 0, & 0 \le k \le n-1, \\ I_3^{n,n} \cong 2^j c_n, & m = 0, \\ I_3^{n,n} = 0, & 0 < m \le 2j. \end{cases} \tag{3.6}$$

Finally, for $k \ge 2$

$$\begin{aligned} I_4^{k,n} = d_k \{h_n^{(\alpha+j,\beta+j)}\}^{-1/2} \{h_{k-2}^{(\alpha+2,\beta+2)}\}^{-1/2} \sum_{m=0}^{2j} b_{m,j}(\alpha,\beta,n) \\ \times \int_{-1}^{1} P_{n+m}^{(\alpha,\beta)}(x)(1-x^2) P_{k-2}^{(\alpha+2,\beta+2)}(x)\omega_{\alpha,\beta}(x)dx. \end{aligned} \tag{3.7}$$

On the other hand, from [12, (2.8)] and [14, (4.5.3)] we have the following relations

$$\begin{aligned} (1-x^2)P_{k-2}^{(\alpha+2,\beta+2)}(x) = b_{2,1}(\alpha+1,\beta+1,n)P_k^{(\alpha+1,\beta+1)}(x) \\ + b_{1,1}(\alpha+1,\beta+1,n)P_{k-1}^{(\alpha+1,\beta+1)}(x) \\ + b_{0,1}(\alpha+1,\beta+1,n)P_{k-2}^{(\alpha+1,\beta+1)}(x) \end{aligned}$$

and

$$P_n^{(\alpha+1,\beta+1)}(x) = \frac{2n+\alpha+\beta+1}{n+\alpha+\beta+1} P_n^{(\alpha,\beta+1)}(x) + \sum_{k=0}^{n-1} C_{n,k} P_k^{(\alpha,\beta+1)}(x).$$

By using these relations and (3.4) we also find from (3.7) that

$$\begin{cases} I_4^{k,n} = 0, & 0 \le k \le n-1, \\ I_4^{n,n} \cong -4 \cdot 2^j d_n, & m = 0, \\ I_4^{n,n} = 0, & 0 < m \le 2j. \end{cases} \tag{3.8}$$

As a conclusion, by using (3.3), (3.5), (3.6), (3.8), and Corollary 2.1 we find that:

i) If $M > 0,\ N = 0$, then

$$\begin{cases} (g_n^{\alpha,\beta,j})\hat{}\,(k) = 0, & 0 \le k \le n-1, \\ (g_n^{\alpha,\beta,j})\hat{}\,(n) \cong 2^j, \end{cases} \tag{3.9}$$

ii) If $M > 0,\ N > 0$, then

$$\begin{cases} (g_n^{\alpha,\beta,j})\hat{}\,(k) = 0, & 0 \le k \le n-1, \\ (g_n^{\alpha,\beta,j})\hat{}\,(n) \cong 4 \cdot 2^j. \end{cases} \tag{3.10}$$

On the other hand, from [12, (3.1)]

$$\|g_n^{\alpha,\beta,j}\|^p_{L^p(d\mu)} = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \|g_n^{\alpha,\beta,j}\|^p_{L^p(\omega_{\alpha,\beta}dx)} \le c \tag{3.11}$$

for $j > \alpha + \frac{1}{2} - (2\alpha+2)/p > \beta + \frac{1}{2} - (2\beta+2)/p$ and $q_0 \le p \le \infty$.

We are now in position to prove Theorem.

Proof of Theorem. Let $\alpha \ge \beta \ge -1/2$ and $\alpha > -1/2$. We assume that $q_0 \le p \le b$, where

$$\begin{cases} b = \infty & \text{if } M > 0,\ N = 0, \\ b < \infty & \text{if } M > 0,\ N > 0. \end{cases}$$

The assertion for $a \le p \le p_0$, a is the index dual to b, then follows by duality. From (3.1), (3.9), (3.10), and (3.11) we have

$$\begin{aligned} \|T_n^{\alpha,\beta,M,N}\|_{[L^p(d\mu)]} &\ge [\|g_n^{\alpha,\beta,j}\|_{L^p(d\mu)}]^{-1} \|T_n^{\alpha,\beta,M,N} g_n^{\alpha,\beta,j}\|_{L^p(d\mu)} \\ &\ge c|c_{n,n}|\ \|p_n^{(\alpha,\beta,M,N)}\|_{L^p(d\mu)}. \end{aligned} \tag{3.12}$$

By Proposition 2.4 we have

$$\|p_n^{(\alpha,\beta,M,N)}\|_{L^p(d\mu)} \ge c \begin{cases} (\log n)^{1/p} & \text{if } p = q_0, \\ n^{(2\alpha+1)/2-(2\alpha+2)/p} & \text{if } q_0 < p < \infty. \end{cases}$$

On combining this and [5, (13)] with (3.12), the statement is seen to be true. □

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