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# *Logarithmic Abelian Varieties, Part I: Complex Analytic Theory*

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*Dedicated to Professor Luc Illusie on his sixtieth birthday*

**Abstract.** We introduce the notions log complex torus and log abelian variety over $\mathbb{C}$, which are new formulations of degenerations of complex torus and abelian variety over $\mathbb{C}$, and which have group structures. We compare them with the theory of log Hodge structures. A main result is that the category of the log complex tori (resp. log abelian varieties) is equivalent to that of the log Hodge structures (resp. fiberwise-polarizable log Hodge structures) of type $(-1,0)+(0,-1)$. The toroidal compactifications of the Siegel spaces are the fine moduli of polarized log abelian varieties with level structure and with the fixed type of local monodromy with respect to the corresponding cone decomposition. In virtue of the fact that log abelian varieties have group structures, we can also show this with a fixed coefficient (rigidified) ring of endomorphisms. The Satake-Baily-Borel compactifications are, in a sense, the coarse moduli. Classical theories of semi-stable degenerations of abelian varieties over $\mathbb{C}$ can be regarded in our theory as theories of proper models of log abelian varieties.

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## Introduction

This is Part I of our series of papers on log abelian varieties.

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Log abelian varieties are some kind of degenerations of abelian varieties. In the usual geometry, degenerations of abelian varieties have singularities and can not have group laws. However log abelian varieties do not live in the world of usual geometry, but they are group objects which live in the world of log geometry in the sense of Fontaine-Illusie.

In this Part I, we consider the complex analytic theory. A main subject is to compare log abelian varieties with log Hodge structures.

In the classical analytic theory, there are equivalences of categories

$$\text{(1)} \qquad \{\text{complex torus } A \text{ over } \mathbb{C}\} \simeq \{\text{Hodge structure } H \text{ of weight } -1 \text{ such that } F^{-1}H_{\mathbb{C}} = H_{\mathbb{C}},\ F^1H_{\mathbb{C}} = 0\},$$

$$\text{(2)} \qquad \{\text{abelian variety } A \text{ over } \mathbb{C}\} \simeq \{\text{polarizable Hodge structure } H \text{ of weight } -1 \text{ such that } F^{-1}H_{\mathbb{C}} = H_{\mathbb{C}},\ F^1H_{\mathbb{C}} = 0\}.$$

(1) and (2) are given by

$$\text{(3)} \qquad A = H_{\mathbb{Z}}\backslash H_{\mathbb{C}}/F^0H_{\mathbb{C}}.$$

We can write the correspondence also as

$$\text{(4)} \qquad A = \operatorname{Ext}^1(\mathbb{Z}, H),$$

where $\operatorname{Ext}^1$ is taken for the category of mixed Hodge structures (see for example 3.1.1).

We generalize the equivalences (1) and (2) to the logarithmic situation, replacing complex torus by log complex torus, abelian variety by log abelian variety, Hodge structure of weight $-1$ by log Hodge structure of weight $-1$, and $\operatorname{Ext}^1$ in (4) by $\operatorname{Ext}^1$ for the category of log mixed Hodge structures. In the log situation, the presentation (3) is replaced by a multiplicative presentation of $A$. Logarithmic Hodge structures are some kind of degenerations of Hodge structures. In [9] 1.8.15, Deligne explained his philosophy on the formulation of “good” degenerations of Hodge structures, and Steenbrink and Zucker realized this philosophy in [34]. The notion of log Hodge structures is a kind of refinement of the notion of “good” degenerations of

Hodge structures considered in [34]. We will prove the following (§3 Theorem 3.1.5). (See [16] for the definition and fundamental properties of fs log analytic spaces.)

0.1. Theorem. *Let $S$ be an fs log analytic space. Then we have an equivalence of categories*

$$\{\textit{log complex torus } A \textit{ over } S\} \simeq \{\textit{log Hodge structure}\, H \textit{ of weight } -1 \textit{ over } S \textit{ such that}\, F^{-1}H_{\mathcal{O}} = H_{\mathcal{O}},\ F^{1}H_{\mathcal{O}} = 0\},$$

$$\{\textit{log abelian variety } A \textit{ over } S\} \simeq \{\textit{fiberwise-polarizable log Hodge structure}\, H \textit{ of weight } -1 \textit{ such that}\, F^{-1}H_{\mathcal{O}} = H_{\mathcal{O}},\ F^{1}H_{\mathcal{O}} = 0\},$$

*which are given by*

$$A = \mathcal{E}xt^1(\mathbb{Z}, H).$$

*Here $\mathcal{E}xt^1$ is taken for the category of log mixed Hodge structures.*

The elliptic curve $\{\mathbb{C}^\times/q^{\mathbb{Z}}\}_{q\in\Delta-\{0\}}$ over $\Delta-\{0\}$ (the standard family of elliptic curves parametrized by $\Delta-\{0\}$, where $\Delta=\{q\in\mathbb{C}\,|\,|q|<1\}$) does not extend to an elliptic curve over $\Delta$, for it degenerates at $q=0$. The theory of degeneration is complicated, in the sense that there are many proper models of this family over $\Delta$. This family, however, extends uniquely to a log elliptic curve (= 1-dimensional log abelian variety) over $\Delta$, where $\Delta$ is endowed with the log structure associated to $\{0\}\subset\Delta$. Furthermore, proper models over $\Delta$ can be found in this uniquely extended log elliptic curve. See §1.1 for this, and see §5 for the generalization of this to log abelian varieties. With the group structure and with such uniqueness, we present a new theory of degenerations of abelian varieties, and our analytic theory of log abelian varieties has essentially the same simple form as the analytic theory of abelian varieties in Chapter 1 of the textbook [21] of Mumford.

We will also prove that the toroidal compactification of the moduli space of polarized abelian varieties with respect to a fixed admissible cone decomposition is the moduli space of polarized log abelian varieties with a fixed type of local monodromy. In virtue of the fact that log abelian varieties have group structures, we can do this with a fixed coefficient (rigidified) ring of endomorphisms. We prove that the Satake-Baily-Borel compactification of the moduli space of polarized abelian varieties is, in a sense, the coarse moduli space of polarized log abelian varieties (without fixing the type of local monodromy). See §4.

From Part II, we will develop the algebraic theory (not the analytic theory) of log abelian varieties.

We mention some related works. M. C. Olsson [29] proves an equivalence of the category of log elliptic curves in his sense and that of log Hodge structures. See also [30]. See also N. Nakayama's study [24] of elliptic fibrations.

We are very happy to dedicate our series of papers on log abelian varieties to Professor Illusie, who is a pioneer of log geometry and who encouraged us constantly. We started to write the series hoping to dedicate them to his 60th birthday, but we are sorry that we completed this Part I rather late. We are very thankful to the referees for giving us many valuable comments. The first and the third authors are partly supported by the Grants-in-Aid for Encouragement of Young Scientists, the Ministry of Education, Culture, Sports, Science and Technology, Japan.

**Terminology.** In this paper, a *monoid* means a commutative semigroup with a unit element. A homomorphism of monoids is assumed to respect the unit elements, and a submonoid of a monoid is assumed to share the unit element.

An *ideal* of a monoid $\mathcal{S}$ is a subset $I$ of $\mathcal{S}$ such that $si \in I$ for any $s \in \mathcal{S}$ and $i \in I$. A *prime ideal* of $\mathcal{S}$ is an ideal of $\mathcal{S}$ whose complement is a submonoid of $\mathcal{S}$. The complement of a prime ideal of $\mathcal{S}$ is called a *face* of $\mathcal{S}$.

The *interior* of $\mathcal{S}$ is the intersection of all nonempty prime ideals of $\mathcal{S}$.

See [16] or Appendix of [15] for basic terms in (analytic) log geometry (fs monoids, fs log analytic spaces and so on). See also [14].

## 1. Log Abelian Varieties and Log Complex Tori

In §1.1, we give an example of a log elliptic curve (= 1-dimensional log abelian variety). After preparations in §1.2, we will define "log complex torus" and "log abelian variety (in the complex analytic situation)" in §1.3.

### 1.1. An example of a log elliptic curve (= 1-dimensional log abelian variety)

**1.1.1.** Let

$$\Delta = \{q \in \mathbb{C} \mid |q| < 1\}$$

be the unit disc. Let $E = \{\mathbb{C}^\times/q^{\mathbb{Z}}\}_{q\in\Delta-\{0\}} \to \Delta - \{0\}$ be the well-known family of elliptic curves over $\Delta - \{0\}$. We say "$E$ is an elliptic curve over $\Delta - \{0\}$". We will identify $E$ with the functor represented by $E$

$$\mathrm{Mor}_{\Delta-\{0\}}(?, E) : (\mathrm{an}/(\Delta - \{0\})) \to (\mathrm{Set}),$$

which is isomorphic to the quotient sheaf $\mathbb{G}_m/q^{\mathbb{Z}}$ on $(\mathrm{an}/(\Delta - \{0\}))$. Here $(\mathrm{an}/(\Delta - \{0\}))$ denotes the category of the analytic spaces over $\Delta - \{0\}$, $\mathbb{G}_m$ denotes the sheaf on $(\mathrm{an}/(\Delta - \{0\}))$ defined by

$$\mathbb{G}_m(T) = \Gamma(T, \mathcal{O}_T^\times),$$

$q$ in the notation $\mathbb{G}_m/q^{\mathbb{Z}}$ denotes the coordinate function of $\Delta$, and hence $\mathbb{G}_m/q^{\mathbb{Z}}$ denotes the sheaf defined by $(\mathbb{G}_m/q^{\mathbb{Z}})(T) = \Gamma(T, \mathcal{O}_T^\times/q^{\mathbb{Z}})$.

As is well-known, $E$ does not extend to an elliptic curve over $\Delta$.

**1.1.2.** Now endow $\Delta$ with the log structure $M_\Delta$ corresponding to the origin of $\Delta$. That is, $M_\Delta = \mathcal{O}_\Delta^\times \cdot \{q^n \mid n \geq 0\} \subset \mathcal{O}_\Delta$. Let $(\mathrm{fs}/\Delta)$ be the category of fs log analytic spaces over $\Delta$.

Then by the method of this paper, $E$ extends uniquely to a log elliptic curve (= 1-dimensional log abelian variety) $\bar{E}$ over $\Delta$, as follows: $\bar{E}$ is defined to be the sheaf $\mathbb{G}_{m,\log}^{(q)}/q^{\mathbb{Z}}$ of abelian groups on the category $(\mathrm{fs}/\Delta)$, where $\mathbb{G}_{m,\log}$ is the sheaf on $(\mathrm{fs}/\Delta)$ defined by

$$\mathbb{G}_{m,\log}(T) = \Gamma(T, M_T^{\mathrm{gp}}),$$

and $\mathbb{G}_{m,\log}^{(q)}$ denotes the subsheaf of $\mathbb{G}_{m,\log}$ defined by

$$\mathbb{G}_{m,\log}^{(q)}(T) = \{f \in \Gamma(T, M_T^{\mathrm{gp}}) \mid \text{locally on } T, \text{ there exist } m, n \in \mathbb{Z} \text{ such that } q^m|f|q^n\}.$$

Here $M_T^{\mathrm{gp}}$ is the sheaf of abelian groups associated to $M_T$ and for local sections $f, g \in M_T^{\mathrm{gp}}$, $f|g$ means $f^{-1}g \in M_T$. See §1.2 for details.

**1.1.3.** Examples of models in $\bar{E}$. $\bar{E}$ is not representable, but it has many representable big subsheaves $P, P', \dots$, which are proper models of $E$ over $\Delta$ in the usual algebraic geometry. Let $P = \tilde{P}/q^{\mathbb{Z}}$ (resp. $P' = \tilde{P}'/q^{\mathbb{Z}}$), where $\tilde{P}$ (resp. $\tilde{P}'$) is the subsheaf of $\mathbb{G}_{m,\log}$, which is stable under the action of $q^{\mathbb{Z}}$, defined by

$$\tilde{P}(T) = \{f \in \Gamma(T, M_T^{\mathrm{gp}}) \,|\, \text{locally on } T, \text{ there exists } n \in \mathbb{Z} \text{ such that } q^n|f|q^{n+1}\}$$

$$\tilde{P}'(T) = \{f \in \Gamma(T, M_T^{\mathrm{gp}}) \,|\, \text{locally on } T, \text{ there exists } n \in \mathbb{Z} \text{ such that } q^n|f^2|q^{n+1}\}.$$

Then the inclusions $\tilde{P}' \subset \tilde{P} \subset \mathbb{G}_{m,\log}^{(q)}$ induce inclusions $P' \subset P \subset \bar{E}$. Here the inclusion $\tilde{P}' \subset \tilde{P}$ is shown as

$$q^n|f^2|q^{n+1} \Rightarrow q^{2[n/2]}|f^2|q^{2([n/2]+1)} \Rightarrow q^{[n/2]}|f|q^{[n/2]+1}.$$

These $P$ and $P'$ are represented by fs log analytic spaces over $\Delta$ which are proper over $\Delta$. The inverse images to $\Delta - \{0\}$ of $P$ and of $P'$ are identified with $E$, the fiber of $P$ over $0 \in \Delta$ is isomorphic to the quotient of $\mathbb{P}^1_{\mathbb{C}}$ by identifying $0$ and $\infty$ (this quotient of $\mathbb{P}^1_{\mathbb{C}}$ is regarded as the limit of $\mathbb{C}^\times/q^{\mathbb{Z}}$ for $q \to 0$), $P'$ is the blowing-up of $P$ at this point $0 = \infty$, and $P$ and $P'$ are endowed with the log structures associated to the fibers over $0 \in \Delta$. (This is easily seen by the fact that, for a fixed $n$, the subsheaf $T \mapsto \{f \in \Gamma(T, M_T^{\mathrm{gp}}) \,|\, q^n|f|q^{n+1}$ (resp. $q^n|f^2|q^{n+1}$)$\}$ of $\tilde{P}$ (resp. $\tilde{P}'$) is represented by the fs log analytic space $(\operatorname{Spec} \mathbb{C}[N_n] \times_{\operatorname{Spec} \mathbb{C}[\mathbb{N}]} \Delta, N_n^a)$, where $N_n$ is the free monoid $\{(a, b) \in \mathbb{Z}^2 \,|\, a + nb \geq 0, a + (n+1)b \geq 0$ (resp. $2a + nb \geq 0, 2a + (n+1)b \geq 0$)$\}$, $\mathbb{N} \to \mathcal{O}_\Delta$ sends 1 to $q$, $\mathbb{N} \to N_n$ sends 1 to $(1, 0)$, and $N_n^a$ denotes the log structure associated to the pre-log structure induced by the natural homomorphism $N_n \to \mathbb{C}[N_n]$.)

A remarkable thing is that, though the map $P' \to P$ is not injective as a map of analytic spaces, the map $P' \to P$ is injective as a map of sheaves on $(\mathrm{fs}/\Delta)$. By putting log structures, the sets of morphisms to these $P$ and

$P'$ differ from those of morphisms in the category of analytic spaces, and this is the reason why degenerate objects can behave in log geometry like non-degenerate objects.

$E$ extends to many proper models $P$, $P', \dots$ over $\Delta$, but extends uniquely to a log elliptic curve $\bar{E}$ over $\Delta$. Furthermore, though $P$, $P', \dots$ are not group objects, $\bar{E}$ is a sheaf of abelian groups. In these points, our theory simplifies the usual theory of degeneration of abelian varieties.

**1.1.4.** Here we remark that our moduli problems are considered in the category of log analytic spaces, so they differ from the usual moduli problems in the category of analytic spaces. A reader who is familiar with moduli problems in the category of analytic spaces (with no log structure) may have the following questions, for example.

Question 1. Although the examples above show only the one-dimensional case, how does our theory work in the higher dimensional case, for example, that of dimension 2?

Question 2. Let $p$ be a point of the underlying analytic space of the moduli space of log abelian varieties (with additional structures for fixed data), and endow $p$ with the pull-back log structure from the moduli space. Denote by $p'$ this fs log point to avoid confusion. Denote by $A_{p'}$ the log abelian variety (with an additional structure) over $p'$ which is the pull-back of the universal family over the moduli space with respect to the natural inclusion morphism. Are there any morphisms from $p'$ to the moduli space whose images equal $\{p\}$ other than the natural inclusion morphism? If so, do they give log abelian varieties over $p'$ (with additional structures) which are not isomorphic to $A_{p'}$?

Here is our answer to Question 1. Our canonical degenerated objects, i.e., log abelian varieties are abelian sheaves defined by means of log structures, such as $\mathbb{G}_{m,\log}^{(q)}/q^{\mathbb{Z}}$ in the case of log Tate curves in 1.1.2. Even in the higher dimensional cases, log abelian varieties are very similar to log Tate curves. For example, consider a two-dimensional log abelian variety $A := (\mathbb{G}_{m,\log}^{(q)}/q^{\mathbb{Z}}) \times (\mathbb{G}_{m,\log}^{(q)}/q^{\mathbb{Z}}) = (\mathbb{G}_{m,\log}^{\oplus 2})^{(q,q)}/q^{\mathbb{Z}} \oplus q^{\mathbb{Z}}$ over $\Delta$, where

$(\mathbb{G}_{m,\log}^{\oplus 2})^{(q,q)}$ is the subsheaf of $\mathbb{G}_{m,\log}^{\oplus 2}$ defined by

$$(\mathbb{G}_{m,\log}^{\oplus 2})^{(q,q)}(T) = \{(f_1, f_2) \in \Gamma(T, M_T^{\mathrm{gp}})^2;\ \text{locally on } T,\ \text{there exist } m, n \in \mathbb{Z} \text{ such that } q^m | f_1 | q^n \text{ and } q^m | f_2 | q^n\}$$

on which $\mathbb{Z}^2$ acts as $(f_1, f_2) \mapsto (q^{y_1} f_1, q^{y_2} f_2)$ for $(f_1, f_2) \in (\mathbb{G}_{m,\log}^{\oplus 2})^{(q,q)}$, and $(y_1, y_2) \in \mathbb{Z}^2$. As is similar to the one-dimensional case, the abelian surface $(\mathbb{G}_m/q^{\mathbb{Z}}) \times (\mathbb{G}_m/q^{\mathbb{Z}})$ over $\Delta - \{0\}$ extends uniquely to the log abelian surface $A$. Clearly, this log abelian surface depends only on $q$, and does not need any additional structures such as relatively complete models in the Mumford construction. It is clear that the products $P \times P$, $P \times P'$, $P' \times P$, $P' \times P'$ are contained in $A$, where $P$ and $P'$ are the proper models in the one-dimensional case in 1.1.3. Besides them, it is easily seen that $A$ also contains another degenerate abelian surface $P'' = \tilde{P}''/q^{\mathbb{Z}} \oplus q^{\mathbb{Z}}$. Here $\tilde{P}''$ is the subsheaf of $\mathbb{G}_{m,\log}^{\oplus 2}$ defined by

$$\begin{aligned}\tilde{P}''(T) = \{&(f_1, f_2) \in \Gamma(T, M_T^{\mathrm{gp}})^2;\ \text{locally on } T, \text{ there exist } n_1, n_2 \in \mathbb{Z} \\ &\text{such that } q^{n_i} | f_i | q^{n_i+1} (i = 1, 2),\ \text{and that } q^{n_1+n_2} | f_1 f_2 | q^{n_1+n_2+1}\} \\ \cup\, \{&(f_1, f_2) \in \Gamma(T, M_T^{\mathrm{gp}})^2;\ \text{locally on } T, \text{ there exist } n_1, n_2 \in \mathbb{Z} \\ &\text{such that } q^{n_i} | f_i | q^{n_i+1} (i = 1, 2),\ \text{and that } q^{n_1+n_2+1} | f_1 f_2 | q^{n_1+n_2+2}\}.\end{aligned}$$

The above proper models can be described in terms of fans in a certain space. See Section 5 for the detail.

Here is our answer to Question 2. The answer to the former question is "yes" as soon as the log structure of $p'$ is not trivial, because in the category of log analytic spaces, (fs log) points can have non-trivial automorphisms. Since our space is the fine moduli, the answer to the latter is also "yes." Let us explain it more concretely. For example, if $s$ is the log point that is the origin of $\Delta$ in 1.1.2 endowed with the inverse image log structure of that of $\Delta$, then, the pull-back $\bar{E}_s$ of $\bar{E}$ in 1.1.2 to $s$ is a log elliptic curve over $s$. The log point $s$ has non-trivial automorphisms, and the automorphism group $\mathrm{Aut}\,(s)$ is isomorphic to $\mathbb{C}^\times$. If $g \in \mathrm{Aut}\,(s)$ is a non-trivial automorphism, then the pull-back of $\bar{E}_s$ by $g$ is not isomorphic to $\bar{E}_s$ over $s$, as is easily seen via the equivalence 3.1.5 (see also 2.4).

## 1.2. Pairings into $\mathbb{G}_{m,\log}$

**1.2.1.** We recall multiplicative presentations of complex tori and abelian varieties over $\mathbb{C}$.

A commutative Lie group $A$ over $\mathbb{C}$ of dimension $g$ is a complex torus if and only if there exist finitely generated free $\mathbb{Z}$-modules $X, Y$ of rank $g$ and a $\mathbb{Z}$-bilinear form $\langle\ ,\ \rangle : X \times Y \to \mathbb{C}^\times$ such that

$$\mathbb{R} \otimes X \times \mathbb{R} \otimes Y \to \mathbb{R}\ ;\ (x, y) \mapsto \log(|\langle x, y\rangle|) \quad (x \in X, y \in Y)$$

is a non-degenerate pairing of $\mathbb{R}$-vector spaces, and $A$ is isomorphic to the cokernel of $Y \to \mathrm{Hom}\,(X, \mathbb{C}^\times)$.

A commutative Lie group $A$ over $\mathbb{C}$ of dimension $g$ is an abelian variety over $\mathbb{C}$ if and only if there exist $X, Y, \langle\ ,\ \rangle$ as above and an injective homomorphism $p : Y \to X$ (polarization) such that

$$\mathbb{R} \otimes Y \times \mathbb{R} \otimes Y; (y, z) \mapsto \log(|\langle p(y), z\rangle|) \quad (y, z \in Y)$$

is symmetric and negative definite.

**1.2.2.** Let $G$ be an abelian group or a sheaf of abelian groups on a site. By a pairing into $G$, we mean a triple $(X, Y, \langle\ ,\ \rangle)$, where $X$ and $Y$ are finitely generated free abelian groups, and $\langle\ ,\ \rangle$ is a $\mathbb{Z}$-bilinear form

$$\langle\ ,\ \rangle \colon X \times Y \to G.$$

Let $S$ be an fs log analytic space ([16]). Let $(\mathrm{fs}/S)$ be the category of fs log analytic spaces over $S$ endowed with the usual topology. That is, a covering of an fs log analytic space $T$ over $S$ is a usual open covering $(U_\lambda)_\lambda$ of the underlying topological space of $T$, where each $U_\lambda$ is endowed with the inverse image of $\mathcal{O}_T$ and the inverse image of the log structure of $T$.

In the next subsection, for an fs log analytic space $S$, we will define the notions of log complex torus over $S$ and log abelian variety over $S$ as sheaves of commutative groups on $(\mathrm{fs}/S)$, by using pairings into $\mathbb{G}_{m,\log,S}$. Here $\mathbb{G}_{m,\log,S}$ denotes the sheaf $T \mapsto \Gamma(T, M_T^{\mathrm{gp}})$ on $(\mathrm{fs}/S)$ (hence giving a pairing into the sheaf $\mathbb{G}_{m,\log,S}$ on $(\mathrm{fs}/S)$ is equivalent to giving a pairing into the sheaf $M_S^{\mathrm{gp}}$ on $S$). In this subsection, we give necessary preparations on such pairings.

**1.2.3.** Admissible pairings. Let $\mathcal{S}$ be an fs monoid and let $(X, Y, \langle\ ,\ \rangle)$ be a pairing into $\mathcal{S}^{\mathrm{gp}}$.

(1) For a face $\sigma$ of $\mathcal{S}$, we define

$$X_\sigma = \{x \in X \mid \langle x, y\rangle \in \sigma^{\mathrm{gp}} \text{ for all } y \in Y\},$$

$$Y_\sigma = \{y \in Y \mid \langle x, y\rangle \in \sigma^{\mathrm{gp}} \text{ for all } x \in X\}.$$

(2) $(X, Y, \langle\ ,\ \rangle)$ is said to be $\mathcal{S}$-admissible if the following (i) is satisfied.

(i) For any face $\sigma$ of $\mathcal{S}$ and any homomorphism $N : \mathcal{S} \to \mathbb{R}_{\geq 0}$ into the additive monoid $\mathbb{R}_{\geq 0} = \mathbb{R}_{\geq 0}^{(+)} = \{r \in \mathbb{R} \mid r \geq 0\}$, if we denote the face $\sigma \cap \mathrm{Ker}\,(N)$ of $\mathcal{S}$ by $\tau$, then the pairing of $\mathbb{R}$-linear spaces

$$\mathbb{R} \otimes (X_\sigma/X_\tau) \times \mathbb{R} \otimes (Y_\sigma/Y_\tau) \to \mathbb{R}; \quad (x, y) \mapsto N(\langle x, y\rangle)$$

is non-degenerate. Here $\mathrm{Ker}\,(N) = \{a \in \mathcal{S} \mid N(a) = 0\}$, and the group homomorphism $\mathcal{S}^{\mathrm{gp}} \to \mathbb{R}$ induced from $N$ is also denoted by $N$ by abuse of notation.

We sometimes simply say that $(X, Y, \langle\ ,\ \rangle)$ is admissible if it is $\mathcal{S}$-admissible and $\mathcal{S}$ is clear from the context.

1.2.4. DEFINITION. Let $S$ be an fs log analytic space. A pairing $(X, Y, \langle\ ,\ \rangle)$ into $M_S^{\mathrm{gp}}/\mathcal{O}_S^\times$ is said to be admissible if the induced pairing into $M_{S,s}^{\mathrm{gp}}/\mathcal{O}_{S,s}^\times$ is $M_{S,s}/\mathcal{O}_{S,s}^\times$-admissible for any $s \in S$.

1.2.5. LEMMA. *Let $\mathcal{S}$ be an fs monoid and let $(X, Y, \langle\ ,\ \rangle)$ be a pairing into $\mathcal{S}^{\mathrm{gp}}$. Let $p : Y \to X$ be a homomorphism satisfying the following conditions (i)–(iii).*

(i) *The cokernel of $p$ is finite.*

(ii) *$\langle p(y), z\rangle = \langle p(z), y\rangle$ for any $y, z \in Y$.*

(iii) *$\langle p(y), y\rangle \in \mathcal{S}$ for any $y \in Y$.*

*Then $(X, Y, \langle\ ,\ \rangle)$ is $\mathcal{S}$-admissible.*

PROOF. We first show

(1) $$Y_\sigma = p^{-1}(X_\sigma) \quad \text{for any face } \sigma \text{ of } \mathcal{S}.$$

In fact, by the assumption (i), for $y \in Y$, $y \in Y_\sigma$ if and only if $\langle p(z), y\rangle \in \sigma^{\mathrm{gp}}$ for all $z \in Y$. Since $\langle p(z), y\rangle = \langle p(y), z\rangle$ by the assumption (ii), the last condition is equivalent to $p(y) \in X_\sigma$. This proves (1). Next, for a face $\sigma$ of $\mathcal{S}$, let $\sigma(\mathbb{Q}_{\geq 0})$ (resp. $\sigma(\mathbb{R}_{\geq 0})$) be the subset of $\mathbb{Q} \otimes \mathcal{S}^{\mathrm{gp}}$ (resp. $\mathbb{R} \otimes \mathcal{S}^{\mathrm{gp}}$) consisting of all linear combinations of elements of $\sigma$ with coefficients in $\mathbb{Q}_{\geq 0}$ (resp. $\mathbb{R}_{\geq 0}$). Then $\sigma(\mathbb{R}_{\geq 0})$ is the closure of $\sigma(\mathbb{Q}_{\geq 0})$. Since $\langle p(y), y\rangle \in \mathcal{S}(\mathbb{Q}_{\geq 0})$ for any $y \in \mathbb{Q} \otimes Y$ by (iii), we have

$$\langle p(y), y\rangle \in \mathcal{S}(\mathbb{R}_{\geq 0}) \quad \text{for any } y \in \mathbb{R} \otimes Y$$

by the continuity.

To prove the lemma, it is enough to show that, for every face $\sigma$ of $\mathcal{S}$ and every homomorphism $N \colon \mathcal{S} \to \mathbb{R}^{(+)}_{\geq 0}$, the induced pairing

$$\mathbb{R} \otimes Y_\sigma / Y_\tau \times \mathbb{R} \otimes Y_\sigma / Y_\tau \longrightarrow \mathbb{R};\ (y, z) \mapsto N(\langle p(y), z\rangle)$$

is non-degenerate. Here $\tau$ denotes $\sigma \cap \operatorname{Ker} N$. Let us take $y \in \mathbb{R} \otimes Y_\sigma$, and suppose that $N(\langle p(y), z\rangle) = 0$ for all $z \in Y_\sigma$. Since $N(\langle p(y), y\rangle) = 0$, we have $\langle p(y), y\rangle \in \operatorname{Ker}(N \colon \mathbb{R} \otimes \mathcal{S}^{\mathrm{gp}} \longrightarrow \mathbb{R}) \cap \sigma(\mathbb{R}_{\geq 0}) = \tau(\mathbb{R}_{\geq 0})$. To show $y \in \mathbb{R} \otimes (Y_\tau)$, it is enough to show that $\langle p(y), z\rangle \in \mathbb{R} \otimes \tau^{\mathrm{gp}}$ for any $z \in \mathbb{R} \otimes Y$. For this, it is enough to show $N'(\langle p(y), z\rangle) = 0$ for any element $N'$ of $\operatorname{Hom}(\mathcal{S}, \mathbb{R}_{\geq 0})$ such that $\operatorname{Ker}(N') \supset \tau$. Since, for any $z \in \mathbb{R} \otimes Y$,

$$0 \leq N'(\langle p(y^n z), y^n z\rangle) = 2n \cdot N'(\langle p(z), y\rangle) + N'(\langle p(z), z\rangle)$$

for all $n \in \mathbb{Z}$, we have $N'(\langle p(z), y\rangle) = 0$. □

1.2.6. Definition. Non-degenerate pairings into $\mathbb{G}_{m,\log,S}$. A pairing $(X, Y, \langle\ ,\ \rangle)$ into $\mathbb{G}_{m,\log,S}$ is said to be non-degenerate if the following (i) and (ii) are satisfied.

(i) The induced pairing into $M_S^{\mathrm{gp}}/\mathcal{O}_S^\times$ is admissible in the sense of 1.2.4.

(ii) Let $s$ be any point of $S$ and let $\sigma$ be the face $\{1\}$ of $M_{S,s}/\mathcal{O}_{S,s}^\times$. Then the pairing of $\mathbb{R}$-linear spaces

$$\mathbb{R} \otimes X_\sigma \times \mathbb{R} \otimes Y_\sigma \to \mathbb{R}; \quad (x, y) \mapsto \log(|\langle x, y\rangle(s)|)$$

is non-degenerate. Here $X_\sigma$ and $Y_\sigma$ are with respect to the induced pairing into $M_{S,s}^{\mathrm{gp}}/\mathcal{O}_{S,s}^\times$, and $\langle x, y\rangle(s) \in \mathbb{C}^\times$ is the value of $\langle x, y\rangle_s \in \mathcal{O}_{S,s}^\times$ at $s$.

1.2.7. Definition. (1) A polarization on a pairing $(X, Y, \langle\ ,\ \rangle)$ into $\mathbb{G}_{m,\log,S}$ is a homomorphism $p : Y \to X$ satisfying the following three conditions (i)–(iii).

(i) $p$ is injective and the cokernel of $p$ is finite.

(ii) $\langle p(y), z\rangle = \langle p(z), y\rangle$ for any $y, z \in Y$.

(iii) For any $y \in Y$, $\langle p(y), y\rangle \in M_S$ in $M_S^{\mathrm{gp}}$. For any $y \in Y - \{0\}$, the map $\alpha : M_S \to \mathcal{O}_S$ sends $\langle p(y), y\rangle$ to a function on $S$ whose values are always of absolute value $< 1$.

(2) A pairing into $\mathbb{G}_{m,\log,S}$ is said to be polarizable if it has a polarization.

1.2.8. Proposition. *A polarizable pairing into $\mathbb{G}_{m,\log,S}$ is non-degenerate.*

Proof. Let $p\colon Y \to X$ be a polarization on a pairing $(X, Y, \langle\ ,\ \rangle)$ into $\mathbb{G}_{m,\log,S}$ and let $s \in S$. By 1.2.5, the condition (i) in 1.2.6 is satisfied. Let $\sigma$ be the face $\{1\}$ of $M_{S,s}/\mathcal{O}_{S,s}^{\times}$. Then we have $p(Y_\sigma) \subset X_\sigma$ ((1) in the proof of 1.2.5). By 1.2.7 (i), $X$ and $Y$ have the same rank so that $X_\sigma$ and $Y_\sigma$ also have the same rank by the admissibility. Thus $\mathbb{R} \otimes Y_\sigma \cong \mathbb{R} \otimes X_\sigma$ via $p$. On the other hand, for any $y \in Y_\sigma - \{0\}$, we have $0 < |\langle p(y), y\rangle(s)| < 1$ and hence $\log|\langle p(y), y\rangle(s)| < 0$. This shows that the pairing of $\mathbb{R}$-vector spaces $\mathbb{R} \otimes Y_\sigma \times \mathbb{R} \otimes Y_\sigma \to \mathbb{R}$ induced by $Y \times Y \to \mathbb{R}; (y, z) \mapsto \log(|\langle p(y), z\rangle(s)|)$ is negative definite. (Note that a symmetric bilinear form over $\mathbb{Q}$ which is definite over $\mathbb{Q}$ is definite over $\mathbb{R}$.) Hence the condition (ii) in 1.2.6 is satisfied. □

1.2.9. Lemma. *(1) Let $\langle\ ,\ \rangle\colon X \times Y \to \mathcal{S}^{\mathrm{gp}}$ be an admissible pairing and let $h\colon \mathcal{S} \to \mathcal{S}'$ be a homomorphism of fs monoids. Then the induced pairing $X \times Y \to (\mathcal{S}')^{\mathrm{gp}}$; $(x, y) \mapsto h(\langle x, y\rangle)$ is also admissible.*

*(2) Let $(X, Y, \langle\ ,\ \rangle)$ be a non-degenerate (resp. admissible) pairing into $\mathbb{G}_{m,\log,S}$ (resp. $M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times}$) and let $T$ be an fs log analytic space over $S$. Then the induced pairing into $\mathbb{G}_{m,\log,T}$ (resp. $M_T^{\mathrm{gp}}/\mathcal{O}_T^{\times}$) is non-degenerate (resp. admissible).*

Proof. We prove (1). Let $\sigma$ be a face of $\mathcal{S}'$ and let $\tau = h^{-1}(\sigma)$. Then $\tau$ is a face of $\mathcal{S}$. It is enough to show that

$$X_\sigma = X_\tau. \tag{$*$}$$

The inclusion $X_\sigma \supset X_\tau$ is clear. Let $N$ be an element of $\mathrm{Hom}\,(\mathcal{S}', \mathbb{R}^{(+)}_{\geq 0})$ with kernel $\sigma$. Then the kernel of $N \circ h \in \mathrm{Hom}\,(\mathcal{S}, \mathbb{R}^{(+)}_{\geq 0})$ is $\tau$. Hence by the admissibility of $(X, Y, \langle\ ,\ \rangle)$, the pairing

$$N \circ h(\langle\ ,\ \rangle) : \mathbb{R} \otimes X/X_\tau \times \mathbb{R} \otimes Y/Y_\tau \to \mathbb{R}$$

is non-degenerate. Since $N \circ h(\langle X_\sigma, Y\rangle) = 0$, this shows $X_\sigma \subset \mathbb{R} \otimes X_\tau$ and thus $X_\sigma \subset X \cap (\mathbb{R} \otimes X_\tau) = X_\tau$.

The part of (2) concerning a pairing into $M_S^{\mathrm{gp}}/\mathcal{O}_S^\times$ follows easily from (1). The part of (2) concerning a pairing into $\mathbb{G}_{m,\log,S}$ follows from it and the equality $(*)$ with $\sigma = \{1\}$ in the proof of (1). $\square$

## 1.3. Log complex tori and log abelian varieties

Let $S$ be an fs log analytic space (see [16] for its definition).

**1.3.1.** For a pairing $(X, Y, \langle\ ,\ \rangle)$ into $\mathbb{G}_{m,\log,S}$, $\mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)}$ denotes the sheaf associating, with each $T \in (\mathrm{fs}/S)$, the abelian group $\{\varphi \in \mathrm{Hom}\,(X, \mathbb{G}_{m,\log,T})$; for each $x \in X$, locally on $T$, there exist $y, y' \in Y$ such that $\langle x, y\rangle | \varphi(x) | \langle x, y'\rangle\}$. Note that we have a natural homomorphism $Y \longrightarrow \mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)}; y \mapsto \langle -, y\rangle$.

**1.3.2.** For a pairing $(X, Y, \langle\ ,\ \rangle)$ into $\mathbb{G}_{m,\log,S}$, the sheaf $\mathrm{Coker}\,(Y \to \mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)})$ of abelian groups is called the quotient associated to $(X, Y, \langle\ ,\ \rangle)$.

**1.3.3.** Example. Let the notation be as in §1.1, and let $S = \Delta$. Recall that $\Delta$ is endowed with the log structure associated to $\{0\} \subset \Delta$. Consider the pairing $(X, Y, \langle\ ,\ \rangle)$ into $\mathbb{G}_{m,\log,S}$, where

$$X = Y = \mathbb{Z}, \quad \langle m, n\rangle = q^{mn}.$$

Then $\mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)}$ coincides with $\mathbb{G}^{(q)}_{m,\log}$ in §1.1, and the quotient associated to $(X, Y, \langle\ ,\ \rangle)$ coincides with $\bar{E}$ in §1.1.

1.3.4. Definition. Log complex tori. A log complex torus over $S$ is a sheaf $A$ of abelian groups on $(\mathrm{fs}/S)$ satisfying the following condition (i) locally on $S$.

(i) There exists a non-degenerate pairing $(X, Y, \langle\ ,\ \rangle)$ into $\mathbb{G}_{m,\log,S}$ such that $A$ is isomorphic to the quotient associated to $(X, Y, \langle\ ,\ \rangle)$.

1.3.5. DEFINITION. Log abelian varieties. A log abelian variety over $S$ is a log complex torus $A$ over $S$ which satisfies the following condition. For any $s \in S$, the pull back of $A$ to $(\mathrm{fs}/s)$ is isomorphic to the quotient associated to a polarizable pairing into $\mathbb{G}_{m,\log}$. Here in $(\mathrm{fs}/s)$, $s$ is endowed with the ring $\mathbb{C}$ and with the inverse image of the log structure of $S$.

**1.3.6.** By 1.2.1, in the case the log structure of $S$ is trivial, a log complex torus over $S$ is nothing but a complex torus over $S$ in the usual sense (i.e., a proper smooth family of complex tori over $S$), and a log abelian variety over $S$ is nothing but a complex torus over $S$ whose all fibers are abelian varieties.

**1.3.7.** For a morphism $T \to S$ of fs log analytic spaces, the pull back of a log complex torus (resp. a log abelian variety) over $S$ to $(\mathrm{fs}/T)$ is a log complex torus (resp. log abelian variety) over $T$. This follows from 1.2.9 (2).

By 1.2.8, we have

1.3.8. PROPOSITION. *Let $A$ be a sheaf of abelian groups on* $(\mathrm{fs}/S)$*. Assume that locally on $S$, there is a polarizable pairing into $\mathbb{G}_{m,\log,S}$ whose associated quotient is isomorphic to $A$. Then $A$ is a log abelian variety over $S$.*

**1.3.9.** We call a log abelian variety having the property in 1.3.8 a locally polarizable log abelian variety.

**1.3.10.** The $\bar{E}$ in §1.1 is a log abelian variety over $\Delta$. In fact, for $X = Y = \mathbb{Z}$ and $\langle\ ,\ \rangle$ as in 1.3.3, the identity map $Y \to X$ is a polarization on $(X, Y, \langle\ ,\ \rangle)$ since $q^{n^2} \in M_\Delta$ for $n \in \mathbb{Z}$ and the values of $q^{n^2}$ on $\Delta$ for $n \in \mathbb{Z} - \{0\}$ are of absolute values $< 1$.

**1.3.11.** We consider the Lie sheaf $\mathrm{Lie}\,(A)$ of a log complex torus $A$.

For a group sheaf $G$ on $(\mathrm{fs}/S)$, we define a group sheaf $\mathrm{Lie}\,(G)$ on $(\mathrm{fs}/S)$, called the Lie sheaf of $G$, by

$$\mathrm{Lie}\,(G)(T) = \mathrm{Ker}\,(G(T[\epsilon]/(\epsilon^2)) \to G(T)),$$

where $T[\epsilon]/(\epsilon^2)$ denotes the fs log analytic space whose underlying topological space is that of $T$, whose sheaf of rings is $\mathcal{O}_T[\epsilon]/(\epsilon^2)$ ($\epsilon$ denotes an indeterminate), and whose log structure is the inverse image of that of $T$.

We denote the structure sheaf of (fs/$S$), defined by $\mathcal{O}_S(T) = \mathcal{O}(T)$, simply by $\mathcal{O}_S$.

We have an action of $\mathcal{O}_S$ on Lie$\,(G)$. For $a \in \mathcal{O}_S(T) = \mathcal{O}(T)$, the action of $a$ on Lie$\,(G)(T)$ is defined to be the map induced by $\epsilon \mapsto a\epsilon$. In general, this is just an action of $\mathcal{O}_S$ as a multiplicative monoid. But usually Lie$\,(G)$ becomes an $\mathcal{O}_S$-module via this action. If this is the case, we say Lie$\,(G)$ is a module. For example, we can prove easily that Lie$(G)$ is a module if $G$ is represented by an fs log analytic space over $S$. Also for example, Lie$\,(\mathbb{G}_{m,\log,S})$ is a module, and is identified with Lie$(\mathbb{G}_m) = \mathcal{O}_S$.

For a log complex torus $A$ over $S$, Lie$\,(A)$ is a module and is locally free of finite rank as an $\mathcal{O}_S$-module. This can be seen as, locally on $S$,

$$\begin{aligned}\mathrm{Lie}\,(\mathcal{H}om(X, \mathbb{G}_{m,\log,S})^{(Y)}) &= \mathrm{Lie}\,(\mathcal{H}om(X, \mathbb{G}_m)) \\ &= \mathcal{H}om(X, \mathrm{Lie}\,(\mathbb{G}_m)) = \mathcal{H}om(X, \mathcal{O}_S).\end{aligned}$$

(Here we use the above Lie$\,(\mathbb{G}_{m,\log,S})$ = Lie$\,(\mathbb{G}_m)$ for the first equality.) We define the dimension of $A$ to be the rank of Lie$\,(A)$ as an $\mathcal{O}_S$-module, which is a locally constant function on $S$. If $A$ comes from a non-degenerate pairing $(X, Y, \langle\ ,\ \rangle)$ into $\mathbb{G}_{m,\log,S}$, it is just the rank of $X$ (= rank of $Y$) as a $\mathbb{Z}$-module.

## 2. Log Hodge Structures

### 2.1. $S^{\log}$ and local monodromy

For the proofs of statements in this subsection and the next, see [16] and [12].

**2.1.1.** Let $S$ be an fs log analytic space. Then $S^{\log}$ is the space of all pairs $(s, h)$, where $s \in S$ and $h$ is a homomorphism

$$M_{S,s}^{\mathrm{gp}} \to \mathbb{S}^1 = \{z \in \mathbb{C}^\times \mid |z| = 1\}$$

which extends $\mathcal{O}_{S,s}^\times \to \mathbb{S}^1\,;\ f \mapsto f(s)/|f(s)|$.

The topology of $S^{\log}$ is defined as in [16]. As a topological space over $S$, $S^{\log}$ represents the functor

$$T \mapsto \{\text{homomorphism}\, h : M_S^{\mathrm{gp}}|_T \to \mathrm{Cont}\,(?, \mathbb{S}^1) \mid h(f) = f/|f| \quad \text{for all } f \in \mathcal{O}_S^\times|_T\}.$$

Here $|_T$ means the inverse image on $T$, and $\mathrm{Cont}(?, \mathbb{S}^1)$ means the sheaf of continuous maps into $\mathbb{S}^1$ ([12]). The canonical map

$$\tau : S^{\log} \to S\ ; \quad (s,h) \mapsto s$$

is proper and surjective.

**2.1.2.** For $s \in S$, $M_{S,s}^{\mathrm{gp}}/\mathcal{O}_{S,s}^\times \simeq \mathbb{Z}^{\oplus r}$ for some $r$, and $\tau^{-1}(s)$ is homeomorphic to the product of $r$ copies of $\mathbb{S}^1$. In fact, fixing an element $h_0 : M_{S,s}^{\mathrm{gp}} \to \mathbb{S}^1$ of $\tau^{-1}(s)$, we have a homeomorphism

$$\tau^{-1}(s) \simeq \mathrm{Hom}\,(M_{S,s}^{\mathrm{gp}}/\mathcal{O}_{S,s}^\times, \mathbb{S}^1)\ ;\ h \mapsto hh_0^{-1}. \tag{1}$$

We have a canonical isomorphism

$$\pi_1(\tau^{-1}(s)) \simeq \mathrm{Hom}\,(M_{S,s}^{\mathrm{gp}}/\mathcal{O}_{S,s}^\times, \mathbb{Z}(1)) \tag{2}$$

which is induced by the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}\,(M_{S,s}^{\mathrm{gp}}/\mathcal{O}_{S,s}^\times, \mathbb{Z}(1)) &\longrightarrow \mathrm{Hom}\,(M_{S,s}^{\mathrm{gp}}/\mathcal{O}_{S,s}^\times, \mathbb{R}(1)) \\ &\xrightarrow{\exp} \mathrm{Hom}\,(M_{S,s}^{\mathrm{gp}}/\mathcal{O}_{S,s}^\times, \mathbb{S}^1) \longrightarrow 0, \end{aligned}$$

where $\mathbb{Z}(1) = \mathbb{Z}\cdot 2\pi i$, $\mathbb{R}(1) = \mathbb{R}\otimes_{\mathbb{Z}} \mathbb{Z}(1)$, and the middle term is regarded as a universal covering of $\tau^{-1}(s) \overset{(1)}{\simeq} \mathrm{Hom}\,(M_{S,s}^{\mathrm{gp}}/\mathcal{O}_{S,s}^\times, \mathbb{S}^1)$. The isomorphism (2) is independent of the choice of $h_0$.

We will use also the isomorphism

$$\pi_1(\tau^{-1}(s)) \simeq \mathrm{Hom}\,(M_{S,s}^{\mathrm{gp}}/\mathcal{O}_{S,s}^\times, \mathbb{Z}) \tag{3}$$

which is induced from (2) via the isomorphism $\mathbb{Z} \cong \mathbb{Z}(1)$; $1 \mapsto 2\pi i$.

We will call the cone

$$C(s) = \mathrm{Hom}\,(M_{S,s}/\mathcal{O}_{S,s}^\times, \mathbb{R}_{\geq 0}) \subset \mathrm{Hom}\,(M_{S,s}^{\mathrm{gp}}/\mathcal{O}_{S,s}^\times, \mathbb{R}) = \mathbb{R}\otimes_{\mathbb{Z}} \pi_1(\tau^{-1}(s))$$

the monodromy cone at $s$, where the last equality comes from (3).

**2.1.3.** For a locally constant sheaf $L$ on $S^{\log}$ and for $s \in S$, $t \in \tau^{-1}(s)$, the action of $\pi_1(\tau^{-1}(s))$ on the stalk $L_t$ is called the local monodromy of $L$.

2.1.4. Definition. Admissibility of local monodromy. Let $L$ be a locally constant sheaf of $\mathbb{Q}$-vector spaces over $S^{\log}$ endowed with an increasing filtration $W = (W_k)_{k \in \mathbb{Z}}$ consisting of locally constant $\mathbb{Q}$-subsheaves.

For $s \in S$, we say that the local monodory of $L$ at $s$ is admissible with respect to $W$ if the action of any element of $\pi_1(\tau^{-1}(s))$ on any stalk of $L$ on $\tau^{-1}(s)$ is unipotent and if there exists an increasing filtration $(W(\sigma)_k)_{k \in \mathbb{Z}}$ consisting of locally constant $\mathbb{Q}$-subsheaves on the pull back $L|_s$ of $L$ to $\tau^{-1}(s)$ given for each face $\sigma$ of the monodromy cone $C(s)$ satisfying the following conditions (i)–(iii).

(i) $W(\{0\}) = W$.

(ii) If $\sigma$ is a face of $C(s)$ and $h \in \sigma$, then $\log(h) : \mathbb{R} \otimes_{\mathbb{Q}} L_t \to \mathbb{R} \otimes_{\mathbb{Q}} L_t$ for $t \in \tau^{-1}(s)$ satisfies

$$\log(h)(\mathbb{R} \otimes_{\mathbb{Q}} W(\sigma)_{k,t}) \subset \mathbb{R} \otimes_{\mathbb{Q}} W(\sigma)_{k-2,t} \quad \text{for all } k \in \mathbb{Z}.$$

(iii) Let $\sigma$ and $\sigma'$ be faces of $C(s)$ such that $\sigma \subset \sigma'$. Let $h \in \sigma'$, and assume that there exists $a \in \sigma$ for which $h + a$ belongs to the interior of $\sigma'$. (Here we denote the semi-group law of $C(s)$ additively.) Then for $t \in \tau^{-1}(s)$, we have isomorphisms

$$\log(h)^{\ell} : \mathbb{R} \otimes_{\mathbb{Q}} \mathrm{gr}^{W(\sigma')}_{k+\ell}(\mathrm{gr}^{W(\sigma)}_{k,t}) \xrightarrow{\simeq} \mathbb{R} \otimes_{\mathbb{Q}} \mathrm{gr}^{W(\sigma')}_{k-\ell}(\mathrm{gr}^{W(\sigma)}_{k,t}) \quad \text{for all } \ell \geq 0.$$

Such family $(W(\sigma))_{\sigma}$ was considered by Deligne ([9]). Note that by [9] 1.6.13, $(W(\sigma))_{\sigma}$ is unique if it exists.

We say the local monodromy of $L$ is admissible with respect to $W$ if the local monodromy of $L$ at $s$ is admissible with respect to $W$ for any $s \in S$.

In the case there exists $j \in \mathbb{Z}$ such that $W_j = L$ and $W_{j-1} = 0$, we call the admissibility of $L$ with respect to $W$ just the admissibility of $L$.

**2.1.5.** Let $T \longrightarrow S$ be a morphism of fs log analytic spaces. Let $L$ be a locally constant sheaf of $\mathbb{Q}$-vector spaces over $S^{\log}$ endowed with an increasing filtration $W = (W_k)_{k\in\mathbb{Z}}$ consisting of locally constant $\mathbb{Q}$-subsheaves. If the local monodromy of $L$ is admissible with respect to $W$, then the local monodromy of the pull back of $L$ to $T^{\log}$ is admissible with respect to the pull back of $W$.

This is seen as follows. Let $t \in T$ with its image $s \in S$. We will define an increasing filtration $W(\tau)$ on the pull back $L|_t$ of $L$ to $\tau^{-1}(t)$ for each face $\tau$ of $C(t)$. Let $\sigma$ be the smallest face of $C(s)$ containing the image of $\tau$. Let $W(\tau)$ be the pull back of $W(\sigma)$. Then as is easily seen, these $W(\tau)$ satisfy the required conditions of the admissibility.

## 2.2. $\mathcal{O}_S^{\log}$

We review the sheaf of rings $\mathcal{O}_S^{\log}$ on $S^{\log}$.

**2.2.1.** Define a sheaf of abelian groups $\mathcal{L}_S$ on $S^{\log}$ by the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}(1) & \to & \mathcal{L}_S & \overset{\exp}{\to} & M_S^{\mathrm{gp}} & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbb{Z}(1) & \to & \mathrm{Cont}(?, \mathbb{R}(1)) & \overset{\exp}{\to} & \mathrm{Cont}(?, \mathbb{S}^1) & \to & 0. \end{array}$$

Here $M_S^{\mathrm{gp}}$ denotes the inverse image of $M_S^{\mathrm{gp}}$ on $S^{\log}$. (In this way, we sometimes use the same notation $\mathcal{F}$ for the inverse image of a sheaf $\mathcal{F}$.) That is, $\mathcal{L}_S$ is defined to be the fiber product of $M_S^{\mathrm{gp}} \to \mathrm{Cont}(?, \mathbb{S}^1) \overset{\exp}{\leftarrow} \mathrm{Cont}(?, \mathbb{R}(1))$. We call $\mathcal{L}_S$ the sheaf of logarithms of $M_S^{\mathrm{gp}}$.

**2.2.2.** We define an embedding

$$\iota : \mathcal{O}_S \to \mathcal{L}_S$$

by

$$\exp : \mathcal{O}_S \to \mathcal{O}_S^\times \subset M_S^{\mathrm{gp}},$$

$$\mathcal{O}_S \to \mathrm{Cont}(?, \mathbb{R}(1))\,;\; f \mapsto \frac{1}{2}(f - \bar{f}).$$

Then the usual exponential sequence $0 \to \mathbb{Z}(1) \to \mathcal{O}_S \overset{\exp}{\to} \mathcal{O}_S^\times \to 0$ is embedded in the upper exact sequence in 2.2.1.

**2.2.3.** We define $\mathcal{O}_S^{\log}$ by

$$\mathcal{O}_S^{\log} = (\mathcal{O}_S \otimes_{\mathbb{Z}} \mathrm{Sym}_{\mathbb{Z}}(\mathcal{L}_S))/\mathfrak{a},$$

where $\mathrm{Sym}_{\mathbb{Z}}$ is the symmetric algebra as a $\mathbb{Z}$-module and $\mathfrak{a}$ is the ideal of $\mathcal{O}_S \otimes_{\mathbb{Z}} \mathrm{Sym}_{\mathbb{Z}}(\mathcal{L}_S)$ generated by the image of

$$\mathcal{O}_S \to \mathcal{O}_S \otimes_{\mathbb{Z}} \mathrm{Sym}_{\mathbb{Z}}(\mathcal{L}_S)\,;\; f \mapsto f \otimes 1 - 1 \otimes \iota(f).$$

**2.2.4.** For $s \in S$ and $t \in \tau^{-1}(s)$, the stalk $\mathcal{O}_{S,t}^{\log}$ of $\mathcal{O}_S^{\log}$ at $t$ has the following structure. If $\ell_j\,(1 \le j \le r)$ denotes elements of $\mathcal{L}_{S,t}$ such that $\exp(\ell_j) \bmod \mathcal{O}_{S,s}^{\times}$ $(1 \le j \le r)$ form a $\mathbb{Z}$-basis of $M_{S,s}^{\mathrm{gp}}/\mathcal{O}_{S,s}^{\times}$, then $\mathcal{O}_{S,t}^{\log}$ is isomorphic to the polynomial ring over $\mathcal{O}_{S,s}$ in $r$ variables by

$$\mathcal{O}_{S,s}[T_1, \cdots, T_r] \xrightarrow{\simeq} \mathcal{O}_{S,t}^{\log}\,;\; T_j \mapsto \ell_j.$$

**2.2.5.** Note that by 2.2.4, $(S^{\log}, \mathcal{O}_S^{\log})$ is usually not a local ringed space.

For $s \in S$ and $t \in \tau^{-1}(s)$, by a specialization at $t$, we mean a ring homomorphism $\mathcal{O}_{S,t}^{\log} \to \mathbb{C}$ which extends $\mathcal{O}_{S,s} \to \mathbb{C}\,;\; f \mapsto f(s)$.

## 2.3. Log mixed Hodge structures

For the details of what treated in this subsection, see [17] §2. Let $S$ be an fs log analytic space.

**2.3.1.** By a pre-log Hodge structure over $S$, we mean a triple $(H_{\mathbb{Z}}, H_{\mathcal{O}}, \iota)$, where $H_{\mathbb{Z}}$ is a locally constant sheaf of finitely generated free $\mathbb{Z}$-modules on $S^{\log}$, $H_{\mathcal{O}}$ is a sheaf of $\mathcal{O}_S$-modules on $S$ endowed with a decreasing filtration $(F^p H_{\mathcal{O}})_{p \in \mathbb{Z}}$ such that the $\mathcal{O}_S$-modules $H_{\mathcal{O}}$, $F^p H_{\mathcal{O}}$ and $H_{\mathcal{O}}/F^p H_{\mathcal{O}}$ for all $p$ are locally free of finite rank, and where

$$\iota : \mathcal{O}_S^{\log} \otimes_{\mathbb{Z}} H_{\mathbb{Z}} \simeq \mathcal{O}_S^{\log} \otimes_{\tau^{-1}(\mathcal{O}_S)} \tau^{-1}(H_{\mathcal{O}})$$

is an isomorphism of $\mathcal{O}_S^{\log}$-modules. We will identify $\mathcal{O}_S^{\log} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$ and $\mathcal{O}_S^{\log} \otimes_{\tau^{-1}(\mathcal{O}_S)} \tau^{-1}(H_{\mathcal{O}})$ via $\iota$.

**2.3.2.** For a pre-log Hodge structure over $S$, the local monodromy of $H_{\mathbb{Z}}$ is unipotent, and we have

$$H_{\mathcal{O}} = \tau_*(\mathcal{O}_S^{\log} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}).$$

**2.3.3.** By a pre-log mixed Hodge structure over $S$, we mean a pair $(H, W)$, where $H$ is a pre-log Hodge structure over $S$ and $W = (W_k)_{k \in \mathbb{Z}}$ is an increasing filtration on $H_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$ consisting of locally constant $\mathbb{Q}$-subsheaves.

**2.3.4.** Let $(H, W)$ be a pre-log mixed Hodge structure. For $s \in S$, $t \in \tau^{-1}(s)$ and for a specialization $a : \mathcal{O}_{S,t}^{\log} \to \mathbb{C}$ at $t$ (2.2.5), we have a 4-ple

$$(H, W)(a) = (H_{\mathbb{Z},t}, H_{\mathcal{O}}(s), \iota(a), W_t),$$

where $H_{\mathcal{O}}(s)$ is the $\mathbb{C}$-vector space $\mathbb{C} \otimes_{\mathcal{O}_{S,s}} H_{\mathcal{O},s}$ endowed with a decreasing filtration $\mathbb{C} \otimes_{\mathcal{O}_{S,s}} F^{\cdot} H_{\mathcal{O},s}$ (here $\mathcal{O}_{S,s} \to \mathbb{C}$ is $f \mapsto f(s)$), and $\iota(a)$ is the isomorphism of $\mathbb{C}$-vector spaces $\mathbb{C} \otimes_{\mathbb{Z}} H_{\mathbb{Z},t} \simeq H_{\mathcal{O}}(s)$ induced by $\iota$. We call $(H, W)(a)$ the specialization of $(H, W)$ at $a$.

**2.3.5.** We define the notion of "log mixed Hodge structure over $S$." We first assume that $S$ is an fs log point, that is, $S$ is a one point set $\{s\}$ and the ring of $S$ is $\mathbb{C}$. In this case, by a log mixed Hodge structure over $S$, we mean a pre-log mixed Hodge structure $(H, W)$ over $S$ satisfying the following conditions (i)–(iii).

(i) The local monodromy of $H_{\mathbb{Q}}$ is admissible with respect to $W$.

(ii) (For the definition of sufficiently shifted specialization, see below.) Let $\sigma$ be a face of $C(s)$ and $t \in \tau^{-1}(s) = S^{\log}$. Then, for any sufficiently shifted specialization $a : \mathcal{O}_{S,t}^{\log} \to \mathbb{C}$ at $t$, the specialization $(H, W(\sigma))(a)$ of $(H, W(\sigma))$ at $a$ (2.3.4) is a mixed Hodge structure in the usual sense.

(iii) Let $\omega_S^1$ be the logarithmic differential module of $S$. Then the homomorphism $\nabla : H_{\mathcal{O}} \to \omega_S^1 \otimes_{\mathcal{O}_S} H_{\mathcal{O}}$ induced from $d : \mathcal{O}_S^{\log} \to \omega_S^1 \otimes_{\mathcal{O}_S} \mathcal{O}_S^{\log}$ satisfies

$$\nabla(F^p H_{\mathcal{O}}) \subset \omega_S^1 \otimes_{\mathcal{O}_S} F^{p-1} H_{\mathcal{O}} \quad \text{for all } p \in \mathbb{Z}.$$

In (ii) and (iii), note $\mathcal{O}_S = \mathbb{C}$ by the assumption. In (ii), take a finite family $(\ell_j)_{1 \le j \le n}$ of elements of $\mathcal{L}_{S,t}$ such that $\exp(\ell_j) \bmod \mathcal{O}_{S,s}^{\times}$ $(1 \le j \le n)$

generate the monoid $M_{S,s}/\mathcal{O}_{S,s}^{\times}$ and such that $\ell_j \notin \mathcal{O}_{S,s}$ for $1 \le j \le n$. Then, by a "sufficiently shifted specialization", we mean a specialization $a : \mathcal{O}_{S,t}^{\log} \to \mathbb{C}$ at $t$ such that $\exp(a(\ell_j))$ $(1 \le j \le n)$ are sufficiently near to 0. This definition is independent of the choice of the family $(\ell_j)_j$. In certain cases (see 2.4), $a((2\pi i)^{-1}\ell_j)$ are regarded as coordinates of the specialization $a$. The condition that $a$ is sufficiently shifted is equivalent to the condition that the imaginary parts of $a((2\pi i)^{-1}\ell_j)$ are sufficiently large, in other words, the coordinates of $a$ are in the upper direction on the upper half plane.

**2.3.6.** We define the notion of log mixed Hodge structure in general. By a log mixed Hodge structure over $S$, we mean a pre-log mixed Hodge structure over $S$ whose pull back to any point $s$ of $S$ is a log mixed Hodge structure over $s$ in the above sense. Here the ring of $s$ is $\mathbb{C}$ and the log structure of $s$ is the pull back of that of $S$.

2.3.7. Remark. The above definition of log mixed Hodge structures follows the ideas of "good" degenerations of variations of mixed Hodge structures by Deligne [9] and Steenbrink-Zucker [34]. See 3.3.4.

**2.3.8.** By a log Hodge structure over $S$ of weight $w$, we mean a log mixed Hodge structure $(H, W)$ satisfying $W_w = H_{\mathbb{Q}}$ and $W_{w-1} = 0$.

**2.3.9.** For a morphism $T \to S$ of fs log analytic spaces, the pull back of a log mixed Hodge structure over $S$ to $T$ is a log mixed Hodge structure over $T$.

This is proved easily by 2.1.5 and by the fact that the condition (ii) in 2.3.5 is independent of the choice of the $(\ell_j)_j$, which was remarked in the above.

## 2.4. Example

We give a log Hodge structure of rank 2 over the unit disc $\Delta$ with the log structure along the origin. As we will see later, in the correspondence between log abelian varieties and log Hodge structures in the next section, this log Hodge structure corresponds to the log elliptic curve $\mathbb{G}_{m,\log}^{(q)}/q^{\mathbb{Z}}$ over $\Delta$ in §1.1.

Let $H_{\mathcal{O}}$ be the free $\mathcal{O}_{\Delta}$-module of rank 2 on $\Delta$ with base $(e, f)$ endowed with the filtration defined by

$$F^p H_{\mathcal{O}} = H_{\mathcal{O}} \text{ for } p \le -1, \quad F^0 H_{\mathcal{O}} = \mathcal{O}_{\Delta} f, \quad F^p H_{\mathcal{O}} = 0 \text{ for } p \ge 1.$$

Let $H_{\mathbb{Z}}$ be the $\mathbb{Z}$-subsheaf of $\mathcal{O}_{\Delta}^{\log} \otimes_{\tau^{-1}(\mathcal{O}_{\Delta})} \tau^{-1}(H_{\mathcal{O}})$ on $\Delta^{\log}$ generated locally by $e$ and $(2\pi i)^{-1}\log(q)e - f$, where $q$ is the coordinate function of $\Delta$ and $\log(q)$ denotes a local section of $\mathcal{L}_{\Delta}$ whose exp in $\tau^{-1}(M_{\Delta}^{\mathrm{gp}})$ coincides with $q$ (then $H_{\mathbb{Z}}$ is independent of the local choice of $\log(q)$). This sheaf $H_{\mathbb{Z}}$ is locally isomorphic to $\mathbb{Z}^2$, $e$ is a global section of $H_{\mathbb{Z}}$, the stalk of $H_{\mathbb{Z}}$ at a point of $\Delta^{\log}$ has a basis $(e, e')$, where $e'$ is the stalk of a local section $(2\pi i)^{-1}\log(q)e - f$, and the local monodromy around 0 with the standard direction sends $e$ to $e$ and $e'$ to $e + e'$. We see easily that the canonical homomorphism $\iota : \mathcal{O}_{\Delta}^{\log} \otimes_{\mathbb{Z}} H_{\mathbb{Z}} \to \mathcal{O}_{\Delta}^{\log} \otimes_{\tau^{-1}(\mathcal{O}_{\Delta})} \tau^{-1}(H_{\mathcal{O}})$ is an isomorphism.

We show that the pre-log Hodge structure $(H_{\mathbb{Z}}, H_{\mathcal{O}}, \iota)$ is a log Hodge structure of weight $-1$.

Let $W$ be the increasing filtration of $H_{\mathbb{Q}}$ defined by $W_{-1} = H_{\mathbb{Q}}$, $W_{-2} = (0)$. Then the local monodromy of $H_{\mathbb{Q}}$ is admissible with respect to $W$. In fact, the increasing filtration $(W(\sigma)_k)_{k\in\mathbb{Z}}$ of $H_{\mathbb{Q}}|_0$ for the maximal face $\sigma := \mathbb{R}_{\geq 0}$ of the monodromy cone at $0 \in \Delta$ is given as

$$(0) = W(\sigma)_{-3} \subset \mathbb{Q}e =: W(\sigma)_{-2} = W(\sigma)_{-1} \subset W(\sigma)_0 := H_{\mathbb{Q}}|_0.$$

We next check 2.3.5 (ii) for the pull back of $(H, W)$ to each point $q \in \Delta$. If $q \neq 0$, the pull back is a Hodge structure of weight $-1$ because the imaginary part of $(2\pi i)^{-1}\log(q)$ is positive by $|q| < 1$.

In the case of $q = 0$, we take $\log q$ as $l_1$ in 2.3.5. Then, the set of the specializations $a\colon \mathcal{O}_{\Delta,t}^{\log} \to \mathbb{C}$ at $t \in \tau^{-1}(0)$ bijectively corresponds to the set $\mathbb{C}$ of the complex numbers $z$ via $z = a((2\pi i)^{-1}l_1)$, and for any $a$, the specialization by $a$ induces the isomorphism

$$\begin{aligned} \mathbb{C} \otimes_{\mathbb{Z}} H_{\mathbb{Z},t} &\simeq \mathbb{C} \otimes_{\mathcal{O}_{\Delta,0}} H_{\mathcal{O},0}; \\ 1 \otimes e &\mapsto 1 \otimes e; \\ 1 \otimes e' &\mapsto z \otimes e - 1 \otimes f. \end{aligned}$$

For the maximal cone $\sigma = \mathbb{R}_{\geq 0}$ of $C(0)$, we have

$$\mathbb{C} \otimes \mathrm{gr}_{-2}^{W(\sigma)} H_{\mathbb{Q},t} = \mathbb{C}e, \quad \mathbb{C} \otimes \mathrm{gr}_{-1}^{W(\sigma)} H_{\mathbb{Q},t} = 0,$$
$$\mathbb{C} \otimes \mathrm{gr}_{0}^{W(\sigma)} H_{\mathbb{Q},t} = (\mathbb{C}e + \mathbb{C}e')/\mathbb{C}e$$

for each $t \in \tau^{-1}(0)$. We can easily verify that the filtrations of $\mathbb{C} \otimes \mathrm{gr}_k H_{\mathbb{Q},t}$ $(k = -2, -1, 0)$ induced by $F^p H_{\mathcal{O}}$ make them Hodge structures of weight $k$.

For the minimal cone (0) of $C(0)$, the imaginary part of $z = a((2\pi i)^{-1}\log q)$ is positive if $\exp(a(\log q))$ is sufficiently near to 0. Hence we can prove this case as in the case $q \neq 0$.

Since $F^{-1}H_{\mathcal{O}} = H_{\mathcal{O}}$ and $F^1 H_{\mathcal{O}} = (0)$, the condition 2.3.5 (iii) is trivial. Therefore, $((H_{\mathbb{Z}}, H_{\mathcal{O}}, \iota), W)$ is a log Hodge structure of weight $-1$ over $\Delta$.

## 2.5. Polarization

We review the definition of polarization in the log Hodge theory (see [17] §2).

2.5.1. DEFINITION. Let $H$ be a pre-log Hodge structure of weight $w$ (i.e., a pre-log mixed Hodge structure such that $W_w = H_{\mathbb{Q}}$ and $W_{w-1} = 0$) over an fs log analytic space $S$.

(1) A polarization on $H$ is a homomorphism $p : H \otimes H \to \mathbb{Z}(-w)$ of pre-log Hodge structures having the following property (i) for each $s \in S$ and each $t \in \tau^{-1}(s) \subset S^{\log}$.

(i) For any sufficiently shifted specialization $a : \mathcal{O}^{\log}_{S,t} \to \mathbb{C}$ at $t$, the specialization $H(a)$ of $H$ at $a$ (2.3.4) is a polarized Hodge structure of weight $w$ with respect to the pairing $p_t$ in the usual sense.

(2) We say $H$ is polarizable if $H$ has a polarization.

We will denote a polarization $p : H \otimes H \to \mathbb{Z}(-w)$ also as $H \to H^*(-w); h \mapsto p(- \otimes h)$, where $H^*$ denotes the $\mathbb{Z}$-dual of $H$.

**2.5.2.** By Cattani and Kaplan [6] and Schmid [32], a pre-log Hodge structure of weight $w$ satisfying (iii) in 2.3.5 and having a polarization is a log Hodge structure of weight $w$. (The condition (i) in 2.3.5 is by [6] and the condition (ii) is by [32].)

**2.5.3.** For example, for the log Hodge structure $H$ of weight $-1$ in §2.4, the homomorphism $p\colon H \otimes H \to \mathbb{Z}(1)$ defined by

$$p(e \otimes e') = -p(e' \otimes e) = 2\pi i, \quad p(e \otimes e) = p(e' \otimes e') = 0$$

is a polarization as is easily seen.

## 3. Equivalences of Categories

### 3.1. Statements of equivalences of categories

**3.1.1.** We recall the well-known isomorphism

$$\mathrm{Ext}^1(\mathbb{Z}, H) \simeq H_{\mathbb{Z}} \backslash H_{\mathbb{C}}/F^0 H_{\mathbb{C}}$$

for a Hodge structure $H$ of weight $-1$ such that $F^{-1}H_{\mathbb{C}} = H_{\mathbb{C}}$ and $F^1 H_{\mathbb{C}} = 0$, where $\mathrm{Ext}^1$ is taken for the category of mixed Hodge structures and $\mathbb{Z}$ is the unit Hodge structure of weight 0. For $z \in H_{\mathbb{C}}$, the extension $H'$ of $\mathbb{Z}$ by $H$ corresponding to the class of $z \in H_{\mathbb{C}}$ in $H_{\mathbb{Z}} \backslash H_{\mathbb{C}}/F^0 H_{\mathbb{C}}$ is given as follows.

$$H'_{\mathbb{Z}} = \mathbb{Z} \oplus H_{\mathbb{Z}},$$

$$F^p H'_{\mathbb{C}} = H'_{\mathbb{C}} \text{ for } p \leq -1, \quad F^p H'_{\mathbb{C}} = 0 \text{ for } p \geq 1,$$

$$F^0 H'_{\mathbb{C}} = \mathbb{C}(1, -z) + (0, F^0 H_{\mathbb{C}}).$$

Conversely, if we have an extension $H'$ of $\mathbb{Z}$ by $H$, we have exact sequences

$$0 \to H_{\mathbb{Z}} \to H'_{\mathbb{Z}} \to \mathbb{Z} \to 0, \quad 0 \to F^0 H_{\mathbb{C}} \to F^0 H'_{\mathbb{C}} \to \mathbb{C} \to 0,$$

and hence there exist an element $a$ of $H'_{\mathbb{Z}}$ whose image in $\mathbb{Z}$ is 1, and an element $b$ of $F^0 H'_{\mathbb{C}}$ whose image in $\mathbb{C}$ is 1. Let $z = a - b \in H_{\mathbb{C}}$. Then the class of $z$ in $H_{\mathbb{Z}} \backslash H_{\mathbb{C}}/F^0 H_{\mathbb{C}}$ is independent of the choices of $a$ and $b$. These $z \mapsto H'$ and $H' \mapsto z$ give mutually inverse isomorphisms between $H_{\mathbb{Z}} \backslash H_{\mathbb{C}}/F^0 H_{\mathbb{C}}$ and $\mathrm{Ext}^1(\mathbb{Z}, H)$.

**3.1.2.** We review the classical theory of correspondences between complex tori and Hodge structures, and between abelian varieties and polarizable Hodge structures.

Let $\mathcal{A}$ be the category of complex tori and let $\mathcal{A}^+$ be the category of abelian varieties. On the other hand, let $\mathcal{H}$ be the category of Hodge structures $H$ of weight $-1$ such that $F^{-1}H_{\mathbb{C}} = H_{\mathbb{C}}$ and $F^1 H_{\mathbb{C}} = 0$, and let $\mathcal{H}^+$ be the full subcategory of $\mathcal{H}$ consisting of polarizable objects. We have equivalences of categories

$$\mathcal{H} \simeq \mathcal{A}, \quad \mathcal{H}^+ \simeq \mathcal{A}^+.$$

The complex torus $A$ corresponding to an object $H$ of $\mathcal{H}$ is given by

$$A = H_{\mathbb{Z}} \setminus H_{\mathbb{C}}/F^0 H_{\mathbb{C}} = \mathrm{Ext}^1(\mathbb{Z}, H),$$

and the equivalence $\mathcal{H}^+ \simeq \mathcal{A}^+$ is induced from $\mathcal{H} \simeq \mathcal{A}$.

The inverse functor $\mathcal{A} \to \mathcal{H}$ is given by $A \mapsto H$, where

$$H_{\mathbb{Z}} = H_1(A(\mathbb{C}), \mathbb{Z}), \quad F^0 H_{\mathbb{C}} = \mathrm{Ker}\,(H_{\mathbb{C}} \to \mathrm{Lie}(A));$$

$$F^p H_{\mathbb{C}} = H_{\mathbb{C}} \quad \text{for} \quad p \le -1, \quad F^p H_{\mathbb{C}} = 0 \quad \text{for} \quad p \ge 1.$$

Here $H_{\mathbb{C}} \to \mathrm{Lie}(A)$ is the surjection which comes from the exact sequence $0 \to H_{\mathbb{Z}} \to \mathrm{Lie}(A) \to A(\mathbb{C}) \to 0$.

The following theorem 3.1.5 gives logarithmic generalizations of these equivalences.

**3.1.3.** Let $S$ be an fs log analytic space.

Let $\mathcal{A}_S$ be the category of log complex tori over $S$, and let $\mathcal{A}_S^+$ be the category of log abelian varieties over $S$.

On the other hand, let $\mathcal{H}_S$ be the category of log Hodge structures $H$ over $S$ of weight $-1$ satisfying

$$F^{-1} H_{\mathcal{O}} = H_{\mathcal{O}}, \quad F^1 H_{\mathcal{O}} = 0.$$

Let $\mathcal{H}_S^+$ be the full subcategory of $\mathcal{H}_S$ consisting of all objects whose pull backs to $s$ for any $s \in S$ are polarizable.

**3.1.4.** For an object $H$ of $\mathcal{H}_S$, we define the sheaf of abelian groups $\mathcal{E}xt^1(\mathbb{Z}, H)$ on (fs/$S$) by

$$\mathcal{E}xt^1(\mathbb{Z}, H)(T) = \mathrm{Ext}^1(\mathbb{Z}, H_T)$$

for fs log analytic spaces $T$ over $S$, where $H_T$ denotes the pull back of $H$ to $T$, and $\mathrm{Ext}^1$ is taken for the category of log mixed Hodge structures over $T$. Note that the category of log mixed Hodge structures has the evident definitions of "exact sequence" and "extension (short exact sequence)." We consider $\mathrm{Ext}^1$ as the set of isomorphism classes of extensions, with the group structure given by Baer sums.

The aim of this section is to prove the following theorem.

3.1.5. THEOREM. *(1) For an object $H$ of $\mathcal{H}_S$, $\mathcal{E}xt^1(\mathbb{Z}, H)$ is a log complex torus over $S$.*

*(2) $H \mapsto \mathcal{E}xt^1(\mathbb{Z}, H)$ defines an equivalence of categories*

$$\mathcal{H}_S \xrightarrow{\cong} \mathcal{A}_S.$$

*(3) The equivalence in (2) induces an equivalence of full subcategories*

$$\mathcal{H}_S^+ \xrightarrow{\cong} \mathcal{A}_S^+.$$

**3.1.6.** The inverse functor $\mathcal{A}_S \to \mathcal{H}_S$ is defined in the following way. Here only the method of the definition is described. See 3.7 for the details.

For an fs log analytic space $S$, let $(\mathrm{fs}/S)^{\log}$ be the following site. An object of $(\mathrm{fs}/S)^{\log}$ is a pair $(U, T)$, where $T$ is an fs log analytic space over $S$ and $U$ is an open set of $T^{\log}$. The morphisms are defined in the evident way. A covering is a family of morphisms $((U_\lambda, T_\lambda) \to (U, T))_\lambda$, where each $T_\lambda \to T$ is an open immersion and the log structure of $T_\lambda$ is the inverse image of that of $T$, and $(U_\lambda)_\lambda$ is an open covering of $U$.

We have a commutative diagram of topoi

$$\begin{array}{ccc} \{\text{sheaf on } (\mathrm{fs}/S)^{\log}\} & \xrightarrow{\beta_S^{\log}} & \{\text{sheaf on } S^{\log}\} \\ \tau \downarrow & & \downarrow \tau \\ \{\text{sheaf on } (\mathrm{fs}/S)\} & \xrightarrow{\beta_S} & \{\text{sheaf on } S\}. \end{array}$$

Here the right vertical arrow $\tau$ is the morphism of topoi induced by the map $\tau : S^{\log} \to S$, and $\beta_S$ and $\beta_S^{\log}$ are the evident ones. The left vertical arrow which we denote also by $\tau$ is defined as follows. For a sheaf $F$ on $(\mathrm{fs}/S)^{\log}$, the image $\tau_*(F)$ on $(\mathrm{fs}/S)$ is defined by $\tau_*(F)(T) = F(T^{\log}, T)$. For a sheaf $F$ on $(\mathrm{fs}/S)$, the inverse image $\tau^{-1}(F)$ on $(\mathrm{fs}/S)^{\log}$ is defined as follows. For an object $(U, T)$ of $(\mathrm{fs}/S)^{\log}$, the restriction of $\tau^{-1}(F)$ to the usual site of open sets of $U$ (i.e., the restriction to the site consisting of $(U', T)$ for all open sets $U'$ of $U$) coincides with the inverse image of the restriction of $F$ to the site of open sets of $T$ under the map $U \to T$. In the above diagram, $\beta_S$ and $\beta_S^{\log}$ are exact, and $\beta_{S*}\beta_S^{-1}$ and $(\beta_S^{\log})_*(\beta_S^{\log})^{-1}$ are isomorphic to the identity functors. Furthermore, for the both $\tau$ in the above diagram, $\tau_*\tau^{-1}$

is isomorphic to the identity functor. This follows from that $T^{\log} \to T$ is proper with connected fibers for fs log analytic spaces $T$.

We will denote the sheaf $(U, T) \mapsto \mathcal{O}_T^{\log}(U)$ on $(\mathrm{fs}/S)^{\log}$ simply by $\mathcal{O}_S^{\log}$.

Now we describe the inverse functor $\mathcal{A}_S \to \mathcal{H}_S; A \mapsto H$. As we will see in 3.7.4, for a log complex torus $A$ over $S$, the $\mathrm{Ext}^1$ sheaf $\mathcal{E}xt^1(\tau^{-1}(A), \mathbb{Z})$ on $(\mathrm{fs}/S)^{\log}$ for the inverse image $\tau^{-1}(A)$ of $A$ on $(\mathrm{fs}/S)^{\log}$ is a locally constant sheaf of finitely generated free abelian groups. We define

$$H_{\mathbb{Z}} = \mathcal{H}om_{\mathbb{Z}}(\mathcal{E}xt^1(\tau^{-1}(A), \mathbb{Z}), \mathbb{Z}).$$

Next we define

$$H_{\mathcal{O}} = \tau_*(\mathcal{O}_S^{\log} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}).$$

As we will see in 3.7.4, the canonical homomorphism $\mathcal{O}_S^{\log} \otimes_{\tau^{-1}(\mathcal{O}_S)} \tau^{-1}(H_{\mathcal{O}}) \to \mathcal{O}_S^{\log} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$ is an isomorphism. Furthermore, there is a canonical surjective homomorphism $H_{\mathcal{O}} \to \mathrm{Lie}(A)$ of $\mathcal{O}_S$-modules. We will define $F^p H_{\mathcal{O}}$ to be $H_{\mathcal{O}}$ if $p \geq -1$, $\mathrm{Ker}\,(H_{\mathcal{O}} \to \mathrm{Lie}(A))$ if $p = 0$, and $0$ if $p \geq 1$. Then this gives an object $H$ of $\mathcal{H}_S$.

**3.1.7.** The plan of the rest of §3 is as follows. In §3.2 and §3.3, we prove certain "simpler" categorical equivalences $\hat{\mathcal{H}}_S \simeq \hat{\mathcal{A}}_S$ and $\tilde{\mathcal{H}}_S \simeq \tilde{\mathcal{A}}_S$, which are closely related to the equivalence $\mathcal{H}_S \simeq \mathcal{A}_S$ in 3.1.5. After preliminaries in §3.4 and §3.5, in §3.6, we study $\mathcal{E}xt^1(\mathbb{Z}, H)$ for an object $H$ of $\mathcal{H}_S$, and prove that it is a log complex torus. Then, we prove the equivalence $\mathcal{H}_S \simeq \mathcal{A}_S$ in 3.1.5 in §3.7 by constructing the inverse functor. In §3.8, we consider dual log complex tori and polarizations of log abelian varieties, and prove the equivalence $\mathcal{H}_S^+ \simeq \mathcal{A}_S^+$ in 3.1.5. In §3.9 and §3.10, we prove some related results, which we will use later.

### 3.2. The equivalence $\hat{\mathcal{H}}_S \simeq \hat{\mathcal{A}}_S$

Let $S$ be an fs log analytic space. We define categories $\hat{\mathcal{A}}_S$ and $\hat{\mathcal{H}}_S$, which are closely related to $\mathcal{A}_S$ and to $\mathcal{H}_S$, respectively, and prove an equivalence $\hat{\mathcal{H}}_S \simeq \hat{\mathcal{A}}_S$.

**3.2.1.** Let $\hat{\mathcal{A}}_S$ be the category of pairings into $\mathbb{G}_{m,\log,S}$. That is, an object of $\hat{\mathcal{A}}_S$ is a triple $(X, Y, \langle\ ,\ \rangle)$, where $X$ and $Y$ are finitely generated free abelian groups and $\langle\ ,\ \rangle$ is a $\mathbb{Z}$-bilinear form $X \times Y \to \mathbb{G}_{m,\log,S}$. A morphism $(X_1, Y_1, \langle\ ,\ \rangle_1) \to (X_2, Y_2, \langle\ ,\ \rangle_2)$ of $\hat{\mathcal{A}}_S$ is a pair $(f, g)$ of homomorphisms

$f : X_2 \to X_1$ and $g : Y_1 \to Y_2$ satisfying $\langle f(x), y\rangle_1 = \langle x, g(y)\rangle_2$ for any $x \in X_2$ and $y \in Y_1$.

**3.2.2.** Let $\hat{\mathcal{H}}_S$ be the category of 4-ples $(H, X, Y, e)$, where $H = (H_{\mathbb{Z}}, H_{\mathcal{O}}, \iota)$ is a pre-log Hodge structure over $S$, $X, Y$ are finitely generated free abelian groups, and $e$ is an exact sequence of local systems

$$0 \to \mathrm{Hom}\,(X, \mathbb{Z}(1)) \to H_{\mathbb{Z}} \to Y \to 0$$

on $S^{\log}$ satisfying the following condition (i).

(i) $F^{-1}H_{\mathcal{O}} = H_{\mathcal{O}}, \quad F^1H_{\mathcal{O}} = 0, \quad H_{\mathcal{O}} = \mathcal{H}om(X, \mathcal{O}_S) \oplus F^0H_{\mathcal{O}}$.

Here we embed $\mathcal{H}om(X, \mathcal{O}_S)$ in $H_{\mathcal{O}}$ via $e$.

A morphism $(H, X, Y, e) \to (H', X', Y', e')$ of $\hat{\mathcal{H}}_S$ is a triple $(h_0, h_1, h_2)$, where $h_0$ is a morphism $H \to H'$, $h_1$ is a homomorphism $X' \to X$, and $h_2$ is a homomorphism $Y \to Y'$ which are compatible with the exact sequences $e$ and $e'$.

3.2.3. PROPOSITION. *We have an equivalence of categories*

$$\hat{\mathcal{H}}_S \simeq \hat{\mathcal{A}}_S.$$

**3.2.4.** We define a functor $\hat{\mathcal{H}}_S \to \hat{\mathcal{A}}_S$.

Let $(H, X, Y, e)$ be an object of $\hat{\mathcal{H}}_S$.

We define a $\mathbb{Z}$-bilinear form $\langle\ ,\ \rangle : X \times Y \to \mathbb{G}_{m,\log}$ as follows. We have a projection

$$H_{\mathcal{O}} = \mathcal{H}om(X, \mathcal{O}_S) \oplus F^0H_{\mathcal{O}} \to \mathcal{H}om(X, \mathcal{O}_S),$$

and hence $H_{\mathbb{Z}} \to \mathcal{H}om(X, \mathcal{O}_S^{\log})$ which induces the identity map on $\mathrm{Hom}\,(X, \mathbb{Z}(1))$. Hence we have a homomorphism $Y = H_{\mathbb{Z}}/\mathrm{Hom}\,(X, \mathbb{Z}(1)) \to \mathcal{H}om(X, \mathcal{O}_S^{\log}/\mathbb{Z}(1))$. By taking $\tau_*$ and by using the following lemma, we obtain a homomorphism

$$Y \to \mathcal{H}om(X, \tau_*(\mathcal{O}_S^{\log}/\mathbb{Z}(1))) \simeq \mathcal{H}om(X, M_S^{\mathrm{gp}}),$$

and hence a $\mathbb{Z}$-bilinear form $\langle\ ,\ \rangle : X \times Y \to \mathbb{G}_{m,\log}$. This gives a functor $\hat{\mathcal{H}}_S \to \hat{\mathcal{A}}_S$.

3.2.5. Lemma. *The isomorphism* $\exp : \mathcal{L}_S/\mathbb{Z}(1) \xrightarrow{\simeq} \tau^{-1}(M_S^{\mathrm{gp}})$ *and the map* $\mathcal{L}_S/\mathbb{Z}(1) \to \mathcal{O}_S^{\log}/\mathbb{Z}$ *induces an isomorphism*

$$M_S^{\mathrm{gp}} \simeq \tau_*(\mathcal{O}_S^{\log}/\mathbb{Z}(1)).$$

Proof. This follows from the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 \to & \mathbb{Z}(1) & \to & \mathcal{O}_S & \overset{\exp}{\to} & M_S^{\mathrm{gp}} & \to & M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times} & \to & 0 \\ & \| & & \| & & \downarrow & & \wr| & & \\ 0 \to & \mathbb{Z}(1) & \to & \mathcal{O}_S & \to & \tau_*(\mathcal{O}_S^{\log}/\mathbb{Z}(1)) & \to & R^1\tau_*\mathbb{Z}(1) & \to & 0, \end{array}$$

where the lower sequence is obtained from $0 \to \mathbb{Z}(1) \to \mathcal{O}_S^{\log} \to \mathcal{O}_S^{\log}/\mathbb{Z}(1) \to 0$ by $R^1\tau_*\mathcal{O}_S^{\log} = 0$ ([20] 4.6, [12] Proposition (3.7) (3)). Here the right vertical isomorphism is by [16] (1.5). □

**3.2.6.** Next in 3.2.6–3.2.8, we define a functor $\hat{\mathcal{A}}_S \to \hat{\mathcal{H}}_S$.

Let $S$ be an fs log analytic space. Let $(X, Y, \langle\ ,\ \rangle)$ be an object of $\hat{\mathcal{A}}_S$.

We define first the Betti realization $H_{\mathbb{Z}}$ of $(X, Y, \langle\ ,\ \rangle)$, which is a locally constant sheaf of abelian groups on $S^{\log}$, by the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}\,(X, \mathbb{Z}(1)) & \to & H_{\mathbb{Z}} & \to & Y & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Hom}\,(X, \mathbb{Z}(1)) & \to & \mathcal{H}om(X, \mathcal{L}_S) & \to & \mathcal{H}om(X, \tau^{-1}(M_S^{\mathrm{gp}})) & \to & 0. \end{array}$$

Here the lower row is induced by the exact sequence $0 \to \mathbb{Z}(1) \to \mathcal{L}_S \overset{\exp}{\to} \tau^{-1}(M_S^{\mathrm{gp}}) \to 0$.

**3.2.7.** The exact sequence of $\mathcal{O}_S^{\log}$-modules

$$0 \to \mathcal{H}om(X, \mathcal{O}_S^{\log}) \to \mathcal{O}_S^{\log} \otimes_{\mathbb{Z}} H_{\mathbb{Z}} \to \mathcal{O}_S^{\log} \otimes_{\mathbb{Z}} Y \to 0 \tag{1}$$

splits canonically, by the $\mathcal{O}_S^{\log}$-homomorphism $\mathcal{O}_S^{\log} \otimes_{\mathbb{Z}} H_{\mathbb{Z}} \to \mathcal{H}om(X, \mathcal{O}_S^{\log})$ induced by the map $H_{\mathbb{Z}} \to \mathcal{H}om(X, \mathcal{L}_S)$ in 3.2.6.

**3.2.8.** Let $H_{\mathcal{O}} = \tau_*(\mathcal{O}_S^{\log} \otimes_{\mathbb{Z}} H_{\mathbb{Z}})$. By 3.2.7, we have a direct decomposition

$$H_{\mathcal{O}} \simeq \mathcal{H}om(X, \mathcal{O}_S) \oplus \mathcal{O}_S \otimes_{\mathbb{Z}} Y.$$

This shows that the $\mathcal{O}_S$-module $H_{\mathcal{O}}$ is locally free of finite rank and the canonical homomorphism $\iota : \mathcal{O}_S^{\log} \otimes_{\tau^{-1}(\mathcal{O}_S)} \tau^{-1}(H_{\mathcal{O}}) \to \mathcal{O}_S^{\log} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$ is an isomorphism.

Define a decreasing filtration on $H_{\mathcal{O}}$ by

$$F^0 H_{\mathcal{O}} = \mathcal{O}_S \otimes_{\mathbb{Z}} Y = \mathrm{Ker}\,(H_{\mathcal{O}} \to \mathcal{H}om(X, \mathcal{O}_S))$$

in the above direct decomposition, and $F^p H_{\mathcal{O}} = H_{\mathcal{O}}$ for $p \le -1$, $F^p H_{\mathcal{O}} = 0$ for $p \ge 1$.

Then we have $H_{\mathcal{O}} = \mathcal{H}om(X, \mathcal{O}_S) \oplus F^0 H_{\mathcal{O}}$ and hence we obtain an object $(H, X, Y, e)$ of $\hat{H}_S$. This gives a functor $\hat{\mathcal{A}}_S \to \hat{\mathcal{H}}_S$.

**3.2.9.** The above two functors are the inverses of each other. In fact, to see that the composition $\hat{\mathcal{H}}_S \to \hat{\mathcal{A}}_S \to \hat{\mathcal{H}}_S$ is equivalent to the identity, it is enough to construct the diagram as in 3.2.6 for an object $(H, X, Y, e)$ of $\hat{\mathcal{H}}_S$. We have a homomorphism

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Hom}\,(X, \mathbb{Z}(1)) & \to & H_{\mathbb{Z}} & \to & Y & \to & 0 \\
 & & \| & & \downarrow a & & \downarrow b & & \\
0 & \to & \mathrm{Hom}\,(X, \mathbb{Z}(1)) & \to & \mathcal{H}om(X, \mathcal{O}_S^{\log}) & \to & \mathcal{H}om(X, \mathcal{O}_S^{\log}/\mathbb{Z}(1)) & \to & 0
\end{array}$$

of exact sequences. Since the map $b$ factors through $\mathcal{H}om(X, \tau^{-1}(M_S^{\mathrm{gp}}))$ by 3.2.5, the map $a$ factors through $\mathcal{H}om(X, \mathcal{L}_S)$, which gives the desired diagram.

To see that the composition $\hat{\mathcal{A}}_S \to \hat{\mathcal{H}}_S \to \hat{\mathcal{A}}_S$ is equivalent to the identity, take an object $(X, Y, \langle\, ,\, \rangle)$ of $\hat{\mathcal{A}}_S$. Then the associated diagram in 3.2.6 induces the diagram as in the above, the map $a$ induces the splitting in 3.2.7, and this $a$ is to be used to obtain the new pairing. Hence $(X, Y, \langle\, ,\, \rangle)$ coincides with the new pairing.

Thus we get $\hat{\mathcal{H}}_S \simeq \hat{\mathcal{A}}_S$.

## 3.3. The equivalence $\tilde{\mathcal{H}}_S \simeq \tilde{\mathcal{A}}_S$

We define categories $\tilde{\mathcal{A}}_S$ and $\tilde{\mathcal{H}}_S$, which are closely related to $\mathcal{A}_S$ and $\mathcal{H}_S$, respectively, and prove an equivalence

$$\tilde{\mathcal{H}}_S \simeq \tilde{\mathcal{A}}_S.$$

**3.3.1.** Let $\tilde{\mathcal{A}}_S$ be the full subcategory of $\hat{\mathcal{A}}_S$ consisting of non-degenerate pairings (1.2.6).

**3.3.2.** Let $\tilde{\mathcal{H}}_S$ be the full subcategory of $\hat{\mathcal{H}}_S$ consisting of all objects $(H, X, Y, e)$ such that $H$ is a log Hodge structure of weight $-1$.

In other words, $\tilde{\mathcal{H}}_S$ is the category of 4-ples $(H, X, Y, e)$, where $H$ is an object of $\mathcal{H}_S$, $X$ and $Y$ are finitely generated free abelian groups, and $e$ is an exact sequence

$$0 \to \mathrm{Hom}\,(X, \mathbb{Z}(1)) \to H_{\mathbb{Z}} \to Y \to 0$$

on $S^{\log}$ satisfying $H_{\mathcal{O}} = \mathcal{H}om(X, \mathcal{O}_S) \oplus F^0 H_{\mathcal{O}}$.

The aim of this subsection is to prove

3.3.3. Proposition. *The equivalence of categories in 3.2.3 induces an equivalence of the full subcategories*

$$\tilde{\mathcal{H}}_S \simeq \tilde{\mathcal{A}}_S.$$

In the course of the proof of 3.3.3, we prove the following.

3.3.4. Proposition. *Let $H$ be a pre-log Hodge structure over $S$ such that $F^{-1}H_{\mathcal{O}} = H_{\mathcal{O}}$ and $F^1 H_{\mathcal{O}} = 0$. Then $H$ is a log Hodge structure of weight $-1$ if and only if the following conditions* (i) *and* (ii) *are satisfied.*

(i) *The local monodromy of $H_{\mathbb{Q}}$ is admissible.*

(ii) *For each $s \in S$, $\mathrm{gr}_j^{W(C(s))}$ is a Hodge structure of weight $j$ for any $j$.*

*Note here that $\mathrm{gr}_j^{W(C(s))} H_{\mathbb{Z}}$ in* (ii) *is a constant sheaf by* (i).

The only if part is clear. We prove below the if part.

To prove 3.3.3, we first compare the admissibility of a pairing and the admissibility of the local monodromy.

3.3.5. Proposition. *Let $S$ be an fs log analytic space. Assume that we are given a locally constant sheaf of abelian groups $L$ on $S^{\log}$, finitely generated free abelian groups $X$, $Y$, and an exact sequence*

$$0 \to \mathrm{Hom}\,(X, \mathbb{Z}(1)) \to L \to Y \to 0 \qquad (1)$$

*of sheaves of abelian groups on $S^{\log}$. Then there exists a unique $\mathbb{Z}$-bilinear form $\langle\ ,\ \rangle : X \times Y \to \mathbb{G}_{m,\log}/\mathbb{G}_m$ which describes the local monodromy of $L$ as follows: for any $s \in S$ and $g \in \pi_1(\tau^{-1}(s)) = \mathrm{Hom}\,(M^{\mathrm{gp}}_{S,s}/\mathcal{O}^{\times}_{S,s}, \mathbb{Z}(1))$, the action of $g-1$ on $L|_{s^{\log}}$ induces $Y \ni y \mapsto (x \mapsto g(\langle x, y\rangle)) \in \mathrm{Hom}\,(X, \mathbb{Z}(1))$.*

PROOF. The uniqueness is clear. We show the existence of $\langle\ ,\ \rangle$.

By applying $\tau_*$ to the exact sequence (1), we obtain an exact sequence

$$0 \to \mathrm{Hom}\,(X, \mathbb{Z}(1)) \to \tau_*(L) \to Y \to R^1\tau_*\mathrm{Hom}\,(X, \mathbb{Z}(1)). \tag{2}$$

By

$$R^1\tau_*\mathbb{Z}(1) \simeq M^{\mathrm{gp}}_S/\mathcal{O}^{\times}_S \tag{3}$$

([16] (1.5)), we have

$$R^1\tau_*\mathrm{Hom}\,(X, \mathbb{Z}(1)) \simeq \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m).$$

Hence (2) gives a homomorphism $Y \to \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)$, i.e., a $\mathbb{Z}$-bilinear form $\langle\ ,\ \rangle : X \times Y \to \mathbb{G}_{m,\log}/\mathbb{G}_m$. We have to prove that this pairing is the desired one. We may assume that $S$ is the standard log point $(\mathrm{Spec}\,\mathbb{C}, \mathbb{C}^{\times} \oplus \mathbb{N})$ and that $X = Y = \mathbb{Z}$. Then (1) is obtained by the pull back of the exact sequence $0 \to \mathbb{Z}(1) \to \mathcal{L}_S \to \tau^{-1}(M^{\mathrm{gp}}_S) \to 0$ with respect to a homomorphism $Y = \mathbb{Z} \to \tau^{-1}(M^{\mathrm{gp}}_S)$. Since the isomorphism (3) is induced from the last exact sequence, we have the desired compatibility. □

3.3.6. PROPOSITION. *Let the notation be as in Proposition 3.3.5. Let $s \in S$. Then the local monodromy of $L_{\mathbb{Q}}$ is admissible at $s$ if and only if the pairing $\langle\ ,\ \rangle : X \times Y \to M^{\mathrm{gp}}_{S,s}/\mathcal{O}^{\times}_{S,s}$ is admissible.*

PROOF. There exists a bijection between the set of faces of $M_{S,s}/\mathcal{O}^{\times}_{S,s}$ and the set of faces of the monodromy cone $C(s)$ given by $\sigma \leftrightarrow \sigma'$, where $\sigma$ and $\sigma'$ are the annihilators of each other.

If the pairing is admissible, $W(\sigma')$ is defined as follows (we are describing after pulling back to $\tau^{-1}(s)$):

$W(\sigma')_0 = L_{\mathbb{Q}}$, $W(\sigma')_{-1} \cap L$ is the inverse image of $Y_\sigma$ under $L \to Y$, $W(\sigma')_{-2} \cap L$ is the kernel of $\mathrm{Hom}\,(X, \mathbb{Z}(1)) \to \mathrm{Hom}\,(X_\sigma, \mathbb{Z}(1))$, and $W(\sigma')_{-3} = 0$. Then, for $h \in \sigma'$, $(h-1)(L_{\mathbb{Q}}) \subset W(\sigma')_{-2}$ and $(h-1)(W(\sigma')_{-1}) = \{0\}$ since $h(\langle X_\sigma, Y\rangle) = h(\langle X, Y_\sigma\rangle) = 0$. Furthermore, for an element $h$ of $C(s)$ and for faces $\sigma, \tau$ of $M_{S,s}/\mathcal{O}_{S,s}^{\times}$ such that $\tau \subset \sigma$, the condition $\tau = \sigma \cap \mathrm{Ker}\,(h)$ is equivalent to the condition that $h \in \tau'$ and there exists $a \in \sigma'$ such that $a + h$ belongs to the interior of $\tau'$. Since $\log(h)\colon \mathbb{R} \otimes_{\mathbb{Q}} \mathrm{gr}_0^{W(\tau')}(\mathrm{gr}_{-1,t}^{W(\sigma')}) \longrightarrow \mathbb{R} \otimes_{\mathbb{Q}} \mathrm{gr}_{-2}^{W(\tau')}(\mathrm{gr}_{-1,t}^{W(\sigma')})$, $t \in \tau^{-1}(s)$, is isomorphic to $h - 1\colon \mathbb{R} \otimes (Y_\sigma/Y_\tau) \longrightarrow \mathrm{Hom}\,(\mathbb{R} \otimes (X_\sigma/X_\tau), \mathbb{R})$, it is an isomorphism if $\tau = \sigma \cap \mathrm{Ker}\,(h)$. Hence the local monodromy of $L_{\mathbb{Q}}$ is admissible. (Note that we have $(h_1 - 1)(h_2 - 1) = 0$ on $L_{\mathbb{Q}}$ for any $h_1, h_2 \in \pi_1(\tau^{-1}(s))$. Hence $\log(h) = h - 1$ for any $h \in \mathbb{R} \otimes \pi_1(\tau^{-1}(s))$.)

Conversely, assume that the local monodromy of $L_{\mathbb{Q}}$ is admissible. Then for any face $\sigma'$ of $C(s)$ and for any element $h$ of the interior of $\sigma'$, $W(\sigma')_k = L_{\mathbb{Q}}$ for $k \geq 0$, $\mathbb{R} \otimes W(\sigma')_{-1} = \mathrm{Ker}\,(h - 1 : L_{\mathbb{R}} \to L_{\mathbb{R}})$, $\mathbb{R} \otimes W(\sigma')_{-2} = \mathrm{Image}(h - 1 : L_{\mathbb{R}} \to L_{\mathbb{R}})$, $W(\sigma')_{-3} = 0$. Furthermore, we have $W(\sigma')_{-2} \subset \mathrm{Hom}\,(X, \mathbb{Q}(1)) \subset W(\sigma')_{-1}$. Let $X(\sigma') \subset X$ be the annihilator of $W(\sigma')_{-2}$, and let $Y(\sigma') \subset Y$ be the image of $L \cap W(\sigma')_{-1}$ under $L \to Y$.

We show that for any face $\sigma$ of $M_{S,s}/\mathcal{O}_{S,s}^{\times}$, if we denote the annihilator of $\sigma$ in $C(s)$ as $\sigma'$, then $X_\sigma = X(\sigma')$ and $Y_\sigma = Y(\sigma')$. It is clear that $X_\sigma \subset X(\sigma')$. We prove $X(\sigma') \subset X_\sigma$. Let $\mathcal{S}$ be an fs monoid. As is well-known, if $\mathcal{S}^\times = \{1\}$, the set of homomorphisms $N\colon \mathcal{S} \to \mathbb{N}$ with $\mathrm{Ker}\,(N) = \{1\}$ generates the group $\mathrm{Hom}\,(\mathcal{S}^{\mathrm{gp}}, \mathbb{Z})$. In general, since $(\mathcal{S}\sigma^{\mathrm{gp}}/\sigma^{\mathrm{gp}})^\times = \{1\}$ and $\mathcal{S} \cap \sigma^{\mathrm{gp}} = \sigma$, this shows that the intersection of $\mathrm{Ker}\,(\mathcal{S}^{\mathrm{gp}} \to \mathbb{Z})$ for all homomorphisms $N\colon \mathcal{S} \to \mathbb{N}$ with $\mathrm{Ker}\,(N) = \sigma$ coincides with $\sigma^{\mathrm{gp}}$. Let $x \in X(\sigma')$. Then, by the above description of $W(\sigma')$, $\langle x, Y\rangle \subset (M_{S,s}/\mathcal{O}_{S,s}^{\times})^{\mathrm{gp}}$ is contained in $\mathrm{Ker}\,(N^{\mathrm{gp}})$ for any $N \in \sigma'$, and hence, by the previous observation, is contained in $\sigma^{\mathrm{gp}}$. Hence $x \in X_\sigma$. The proof for $Y_\sigma = Y(\sigma')$ is the same.

By definition of $X(\sigma')$ and $Y(\sigma')$, for any $h \in C(s)$ and any face $\sigma$ of $M_{S,s}/\mathcal{O}_{S,s}^{\times}$, the isomorphism $\log(h)\colon \mathbb{R} \otimes_{\mathbb{Q}} \mathrm{gr}_0^{W(\tau')}(\mathrm{gr}_{-1,t}^{W(\sigma')}) \longrightarrow \mathbb{R} \otimes_{\mathbb{Q}} \mathrm{gr}_{-2}^{W(\tau')}(\mathrm{gr}_{-1,t}^{W(\sigma')})$, $t \in \tau^{-1}(s)$, is naturally isomorphic to $h - 1\colon \mathbb{R} \otimes (Y(\sigma')/Y(\tau')) \longrightarrow \mathrm{Hom}\,(\mathbb{R} \otimes (X(\sigma')/X(\tau')), \mathbb{R})$, where $\tau = \sigma \cap \mathrm{Ker}\,(h)$. Therefore $h - 1\colon \mathbb{R} \otimes (Y_\sigma/Y_\tau) \longrightarrow \mathrm{Hom}\,(\mathbb{R} \otimes (X_\sigma/X_\tau), \mathbb{R})$ is an isomorphism. Thus $\langle\ ,\ \rangle$ is admissible. □

3.3.7. Lemma. *For an object $(X, Y, \langle\ ,\ \rangle)$ of $\hat{\mathcal{A}}_S$, the pairing into $\mathbb{G}_{m,\log}/\mathbb{G}_m$ induced by $(X, Y, \langle\ ,\ \rangle)$ coincides with the pairing of the local monodromy of $H_{\mathbb{Z}}$ defined in 3.3.5 above. Here $H_{\mathbb{Z}}$ is the object of $\hat{\mathcal{H}}_S$ corresponding to $(X, Y, \langle\ ,\ \rangle)$.*

Proof. Let $(X, Y, \langle\ ,\ \rangle)$ be an object of $\hat{\mathcal{A}}_S$. From the commutative diagram in 3.2.6, we have a commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & R^1\tau_*\mathrm{Hom}\,(X, \mathbb{Z}(1)) \arrow[d, "\wr"'] \\ \mathcal{H}om(X, M_S^{\mathrm{gp}}) \arrow[r] & \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m). \end{tikzcd}$$

The composite $Y \to \mathcal{H}om(X, M_S^{\mathrm{gp}}) \to \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ gives the induced pairing into $\mathbb{G}_{m,\log}/\mathbb{G}_m$ from the original pairing. On the other hand, the composite $Y \to R^1\tau_*\mathrm{Hom}\,(X, \mathbb{Z}(1)) \to \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ is the pairing from the local monodromy of $H_{\mathbb{Z}}$ as is seen in the proof of 3.3.5. Thus the both coincide. $\square$

This lemma together with 3.3.6 implies that the condition (i) in 1.2.6 for $(X, Y, \langle\ ,\ \rangle)$ is equivalent to the admissibility of the local monodromy of $H_{\mathbb{Q}}$.

We next prove the "trivial base case" of Proposition 3.3.3.

3.3.8. Lemma. *Proposition 3.3.3 is true in the case where $S$ is* Spec$(\mathbb{C})$ *with the trivial log structure.*

Proof. This lemma should be well-known. For the convenience of readers, we describe a proof. Let $(X, Y, \langle\ ,\ \rangle)$ be a pairing into $\mathbb{C}^\times$. Then the corresponding $H$ is defined as follows: $H_{\mathbb{Z}}$ is the fiber product of $\mathrm{Hom}\,(X, \mathbb{C}) \overset{\exp}{\to} \mathrm{Hom}\,(X, \mathbb{C}^\times) \leftarrow Y$, and $F^0H_{\mathbb{C}}$ is the kernel of the canonical map $H_{\mathbb{C}} \to \mathrm{Hom}\,(X, \mathbb{C})$.

It is sufficient to prove that $H$ is a Hodge structure of weight $-1$ if and only if

$$b : \mathbb{R} \otimes_{\mathbb{Z}} X \times \mathbb{R} \otimes_{\mathbb{Z}} Y \to \mathbb{R}\,;\ (x, y) \mapsto -\log(|\langle x, y\rangle|)$$

$(x \in X, y \in Y)$ is non-degenerate.

In fact, $H$ is a Hodge structure of weight $-1$ if and only if $H_{\mathbb{C}}$ is the direct sum of $F^0H_{\mathbb{C}}$ and its complex conjugate. Since $H_{\mathbb{C}} = F^0H_{\mathbb{C}} \oplus \mathrm{Hom}\,(X, \mathbb{C})$, the last condition is equivalent to the condition that the homomorphism

$$c : F^0H_{\mathbb{C}} \to \mathrm{Hom}\,(X, \mathbb{C}); f \mapsto (\bar{f} \bmod F^0H_{\mathbb{C}})$$

is an isomorphism. Here $\bar{f}$ is the complex conjugate. As is shown below, $c$ coincides with the composite $F^0H_{\mathbb{C}} \overset{\simeq}{\to} \mathbb{C} \otimes_{\mathbb{Z}} Y \to \mathrm{Hom}\,(X, \mathbb{C})$, where the second map is $z \otimes y \mapsto (x \mapsto -2\bar{z}b(x,y))$. Hence $c$ is an isomorphism if and only if $b$ is non-degenerate.

We prove the above description of $c$ by $b$. Let $h \in H_{\mathbb{Z}}$ and write $h = f+g$ with $f \in F^0 H_{\mathbb{C}}$ and $g \in \mathrm{Hom}\,(X, \mathbb{C})$. Let $y$ be the image of $h$ in $Y$. Then for $x \in X$, $b(x,y) = -\Re(g(x))$, where $\Re$ denotes the real part. On the other hand,

$$\bar{f} = h - \bar{g} = f + g - \bar{g} \equiv g - \bar{g} \bmod F^0 H_{\mathbb{C}}.$$

Since the complex conjugate of $H_{\mathbb{C}}$ induces the identity map on $\mathrm{Hom}\,(X, \mathbb{Z}(1))$ (not on $\mathrm{Hom}\,(X, \mathbb{Z})$), $\bar{g}$ sends $x \in X$ to the complex conjugate of $-g(x)$. Hence $c(f)(x) = (g - \bar{g})(x) = 2\Re(g(x))$. $\square$

Now we will prove the general case of 3.3.3 together with 3.3.4. We explain the plan. Let $(H, X, Y, e)$ be an object of $\hat{\mathcal{H}}_S$, and let $(X, Y, \langle\ ,\ \rangle)$ be its corresponding object of $\hat{\mathcal{A}}_S$.

First, as noted before, it is clear that if $(H, X, Y, e)$ belongs to $\tilde{\mathcal{H}}_S$, then $H$ satisfies the conditions (i) and (ii) in 3.3.4.

Second, in 3.3.9 below, we will show that $H$ satisfies the conditions (i) and (ii) if and only if the corresponding $(X, Y, \langle\ ,\ \rangle)$ belongs to $\tilde{\mathcal{A}}_S$.

Third, from 3.3.10, we will show that if $(X, Y, \langle\ ,\ \rangle)$ belongs to $\tilde{\mathcal{A}}_S$, then the corresponding $(H, X, Y, e)$ belongs to $\tilde{\mathcal{H}}_S$, which completes the proofs of 3.3.3 and 3.3.4 simultaneously.

**3.3.9.** Let $(X, Y, \langle\ ,\ \rangle)$ be an object of $\hat{\mathcal{A}}_S$, and let $(H, X, Y, e)$ be the corresponding object of $\hat{\mathcal{H}}_S$. We show that $H$ satisfies the conditions (i) and (ii) of 3.3.4 if and only if $(X, Y, \langle\ ,\ \rangle)$ belongs to $\tilde{\mathcal{A}}_S$.

As noted before, (i) in 3.3.4 is equivalent to the admissibility of $X \times Y \to \mathbb{G}_{m,\log}/\mathbb{G}_m$ by 3.3.6 and 3.3.7. Let $s \in S$. Since $W(C(s))_{-2} \subset \mathrm{Hom}\,(X, \mathbb{Q}(1)) \subset W(C(s))_{-1}$, $\mathrm{gr}_j^{W(C(s))}$ is a Hodge structure unless $j = -1$. It remains to show that $\mathrm{gr}_{-1}^{W(C(s))}$ is a Hodge structure of weight $-1$ if and only if for the pull back of $(X, Y, \langle\ ,\ \rangle)$ to $s$, the pairing $\mathbb{R} \otimes_{\mathbb{Z}} X_{\{1\}} \times \mathbb{R} \otimes_{\mathbb{Z}} Y_{\{1\}} \to \mathbb{R}$ is non-degenerate. For this, by replacing $X$ by $X_{\{1\}}$, $Y$ by $Y_{\{1\}}$, and $H$ by $W(C(s))_{-1}/W(C(s))_{-2}$ of $H$, we may assume that $H$ is the pull back of a usual Hodge structure over $\mathrm{Spec}(\mathbb{C})$ with the trivial log structure, and hence we are reduced to 3.3.8.

**3.3.10.** Let $(X, Y, \langle\ ,\ \rangle)$ be an object of $\hat{\mathcal{A}}_S$, and let $(H, X, Y, e)$ be the corresponding object of $\hat{\mathcal{H}}_S$. Assume that $(X, Y, \langle\ ,\ \rangle)$ belongs to $\tilde{\mathcal{A}}_S$. We prove that $(H, X, Y, e)$ belongs to $\tilde{\mathcal{H}}_S$, that is, $H$ is a log Hodge structure of weight $-1$.

The proof is long and occupies the rest of this subsection.

With 3.3.9, this proves Proposition 3.3.3, and also proves Proposition 3.3.4.

The admissibility of the local monodromy of $H_{\mathbb{Q}}$ follows from 3.3.6.

Since $F^{-1}H_{\mathcal{O}} = H_{\mathcal{O}}$ and $F^1 H_{\mathcal{O}} = 0$, the Griffiths transversality is automatically satisfied.

It remains to prove the following. Let $s \in S$, $\sigma'$ a face of $C(s)$, $j \in \mathbb{Z}$, and $t \in \tau^{-1}(s)$. Then for any sufficiently shifted specialization $a : \mathcal{O}_{S,t}^{\log} \to \mathbb{C}$ at $t$, $(\mathrm{gr}_j^{W(\sigma')}, F)(a)$ is a Hodge structure of weight $j$. Note that $\mathrm{gr}_j^{W(\sigma')} = 0$ unless $j \in \{0, -1, -2\}$.

**3.3.11.** Assume $j = 0$.

By the direct decomposition $H_{\mathcal{O}} = \mathcal{H}om(X, \mathcal{O}_S) \oplus F^0 H_{\mathcal{O}}$ and by the fact $\mathcal{H}om(X, \mathcal{O}_S) \subset W(\sigma')_{-1} H_{\mathcal{O}}$, the canonical map $F^0 H_{\mathcal{O}}(a) \to \mathrm{gr}_0^{W(\sigma')}(H_{\mathcal{O}})(a)$ is surjective, and hence $(\mathrm{gr}_0^{W(\sigma')}, F)(a)$ is a Hodge structure of weight 0 for any specialization $a$ at $t$.

**3.3.12.** By duality, $(\mathrm{gr}_{-2}^{W(\sigma')}, F)(a)$ is a Hodge structure of weight $-2$ for any specialization $a$ at $t$.

**3.3.13.** It remains to consider the case $j = -1$. We may assume that the underlying analytic space of $S$ is $\operatorname{Spec} \mathbb{C} = \{s\}$.

Let $\sigma$ be the face of $M_{S,s}/\mathcal{O}_{S,s}^{\times}$ corresponding to $\sigma'$. Then $(X_\sigma, Y_\sigma, \langle\ ,\ \rangle_\sigma)$, where $\langle\ ,\ \rangle_\sigma$ is the restriction of $\langle\ ,\ \rangle$ to $X_\sigma \times Y_\sigma$, belongs to $\tilde{\mathcal{A}}_S$. Hence by replacing $(X_\sigma, Y_\sigma, \langle\ ,\ \rangle_\sigma)$ by $(X, Y, \langle\ ,\ \rangle)$, we see that it is sufficient to prove that for a sufficiently shifted specialization $a$, $H(a)$ is a Hodge structure of weight $-1$.

Let

$$b_a : \mathbb{R} \otimes_{\mathbb{Z}} X \times \mathbb{R} \otimes_{\mathbb{Z}} Y \to \mathbb{R}$$

be the pairing defined by composing $\langle\ ,\ \rangle : X \times Y \to M_{S,s}^{\mathrm{gp}}$,

$$M_{S,s}^{\mathrm{gp}} \to \mathbb{C}^{\times}\ ;\ f \mapsto \exp(a(\log(f)))$$

(independent of the choice of $\log(f) \in \mathcal{L}_{S,t}$ which is determined only modulo $\mathbb{Z}(1)$), and $-\log(|?|) : \mathbb{C}^\times \to \mathbb{R}$. We have a commutative diagram

$$\begin{array}{ccc} \hat{\mathcal{H}}_S & \simeq & \hat{\mathcal{A}}_S \\ \downarrow & & \downarrow \\ \hat{\mathcal{H}}_{\mathrm{Spec}(\mathbb{C})} & \simeq & \hat{\mathcal{A}}_{\mathrm{Spec}(\mathbb{C})}, \end{array}$$

where $\mathrm{Spec}(\mathbb{C})$ is endowed with the trivial log structure and the vertical arrows are defined by the specialization $a$; the left vertical arrow is given by $M_{S,s} \to \mathbb{C}^\times$ ; $f \mapsto \exp(a(\log(f)))$. By 3.3.8, $H(a)$ is a Hodge structure of weight $-1$ if and only if $b_a$ is non-degenerate.

3.3.14. Lemma. *Fix a specialization $a$. Let $N_j : X \times Y \to \mathbb{Z}$ $(1 \le j \le n)$ be the pairings induced by $\langle\, , \rangle$ and a set of elements of the monodromy cone $C(s)$ which generates $C(s)$. Then $b_{a'}$ for a sufficiently shifted $a'$ is $b_a + y_1 N_1 + \cdots + y_n N_n$ with $y_j >> 0$.*

Proof. Fix a splitting $\mathcal{L}_{S,t} \cong P^{\mathrm{gp}} \oplus \mathbb{C}$, where $P := M_{S,s}/\mathcal{O}^\times_{S,s}$. Then we identify the set of specializations with the set $\mathrm{Hom}\,(P^{\mathrm{gp}}, \mathbb{C})$, and $b_a = b_{a'}$ if the corresponding homomorphisms of $a, a'$ have the same real part. Take a set of generators $(l_k)_k$ of $P$. Assume that $l_k \neq 1$ for any $k$. It is enough to show that for any $a, N_1, \cdots, N_n \in \mathrm{Hom}\,(P^{\mathrm{gp}}, \mathbb{R})$ such that $N_1, \cdots, N_n$ generate $\mathrm{Hom}\,(P, \mathbb{R}_{\ge 0})$, and for any $C$, there exists $C'$ such that $\{a + \sum y_j N_j | y_j \ge C\} \supset \{N | N(l_k) \ge C'$ for any $k\}$. But it is enough to take $C' := \max_k (a + C \sum N_j)(l_k)$. $\square$

For $y_1, \cdots, y_n > 0$, let $f(y_1, \cdots, y_n) = b_a + y_1 N_1 + \cdots + y_n N_n : \mathbb{R} \otimes_{\mathbb{Z}} X \times \mathbb{R} \otimes_{\mathbb{Z}} Y \to \mathbb{R}$.

We prove that there exists $C > 0$ such that if $y_j > C$ $(1 \le j \le n)$, then $f(y_1, \cdots, y_n)$ is non-degenerate.

It is enough to prove this under the assumption $y_1 \ge \cdots \ge y_n \ge 1$.

Let $[0, 1]$ be the closed interval $\{x \in \mathbb{R} \,|\, 0 \le x \le 1\}$.

3.3.15. Lemma. *Let $\alpha \in [0, 1]^n$ and assume $\alpha_n = 0$. Then there exists a neighborhood $U_\alpha$ of $\alpha$ in $[0, 1]^n$ such that $f(y_1, \cdots, y_n)$ is non-degenerate for any $y_j$ $(1 \le j \le n)$ such that $(y_{j+1}/y_j)_{1 \le j \le n} \in U_\alpha$, where $y_{n+1}$ denotes $1$.*

Assume 3.3.15. For each $\beta \in [0,1]^{n-1}$, take an open neighborhood $V_\beta$ of $\beta$ in $[0,1]^{n-1}$ and $C_\beta > 0$ such that $V_\beta \times [0, C_\beta^{-1})$ is contained in $U_{(\beta,0)}$. Since $[0,1]^{n-1} = \cup_\beta V_\beta$ and $[0,1]^{n-1}$ is compact, there is a finite subset $\{\beta_1, \cdots, \beta_r\}$ of $[0,1]^{n-1}$ such that $[0,1]^{n-1} = \cup_{1\le j\le r} V_{\beta_j}$. Let $C = \max\{C_{\beta_j} \mid 1 \le j \le r\}$. Then $f(y_1, \cdots, y_n)$ is non-degenerate whenever $y_j > C$ $(1 \le j \le n)$.

We prove 3.3.15.

Let $T = \{t \mid 1 \le t \le n, \alpha_t = 0\}$. For $t \in T$, let

$$J(t) = \{j \mid 1 \le j \le n, t = \min\{t' \in T \mid j \le t'\}\}.$$

Then $\{1, \cdots, n\}$ is the disjoint union of $J(t)$ $(t \in T)$.

For $t \in T$, let $\sigma'(t)$ be the smallest face of the monodromy cone $C(s)$ containing $N_j$ for any $j \le t$, and let $\sigma(t)$ be the corresponding face of $M_{S,s}/\mathcal{O}_{S,s}^\times$. Then if $t_1, t_2 \in T$ and $t_1 \le t_2$, we have $\sigma'(t_1) \subset \sigma'(t_2)$, $\sigma(t_1) \supset \sigma(t_2)$, $Y_{\sigma(t_1)} \supset Y_{\sigma(t_2)}$. For $t \in T$, fix an $\mathbb{R}$-subspace $V_t$ of $\mathbb{R} \otimes_\mathbb{Z} Y$ such that

$$\mathbb{R} \otimes_\mathbb{Z} Y = (\oplus_{t\in T} V_t) \oplus \mathbb{R} \otimes_\mathbb{Z} Y_{\{1\}},$$

$$Y_{\sigma(t)} = (\oplus_{t'\in T, t'>t} V_{t'}) \oplus \mathbb{R} \otimes_\mathbb{Z} Y_{\{1\}} \quad \text{for all } t \in T.$$

Define $g(y_1, \cdots, y_n) : \mathbb{R} \otimes_\mathbb{Z} X \times \mathbb{R} \otimes_\mathbb{Z} Y \to \mathbb{R}$ by

$$g(y_1, \cdots, y_n)(x, (\sum_{t\in T} v_t) + w) = f(y_1, \cdots, y_n)(x, (\sum_{t\in T} y_t^{-1} v_t) + w)$$

$(x \in \mathbb{R} \otimes X, v_t \in V_t, w \in \mathbb{R} \otimes_\mathbb{Z} Y_{\{1\}})$.

Then $g(y_1, \cdots, y_n)$ depends only on $(y_{j+1}/y_j)_{1\le j\le n} \in [0,1]^n$ as well as $f(y_1, \cdots, y_n)$ does. ($y_{n+1}$ denotes 1.) We have

$$\det(f(y_1, \cdots, y_n)) = (\prod_{t\in T} y_t^{d(t)}) \det(g(y_1, \cdots, y_n)).$$

Here $d(t) = \dim(V_t)$. Hence

3.3.16. LEMMA. *$f(y_1, \cdots, y_n)$ is non-degenerate if and only if $g(y_1, \cdots, y_n)$ is non-degenerate.*

Define $g_\alpha : \mathbb{R} \otimes_\mathbb{Z} X \times \mathbb{R} \otimes_\mathbb{Z} Y \to \mathbb{R}$ by

$$g_\alpha(x, (\sum_{t\in T} v_t) + w) = (\sum_{t\in T} \sum_{j\in J(t)} (\prod_{j\le k<t} \alpha_k^{-1}) N_j(x, v_t)) + b_a(x, w)$$

$(v_t \in V_t, w \in \mathbb{R} \otimes_{\mathbb{Z}} Y_{\{1\}})$. Then as is easily seen,

3.3.17. Lemma. *When* $(y_{j+1}/y_j)_{1\leq j\leq n}$ *converges to* $\alpha$*,* $g(y_1, \cdots, y_n)$ *converges to* $g_\alpha$*.*

3.3.18. Lemma. $g_\alpha$ *is non-degenerate.*

Proof. This is because $(X, Y, \langle\, ,\, \rangle)$ is non-degenerate. More precisely, let $N'_t = \sum_{j\in J(t)} (\prod_{j\leq k<t} \alpha_k^{-1}) N_j$ for any $t \in T$. Then $\sigma(t) \cap \mathrm{Ker}\,(N'_{t_1}) = \sigma(t_1)$ if $t \in T$ and $t_1 = \min\{t' \in T | t < t'\}$. Let $x \in \mathbb{R} \otimes X$. Assume that $g_\alpha(x, y) = 0$ for any $y \in \mathbb{R}\otimes Y$, that is, that $N'_t(x, v_t) = b_a(x, w) = 0$ for any $t \in T$, $v_t \in V_t$, $w \in \mathbb{R} \otimes Y_{\{1\}}$. Then we see inductively $x \in \mathbb{R} \otimes X_{\sigma(t)}$ for any $t \in T$. Thus $x \in \mathbb{R} \otimes X_{\{1\}}$ and $x = 0$. Similarly, let $v_t \in V_t \quad (t \in T)$ and $w \in \mathbb{R}\otimes Y_{\{1\}}$. Assume that $\sum_{t\in T} N'_t(x, v_t) + b_a(x, w) = 0$ for any $x \in \mathbb{R}\otimes X$, then $b_a(x, w) = 0$ for any $x \in \mathbb{R} \otimes X_{\{1\}}$, hence $w = 0$. Inductively all the $v_t$'s are also zeros. $\square$

These 3.3.16, 3.3.17, 3.3.18 prove the existence of $U_\alpha$.

### 3.4. On the cone $C$ and its subcones

In this subsection and in the next subsection, we consider cone decompositions and toric geometry which will be used to complete the proof of $\mathcal{H}_S \simeq \mathcal{A}_S$. The results of these two subsections will be also used in §4 to prove Theorem 4.6.5 and in §5 to construct models of log complex tori.

**3.4.1.** In this subsection, let $\mathcal{S}$ be an fs monoid, let $X$ and $Y$ be finitely generated free abelian groups, and let $\langle\, ,\, \rangle : X \times Y \to \mathcal{S}^{\mathrm{gp}}$ be an $\mathcal{S}$-admissible pairing.

For a finitely generated abelian group $L$ and a subset $P$ of $L$, we say $P$ is a cone of $L$ if $P$ is a submonoid of $L$ and if $P$ is saturated in $L$ (that is, for $a \in L$ such that $a^n \in P$ for some $n \geq 1$, we have $a \in P$).

**3.4.2.** We define a subset $C$ of $\mathrm{Hom}\,(\mathcal{S}, \mathbb{N}) \times \mathrm{Hom}\,(X, \mathbb{Z})$. Let

$$C := \left\{(N, l) \in \mathrm{Hom}\,(\mathcal{S}, \mathbb{N}) \times \mathrm{Hom}\,(X, \mathbb{Z}); l(X_{\mathrm{Ker}\,(N)}) = \{0\}\right\}.$$

Here $X_\sigma$ with $\sigma = \mathrm{Ker}\,(N)$ is as in 1.2.3.

3.4.3. Lemma. *(1) $C$ is a cone of* $\mathrm{Hom}\,(\mathcal{S}, \mathbb{N}) \times \mathrm{Hom}\,(X, \mathbb{Z})$*.*

*(2) The group of the invertible elements $C^\times$ of $C$ is trivial.*

We will prove this lemma in 3.4.6.

3.4.4. REMARK. We remark that the monoid $C$ is not necessarily finitely generated.

We denote the semi-group law of $C$ additively.

3.4.5. LEMMA. *Let $\langle\ ,\ \rangle\colon X \times Y \to \mathcal{S}^{\mathrm{gp}}$ be an admissible pairing.*

*(1) For faces $\sigma$ and $\tau$ of $\mathcal{S}$, we have*

$$X_\sigma \cap X_\tau = X_{\sigma\cap\tau},\ Y_\sigma \cap Y_\tau = Y_{\sigma\cap\tau}.$$

*(2) Let $x \in X$ (resp. $y \in Y$). Then the set of all faces $\sigma$ of $\mathcal{S}$ such that $\langle x, Y\rangle \subset \sigma^{\mathrm{gp}}$ (resp. $\langle X, y\rangle \subset \sigma^{\mathrm{gp}}$) has the smallest element.*

*(3) Let $x \in X$ (resp. $y \in Y$) and let $\sigma$ be the smallest face of $\mathcal{S}$ satisfying the property in (2). Then there exists an element $y \in Y_\sigma$ (resp. $x \in X_\sigma$) such that $\langle x, y\rangle$ belongs to the interior (see* Terminology*) of $\sigma$.*

*(4) If there exists a homomorphism $p$ satisfying the conditions in 1.2.5, then, for $y \in Y$, a face $\sigma$ of $\mathcal{S}$ is the minimal face satisfying $y \in Y_\sigma$ if and only if $\langle p(y), y\rangle$ is in the interior of $\sigma$.*

PROOF. (1) By symmetry, it is enough to show the first equality. There exist homomorphisms $N, N'\colon \mathcal{S} \to \mathbb{N}$ such that $\mathrm{Ker}\,(N) = \sigma$ and $\mathrm{Ker}\,(N') = \tau$. Let $N'' = N + N'$. Then $\mathrm{Ker}\,(N'') = \sigma \cap \tau$. We show $X_\sigma \cap X_\tau = X_{\sigma\cap\tau}$. The inclusion $X_{\sigma\cap\tau} \subset X_\sigma \cap X_\tau$ is clear. We prove the converse inclusion. Let $x \in X_\sigma \cap X_\tau$. Since $N(\langle x, Y\rangle) = 0$ and $N'(\langle x, Y\rangle) = 0$, we have $N''(\langle x, Y\rangle) = 0$. Since $\mathrm{Ker}\,(N'') = \sigma \cap \tau$, by the admissibility, $x$ belongs to $X \cap (\mathbb{R} \otimes X_{\sigma\cap\tau}) = X_{\sigma\cap\tau}$.

(2) follows from (1).

(3) By symmetry, it is enough to prove the statement for $x \in X$. Let $H = \langle x, Y_\sigma\rangle \subset \sigma^{\mathrm{gp}}$, and assume that $H$ does not meet the interior of $\sigma$. Let $\overline{\sigma}$ be the image of $\sigma$ in $\sigma^{\mathrm{gp}}/H$. Then if $a$ is an element of the interior of $\sigma$, the image $\overline{a}$ of $a$ in $\overline{\sigma}$ is not invertible in $\overline{\sigma}$. (Indeed, if $\overline{a} \cdot \overline{b} = 1$ in $\overline{\sigma}$ for some $b \in \sigma$, we have $ab \in H$. This contradicts the fact that $H$ does not meet the interior of $\sigma$ since $ab$ is in the interior of $\sigma$.) Thus $\overline{\sigma}$ is not a group. Hence there exists a non-trivial homomorphism $\overline{\sigma} \to \mathbb{N}$. That is, there exists a non-trivial homomorphism $h\colon \sigma \to \mathbb{N}$ such that the induced

map $\sigma^{\mathrm{gp}} \to \mathbb{Z}$ kills $H$. Since $\sigma$ is a face of $\mathcal{S}$, the homomorphism $h$ extends to $N\colon \mathcal{S} \to (1/m)\mathbb{N}$ for some $m \geq 1$. Let $\tau = \sigma \cap \mathrm{Ker}\,(N)$. Then $x$ belongs to $X_\tau$. The minimality of $\sigma$ implies that $\sigma = \tau$. This contradicts the fact that $N$ does not kill $\sigma$.

(4) It is enough to show that for a face $\sigma$ of $\mathcal{S}$, $\langle p(y), y\rangle \in \sigma$ if and only if $y \in Y_\sigma$. By (iii) in 1.2.5, $y \in Y_\sigma$ implies $\langle p(y), y\rangle \in \mathcal{S} \cap \sigma^{\mathrm{gp}} = \sigma$. We will prove the converse. Assume that $\langle p(y), y\rangle \in \sigma$. Take a homomorphism $N\colon \mathcal{S} \longrightarrow \mathbb{R}^{(+)}_{\geq 0}$ whose kernel is $\sigma$. To prove $y \in Y_\sigma$, it is enough to show $N(\langle p(y), z\rangle) = 0$ for any $z \in Y$. This can be seen as in the last part of the proof of 1.2.5. □

**3.4.6.** We prove 3.4.3.

Proof. (1) Let $(N, l), (N', l') \in C$. Since

$$X_{\mathrm{Ker}\,(N+N')} = X_{\mathrm{Ker}\,(N) \cap \mathrm{Ker}\,(N')} = X_{\mathrm{Ker}\,(N)} \cap X_{\mathrm{Ker}\,(N')}$$

(see 3.4.5 (1)), the homomorphism $l+l'$ kills $X_{\mathrm{Ker}\,(N+N')}$. Hence $(N+N', l+l') \in C$.

It is easy to see that $C$ is saturated in $\mathrm{Hom}\,(\mathcal{S}, \mathbb{N}) \times \mathrm{Hom}\,(X, \mathbb{Z})$.

(2) Assume $(N, l), (-N, -l) \in C$. Then $N = 0$. Hence $l$ kills $X_{\mathcal{S}} = X$. Hence $l = 0$. □

3.4.7. Lemma. *Let $\Delta$ be a finitely generated subcone of $C$, and let $\Delta^\vee$ be the dual cone of $\Delta$ in $\mathcal{S}^{\mathrm{gp}} \times X$, i.e.,*

$$\Delta^\vee := \{(\mu, x) \in \mathcal{S}^{\mathrm{gp}} \times X; N(\mu) + l(x) \geq 0 \textit{ for all } (N, l) \in \Delta\}\,.$$

*(1) Identify $\mathcal{S}$ with $\mathcal{S} \times \{1\}$ in $\mathcal{S} \times X$. Then $\mathcal{S} \subset \Delta^\vee$.*

*(2) $(\Delta^\vee)^{\mathrm{gp}} = \mathcal{S}^{\mathrm{gp}} \times X$.*

*(3) Let $\sigma$ be a face of $\mathcal{S}$ and let $x \in X_\sigma$. Then there exists $\mu \in \sigma$ such that $(\mu, x) \in \Delta^\vee$.*

Proof. (1) is clear because $\Delta \subset C$. (2) follows from $\Delta^\times = C^\times = \{0\}$ by 3.4.3 (2).

We prove (3). Let $(N_i, \ell_i)$ $(1 \leq i \leq n)$ be elements of $\Delta$ which generate the cone $\Delta$. For each $i$, take $\mu_i \in \sigma$ as follows. If $N_i(\sigma) = \{0\}$, take any element $\mu_i$ of $\sigma$. If $N_i(\sigma) \neq \{0\}$, take any element $\mu_i$ of $\sigma$ such that

$N_i(\mu_i) + \ell_i(x) \geq 0$ (such $\mu_i$ exists; take any $\mu_i' \in \sigma$ such that $N_i(\mu_i') > 0$, and multiply $\mu_i'$ to get $\mu_i$). Let $\mu = \prod_{1 \leq i \leq n} \mu_i \in \sigma$. We show $(\mu, x) \in \Delta^\vee$. In fact, if $N_i(\sigma) = \{0\}$, then $\ell_i(x) = 0$ and hence $N_i(\mu) + \ell_i(x) = N_i(\mu) \geq 0$. If $N_i(\sigma) \neq \{0\}$, then $N_i(\mu) + \ell_i(x) \geq N_i(\mu_i) + \ell_i(x) \geq 0$. $\square$

In the rest of this subsection, we prove the following proposition.

3.4.8. PROPOSITION. *Let $\Delta$ be a finitely generated subcone of $C$. Then there exists a finitely generated subcone $\Delta'$ of $C$ containing $\Delta$ and satisfying the following conditions* (i) *and* (ii).

(i) *If $(\mu, x) \in (\Delta')^\vee$ ($\mu \in \mathcal{S}^{\mathrm{gp}}, x \in X$), then $\mu \in \mathcal{S}$;*

(ii) *Let $\sigma$ be a face of $\mathcal{S}$. If $(\mu, x) \in (\Delta')^\vee$ and $x \notin X_\sigma$, then there exist $\mu', \mu'' \in \mathcal{S}$ such that $\mu = \mu'\mu''$ and such that $(\mu', x) \in \Delta^\vee$ and $\mu'' \notin \sigma$.*

**3.4.9.** For each face $\sigma$ of $\mathcal{S}$, fix an element $s_\sigma$ of the interior of $\sigma$ and fix a $\mathbb{Z}$-basis $(x_{\sigma,i})_i$ of $X_\sigma$. For an integer $a \geq 0$, let

$$C(a) := \{(N, l) \in \mathrm{Hom}\,(\mathcal{S}, \mathbb{N}) \times \mathrm{Hom}\,(X, \mathbb{Z}); \\ a \cdot N(s_\sigma) \geq |l(x_{\sigma,i})| \text{ for all } \sigma \text{ and } i\}\,.$$

3.4.10. LEMMA. *(1) $C(a)$ is a finitely generated subcone of $C$.*
*(2) $\bigcup_a C(a) = C$.*

PROOF. (1) It is easily seen that $C(a)$ is a finitely generated monoid. We prove $C(a) \subset C$. Let $(N, l) \in C(a)$, and let $\sigma = \mathrm{Ker}\,(N)$. Then $N(s_\sigma) = 0$ and hence $l(x_{\sigma,i}) = 0$ for all $i$. This shows $l(X_\sigma) = 0$ and hence $(N, l) \in C$.

(2) Let $(N, l) \in C$. We prove $(N, l) \in C(a)$ if $a$ is sufficiently large. Let $\sigma$ be a face of $\mathcal{S}$. If $N(\sigma) = 0$, then by $(N, l) \in C$, we have $l(X_\sigma) = 0$ and hence the inequality $a \cdot N(s_\sigma) \geq |l(x_{\sigma,i})|$ holds for any $a \geq 0$. If $N(\sigma) \neq 0$, then $N(s_\sigma) > 0$ and hence the inequality $a \cdot N(s_\sigma) \geq |l(x_{\sigma,i})|$ holds if $a$ is sufficiently large. $\square$

**3.4.11.** We now prove Proposition 3.4.8.

Let $s_\sigma$, $(x_{\sigma,i})_i$, and $C(a)$ be as in 3.4.9. Further, for each face $\sigma$ of $\mathcal{S}$, fix a homomorphism $N_\sigma\colon \mathcal{S} \to \mathbb{N}$ such that $\mathrm{Ker}\,(N_\sigma) = \sigma$. Fix also a finite subset $B_\sigma$ of $\mathrm{Hom}\,(X,\mathbb{Z})$ which generates the abelian group $\{l \in \mathrm{Hom}\,(X,\mathbb{Z}) \,|\, l(X_\sigma)=0\}$. By $\bigcup_a C(a) = C$ (3.4.10 (2)), the following condition $(*)$ is satisfied if $a$ is large enough:

$$(*) \qquad (N_\sigma, l) \in C(a) \text{ for any } \sigma \text{ and } l \in B_\sigma.$$

Since $\Delta \subset C(a)$ for some of such $a > 0$, we may assume $\Delta = C(a)$ for some $a > 0$ satisfying $(*)$.

Fix a finite set of generators $\{(N_i, l_i)\}_{1\le i\le r}$ of $\Delta = C(a)$, and fix also a finite set of generators $J_\sigma$ of the ideal $I_\sigma := \mathcal{S} - \sigma$ (see Terminology) of $\mathcal{S}$ for every $\sigma$. Now let $m \ge 1$ be a sufficiently large integer satisfying the following $(**)$:

$(**)$ $m > \max\{N_i(\mu); 1 \le i \le r, \mu \in J_\sigma, \sigma \text{ is a face of } \mathcal{S}\}$.

We prove that $\Delta' = C(am)$ has the property stated in 3.4.8. Let $(\mu, x) \in (\Delta')^\vee$.

First, since $(N_i, \pm m l_i) \in \Delta'$, we have $N_i(\mu) \ge |m l_i(x)| \ge 0$ for all $i$, which implies (i). We prove (ii). Let $\sigma$ and $(\mu, x)$ be as in (ii).

Take $l \in B_\sigma$ such that $l(x) \ne 0$. Since $(N_\sigma, \pm l) \in C(a)$ by $(*)$, $aN_\sigma(\mu) \ge |l(x)| > 0$. Then, this shows $\mu \in I_\sigma$. Hence $\mu = \mu' \cdot \mu''$ for some $\mu' \in \mathcal{S}$ and $\mu'' \in J_\sigma$. We prove $(\mu', x) \in \Delta^\vee$. It is sufficient to prove $N_i(\mu') + l_i(x) \ge 0$ for $1 \le i \le r$. If $l_i(x) = 0$, we have nothing to prove. If $l_i(x) \ne 0$,

$$\begin{aligned} N_i(\mu') + l_i(x) &= N_i(\mu) - N_i(\mu'') + l_i(x) \\ &\ge m|l_i(x)| - \max(\{N_i(\mu'')\}_i) + l_i(x) \\ &\quad \text{(this follows from } (N_i, \pm m l_i) \in \Delta') \\ &\ge m - 1 - \max(\{N_i(\mu'')\}_i) \ge 0 \text{ (by } (**)). \end{aligned}$$

### 3.5. Study of $\mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)}$

**3.5.1.** Let $\langle\, ,\rangle \colon X \times Y \to \mathcal{S}^{\mathrm{gp}}$ be as in 3.4. In this subsection, let $S$ be an fs log analytic space and assume that we are given a homomorphism $\mathcal{S} \to M_S/\mathcal{O}_S^\times$.

The induced pairing $X \times Y \to \mathbb{G}_{m,\log,S}/\mathbb{G}_{m,S}$ defines a subgroup sheaf $\mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)}$ as in 1.3.1. In fact in 1.3.1, we defined $\mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)}$ for a pairing $X \times Y \to \mathbb{G}_{m,\log,S}$, but the definition relies only on the induced pairing into $\mathbb{G}_{m,\log,S}/\mathbb{G}_{m,S}$.

For $s \in S$, let $\sigma(s)$ be the kernel of $\mathcal{S} \to M_{S,s}/\mathcal{O}_{S,s}^\times$, which is a face of $\mathcal{S}$.

**3.5.2.** For a finitely generated subcone $\Delta$ of the cone $C$ (3.4.2), define a subsheaf $V(\Delta)$ of $\mathcal{H}om(X, \mathbb{G}_{m,\log})$ on (fs/$S$) as follows. For an object $U$ of (fs/$S$), let

$$V(\Delta)(U) := \big\{\varphi \in \operatorname{Hom}(X, M_U^{\mathrm{gp}}); \mu \cdot (\varphi(x) \bmod \mathcal{O}_U^\times) \in M_U/\mathcal{O}_U^\times \\ \text{for every } (\mu, x) \in \Delta^\vee\big\}.$$

If $\Delta'$ is a finitely generated subcone of $\Delta$, we have $V(\Delta') \subset V(\Delta)$.

3.5.3. Lemma. *Let $\Delta$ be a finitely generated subcone of $C$.*

*(1) $V(\Delta)$ is represented by a log smooth fs log analytic space over $S$. (In the following, we will identify $V(\Delta)$ with the fs log analytic space which represents it.)*

*(2) If $\Delta'$ is a face of $\Delta$, then $V(\Delta')$ is an open fs log analytic subspace in $V(\Delta)$.*

*(3) If $\Delta'$ is a finitely generated subcone of $C$, then we have*

$$V(\Delta) \cap V(\Delta') = V(\Delta \cap \Delta').$$

Proof. (1) We may assume that there exists a lifting $\mathcal{S} \to M_S$ of $\mathcal{S} \to M_S/\mathcal{O}_S^\times$. Fix such a lifting. Then $V(\Delta)$ is represented by the fiber product

$$S \times_{(\operatorname{Spec}\mathbb{C}[\mathcal{S}])_{\mathrm{an}}} (\operatorname{Spec}\mathbb{C}[\Delta^\vee])_{\mathrm{an}},$$

where $\operatorname{Spec}\mathbb{C}[\mathcal{S}]$ and $\operatorname{Spec}\mathbb{C}[\Delta^\vee]$ are endowed with the canonical log structures associated to $\mathcal{S}$ and $\Delta^\vee$, respectively.

(2) As is well-known in toric geometry, if $\Delta'$ is a face of $\Delta$, then $(\operatorname{Spec}\mathbb{C}[(\Delta')^\vee])_{\mathrm{an}}$ is an open fs log analytic subspace of $(\operatorname{Spec}\mathbb{C}[\Delta^\vee])_{\mathrm{an}}$.

(3) This follows from the fact that $(\Delta \cap \Delta')^\vee$ coincides with

$$\big\{t \in \mathcal{S}^{\mathrm{gp}} \times X; t^n = ab \text{ for some } n \geq 1, a \in \Delta^\vee, b \in (\Delta')^\vee\big\}. \square$$

3.5.4. Proposition. *We have*

$$\mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)} = \bigcup_{\Delta} V(\Delta),$$

*where $\Delta$ ranges over all finitely generated subcones of $C$.*

PROOF. We first prove $V(\Delta) \subset \mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)}$ for any finitely generated subcone $\Delta$ of $C$. Let $\{(N_i, l_i)\}_i$ be a finite set of generators of $\Delta$, $x \in X$ an arbitrary element, and $\sigma$ the minimal face of $\mathcal{S}$ with $x \in X_\sigma$ (see 3.4.5 (2)). Let us take $y \in Y_\sigma$ such that $\langle x, y\rangle$ is in the interior of $\sigma$ (see 3.4.5 (3)). Since $N_i(\langle x, y\rangle) = 0$ (resp. $\neq 0$) implies $N_i(\sigma) = 0$ and $l_i(x) = 0$ (resp. $N_i(\langle x, y\rangle) > 0$), there exists a sufficiently large integer $m \in \mathbb{Z}_{\geq 0}$ which satisfies $N_i(\langle x, my\rangle) \geq l_i(x) \geq N_i(\langle x, -my\rangle)$ for each $i$. Hence $(\langle x, my\rangle, -x), (\langle x, my\rangle, x) \in \Delta^\vee$. Hence if $\varphi : X \to \mathbb{G}_{m,\log}$ belongs to $V(\Delta)$, then $\langle x, my\rangle\varphi(x)^{-1}$ and $\langle x, my\rangle\varphi(x)$ belong to $M/\mathcal{O}^\times$ in $M^{\mathrm{gp}}/\mathcal{O}^\times$, and hence $\langle x, -my\rangle | (\varphi(x) \bmod \mathcal{O}^\times) | \langle x, my\rangle$.

We next prove $\mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)} \subset \bigcup_\Delta V(\Delta)$. Let us take a finite family $(x_i \in X)_i$ such that, for every face $\sigma$ of $\mathcal{S}$, some $x_i$'s generate $X_\sigma$. Let $\varphi$ be a section of $\mathcal{H}om(X, \mathbb{G}_{m\log})^{(Y)}$. Then locally on $S$, for each $i$, there exist $y_i$ and $y_i'$ such that $\langle x_i, y_i'\rangle | (\varphi(x_i) \bmod \mathcal{O}^\times) | \langle x_i, y_i\rangle$. Define

$$\Delta := \{(N, l) \in \mathrm{Hom}\,(\mathcal{S}, \mathbb{N}) \times \mathrm{Hom}\,(X, \mathbb{Z}); \\ N(\langle x_i, y_i\rangle) \geq l(x_i) \geq N(\langle x_i, y_i'\rangle) \text{ for all } i\}.$$

Then $\Delta \subset C$ since $\langle x_i, Y\rangle \subset (\mathrm{Ker}\, N)^{\mathrm{gp}}$ implies $l(x_i) = 0$. Since $\Delta^\vee$ is generated (over $\mathbb{Q}_{\geq 0}$) by $\mathcal{S}, (\langle x_i, y_i\rangle, -x_i)$, and $(\langle x_i, -y_i'\rangle, x_i)$, we have $\varphi \in V(\Delta)$ because of the choice of $y_i$ and $y_i'$. □

**3.5.5.** Assume that $\Delta \supset \mathcal{S}^\vee \times \{0\}$. In $\mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)}$, $V(\Delta)$ contains the torus $\mathcal{H}om(X, \mathbb{G}_m)$, and stable under the translation by $\mathcal{H}om(X, \mathbb{G}_m)$.

Since $V(\Delta) \subset \mathcal{H}om(X, \mathbb{G}_{m,\log})$, we have a canonical homomorphism $X \to M^{\mathrm{gp}}_{V(\Delta)}$, and hence a canonical continuous map $V(\Delta)^{\log} \to \mathrm{Hom}\,(X, \mathbb{S}^1)$. The composition $S^{\log} \times \mathrm{Hom}\,(X, \mathbb{C}^\times) \to V(\Delta)^{\log} \to \mathcal{H}om(X, \mathbb{S}^1)$, where the first arrow is induced by $\mathcal{H}om(X, \mathbb{G}_m) \to V(\Delta)$, coincides with the canonical projection induced by $\mathbb{C}^\times \to \mathbb{S}^1; z \mapsto z/|z|$.

If $\sigma$ is a face of $\mathcal{S}$ contained in the kernel of $\mathcal{S} \to M_S/\mathcal{O}_S^\times$, then any section of $\mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)}$ sends $X_\sigma$ into $\mathbb{G}_m$. In fact, if $S'$ is an fs log analytic space over $S$ and $\varphi : X \to M^{\mathrm{gp}}_{S'}$ belongs to $\mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)}(S')$, and if $x \in X_\sigma$, then locally on $S'$, there are $y, y' \in Y$ such that $\langle x, y\rangle | (\varphi(x) \bmod \mathcal{O}^\times_{S'}) | \langle x, y'\rangle$ in $M^{\mathrm{gp}}_{S'}/\mathcal{O}^\times_{S'}$, and since $\langle x, y\rangle$ and $\langle x, y'\rangle$ are trivial in $M^{\mathrm{gp}}_{S'}/\mathcal{O}^\times_{S'}$

by $x \in X_\sigma$, we have $\varphi(x) \in \mathcal{O}_{S'}^\times$. Hence we have a homomorphism $\mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)} \to \mathcal{H}om(X_\sigma, \mathbb{G}_m)$. The induced morphism $V(\Delta) \to \mathcal{H}om(X_\sigma, \mathbb{G}_m)$ is described also as follows. Locally on $S$, take a lifting $\mathcal{S} \to M_S$ of $\mathcal{S} \to M_S/\mathcal{O}_S^\times$. Let $x \in X_\sigma$. Then by 3.4.7 (3), there is $\mu \in \sigma$ such that $(\mu, x) \in \Delta^\vee$. The above morphism is induced from the ring homomorphism $\mathbb{C}[X_\sigma] \to \mathcal{O}_S \otimes_{\mathbb{C}[\mathcal{S}]} \mathbb{C}[\Delta^\vee]$ which sends $x$ to $\mu^{-1} \otimes (\mu, x)$ (which is independent of the choice of $\mu$).

In particular, for each $s \in S$, we have a canonical morphism of analytic spaces $V(\Delta) \times_S s \to \operatorname{Hom}(X_{\sigma(s)}, \mathbb{C}^\times)$ which is compatible with the canonical projection $\operatorname{Hom}(X, \mathbb{C}^\times) \to \operatorname{Hom}(X_{\sigma(s)}, \mathbb{C}^\times)$.

For $s \in S$, the diagram

$$\begin{array}{ccc} V(\Delta)^{\log} \times_{S^{\log}} s^{\log} & \to & \operatorname{Hom}(X_{\sigma(s)}, \mathbb{C}^\times) \\ \downarrow & & \downarrow \\ \operatorname{Hom}(X, \mathbb{S}^1) & \to & \operatorname{Hom}(X_{\sigma(s)}, \mathbb{S}^1) \end{array}$$

is commutative.

The map $V(\Delta)^{\log} \times_{S^{\log}} s^{\log} \to \operatorname{Hom}(X_{\sigma(s)}, \mathbb{R}_{>0}) \times \operatorname{Hom}(X, \mathbb{S}^1) \times s^{\log}$ is surjective, where the map $V(\Delta)^{\log} \times_{S^{\log}} s^{\log} \to \operatorname{Hom}(X_{\sigma(s)}, \mathbb{R}_{>0})$ is obtained from $V(\Delta)^{\log} \times_{S^{\log}} s^{\log} \to \operatorname{Hom}(X_{\sigma(s)}, \mathbb{C}^\times)$ by $|\ \ | : \mathbb{C}^\times \to \mathbb{R}_{>0}$.

3.5.6. PROPOSITION. *Let $\Delta$ be a finitely generated subcone of $C$.*

*(1) There exists a finitely generated subcone $\Delta'$ of $C$ containing $\Delta \cup (\mathcal{S}^\vee \times \{0\})$ and satisfying the following conditions* (i)–(iii).

(i) *For each $s \in S$, the underlying morphism of analytic spaces $V(\Delta) \times_S s \to V(\Delta') \times_S s$ (here we forget the log structures) factors through the canonical surjection $V(\Delta) \times_S s \to \operatorname{Hom}(X_{\sigma(s)}, \mathbb{C}^\times)$.*

(ii) *For each $s \in S$, the morphism $V(\Delta)^{\log} \times_{S^{\log}} s^{\log} \to V(\Delta')^{\log} \times_{S^{\log}} s^{\log}$ of topological spaces factors through the canonical surjection $V(\Delta)^{\log} \times_{S^{\log}} s^{\log} \to \operatorname{Hom}(X_{\sigma(s)}, \mathbb{R}_{>0}) \times \operatorname{Hom}(X, \mathbb{S}^1) \times s^{\log}$.*

(iii) *The canonical map $V(\Delta)^{\log} \to V(\Delta')^{\log}$ is homotope with the composite $V(\Delta)^{\log} \to S^{\log} \times \operatorname{Hom}(X, \mathbb{S}^1) \to S^{\log} \times \operatorname{Hom}(X, \mathbb{C}^\times) \to V(\Delta')^{\log}$.*

*(2) Assume that $S$ is a one point set $\{s\}$. Then there exists a finitely generated subcone $\Delta'$ of $C$ containing $\Delta$ and satisfying the following condition. The underlying morphism of analytic spaces $V(\Delta) \to V(\Delta')$ (here we forget the log structures) factors through $V(\Delta) \to \mathcal{H}om(X_{\sigma(s)}, \mathbb{G}_m)$.*

PROOF. We prove (1). Take $\Delta'$ as in 3.4.8. We prove that the conditions (i)–(iii) are satisfied.

We prove (i). Take a lifting $\mathcal{S} \to M_{S,s}$ of $\mathcal{S} \to M_{S,s}/\mathcal{O}_{S,s}^{\times}$. It is sufficient to prove that the homomorphism of $\mathbb{C}$-algebras $\mathbb{C}[(\Delta')^{\vee}] \otimes_{\mathbb{C}[\mathcal{S}]} \mathbb{C} \to \mathbb{C}[\Delta^{\vee}] \otimes_{\mathbb{C}[\mathcal{S}]} \mathbb{C}$ factors through the injection $\mathbb{C}[X_{\sigma(s)}] \to \mathbb{C}[\Delta^{\vee}] \otimes_{\mathbb{C}[\mathcal{S}]} \mathbb{C}$. Let $(\mu, x) \in (\Delta')^{\vee}$. If $x \in X_{\sigma(s)}$, then by 3.4.7 (3), there is $\mu' \in \sigma(s)$ such that $(\mu', x) \in \Delta^{\vee}$, and the image of $(\mu, x)$ in $\mathbb{C}[\Delta^{\vee}] \otimes_{\mathbb{C}[\mathcal{S}]} \mathbb{C}$ is $\mu(s)\mu'(s)^{-1}(\mu', x)$ and it is contained in the image of $\mathbb{C}[X_{\sigma(s)}]$. Here $\mu(s)$ denotes the image of $\mu$ under $\mathcal{S} \to \mathcal{O}_{S,s} \to \mathbb{C}$. If $x \notin X_{\sigma(s)}$, by 3.4.8, there are $\mu', \mu'' \in \mathcal{S}$ such that $\mu = \mu'\mu''$, $(\mu', x) \in \Delta^{\vee}$ and $\mu'' \notin \sigma(s)$. Since $\mu''(s) = 0$, the image of $(\mu, x)$ in $\mathbb{C}[\Delta^{\vee}] \otimes_{\mathbb{C}[\mathcal{S}]} \mathbb{C}$ is $\mu''(s)(\mu', x) = 0$.

We prove (ii). It is enough to show the set-theoretical factorization. Take a lifting $\mathcal{S} \to M_{S,s}$ of $\mathcal{S} \to M_{S,s}/\mathcal{O}_{S,s}^{\times}$. Then $V(\Delta)^{\log} \times_{S^{\log}} s^{\log}$ is identified with the set of all triples $(t, \psi, h)$, where $t \in s^{\log}$, $h$ is a homomorphism $X \to \mathbb{S}^1$ and $\psi$ is a homomorphism $\Delta^{\vee} \to \mathbb{R}_{\geq 0}$ for the multiplicative structure of $\mathbb{R}_{\geq 0}$ such that $\psi(\mu) = |\mu(s)|$ for any $\mu \in \mathcal{S}$. The image of $(t, \psi, h)$ in $V(\Delta')^{\log}$ is $(t, \psi', h)$, where $\psi'$ is the composite $(\Delta')^{\vee} \to \Delta^{\vee} \xrightarrow{\psi} \mathbb{R}_{\geq 0}$. Fix $(\mu, x) \in (\Delta')^{\vee}$. It is sufficient to prove that $\psi'(\mu, x)$ depends only on the image of $\psi$ in $\mathrm{Hom}\,(X_{\sigma(s)}, \mathbb{R}_{>0})$. If $x \in X_{\sigma(s)}$, by 3.4.7 (3), there is $\mu' \in \sigma(s)$ such that $(\mu', x) \in \Delta^{\vee}$. We have $\psi'(\mu, x) = |\mu(s)||\mu'(s)|^{-1}\psi(\mu', x)$ and $|\mu'(s)|^{-1}\psi(\mu', x)$ is nothing but the value at $x$ of the image of $\psi$ in $\mathrm{Hom}\,(X_{\sigma(s)}, \mathbb{R}_{>0})$. If $x \notin X_{\sigma(s)}$, we have $\mu = \mu'\mu''$ with $\mu', \mu'' \in \mathcal{S}$ such that $(\mu', x) \in \Delta^{\vee}$ and $\mu'' \notin \sigma(s)$. Since $\mu''(s) = 0$, we have $\psi'(\mu, x) = \mu''(s)\psi(\mu', x) = 0$.

We prove that (iii) is satisfied. Let $[0,1] \subset \mathbb{R}$ be the closed interval between 0 and 1. We define a map

$$H : [0,1] \times V(\Delta)^{\log} \to V(\Delta')^{\log}$$

as follows. On the inverse image $[0,1] \times V(\Delta)^{\log} \times_{S^{\log}} s^{\log}$ of $s \in S$ in $[0,1] \times V(\Delta)^{\log}$, we define it as the composite

$$[0,1] \times V(\Delta)^{\log} \times_{S^{\log}} s^{\log} \to [0,1] \times \mathrm{Hom}\,(X_{\sigma(s)}, \mathbb{R}_{>0}) \times \mathrm{Hom}\,(X, \mathbb{S}^1) \times s^{\log}$$

$$\to \mathrm{Hom}\,(X_{\sigma(s)}, \mathbb{R}_{>0}) \times \mathrm{Hom}\,(X, \mathbb{S}^1) \times s^{\log} \to V(\Delta')^{\log},$$

where the last arrow is by the condition (ii) and the second arrow is induced by

$$[0,1] \times \mathrm{Hom}\,(X_{\sigma(s)}, \mathbb{R}_{>0}) \to \mathrm{Hom}\,(X_{\sigma(s)}, \mathbb{R}_{>0}); \quad (u, \psi) \mapsto (a \mapsto \psi(a)^u).$$

The restriction of $H$ to $u = 1$ gives the canonical map $V(\Delta)^{\log} \to V(\Delta')^{\log}$ and the restriction to $u = 0$ gives the composite in the condition (iii). Hence it is sufficient to prove that $H$ is continuous. The problem is local on $S$, and hence we may assume that there is a lifting $\mathcal{S} \to M_S$ of $\mathcal{S} \to M_S/\mathcal{O}_S^\times$. Then $V(\Delta)^{\log}$ is identified with the set of all triples $(t, \psi, h)$, where $t \in S^{\log}$, $h$ is a homomorphism $X \to \mathbb{S}^1$, and $\psi$ is a homomorphism $\Delta^\vee \to \mathbb{R}_{\geq 0}$ with respect to the multiplicative structure of $\mathbb{R}_{\geq 0}$ such that $\psi(\mu) = |\mu(s)|$ for all $\mu \in \mathcal{S}$ ($s$ denotes the image of $t$ in $S$). For $(t, \psi, h) \in V(\Delta)^{\log}$ and $u \in [0,1]$, write $H(u, t, \psi, h) \in V(\Delta')^{\log}$ as $(t, \psi_{t,u}, h)$. Then $\psi_{t,u}$ is described as follows. Let $(\mu, x) \in (\Delta')^\vee$ and let $s$ be the image of $t$ in $S$. Then if $x \in X_{\sigma(s)}$, there exists $\mu' \in \sigma(s)$ such that $(\mu', x) \in \Delta^\vee$. We have $\psi_{t,u}(\mu, x) = |\mu(s)||\mu'(s)|^{-u}\psi(\mu', x)^u$. If $x \notin X_{\sigma(s)}$, then $\psi_{t,u}(\mu, x) = 0$. Let $u_0 \in [0,1], (t_0, \psi_0, h_0) \in V(\Delta)^{\log}$, and assume $(u, t, \psi, h) \in [0,1] \times V(\Delta)^{\log}$ converges to $(u_0, t_0, \psi_0, h_0)$. It is sufficient to prove that for each $(\mu, x) \in (\Delta')^\vee$, $\psi_{t,u}(\mu, x)$ converges to $(\psi_0)_{t_0,u_0}(\mu, x)$. Let $s_0$ be the image of $t_0$ in $S$, and let $s$ be the image of $t$ in $S$. First assume $x \in X_{\sigma(s_0)}$. Then if $s$ is sufficiently near to $s_0$, we have $\sigma(s) \supset \sigma(s_0)$ and hence $x \in X_{\sigma(s)}$. By 3.4.7 (3), there exists $\mu' \in \sigma(s_0)$ such that $(\mu', x) \in \Delta^\vee$. We have $\psi_{t,u}(\mu, x) = |\mu(s)||\mu'(s)|^{-u}\psi(\mu', x)^u$, and $\mu(s)$ converges to $\mu(s_0)$, $\mu'(s)$ converges to $\mu'(s_0) \neq 0$ and $\psi(\mu', x)$ converges to $\psi_0(\mu', x) \in \mathbb{R}_{>0}$. Hence $\psi_{t,u}(\mu, x)$ converges to $(\psi_0)_{t_0,u_0}(\mu, x)$. Next assume $x \notin X_{\sigma(s_0)}$. Then $(\psi_0)_{t_0,u_0}(\mu, x) = 0$. We prove that $\psi_{t,u}(\mu, x)$ converges to 0. Take $\mu', \mu'' \in \mathcal{S}$ such that $\mu = \mu'\mu''$, $(\mu', x) \in \Delta^\vee$ and $\mu'' \notin \sigma(s_0)$. If $x \notin X_{\sigma(s)}$, then $\psi_{t,u}(\mu, x) = 0$. Assume $x \in X_{\sigma(s)}$. Then by 3.4.7 (3), there exists $\mu_s$ such that $\mu_s \in \sigma(s)$ and $(\mu, x) \in \Delta^\vee$. We have $\psi_{t,u}(\mu, x) = |\mu''(s)||\mu'(s)||\mu_s(s)|^{-u}\psi(\mu_s, x)^u$. Since $|\mu_s(s)|^{-u}\psi(\mu_s, x)^u$ belongs to the interval between 1 and $|\mu_s(s)|^{-1}\psi(\mu_s, x)$, $|\mu'(s)||\mu_s(s)|^{-u}\psi(\mu_s, x)^u$ belongs to the interval between $|\mu'(s)|$ and $|\mu'(s)||\mu_s(s)|^{-1}\psi(\mu_s, x) = \psi(\mu', x)$. Since $\mu'(s)$ converges to $\mu'(s_0)$ and $\psi(\mu', x)$ converges to $\psi_0(\mu', x)$, $|\mu'(s)||\mu_s(s)|^{-u}\psi(\mu_s, x)^u$ is bounded in $\mathbb{R}$. By this and by the fact that $\mu''(s)$ converges to $\mu''(s_0) = 0$, we see that $\psi_{t,u}(\mu, x) = |\mu''(s)||\mu'(s)||\mu_s(s)|^{-u}\psi(\mu_s, x)^u$ converges to 0.

Finally we prove (2). Take a lifting $\mathcal{S} \to M_{S,s}$ of $\mathcal{S} \to M_{S,s}/\mathcal{O}_{S,s}^{\times}$. Let $n \geq 1$ be an integer such that the $n$-th power of the maximal ideal $m$ of $\mathcal{O}_{S,s}$ is zero. Take $\Delta'$ as in 3.4.8, and by replacing $\Delta$ by $\Delta'$, we take $\Delta'$ again and so on, and we have $\Delta'$ satisfying the following condition. If $(\mu, x) \in (\Delta')^{\vee}$ and $x \notin \sigma(s)$, then there exist elements $\mu'$ and $\mu_i''$ $(1 \leq i \leq n)$ of $\mathcal{S}$ such that $(\mu', x) \in \Delta^{\vee}$ and $\mu_i'' \notin \sigma(s)$. Let $\mu'' = \prod_{1 \leq i \leq n} \mu_i''$. The image of $\mu_i''$ in $\mathcal{O}_{S,s}$ belongs to the maximal ideal, and hence the image of $\mu''$ in $\mathcal{O}_{S,s}$ is zero. The rest of the proof goes in the same way as the proof of the condition (i) in (1). □

### 3.6. From $\mathcal{H}_S$ to $\mathcal{A}_S$

In this subsection, we prove that for an fs log analytic space $S$ and for an object $H$ of $\mathcal{H}_S$, $\mathcal{E}xt^1(\mathbb{Z}, H)$ is a log complex torus over $S$. For this, we study $\mathcal{E}xt^1(\mathbb{Z}, H)$ by embedding it in a bigger group $\mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H)$.

**3.6.1.** For an fs log analytic space $S$ and for an object $H$ of $\mathcal{H}_S$, define a sheaf of abelian groups $\mathcal{V}_H$ on $(\text{fs}/S)^{\log}$ by

$$\mathcal{V}_H = \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} (H_{\mathcal{O}}/F^0 H_{\mathcal{O}}).$$

3.6.2. Proposition. *Let $\mathcal{E}xt^1_{\text{naive}}$ be the $\mathcal{E}xt^1$ for the category of pre-log mixed Hodge structures. Then we have a canonical isomorphism*

$$\mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H) \simeq \tau_*(H_{\mathbb{Z}} \backslash \mathcal{V}_H).$$

Proof. We will prove this by an explicit construction of the extension in the same way as in the classical case explained in 3.1.1.

For $z \in \tau_*(H_{\mathbb{Z}} \backslash \mathcal{V}_H)$, the extension $H'$ of $\mathbb{Z}$ by $H$ corresponding to $z$ is given as follows. Let $0 \to H_{\mathbb{Z}} \to H'_{\mathbb{Z}} \to \mathbb{Z} \to 0$ be the extension on $(\text{fs}/S)^{\log}$ obtained as the pull back of $0 \to H_{\mathbb{Z}} \to \mathcal{V}_H \to H_{\mathbb{Z}} \backslash \mathcal{V}_H \to 0$ by $\mathbb{Z} \to H_{\mathbb{Z}} \backslash \mathcal{V}_H; 1 \mapsto z$. Let $F^0(\mathcal{O}_S^{\log} \otimes H'_{\mathbb{Z}})$ be the kernel of $\mathcal{O}_S^{\log} \otimes H'_{\mathbb{Z}} \to \mathcal{V}_H$. We give the local description of this construction. Lift $z$ locally on $S^{\log}$ to sections $z_i \in (\mathcal{O}_S^{\log} \otimes H_{\mathcal{O}})(U_i)$, where $(U_i \to S^{\log})$ is an open covering. On $U_i$, we have then

$$H'_{\mathbb{Z}} = \mathbb{Z} \oplus H_{\mathbb{Z}},$$

$$F^p(\mathcal{O}_S^{\log} \otimes H'_{\mathbb{Z}}) = \mathcal{O}_S^{\log} \otimes H'_{\mathbb{Z}} \text{ for } p \le -1, \quad F^p(\mathcal{O}_S^{\log} \otimes H'_{\mathbb{Z}}) = 0 \text{ for } p \ge 1,$$

$$F^0(\mathcal{O}_S^{\log} \otimes H'_{\mathbb{Z}}) = \mathcal{O}_S^{\log}(1, -z_i) + (0, F^0(\mathcal{O}_S^{\log} \otimes H_{\mathbb{Z}})).$$

(Note that $H'_{\mathbb{Z}}$ does not necessarily coincide with the direct sum $\mathbb{Z} \oplus H_{\mathbb{Z}}$ locally on $S$.) By this description, we see that there is an exact sequence $0 \to F^0(\mathcal{O}_S^{\log} \otimes H_{\mathbb{Z}}) \to F^0(\mathcal{O}_S^{\log} \otimes H'_{\mathbb{Z}}) \to \mathcal{O}_S^{\log} \to 0$, and that $H'$ is a pre-log mixed Hodge structure.

Conversely, if we have an extension $H'$ of $\mathbb{Z}$ by $H$, we have exact sequences

$$0 \to H_{\mathbb{Z}} \to H'_{\mathbb{Z}} \to \mathbb{Z} \to 0, \quad 0 \to F^0 H_{\mathcal{O}} \to F^0 H'_{\mathcal{O}} \to \mathcal{O}_S \to 0,$$

and hence, locally on $S^{\log}$, there exist a section $a$ of $H'_{\mathbb{Z}}$ whose image in $\mathbb{Z}$ is 1, and a section $b$ of $F^0 H'_{\mathcal{O}}$ whose image in $\mathcal{O}_S$ is 1. Let $z = a - b \in \mathcal{O}_S^{\log} \otimes H_{\mathcal{O}}$. Then the class of $z$ in $H_{\mathbb{Z}} \backslash \mathcal{V}_H$ is independent of the choices of $a$ and $b$, and gives a global section of $H_{\mathbb{Z}} \backslash \mathcal{V}_H$. These $z \mapsto H'$ and $H' \mapsto z$ give mutually inverse isomorphisms between $\tau_*(H_{\mathbb{Z}} \backslash \mathcal{V}_H)$ and $\mathcal{E}xt^1_{\mathrm{naive}}(\mathbb{Z}, H)$. □

**3.6.3.** By 3.6.2, $\tau^{-1}(\mathcal{E}xt^1_{\mathrm{naive}}(\mathbb{Z}, H))$ is regarded as a subsheaf of $H_{\mathbb{Z}} \backslash \mathcal{V}_H$, and hence $\tau^{-1}(\mathcal{E}xt^1(\mathbb{Z}, H))$ is also regarded as a subsheaf of $H_{\mathbb{Z}} \backslash \mathcal{V}_H$. Let $\mathcal{U}_H \subset \mathcal{V}_H$ be the inverse image of $\tau^{-1}(\mathcal{E}xt^1(\mathbb{Z}, H))$. We have an exact sequence

$$0 \to H_{\mathbb{Z}} \to \mathcal{U}_H \to \tau^{-1}(\mathcal{E}xt^1(\mathbb{Z}, H)) \to 0.$$

3.6.4. PROPOSITION. *Assume that $H$ comes from an object $(H, X, Y, e)$ of $\tilde{\mathcal{H}}_S$. Let $(X, Y, \langle\, , \rangle)$ be the corresponding pairing into $\mathbb{G}_{m,\log}$, and let $\mathcal{H}om(X, \mathcal{L})^{(Y)} \subset \mathcal{H}om(X, \mathcal{L})$ be the inverse image of $\tau^{-1}(\mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)})$. Then the composite*

$$\mathcal{H}om(X, \mathcal{L})^{(Y)} \to \mathcal{H}om(X, \mathcal{O}_S^{\log}) \to \mathcal{O}_S^{\log} \otimes_{\mathbb{Z}} H_{\mathbb{Z}} \to \mathcal{V}_H$$

*is injective and the image coincides with $\mathcal{U}_H$ so that it induces an isomorphism*

$$\mathcal{H}om(X, \mathcal{L})^{(Y)} \xrightarrow{\simeq} \mathcal{U}_H.$$

PROOF. The injectivity of $\mathcal{H}om(X,\mathcal{L})^{(Y)} \to \mathcal{V}_H$ is clear by $\mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} H_{\mathcal{O}} = \mathcal{H}om(X, \mathcal{O}_S^{\log}) \oplus \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} F^0 H_{\mathcal{O}}$.

Let $v$ be a section of $\tau_*(H_{\mathbb{Z}} \backslash \mathcal{V}_H)$. It is sufficient to prove that $v$ belongs to $\mathcal{E}xt^1(\mathbb{Z}, H)$ if and only if $v$ belongs to the image of $\mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)}$. Let $0 \to H_{\mathbb{Z}} \to L \to \mathbb{Z} \to 0$ be the extension on $(\mathrm{fs}/S)^{\log}$ obtained as the pull back of $0 \to H_{\mathbb{Z}} \to \mathcal{V}_H \to H_{\mathbb{Z}} \backslash \mathcal{V}_H \to 0$ by $v : \mathbb{Z} \to H_{\mathbb{Z}} \backslash \mathcal{V}_H$. For $s \in S$ and for a face $\sigma$ of the monodromy cone at $s$, define an increasing filtration on $L_{\mathbb{Q}} = L_{\mathbb{Q}}|_s$ by $W(\sigma)_k(L_{\mathbb{Q}}) = L_{\mathbb{Q}}$ for $k \geq 0$ and $W(\sigma)_k(L_{\mathbb{Q}}) = W(\sigma)_k(H_{\mathbb{Q}})$ for $k \leq -1$. Then as is easily checked, if $v$ belongs to $\mathcal{E}xt^1(\mathbb{Z}, H)$, then the family of weight filtrations on $L_{\mathbb{Q}}$ associated to faces of $C(s)$ must coincide with this. Furthermore it is easily checked that $v$ belongs to $\mathcal{E}xt^1(\mathbb{Z}, H)$ if and only if for any $s \in S$ and for any homomorphism $N : M_{S,s}/\mathcal{O}_{S,s}^{\times} \to \mathbb{N}$, if we denote the kernel of $N$ by $\sigma$, then the image of the log of the local monodromy on $L_{\mathbb{Q}}$ corresponding to $N$ is contained in $W(\sigma')_{-2}(L_{\mathbb{Q}}) = \mathrm{Hom}\,(X/X_{\sigma}, \mathbb{Q})$. Here $\sigma'$ is the annihilator of $\sigma$.

On the other hand, we show that if $v$ belongs to $\mathcal{E}xt^1(\mathbb{Z}, H)$, then $v$ belongs to the image of $\mathcal{H}om(X, \mathbb{G}_{m,\log})$. In fact, the exact sequence $0 \to Y \to \mathcal{H}om(X, \mathcal{O}_S^{\log}/\mathbb{Z}) \to H_{\mathbb{Z}} \backslash \mathcal{V}_H \to 0$ gives an exact sequence $\mathcal{H}om(X, \mathbb{G}_{m,\log}) \to \mathcal{E}xt^1_{\mathrm{naive}}(\mathbb{Z}, H) \xrightarrow{\partial} Y \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m$ via $\tau_*(\mathcal{O}_S^{\log}/\mathbb{Z}) = \tau_*(\mathcal{L}/\mathbb{Z}) = \mathbb{G}_{m,\log}$ (3.2.5). Further, we see $\partial(\mathcal{E}xt^1(\mathbb{Z}, H)) = 0$ because for $s \in S$ and for $t \in S^{\log}$ lying over $s$, and for an element $e$ of $L_t$ which lifts $1 \in \mathbb{Z} = L_t/H_{\mathbb{Z},t}$, the image of $e$ under the logarithm $N$ of any local monodromy is an element of $\mathrm{Hom}\,(X, \mathbb{Z}(1)) \subset H_{\mathbb{Z},t}$. Hence $v$ comes from $\mathcal{H}om(X, \mathbb{G}_{m,\log})$.

Now the above element of $\mathrm{Hom}\,(X, \mathbb{Z}(1))$ is obtained as the image of a lifting $\varphi$ of $v$ in $\mathcal{H}om(X, \mathbb{G}_{m,\log})_s$ under the composite

$$\mathrm{Hom}\,(X, M_{S,s}^{\mathrm{gp}}) \to \mathrm{Hom}\,(X, M_{S,s}^{\mathrm{gp}}/\mathcal{O}_{S,s}^{\times}) \xrightarrow{N} \mathrm{Hom}\,(X, \mathbb{Z}).$$

Hence $v$ belongs to $\mathcal{E}xt^1(\mathbb{Z}, H)$ if and only if for any $s$, $N$, $\sigma$ as above, the homomorphism $\varphi\colon X \to M_{S,s}^{\mathrm{gp}}$ corresponding to $v$ satisfies $N\varphi(X_{\sigma}) = 0$.

Hence it is sufficient to prove that for $s \in S$ and for a homomorphism $\varphi\colon X \to M_{S,s}^{\mathrm{gp}}$, $\varphi$ belongs to $\mathcal{H}om(X, \mathbb{G}_{m,\log})_s^{(Y)}$ if and only if $N\varphi(X_{\sigma}) = 0$ for any homomorphism $N\colon M_{S,s}/\mathcal{O}_{S,s}^{\times} \to \mathbb{N}$, where $\sigma = \mathrm{Ker}\,(N)$. Assume that $\varphi$ belongs to $\mathcal{H}om(X, \mathbb{G}_{m,\log})_s^{(Y)}$. Let $x \in X_{\sigma}$. Then there exists $y, y' \in Y$ such that $\langle x, y \rangle | \varphi(x) | \langle x, y' \rangle$ at $s$. Since $\langle x, y \rangle, \langle x, y' \rangle \bmod \mathcal{O}_{S,s}^{\times}$ belong to

$\sigma^{\mathrm{gp}}$ and since $N$ annihilates $\sigma$, this shows $N\varphi(x) = 0$. Conversely assume that $N\varphi(X_\sigma) = 0$ for any $N\colon M_{S,s} \to \mathbb{N}$ ($\sigma = \mathrm{Ker}\,(N)$). Take a chart $\mathcal{S} \to M_S$ at $s$ such that $\mathcal{S} \xrightarrow{\simeq} M_{S,s}/\mathcal{O}_{S,s}^\times$, and regard the pairing into $M_{S,s}^{\mathrm{gp}}/\mathcal{O}_{S,s}^\times$ as a pairing into $\mathcal{S}^{\mathrm{gp}}$. Let $C$ be as in 3.4.2. Then the assumption on $\varphi$ shows $(N, N\varphi) \in C$ for any $N\colon M_{S,s}/\mathcal{O}_{S,s}^\times \to \mathbb{N}$. Let $\Delta$ be the saturation of $\{(N, N\varphi) \mid N\colon M_{S,s}/\mathcal{O}_{S,s}^\times \to \mathbb{N}\}$ in $\mathrm{Hom}\,(\mathcal{S}, \mathbb{N}) \times \mathrm{Hom}\,(X, \mathbb{Z})$. Then $\Delta$ is a finitely generated subcone of $C$. Furthermore we have $\varphi \in V(\Delta)$. Indeed, if $(\mu, x) \in \Delta^\vee$, then $\mu\varphi(x) \in M_{S,s}^{\mathrm{gp}}$ belongs to $M_{S,s}$ because for any homomorphism $N\colon M_{S,s}/\mathcal{O}_{S,s}^\times \to \mathbb{N}$, we have $N(\mu\varphi(x)) = N(\mu) + N\varphi(x) \geq 0$ since $(N, N\varphi) \in \Delta$. By 3.5.4, we have $\varphi \in \mathcal{H}om(X, \mathbb{G}_{m,\log})_s^{(Y)}$. $\square$

Now we can prove

3.6.5. PROPOSITION. *(1) For an object $H$ of $\mathcal{H}_S$, $\mathcal{E}xt^1(\mathbb{Z}, H)$ is a log complex torus.*

*(2) We have a commutative diagram of categories*

$$\begin{array}{ccc} \tilde{\mathcal{H}}_S & \simeq & \tilde{\mathcal{A}}_S \\ \downarrow & & \downarrow \\ \mathcal{H}_S & \to & \mathcal{A}_S. \end{array}$$

*Here the right vertical arrow is to take the associated quotient, and the lower horizontal arrow is $H \mapsto \mathcal{E}xt^1(\mathbb{Z}, H)$.*

*(3) An object of $\mathcal{A}_S$ comes locally on $S$, from $\mathcal{H}_S$.*

PROOF. Let $(H, X, Y, e)$ be an object of $\tilde{\mathcal{H}}_S$ and let $(X, Y, \langle\ ,\ \rangle)$ be the corresponding object of $\tilde{\mathcal{A}}_S$. Then by 3.6.4,

$$\mathcal{E}xt^1(\mathbb{Z}, H) \simeq H_{\mathbb{Z}} \backslash \mathcal{H}om(X, \mathcal{L})^{(Y)} \simeq Y \backslash \mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)}.$$

By the definition of log complex torus, any object of $\mathcal{A}_S$ comes locally on $S$ from $\tilde{\mathcal{A}}_S$.

Hence it remains to show that any object $H$ of $\mathcal{H}_S$ comes locally on $S$ from an object $(H, X, Y, e)$ of $\tilde{\mathcal{H}}_S$. It is enough to find $X$ and $Y$ at each point $s$ of $S$. Consider $H' := \mathrm{gr}_{-1}^{W(C(s))} H(s)$. Then $H'$ is a Hodge structure of weight $-1$ so that there is a sublattice $L'$ of $H'_{\mathbb{Z}}$ such that $H'_{\mathbb{Z}}/L'$ is torsion free and such that $H'_{\mathbb{C}} = L'_{\mathbb{C}} \oplus F^0$. Let $L$ be the inverse image of $L'$ in $H_{\mathbb{Z}}$. Then $X := \mathrm{Hom}\,(L, \mathbb{Z}(1))$ and $Y := H_{\mathbb{Z}}/L$ are the desired ones. (Note that the above construction gives $X$ and $Y$ which are constant locally on $S$, not only on $S^{\log}$). $\square$

## 3.7. From $\mathcal{A}_S$ to $\mathcal{H}_S$

In this subsection, we prove the equivalence $\mathcal{H}_S \simeq \mathcal{A}_S$ by constructing the inverse functor $\mathcal{A}_S \to \mathcal{H}_S$.

3.7.1. LEMMA. *Let $S$ be an fs log analytic space, let $V$ be an fs log analytic space over $S$, and let $\tau^{-1}(V)$ be the inverse image on $(\mathrm{fs}/S)^{\log}$ of the sheaf $V$ on $(\mathrm{fs}/S)$. Then we have an equivalence of topoi*

$$\{\textit{sheaf on } \tau^{-1}(V)\} \simeq \{\textit{sheaf on } (\mathrm{fs}/V)^{\log}\}$$

*defined as follows. Here the left category is the category of sheaves on the site of objects $(U, T)$ of $(\mathrm{fs}/S)^{\log}$ endowed with a section of $\tau^{-1}(V)$ on $(U, T)$. For a sheaf $F$ on $\tau^{-1}(V)$, the corresponding sheaf on $(\mathrm{fs}/V)^{\log}$ is defined by $(U, T) \mapsto F(U, T)$, where we endow $(U, T)$ with the canonical section of $\tau^{-1}(V)$ on $(U, T)$.*

PROOF. In general, let $C$ be a site, let $Z$ be a pre-sheaf on $C$, and let $Z^a$ be the associated sheaf on $C$. Let $C/Z$ (resp. $C/Z^a$) be the site of objects of $C$ endowed with a section of $Z$ (resp. $Z^a$). Then the evident functor gives an equivalence of topoi

$$\{\text{sheaf on}\, C/Z^a\} \xrightarrow{\simeq} \{\text{sheaf on}\, C/Z\}.$$

In the case $C = (\mathrm{fs}/S)^{\log}$ and $Z$ is the presheaf represented by the pair $(V^{\log}, V)$, we have $Z^a = \tau^{-1}(V)$ and $C/Z = (\mathrm{fs}/V)^{\log}$. □

**3.7.2.** Let the notation be as in 3.7.1. Denote the structure morphism $V \to S$ by $f$, and let

$$f^{\log} : \{\text{sheaf on}\, \tau^{-1}(V)\} \simeq \{\text{sheaf on}\, (\mathrm{fs}/V)^{\log}\} \to \{\text{sheaf on}\, (\mathrm{fs}/S)^{\log}\}$$

be the canonical morphism of topoi. We will often denote the derived functor $R^m f_*^{\log}$ by $\mathcal{H}^m(\tau^{-1}(V),\ \ )$. We regard it as the $m$-th cohomology sheaf of the "space" $\tau^{-1}(V)$.

Let $S$ be an fs log analytic space.

3.7.3. PROPOSITION. *Let $H$ be an object of $\mathcal{H}_S$, and let $A$ be the corresponding log complex torus over $S$. Then the exact sequence*

$$0 \to H_{\mathbb{Z}} \to \mathcal{U}_H \to \tau^{-1}(A) \to 0$$

*in 3.6.3 induces an isomorphism*

$$\mathcal{H}om(H_{\mathbb{Z}}, \mathbb{Z}) \overset{\cong}{\to} \mathcal{E}xt^1(\tau^{-1}(A), \mathbb{Z}).$$

PROOF. Since the problem is local on $S$, we may assume that $H$ comes from an object $(H, X, Y, e)$ of $\tilde{\mathcal{H}}_S$. Let $(X, Y, \langle\, ,\, \rangle)$ be the corresponding pairing into $\mathbb{G}_{m,\log}$. Write $\Psi = \mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)}$. Then the exact sequence $0 \to \mathrm{Hom}\,(X, \mathbb{Z}(1)) \to \mathcal{H}om(X, \mathcal{L})^{(Y)} \to \tau^{-1}(\Psi) \to 0$ gives a canonical homomorphism $X(-1) \to \mathcal{E}xt^1(\tau^{-1}(\Psi), \mathbb{Z})$. By the exact sequence $0 \to Y \to \Psi \to A \to 0$, we have the lower exact sequence of the commutative diagram on $S^{\log}$

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Hom}\,(Y, \mathbb{Z}) & \to & \mathcal{H}om(H_{\mathbb{Z}}, \mathbb{Z}) & \to & X(-1) & \to 0 & \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}\,(Y, \mathbb{Z}) & \to & \mathcal{E}xt^1(\tau^{-1}(A), \mathbb{Z}) & \to & \mathcal{E}xt^1(\tau^{-1}(\Psi), \mathbb{Z}) & \to & 0.
\end{array}$$

Hence it is sufficient to prove that the map $X(-1) \to \mathcal{E}xt^1(\tau^{-1}(\Psi), \mathbb{Z})$ is an isomorphism. We define the inverse map of it as follows. Let

$$T = \mathcal{H}om(X, \mathbb{G}_m) \subset \Psi.$$

Then with the notation in 3.7.2, we have homomorphisms

$$\mathcal{E}xt^1(\tau^{-1}(\Psi), \mathbb{Z}) \to \mathcal{E}xt^1(\tau^{-1}(T), \mathbb{Z}) \to \mathcal{H}^1(\tau^{-1}(T), \mathbb{Z}) \simeq X(-1).$$

The composition $X(-1) \to \mathcal{E}xt^1(\tau^{-1}(\Psi), \mathbb{Z}) \to X(-1)$ coincides with the identity map. Hence we are reduced to proving the injectivity of $\mathcal{E}xt^1(\tau^{-1}(\Psi), \mathbb{Z}) \to \mathcal{H}^1(\tau^{-1}(T), \mathbb{Z})$.

Let $U$ be an open set of $S^{\log}$ and let $(\mathrm{fs}/S)^{\log}_U$ be the full subcategory of $(\mathrm{fs}/S)^{\log}$ consisting of all objects over $(U, S)$, and let $\tau^{-1}(\Psi)_U$ be the restriction of $\tau^{-1}(\Psi)$ to $(\mathrm{fs}/S)^{\log}_U$. Let $0 \to \mathbb{Z} \to F \to \tau^{-1}(\Psi)_U \to 0$ be an exact sequence whose extension class becomes $0$ in $H^1(T^{\log}_U, \mathbb{Z})$, where $T^{\log}_U = T^{\log} \times_{S^{\log}} U$. It is sufficient to prove that this sequence splits locally on $U$. We may work locally on $S$, and hence we may assume that there is an fs monoid $\mathcal{S}$, a pairing $X \times Y \to \mathcal{S}^{\mathrm{gp}}$, and a homomorphism $\mathcal{S} \to M_S/\mathcal{O}_S^\times$ such that the induced pairing $X \times Y \to M_S^{\mathrm{gp}}/\mathcal{O}_S^\times$ coincides with the original pairing $X \times Y \to \mathbb{G}_{m,\log}$ modulo $\mathbb{G}_m$. Fix a splitting $s_0$ of the induced extension on $T^{\log}_U$. Let $C$ be as in 3.4.2. By 3.5.4, $\tau^{-1}(\Psi)_U = \bigcup_\Delta \tau^{-1}(V(\Delta))_U$, where $\Delta$ ranges over all finitely generated subcones of $C$. Let $\Delta$ be a finitely generated subcone of $C$ containing $\mathcal{S}^\vee \times \{0\}$, and take a finitely generated subcone $\Delta'$ of $C$ containing $\Delta$ and satisfying the conditions in 3.5.6 (1). Then by 3.5.6 (1)(iii), the homomorphism $H^1(V(\Delta')_U^{\log}, \mathbb{Z}) \to H^1(V(\Delta)_U^{\log}, \mathbb{Z})$ factors through $H^1(T_U^{\log}, \mathbb{Z})$, the image of the extension class in $H^1(V(\Delta)_U^{\log}, \mathbb{Z}) = H^1(\tau^{-1}(V(\Delta))_U, \mathbb{Z})$ vanishes, and $s_0$ induces a morphism $s_\Delta : \tau^{-1}(V(\Delta))_U \to F$ whose composite with $F \to \tau^{-1}(\Psi)_U$ coincides with the inclusion morphism $\tau^{-1}(V(\Delta))_U \to \tau^{-1}(\Psi)_U$. This $s_\Delta$ is independent of the choice of $\Delta'$. Then $s_\Delta$'s for all $\Delta$ are compatible and give a morphism $s : \tau^{-1}(\Psi)_U = \bigcup_\Delta \tau^{-1}(V(\Delta))_U \to F$. Replacing $s$ by $s - s(e)$, where $e$ is the unit section of $\tau^{-1}(T)_U$, we may assume $s(e) = 0$. We prove that $s$ is a homomorphism. In fact, $h : \tau^{-1}(\Psi)_U \times \tau^{-1}(\Psi)_U \to F; (x, y) \mapsto s(x + y) - s(x) - s(y)$ has values in $\mathbb{Z}$ and is zero on the unit section. We prove $h = 0$. For each $\Delta$, by taking a big $\Delta'$ as above, we see that $h$ induces the zero map on $\tau^{-1}(V(\Delta))_U \times \tau^{-1}(V(\Delta))_U$. This shows $h = 0$. □

3.7.4. Proposition. *Let $A$ be a log complex torus over an fs log analytic space $S$. Then*:

*(1) $\mathcal{E}xt^1(\tau^{-1}(A), \mathbb{Z})$ is a locally constant sheaf of finitely generated free abelian groups on* $(\mathrm{fs}/S)^{\log}$.

*(2) Define $H_\mathbb{Z} = \mathcal{H}om_\mathbb{Z}(\mathcal{E}xt^1(\tau^{-1}(A), \mathbb{Z}), \mathbb{Z})$. If $A = \mathcal{E}xt^1(\mathbb{Z}, H')$ for an object $H'$ of $\mathcal{H}_S$, we have a canonical isomorphism $H_\mathbb{Z} \simeq H'_\mathbb{Z}$.*

*(3) Define $H_\mathcal{O} = \tau_*(\mathcal{O}_S^{\log} \otimes_\mathbb{Z} H_\mathbb{Z})$. Then the $\mathcal{O}_S$-module $H_\mathcal{O}$ is locally free of finite rank, and*

$$\mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} H_\mathcal{O} \xrightarrow{\simeq} \mathcal{O}_S^{\log} \otimes_\mathbb{Z} H_\mathbb{Z}.$$

Proof. By 3.6.5 (3), $A$ comes from $\mathcal{H}_S$ locally on $S$.

To prove (1), we may work locally on $S$ and hence we may assume that $A$ comes from $\mathcal{H}_S$. Then (1) was shown in 3.7.3.

(2) is clear from the construction.

To prove (3), we may work locally on $S$ and we may assume that $A$ comes from an object $H'$ of $\mathcal{H}_S$. Then $H_\mathbb{Z} \simeq H'_\mathbb{Z}$ and (3) is evidently true if we replace $H_\mathbb{Z}$ by $H'_\mathbb{Z}$. □

**3.7.5.** Let $A$ be a log complex torus over an fs log analytic space $S$, and let $H_{\mathbb{Z}}$ be as in 3.7.4. Then

$$\begin{aligned}\mathcal{E}xt^1(\tau^{-1}(A), H_{\mathbb{Z}}) &\simeq \mathcal{E}xt^1(\tau^{-1}(A), \mathbb{Z}) \otimes_{\mathbb{Z}} H_{\mathbb{Z}} \simeq \mathcal{H}om(H_{\mathbb{Z}}, \mathbb{Z}) \otimes_{\mathbb{Z}} H_{\mathbb{Z}} \\ &\simeq \mathcal{H}om_{\mathbb{Z}}(H_{\mathbb{Z}}, H_{\mathbb{Z}})\end{aligned}$$

and we have a canonical element of $\mathcal{E}xt^1(\tau^{-1}(A), H_{\mathbb{Z}})$ corresponding to the identity of $H_{\mathbb{Z}}$. The exact sequence

$$0 \to H_{\mathbb{Z}} \to \mathcal{U}_A \to \tau^{-1}(A) \to 0$$

corresponding to this element is defined up to canonical isomorphisms because

$$\mathcal{H}om(\tau^{-1}(A), H_{\mathbb{Z}}) = 0$$

as is shown below.

If $A$ comes from $H'$, then $\mathcal{U}_A$ is identified with $\mathcal{U}_{H'}$ in 3.6.3.

We prove $\mathcal{H}om(\tau^{-1}(A), H_{\mathbb{Z}}) = 0$. Since $\mathcal{H}om(\tau^{-1}(A), H_{\mathbb{Z}}) \simeq \mathcal{H}om(\tau^{-1}(A), \mathbb{Z}) \otimes H_{\mathbb{Z}}$, it is sufficient to prove $\mathcal{H}om(\tau^{-1}(A), \mathbb{Z}) = 0$. Hence it is sufficient to prove $\mathcal{H}om(\tau^{-1}(\Psi), \mathbb{Z}) = 0$ with $\Psi = \mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)}$ for a non-degenerate pairing $X \times Y \to \mathbb{G}_{m,\log}$. By using $\tau^{-1}(\Psi) = \bigcup_{\Delta} \tau^{-1}(V(\Delta))$ and taking a big $\Delta'$ for each $\Delta$, we can prove it by arguments as in the proof of 3.7.3.

**3.7.6.** To complete the definition of $\mathcal{A}_S \to \mathcal{H}_S$, we need preparations on Lie.

For an fs log analytic space $S$ and for a sheaf of groups $F$ on $(\mathrm{fs}/S)^{\log}$, let $\mathrm{Lie}(F)$ be the sheaf of groups on $(\mathrm{fs}/S)^{\log}$ defined as follows. For an object $(U, T)$ of $(\mathrm{fs}/S)^{\log}$,

$$\mathrm{Lie}(F)(U, T) = \mathrm{Ker}\,(F(U, T[\epsilon]/(\epsilon^2)) \to F(U, T)),$$

where $T[\epsilon]/(\epsilon^2)$ is as in 1.3.11.

We have $\mathrm{Lie}(\tau^{-1}(A)) = \tau^{-1}(\mathrm{Lie}(A))$.

3.7.7. PROPOSITION. *Let $A$ and $\mathcal{U}_A$ be as above. Then there is a unique homomorphism of sheaves of groups*

$$\ell : \mathcal{U}_A \to \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} \mathrm{Lie}(A)$$

*such that* $\mathrm{Lie}(\ell)\colon \mathrm{Lie}(\mathcal{U}_A) \to \mathrm{Lie}(\mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} \mathrm{Lie}(A)) = \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} \mathrm{Lie}(A)$ *is the homomorphism induced by the canonical isomorphism* $\mathrm{Lie}(\mathcal{U}_A) \to \mathrm{Lie}(A)$.

Proof. We first prove the uniqueness of $\ell$. We may assume that $A$ comes from an object $(H', X, Y, e)$ of $\tilde{\mathcal{H}}_S$. Then $\mathcal{U}_A \simeq \mathcal{H}om(X, \mathcal{L})^{(Y)}$. Any $\ell$ must coincide on $\mathcal{H}om(X, \mathcal{O}_S)$. Hence it is sufficient to prove that any homomorphism $\mathcal{H}om(X, \mathcal{L})^{(Y)}/\mathcal{H}om(X, \mathcal{O}_S) \to \mathcal{O}_S^{\log}$ is zero. Since $\mathcal{H}om(X, \mathcal{L})^{(Y)}/\mathcal{H}om(X, \mathcal{O}_S) = \tau^{-1}(\Psi/T)$ and $\tau_*(\mathcal{O}_S^{\log}) = \mathcal{O}_S$, we are reduced to proving that any homomorphism $h : \Psi/T \to \mathcal{O}_S$ is zero. Since $\mathcal{O}_{S,s}$ for $s \in S$ injects to the inverse limit of $\mathcal{O}_{S,s}/m_s^n\mathcal{O}_{S,s}$, we can replace $S$ by the fs log analytic space whose underlying set is $\{s\}$, whose sheaf of rings is $\mathcal{O}_{S,s}/m_{S,s}^n$, and whose log structure is the inverse image of that of $S$. Hence we may assume that the underlying set of $S$ is a one point set $\{s\}$. Since $\Psi = \bigcup_\Delta V(\Delta)$, it is sufficient to prove that the composition $V(\Delta) \to \Psi/T \to \mathcal{O}_S$ is zero for each finitely generated subcone $\Delta$ of $C$. Take $\Delta'$ for which the morphism of analytic spaces $V(\Delta) \to V(\Delta')$ factors through $V(\Delta) \to T_{\sigma(s)}$ (3.5.6 (2)). Since $T \to \Psi/T \to \mathcal{O}_S$ is zero, $T_{\sigma(s)} \to \mathcal{O}_S$ is also zero. Hence $V(\Delta) \to \mathcal{O}_S$ is zero.

We prove the existence of $\ell$. By the uniqueness, we may work locally on $S$ and hence we may assume that $A$ comes from an object $(H', X, Y, e)$ of $\tilde{\mathcal{H}}_S$. In this case, $\mathcal{U}_A$ is identified with $\mathcal{H}om(X, \mathcal{L})^{(Y)}$, and $\mathrm{Lie}(\mathcal{U}_A) = \mathcal{H}om(X, \mathcal{O}_S)$ and $\mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} \mathrm{Lie}(\mathcal{U}_A) = \mathcal{H}om(X, \mathcal{O}_S^{\log})$. Then we have the existence of $\ell$, which is the canonical map induced by the inclusion $\mathcal{L} \to \mathcal{O}_S^{\log}$. $\square$

**3.7.8.** Let $A$ be a log complex torus over an fs log analytic space $S$, and define $H_{\mathbb{Z}}$ and $H_{\mathcal{O}}$ be as above. We define a canonical $\mathcal{O}_S$-homomorphism $H_{\mathcal{O}} \to \mathrm{Lie}(A)$. The homomorphism $\ell$ in 3.7.7 induces $\mathcal{O}_S^{\log} \otimes_{\mathbb{Z}} H_{\mathbb{Z}} \to \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} \mathrm{Lie}(A)$. (We omitted $\tau^{-1}$ on the right hand side.) By applying $\tau_*$, we obtain $H_{\mathcal{O}} \to \mathrm{Lie}(A)$ because $\tau_*(\mathcal{O}_S^{\log}) = \mathcal{O}_S$.

3.7.9. Proposition. *Let* $A$ *be a log complex torus over an fs log analytic space* $S$, *and define* $H_{\mathbb{Z}}$, $H_{\mathcal{O}}$, *and* $H_{\mathcal{O}} \to \mathrm{Lie}(A)$ *as above.*

*(1) The homomorphism* $H_{\mathcal{O}} \to \mathrm{Lie}(A)$ *is surjective.*

*(2) If* $A$ *comes from an object* $H'$ *of* $\mathcal{H}_S$ *(then* $H'_{\mathcal{O}} = H_{\mathcal{O}}$*), the kernel of* $H_{\mathcal{O}} \to \mathrm{Lie}(A)$ *coincides with* $F^0H'_{\mathcal{O}}$.

*(3) Define* $F^pH_{\mathcal{O}} = H_{\mathcal{O}}$ *if* $p \le -1$, $F^0H_{\mathcal{O}} = \mathrm{Ker}\,(H_{\mathcal{O}} \to \mathrm{Lie}(A))$, *and* $F^pH_{\mathcal{O}} = 0$ *for* $p \ge 1$. *Then* $H$ *becomes an object of* $\mathcal{H}_S$.

PROOF. (1), (2) and (3) are shown by assuming that $A$ comes from an object $H'$ of $\mathcal{H}_S$. In this case $H_{\mathbb{Z}} = H'_{\mathbb{Z}}$ and $H_{\mathcal{O}} = H'_{\mathcal{O}}$. It suffices to show that $H'_{\mathcal{O}} \to \mathrm{Lie}(A)$ induces an isomorphism $H'_{\mathcal{O}}/F^0H'_{\mathcal{O}} \xrightarrow{\cong} \mathrm{Lie}(A)$. By the definition of $\ell$, $H_{\mathcal{O}} \to \mathrm{Lie}(A)$ kills $F^0H'_{\mathcal{O}}$. Further, the composite $\mathcal{H}om(X, \mathcal{O}_S) \simeq H'_{\mathcal{O}}/F^0H'_{\mathcal{O}} \to \mathrm{Lie}(A)$ is an isomorphism. Hence $H'_{\mathcal{O}}/F^0H'_{\mathcal{O}} \to \mathrm{Lie}(A)$ is an isomorphism. □

**3.7.10.** Now we prove the equivalence $\mathcal{H}_S \xrightarrow{\simeq} \mathcal{A}_S$ . We have defined a functor $\mathcal{A}_S \to \mathcal{H}_S$ above. It is clear that the composite $\mathcal{H}_S \to \mathcal{A}_S \to \mathcal{H}_S$ is isomorphic to the identity functor. Since any object of $\mathcal{A}_S$ comes from $\mathcal{H}_S$ locally on $S$, this shows that $\mathcal{A}_S \to \mathcal{H}_S$ is fully faithful. Hence $\mathcal{H}_S \to \mathcal{A}_S$ and $\mathcal{A}_S \to \mathcal{H}_S$ are equivalences of categories.

**3.7.11.** We have a commutative diagram of categories

$$\begin{array}{ccc} \tilde{\mathcal{H}}_S & \simeq & \tilde{\mathcal{A}}_S \\ \downarrow & & \downarrow \\ \mathcal{H}_S & \simeq & \mathcal{A}_S. \end{array}$$

3.7.12. REMARK. Let $\mathcal{H}^*_S$ be the category of log Hodge structures of weight 1 over $S$ satisfying $F^0H_{\mathcal{O}} = H_{\mathcal{O}}$ and $F^2H_{\mathcal{O}} = 0$. Then we have an anti-equivalence between $\mathcal{H}_S$ and $\mathcal{H}^*_S$ given by the $\mathbb{Z}$-dual. By composing with the equivalence $\mathcal{H}_S \simeq \mathcal{A}_S$, we have an anti-equivalence between $\mathcal{H}^*_S$ and $\mathcal{A}_S$. It is given by $\mathcal{H}^*_S \to \mathcal{A}_S; H \mapsto \mathcal{E}xt^1(H, \mathbb{Z})$, and the inverse functor $A \mapsto H$ has the property $H_{\mathbb{Z}} = \mathcal{E}xt^1(\tau^{-1}(A), \mathbb{Z})$. Here is a beautiful symmetry between $H \mapsto \mathcal{E}xt^1(H, \mathbb{Z})$ and $A \mapsto \mathcal{E}xt^1(\tau^{-1}(A), \mathbb{Z})$.

### 3.8. The dual log complex tori and polarizations

Here we define the dual log complex torus of a log complex torus, we define and consider a polarization of a log complex torus, and then prove the equivalence $\mathcal{A}^+_S \simeq \mathcal{H}^+_S$ in 3.1.5.

Let $A$ be a log complex torus over an fs log analytic space $S$.

**3.8.1.** We define the dual log complex torus $A^*$ of $A$ as follows. Let $H$ be the log Hodge structure of weight $-1$ corresponding to $A$, and let $H^*$

be the $\mathbb{Z}$-dual of $H$ (so $H^*$ is of weight 1). Then $A^*$ is defined to be the log complex torus corresponding to the twist $H^*(1)$ of $H^*$.

**3.8.2.** We define a polarization of a log complex torus as a homomorphism $p : A \to A^*$ corresponding to a polarization $H \to H^*(1)$ in the sense of 2.5.1.

In 3.8.10, we will give a criterion on a homomorphism $p : A \to A^*$ to be a polarization, not going to the log Hodge side but staying in the geometric side.

**3.8.3.** Let $(X, Y, \langle\ ,\ \rangle)$ be an object of $\hat{\mathcal{A}}_S$, that is, a pairing $\langle\ ,\ \rangle : X \times Y \to \mathbb{G}_{m,\log}$.

We call the object $(Y, X, {}^t\langle\ ,\ \rangle)$ the dual of $(X, Y, \langle\ ,\ \rangle)$. Here ${}^t\langle\ ,\ \rangle$ is the transpose of $\langle\ ,\ \rangle$, that is, the pairing $Y \times X \to \mathbb{G}_{m,\log}; (y, x) \mapsto \langle x, y\rangle$. Via the equivalence $\hat{\mathcal{A}}_S \simeq \hat{\mathcal{H}}_S$, this duality in $\hat{\mathcal{A}}_S$ corresponds to the duality $(H, X, Y, e) \mapsto (H^*(1), Y, X, e^*(1))$ in $\hat{\mathcal{H}}_S$, where $e^*(1)$ denotes the exact sequence $0 \to \mathcal{H}om(Y, \mathbb{Z}(1)) \xrightarrow{i} H^*_{\mathbb{Z}}(1) \to X \to 0$. Here the injection $i$ is $-1$ times of the canonical one. Hence in the case $(X, Y, \langle\ ,\ \rangle)$ belongs to $\tilde{\mathcal{A}}_S$ and $A$ is the associated log complex torus, $A^*$ is the log complex torus associated to $(Y, X, {}^t\langle\ ,\ \rangle)$.

3.8.4. Proposition. *Let $(X, Y, \langle\ ,\ \rangle)$ be an object of $\hat{\mathcal{A}}_S$, and let $(H, X, Y, e)$ be the corresponding object of $\hat{\mathcal{H}}_S$. Let $p : Y \to X$ be a homomorphism such that $\langle p(y), z\rangle = \langle p(z), y\rangle$ for all $y, z \in Y$, and let $p_H : (H, X, Y, e) \to (H^*(1), Y, X, e^*(1))$ be the morphism of $\hat{\mathcal{H}}_S$ corresponding to the morphism $(X, Y, \langle\ ,\ \rangle) \to (Y, X, {}^t\langle\ ,\ \rangle)$ in $\hat{\mathcal{A}}_S$ induced by $p$. Then $p$ is a polarization (1.2.7) if and only if $p_H : H \to H^*(1)$ is a polarization.*

Remark. The homomorphism $p$ having the above symmetry corresponds bijectively to an anti-symmetric homomorphism $H \otimes H \to \mathbb{Z}(1)$ which kills $\mathcal{H}om(X, \mathcal{O}_S) \times \mathcal{H}om(X, \mathcal{O}_S)$. Thus a polarization of a pairing corresponds bijectively to a polarization $H \otimes H \to \mathbb{Z}(1)$ which kills $\mathcal{H}om(X, \mathcal{O}_S) \times \mathcal{H}om(X, \mathcal{O}_S)$.

3.8.5. Corollary. *For an object of $\tilde{\mathcal{A}}_S$, and for $p$ as above, $p$ is a polarization if and only if the induced homomorphism $A \to A^*$ is a polarization, where $A$ is the associated quotient.*

We first prove

3.8.6. LEMMA. *Proposition 3.8.4 is true in the case where $S$ is* $\mathrm{Spec}(\mathbb{C})$ *with the trivial log structure.*

PROOF. Let $(X, Y, \langle\ ,\ \rangle)$ be a pairing into $\mathbb{C}^\times$ and let $p : Y \to X$ be a homomorphism such that $\langle p(y), z\rangle = \langle p(z), y\rangle$ for all $y, z \in Y$. We prove that $p$ is a polarization if and only if $p_H$ is a polarization. Recall that $p_H$ is a polarization if and only if the Hermitian form $c : (f_1, f_2) \mapsto p_H(f_1 \otimes \bar{f}_2)$ on $F^0 H_{\mathbb{C}}$ is positive definite. As is shown below, $c$ coincides with the composite

$$F^0 H_{\mathbb{C}} \times F^0 H_{\mathbb{C}} \to \mathbb{C} \otimes_{\mathbb{Z}} Y \times \mathbb{C} \otimes_{\mathbb{Z}} Y \to \mathbb{C},$$

where the last arrow is $(u \otimes y, v \otimes z) \mapsto 2u\bar{v}b(p(y), z)$ and $b$ is the pairing $\mathbb{R} \otimes_{\mathbb{Z}} X \times \mathbb{R} \otimes_{\mathbb{Z}} Y \to \mathbb{R}$ in 3.3.8. Hence $c$ is positive definite if and only if $b(p(-), -)$ is positive definite.

We prove the above description of $c$ by $b$.

Let $h_j \in H_{\mathbb{Z}}$ $(j = 1, 2)$ and write $h_j = f_j + g_j$ with $f_j \in F^0 H_{\mathbb{C}}$ and $g_j \in \mathrm{Hom}\,(X, \mathbb{C})$. Let $y_j$ be the image of $h_j$ in $Y$. Then for $x \in X$, $b(x, y_j) = -\Re(g_j(x))$. From this, we obtain $b(p(y_1), y_2) = -\Re(g_2(p(y_1)))$. On the other hand, by using $p_H(f_1, f_2) = 0$ and by using the fact that $\mathrm{Hom}\,(X, \mathbb{C})$ is orthogonal to itself with respect to the pairing $p_H$, we have

$$p_H(f_1, \bar{f}_2) = p_H(f_1, \bar{f}_2 - f_2) = p_H(h_1 - g_1, g_2 - \bar{g}_2)$$

$$= p_H(h_1, g_2 - \bar{g}_2) = -(g_2 - \bar{g}_2)(p(y_1)) = -2\Re(g_2(p(y_1))). \ \square$$

We consider the general case of 3.8.4. Note that by 1.2.8 and 3.8.6, we may assume that the local monodromy of $H_{\mathbb{Q}}$ is admissible. We may assume further that $S = \{s\}$ is an fs log point. By 3.8.6 applied to $\mathrm{gr}_{-1}^{W(C(s))}$, $p$ is a polarization if and only if $H$ with $p_H$ is a polarized mixed Hodge structure with respect to $W(C(s))$ in the sense of Cattani-Kaplan-Schmid [7]. By the theorem [7] (4.66), the last condition is equivalent to that it generates a nilpotent orbit, and hence, by [19] Proposition 2.5.5, equivalent to that it is a polarized log Hodge structure. Thus 3.8.4 is proved.

3.8.7. LEMMA. *A polarization on an object of $\mathcal{H}_S$ comes, locally on $S$, from a polarization of a pairing into $\mathbb{G}_{m,\log}$.*

PROOF. We have to show that for any object $H$ of $\mathcal{H}_S$ and any polarization $p$ on it, locally over the base, there is $(X, Y, e)$ such that $(H, X, Y, e)$ belongs to $\tilde{\mathcal{H}}_S$ and such that $p$ kills $\mathcal{H}om(X, \mathcal{O}_S) \times \mathcal{H}om(X, \mathcal{O}_S)$ (cf. Remark after 3.8.4). It is shown in the same way as in the proof of 3.6.5. In fact, at a point of $S$, consider $W_{-1}$ and $W_{-2}$ of the weight filtration associated to the whole monodromy cone. Then the intersection form on $H_{\mathbb{Z}}$ induces an anti-symmetric $\mathbb{Q}$-bilinear form on $W_{-1}/W_{-2}$. Take a totally isotropic subspace $V$ of $W_{-1}/W_{-2}$ whose dimension is the half of that of $W_{-1}/W_{-2}$. Let $V'$ be the inverse image of $V$ in $H_{\mathbb{Q}}$ and let $X = \mathrm{Hom}\,(V' \cap H_{\mathbb{Z}}, \mathbb{Z}(1))$, $Y = H_{\mathbb{Z}}/(H_{\mathbb{Z}} \cap V')$. Then we have an exact sequence $0 \to \mathrm{Hom}\,(X, \mathbb{Z}(1)) \to H_{\mathbb{Z}} \to Y \to 0$ and $p$ kills $\mathcal{H}om(X, \mathcal{O}_S) \times \mathcal{H}om(X, \mathcal{O}_S)$. $\square$

**3.8.8.** By 3.8.4 and by 3.8.7, we see that the equivalence $\mathcal{H}_S \simeq \mathcal{A}_S$ induces an equivalence $\mathcal{H}_S^+ \simeq \mathcal{A}_S^+$. We also see that in this equivalence, an object of $\mathcal{A}_S^+$ is locally polarizable in the sense of 1.3.9 if and only if the corresponding object of $\mathcal{H}_S^+$ is locally polarizable.

**3.8.9.** To state the next proposition, we make some preliminaries. Assume that the underlying space of the base $S$ is $\{s\} = \mathrm{Spec}\,\mathbb{C}$. Let $\mathcal{S} = M_{S,s}/\mathcal{O}_{S,s}^\times$. Then to a log complex torus $A$ over $S$, we associate finitely generated free abelian groups $X_A$ and $Y_A$, the canonical pairing $(X_A, Y_A, \langle\ ,\ \rangle_A)$ into $\mathcal{S}^{\mathrm{gp}}$, and the abelian variety $B$ over $\mathbb{C}$ as follows. Let $(X, Y, \langle\ ,\ \rangle)$ be a non-degenerate pairing such that $A$ is its associated quotient. Let $\sigma = \{1\}$ be the minimal face of $\mathcal{S}$. Let $X_A := X/X_\sigma$, $Y_A := Y/Y_\sigma$, $\langle\ ,\ \rangle_A$ the induced pairing, and $B := \mathrm{Hom}\,(X_\sigma, \mathbb{C}^\times)/Y_\sigma$.

These definitions are independent of choices of pairings. We will prove this fact in the proof of the next proposition together with the fact that a homomorphism $A \to A^*$ induces homomorphisms $B \to B^*$ and $Y_A \to X_A$ naturally.

3.8.10. PROPOSITION. *Let $A$ be a log complex torus over $S$. A homomorphism $p\colon A \to A^*$ is a polarization if and only if its pull back to any $s \in S$ is a polarization. When the underlying analytic space of $S$ is* $\mathrm{Spec}\,\mathbb{C} = \{s\}$, *$p$ is a polarization if and only if the induced $B \to B^*$ is a polarization in the usual sense, and the induced $Y_A \to X_A$ satisfies the condition that for any $y \in Y_A - \{1\}$, $\langle p(y), y\rangle_A \in M_{S,s}/\mathcal{O}_{S,s}^\times - \{1\}$.*

PROOF. First we prove the statements before the proposition. Let the base be an fs log point.

Let $H$ be the log Hodge structure of weight $-1$ corresponding to $A$. Then

$$Y_A = H_{\mathbb{Z}}/(H_{\mathbb{Z}} \cap W(C(s))_{-1}H_{\mathbb{Q}}) = H_{\mathbb{Z}}/\tau^{-1}\tau_* H_{\mathbb{Z}},$$

$$X_A = \mathrm{Hom}\,(H_{\mathbb{Z}} \cap W(C(s))_{-2}H_{\mathbb{Q}}, \mathbb{Z}(1)) = (H^*_{\mathbb{Z}}/\tau^{-1}\tau_* H^*_{\mathbb{Z}})(1),$$

and $B$ is the abelian variety corresponding to the (classical) Hodge structure induced on $\mathrm{gr}^{W(C(s))}_{-1} H$. This shows the statements before the proposition.

Now we prove the proposition. The former statement is clear. For the latter, we take $p_0 \colon Y \to X$ satisfying $\langle p_0(y), z\rangle = \langle y, p_0(z)\rangle$ for any $y, z \in Y$ which induces $p$. This is seen possible as in the proof of 3.8.7. By 3.8.4, it is enough to show that $p_0$ is a polarization if and only if $p_1 \colon Y_\sigma \to X_\sigma$ is a polarization and $p_2 \colon Y_A = Y/Y_\sigma \to X/X_\sigma = X_A$ satisfies

($*$) For any $y \in Y/Y_\sigma - \{1\}$, $\langle p_2(y), y\rangle_A \in M_{S,s}/\mathcal{O}^\times_{S,s} - \{1\}$.

We prove the if part. First note that ($*$) implies that $p_2$ is injective. Since $\mathrm{rank}\, Y = \mathrm{rank}\, X$ and $\mathrm{rank}\, Y_\sigma = \mathrm{rank}\, X_\sigma$, this implies that $p_0$ is injective and its cokernel is finite. Let $y \in Y - \{1\}$. If $y \notin Y_\sigma$, then ($*$) implies $\langle p(y), y\rangle \in M_{S,s} - \mathcal{O}^\times_{S,s}$ and the canonical map $\alpha \colon M_S \to \mathcal{O}_S$ sends $\langle p(y), y\rangle$ to 0. Otherwise, $\langle p(y), y\rangle \in M_{S,s}$ and $|\alpha(\langle p(y), y\rangle)| < 1$ because $p_1$ is a polarization.

We prove the only if part. It is clear that if $p_0$ is a polarization, then $p_1$ is a polarization. If $\langle p(y), y\rangle \in \mathcal{O}^\times_{S,s}$, then $y \in Y_\sigma$ by 3.4.5 (4). $\square$

3.8.11. REMARK. By 3.8.7, at least locally on the base, a polarization of $A$ comes from a polarization of a pairing into $\mathbb{G}_{m,\log}$.

3.8.12. REMARK. As we will see in a forthcoming paper ([13] Theorem 6.1), the expression of the dual complex torus $A^* \simeq \mathcal{E}xt^1(A, \mathbb{G}_m)$ in the usual algebraic geometry has the log version

$$\mathcal{E}xt^1(A, \mathbb{G}_m) \subset A^* \subset \mathcal{E}xt^1(A, \mathbb{G}_{m,\log})$$

for the dual log complex torus $A^*$ of a log complex torus $A$.

### 3.9. Extensions from open sets

As an application of main results in this section, here we discuss when a log complex torus or a log abelian variety on an open subspace of a log smooth fs log analytic space can extend to the ambient space.

3.9.1. PROPOSITION. *Let $S$ be a log smooth fs log analytic space. Let $S_{\mathrm{triv}}$ be the largest open subspace where the log structure is trivial. Then the restriction to $S_{\mathrm{triv}}$ gives a fully faithful functor from the category of log complex tori on $S$ to the category of complex tori on $S_{\mathrm{triv}}$.*

PROOF. We consider the Hodge side via 3.1.5. The local system on $S_{\mathrm{triv}}$ extends uniquely to $S^{\log}$. If a homomorphism $H_{\mathbb{Z}} \to H'_{\mathbb{Z}}$ preserves the Hodge filtration after restricted to $U = S_{\mathrm{triv}}$, it preserves the filtration because $H'_{\mathcal{O}}/F'^0 \to j_*((H'_{\mathcal{O}}/F'^0)|_U)$ is injective. Here $j\colon U \hookrightarrow S$ is the inclusion. $\square$

3.9.2. PROPOSITION. *A polarized abelian variety on $S_{\mathrm{triv}}$ with unipotent local monodromy extends to a polarized log abelian variety on $S$. Here we say the local monodromy is unipotent when the extension to $S^{\log}$ of the corresponding local system $H_{\mathbb{Z}}$ has the unipotent local monodromy.*

Again via 3.1.5, this follows from the nilpotent orbit theorem of W. Schmid ([32]) on the Hodge side. See [15] 2.4 and 2.5 for the details.

**3.9.3.** Example of a complex torus on $(\Delta^*)^2$ which does not extend to a log complex torus over $\Delta^2$.

Let $S = \Delta^2$ with the coordinate functions $q_1, q_2$, endowed with the log structure by the divisor $\{q_1 = 0\} \cup \{q_2 = 0\}$. Let $X = Y = \mathbb{Z}^2$. Let $\langle\, ,\, \rangle : X \times Y \to \mathbb{G}_{m,\log}$ be the pairing defined by $\langle e_1, e_1\rangle = 1$, $\langle e_1, e_2\rangle = q_1$, $\langle e_2, e_1\rangle = q_1$, $\langle e_2, e_2\rangle = q_2$. Let $\sigma$ be the face of $M_{S,0}/\mathcal{O}^{\times}_{S,0}$ generated by the image of $q_1$. Then $X_\sigma$ is generated by $e_1$ and $Y_\sigma$ is also generated by $e_1$, and the restriction of $\langle\, ,\, \rangle$ to $X_\sigma \times Y_\sigma$ is trivial (cf. 1.2.3). Hence the induced pairing into $\mathbb{G}_{m,\log}/\mathbb{G}_m$ is not admissible.

On $(\Delta^*)^2$, it gives a complex torus. This is because the matrix $\begin{pmatrix} 0 & \log(|q_1|) \\ \log(|q_1|) & \log(|q_2|) \end{pmatrix}$ has determinant $< 0$ on $(\Delta^*)^2$ since $|q_1| < 1$ there. This complex torus on $(\Delta^*)^2$ can not extend to a log complex torus over $\Delta^2$.

## 3.10. Liftings from closed sets

Here we prove that a log complex torus or a locally polarizable log abelian variety on a closed subspace of an fs log analytic space can lift to the ambient space at least locally.

3.10.1. PROPOSITION. *Let $S$ be an fs log analytic space. Then the admissibility of a given pairing into $\mathbb{G}_{m,\log}/\mathbb{G}_m$ over $S$ and the non-degeneration of a given pairing into $\mathbb{G}_{m,\log}$ over $S$ are open conditions.*

PROOF. The former is by 1.2.9.

Let $(X, Y, \langle\ ,\ \rangle)$ be a pairing into $\mathbb{G}_{m,\log}$ and assume that its pull back to a point $s \in S$ is non-degenerate. We prove that the pull backs to the points $z$ near $s$ are also non-degenerate. By the former, we may assume that the induced pairing into $\mathbb{G}_{m,\log}/\mathbb{G}_m$ is admissible. We may assume that there is a chart $\beta\colon \mathcal{S} \to M_S$ such that $\mathcal{S} \to M_{S,s}/\mathcal{O}^\times_{S,s}$ is bijective. Let $\sigma$ be a face of $\mathcal{S}$. It is enough to show that there exists a neighborhood $U$ of $s$ such that for any $z \in U \cap \{z \,|\, \beta^{-1}(\mathcal{O}^\times_{S,z}) = \sigma\}$, the pairing $f(z) := -\log|\langle\ ,\ \rangle(z)|\colon \mathbb{R}\otimes X_\sigma \times \mathbb{R}\otimes Y_\sigma \to \mathbb{R}$ is non-degenerate. We may assume that $\sigma = \mathcal{S}$ because the induced pairing $X_\sigma \times Y_\sigma \to \mathbb{G}_{m,\log,S'}$ is also non-degenerate at $s$, where $S'$ is the fs log analytic space whose underlying space is the same as $S$ and whose log structure is given by $\sigma$. Denote by $\langle\ ,\ \rangle_1\colon X \times Y \to M^{\mathrm{gp}}_{S,s}/\mathcal{O}^\times_{S,s} \cong \mathcal{S}^{\mathrm{gp}} \hookrightarrow M^{\mathrm{gp}}_S$ the induced pairing and by $\langle\ ,\ \rangle_2\colon X \times Y \to \mathcal{O}^\times_S$ the pairing such that $\langle\ ,\ \rangle = \langle\ ,\ \rangle_1 \cdot \langle\ ,\ \rangle_2$. Then we have $f(z) = -\log|\langle\ ,\ \rangle_1(z)| - \log|\langle\ ,\ \rangle_2(z)|$.

Now for each $z \in S_{\mathrm{triv}}$, we have a homomorphism $z\colon \mathcal{S} \to \mathbb{C}^\times$. Write as $N'(z)$ the induced homomorphism $-\log(|\quad|(z))\colon \mathcal{S}^{\mathrm{gp}} \to \mathbb{R}$. Then $-\log|\langle\ ,\ \rangle_1(z)|$ coincides with the pairing $X\times Y \to M^{\mathrm{gp}}_{S,s}/\mathcal{O}^\times_{S,s} \cong \mathcal{S}^{\mathrm{gp}} \xrightarrow{N'(z)} \mathbb{R}$. Take a set of generators $(N'_j)_{1\le j\le n}$ of the monodromy cone $C(s)$ and denote by $N_j$ $(1 \le j \le n)$ the induced pairing $X \times Y \to M^{\mathrm{gp}}_{S,s}/\mathcal{O}^\times_{S,s} \xrightarrow{N'_j} \mathbb{R}$. Then, since the $N'(z)$ for $z$ near $s$ is written as $\sum y_j N'_j$ for $y_1, \cdots, y_n \gg 0$, the $-\log|\langle\ ,\ \rangle_1(z)|$ for $z$ near $s$ is written as $\sum y_j N_j$ for $y_1, \cdots, y_n \gg 0$.

On the other hand, take a compact set $B$ of pairings $\mathbb{R}\otimes X \times \mathbb{R}\otimes Y \to \mathbb{R}$ satisfying the following conditions (i) and (ii).

(i) The restriction $\mathbb{R}\otimes X_{\{1\}} \times \mathbb{R}\otimes Y_{\{1\}} \to \mathbb{R}$ of each member of $B$ is non-degenerate.

(ii) For each $z$ near $s$, the pairing $-\log|\langle\ ,\ \rangle_2(z)|$ belongs to $B$.

Such a $B$ exists because the pairing $\mathbb{R}\otimes X_{\{1\}}\times\mathbb{R}\otimes Y_{\{1\}}\to\mathbb{R}$ induced by $-\log|\langle\ ,\ \rangle_2(s)|$ is non-degenerate by the assumption.

Let $f_b(y_1,\cdots,y_n)=b+\sum y_jN_j$, where $b\in B$ and $y_1,\cdots,y_n>0$. Then the $f(z)$ for $z$ near $s$ is $f_b(y_1,\cdots,y_n)$ for some $b\in B$ and $y_1,\cdots,y_n\gg 0$. Now the rest is to show that there exists $C>0$ such that if $y_j>C$ $(1\le j\le n)$ and $b\in B$, then $f_b(y_1,\cdots,y_n)$ is non-degenerate; the same proof as that for the statement before 3.3.15, which can be regarded as a special case of the above statement where $B$ consists of a single pairing, works for it. In fact, let $\alpha\in[0,1]^n$ with $\alpha_n=0$ and write as $g_b(y_1,\cdots,y_n)$ and $g_{\alpha,b}$ the pairings defined from $b,N_1,\cdots,N_n$ exactly in the same way as $g(y_1,\cdots,y_n)$ and $g_\alpha$ are defined from $b_a,N_1,\cdots,N_n$ in 3.3.15 and 3.3.16. Then $f_b(y_1,\cdots,y_n)$ is non-degenerate if and only if $g_b(y_1,\cdots,y_n)$ is non-degenerate. Further, $g_{\alpha,b}$ is non-degenerate. (The proofs are the same.) In the following, taking bases of $X$ and $Y$, we regard $\mathrm{Hom}\,(X\otimes Y,\mathbb{R})$ as a metric space. Since $B$ is compact, we easily see that $g_b(y_1,\cdots,y_n)$ converges to $g_{\alpha,b}$ uniformly with respect to $b\in B$ when $(y_{j+1}/y_j)_{1\le j\le n}$ converges to $\alpha$. Since $\{g_{\alpha,b}\,|\,b\in B\}$ is also compact, we see then that there exists a neighborhood $U_\alpha$ of $\alpha$ in $[0,1]^n$ such that $f_b(y_1,\cdots,y_n)$ is non-degenerate for any $b\in B$ and $y_j$ $(1\le j\le n)$ such that $(y_{j+1}/y_j)_{1\le j\le n}\in U_\alpha$, where $y_{n+1}$ denotes 1. From this, we have the desired $C>0$. $\square$

3.10.2. Proposition. *Let $S$ be an fs log analytic space. Let $(X,Y,\langle\ ,\ \rangle)$ be a pairing into $\mathbb{G}_{m,\log}$ over $S$. Let $p\colon Y\to X$ be a homomorphism satisfying the conditions* (i) *and* (ii) *in the definition of polarization of a pairing into $\mathbb{G}_{m,\log,S}$ in 1.2.7. Then* (iii) *in 1.2.7 for $p$ is an open condition.*

Proof. The proof is similar to the previous one. Assume that $p$ is a polarization at $s$ and prove that it is so near $s$. Take $\beta$ and $\sigma$ as before. If $y\in Y-Y_\sigma$, then $\langle p(y),y\rangle\in\mathcal{S}-\sigma$ (the proof of 3.4.5 (4)), and $|\alpha(\langle p(y),y\rangle)|=0$ on $S_\sigma:=\{z\,|\,\beta^{-1}(\mathcal{O}^\times_{S,z})=\sigma\}$. Hence it is enough to show that there exists a neighborhood $U$ of $s$ such that for any $z\in U\cap S_\sigma$, the pairing $f(z):=-\log|\langle\ ,\ \rangle(z)|$ is positive definite after composed with $p\times\mathrm{id}$. We may assume that $\sigma=\mathcal{S}$. (Note that $p$ induces $Y_\sigma\to X_\sigma$ by (1) in the proof of 1.2.5.) Let $\langle\ ,\ \rangle_1$, $\langle\ ,\ \rangle_2$, $N_1,\cdots,N_n$, and $B$ be also as in the proof of the previous proposition. In this time, we may assume that the restrictions $\mathbb{R}\otimes X_{\{1\}}\times\mathbb{R}\otimes Y_{\{1\}}\to\mathbb{R}$ of the members of $B$ are positive definite. (Here and hereafter we identify $X_\mathbb{R}$ and $Y_\mathbb{R}$ via $p$.) Now the rest is to show that

$b + \sum y_j N_j$ is positive definite for any $b \in B$ and $y_1, \cdots, y_n \gg 0$. Fix $y_0 \in Y_{\mathbb{R}}$. Since $Y_{\mathbb{R}}/\mathbb{R}^{\times}$ is compact, it is enough to show that there exists a neighborhood $V$ of $y_0$ such that $(b + \sum y_j N_j)(y, y) > 0$ for $y \in V$, $b \in B$ and $y_1 \cdots, y_n \gg 0$. Assume first that $y_0 \notin Y_{\{1\}}$. Then, by the assumption, $(\sum N_j)(y_0, y_0) > 0$. Hence for some $j$, there exists a neighborhood $V$ of $y_0$ such that $N_j(y, y) > 0$ for $y \in V$. Since $B$ is compact, this $V$ is sufficient. In case $y_0 \in Y_{\{1\}}$, the value $b(y_0, y_0)$ is positive. Again by the compactness of $B$, we can find a $V$ such that $b(y, y) > 0$ for $y \in V$ and $b \in B$. □

3.10.3. Proposition. *Let $S \subset T$ be an exact closed immersion of fs log analytic spaces. Let $A$ be a log complex torus (resp. a locally polarizable log abelian variety) over $S$. Then for each $s \in S$, there exists an open neighborhood $U$ of $s$ in $T$ such that the restriction of $A$ to $U \cap S$ extends to a log complex torus (resp. a polarizable log abelian variety) over $U$.*

Proof. On an open neighborhood $V$ of $s$ in $S$, take a pairing $(X, Y, \langle\ ,\ \rangle)$ into $\mathbb{G}_{m,\log,S}$ which induces $A|_V$. If $A$ is locally polarizable, we may assume that there is a polarization $p\colon Y \to X$ on $A|_V$. Since $M^{\mathrm{gp}}_{T,s} \to M^{\mathrm{gp}}_{S,s}$ is surjective, there is an open neighborhood $U'$ of $s$ in $T$ such that $U' \cap S \subset V$ and a lifting $\langle\ ,\ \rangle : X \times Y \to \mathbb{G}_{m,\log,U'}$ of $\langle\ ,\ \rangle|_{U' \cap S}$. If $A$ is locally polarizable, we may also assume that $p$ satisfies the conditions (i) and (ii) in 1.2.7 on $U'$. Then the quotient associated to $(X, Y, \langle\ ,\ \rangle)$ is a log complex torus (resp. a polarizable log abelian variety) in an open neighborhood $U$ of $s$ in $U'$. This follows from 3.10.1 (resp. 3.10.2). □

## 4. Moduli of Log Abelian Varieties

The aim of this section is to prove that the toroidal compactifications and the Satake-Baily-Borel compactifications of the moduli spaces over $\mathbb{C}$ of abelian varieties (with additional structures) are moduli spaces of log abelian varieties (with additional structures).

These compactifications were constructed, not necessarily as moduli spaces. Here we have moduli interpretations of these compactifications. We prove that the toroidal compactifications are the fine moduli of polarized log abelian varieties (with additional structures) (Theorem 4.4.4) and the Satake-Baily-Borel compactifications are, in a sense, the coarse moduli (Theorem 4.6.3, 4.6.4, 4.6.5, cf. 4.6.10, 4.7.8). The subsequent Parts of this series of papers will mainly concern the algebraic theory of log abelian varieties. There we will prove the arithmetic versions of the above results, which are closely related to the works [10], [11] (cf. 4.1.10).

The result for the Satake-Baily-Borel compactifications is reduced to that for the toroidal compactifications. For the latter, we have two proofs. One proof is as follows. In [19], moduli spaces of polarized log Hodge structures were studied. In particular, [19] contains the corresponding result that the toroidal compactifications are the fine moduli of polarized log Hodge structures of a specific Hodge type. We can deduce from this the result in this section on the relation between toroidal compactifications and moduli of log abelian varieties, through the equivalence proved in §3 between the category of log abelian varieties and the category of log Hodge structures of the specific Hodge type (cf. 4.1.9). This is one proof. But, actually, we give another, direct proof in this section by using only basic properties of log abelian varieties, not via [19].

## 4.1. Introduction to §4

In this subsection, to present the pictures of our ideas and results, we first introduce the results of this section, for simplicity, in a special case (the case of principal polarization, standard level structures, with no coefficient ring of endomorphisms), and then describe the organization of this section. The general results and the proofs are given in later subsections.

**4.1.1.** Let $g \geq 1$ and let $\mathfrak{H}_g$ be the Siegel upper half space of degree $g$ consisting of all complex symmetric $(g, g)$-matrices whose imaginary parts are positive definite. The group $\mathrm{Sp}\,(2g, \mathbb{R})$ of real symplectic $(2g, 2g)$-matrices acts on $\mathfrak{H}_g$.

For $n \geq 1$, let

$$\Gamma(n) = \mathrm{Ker}\,(\mathrm{Sp}\,(2g, \mathbb{Z}) \to \mathrm{Sp}\,(2g, \mathbb{Z}/n\mathbb{Z})).$$

Then the toroidal compactifications (Mumford compactifications) of $\Gamma(n) \setminus \mathfrak{H}_g$ ([3], [5], [25]), which we denote by $\bar{D}_{g,n,\Sigma}$, and the Satake-Baily-Borel compactification of $\Gamma(n) \setminus \mathfrak{H}_g$ ([31]), which we denote by $\bar{D}_{g,n}$, are defined. They are compact normal analytic spaces containing $\Gamma(n) \setminus \mathfrak{H}_g$ as an open analytic subspace. The toroidal compactifications $\bar{D}_{g,n,\Sigma}$ are defined depending on choices of admissible cone decomposition $\Sigma$ ([3], cf. 4.7.1) of the space of semi-positive definite real symmetric $(g, g)$-matrices. In the case $g = 1$, $\Sigma$ is unique, $\bar{D}_{1,n,\Sigma} = \bar{D}_{1,n}$, and this space is the modular curve with cusps corresponding to $\Gamma(n)$ which is the unique compactification of $\Gamma(n) \backslash \mathfrak{H}_1$. For $g \geq 2$, there are many $\Sigma$. There is a canonical surjective morphism $\bar{D}_{g,n,\Sigma} \to \bar{D}_{g,n}$ which is not an isomorphism, and which induces the identity morphism on $\Gamma(n) \backslash \mathfrak{H}_g$.

**4.1.2.** As is well-known, for $n \geq 3$, the quotient analytic space $\Gamma(n) \backslash \mathfrak{H}_g$ is the fine moduli space of principally polarized abelian varieties of dimension $g$ with $n$-level structures, as we will state more precisely soon.

By using the group structures of log abelian varieties, we can generalize the notions principal polarization and $n$-level structure, to log abelian varieties, as follows.

Let $S$ be an fs log analytic space and let $A$ be a log abelian variety over $S$.

A polarization $p : A \to A^*$ of $A$ is called a principal polarization if $p$ is an isomorphism.

For $n \geq 1$, an $n$-level structure on $A$ is an isomorphism ${}_nA \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$ of sheaves of abelian groups, where ${}_nA = \mathrm{Ker}\,(n : A \to A)$ and $g = \dim(A)$.

**4.1.3.** Let (an) be the category of analytic spaces. For $n \geq 1$, let

$$\Phi_{g,n} : (\mathrm{an}) \to (\mathrm{Set})$$

be the following contravariant functor. For an analytic space $S$, $\Phi_{g,n}(S)$ is the set of isomorphism classes of triples $(A, p, l)$, where $A$ is an abelian variety over $S$, $p$ is a principal polarization of $A$, and $l$ is an $n$-level structure of $A$ satisfying the following condition (i).

(i) The composition

$${}_nA \times {}_nA \xrightarrow{1\times p} {}_nA \times {}_n(A^*) \to (\mathbb{Z}/n\mathbb{Z})(1),$$

where the last arrow is the Weil pairing, is compatible via $l$ with the standard symplectic pairing $(\mathbb{Z}/n\mathbb{Z})^{2g} \times (\mathbb{Z}/n\mathbb{Z})^{2g} \to (\mathbb{Z}/n\mathbb{Z})(1)$ which sends $(e_j, e_k)$ to $2\pi i \bmod n\mathbb{Z}(1)$ if $k = j - g$, to $-2\pi i \bmod n\mathbb{Z}(1)$ if $k = j + g$, and to 0 otherwise.

As is said in the above, if $n \geq 3$, $\Phi_{g,n}$ is represented by $\Gamma(n) \backslash \mathfrak{H}_g$.

Let (fs) be the category of fs log analytic spaces. For $n \geq 1$, we denote the composition

$$(\mathrm{fs}) \to (\mathrm{an}) \xrightarrow{\Phi_{g,n}} (\mathrm{Set}),$$

where the first arrow is forgetting the log structure, by the same letter $\Phi_{g,n}$. Then, for $n \geq 3$, this $\Phi_{g,n}$ is represented by $\Gamma(n) \setminus \mathfrak{H}_g$ with the trivial log structure.

For $n \geq 1$, let

$$\bar{\Phi}_{g,n} : (\mathrm{fs}) \to (\mathrm{Set})$$

be the following contravariant functor. For an fs log analytic space $S$, $\bar{\Phi}_{g,n}(S)$ is the set of isomorphism classes of triples $(A, p, l)$, where $A$ is a log abelian variety over $S$, $p$ is a principal polarization of $A$ and $l$ is an $n$-level structure on $A$ satisfying the same condition (i) as above. (See 4.4.2 for the definition of the Weil pairing on a log abelian variety.)

We have

$$\Phi_{g,n} \subset \bar{\Phi}_{g,n}.$$

**4.1.4.** We endow $\bar{D}_{g,n,\Sigma}$ and $\bar{D}_{g,n}$ with the log structures consisting of all holomorphic functions which are invertible on $\Gamma(n) \setminus \mathfrak{H}_g$. If $g \geq 2$, the log structure of $\bar{D}_{g,n}$ is trivial because the complement of $\Gamma(n) \setminus \mathfrak{H}_g$ is of codimension $\geq 2$. On the other hand, the complement of $\Gamma(n) \setminus \mathfrak{H}_g$ in $\bar{D}_{g,n,\Sigma}$ is a divisor, the log structure of $\bar{D}_{g,n,\Sigma}$ is not trivial, and $\bar{D}_{g,n,\Sigma}$ is a log smooth fs log analytic space.

If $g = 1$ and $n \geq 3$, $\bar{\Phi}_{1,n}$ is represented by the unique compactification $\bar{D}_{1,n}$ of $\Gamma(n) \setminus \mathfrak{H}_1$, the modular curve of full level $n$ structure with cusps. If $g \geq 2$, $\bar{\Phi}_{g,n}$ is not representable but still it is closely related with $\bar{D}_{g,n}$ as in the following theorem, which is a rough version of the result in 4.6. See 4.6 for the precise formulation. Cf. 4.6.10, 4.7.8 for a discussion about coarse moduli spaces.

4.1.5. Theorem. *For $g, n \geq 1$, endow $\bar{D}_{g,n}$ with the log structure as above. Then the fs log analytic space $\bar{D}_{g,n}$ is described in terms of $\bar{\Phi}_{g,n}$. Further, $\bar{D}_{g,n}$ is the universal one among all Hausdorff fs log analytic spaces endowed with a morphism from $\bar{\Phi}_{g,n}$.*

**4.1.6.** Next, fix an admissible cone decomposition $\Sigma$ of the space of semi-positive definite real symmetric $(g, g)$-matrices. Then we have a subfunctor $\bar{\Phi}_{g,n,\Sigma}$ of $\bar{\Phi}_{g,n}$ for each $n \geq 1$ such that

$$\Phi_{g,n} \subset \bar{\Phi}_{g,n,\Sigma} \subset \bar{\Phi}_{g,n}.$$

By definition, the class of a triple $(A, p, l)$ in $\bar{\Phi}_{g,n}(S)$ belongs to $\bar{\Phi}_{g,n,\Sigma}(S)$ if and only if the pull back of $(A, p, l)$ to $(\mathrm{fs}/s)$ for any $s \in S$ satisfies the following condition (i). (When this condition is satisfied, we say that the local monodromies of $A$ are in the direction of $\Sigma$. See 4.1.9 below.)

(i) $(A, p)$ comes from a polarized object $(Y_0, Y_0, \langle\, ,\, \rangle)$ in $\tilde{\mathcal{A}}_s$ ($\langle\, ,\, \rangle : Y_0 \times Y_0 \to \mathbb{G}_{m,\log,s}$, the identity map $\mathrm{id} : Y_0 \to Y_0$ is assumed to be a polarization) such that there exists $\sigma \in \Sigma$ having the following property: For any $N$ in the monodromy cone $C(s)$, the composition

$$Y_0 \times Y_0 \to M^{\mathrm{gp}}_{S,s}/\mathcal{O}^{\times}_{S,s} \xrightarrow{N} \mathbb{R}$$

belongs to $\sigma$. (See 3.3.1 for $\tilde{\mathcal{A}}_s$.)

4.1.7. THEOREM. *Endow $\bar{D}_{g,n,\Sigma}$ with the log structure as in 4.1.4. If $n \geq 3$, $\bar{\Phi}_{g,n,\Sigma}$ is represented by $\bar{D}_{g,n,\Sigma}$.*

**4.1.8.** We can generalize this theorem to the case of non-principal polarization, a $\Gamma$-level structure, and to the case with a fixed ring of endomorphisms. Note that since log abelian varieties have group structures, the notion of homomorphism between log abelian varieties and the notion of endomorphism are defined. The main result is stated in 4.4.4.

**4.1.9.** As is said before, we can easily reduce this 4.4.4 to a part of [19] via the equivalence proved in §3. As an illustration, we give here a proof of the special case 4.1.7. First we remark that the condition (i) in 4.1.6 is equivalent to the following condition (i′), which says that the local monodromy of the corresponding log Hodge structure to $A$ is in the direction of $\Sigma$. Let $(H, p_H)$ be the polarized log Hodge structure over $s$ corresponding to $(A, p)$.

(i′) There exist $\sigma \in \Sigma$ and a surjection $f \colon H_{\mathbb{Z}} \to Y_0$ of local systems on $\tau^{-1}(s)$ satisfying the following conditions (a) and (b).

(a) The kernel $L$ of $f$ is contained in $W_{-1}H_{\mathbb{Z}}$, contains $W_{-2}H_{\mathbb{Z}}$ and satisfies $p_H(L, L) = 0$.

(b) For any $\gamma \in C(s)$, the pairing $Y_0 \times Y_0 \to \mathbb{R}$ induced via $f$ by

$$H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{R}\,;\ (h_1, h_2) \mapsto \frac{1}{2\pi i} p_H(N_\gamma(h_1), h_2)$$

belongs to $\sigma$. Here $N_\gamma$ is the logarithm of the action of $\gamma$.

Let $w = -1, h^{-1,0} = h^{0,-1} = g, h^{p,q} = 0$ if $(p, q) \neq (-1, 0), (0, -1), H_0 = \mathbb{Z}^{2g}$, and let $\langle\ ,\ \rangle_0$ be the non-degenerate and anti-symmetric pairing on $H_0$ defined by the matrix $\begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}$. Then [19] shows that the functor associating, with each $S \in$ (fs), the isomorphism classes of PLH on $S$ of type $(w, (h^{p,q})_{p+q=w}, H_0, \langle\ ,\ \rangle_0, \Gamma(n), \Sigma)$ is represented by $\bar{D}_{g,n,\Sigma}$. By the equivalence of (i) and (i′) remarked above, this functor coincides with $\bar{\Phi}_{g,n,\Sigma}$ via our 3.1.5. This completes the proof of 4.1.7.

As is said before, in the following in this section, we give a direct proof for the general case 4.4.4 including 4.1.7 without appealing to [19].

**4.1.10.** Toroidal compactifications of moduli spaces of abelian varieties over schemes were constructed by Faltings and Chai [10]. The theory for abelian varieties with coefficients was established in Fujiwara's work [11]. In the sequel of this paper, we will give interpretations of their compactifications as the moduli spaces of log abelian varieties. (See [1], [23], [30] for other moduli interpretations.)

**4.1.11.** Organization of this section is as follows. In 4.2, we review the theory of moduli without degeneration in a general case, that is, the case of non-principal polarization, Γ-level structure, and with a coefficient ring of endomorphisms. In 4.3, we review the theory of toroidal compactifications and the Satake-Baily-Borel compactifications in a general case in the sense as above. In 4.4, we describe the moduli problems of log abelian varieties and state one of the main results 4.4.4 in this section, which includes the above 4.1.7 as its special case. In 4.5, we prove 4.4.4 by a different method from that explained in 4.1.9. In 4.6, we state the other main results, which concern the Satake-Baily-Borel compactifications, precisely. They include the above 4.1.5. In 4.6–4.7, we prove them.

**4.1.12.** For such generality explained in 4.1.11, it is better to formulate fans in the space End $(H_{\mathbb{R}})$ as in [3], [19], not in the space of $(g,g)$-matrices as in this subsection. See 4.3.3 for the definition of fans. A fan $\Sigma$ in the sense of this subsection induces a fan $\Sigma'$ in the sense of 4.3.3. See 4.3.12 for details. We remark that, as suggested in (i′) in 4.1.9, this formulation of fans enables us to replace the condition (i) in 4.1.6 with the following simpler equivalent condition (i″) when we describe moduli problems of log abelian varieties in 4.4. See 4.4.3 for details.

(i″) Let $t \in s^{\log}$ and take an isomorphism $f\colon \mathcal{H}_1(\tau^{-1}(A),\mathbb{Z})_t \simeq \mathbb{Z}^{2g}$ which preserves the intersection forms. Then there is a $\sigma \in \Sigma'$ such that via $f$, the logarithms of all positive local monodromy of $\mathcal{H}_1(\tau^{-1}(A),\mathbb{Z})$ at $t$ belong to $\sigma$.

## 4.2. Moduli spaces of abelian varieties (review)

In this subsection, we review the moduli theory without degeneration, that is the theory of Shimura's families ([33]), based on [8].

**4.2.1.** We fix a finite dimensional $\mathbb{Q}$-vector space $V$ and a semi-simple $\mathbb{Q}$-subalgebra $L$ of $\mathrm{End}_{\mathbb{Q}}(V)$. Assume that we are given an anti-symmetric non-degenerate $\mathbb{Q}$-bilinear form

$$\psi : V \times V \to \mathbb{Q}(1)$$

such that for any $\ell \in L$, there is $\ell^* \in L$ satisfying

$$\psi(\ell x, y) = \psi(x, \ell^* y) \quad \text{for all } x, y \in V.$$

We will sometimes fix a finitely generated $\mathbb{Z}$-submodule $V_{\mathbb{Z}}$ of $V$ such that $\mathbb{Q} \otimes V_{\mathbb{Z}} = V$ and $\psi(V_{\mathbb{Z}}, V_{\mathbb{Z}}) \subset \mathbb{Z}(1)$.

**4.2.2.** For $R = \mathbb{R}, \mathbb{C}$, let $V_R = R \otimes_{\mathbb{Q}} V$, $L_R = R \otimes_{\mathbb{Q}} L$. Let $D$ be the set of all $L_{\mathbb{C}}$-submodules $F$ of $V_{\mathbb{C}}$ satisfying the following three conditions (i)–(iii).

(i) $\psi(F, F) = 0$.

(ii) $V_{\mathbb{C}} = F \oplus \bar{F}$.

Here $\bar{F}$ denotes $\{\bar{v} \mid v \in F\}$ in which $\bar{v}$ means the image of $v$ under $V_{\mathbb{C}} \to V_{\mathbb{C}}; a \otimes x \mapsto \bar{a} \otimes x$ $(a \in \mathbb{C}, x \in V)$.

(iii) The hermitian form

$$F \times F \to \mathbb{C}\ ;\ (x, y) \mapsto \psi(x, \bar{y})$$

is positive definite. Here we denote the $\mathbb{C}$-bilinear form $V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ induced by $\psi$, by the same letter $\psi$.

4.2.3. *Example.* In the case $L = \mathbb{Q}$, there is a $\mathbb{Q}$-basis $(e_j)_{1 \leq j \leq 2g}$ of $V$ such that

$$\psi(e_j, e_k) = 2\pi i \quad \text{if } j = k + g, \quad \psi(e_j, e_k) = -2\pi i \quad \text{if } k = j + g,$$
$$\psi(e_j, e_k) = 0 \quad \text{otherwise.}$$

In this case, if we fix such basis, $D$ is identified with Siegel's upper half space $\mathfrak{H}_g$ of degree $g$ consisting of all symmetric $(g, g)$-matrices whose imaginary parts are positive definite. For $z \in \mathfrak{H}_g$, the corresponding $F$ is the $\mathbb{C}$-subspace of $V_{\mathbb{C}}$ generated by $e_{j+g} + ze_j$ $(1 \leq j \leq g)$.

4.2.4. Remark. Let $L = \prod_j L_j$ be the presentation of $L$ as a product of simple $\mathbb{Q}$-algebras, and let $V = \oplus_j V_j$ be the corresponding direct decomposition. If $D$ is not empty, then the following (i)–(iii) hold.

(i) $\psi(x, y) = 0$ if $x \in V_j$, $y \in V_k$ with $j \neq k$.

(ii) For any $j$ and $\ell \in L_j$, $\ell^*$ belongs to $L_j$ in the decomposition of $L$.

(iii) $\mathrm{Tr}_{L_j/\mathbb{Q}}(xx^*) > 0$ for any $j$ and for any non-zero element $x$ of $L_j$. Here $\mathrm{Tr}_{L_j/\mathbb{Q}}$: $L_j \to \mathbb{Q}$ is the trace map.

**4.2.5.** If we fix $V_{\mathbb{Z}}$ as in 4.2.1, $D$ is identified with the set of all decreasing filtrations $(F^p)_p$ on $V_{\mathbb{C}}$ by $L_{\mathbb{C}}$-submodules such that the $\mathbb{Z}$-module $V_{\mathbb{Z}}$ with the filtration $(F^p)_p$ and with the pairing $\psi$ is a polarized Hodge structure of weight $-1$. Here we identify $F \in D$ with the filtration defined by $F^p = V_{\mathbb{C}}$ for $p \leq -1$, $F^0 = F$, $F^p = 0$ for $p \geq 1$.

**4.2.6.** By 4.2.5 and by the correspondence between abelian varieties and Hodge structures, we see that in the category (an) of analytic spaces, $D$ represents the following functor

$$\Phi : (\mathrm{an}) \to (\mathrm{Set}).$$

First, we give the definition of $\Phi$ fixing $V_{\mathbb{Z}}$.

For an analytic space $S$, we define $\Phi(S)$ to be the set of all isomorphism classes of 4-ples $(A, i, p, k)$, where $A$ is an abelian variety over $S$, $i$ is a homomorphism of $\mathbb{Q}$-algebras $L \to \mathbb{Q} \otimes \mathrm{End}(A)$, $p$ is a polarization of $A$, and $k$ is an isomorphism $\mathcal{H}_1(A, \mathbb{Z}) \simeq V_{\mathbb{Z}}$ which is compatible with the actions of $L$ after $\otimes\mathbb{Q}$ and which sends the pairing $\psi_p \colon \mathcal{H}_1(A, \mathbb{Z}) \times \mathcal{H}_1(A, \mathbb{Z}) \to \mathbb{Z}(1)$ induced by $p$ to $\psi$. Here $\mathcal{H}_1(A, \mathbb{Z})$ denotes the homology sheaf of $A/S$ of degree one, that is, $\mathcal{H}om_{\mathbb{Z}}(R^1(A \to S)_*\mathbb{Z}, \mathbb{Z})$, and "induced by $p$"means that $\psi_p$ is the composite of $\mathrm{id}\times p$ and the canonical pairing.

Next, if we do not fix $V_{\mathbb{Z}}$, the above functor $\Phi$ for any fixed $V_{\mathbb{Z}}$ is canonically identified with the following functor.

For an analytic space $S$, $\Phi(S)$ is the set of all isomorphism classes of 4-ples $(A, i, p, k)$, where $A$ is an object of the category $\mathbb{Q} \otimes \mathcal{A}_S$ of abelian varieties over $S$ mod isogeny, $i$ is a homomorphism of $\mathbb{Q}$-algebras $L \to \mathrm{End}_{\mathbb{Q}\otimes\mathcal{A}_S}(A)$, $p$ is a polarization of $A$ in $\mathbb{Q}\otimes\mathcal{A}_S$ (locally on $S$, it has the form $(1/n)\otimes p'$ with $n \geq 1$ and with $p'$ a usual polarization of the abelian variety $A$ over $S$), and $k$ is an isomorphism $\mathcal{H}_1(A, \mathbb{Q}) \simeq V$ which is compatible with the actions of $L$ and which sends the pairing $\psi_p \colon \mathcal{H}_1(A, \mathbb{Q}) \times \mathcal{H}_1(A, \mathbb{Q}) \to \mathbb{Q}(1)$ induced by $p$ to $\psi$.

**4.2.7.** Let $G(\mathbb{Q})$ be the group of all automorphisms of the $L$-module $V$ which preserve $\psi$. In the case where we fix $V_{\mathbb{Z}}$, we denote the subgroup $\{\gamma \in G(\mathbb{Q}) \mid \gamma V_{\mathbb{Z}} = V_{\mathbb{Z}}\}$ of $G(\mathbb{Q})$ by $G(\mathbb{Z})$.

We will consider a subgroup $\Gamma$ of $G(\mathbb{Q})$ satisfying the following condition (C).

(C) $\Gamma \subset G(\mathbb{Z})$ for some choice of $V_{\mathbb{Z}}$ (that is, $\Gamma$ preserves $V_{\mathbb{Z}}$ for some choice of $V_{\mathbb{Z}}$).

If $\Gamma$ is a subgroup of $G(\mathbb{Z})$ of finite index for some choice of $V_{\mathbb{Z}}$, we call $\Gamma$ an arithmetic subgroup of $G(\mathbb{Q})$.

We say $\Gamma$ is neat if for any $\gamma \in \Gamma$, the subgroup of $\mathbb{C}^{\times}$ generated by all the eigenvalues of the action on $V_{\mathbb{C}}$ of $\gamma$ is torsion free.

It can be shown that for any subgroup $\Gamma$ of $G(\mathbb{Q})$ satisfying (C), there is a neat subgroup of $\Gamma$ of finite index.

4.2.8. *Example.* Assume $L = \mathbb{Q}$, and fix a basis $(e_j)$ as in 4.2.3. Then $G(\mathbb{Q})$ is identified with $\mathrm{Sp}(2g, \mathbb{Q})$. For $V_{\mathbb{Z}} = \oplus_j \mathbb{Z}e_j$, $G(\mathbb{Z})$ is identified with

$\mathrm{Sp}(2g, \mathbb{Z})$. Define an arithmetic subgroup $\Gamma(n)$ $(n \geq 1)$ of $G(\mathbb{Q})$ by

$$\Gamma(n) = \mathrm{Ker}\,(\mathrm{Sp}(2g, \mathbb{Z}) \to \mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})).$$

Then $\Gamma(n)$ is neat if $n \geq 3$.

**4.2.9.** (Shimura's families.) Let $\Gamma$ be a subgroup of $G(\mathbb{Q})$ satisfying (C). Then $\Gamma \setminus D$ is regarded as an analytic space in the usual way. That is, the topology of $\Gamma \setminus D$ is the quotient of the topology of $D$. For an open set $U$ of $\Gamma \setminus D$, $\mathcal{O}(U)$ is the set of all $\mathbb{C}$-valued functions on $U$ whose pull backs on the inverse image of $U$ in $D$ are holomorphic.

If $\Gamma$ is neat, $D \to \Gamma \setminus D$ is locally an isomorphism of analytic spaces.

If $\Gamma$ is neat, $\Gamma \setminus D$ represents the following functor

$$\Phi_\Gamma : (\mathrm{an}) \to (\mathrm{Set}),$$

which is defined for any $\Gamma$ satisfying (C), not necessarily neat.

First, fix $V_\mathbb{Z}$ which is stable under $\Gamma$.

For an analytic space $S$, we define $\Phi_\Gamma(S)$ to be the set of all isomorphism classes of 4-ples $(A, i, p, k)$, where $A$ is an abelian variety over $S$, $i$ is a homomorphism of $\mathbb{Q}$-algebras $L \to \mathbb{Q} \otimes_\mathbb{Z} \mathrm{End}(A)$, $p$ is a polarization of $A$, and $k$ is a section of the quotient sheaf $\Gamma \setminus I$ of the following sheaf $I$ on $S$. Here $I$ is the sheaf of isomorphisms $\mathcal{H}_1(A, \mathbb{Z}) \simeq V_\mathbb{Z}$ which are compatible with the actions of $L$ after $\mathbb{Q}\otimes$ and which send the pairing $\psi_p \colon \mathcal{H}_1(A, \mathbb{Z}) \times \mathcal{H}_1(A, \mathbb{Z}) \to \mathbb{Z}(1)$ induced by $p$ to $\psi$, and $\gamma \in \Gamma$ acts on $I$ by $h \mapsto \gamma \circ h$ for a local section $h : \mathcal{H}_1(A, \mathbb{Z}) \to V_\mathbb{Z}$ of $I$.

Next, if we do not fix $V_\mathbb{Z}$, the above functor $\Phi_\Gamma$ for any fixed $V_\mathbb{Z}$ which is stable under $\Gamma$ is canonically identified with the following functor.

For an analytic space $S$, $\Phi_\Gamma(S)$ is the set of all isomorphism classes of 4-ples $(A, i, p, k)$, where $A$ is an object of the category $\mathbb{Q} \otimes \mathcal{A}_S$ of abelian varieties over $S$ mod isogeny, $i$ is a homomorphism of $\mathbb{Q}$-algebras $L \to \mathrm{End}_{\mathbb{Q}\otimes\mathcal{A}_S}(A)$, $p$ is a polarization of $A$ in $\mathbb{Q} \otimes \mathcal{A}_S$, and $k$ is a section of the quotient sheaf $\Gamma \setminus I$ of the following sheaf $I$ on $S$. Here $I$ is the sheaf of isomorphisms $\mathcal{H}_1(A, \mathbb{Q}) \simeq V$ which are compatible with the actions of $L$ and which send the pairing $\psi_p \colon \mathcal{H}_1(A, \mathbb{Q}) \times \mathcal{H}_1(A, \mathbb{Q}) \to \mathbb{Q}(1)$ induced by $p$ to $\psi$.

The above $k$ (whether we fix $V_\mathbb{Z}$ or not) is called a $\Gamma$-level structure.

4.2.10. *Example*. Assume $L = \mathbb{Q}$, fix a basis $(e_j)$ as in 4.2.3, and take $\Gamma = \Gamma(n) \subset G(\mathbb{Q})$ with $n \geq 1$. Then $\Phi_\Gamma$ is identified with the functor $\Phi_{g,n}$ in 4.1.3.

This identification follows from the fact that via the canonical homomorphism ${}_nA \simeq \mathcal{H}_1(A, \mathbb{Z})/n\mathcal{H}_1(A, \mathbb{Z}) \simeq \mathcal{H}_1(A, \mathbb{Z}/n\mathbb{Z})$, the Weil pairing corresponds to the canonical perfect pairing $\mathcal{H}_1(A, \mathbb{Z}/n\mathbb{Z}) \times \mathcal{H}_1(A^*, \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}(1)$.

## 4.3. Compactifications (review)

We review the toroidal compactifications and the Satake-Baily-Borel compactifications ([3], [5], [25], [31]).

We fix $V$, $L$, and $\psi$ as in §4.2.

We first review toroidal compactifications of $\Gamma \setminus D$ and then the Satake-Baily-Borel compactification of $\Gamma \setminus D$ defined for arithmetic subgroups $\Gamma$ of $G(\mathbb{Q})$.

First, toroidal compactifications depend on choices of fans (4.3.3).

4.3.1. DEFINITION. A monodromy cone is a set $\sigma$ of $L_\mathbb{R}$-linear maps $V_\mathbb{R} \to V_\mathbb{R}$ satisfying the following conditions (i)–(iv).

(i) There is a finite family $(N_j)_{l \leq j \leq n}$ of $\mathbb{Q}$-linear maps $V \to V$ such that

$$\sigma = \{ \sum_{1 \leq j \leq n} a_j N_j \mid a_j \in \mathbb{R}, a_j \geq 0 \}.$$

(ii) $\psi(N(x), y) + \psi(x, N(y)) = 0$ for any $N \in \sigma$ and any $x, y \in V_\mathbb{R}$.

(iii) $NN' = 0$ for any $N, N' \in \sigma$.

(iv) $i\psi(N(x), x) \geq 0$ for any $N \in \sigma$ and any $x \in V_\mathbb{R}$.

For a monodromy cone $\sigma$ and for $R = \mathbb{R}, \mathbb{C}$, we denote by $\sigma_R$ the $R$-subspace of $\mathrm{End}_R(V_R)$ generated by $\sigma$.

4.3.2. DEFINITION. For a monodromy cone $\sigma$, a subset $\tau$ of $\sigma$ is called a face of $\sigma$ if the following conditions (i)–(iii) are satisfied.

(i) $0 \in \tau$.

(ii) If $a, a' \in \mathbb{R}$, $a, a' \geq 0$, $N, N' \in \tau$, then $aN + a'N' \in \tau$.

(iii) If $N, N' \in \sigma$ and $N + N' \in \tau$, then $N, N' \in \tau$.

There are only finitely many faces of $\sigma$.

A face of a monodromy cone is also a monodromy cone.

4.3.3. Definition. By a fan, we mean a non-empty set $\Sigma$ of monodromy cones satisfying the following conditions (i) and (ii).

(i) If $\sigma \in \Sigma$, any face of $\sigma$ belongs to $\Sigma$.

(ii) If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of $\sigma$ and is also a face of $\tau$.

**4.3.4.** Let $\Gamma$ be a subgroup of $G(\mathbb{Q})$ satisfying (C) in 4.2.7 and let $\Sigma$ be a fan.

We say $\Gamma$ and $\Sigma$ are compatible if the following condition (i) is satisfied.

(i) $\gamma\sigma\gamma^{-1} \in \Sigma$ for any $\sigma \in \Sigma$ and any $\gamma \in \Gamma$.

We say $\Gamma$ and $\Sigma$ are strongly compatible if they are compatible and if the following condition (ii) is satisfied for any $\sigma \in \Sigma$. Let

$$\Gamma(\sigma) = \Gamma \cap \exp(\sigma),$$

where the intersection is taken in the set of all $\mathbb{R}$-automorphisms of $V_{\mathbb{R}}$.

(ii) $\sigma$ coincides with the set of all linear combinations with non-negative real coefficients of $\log(\gamma)$ for all $\gamma \in \Gamma(\sigma)$.

The following fact is shown easily. If $\Gamma$ is an arithmetic subgroup of $G(\mathbb{Q})$ and if $\Gamma$ and $\Sigma$ are compatible, then $\Gamma$ and $\Sigma$ are strongly compatible.

**4.3.5.** Let $\Gamma$ be a subgroup of $G(\mathbb{Q})$ satisfying (C), let $\Sigma$ be a fan, and assume that they are strongly compatible. Then the partial toroidal compactification $\bar{D}_{\Gamma,\Sigma}$ of $\Gamma \setminus D$ with respect to $\Sigma$ is defined.

It is an analytic space containing $\Gamma \setminus D$ as a dense open analytic subspace. It is called a toroidal compactification if it is compact. For the compactness, see 4.3.10.

**4.3.6.** For a monodromy cone $\sigma$, define the increasing filtration $W(\sigma)$ on $V$ associated to $\sigma$ as follows. The subspace $\mathrm{Ker}\,(N)$ (resp. $\mathrm{Image}(N)$) of $V$ for $N$ in the interior of $\sigma$ is independent of $N$. Define $W_m(\sigma) = V$

for $m \geq 0$, $W(\sigma) = 0$ for $m \leq -3$, and $W_{-1}(\sigma) = \mathrm{Ker}\,(N)$, $W_{-2}(\sigma) = \mathrm{Image}(N)$ for such $N$.

**4.3.7.** We describe the partial toroidal compactification $\bar{D}_{\Gamma,\Sigma}$ of $\Gamma \setminus D$ with respect to $\Sigma$ first in a special local situation in which the theory of toroidal embeddings plays an important role.

We say the pair $(\Gamma, \Sigma)$ is local if there is a monodromy cone $\sigma$ such that $\Sigma$ is the fan consisting of all faces of $\sigma$, and $\Gamma = \Gamma(\sigma)^{\mathrm{gp}} = \{ab^{-1} \mid a, b \in \Gamma(\sigma)\}$.

In this case, $\Gamma$ is a finitely generated free abelian group, and $\Gamma$ is neat since the actions of elements of $\Gamma$ on $V$ are unipotent.

Assume that $(\Gamma, \Sigma)$ is local. Consider the analytic space $\exp(\sigma_{\mathbb{C}})D$ of $L_{\mathbb{C}}$-submodules of $V_{\mathbb{C}}$ obtained by applying $\exp(N)$ for $N \in \sigma_{\mathbb{C}}$ to elements of $D$. The quotient spaces $\Gamma \setminus \exp(\sigma_{\mathbb{C}})D$ and $\exp(\sigma_{\mathbb{C}}) \setminus \exp(\sigma_{\mathbb{C}})D$ have the following structures of analytic spaces. The topologies are the quotients of the topology of $\exp(\sigma_{\mathbb{C}})D$. For an open set $U$, $\mathcal{O}(U)$ is the set of all $\mathbb{C}$-valued functions whose pull backs on the inverse image of $U$ in $\exp(\sigma_{\mathbb{C}})D$ are holomorphic.

The projection $\exp(\sigma_{\mathbb{C}})D \to \Gamma \setminus \exp(\sigma_{\mathbb{C}})D$ is locally an isomorphism of analytic spaces.

Let

$$T = \Gamma \setminus \exp(\sigma_{\mathbb{C}}) \simeq \mathbb{C}^{\times} \otimes \Gamma.$$

Then $\Gamma \setminus \exp(\sigma_{\mathbb{C}})D$ is a $T$-torsor over $\exp(\sigma_{\mathbb{C}}) \setminus \exp(\sigma_{\mathbb{C}})D$.

Let $\Gamma(\sigma)^{\vee}$ be the dual fs monoid of $\Gamma(\sigma)$. That is, $\Gamma(\sigma)^{\vee}$ is the monoid of all homomorphisms $\Gamma(\sigma) \to \mathbb{N}$. Consider the toric variety $\bar{T} = \mathrm{Spec}(\mathbb{C}[\Gamma(\sigma)^{\vee}])_{\mathrm{an}}$. Then $T$ acts on $\bar{T} = \mathrm{Hom}\,(\Gamma(\sigma)^{\vee}, \mathbb{C}_{\mathrm{mult}})$ ($\mathbb{C}_{\mathrm{mult}}$ denotes $\mathbb{C}$ regarded as a multiplicative monoid) in the natural way. For $q \in \bar{T}$, let $\sigma(q)$ be the face of $\sigma$ characterized as follows. For $a \in \sigma_{\mathbb{C}}$, $\exp(a)q = q$ if and only if $a \in \sigma(q)_{\mathbb{C}}$.

We now perform the toroidal embedding

$$\Gamma \setminus \exp(\sigma_{\mathbb{C}})D \xrightarrow{\subset} (\Gamma \setminus \exp(\sigma_{\mathbb{C}})D) \times^{T} \bar{T}.$$

Here, when $T$ acts on sets $P, Q$, $P \times^{T} Q$ denotes the quotient of $P \times Q$ by the relation $(tx, y) \sim (x, ty)$ $(x \in P, y \in Q, t \in T)$. Then $(\Gamma \setminus \exp(\sigma_{\mathbb{C}})D) \times^{T} \bar{T}$ is a $\bar{T}$-bundle over $\exp(\sigma_{\mathbb{C}}) \setminus \exp(\sigma_{\mathbb{C}})D$.

In the case $(\Gamma, \Sigma)$ is local, $\bar{D}_{\Gamma,\Sigma}$ is defined to be the subset of $(\Gamma \setminus \exp(\sigma_{\mathbb{C}})D) \times^{T} \bar{T}$ consisting of all classes of $(F, q)$ $(F \in \exp(\sigma_{\mathbb{C}})D, q \in \bar{T})$ satisfying the following conditions (i) and (ii).

(i) $F$ is a mixed Hodge structure for the weight filtration $W(\sigma(q))$.

(ii) The decreasing filtration on $\mathrm{gr}_{-1}^{W(\sigma(q))}(V)_{\mathbb{C}}$ induced by $F$ is polarized by $\mathrm{gr}_{-1}^{W(\sigma(q))}(\psi)$.

(If $(F,q) \sim (F',q')$, the condition (i) (resp. (ii)) is satisfied by $(F,q)$ if and only if it is satisfied by $(F',q')$.)

Then $\bar{D}_{\Gamma,\Sigma}$ is open in $(\Gamma \setminus \exp(\sigma_{\mathbb{C}})D) \times^T \bar{T}$. Hence we have an analytic structure on $\bar{D}_{\Gamma,\Sigma}$ as an open analytic subspace of $(\Gamma \setminus \exp(\sigma_{\mathbb{C}})D) \times^T \bar{T}$.

We remark that for an element of $\bar{D}_{\Gamma,\Sigma}$, we have a representative $(F,q)$ such that $F \in D$.

**4.3.8.** We describe the partial toroidal compactification $\bar{D}_{\Gamma,\Sigma}$ of $\Gamma \setminus D$ with respect to $\Sigma$ in general.

For each $\sigma \in \Sigma$, let $\Sigma(\sigma)$ be the set of all faces of $\sigma$. Then $(\Gamma(\sigma)^{\mathrm{gp}}, \Sigma(\sigma))$ is local in the sense of 4.3.7. Hence we have an analytic space $\bar{D}_{\Gamma(\sigma)^{\mathrm{gp}},\Sigma(\sigma)}$, which we denote by $U_\sigma$ for simplicity.

We define $\bar{D}_{\Gamma,\Sigma}$ as the quotient of the disjoint union of $U_\sigma$ for all $\sigma \in \Sigma$ by the following equivalence relation. For $\sigma, \sigma' \in \Sigma$, an element $a$ of $U_\sigma$ and an element $a'$ of $U_{\sigma'}$ are equivalent if and only if there are $\tau \in \Sigma$, $\gamma, \gamma' \in \Gamma$, $b \in U_\tau$ such that $\gamma\tau\gamma^{-1} \subset \sigma$, $\gamma'\tau(\gamma')^{-1} \subset \sigma'$, the unique analytic map $U_\tau \to U_\sigma$ which is compatible with $\gamma : D \to D$ sends $b$ to $a$, and the unique analytic map $U_\tau \to U_{\sigma'}$ which is compatible with $\gamma' : D \to D$ sends $b$ to $a'$.

The topology of $\bar{D}_{\Gamma,\Sigma}$ is the quotient topology. For an open set $U$ of $\bar{D}_{\Gamma,\Sigma}$, $\mathcal{O}(U)$ is the set of all $\mathbb{C}$-valued functions on $U$ whose pull backs to the inverse image of $U$ in $U_\sigma$ are holomorphic for any $\sigma \in \Sigma$.

If $\Gamma$ is neat, the canonical projections $\bar{D}_{\Gamma(\sigma)^{\mathrm{gp}},\Sigma(\sigma)} \to \bar{D}_{\Gamma,\Sigma}$ are locally isomorphisms of analytic spaces.

4.3.9. *Example.* In the situation as in 4.2.10, let $\Sigma$ be an admissible cone decomposition. Then $\Sigma$ corresponds to a fan $\Sigma'$ which is strongly compatible with $\Gamma(n)$ (4.3.12 below), and $\bar{D}_{\Gamma(n),\Sigma'}$ is $\bar{D}_{g,n,\Sigma}$ in 4.1.1.

**4.3.10.** Concerning the compactness:

Let $\Sigma$ be a fan. We say $\Sigma$ is complete if for any monodromy cone $\tau$, the set $\{\tau \cap \sigma \mid \sigma \in \Sigma\}$ makes a finite subdivision of $\tau$. If $\Sigma$ is complete, then its support $\bigcup_{\sigma\in\Sigma} \sigma$ coincides with the union of all monodromy cones, that is, the set of all $L_{\mathbb{R}}$-linear maps $N \colon V_{\mathbb{R}} \to V_{\mathbb{R}}$ satisfying the following (i)–(iv).

(i) $\psi(N(x), y) + \psi(x, N(y)) = 0$ for any $x, y \in V_{\mathbb{R}}$.
(ii) $N^2 = 0$.
(iii) $i\psi(N(x), x) \geq 0$ for any $x \in V_{\mathbb{R}}$.
(iv) The kernel of $N$ is $\mathbb{Q}$-rational.

Let $\Sigma$ be a complete fan. Let $\Gamma$ be a subgroup of $G(\mathbb{Z})$ for some choice of $V_{\mathbb{Z}}$ and assume that $\Gamma$ and $\Sigma$ are strongly compatible. Then the partial toroidal compactification of $\Gamma \setminus D$ with respect to $\Sigma$ is compact and $\Gamma$ is of finite index in $G(\mathbb{Z})$.

**4.3.11.** If $\Gamma$ is an arithmetic subgroup of $G(\mathbb{Q})$, there exists a complete fan which is strongly compatible with $\Gamma$.

**4.3.12.** It is sometimes simpler to formulate facts about fans in a different way.

Take a maximal one among all $L$-submodules $V'$ of $V$ such that the restriction of $\psi$ to $V' \times V'$ is trivial. Any two maximal $V'$ are $G(\mathbb{Q})$-equivalent.

Let $V'' \subset V$ be the annihilator of $V'$ for $\psi$. Consider monodromy cones $\sigma$ such that $N(V'') = 0$ and $N(V) \subset V'$ for any $N \in \sigma$. Since a $\mathbb{Q}$-linear map $N : V \to V$ satisfying this condition and the condition $\psi(N(x), y) + \psi(x, N(y)) = 0$ for any $x, y \in V$ is identified with a symmetric bilinear form $V/V'' \times V/V'' \to \mathbb{Q}(1)$ ($N$ is identified with the bilinear form $(x, y) \mapsto \psi(x, N(y))$), we can define a monodromy cone as a rational polyhedral cone in the space of the positive semi-definite symmetric bilinear forms on $V/V''$. Further, we can formulate a fan as a set of monodromy cones in this sense. It is clear that a fan in the sense of 4.3.3 induces a fan in this sense (by taking only cones contained in the set $\{N \mid N(V'') = 0, N(V) \subset V'\}$). Conversely, if $\Gamma$ is a subgroup of $G(\mathbb{Q})$ satisfying (C) and if any two maximal totally isotropic subspaces of $\psi$ are $\Gamma$-equivalent, then a fan in the sense of 4.3.3 which is compatible with $\Gamma$ is determined by the induced fan in the sense here. For example, in the situation as in 4.2.10, for a given admissible cone decomposition $\Sigma$ on the space of symmetric bilinear forms $\mathrm{Sym}^2(\mathbb{Z}^g, \mathbb{Z})$, there is a unique fan which is compatible with $\Gamma(1)$ and which induces $\Sigma$.

**4.3.13.** Next, we review the Satake-Baily-Borel compactification of $\Gamma \setminus D$.

For an arithmetic subgroup $\Gamma$ of $G(\mathbb{Q})$, the Satake-Baily-Borel compactification of $\Gamma \setminus D$, which we denote by $\bar{D}_{\Gamma}$, is defined as follows.

Let $\bar{D}$ be the set of all pairs $(W, F)$, where $W$ is an increasing filtration on $V$ by $L$-submodules such that $W_0 = V$, $W_{-3} = 0$, and $W_{-1}$ coincides with the annihilator of $W_{-2}$ under $\psi$, and $F$ is an $L_{\mathbb{C}}$-submodule of $\mathrm{gr}^W_{-1}(V)_{\mathbb{C}}$ satisfying the following conditions (i)–(iii).

(i) For the $\mathbb{Q}$-bilinear form $\mathrm{gr}^W_{-1}(\psi) : \mathrm{gr}^W_{-1}(V) \times \mathrm{gr}^W_{-1}(V) \to \mathbb{Q}(1)$ induced by $\psi$, we have $\mathrm{gr}^W_{-1}(\psi)(F, F) = 0$.

(ii) $\mathrm{gr}^W_{-1}(V)_{\mathbb{C}} = F \oplus \bar{F}$.

(iii) The hermitian form $F \times F \to \mathbb{C}$ ; $(x, y) \mapsto \mathrm{gr}^W_{-1}(\psi)(x, \bar{y})$ is positive definite.

We have the embedding

$$D \overset{\subset}{\to} \bar{D} \ ; \ F \mapsto (W^{\mathrm{triv}}, F),$$

where $W^{\mathrm{triv}}$ is the increasing filtration defined by $W^{\mathrm{triv}}_{-1} = V$ and $W^{\mathrm{triv}}_{-2} = 0$.

The action of $\Gamma$ on $D$ naturally extends to an action of $\Gamma$ on $\bar{D}$.

As a set,

$$\bar{D}_{\Gamma} = \Gamma \setminus \bar{D}.$$

The analytic structure of $\bar{D}_{\Gamma}$ is as follows.

For any local pair $(\Gamma', \Sigma(\sigma))$ with $\Gamma' \subset \Gamma$, we have a canonical map

$$\bar{D}_{\Gamma', \Sigma(\sigma)} \to \bar{D}_{\Gamma} \ ; \ (F, q) \mapsto (W(\sigma(q)), \mathrm{gr}^{W(\sigma(q))}_{-1}(F)),$$

where $\mathrm{gr}^{W(\sigma(q))}_{-1}(F)$ denotes the decreasing filtration on $\mathrm{gr}^{W(\sigma(q))}_{-1}(V)_{\mathbb{C}}$ induced by $F$.

Fix any strongly compatible pair $(\Gamma, \Sigma)$ with $\Sigma$ complete. Then these maps given for the local pairs $(\Gamma(\sigma)^{\mathrm{gp}}, \Sigma(\sigma))$ for all $\sigma \in \Sigma$ glue to a map

$$\bar{D}_{\Gamma, \Sigma} \to \bar{D}_{\Gamma}.$$

This map is surjective. The topology of $\bar{D}_{\Gamma}$ is the quotient of the topology of $\bar{D}_{\Gamma, \Sigma}$ with respect to this surjection. For any open set $U$ of $\bar{D}_{\Gamma}$, $\mathcal{O}(U)$ is the set of all $\mathbb{C}$-valued functions on $U$ whose pull backs on the inverse image of $U$ in $\bar{D}_{\Gamma, \Sigma}$ are holomorphic. This analytic structure does not depend on the choices of $\Sigma$. We endow $\bar{D}_{\Gamma}$ with the log structure consisting of all holomorphic functions which are invertible on $\Gamma \setminus D$.

4.3.14. *Example*. In the situation as in 4.2.10, $\bar{D}_{\Gamma(n)}$ is nothing but $\bar{D}_{g,n}$ in 4.1.1. In this case, the above description is specialized as follows.

Let $(H_0, \langle\ ,\ \rangle_0)$ be as in 4.1.9. Let $\bar{\mathfrak{H}}_g$ be the set of all pairs $(W, F)$, where $W$ is an increasing filtration on $H_{0,\mathbb{Q}}$ such that $W_0 = H_{0,\mathbb{Q}}$, $W_{-3} = 0$, and $W_{-1}$ is the annihilator of $W_{-2}$ with respect to the pairing $\langle\ ,\ \rangle_0 : H_{0,\mathbb{Q}} \times H_{0,\mathbb{Q}} \to \mathbb{Q}(1)$, and $F$ is a decreasing filtration on $\mathbb{C} \otimes_{\mathbb{Q}} W_{-1}/W_{-2}$ such that $((H_0 \cap W_{-1})/(H_0 \cap W_{-2}), F)$ is a polarized Hodge structure with respect to the pairing $W_{-1}/W_{-2} \times W_{-1}/W_{-2} \to \mathbb{Q}(1)$ induced by $\langle\ ,\ \rangle_0$. We have

$$\mathfrak{H}_g \subset \bar{\mathfrak{H}}_g$$

by $F \mapsto (W, F)$, where $W$ is the increasing filtration defined by $W_{-1} = H_{0,\mathbb{Q}}$ and $W_{-2} = 0$.

As a set,

$$\bar{D}_{g,n} = \Gamma(n) \setminus \bar{\mathfrak{H}}_g.$$

## 4.4. Log abelian varieties and toroidal compactifications

Fix $V, L, \psi$ as in 4.2.1.

**4.4.1.** Let $\Gamma$ be a subgroup of $G(\mathbb{Q})$ satisfying (C).

We denote the composition

$$(\mathrm{fs}) \to (\mathrm{an}) \xrightarrow{\Phi_\Gamma} (\mathrm{Set}),$$

where the first arrow is to forget the log structure, by the same letter $\Phi_\Gamma$. Then, if $\Gamma$ is neat, this $\Phi_\Gamma$ is represented by $\Gamma \setminus D$ with the trivial log structure.

We define a moduli functor of log abelian varieties

$$\bar{\Phi}_\Gamma : (\mathrm{fs}) \to (\mathrm{Set})$$

such that $\Phi_\Gamma \subset \bar{\Phi}_\Gamma$ as follows.

As a preliminary, we define homology sheaves of log abelian varieties. Let $A$ be a log abelian variety over an fs log analytic space $S$. We define $\mathcal{H}_1(\tau^{-1}(A), \mathbb{Z}) = \mathcal{H}om_{\mathbb{Z}}(\mathcal{E}xt^1(\tau^{-1}(A), \mathbb{Z}), \mathbb{Z})$ (cf. 3.1.6). We remark that, as seen in [13], $\mathcal{E}xt^1(\tau^{-1}(A), \mathbb{Z}) = \mathcal{H}^1(\tau^{-1}(A), \mathbb{Z})$, where, in general, for an fs log analytic space $S$ and a sheaf $F$ on $(\mathrm{fs}/S)^{\log}$, we denote by $\mathcal{H}^m(F, -)$ the right derived functor of the direct image functor (abelian sheaf on $(\mathrm{fs}/S)^{\log}/F) \to$ (abelian sheaf on $(\mathrm{fs}/S)^{\log}$). Thus we can call $\mathcal{H}_1(\tau^{-1}(A), \mathbb{Z})$ the homology sheaf of $A/S$ of degree one.

Now we define $\bar{\Phi}_\Gamma$.

First, fix $V_\mathbb{Z}$ which is stable under $\Gamma$.

For an fs log analytic space $S$, we define $\bar{\Phi}_\Gamma(S)$ to be the set of all isomorphism classes of 4-ples $(A, i, p, k)$, where $A$ is a log abelian variety over $S$, $i$ is a homomorphism of $\mathbb{Q}$-algebras $L \to \mathbb{Q} \otimes_\mathbb{Z} \mathrm{End}(A)$, $p$ is a polarization of $A$, and $k$ is a section of the quotient sheaf $\Gamma \setminus I$ on $S^{\log}$. Here $I$ is the sheaf on $S^{\log}$ of isomorphisms $\mathcal{H}_1(\tau^{-1}(A), \mathbb{Z}) \simeq V_\mathbb{Z}$ which are compatible with the actions of $L$ after $\mathbb{Q}\otimes$ and which send the pairing $\psi_p \colon \mathcal{H}_1(\tau^{-1}(A), \mathbb{Z}) \times \mathcal{H}_1(\tau^{-1}(A), \mathbb{Z}) \to \mathbb{Z}(1)$ induced by $p$ to $\psi$.

Next, we do not fix $V_\mathbb{Z}$. The above functor $\bar{\Phi}_\Gamma$ for any fixed $V_\mathbb{Z}$ which is stable under $\Gamma$ is canonically identified with the following functor.

For an fs log analytic space $S$, $\bar{\Phi}_\Gamma(S)$ is the set of all isomorphism classes of 4-ples $(A, i, p, k)$, where $A$ is an object of the category $\mathbb{Q} \otimes \mathcal{A}_S$ of log abelian varieties over $S$ mod isogeny, $i$ is a homomorphism of $\mathbb{Q}$-algebras $L \to \mathrm{End}_{\mathbb{Q}\otimes\mathcal{A}_S}(A)$, $p$ is a polarization of $A$ in $\mathbb{Q} \otimes \mathcal{A}_S$, and $k$ is a section of the quotient sheaf $\Gamma \setminus I$ on $S^{\log}$. Here $I$ is the sheaf on $S^{\log}$ of isomorphisms $\mathcal{H}_1(\tau^{-1}(A), \mathbb{Q}) \simeq V$ which are compatible with the actions of $L$ and which send the pairing $\psi_p \colon \mathcal{H}_1(\tau^{-1}(A), \mathbb{Q}) \times \mathcal{H}_1(\tau^{-1}(A), \mathbb{Q}) \to \mathbb{Q}(1)$ induced by $p$ to $\psi$.

4.4.2. *Example*. Assume $L = \mathbb{Q}$, fix a basis $(e_j)$ as in 4.2.3, and take $\Gamma = \Gamma(n) \subset G(\mathbb{Q})$ with $n \geq 1$. Then $\bar{\Phi}_\Gamma$ is identified with the functor $\bar{\Phi}_{g,n}$ in 4.1.3. Here we give the definition of the Weil pairing of a log abelian variety $A$. Let $H$ be the log Hodge structure corresponding to $A$. Then the exact sequence $0 \to H_\mathbb{Z} \to \mathcal{U}_H \to \tau^{-1}(A) \to 0$ in 3.6.3 induces a homomorphism $\tau^{-1}({}_nA) \to H_\mathbb{Z}/nH_\mathbb{Z}$ and hence a homomorphism ${}_nA \to \tau_*(H_\mathbb{Z}/nH_\mathbb{Z})$. Similarly we have ${}_nA^* \to \tau_*(H^*(1)_\mathbb{Z}/nH^*(1)_\mathbb{Z})$. The pairing $H_\mathbb{Z} \times H^*(1)_\mathbb{Z} \to \mathbb{Z}(1)$ induces a pairing $H_\mathbb{Z}/nH_\mathbb{Z} \times H^*(1)_\mathbb{Z}/nH^*(1)_\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}(1)$ and this induces the Weil pairing ${}_nA \times {}_nA^* \to \mathbb{Z}/n\mathbb{Z}(1)$. (The homomorphism $\tau^{-1}({}_nA) \to H_\mathbb{Z}/nH_\mathbb{Z}$ is always injective since $\mathcal{U}_H$ is torsion free, and it is an isomorphism if $A$ has an $n$-level structure.)

**4.4.3.** Let $\Gamma$ be as in 4.4.1 and let $\Sigma$ be a fan which is strongly compatible with $\Gamma$. We define a functor $\bar{\Phi}_{\Gamma,\Sigma}$ : (fs) $\to$(Set) such that

$$\Phi_\Gamma \subset \bar{\Phi}_{\Gamma,\Sigma} \subset \bar{\Phi}_\Gamma$$

as follows. Consider here the definition of $\bar{\Phi}_\Gamma$ without fixing $V_{\mathbb{Z}}$.

For an fs log analytic space $S$, the class of 4-ple $(A, i, p, k)$ of $\bar{\Phi}_\Gamma(S)$ belongs to $\bar{\Phi}_{\Gamma,\Sigma}(S)$ if and only if it satisfies the following condition (i) for any $t \in S^{\log}$.

(i) Let $a\colon \mathcal{H}_1(\tau^{-1}(A), \mathbb{Q})_t \simeq V$ be a representative of $k$ at $t$. Let $s$ be the image of $t$ in $S$. Then there is $\sigma \in \Sigma$ such that for any element $\gamma$ of $\pi_1(\tau^{-1}(s))$ which belongs to $\mathrm{Hom}\,(M_{S,s}, \mathbb{N}) \subset \pi_1(\tau^{-1}(s))$, the homomorphism $V \to V$ corresponding to the action of $\log(\gamma) = \gamma - 1$ on $\mathcal{H}_1(\tau^{-1}(A), \mathbb{Q})_t$ via $a$ belongs to $\sigma$.

Roughly speaking, this condition (i) says that the local monodromies of $A$ are in the direction of $\Sigma$.

4.4.4. THEOREM. *Let $\Gamma$ be a neat subgroup of $G(\mathbb{Q})$ satisfying the condition (C) in 4.2.7. Let $\Sigma$ be a fan which is strongly compatible with $\Gamma$. Endow the partial toroidal compactification $\bar{D}_{\Gamma,\Sigma}$ with the fs log structure $\{f \in \mathcal{O} \mid f$ is invertible on $\Gamma \setminus D\}$. Then $\bar{\Phi}_{\Gamma,\Sigma}$ is represented by $\bar{D}_{\Gamma,\Sigma}$.*

Thus $\bar{D}_{\Gamma,\Sigma}$ for $(\Gamma, \Sigma)$ as in Theorem 4.4.4 is a fine moduli space of the moduli functor $\bar{\Phi}_{\Gamma,\Sigma}$.

For an arithmetic and neat $\Gamma$ and for $\Sigma$ which is strongly compatible with $\Gamma$, we will have a commutative diagram

$$\begin{array}{ccccc} \Phi_\Gamma & \subset & \bar{\Phi}_{\Gamma,\Sigma} & \subset & \bar{\Phi}_\Gamma \\ \wr\| & & \wr\| & & \downarrow \\ \Gamma \setminus D & \subset & \bar{D}_{\Gamma,\Sigma} & \to & \bar{D}_\Gamma. \end{array}$$

4.4.5. REMARK. In the algebraic context in a forthcoming Part of this series of papers, instead of using $\Gamma$-level structures as above, we are to formulate the moduli problem of log abelian varieties by using $K$-level structures on their Tate modules, where $K$ is a compact open subgroup of $G(\mathbb{A}^f)$. Here $\mathbb{A}^f$ is the ring of finite adeles. There we are to regard the Tate modules of log abelian varieties as sheaves on the Kummer étale sites ([12]) rather than the usual étale ones.

### 4.5. Proof of Theorem 4.4.4

We prove Theorem 4.4.4. How to relate the toroidal embeddings in §4.3 to log abelian varieties is, roughly speaking, the following. We can twist an abelian variety (with additional structures) by a torus action. When our twisting reaches infinity corresponding to a boundary point of a toric variety of the toroidal embedding, we obtain a log abelian variety (cf. 4.5.3).

**4.5.1.** First we reduce 4.4.4 to the local case. Let $\Gamma$ and $\Sigma$ be as in 4.4.4. As in 4.3.8, $\bar{D}_{\Gamma,\Sigma}$ is defined by gluing $\bar{D}_{\Gamma(\sigma)^{\mathrm{gp}},\Sigma(\sigma)}$ ($\sigma \in \Sigma$) in the notation there. On the other hand, we have a

4.5.2. Lemma. *$\bar{\Phi}_{\Gamma,\Sigma}$ is a sheaf.*

Proof. This follows from the assumption that $\Gamma$ is neat. □

By this, $\bar{\Phi}_{\Gamma,\Sigma}$ is obtained by gluing $\bar{\Phi}_{\Gamma(\sigma)^{\mathrm{gp}},\Sigma(\sigma)}$ ($\sigma \in \Sigma$). It is easy to see that the local isomorphisms $\bar{D}_{\Gamma(\sigma)^{\mathrm{gp}},\Sigma(\sigma)} \cong \bar{\Phi}_{\Gamma(\sigma)^{\mathrm{gp}},\Sigma(\sigma)}$ constructed below are compatible so that we have the desired global isomorphism $\bar{D}_{\Gamma,\Sigma} \cong \bar{\Phi}_{\Gamma,\Sigma}$.

**4.5.3.** In the rest of this subsection, we prove the local case of 4.4.4. Assume that $(\Gamma, \Sigma)$ is local in the sense of 4.3.7. We define a morphism $\bar{D}_{\Gamma,\Sigma} \to \bar{\Phi}_{\Gamma,\Sigma}$. Recall that in this case, $\Sigma$ is the set of all faces of a monodromy cone $\sigma$.

We use here the definition of $\bar{\Phi}_{\Gamma,\Sigma}$ given by fixing $V_{\mathbb{Z}}$ which is stable under $\Gamma$, but it will be easy to see that if we use the definition without fixing $V_{\mathbb{Z}}$, this morphism is independent of the choice of $V_{\mathbb{Z}}$.

Let $S$ be an fs log analytic space and assume that we are given a morphism $f : S \to \bar{D}_{\Gamma,\Sigma}$. We define an element of $\bar{\Phi}_{\Gamma,\Sigma}(S)$ corresponding to $f$. In the following, we work locally on $S$. The local sections we will give glue into a section over $S$. Locally on $S$, lift $f$ to a morphism $(F, q) : S \to D \times \bar{T}$. Here we can take $D$ (not $\exp(\sigma_{\mathbb{C}})D$) in the target space by the last remark in 4.3.7. The morphism $F$ is identified with an $\mathcal{O}_S$-submodule of $\mathcal{O}_S \otimes_{\mathbb{Q}} V$ stable under the action of $L$. Let $A$ be an abelian variety $V_{\mathbb{Z}} \backslash (\mathcal{O}_S \otimes_{\mathbb{Q}} V)/F$ over $S$ corresponding to $F$ whose $\mathcal{H}_1(A, \mathbb{Z})$ is identified with $V_{\mathbb{Z}}$.

Let $W(\sigma)_{-2,\mathbb{Z}} = W(\sigma)_{-2} \cap V_{\mathbb{Z}}$ and let $X' = \mathrm{Hom}\,(W(\sigma)_{-2,\mathbb{Z}}, \mathbb{Z}(1))$. Then, for the above $q\colon S \to \bar{T}$, the composition

$$S \xrightarrow{q} \bar{T} \xrightarrow{\subseteq} \Gamma \otimes \mathbb{G}_{m,\log} \to \mathcal{H}om(V_{\mathbb{Z}}, W(\sigma)_{-2,\mathbb{Z}}) \otimes \mathbb{G}_{m,\log},$$

where the third arrow is induced by $\Gamma \to \mathrm{Hom}\,(V_{\mathbb{Z}}, W(\sigma)_{-2,\mathbb{Z}}); \gamma \mapsto (v \mapsto (\gamma - 1)(v))$, defines a homomorphism $h\colon V_{\mathbb{Z}} \to \mathcal{H}om(X', \mathbb{G}_{m,\log})$. The morphism $\mathcal{H}om(X', \mathcal{O}_S) = \mathcal{O}_S \otimes W(\sigma)_{-2} \to \mathrm{Lie}(A) = (\mathcal{O}_S \otimes_{\mathbf{Q}} V)/F$ is injective and hence by $\mathbf{G}_m \simeq \mathcal{O}_S/\mathbf{Z}(1)$, it induces an injective homomorphism $\mathcal{H}om(X', \mathbf{G}_m) \to \mathrm{Lie}(A)/W(\sigma)_{-2,\mathbf{Z}}$. Let $J$ be the push forward of $\mathcal{H}om(X', \mathbf{G}_{m,\log}) \leftarrow \mathcal{H}om(X', \mathbf{G}_m) \to \mathrm{Lie}(A)/W(\sigma)_{-2,\mathbf{Z}}$. Then we have a homomorphism $V_{\mathbf{Z}} \to J$ as the product of the canonical homomorphism $V_{\mathbf{Z}} \to \mathrm{Lie}(A)$ and $h : V_{\mathbf{Z}} \to \mathcal{H}om(X', \mathbf{G}_{m,\log})$. Let $J \to \mathcal{H}om(X', \mathbf{G}_{m,\log}/\mathbf{G}_m)$ be the canonical projection ($\mathrm{Lie}(A)$ is killed here), and let $J^{(V_{\mathbf{Z}})} \subset J$ be the inverse image of the subgroup sheaf $\mathcal{H}om(X', \mathbf{G}_{m,\log}/\mathbf{G}_m)^{(V_{\mathbf{Z}})}$ of $\mathcal{H}om(X', \mathbf{G}_{m,\log}/\mathbf{G}_m)$, and let $A'$ be the cokernel of $V_{\mathbf{Z}} \to J^{(V_{\mathbf{Z}})}$.

CLAIM 1. *$A'$ is a log abelian variety.*

PROOF. To see this, we need not consider the action of $L$. First, choose a $\mathbf{Q}$-subspace $V'$ of $V$ such that $\dim_{\mathbf{Q}}(V') = \dim_{\mathbf{Q}}(V)/2$ and such that $W(\sigma)_{-2} \subset V' \subset W(\sigma)_{-1}$, $\psi(V', V') = 0$. Let $V'_{\mathbf{Z}} = V' \cap V_{\mathbf{Z}}$, $Y = V_{\mathbf{Z}}/V'_{\mathbf{Z}}$, $X = \mathrm{Hom}\,(V'_{\mathbf{Z}}, \mathbf{Z}(1))$. Let $\phi : Y \to X$ be the unique homomorphism such that for any $y \in Y$ and $h \in \mathrm{Hom}\,(X, \mathbf{Z}(1)) = V'_{\mathbf{Z}}$, we have $h(\phi(y)) = \psi(h, y)$. Let $s(X \times Y, \mathbf{G}_m)$ (resp. $s(X \times Y, \mathbf{G}_{m,\log})$) be the sheaf of pairings $b : X \times Y \to \mathbf{G}_m$ (resp. $\mathbf{G}_{m,\log}$) satisfying $b(\phi(y), z) = b(\phi(z), y)$ for any $y, z \in Y$. Then we have a commutative diagram

$$
\begin{array}{ccc}
\Gamma \setminus \exp(\sigma_{\mathbf{C}})D & \longrightarrow & s(X \times Y, \mathbf{G}_m) \\
\cap & & \cap \\
(\Gamma \setminus \exp(\sigma_{\mathbf{C}})D) \times^T \bar{T} & \longrightarrow & s(X \times Y, \mathbf{G}_{m,\log}),
\end{array}
\tag{1}
$$

where the horizontal arrows are defined as follows.

The upper horizontal arrow in (1) is as follows. For an analytic space $S$, a morphism $S \to \exp(\sigma_{\mathbf{C}})D$ defines an $\mathcal{O}_S$-submodule $F$ of $\mathcal{O}_S \otimes_{\mathbf{Q}} V$ such that $\mathcal{O}_S \otimes_{\mathbf{Q}} V = F \oplus \mathcal{H}om(X, \mathcal{O}_S)$. The commutative diagram

$$
\begin{array}{ccccc}
V'_{\mathbf{Z}} & & = & & \mathrm{Hom}\,(X, \mathbf{Z}(1)) \\
\cap & & & & \cap \\
V_{\mathbf{Z}} & \to & (\mathcal{O}_S \otimes_{\mathbf{Q}} V)/F & \xleftarrow{\simeq} & \mathcal{H}om(X, \mathcal{O}_S)
\end{array}
$$

defines a homomorphism $Y = V_{\mathbf{Z}}/V'_{\mathbf{Z}} \to \mathcal{H}om(X, \mathcal{O}_S)/\mathrm{Hom}\,(X, \mathbf{Z}(1)) = \mathcal{H}om(X, \mathbf{G}_m)$ which belongs to $s(X \times Y, \mathbf{G}_m)$.

The lower horizontal arrow in (1) sends $(F, q)$ $(F : S \to \exp(\sigma_{\mathbb{C}})D$, $q : S \to \bar{T})$ to the product of $X \times Y \to \mathbb{G}_m$ defined by $F$ and $X \times Y \to \mathbb{G}_{m,\log}$ induced by $q$ and

$$\bar{T} \overset{\subseteq}{\to} \Gamma \otimes \mathbb{G}_{m,\log} \to \mathrm{Hom}\,(Y, V'_{\mathbb{Z}}) \otimes \mathbb{G}_{m,\log} = \mathcal{H}om(Y, \mathcal{H}om(X, \mathbb{G}_{m,\log})),$$

where the second arrow is induced by $\gamma \mapsto (v \mapsto (\gamma - 1)(v))$ $(\gamma \in \Gamma, v \in V_{\mathbb{Z}})$.

The $A'$ constructed in the above is the quotient associated to the pairing $X \times Y \to \mathbb{G}_{m,\log}$ which is the image of the given $(F, q)$ by the lower horizontal map in (1). From this, we see that $A'$ is a log abelian variety. □

The ring homomorphism $L \to \mathbb{Q} \otimes \mathrm{End}(A')$ is given by the actions of $L$ on $V$, $\mathrm{Lie}(A)$, and $W(\sigma)_{-2}$.

The $\Gamma$-level structure is given as follows. Locally on $S^{\log}$, take a lifting $\tilde{q} \in \Gamma \otimes \mathcal{L}$ of $q \in \Gamma \otimes \mathbb{G}_{m,\log}$. Then this gives $\tilde{h} : V_{\mathbb{Z}} \to \mathcal{H}om(X', \mathcal{L})$. Let $\tilde{J}$ be the push forward of $\mathcal{H}om(X', \mathcal{L}) \leftarrow \mathcal{H}om(X', \mathcal{O}_S) \to \mathrm{Lie}(A)$, and let $\tilde{J}^{(V_{\mathbb{Z}})} \subset \tilde{J}$ be the inverse image of $J^{(V_{\mathbb{Z}})}$. Then we have a homomorphism $V_{\mathbb{Z}} \to \tilde{J}^{(V_{\mathbb{Z}})}$ as the product of the canonical homomorphism $V_{\mathbb{Z}} \to \mathrm{Lie}(A)$ and $\tilde{h}$, and this is a local isomorphism $V_{\mathbb{Z}} \to H_{\mathbb{Z}}$. Here $H_{\mathbb{Z}} = \mathcal{H}_1(\tau^{-1}(A'), \mathbb{Z})$. Such local definition gives a $\Gamma$-level structure globally.

We define a polarization of $A'$. It is $H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}(1)$ obtained from $\psi$ through the $\Gamma$-level structure.

Claim 2. This is a polarization. □

It is easy to see that the map $D \times \bar{T} \to \bar{\Phi}_{\Gamma,\Sigma}$ which we have defined factors through the surjection $D \times \bar{T} \to \bar{D}_{\Gamma,\Sigma}$. Thus we have defined a morphism $\bar{D}_{\Gamma,\Sigma} \to \bar{\Phi}_{\Gamma,\Sigma}$ in the local case.

**4.5.4.** We prove that the morphism $\bar{D}_{\Gamma,\Sigma} \to \bar{\Phi}_{\Gamma,\Sigma}$ we have defined in the local case is in fact an isomorphism, which completes the proof of 4.4.4.

First, we can prove this in the case $L = \mathbb{Q}$ using the commutative diagram (1) in 4.5.3, because this diagram shows that to perform a local toroidal embedding can be interpreted as to extend the space of the pairings into $\mathbb{G}_m$ to the space of the pairings into $\mathbb{G}_{m,\log}$.

Next, we consider the case for general $L$. By the result in the case $L = \mathbb{Q}$, the map $\bar{D}_{\Gamma,\Sigma} \to \bar{\Phi}_{\Gamma,\Sigma}$ is injective as a morphism of sheaves. To prove that it is also surjective, assume that we are given a log abelian variety $A'$ with additional structures. Then, by the result in the case $L = \mathbb{Q}$, we can find an abelian variety $A$ (without the action of $L$) which yields our log abelian variety $A'$ (without the action of $L$) via the morphism constructed in 4.5.3. Since the action of $L$ is compatible with that of $\Gamma$ and with $\Sigma$, and since $L$ acts on $A'$, we see that $L$ acts on $A$. Thus we have proved the desired surjectivity of the morphism $\bar{D}_{\Gamma,\Sigma} \to \bar{\Phi}_{\Gamma,\Sigma}$. $\square$

## 4.6. Log abelian varieties and Satake-Baily-Borel compactifications

The last two subsections of this section are devoted to studying Satake-Baily-Borel compactifications in view of log abelian varieties. In the subsection 4.4, we proved that the toroidal compactifications are described in terms of the moduli functors of log abelian variety (with additional structures): they are nothing but the fine moduli spaces which represent the functors. In this subsection and the next, we show that the Satake-Baily-Borel compactifications are also described as fs log analytic spaces in terms of the moduli functors of log abelian varieties. This result means that Satake-Baily-Borel compactifications are coarse moduli spaces in a sense for the moduli functors of log abelian varieties (with additional structures). (See 4.6.10 and 4.7.8 for a discussion about formulations of coarse moduli spaces.)

**4.6.1.** First, we give preliminary definitions to describe the point set of Satake-Baily-Borel compactifications in terms of the moduli functors.

Let $F : (\mathrm{fs}) \to (\mathrm{Set})$ be a contravariant functor.

Let $|F|$ be the set (not necessarily belonging to the fixed universe) of all equivalence classes of $(a, f)$, where $a$ is an fs log point and $f$ is a morphism $a \to F$. Here we say $(a, f)$ and $(b, g)$ are equivalent if and only if there are fs log points $c_0, c_1, \cdots, c_{2k}$ ($k \geq 0$) with morphisms $h_j : c_j \to F$ ($0 \leq j \leq 2k$) and morphisms $c_0 \to a$, $c_{2k} \to b$, $c_{2j} \to c_{2j+1}$ ($0 \leq j < k$), $c_{2j} \to c_{2j-1}$ ($0 < j \leq k$) over $F$.

Let $P$ be an fs log analytic space. If we regard $P$ as the functor $(\mathrm{fs}) \to (\mathrm{Set})$ represented by $P$, we have clearly $|P| = P$.

**4.6.2.** Consider the moduli functor $F = \bar{\Phi}_\Gamma$ of log abelian varieties with $\Gamma$ an arithmetic subgroup of $G(\mathbb{Q})$. Define $||F||$ as the quotient of $|F|$ by the following equivalence relation: For $j = 1, 2$, let $a_j$ be an fs log point. Let $(A_j, i_j, p_j, k_j) \in F(a_j)$. Let $(H_j, i_j, p_j, k_j)$ be the corresponding polarized log Hodge structure with coefficient and with level structure over $a_j$. Let $t_j \in a_j^{\log}$. Assume that there is an isomorphism between two $\mathbb{Z}$-modules $H_{1,\mathbb{Z},t_1}$ and $H_{2,\mathbb{Z},t_2}$ which respects $i_j, p_j, k_j$, which preserves the weight filtrations $W^{(j)}$ with respect to the whole monodromy cone of $a_j$, and which induces an isomorphism of the polarized Hodge structures with coefficient and with level structure on $\mathrm{gr}^{W^{(j)}}_{-1}$. Then the two classes of $|F|$ defined by $(A_j, i_j, p_j, k_j)$ are equivalent. This definition of the equivalence relation does not depend on choices.

Remark. As is explained in 4.6.10 below, in general, $||F||$ does not necessarily coincide with $|F|$.

4.6.3. Theorem. *The Satake-Baily-Borel compactification $\bar{D}_\Gamma$ is described by the moduli functor $F = \bar{\Phi}_\Gamma$ of log abelian varieties (with additional structures), as follows.*

(1) *As a point set, $\bar{D}_\Gamma$ is $||F||$.*

(2) *The topology of $\bar{D}_\Gamma$ is given as follows. For a subset $U$ of $||F||$, $U$ is open if and only if for any fs log analytic space $P$ and any morphism of functors $P \to F$, the inverse image of $U$ in $P$ under $P = |P| \to |F| \to ||F|| = \bar{D}_\Gamma$ is open.*

(3) *The structure of ringed space of $\bar{D}_\Gamma$ is given as follows. Let $R$ be the sheaf of rings on $\bar{D}_\Gamma$ defined as follows. For an open set $U$ of $||F||$, $R(U)$ is the set of morphisms of functors $F_U \to \mathbb{G}_a$, where $F_U$ denotes the subfunctor of $F$ defined as follows, and $\mathbb{G}_a$ is the sheaf $T \mapsto \mathcal{O}_T(T)$ on* (fs). *Then $R$ is isomorphic to the structure sheaf of $\bar{D}_\Gamma$.*

*The definition of $F_U$ is as follows:*

$F_U(T) = \{a \in F(T) \mid$ *the image of $T \to |F| \to \bar{D}_\Gamma$ induced by $a$ is contained in $U$*$\}$.

(4) *Let $N$ be the sheaf on $\bar{D}_\Gamma$ defined as follows. For an open set $U$ of $||F||$, $N(U)$ is the set of morphisms of functors $F_U \to [M]$, where $[M]$ is the sheaf $T \mapsto M_T(T)$ on* (fs). *Then $N$ is isomorphic to the log structure of $\bar{D}_\Gamma$.*

From this theorem, we have the following consequence (for the reason why the following theorem is a consequence of the above theorem, see 4.6.9 below).

4.6.4. THEOREM. *There is a canonical morphism of functors* $F \to \bar{D}_\Gamma$*, which induces the canonical map* $|F| \to ||F|| = \bar{D}_\Gamma$ *and which has the following universal property.*

*For an fs log analytic space* $Q$*, we have a functorial bijection :*
$\{$*a morphism* $\bar{D}_\Gamma \to Q\}$ $\leftrightarrow$ $\{$*a morphism* $F \to Q$ *such that* $|F| \to Q$ *factors through* $||F||\}$.

In the next section, we prove a stronger theorem in the case with no coefficient.

4.6.5. THEOREM. *Assume that* $L = \mathbb{Q}$ *and* $\Gamma = \Gamma(n)$ *for some* $n \geq 1$ *so that* $\bar{D}_\Gamma = \bar{D}_{g,n}$*. Then, for any Hausdorff fs log analytic space* $Q$ *and any morphism* $F \to Q$*, the map* $|F| \to Q$ *factors through* $||F||$*.*

*Consequently (by 4.6.4), for a Hausdorff fs log analytic space* $Q$*, we have a functorial bijection :*
$\{$*a morphism* $\bar{D}_{g,n} \to Q\}$ $\leftrightarrow$ $\{$*a morphism* $F \to Q\}$.

We prove the first theorem 4.6.3. For this, we prepare a lemma.

4.6.6. LEMMA. *Let* $F$ *be a sheaf on the category* (fs) *of fs log analytic spaces, and assume that we are given fs log analytic spaces* $P_j$ $(j = 1, 2)$ *and morphisms* $a_j : P_j \to F$ $(j = 1, 2)$.

*(1) Assume that the following (i)–(iii) are satisfied.*

*(i)* $a_1$ *is relatively represented by morphisms which are locally base changes of birational proper equivariant morphisms of toric varieties, that is, for any fs log analytic space* $T$ *and any morphism* $T \to F$*, the fiber product* $P_1 \times_F T$ *is represented and the base change* $P_1 \times_F T \to T$ *of* $a_1$ *is, locally on* $T$*, a base change of a birational proper equivariant morphism of toric varieties.*

*(ii)* $a_2$ *is a surjection of sheaves.*

*(iii)* $P_2$ *is log smooth.*

*Then, if $P$ is an fs log analytic space and if two morphisms $f, g : F \to P$ satisfy $f \circ a_1 = g \circ a_1$, we have $f = g$.*

*(2) Assume that we are given an fs log analytic space $P$ and a morphism $p : P_1 \to P$, and assume that the above conditions (i)–(iii) and the following conditions (iv) and (v) are satisfied. (Note that, in (v), the map $P_1 \to |F|$ induced by $a_1 \colon P_1 \to F$ is surjective by (i).)*

*(iv) The morphism $a_2$ is relatively represented by log smooth morphisms.*

*(v) The map of the underlying sets $P_1 \to P$ of $p$ factors through the surjection $P_1 \to |F|$.*

*Then there exists a unique morphism $q : F \to P$ such that $p = q \circ a_1$.*

*(3) Let the assumptions be as in (2), and assume further the following (vi).*

*(vi) $p$ is proper and surjective, and the $\mathcal{O}$ and the $M$ of $P$ are the direct images of $\mathcal{O}$ and $M$ of $P_1$ under $p$, respectively.*

*Then the fs log analytic space structure of $P$ is described in terms of $F$ as follows.*

*(a) A subset $U$ of $P$ is open if and only if for any fs log analytic space $Q$ and any morphism $Q \to F$, the inverse image of $U$ in $Q$ is open.*

*(b) For an open set $U$ of $P$, let $F_U$ be the subsheaf $F \times_P U$ of $F$, that is,*

$$F_U(T) = \{a \in F(T) \mid \text{the image of } T \text{ by } a \text{ in } P \text{ is contained in } U\}.$$

*Then*

$$\mathcal{O}_P(U) = \mathrm{Mor}\,(F_U, \mathbb{G}_a), \quad M_P(U) = \mathrm{Mor}\,(F_U, [M]).$$

Proof. We prove (1). By the condition (ii), it is sufficient to prove $f \circ a_2 = g \circ a_2$. Let $P_2' = P_1 \times_F P_2$. By the condition (i), the canonical projection $b : P_2' \to P_2$ has the same property as $a_1$ described in (i). Hence we have:

(vii) $b$ is surjective, and the topology of $P_2$ is the image of the topology of $P_2'$.

Furthermore, by the condition (iii), the fact that $b$ has the same property as $a_1$ in (i) shows:

(viii) The $\mathcal{O}$ and the $M$ of $P_2$ are the direct images of those of $P_2'$ under $b$, respectively.

By (vii), (viii), the fact $f \circ a_2 \circ b = g \circ a_2 \circ b$ shows $f \circ a_2 = g \circ a_2$.

Next we prove (2). The uniqueness of $q$ follows from (1). We prove the existence of $q$. By (v), we have a map of sets $P_2 \to P$ as the composition $P_2 \to |F| \to P$. Since $b : P_2' \to P_2$ satisfies the above (vii) and (viii), there exists a unique morphism $h : P_2 \to P$ whose underlying map is the above map $P_2 \to P$ such that the composition $h \circ b$ coincides with the composition $P_2' \to P_1 \to P$. We will prove that $h$ factors through the surjection of sheaves $a_2 : P_2 \to F$ and hence gives $q \colon F \to P$. For this, it is sufficient to prove $h \circ \mathrm{pr}_1 = h \circ \mathrm{pr}_2$, where $\mathrm{pr}_j$ $(j = 1, 2)$ is the $j$-th projection $R := P_2 \times_F P_2 \to P_2$. Let $R' := P_2' \times_{P_1} P_2' = P_1 \times_F R$, and let $c : R' \to R$ be the canonical morphism. Then $c$ has the same property as $a_1$ in (i). Hence

(ix) $c$ is surjective, and the topology of $R$ is the image of the topology of $R'$.

By (iii) and (iv), $R$ is log smooth. Hence the fact that $c$ has the same property in (i) shows

(x) The $\mathcal{O}$ and the $M$ of $R$ are the direct images of those of $R'$, respectively.

By (ix), (x), the fact $h \circ \mathrm{pr}_1 \circ c = h \circ \mathrm{pr}_2 \circ c$ shows $h \circ \mathrm{pr}_1 = h \circ \mathrm{pr}_2$.

Finally we prove (3). (a) follows from the fact that $P_1 \to P$ is proper and surjective. We prove (b). Let $U$ be an open set of $P$. Then we have maps

$$\mathrm{Mor}(U, \mathbb{G}_a) \to \mathrm{Mor}(F_U, \mathbb{G}_a) \to \mathrm{Mor}(P_1 \times_P U, \mathbb{G}_a),$$

$$\mathrm{Mor}(U, [M]) \to \mathrm{Mor}(F_U, [M]) \to \mathrm{Mor}(P_1 \times_P U, [M])$$

such that the compositions are bijective by (vi). By applying (1) to the case where $F$, $P_1$, $P_2$, $P$ in (1) are $F_U$, $P_1 \times_P U$, $P_2 \times_P U$,$\mathbb{G}_a$ (resp. $[M]$), we have the injectivity of $\mathrm{Mor}(F_U, \mathbb{G}_a) \to \mathrm{Mor}(P_1 \times_P U, \mathbb{G}_a)$ (resp. $\mathrm{Mor}(F_U, [M]) \to \mathrm{Mor}(P_1 \times_P U, [M])$). This shows that $\mathrm{Mor}(U, \mathbb{G}_a) \to \mathrm{Mor}(F_U, \mathbb{G}_a)$ and $\mathrm{Mor}(U, [M]) \to \mathrm{Mor}(F_U, [M])$ are bijective. $\square$

**4.6.7.** Now we prove 4.6.3. We apply 4.6.6. First we assume that $\Gamma$ is neat. The general case will be treated in 4.6.8. Let $F = \bar{\Phi}_\Gamma$, $P_1 = \bar{D}_{\Gamma,\Sigma}$ for some complete fan $\Sigma$ which is strongly compatible with $\Gamma$, $P = \bar{D}_\Gamma$ and let $P_2$ be the disjoint union of $U_\sigma$, where $\sigma$ ranges over the set of all monodromy cones and $U_\sigma = \bar{D}_{\Gamma(\sigma)^{\mathrm{gp}},\Sigma(\sigma)}$ in the notation in 4.3.8. Let $P_1 \to F$ be the canonical morphism (4.3.13) and let $P_2 \to F$ be the morphism whose $\sigma$-component is the composite $U_\sigma \to \bar{\Phi}_{\Gamma(\sigma)^{\mathrm{gp}},\Sigma(\sigma)} \subset \bar{\Phi}_{\Gamma(\sigma)^{\mathrm{gp}}} \to \bar{\Phi}_\Gamma$, where the first homomorphism is the one we constructed in 4.5.3. Then it is easy to see that the conditions (i)–(v) are satisfied.

Hence by 4.6.6 (2), there exists a unique morphism $q : \bar{\Phi}_\Gamma \to \bar{D}_\Gamma$ such that the composition $\bar{D}_{\Gamma,\Sigma} = \bar{\Phi}_{\Gamma,\Sigma} \subset \bar{\Phi}_\Gamma \to \bar{D}_\Gamma$ coincides with the canonical morphism $\bar{D}_{\Gamma,\Sigma} \to \bar{D}_\Gamma$.

By the definition of $\bar{D}_\Gamma$ as a set (4.3.13), (1) of 4.6.3 is clear. To see (2)–(4) of it, it is enough to apply 4.6.6 (3), which is possible because the condition (vi) in 4.6.6 (3) is satisfied (4.3.13). This completes the proof of 4.6.3 in case where $\Gamma$ is neat.

**4.6.8.** We will complete the proof of 4.6.3 by reducing the general case to the case where $\Gamma$ is neat, which is already proved. First, (1) of 4.6.3 is clear. To see (2)–(4), we take a neat subgroup $\Gamma'$ of $\Gamma$ of finite index. Then $\bar{D}_{\Gamma'}$ is described in termes of $\bar{\Phi}_{\Gamma'}$. Since the map $\bar{D}_{\Gamma'} \to \bar{D}_\Gamma$ is closed, we deduce (2) for $\Gamma$. Similarly, we deduce (3) and (4). This completes the proof of 4.6.3.

The second theorem 4.6.4 follows from the first theorem 4.6.3 by the following general lemma.

4.6.9. Lemma. *Let $F$ be a sheaf on* (fs) *and $P$ an fs log analytic space. If the point set of $P$ is a quotient $||F||$ of $|F|$ and if the topology of $P$, the ringed space structure of $P$, and the log structure of $P$ are determined by $F$ as those of $\bar{D}_\Gamma$ are determined by $\bar{\Phi}_\Gamma$ in 4.6.3, then we have a functorial bijection for an fs log analytic space $Q$*:

$\{$*a morphism* $P \to Q\} \leftrightarrow \{$*a morphism* $F \to Q$ *such that* $|F| \to Q$ *factors through* $||F||\}$.

Proof. First we define the canonical morphism $F \to P$. Let $T$ be an fs log analytic space and $T \to F$ a morphism. Then, we have a morphism of sets $T \to |F| \to ||F|| = P$, which is continuous by the assumption. Further, by the assumption, it extends to a morphism of fs log analytic spaces, that is, gives an element of $P(T)$, where we identify $P$ with the functor represented by $P$. We easily see that the above correspondence makes a morphism $F \to P$. By composing this morphism $F \to P$, we can define a map of sets from the left hand side to the right hand side in the desired bijection in the statement of the lemma. Again by the assumption, it is straightforward to verify that this map is bijective. □

**4.6.10.** Discussion about coarse moduli.

First we review the non-log case. Let $F\colon (\mathrm{an}) \to (\mathrm{Set})$ be a contravariant functor. An analytic space $P$ is a coarse moduli space of $F$ if the following (i)$_0$ and (ii)$_0$ are satisfied.

(i)$_0$ As a point set, $P$ is $F(\operatorname{Spec}\mathbb{C})$.

(ii)$_0$ For an analytic space $Q$, there is a functorial bijection :
$\{$a morphism $P \to Q\}$ $\leftrightarrow$ $\{$a morphism $F \to Q\}$.

Since $F(\operatorname{Spec}\mathbb{C})$ is the set of morphisms from a point $\operatorname{Spec}\mathbb{C}$ to $F$, the set $|F|$ which we introduced in 4.6.1 for a contravariant functor $F\colon (\mathrm{fs}) \to (\mathrm{Set})$ is an analogue of it in the log case. However, we do not formulate the coarse moduli using $|F|$ because for the moduli functor $F = \bar{\Phi}_\Gamma$ of log abelian varieties (with additional structures), $|F|$ does not coincide with $||F|| = \bar{D}_\Gamma$. In fact, for example, in case where $g = 2$ and $\Gamma = \Gamma(n)$ for $n \geq 3$, consider the open subfunctor of $F'$ of $F$ which is defined by discarding the totally degenerate locus. Then, $F'$ is represented by the partial Mumford compactification with respect to the fan consisting of all one-dimensional monodromy cones. Hence the subset $|F'|$ of $|F|$ coincides with this partial Mumford compactification. But the map from this partial Mumford compactification to the Satake-Baily-Borel compactification is not injective, which shows that $|F|$ does not coincide with the Satake-Baily-Borel compactification.

Nevertheless, if we define a coarse moduli for a pair $(F, ||F||)$, where $F$ is a contravariant functor $F\colon (\mathrm{fs}) \to (\mathrm{Set})$ and $||F||$ is a quotient of $|F|$, to be the fs log analytic space $P$ which satisfies the following (i) and (ii), then the Satake-Baily-Borel compactification $\bar{D}_\Gamma$ for any arithmetic subgroup $\Gamma$ of $G(\mathbb{Q})$ is the coarse moduli for the pair $(\bar{\Phi}_\Gamma, ||\bar{\Phi}_\Gamma||)$, where $||\bar{\Phi}_\Gamma||$ is the quotient set defined in 4.6.2. This is nothing but 4.6.4.

(i) As a point set, $P$ is $||F||$.

(ii) For an fs log analytic space $Q$, there is a functorial bijection :
$\{$a morphism $P \to Q\}$ $\leftrightarrow$ $\{$a morphism $F \to Q$ such that $|F| \to Q$ factors through $||F||\}$.

We will discuss another formulation in case of no coefficient in 4.7.8 in the next subsection.

## 4.7. Proof of 4.6.5

In this subsection, we prove 4.6.5.

Let $g, n \geq 1$. Let the notation be as in 4.1 (cf. 4.3.14). In this subsection, we work without coefficient rings.

**4.7.1.** To prove 4.6.5, we use the result 4.1.7 for the toroidal compactifications, which is a special case of 4.4.4. We use an admissible cone decomposition, which is precisely as follows.

Let $Y_0$ be a finitely generated free abelian group of rank $g$. Let $\mathrm{Sym}^2(Y_0, \mathbb{R})$ be the set of all symmetric $\mathbb{Z}$-bilinear forms $Y_0 \times Y_0 \to \mathbb{R}$. Let $\Sigma$ be an admissible cone decomposition of $\mathrm{Sym}^2(Y_0, \mathbb{R})$ in the sense of [3], that is, a set of finitely generated (sharp) rational polyhedral cones in $\mathrm{Sym}^2(Y_0, \mathbb{R})$ satisfying the following conditions:

(1) For $\sigma \in \Sigma$, every face of $\sigma$ is in $\Sigma$;

(2) For $\sigma, \tau \in \Sigma$, the intersection $\sigma \cap \tau$ is a face of $\sigma$;

(3) $\Sigma$ is stable under the action of $\mathrm{Aut}_{\mathbb{Z}}(Y_0)$. Here $\alpha \in \mathrm{Aut}_{\mathbb{Z}}(Y_0)$ acts on $\mathrm{Sym}^2(Y_0, \mathbb{R})$ by $\langle\ ,\ \rangle \mapsto \langle \alpha(\cdot), \alpha(\cdot) \rangle$;

(4) The number of the $\mathrm{Aut}_{\mathbb{Z}}(Y_0)$-orbits in $\Sigma$ is finite;

(5) For any $\sigma \in \Sigma$, any element of $\sigma$ is positive semi-definite, i.e., $\langle y, y \rangle \geq 0$ for any $\langle\ ,\ \rangle \in \sigma$ and any $y \in Y_0$;

(6) For each positive definite symmetric bilinear form $\langle\ ,\ \rangle \colon Y_0 \times Y_0 \to \mathbb{R}$, there exists a unique $\sigma \in \Sigma$ for which $\langle\ ,\ \rangle$ is contained in the interior of $\sigma$.

Let $\bar{D}_{g,n,\Sigma}$ be the toroidal compactification associated to $\Sigma$. Then, we have a surjective morphism $\bar{D}_{g,n,\Sigma} \to \bar{D}_{g,n}$ (cf. 4.3.13).

**4.7.2.** We give a preparation for the proof of 4.6.5.

Let $J_g$ be the sheaf on (fs) defined as follows: For an fs log analytic space $T$, $J_g(T)$ is the set of all pairings $\mathbb{Z}^g \times \mathbb{Z}^g \to M_T^{\mathrm{gp}}$ for which the identity map of $\mathbb{Z}^g$ is a polarization.

Let $\mathrm{Sym}^2(\mathbb{Z}^g,\mathbb{Z})$ be the set of all symmetric bilinear forms $\mathbb{Z}^g \times \mathbb{Z}^g \to \mathbb{Z}$, and let $\mathrm{Sym}^2_{(+)}(\mathbb{Z}^g,\mathbb{Z})$ be the subcone of $\mathrm{Sym}^2(\mathbb{Z}^g,\mathbb{Z})$ consisting of all positive semi-definite forms. For a finitely generated subcone $\alpha$ of $\mathrm{Sym}^2_{(+)}(\mathbb{Z}^g,\mathbb{Z})$, let $J_{g,\alpha}$ be the subsheaf of $J_g$ defined as follows. For an fs log analytic space $T$, $J_{g,\alpha}(T)$ is the set of all $b \in J_g(T)$ such that for any $t \in T$ and any homomorphism $N : M_{T,t}/\mathcal{O}^\times_{T,t} \to \mathbb{N}$, the induced pairing $\mathbb{Z}^g \times \mathbb{Z}^g \to M^{\mathrm{gp}}_{T,t}/\mathcal{O}^\times_{T,t} \xrightarrow{N} \mathbb{Z}$ belongs to $\alpha$. If $\alpha^\vee \subset \mathrm{Hom}\,(\mathrm{Sym}^2(\mathbb{Z}^g,\mathbb{Z}),\mathbb{Z}) = \mathrm{Sym}^2_{\mathbb{Z}}(\mathbb{Z}^g)$ denotes the dual cone of $\alpha$ consisting of all elements of $\mathrm{Sym}^2_{\mathbb{Z}}(\mathbb{Z}^g)$ which send $\alpha$ into $\mathbb{N}$, then by Proposition 3.10.2, $J_{g,\alpha}$ is represented by an open subspace of the toric variety $\mathrm{Spec}(\mathbb{C}[\alpha^\vee])_{\mathrm{an}}$ with the natural log structure.

Let $s_n : J_g \to \bar{\Phi}_{g,n}$ be the morphism which sends $b \in J_g(T)$ to $(A,p,l) \in \bar{\Phi}_{g,n}(T)$, where $A$ is the quotient associated to the pairing $\mathbb{Z}^g \times \mathbb{Z}^g \to M^{\mathrm{gp}}_T; (y,z) \mapsto b(y,z)^n$, $p$ is the polarization induced by the identity map of $\mathbb{Z}^g$, and $l$ is the level structure $(e_j)_{1\le j\le 2g}$, where $e_j$ is the $n$-division point of $A$ defined as follows. Let $(f_j)_{1\le j\le g}$ be the standard base of $\mathbb{Z}^g$. Then for $1 \le j \le g$, $e_j$ is the image of $b(f_j,-) : \mathbb{Z}^g \to \mathbb{G}_{m,\log}$ in $A$. If $g < j \le 2g$, $e_j$ is the image of the element of $\mathrm{Hom}\,(\mathbb{Z}^g,\mathbb{C}^\times) \subset \mathrm{Hom}\,(\mathbb{Z}^g,\mathbb{G}_{m,\log})$ which sends $f_j$ to $\exp(2\pi i/n)$ and $f_k$ $(k \ne j)$ to 1.

Then, as is easily seen, when $\gamma$ ranges over $\mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})$, $\bar{\Phi}_{g,n}$ is the union of the images of $\gamma \circ s_n \colon J_g \to \bar{\Phi}_{g,n}$. Furthermore, $J_g$ is the union of $J_{g,\alpha}$, where $\alpha$ ranges over all finitely generated subcones of $\mathrm{Sym}^2_{(+)}(\mathbb{Z}^g,\mathbb{Z})$.

**4.7.3.** Let $Q$ be a Hausdorff fs log analytic space. Let $\bar{\Phi}_{g,n} \to Q$ be a morphism. To prove 4.6.5, since any fs log point has a morphism from the standard log point $p = (\mathrm{Spec}\,\mathbb{C}, \mathbb{C}^\times \oplus \mathbb{N})$, it is sufficient to prove the following:

Let $(A_1,p_1,l_1)$ and $(A_2,p_2,l_2) \in \bar{\Phi}_{g,n}(p)$ be principally polarized log abelian varieties with $n$-level structures over the standard log point $p$. Assume that their images in $\bar{D}_{g,n}(p)$ coincide. Then their images in $Q$ coincide.

Such $(A_1,p_1,l_1)$ and $(A_2,p_2,l_2)$ come from $b_1, b_2 \in J_g(p)$ having the following property $(*)$ below, via $\gamma \circ s_n : J_g \to \bar{\Phi}_{g,n}$ for a common $\gamma$. Let $2g'$ be the rank of $\mathrm{gr}_{-1}$.

$(*)$ The pairing $h_j : \mathbb{Z}^g \times \mathbb{Z}^g \to M^{\mathrm{gp}}_p/\mathcal{O}^\times_p \simeq \mathbb{Z}$ induced by $b_j$ kills $(\mathbb{Z}^{g'} \oplus \{0\}^{g-g'}) \times \mathbb{Z}^g$ for $j = 1, 2$. The pairing $(\{0\}^{g'} \oplus \mathbb{R}^{g-g'}) \times (\{0\}^{g'} \oplus$

$\mathbb{R}^{g-g'}) \to \mathbb{R}$ induced by $h_j$ is non-degenerate for $j = 1, 2$. The pairings $(\mathbb{Z}^{g'} \oplus \{0\}^{g-g'}) \times (\mathbb{Z}^{g'} \oplus \{0\}^{g-g'}) \to \mathcal{O}_p^\times = \mathbb{C}^\times$ induced by $b_1$ and $b_2$ coincide.

It is sufficient to prove that the images of $b_1, b_2$ in $Q$ coincide. We will prove this in 4.7.5 after a preparation in the next subparagraph.

**4.7.4.** We give preparations on finitely generated subcones of $\mathrm{Sym}^2_{(+)}(\mathbb{Z}^g, \mathbb{Z})$.

Let

$$\psi : \mathrm{Sym}^2(\mathbb{Z}^g, \mathbb{Z}) \simeq \mathrm{Sym}^2(\mathbb{Z}^{g'}, \mathbb{Z}) \times \mathrm{Hom}\,(X, \mathbb{Z}) \times \mathrm{Sym}^2(\mathbb{Z}^{g-g'}, \mathbb{Z})$$

with $X = (\mathbb{Z}^{g'})^{g-g'}$, be the canonical isomorphism, where the first component of $\psi$ is defined by the restriction to $(\mathbb{Z}^{g'} \oplus \{0\}^{g-g'}) \times (\mathbb{Z}^{g'} \oplus \{0\}^{g-g'})$, the second component of $\psi$ is defined by the restriction to $(\mathbb{Z}^{g'} \oplus \{0\}^{g-g'}) \times (\{0\}^{g'} \oplus \mathbb{Z}^{g-g'})$, and the third component of $\psi$ is defined by the restriction to $(\{0\}^{g'} \oplus \mathbb{Z}^{g-g'}) \times (\{0\}^{g'} \oplus \mathbb{Z}^{g-g'})$.

Fix a finitely generated subcone $\alpha'$ of $\mathrm{Sym}^2_{(+)}(\mathbb{Z}^{g'}, \mathbb{Z})$ such that $(\alpha')^{\mathrm{gp}} = \mathrm{Sym}^2(\mathbb{Z}^{g'}, \mathbb{Z})$. Let $S = J_{g',\alpha'}$ and let $s \in S$ be the unique point such that the image of any element of $(\alpha')^\vee - \{1\}$ in $\mathcal{O}(s) = \mathbb{C}$ is zero.

Now in 3.4, take $S$ as above and take $Y, X, \mathcal{S}$ as follows. Let $Y = X = (\mathbb{Z}^{g'})^{g-g'}$, $\mathcal{S} = (\alpha')^\vee$ the set of all elements of $\mathrm{Sym}^2_{\mathbb{Z}}(\mathbb{Z}^{g'})$ which send $\alpha'$ into $\mathbb{N}$, and let

$$\langle\, , \,\rangle : (\mathbb{Z}^{g'})^{g-g'} \times (\mathbb{Z}^{g'})^{g-g'} \to \mathrm{Sym}^2_{\mathbb{Z}}(\mathbb{Z}^{g'}) = \mathcal{S}^{\mathrm{gp}}$$

be the canonical pairing $((u_j)_{1 \le j \le g-g'}, (v_j)_{1 \le j \le g-g'}) \mapsto \sum_{1 \le j \le g-g'} u_j \otimes v_j$ $(u_j, v_j \in \mathbb{Z}^{g'})$. Then this pairing is $\mathcal{S}$-admissible. This is because the canonical pairing $\mathbb{Z}^{g'} \times \mathbb{Z}^{g'} \to \mathrm{Sym}^2_{\mathbb{Z}}(\mathbb{Z}^{g'}) = \mathcal{S}^{\mathrm{gp}}$; $(u, v) \mapsto u \otimes v$ is $\mathcal{S}$-admissible, for this pairing is identified with the canonical pairing $\mathbb{Z}^{g'} \times \mathbb{Z}^{g'} \to M^{\mathrm{gp}}_{S,s}/\mathcal{O}^\times_{S,s}$ and $\mathcal{S}$ is identified with $M_{S,s}/\mathcal{O}^\times_{S,s}$.

Let $C \subset \mathrm{Hom}\,(\mathcal{S}, \mathbb{N}) \times \mathrm{Hom}\,(X, \mathbb{Z})$ be the cone defined in 3.4.

Claim 1. Let $N \in \alpha' = \mathrm{Hom}\,(\mathcal{S}, \mathbb{N}) \subset \mathrm{Sym}^2(\mathbb{Z}^{g'}, \mathbb{Z})$ and let $\ell \in \mathrm{Hom}\,(X, \mathbb{Z})$. Then $(N, \ell) \in C$ if and only if there exist $f \in \mathrm{Sym}^2_{(+)}(\mathbb{Z}^g, \mathbb{Z})$ and $m \in \mathrm{Sym}^2(\mathbb{Z}^{g-g'}, \mathbb{Z})$ such that $\psi(f) = (N, \ell, m)$.

Proof of Claim 1. We prove the "if part". Assume that $f, m$ as above exist. For $1 \le j \le g - g'$, let $\ell_j : \mathbb{Z}^{g'} \to \mathbb{Z}$ be the $j$-th component of $\ell$.

Let $x = (x_j)_{1 \le j \le g-g'} \in X = (\mathbb{Z}^{g'})^{g-g'}$ $(x_j \in \mathbb{Z}^{g'})$, and assume $x \in X_{\mathrm{Ker}\,(N)}$. This last assumption means that $\{x_j\} \times \mathbb{Z}^{g'}$ is killed by the pairing $\mathbb{Z}^{g'} \times \mathbb{Z}^{g'} \to \mathcal{S}^{\mathrm{gp}} \xrightarrow{N} \mathbb{Z}$ for any $1 \le j \le g-g'$. To prove $(N, \ell) \in C$, it is sufficient to prove $\ell(x) = 0$. We will prove $\ell_j(x_j) = 0$ for any $j$. Let $(\delta_j)_{1 \le j \le g-g'}$ be the standard base of $\mathbb{Z}^{g-g'}$. For any real number $c$, we have $f(cx_j+\delta_j, cx_j+\delta_j) \ge 0$, where we regard $x_j$ as an element of $\mathbb{Z}^{g'} \oplus \{0\}^{g-g'}$ and $\delta_j$ as an element of $\{0\}^{g'} \oplus \mathbb{Z}^{g-g'}$. Since $f(cx_j+\delta_j, cx_j+\delta_j) = c^2N(x_j, x_j) + 2c\ell_j(x_j) + m(\delta_j, \delta_j)$ and since $N(x_j, x_j) = 0$, we have $2c\ell_j(x_j) + m(\delta_j, \delta_j) \ge 0$. By taking $c \to \infty$, we obtain $\ell_j(x_j) \ge 0$. By taking $c \to -\infty$, we obtain $\ell_j(x_j) \le 0$. Hence $\ell_j(x_j) = 0$.

Next we prove the "only if part". Assume $(N, \ell) \in C$. Fix any positive definite symmetric bilinear form $m_0 : \mathbb{Z}^{g-g'} \times \mathbb{Z}^{g-g'} \to \mathbb{Z}$. Let $Z \subset \mathbb{Z}^{g'}$ be the set of all $z \in \mathbb{Z}^{g'}$ such that $N(z, \mathbb{Z}^{g'}) = 0$. Then since $(N, \ell) \in C$, $\ell_j(Z) = 0$ for any $j$. Since $N$ induces a positive definite symmetric bilinear form $\mathbb{Z}^{g'}/Z \times \mathbb{Z}^{g'}/Z \to \mathbb{Z}$, for any sufficiently large integer $c > 0$, the symmetric bilinear form $f_c : \mathbb{Z}^g \times \mathbb{Z}^g \to \mathbb{Z}$ characterized by $\psi(f_c) = (N, \ell, cm_0)$ induces a positive definite symmetric bilinear form $(\mathbb{Z}^{g'}/Z \oplus \mathbb{Z}^{g-g'}) \times (\mathbb{Z}^{g'}/Z \oplus \mathbb{Z}^{g-g'}) \to \mathbb{Z}$ and in particular belongs to $\mathrm{Sym}^2_{(+)}(\mathbb{Z}^g, \mathbb{Z})$. $\square$

We now consider finitely generated subcones $\alpha$ of $\mathrm{Sym}^2_{(+)}(\mathbb{Z}^g, \mathbb{Z})$ having the following properties (i) and (ii).

(i) The restriction of each member of $\alpha$ to $\mathbb{Z}^{g'} \times \mathbb{Z}^{g'} = (\mathbb{Z}^{g'} \oplus \{0\}^{g-g'}) \times (\mathbb{Z}^{g'} \oplus \{0\}^{g-g'})$ is contained in $\alpha'$.

(ii) Let $\alpha_0$ be the face of $\alpha$ consisting of all elements of $\alpha$ which kill $(\mathbb{Z}^{g'} \oplus \{0\}^{g-g'}) \times \mathbb{Z}^g$, and regard $\alpha_0$ as a set of symmetric $\mathbb{Z}$-bilinear forms $(\{0\}^{g'} \oplus \mathbb{Z}^{g-g'}) \times (\{0\}^{g'} \oplus \mathbb{Z}^{g-g'}) \to \mathbb{Z}$. Then $(\alpha_0)^{\mathrm{gp}}$ coincides with the set of all symmetric $\mathbb{Z}$-bilinear forms $(\{0\}^{g'} \oplus \mathbb{Z}^{g-g'}) \times (\{0\}^{g'} \oplus \mathbb{Z}^{g-g'}) \to \mathbb{Z}$.

CLAIM 2. Let $\alpha$ be a finitely generated subcone of $\mathrm{Sym}^2_{(+)}(\mathbb{Z}^g, \mathbb{Z})$ satisfying the above (i) and (ii), and let $\sigma$ be the image of $\alpha$ in $\mathrm{Sym}^2(\mathbb{Z}^{g'}, \mathbb{Z}) \times \mathrm{Hom}\,(X, \mathbb{Z})$ under the first and the second components of $\psi$. Then $\sigma$ is a finitely generated subcone of $C$.

This follows from the "if part" of Claim 1.

Claim 3. For any finitely generated subcone $\sigma$ of $C$, there is a finitely generated subcone $\alpha$ of $\mathrm{Sym}^2_{(+)}(\mathbb{Z}^g, \mathbb{Z})$ satisfying the above (i) and (ii) whose image in $\mathrm{Sym}^2(\mathbb{Z}^{g'}, \mathbb{Z}) \oplus \mathrm{Hom}\,(X, \mathbb{Z})$ coincides with $\sigma$.

This follows from the "only if part" of Claim 1.

Let $\alpha$ and $\sigma$ be as in Claim 2.

Note that $\alpha_0$ is the kernel of $\alpha \to \sigma$. Let $\alpha^\vee \subset \mathrm{Sym}^2_{\mathbb{Z}}(\mathbb{Z}^g)$ be the dual cone of $\alpha$, let $\sigma^\vee \subset \mathcal{S}^{\mathrm{gp}} \times X$ be the dual cone of $\sigma$ which we identify with the annihilator of $\alpha_0$ in $\alpha^\vee$, and let $I$ be the complement of $\sigma^\vee$ in $\alpha^\vee$. Let $R_\alpha$ be the closed analytic subspace of $J_{g,\alpha}$ defined by the ideal generated by $I$. We endow $R_\alpha$ with the inverse image of the natural log structure of $J_{g,\alpha}$.

The restriction to $(\mathbb{Z}^{g'} \oplus \{0\}^{g-g'}) \times \mathbb{Z}^g$ defines a canonical morphism

$$R_\alpha \to V(\sigma),$$

where $V(\sigma) \to S$ is as in 3.5.

Claim 4. The underlying morphism of analytic spaces of $R_\alpha \to V(\sigma)$ (we forget the log structures here) is an isomorphism.

Proof of Claim 4. This is because $\alpha^\vee$ is the disjoint union of $\sigma^\vee$ and $I$, $R_\alpha$ is the inverse image of $S$ in $\mathrm{Spec}(\mathbb{C}[\alpha^\vee]/(I))_{\mathrm{an}}$, and $V(\sigma)$ is the inverse image of $S$ in $\mathrm{Spec}(\mathbb{C}[\sigma^\vee])_{\mathrm{an}}$. $\square$

**4.7.5.** Now we prove that for $b_1, b_2 \in J_g(p)$ satisfying $(*)$ in 4.7.3, the images of $b_1$ and $b_2$ in $Q$ coincide. (This will complete the proof of Theorem 4.6.5).

Fix $\alpha'$ as in 4.7.4. Let $S$, $s$, $\mathcal{S}$, $X$, $C$ be as in 4.7.4.

Take any $\alpha$ satisfying (i) and (ii) such that $\alpha_0$ contains $h_1$ and $h_2$ in 4.7.3. Let $\sigma$ be the image of $\alpha$ in $\mathrm{Hom}\,(\mathcal{S}, \mathbb{N}) \times \mathrm{Hom}\,(X, \mathbb{Z})$, and let $I \subset \alpha^\vee$ be as in 4.7.4. Then $b_1, b_2 \in J_g(p)$ are contained in $J_{g,\alpha}(p)$. Furthermore, $b_1, b_2 \in R_\alpha(p)$. In fact, $I$ does not kill $\alpha_0$ and hence does not kill any element in the interior of $\alpha_0$. Since $h_j$ $(j = 1, 2)$ induces a positive definite pairing $(\{0\}^{g'} \times \mathbb{R}^{g-g'}) \times (\{0\}^{g'} \times \mathbb{R}^{g-g'}) \to \mathbb{R}$, $h_j$ belongs to the interior of $\alpha_0$, and hence $I$ does not kill $h_j$. This shows that the pull backs of the elements of $I$ in $\mathcal{O}(p) = \mathbb{C}$ are zero. Hence $b_1, b_2$ belong to $R_\alpha(p)$.

Let $F = J_g$. Let $T = R_\alpha \times_S R_\alpha$ (recall that $S = J_{g',\alpha'}$). We have two morphisms $f, g\colon T \rightrightarrows Q$ induced by the two projections $T \rightrightarrows R_\alpha$. It is enough to show that the underlying morphisms of these two morphisms $T \rightrightarrows Q$ coincide. Let $T'$ be the fiber product $T \times_{Q\times Q} Q \subset T$ in the category of analytic spaces, where $T \to Q \times Q$ is $(f, g)$ and $Q \to Q \times Q$ is the diagonal. It is sufficient to prove $T' = T$. Since $Q$ is Hausdorff, $T'$ is closed in $T$. Let $\Lambda$ be the set of all infinitesimal neighborhoods of $s$ in $S$, that is, the set of all closed analytic subspaces of $S$ whose underlying sets coincide with $s$. For $\lambda \in \Lambda$, let $T_\lambda = T \times_S \lambda$. Since $T'$ is a closed analytic subspace of $T$, it is sufficient to show that for any $\lambda$, the morphism $T_\lambda \to T$ factors through a morphism $T_\lambda \to T'$, that is, the underlying morphisms of analytic spaces of $f, g : T_\lambda \rightrightarrows Q$ coincide. Fix $\lambda \in \Lambda$. By Proposition 3.5.6 (2), for a sufficiently large finitely generated subcone $\tau$ of $C$, the underlying morphism of analytic spaces of $V(\sigma) \times_S \lambda \to V(\tau) \times_S \lambda$ factors through the canonical projection $V(\sigma) \times_S \lambda \to \lambda$. By Claim 3, there is a finitely generated subcone $\beta$ of $\mathrm{Sym}^2_{(+)}(\mathbb{Z}^g, \mathbb{Z})$ which contains $\alpha$ and whose image in $\mathrm{Hom}\,(\mathcal{S}, \mathbb{N}) \times \mathrm{Hom}\,(X, \mathbb{Z})$ coincides with $\tau$. By Claim 4, the underlying morphism of analytic spaces of $R_\alpha \times_S \lambda \to R_\beta \times_S \lambda$ factors through the canonical projection $R_\alpha \times_S \lambda \to \lambda$. Hence the underlying morphisms of analytic spaces of the two morphisms $T_\lambda \rightrightarrows R_\beta \times_S \lambda \to Q$ coincide.

4.7.6. Remark. The generalization of 4.6.5 to the case with coefficient rings does not necessarily valid. But, for example, the generalization is valid if there is an $L$-submodule $V'$ of $V$ satisfying $\dim(V') = \dim(V)/2$ and $\psi(V', V') = 0$. The proof is similar and we do not discuss it in detail here.

4.7.7. Remark. The above 4.7.3–4.7.5 are roughly outlined in terms of the previous subsection 4.3 as follows. Let the notation be as in 4.3.7. Assume that $(\Gamma, \Sigma)$ is local. The problem is roughly to show that any fiber of the map from the maximally degenerate locus of $\bar{D}_{\Gamma,\Sigma}$ to the Satake-Baily-Borel compactification maps on one point of a Hausdorff $Q$. Since this map factors through the base $\exp(\sigma_{\mathbb{C}}) \setminus \exp(\sigma_{\mathbb{C}})D$ of the toroidal embedding performed there, and since the underlying morphism of the morphism from the maximally degenerate locus of $\bar{D}_{\Gamma,\Sigma}$ to $\exp(\sigma_{\mathbb{C}}) \setminus \exp(\sigma_{\mathbb{C}})D$ is an isomorphism (cf. Claim 4), it is essential to consider fibers of the factored morphism from $\exp(\sigma_{\mathbb{C}}) \setminus \exp(\sigma_{\mathbb{C}})D$ to the Satake-Baily-Borel compactification. The last map is, as in the subsection 4.5, described as a restriction of the map from the space of the pairings $\mathbb{Z}^g \times \mathbb{Z}^{g'} \to \mathbb{G}_m$ to the space of the pairings $\mathbb{Z}^{g'} \times \mathbb{Z}^{g'} \to \mathbb{G}_m$, its fibers are (usual) abelian varieties $A$ of dimension $g(g-g')$, and it is not at all trivial that each $A$ maps on one point of $Q$. What we did above is to take a log abelian variety $A'$ which is a total degeneration of this $A$ and apply a theory in the preceding section to this $A'$. Then we saw that $A'$ mapped on one point. Then the original $A$ also mapped on one point.

**4.7.8.** Further discussion about coarse moduli. This is a continuation of 4.6.10. Let $F$ : (fs) $\to$ (Set) be an arbitrary contravariant functor. At present, we do not have any appropriate candidate of what should be the point set of $F$. As explained in 4.6.10, we do not think that the naive candidate $|F|$ is good. (However, $|F|$ is useful. For example, we used it in the proof of 4.6.3).

Another candidate is $||F||$ which is defined below. The above proof of 4.6.5 shows that, in case of $F = \bar{\Phi}_{g,n}$, this $||F||$ coincides with the point set of $\bar{D}_{g,n}$, so that if we formulate the coarse moduli for an $F$ with this $||F||$, then we can say that $\bar{D}_{g,n}$ is the coarse moduli for $\bar{\Phi}_{g,n}$. But, as is suggested in 4.7.6, this $||F||$ does not necessarily coincide with the point set of the Satake-Baily-Borel compactification in the case of coefficient rings. So we do not think that $||F||$ is always good, neither.

Here is the definition of $||F||$, where $F$: (fs) $\to$ (Set) is a contravariant functor. Let $||F||$ be the quotient set of $|F|$ by the equivalence generated by the following relation.

Let $T$ be an fs log analytic space, let $f, g : T \to F$ be morphisms and let $p \in T$. Assume that there exists a family $(T_\lambda)$ of closed analytic subspaces of $T$ satisfying the following (i) and (ii). Then the classes of $f : p \to F$ and $g : p \to F$ in $|F|$ are equivalent.

(i) For any closed analytic subspace $T'$ of $T$ such that the morphism of analytic spaces $T_\lambda \to T$ factors through a morphism $T_\lambda \to T'$ for any $\lambda$, we have $p \in T'$.

(ii) For each $\lambda$, there are an fs log analytic space $V_\lambda$, morphisms $f_\lambda, g_\lambda$ : $T_\lambda \to V_\lambda$ and a morphism $h_\lambda : V_\lambda \to F$ such that $h_\lambda \circ f_\lambda$ coincides with the restriction of $f$ to $T_\lambda$, $h_\lambda \circ g_\lambda$ coincides with the restriction of $g$ to $T_\lambda$, and the underlying morphism of analytic spaces of $f_\lambda$ coincides with that of $g_\lambda$.

We add a remark about the condition (i). Consider the condition

(i′) The point $p$ belongs to the closure of the union of all the sets $T_\lambda$ in $T$.

Then we have (i′) $\Rightarrow$ (i). However, the converse is not true. For example, if $T = \mathbb{C}$ and the family $(T_\lambda)$ is the family $(\mathrm{Spec}(\mathbb{C}[x]/(x^n))_{\mathrm{an}})_{n\geq 1}$, then the closure of the union of all the sets $T_\lambda$ is $\{0\}$, but the condition (i) holds for any point $p$ in $\mathbb{C}$.

Note that if $P$ is a Hausdorff fs log analytic space, then the canonical map $P \to ||P||$ is bijective. This is essentially proved in 4.7.5.

## 5. Proper Models

### 5.1. Results

**5.1.1.** Let $(X, Y, \langle\ ,\ \rangle)$ be a non-degenerate pairing into $\mathbb{G}_{m,\log}$ over an fs log analytic space $S$. We denote by $A$ the associated log complex torus. Let $\mathcal{S}$ be an fs monoid. Assume that we are given an $\mathcal{S}$-admissible pairing $X \times Y \to \mathcal{S}^{\mathrm{gp}}$ and a homomorphism $\mathcal{S} \to M_S/\mathcal{O}_S^\times$ such that the induced map $X \times Y \to M_S^{\mathrm{gp}}/\mathcal{O}_S^\times$ coincides with $\langle\ ,\ \rangle$ modulo $\mathbb{G}_m$.

**5.1.2.** Now we consider the cone $C$ (3.4.2). A cone decomposition $\Sigma$ is by definition a fan in $\mathrm{Hom}\,(\mathcal{S}^{\mathrm{gp}} \times X, \mathbb{Q})$ whose support is contained in the cone $C_\mathbb{Q}$ of the non-negative rational linear combinations of elements of $C$. Note that $\Sigma$ may not cover $C$. Assume that $\Sigma$ is stable under the action of $Y$, where $y \in Y$ acts on $C$ by $(N, l) \mapsto (N, l + N(\langle\ , y\rangle))$. Then we define the subsheaf $A^{(\Sigma)}$ of $A$ as $Y \backslash \mathcal{H}om(X, \mathbb{G}_{m,\log,S})^{(\Sigma)}$, where $\mathcal{H}om(X, \mathbb{G}_{m,\log,S})^{(\Sigma)} = \bigcup_{\Delta\in\Sigma} V(\Delta)$ (3.5.2). We show that $A^{(\Sigma)}$ is representable in 5.3, and call the representing object, which is also denoted by $A^{(\Sigma)}$, the model of $A$ associated to $\Sigma$. We say that a model is a proper (resp. projective) model if it is proper (resp. projective) over $S$.

A vague form of the result in this section is the following. See 5.2, 5.3 and 5.4 for the precise statements. Let $A$ be a log complex torus over $S$.

5.1.3. THEOREM. *A "nice" cone decomposition exists at least locally on the base $S$. The model associated to it is a proper model. If $A$ is a polarizable log abelian variety, then at least locally on $S$, there is a "nice" cone decomposition which produces a projective model.*

## 5.2. Cone decomposition

In this section, we show the following theorem.

5.2.1. Theorem. *Let $\langle\ ,\ \rangle\colon X \times Y \to \mathcal{S}^{\mathrm{gp}}$ be an admissible pairing. Then there exists a fan $\Sigma$ in $(\mathcal{S}^{\mathrm{gp}})^*_{\mathbb{Q}} \oplus X^*_{\mathbb{Q}}$ satisfying the following four conditions:*

*(1) The support of $\Sigma$ is equal to the cone $C_{\mathbb{Q}}$ of the non-negative rational linear combinations of elements of $C$ in 3.4.2;*

*(2) The fan $\Sigma$ is stable under the action of $Y$ on $C_{\mathbb{Q}}$ defined by $(N, l) \mapsto (N, l + N(\langle ?, y\rangle))$ for $y \in Y$;*

*(3) $\Sigma$ has only finitely many $Y$-orbits;*

*(4) For any cone $\Delta \in \Sigma$ and $y \in Y$, the intersection $\Delta \cap (\Delta + y)$ coincides with $\{(N, l) \in \Delta; N(\langle ?, y\rangle) = 0\}$.*

*Here we denote by $+$ the $Y$-action, and by $(?)^*_{\mathbb{Q}}$ the set* $\mathrm{Hom}\,(?, \mathbb{Q})$ *of homomorphisms to $\mathbb{Q}$.*

5.2.2. Remark. If $\langle\ ,\ \rangle$ has a polarization $p\colon Y \to X$ as in Lemma 1.2.5, then it yields a natural cone decomposition of $C$ satisfying (1)—(4) above. See [27], [22] and [10]. Cf. also 5.4.

**5.2.3.** By a combinatorial lemma 5.2.14 proved later, to prove Theorem 5.2.1, it is enough to show that there is a finite set $J$ of finitely generated $\mathbb{Q}_{\geq 0}$-subcones of $C_{\mathbb{Q}}$ satisfying the following three conditions:

(1) $C_{\mathbb{Q}} = \bigcup_{\Delta \in J, y \in Y} (\Delta + y)$.

(2) For any $\Delta \in J$ and $y \in Y$, $\Delta \cap (\Delta + y)$ is a face of $\Delta$ and a face of $\Delta + y$. The action $? \mapsto ? + y$ is the identity on $\Delta \cap (\Delta + y)$.

(3) For any $\Delta, \Delta' \in J$, the set of cones $\{\Delta \cap (\Delta' + y); y \in Y\}$ is finite.

We remark that in (2), that $\Delta \cap (\Delta - y)$ is a face of $\Delta$ implies that $(\Delta \cap (\Delta - y)) + y = \Delta \cap (\Delta + y)$ is a face of $\Delta + y$. Thus the phrase "and a face of $\Delta + y$" in (2) could be deleted.

We will construct such a $J$.

5.2.4. Definition. For every face $\sigma$ of $\mathcal{S}$, we fix bases $(x_{\sigma,i})_i$ of $X_\sigma$, and an element $s_\sigma$ in the interior of $\sigma$. Let $a = (a_{\sigma,i})_{\sigma,i}$ and $b = (b_{\sigma,i})_{\sigma,i}$ be collections of rational numbers with $a_{\sigma,i} \leq b_{\sigma,i}$ for each $\sigma$ and $i$. Then we define a (finitely generated) cone $C_{a,b} \subset \mathrm{Hom}\,(\mathcal{S}, \mathbb{N}) \oplus \mathrm{Hom}\,(X, \mathbb{Z})$

$$C_{a,b} := \{(N, l); a_{\sigma,i} N(s_\sigma) \leq l(x_{\sigma,i}) \leq b_{\sigma,i} N(s_\sigma) \text{ for each } \sigma, i\}.$$

In 3.4.9, we made a similar construction. The cone $C(a)$ ($a \in \mathbb{N}$) there is $C_{-a,a}$ here.

In the following, we fix $(x_{\sigma,i})_i$ and $s_\sigma$ as above.

The next is proved similarly as 3.4.10.

5.2.5. LEMMA. *Let the notation be as above.*

*(1)* $C_{a,b} \subset C$ *for every* $a, b$.

*(2)* $C = \bigcup_{-\infty<a\le b<+\infty} C_{a,b}$. *Here* $-\infty < a = (a_{\sigma,i}) \le b = (b_{\sigma,i}) < +\infty$ *means that* $-\infty < a_{\sigma,i} \le b_{\sigma,i} < +\infty$ *for each face* $\sigma$ *of* $\mathcal{S}$ *and* $i$.

PROOF. (1) Suppose $(N, l) \in C_{a,b}$ and set $\sigma := \operatorname{Ker} N$. Then $N(s_\sigma) = 0$ implies that $l(x_{\sigma,i}) = 0$ for all $i$. Hence $l(X_\sigma) = 0$ and $(N, l) \in C$.

(2) Let us take $(N, l) \in C$. We show that there exist (sufficiently small) $a = (a_{\sigma,i})$ and (sufficiently big) $b = (b_{\sigma,i})$ such that

$$a_{\sigma,i}N(s_\sigma) \le l(x_{\sigma,i}) \le b_{\sigma,i}N(s_\sigma) \text{ for all } \sigma, i.$$

If $N(s_\sigma) = 0$, then $l(x_{\sigma,i}) = 0$. If $N(s_\sigma) \ne 0$, then $N(s_\sigma) > 0$ by definition. So, the assertion is clear. □

In the followings, we denote $\mathbb{Q}_{\ge 0} \otimes_{\mathbb{N}} C_{a,b}$ by $C_{a,b}$ by abuse of notation.

Before constructing a $J$, we first propose three lemmas.

5.2.6. LEMMA. *For a sufficiently small* $\varepsilon > 0$*, the following holds: For any collections of rational numbers*

$$a = (a_{\sigma,i})_{\sigma,i},\ b = (b_{\sigma,i})_{\sigma,i},$$
$$a' = (a'_{\sigma,i})_{\sigma,i},\ b' = (b'_{\sigma,i})_{\sigma,i}$$

*such that for any* $\sigma, i$,

$$0 \le b_{\sigma,i} - a_{\sigma,i} < \varepsilon$$
$$0 \le b'_{\sigma,i} - a'_{\sigma,i} < \varepsilon,$$

*the following holds:*

*For any* $y, z \in Y$ *and any* $N \in \operatorname{Hom}(\mathcal{S}, \mathbb{Q}_{\ge 0})$*, if*

$$C_{a,b} \cap (C_{a',b'} + y) \cap (\textit{the fiber of } N) \ne \varnothing \textit{ and}$$
$$C_{a,b} \cap (C_{a',b'} + z) \cap (\textit{the fiber of } N) \ne \varnothing,$$

*then* $y \equiv z \bmod Y_\sigma$*, where* $\sigma := \mathrm{Ker}\,(N)$*. (Here the fiber of* $N$ *means the inverse image of* $N$ *by* $C_{\mathbb{Q}} \longrightarrow \mathrm{Hom}\,(\mathcal{S}, \mathbb{Q}_{\geq 0}); (N, l) \mapsto N$*.)*

5.2.7. Lemma. *For any collections of rational numbers* $a, b, a', b'$*, there exist a finite set of elements* $y_1, \dots, y_k$ *of* $Y$ *such that for any* $y \in Y$ *and* $N \in \mathrm{Hom}\,(\mathcal{S}, \mathbb{Q}_{\geq 0})$*, if* $C_{a,b} \cap (C_{a',b'} + y) \cap$ *(the fiber of* $N$*)* $\neq \varnothing$*, then there exists* $i$ *such that* $y \equiv y_i \bmod Y_\sigma$*, where* $\sigma = \mathrm{Ker}\,(N)$*.*

5.2.8. Lemma. *Let* $W$ *be a compact subset of* $\mathbb{R} \otimes Y$*. Then, for some rational numbers* $a \ll 0$ *and* $b \gg 0$*, the following holds:*
*If* $y \in W \cap (\mathbb{Q} \otimes Y)$*,* $N \in \mathrm{Hom}\,(\mathcal{S}, \mathbb{Q}_{\geq 0})$*, then* $(N, N\langle ?, y\rangle) \in C_{a,b}$*.*
*(Here* $a, b$ *of* $C_{a,b}$ *means the collections* $a_{\sigma,i} = a, b_{\sigma,i} = b$ *for any* $\sigma, i$*.)*

**5.2.9.** We deduce the existence of $J$ from Lemmas 5.2.6, 5.2.7 and 5.2.8. Let the situation be as in Theorem 5.2.1. Take an $\varepsilon > 0$ satisfying the condition in Lemma 5.2.6. On the other hand, take a compact subset $W$ of $\mathbb{R} \otimes Y$ satisfying $\mathbb{R} \otimes Y = \bigcup_{y \in Y}(W + y)$, and take rational numbers $a, b$ satisfying the condition in Lemma 5.2.8 for $W$. Next, take a finite number of rational numbers $c_0, \dots, c_k$ such that

$$a = c_0 \leq c_1 \leq \cdots \leq c_k = b, \; c_i - c_{i-1} < \varepsilon \; (i = 1, \dots, k).$$

For each map $f \colon \{(\sigma, i)\} \longrightarrow \{1, \dots, k\}$, define the finitely generated subcone $C_f$ of $C_{\mathbb{Q}}$ as

$$C_f = C_{p,q},$$

where $p_{\sigma,i} = c_{f(\sigma,i)-1}$ and $q_{\sigma,i} = c_{f(\sigma,i)}$. Let $J$ be the set of all such $C_f$ ($f$ runs). Then this $J$ satisfies the three conditions in 5.2.3. In fact,
(1): By the admissibility, any element of $C_{\mathbb{Q}}$ has the form $(N, N\langle ?, z\rangle)$, $N \in \mathrm{Hom}\,(\mathcal{S}, \mathbb{Q}_{\geq 0})$, $z \in \mathbb{Q} \otimes Y$. Take a $y \in Y$ such that $z - y \in W$. By Lemma 5.2.8, $(N, N\langle ?, z - y\rangle) \in C_{a,b}$. Hence there is an $f$ such that $(N, N\langle ?, z - y\rangle) \in C_f$. (Note that $C_{a,b} = \bigcup_f C_f$.) This implies $(N, N\langle ?, z\rangle) \in C_f + y$.
(2): For $y \in Y$, $N \in \mathrm{Hom}\,(\mathcal{S}, \mathbb{Q}_{\geq 0})$ and $C_f \in J$,

$$C_f \cap (C_f + y) \cap (\text{the fiber of } N) \neq \varnothing$$

implies $y \in Y_\sigma$, where $\sigma = \mathrm{Ker}\,(N)$. This is seen from Lemma 5.2.6 by letting $z = 0$. Hence, taking the minimal face $\sigma$ such that $y \in Y_\sigma$

(Lemma 3.4.5 (2)), we see that $C_f \cap (C_f + y)$ is the inverse image of the face $\{N; N(\sigma) = 0\}$ of $\mathrm{Hom}(\mathcal{S}, \mathbb{Q}_{\geq 0})$ by the natural homomorphism $C_f \longrightarrow \mathrm{Hom}(\mathcal{S}, \mathbb{Q}_{\geq 0})$. Hence this is a face of $C_f$ on which the action of $y$ is trivial.

(3): Let $C_f, C_{f'} \in J$. For each $y \in Y$, let $\sigma_y$ be the minimal element in

$$\{\mathrm{Ker}(N); N \in \mathrm{Image}(C_f \cap (C_{f'} + y) \longrightarrow \mathrm{Hom}(\mathcal{S}, \mathbb{Q}_{\geq 0}))\}.$$

Let $N$ be an element of $\mathrm{Hom}(\mathcal{S}, \mathbb{Q}_{\geq 0})$ with $\mathrm{Ker}(N) = \sigma_y$ that belongs to the image of $C_f \cap (C_{f'} + y)$. From Lemma 5.2.7, there exists an $i$ such that $y \equiv y_i \bmod Y_{\sigma_y}$. Thus $C_f \cap (C_{f'} + y) = C_f \cap (C_{f'} + y_i)$.

**5.2.10.** We prove Lemmas 5.2.6, 5.2.7 and 5.2.8. To prove 5.2.6 (resp. 5.2.7, resp. 5.2.8), it is enough to show the existence of $\varepsilon$ (resp. $y_1, \cdots, y_k$, resp. $a$ and $b$) satisfying the conditions there for every $N \in \mathrm{Hom}(\mathcal{S}, \mathbb{Q}_{\geq 0})$ such that $\mathrm{Ker}\, N$ coincides with a prescribed face $\tau$ of $\mathcal{S}$. In the following, we fix such a $\tau$ and consider such $N$'s only.

Take a set of generators $N_0, \dots, N_n$ of $\{N \in \mathrm{Hom}(\mathcal{S}, \mathbb{Q}_{\geq 0}); N(\tau) = 0\}$. Then we have $\{N \in \mathrm{Hom}(\mathcal{S}, \mathbb{Q}_{\geq 0}); \mathrm{Ker}(N) = \tau\} = \{t_0 N_0 + \cdots + t_n N_n; t_i \in \mathbb{Q}, t_i > 0\}$. It is enough to show the existence of $\varepsilon$ (resp. $y_1, \cdots, y_k$, resp. $a$ and $b$) satisfying the conditions in Lemma 5.2.6 (resp. 5.2.7, resp. 5.2.8) for the $N = t_0 N_0 + \cdots + t_n N_n \in \mathrm{Hom}(\mathcal{S}, \mathbb{Q}_{\geq 0})$ with every $t_0, \dots, t_n \in \mathbb{Q}_{>0}$ such that $t_{h(0)} \geq t_{h(1)} \geq \cdots \geq t_{h(n)}$, where $h\colon \{0, \dots, n\} \longrightarrow \{0, \dots, n\}$ is a prescribed map. Hence we consider only $N = t_0 N_0 + \cdots + t_n N_n$ with $t_0 \geq \cdots \geq t_n$.

For $\alpha \in (0, 1]^n$, put

$$N_\alpha := N_0 + \alpha_1 N_1 + \alpha_1 \alpha_2 N_2 + \cdots + \alpha_1 \cdots \alpha_n N_n.$$

It is enough to show the existence of $\varepsilon$ (resp. $y_1, \cdots, y_k$, resp. $a$ and $b$) satisfying the conditions in 5.2.6 (resp. 5.2.7, resp. 5.2.8) for $tN_\alpha$ with every $\alpha \in ((0, 1] \cap \mathbb{Q})^n$ and $t \in \mathbb{Q}_{>0}$.

Since $[0, 1]^n$ is compact, it is enough to show the existence of $\varepsilon$ (resp. $y_1, \cdots, y_k$, resp. $a$ and $b$) for $tN_\alpha$ with every $\alpha \in ((0, 1] \cap \mathbb{Q})^n$ which is sufficiently near to $\beta$ and $t \in \mathbb{Q}_{>0}$, where $\beta$ is a prescribed element of $[0, 1]^n$.

Let $I = \{(\sigma, i); \sigma \not\subset \tau\}$. For $\alpha \in [0, 1]^n$, we define an $\mathbb{R}$-linear map

$$\varphi_\alpha\colon \mathbb{R} \otimes Y \longrightarrow \mathbb{R}^I$$

as follows: for any $(\sigma, i) \in I$, taking the minimal $k$ such that $N_k(\sigma) \neq 0$, writing

$$N_{\alpha,k} := N_k + \alpha_{k+1}N_{k+1} + \alpha_{k+1}\alpha_{k+2}N_{k+2} + \cdots + (\alpha_{k+1}\cdots\alpha_n)N_n,$$

we define that the $(\sigma, i)$-component of $\varphi_\alpha(y)$ is $\dfrac{N_{\alpha,k}(\langle x_{\sigma,i}, y\rangle)}{N_{\alpha,k}(s_\sigma)}$ $(y \in \mathbb{R} \otimes Y)$. As is easily seen, if $\alpha \in (0,1]^n$, then the $(\sigma, i)$-component of $\varphi_\alpha(y)$ coincides with $\dfrac{N_\alpha(\langle x_{\sigma,i}, y\rangle)}{N_\alpha(s_\sigma)}$. Note that $\varphi_\alpha$ is continuous with respect to $\alpha$, that is, for a fixed $\beta \in [0,1]^n$, when $\alpha \in [0,1]^n$ converges to $\beta$, $\varphi_\alpha$ converges to $\varphi_\beta$.

**5.2.11.** We will complete the proof of Lemma 5.2.8. Fix a $\beta \in [0,1]^n$. It is enough to show that, for some $a \ll 0$ and $b \gg 0$,

$$y \in W \cap (\mathbb{Q} \otimes Y) \Rightarrow (N_\alpha, N_\alpha\langle ?, y\rangle) \in C_{a,b}$$

for every $\alpha \in ((0,1] \cap \mathbb{Q})^n$ sufficiently near to $\beta$. But

$$\begin{aligned} &(N_\alpha, N_\alpha\langle ?, y\rangle) \in C_{a,b} \\ \Longleftrightarrow\ &\text{for any } \sigma, i, aN_\alpha(s_\sigma) \leq N_\alpha\langle x_{\sigma,i}, y\rangle \leq bN_\alpha(s_\sigma) \\ \Longleftrightarrow\ &\text{for any } (\sigma, i) \in I, a \leq \text{the } (\sigma, i)\text{-component of } \varphi_\alpha(y) \leq b, \end{aligned}$$

and we see that by the compactness of $W$, when $a \ll 0$ and $b \gg 0$, the last condition is satisfied for every $\alpha$ sufficiently near to $\beta$.

For the proof of Lemmas 5.2.6 and 5.2.7, we use the next lemma.

5.2.12. Lemma. *Let $\alpha \in [0,1]^n$. Then $\varphi_\alpha$ induces an injection*

$$\mathbb{R} \otimes (Y/Y_\tau) \longrightarrow \mathbb{R}^I.$$

Proof. For $k$, $-1 \leq k \leq n$, we define a face $\sigma_k$ of $\mathcal{S}$ by $\sigma_k = \bigcap_{0 \leq i \leq k} \operatorname{Ker}(N_i)$ $(\sigma_{-1} = \mathcal{S})$. We have $\sigma_n = \tau$. Let $y \in \mathbb{R} \otimes Y$ and assume that $\varphi_\alpha(y) = 0$. By an induction on $k$, we will show $y \in \mathbb{R} \otimes Y_{\sigma_k}$. (Then we get $y \in \mathbb{R} \otimes Y_\tau$ when $k = n$.) It is clear that $y \in \mathbb{R} \otimes Y_{\sigma_{-1}}$. Assuming that $y \in \mathbb{R} \otimes Y_{\sigma_{k-1}}$, we show that $y \in \mathbb{R} \otimes Y_{\sigma_k}$. If $N_k(\sigma_{k-1}) = 0$, then $\sigma_k = \sigma_{k-1}$ so that there is nothing to prove. Hence we may assume that $N_k(\sigma_{k-1}) \neq 0$. Since the $(\sigma_{k-1}, i)$-component of $\varphi_\alpha(y)$ is 0, we see $N_{\alpha,k}(\langle X_{\sigma_{k-1}}, y\rangle) = 0$. Hence, by the admissibility, $y \in \mathbb{R} \otimes Y_{\sigma_{k-1} \cap \mathrm{Ker}\,(N_{\alpha,k})}$. Since $\mathrm{Ker}\,(N_{\alpha,k}) \subset \mathrm{Ker}\,(N_k)$, $y \in \mathbb{R} \otimes Y_{\sigma_{k-1} \cap \mathrm{Ker}\,(N_k)} = \mathbb{R} \otimes Y_{\sigma_k}$. $\square$

**5.2.13.** We will complete the proof of Lemmas 5.2.6 and 5.2.7. Fix a $\beta \in [0,1]^n$. By Lemma 5.2.12, for a sufficiently small $\delta > 0$, the following holds: For every $\alpha \in [0,1]^n$ sufficiently near to $\beta$ and every $y \in Y$, if the absolute values of all components of $\varphi_\alpha(y) < \delta$, then $y \in Y_\tau$. Take such a $\delta$. Let $t \in \mathbb{Q}_{>0}$, $\alpha \in ((0,1] \cap \mathbb{Q})^n$, and assume that $C_{a,b} \cap (C_{a',b'} + y) \cap$ (the fiber of $tN_\alpha$) $\neq \varnothing$. Then for any $(\sigma, i) \in I$,

$$(*) \qquad a_{\sigma,i} - b'_{\sigma,i} \leq \text{the } (\sigma, i)\text{-component of } \varphi_\alpha(y) \leq b_{\sigma,i} - a'_{\sigma,i}.$$

We prove 5.2.6. Let $z \in Y$ and assume that $C_{a,b} \cap (C_{a',b'} + z) \cap$ (the fiber of $tN_\alpha$) $\neq \varnothing$. Then, for any $(\sigma, i) \in I$,

$$(**) \qquad a_{\sigma,i} - b'_{\sigma,i} \leq \text{the } (\sigma, i)\text{-component of } \varphi_\alpha(z) \leq b_{\sigma,i} - a'_{\sigma,i}.$$

By $(*)$ and $(**)$, we have, for any $(\sigma, i) \in I$,

$$|\varphi_\alpha(y) - \varphi_\alpha(z)| < 4\varepsilon.$$

Thus, taking $\varepsilon$ such that $4\varepsilon < \delta$, we have $y \equiv z \bmod Y_\tau$.

We show 5.2.7. By $(*)$, the $(\sigma, i)$-component of $\varphi_\alpha(y)$ is bounded for every such $y$ and $\alpha$ near $\beta$. Hence the set of such $y$ modulo $Y_\tau$ is finite.

The next lemma completes the proof of Theorem 5.2.1. The authors think that it is known, but do not know a suitable reference.

5.2.14. LEMMA. *Let a group $G$ act on a finite dimensional $\mathbb{Q}$-vector space $V$. Let $J$ be a finite set of finitely generated cones in $V$. Assume that the following (1) and (2) hold.*

*(1) For any $\sigma \in J$ and $g \in G$, $\sigma \cap g\sigma$ is a face of $\sigma$ (and a face of $g\sigma$ (cf. a remark in 5.2.3)), on which $g$ acts trivially.*

*(2) For any $\sigma, \tau \in J$, the set of cones $\{\sigma \cap g\tau; g \in G\}$ is finite.*

*Then there is a fan $\Sigma$ satisfying the following five conditions:*

*(a) The support of $\Sigma$ is equal to $\bigcup_{\sigma \in J, g \in G} g\sigma$;*

*(b) The fan $\Sigma$ is stable under the action of $G$;*

*(c) $\Sigma$ has only finitely many $G$-orbits;*

*(d) For any cone $\tau \in \Sigma$ and $g \in G$, the intersection $\tau \cap g\tau$ coincides with $\{t \in \tau; g(t) = t\}$.*

*(e) For each $\sigma \in J$, $\Sigma$ includes a finite subdivision of $\sigma$.*

For the sake of induction, we generalize the statement.

5.2.15. Definition. Let $V$ be a finite dimensional $\mathbb{Q}$-vector space, $\Sigma_1, \Sigma_2$ fans in $V$.

(1) We denote by $\Sigma_1 \sqcap \Sigma_2$ the fan $\{\sigma_1 \cap \sigma_2; \sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2\}$.

(2) If $\Sigma_1 \sqcap \Sigma_2$ is a subfan of $\Sigma_1$ (i.e., $\Sigma_1 \sqcap \Sigma_2 \subset \Sigma_1$), we say that $\Sigma_1$ *respects* $\Sigma_2$, and denote it by $\Sigma_1 \rightsquigarrow \Sigma_2$.

(3) If $\Sigma_1 \sqcap \Sigma_2$ is a subfan of $\Sigma_i$ for $i = 1, 2$, we say that $\Sigma_1$ and $\Sigma_2$ are *compatible*, and denote it by $\Sigma_1 \leftrightsquigarrow \Sigma_2$.

We will soon give equivalent definitions and some properties of these relations.

Now the statement we prove is the following.

5.2.16. Lemma. *Let $G$, $V$ be as in Lemma 5.2.14. Let $\{\Sigma_1, \dots, \Sigma_m\}$ be a finite set of finite fans in $V$.*

*Assume that the following (1) and (2) hold.*

*(1) For any $i$ and $g \in G$, $\Sigma_i$ and $g\Sigma_i$ are compatible, and $g$ acts on $|\Sigma_i| \cap |g\Sigma_i|$ trivially.*

*(2) For any $i, j$, the set of fans $\{\Sigma_i \sqcap g\Sigma_j; g \in G\}$ is finite.*

*Then there is a fan $\Sigma$ satisfying the following five conditions:*

*(a) The support of $\Sigma$ is equal to $\bigcup_{i,g\in G} g|\Sigma_i|$;*

*(b) The fan $\Sigma$ is stable under the action of $G$;*

*(c) $\Sigma$ has only finitely many $G$-orbits;*

*(d) For any cone $\tau \in \Sigma$ and $g \in G$, the intersection $\tau \cap g\tau$ coincides with $\{t \in \tau; g(t) = t\}$.*

*(e) For each $i$, $\Sigma$ includes a finite subdivision of $\Sigma_i$.*

To see this includes 5.2.14, notice that when fans $\Sigma_1$ and $\Sigma_2$ consist of all faces of cones $\sigma_1$ and $\sigma_2$ respectively, $\Sigma_1 \leftrightsquigarrow \Sigma_2$ if and only if $\sigma_1 \cap \sigma_2$ is a face of $\sigma_1$ and a face of $\sigma_2$.

5.2.17. Proposition. *Let $V$, $\Sigma_1$, $\Sigma_2$ be as in Definition 5.2.15.*

*(1) The following (i) and (ii) are equivalent.*

*(i)* $\Sigma_1 \rightsquigarrow \Sigma_2$.

*(ii) For any* $\tau \in \Sigma_2$, $\tau \cap |\Sigma_1|$ *is the union of a set of cones of* $\Sigma_1$.

*(2) The following (i)—(v) are equivalent.*

*(i)* $\Sigma_1 \leftrightsquigarrow \Sigma_2$.

*(ii)* $\Sigma_1 \rightsquigarrow \Sigma_2$ *and* $\Sigma_2 \rightsquigarrow \Sigma_1$.

*(iii)* $\Sigma_1 \cup \Sigma_2$ *is a fan.*

*(iv) For any* $\sigma_1 \in \Sigma_1$ *and* $\sigma_2 \in \Sigma_2$, $\sigma_1 \cap \sigma_2$ *is a face of* $\sigma_1$ *and a face of* $\sigma_2$.

*(v)* $\Sigma_1 \sqcap \Sigma_2 = \Sigma_1 \cap \Sigma_2$.

PROOF. Easy. □

5.2.18. PROPOSITION. *Let* $V$, $\Sigma_1$, $\Sigma_2$ *be as in Definition 5.2.15. Let* $\Sigma_3$ *be another fan in* $V$. *Then the following hold.*

*(1)* $\Sigma_1$ *is a subdivision of* $\Sigma_2$ *if and only if* $\Sigma_1 \rightsquigarrow \Sigma_2$ *and* $|\Sigma_1| = |\Sigma_2|$.

*(2) Assume that* $\Sigma_1 \rightsquigarrow \Sigma_2 \rightsquigarrow \Sigma_3$. *Assume further that* $|\Sigma_1| \cap |\Sigma_3| \subset |\Sigma_2|$ *(for example, that* $\Sigma_1$ *is a subdivision of* $\Sigma_2$ *or that* $\Sigma_2$ *is a subdivision of* $\Sigma_3$*) , then* $\Sigma_1 \rightsquigarrow \Sigma_3$.

*(3) Define* $\Sigma_1 \leqq \Sigma_2 \Leftrightarrow |\Sigma_1| \subset |\Sigma_2|$ *and* $\Sigma_1 \rightsquigarrow \Sigma_2$. *Then* $\leqq$ *is a partial order on the set of the fans in* $V$. *In general,* $\Sigma_1 \sqcap \Sigma_2 = \mathrm{Inf}\,(\Sigma_1, \Sigma_2)$ *with respect to this ordering. If* $\Sigma_1 \leftrightsquigarrow \Sigma_2$, $\Sigma_1 \cup \Sigma_2 = \mathrm{Sup}\,(\Sigma_1, \Sigma_2)$.

*(4) Let* $\Sigma_1, \dots, \Sigma_k$ $(k \geq 2)$ *be finite fans. Then there exists a finite subdivision* $\Sigma_1'$ *of* $\Sigma_1$ *which respects* $\Sigma_2, \dots, \Sigma_k$.

*(5) Assume that* $\Sigma_1 \rightsquigarrow \Sigma_2$ *and* $\Sigma_1 \sqcap \Sigma_2$ *is finite. Then there is a fan* $\widetilde{\Sigma}_1$ *supported by* $|\Sigma_1| \cup |\Sigma_2|$ *which includes* $\Sigma_1$ *and a locally finite subdivision of* $\Sigma_2$.

PROOF. It is easy to see (1)—(3).

(4) We may assume that $k = 2$. Take a complete finite fan $\widetilde{\Sigma_2'}$ including a subdivision of $\Sigma_2$ (cf. [28], the proof of Proposition 2.17). Let $\Sigma_1' := \Sigma_1 \sqcap \widetilde{\Sigma_2'}$. Then $\Sigma_1'$ is a finite subdivision of $\Sigma_1$. Since $\Sigma_1' \rightsquigarrow \widetilde{\Sigma_2'}$, $\Sigma_1' \rightsquigarrow \Sigma_2'$, where $\Sigma_2'$ is the subdivision of $\Sigma_2$ included by $\widetilde{\Sigma_2'}$. Hence $\Sigma_1' \rightsquigarrow \Sigma_2$.

(5) Consider the subfan $\Sigma_3 = \Sigma_1 \sqcap \Sigma_2$ of $\Sigma_1$. By the equivariant compactification theorem ([28] p.17), there is a complete finite fan $\widetilde{\Sigma_3} \supset \Sigma_3$. Then $\Sigma_2' := \widetilde{\Sigma_3} \sqcap \Sigma_2$ is a locally finite subdivision of $\Sigma_2$ and contains $\Sigma_3$. Since $\Sigma_3 \subset \Sigma_1 \cap \Sigma_2'$ and $|\Sigma_3| = |\Sigma_1| \cap |\Sigma_2|$, $\Sigma_1 \leftrightsquigarrow \Sigma_2'$ and $\widetilde{\Sigma}_1 := \Sigma_1 \cup \Sigma_2'$ satisfies the required condition. □

**5.2.19.** We prove Lemma 5.2.16.

First we prove that the condition (1) plus (2) is inherited by the subdivisions, that is, if $\Sigma'_1, \dots, \Sigma'_m$ are subdivisions of $\Sigma_1, \dots, \Sigma_m$ respectively, then $\Sigma'_1, \dots, \Sigma'_m$ satisfy the same conditions. We prove that the condition (1) is inherited first. For this, it is enough to show that (1) is equivalent to

(1)′ For any $i$, $g \in G$, $g$ acts on $|\Sigma_i| \cap g|\Sigma_j|$ trivially and for any $\sigma \in \Sigma_i$, $\{s \in \sigma; g(s) = s\}$ is a face of $\sigma$.

We prove this equivalence. We may work under the assumption

(0) For any $i$, $g \in G$, $g$ acts on $|\Sigma_i| \cap g|\Sigma_j|$ trivially.

Under it, we have $\sigma \cap g\sigma = \{s \in \sigma; g(s) = s\}$ for any $\sigma \in \Sigma_i$. Thus (1) implies (1)′. Further, under the same (0), for any $\sigma, \tau \in \Sigma_i$, $\sigma \cap g\tau = \sigma \cap \tau \cap g(\sigma \cap \tau)$. Since $\sigma > \sigma \cap \tau$, (1)′ implies (1). To prove (1) plus (2) is inherited, it is enough to show that under (0), (2) is inherited. Let $\Sigma'_1, \dots, \Sigma'_m$ be subdivisions of $\Sigma_1, \dots, \Sigma_m$ respectively. Assume that for some $i, j$, $g, h \in G$, $\Sigma_i \sqcap g\Sigma_j = \Sigma_i \sqcap h\Sigma_j$. It is enough to show that $\Sigma'_i \sqcap g\Sigma'_j = \Sigma'_i \sqcap h\Sigma'_j$. Let $\sigma'_i \in \Sigma'_i$, $\sigma'_j \in \Sigma'_j$. Since $g^{-1}h$ trivially acts on $|\Sigma_j| \cap g^{-1}h|\Sigma_j|$, $gh^{-1}$ acts on $g|\Sigma_j| \cap h|\Sigma_j|$ trivially. Hence it acts trivially also on $|\Sigma_i| \cap g|\Sigma_j| = |\Sigma_i| \cap g|\Sigma_j| \cap h|\Sigma_j|$. It implies $\sigma'_i \cap g\sigma'_j = \sigma'_i \cap h\sigma'_j$.

The proof of 5.2.16 goes with an induction on $m$. Before starting it, we note that the conditions (c) and (d) are deduced from the others automatically. (As for (d), use (1)′ as above.)

Now we start the induction. In case where $m = 1$, $g\Sigma_1$, $g \in G$, are pairwise compatible: for any $g, h \in G$, $g\Sigma_1 \leftrightsquigarrow h\Sigma_1$ because $\Sigma_1 \leftrightsquigarrow g^{-1}h\Sigma_1$. Hence $\Sigma := \bigcup_g g\Sigma_1$ satisfies the required condition.

Assume that the statement is valid for $m$, and prove it for $m + 1$. For each $i \le m$, let $\Sigma_i \sqcap g_1\Sigma_{m+1}, \dots, \Sigma_i \sqcap g_t\Sigma_{m+1}$ be all the members of the set $\{\Sigma_i \sqcap g\Sigma_{m+1}; g \in G\}$. By 5.2.18 (4), there is a finite subdivision $\Sigma'_i$ of $\Sigma_i$ which respects $g_k\Sigma_{m+1}$ for all $k$. Since $\Sigma_i \sqcap g\Sigma_{m+1} = \Sigma_i \sqcap g_k\Sigma_{m+1}$ implies $\Sigma'_i \sqcap g\Sigma_{m+1} = \Sigma'_i \sqcap g_k\Sigma_{m+1}$, $\Sigma'_i$ respects $g\Sigma_{m+1}$ for all $g \in G$ also. By what we have shown first, we can apply the inductive hypothesis to $\Sigma'_1, \dots, \Sigma'_m$, and let $\Sigma$ be the resulting fan which includes finite subdivisions $\Sigma''_i$ of all $\Sigma'_i$ $(1 \le i \le m)$.

This $\Sigma$ respects $\Sigma_{m+1}$ because for any $i, g \in G$, $\Sigma''_i \rightsquigarrow g^{-1}\Sigma_{m+1}$ implies $g\Sigma''_i \rightsquigarrow \Sigma_{m+1}$, and $\Sigma$ is the union of such $g\Sigma''_i$. Further $\Sigma \sqcap \Sigma_{m+1}$ is finite because $\{g\Sigma''_i \sqcap \Sigma_{m+1}; g \in G\}$ are finite for all $i$ by the inherited assumption (2). Hence, by 5.2.18 (5), there is a fan $\widetilde{\Sigma}$ supported by $|\Sigma| \cup |\Sigma_{m+1}|$ which includes $\Sigma$ and a finite subdivision $\Sigma'_{m+1}$ of $\Sigma_m$. The rest is to show that $\widetilde{\Sigma} \cup \{g\Sigma'_{m+1}; g \in G\}$ is indeed a fan. But $\widetilde{\Sigma} \leftrightsquigarrow g\Sigma'_{m+1}$ for any $g \in G$ because $g^{-1}\widetilde{\Sigma} = \widetilde{\Sigma} \leftrightsquigarrow \Sigma'_{m+1}$. Further $g\Sigma'_{m+1} \leftrightsquigarrow h\Sigma'_{m+1}$ $(g, h \in G)$ comes from the inherited assumption (1). This completes the proof of Lemma 5.2.16, and hence completes that of Theorem 5.2.1 also.

## 5.3. Proper models

**5.3.1.** Let the notation be as in 5.1.1. For a cone decomposition $\Sigma$ in $C$ (5.1.2), let

$$V(\Sigma) = \bigcup_{\sigma \in \Sigma} V(\sigma) \subset \mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)}.$$

It is represented by (and identified with) an fs log analytic space over $S$ which is log smooth over $S$, and $V(\sigma)$ for $\sigma \in \Sigma$ are open in $V(\Sigma)$.

Assume that $\Sigma$ is stable under the action of $Y$ as in 5.1.2. We define a sheaf $A^{(\Sigma)}$ on (fs/$S$) by

$$A^{(\Sigma)} = V(\Sigma)/Y \subset A.$$

We say $\Sigma$ is complete if it satisfies (1)–(3) of Theorem 5.2.1.

The aim of this subsection is to prove the following theorem.

5.3.2. THEOREM. *(1) $A^{(\Sigma)}$ is represented by an fs log analytic space over $S$ which is log smooth over $S$.*
*(2) If $\Sigma$ is complete, $A^{(\Sigma)}$ is proper over $S$.*

**5.3.3.** To prove this theorem, we define below a continuous map

$$\theta \colon V(\Sigma)_{\mathrm{val}} := \varprojlim{}_{\Sigma'} V(\Sigma') \to \mathbb{R} \otimes Y,$$

where $\Sigma'$ ranges over all subdivisions of $\Sigma$. We define the sheaf of rings on $V(\Sigma)_{\mathrm{val}}$ and the log structure of $V(\Sigma)_{\mathrm{val}}$ as the inductive limit of the inverse images of those of $V(\Sigma')$, respectively. The stalks of the log structure of $V(\Sigma)_{\mathrm{val}}$ are valuative. That is, for $p \in V(\Sigma)_{\mathrm{val}}$ and for two elements $f, g$ in the stalk $M_p$ at $p$ of the log structure of $V(\Sigma)_{\mathrm{val}}$, we have either $f|g$ or $g|f$.

For a finitely generated subcone $\Delta$ of $C$, let $V(\Delta)_{\mathrm{val}} = V(\{\text{face of}\, \Delta\})_{\mathrm{val}}$.

If the support of the cone $\Delta$ is contained in the support of a cone decomposition $\Sigma$, $V(\Delta)_{\mathrm{val}}$ is identified with an open set of $V(\Sigma)_{\mathrm{val}}$. In particular, $V(\Sigma)_{\mathrm{val}}$ for a complete $\Sigma$ is independent of $\Sigma$.

**5.3.4.** Let $p \in V(\Sigma)_{\mathrm{val}}$. Then for elements $f, g \in M_p^{\mathrm{gp}}$, we say $|f| < |g|$ (resp. $|f| \leq |g|$) at $p$ if $g|f$ at $p$ and the evaluation $M_p \to \mathbb{C}$ at $p$ sends $fg^{-1}$ to a complex number of absolute value $< 1$ (resp. $\leq 1$).

5.3.5. PROPOSITION. *(1) Let $p \in V(\Sigma)_{\mathrm{val}}$. Then there exists a unique element $z$ of $\mathbb{R} \otimes Y$ satisfying the following condition for any neighborhood $U$ of $z$ in $\mathbb{R} \otimes Y$.*

*For each $x \in X$, there exists $y, y' \in U \cap (\mathbb{Q} \otimes Y)$ such that $|\langle x, ny\rangle| < |x^n| < |\langle x, ny'\rangle|$ at $p$ for any integer $n \geq 1$ such that $ny, ny' \in Y$.*

*(2) Write the map $V(\Sigma)_{\mathrm{val}} \to \mathbb{R} \otimes Y; p \mapsto z$ defined by (1), by $\theta$. Then $\theta$ is continuous and is compatible with the actions of $Y$.*

*(3) If $\Sigma$ is complete, the induced map $\hat{\theta} : V(\Sigma)_{\mathrm{val}} \to S \times (\mathbb{R} \otimes Y)$ is proper and surjective.*

**5.3.6.** We deduce Theorem 5.3.2 from the above proposition 5.3.5.

In general, let $G$ be a topological group, let $A$ and $B$ be topological spaces endowed with continuous actions of $G$, and let $f : A \to B$ be a continuous map which is compatible with the actions of $G$. Then, if the action of $G$ on $B$ is proper and $A$ is Hausdorff, the action of $G$ on $A$ is proper. If the action of $G$ on $A$ is proper and $f$ is proper and surjective, then the action of $G$ on $B$ is proper. (See [4], Ch. 3, §4.)

We may assume that $S$ is Hausdorff. Since the action of $Y$ on $\mathbb{R} \otimes Y$ is proper, 5.3.5 (2) shows that the action of $Y$ on $V(\Sigma)_{\mathrm{val}}$ is proper. Since $V(\Sigma)_{\mathrm{val}} \to V(\Sigma)$ is proper and surjective, the action of $Y$ on $V(\Sigma)$ is proper. Hence the map from $V(\Sigma)$ to the quotient topological space by $Y$ is a local homeomorphism. This shows that the sheaf $A^{(\Sigma)} = V(\Sigma)/Y$ is represented by this quotient topological space endowed with $\mathcal{O}$ and $M$ transferred from $V(\Sigma)$ via this local homeomorphism.

Assume that $\Sigma$ is complete. Then the properness of $\hat{\theta}$ and the properness of $S \times (\mathbb{R} \otimes Y)/Y$ over $S$ proves the properness of $V(\Sigma)_{\mathrm{val}}/Y$ over $S$. Hence $A^{(\Sigma)}$ is proper over $S$.

**5.3.7.** Before the proof of 5.3.5, we describe the map $\theta$ in the case the log structure of $S$ is trivial. In this case, $V(\Sigma)_{\mathrm{val}} = V(\Delta) = S \times$

$\mathrm{Hom}\,(X,\mathbb{C}^\times)$, and the map $V(\Sigma)_{\mathrm{val}}\to\mathbb{R}\otimes Y$ is given by $\mathrm{Hom}\,(X,\mathbb{C}^\times)\xrightarrow{|\,|}\mathrm{Hom}\,(X,\mathbb{R}_{>0})\overset{\simeq}{\leftarrow}\mathbb{R}\otimes Y$, where the last isomorphism sends $a\otimes y$ for $a\in\mathbb{R}$ and $y\in Y$ to $X\to\mathbb{R}_{>0}; x\mapsto|\langle x,y\rangle|^a$.

**5.3.8.** We prove (1) of 5.3.5. Let

$$\{1\}=\tau_0\subset\tau_1\subset\cdots\subset\tau_m=M_p/\mathcal{O}_p^\times$$

be all the faces of $M_p/\mathcal{O}_p^\times$. For each $j$ $(1\le j\le m)$, there exists an injective homomorphism $v_j:\tau_j^{\mathrm{gp}}/\tau_{j-1}^{\mathrm{gp}}\to\mathbb{R}$ such that for $a\in\tau_j^{\mathrm{gp}}$, $a\in\tau_j\setminus\tau_{j-1}$ if and only if $v_j(a)>0$. Such a $v_j$ is unique up to the multiplication by a positive real constant.

Let $\sigma_j\subset\mathcal{S}$ be the inverse image of $\tau_j\subset M_p/\mathcal{O}_p^\times$, let $\mathrm{gr}_j(X)=X_{\sigma_j}/X_{\sigma_{j-1}}$ for $1\le j\le m$ and let $\mathrm{gr}_0(X)=X_{\sigma_0}$, and define $\mathrm{gr}_j(Y)$ $(0\le j\le m)$ similarly. Then since the pairing is non-degenerate, $\langle X_i,Y_j\rangle\subset\sigma^{\mathrm{gp}}_{\min(i,j)}$ for any $i,j$, and the pairing

$$v_j(\langle\ ,\ \rangle):\mathbb{R}\otimes\mathrm{gr}_j(X)\times\mathbb{R}\otimes\mathrm{gr}_j(Y)\to\mathbb{R}$$

for $1\le j\le m$ and the pairing

$$-\log(|\langle\ ,\ \rangle|):\mathbb{R}\otimes\mathrm{gr}_0(X)\times\mathbb{R}\otimes\mathrm{gr}_0(Y)\to\mathbb{R}$$

are non-degenerate. Hence we have an isomorphism

$$e_p:\mathbb{R}\otimes Y\overset{\simeq}{\to}\oplus_{0\le j\le m}\mathrm{Hom}\,(\mathrm{gr}_j(X),\mathbb{R})$$

whose $j$-th component is defined by $v_j(\langle\,,\,\rangle)$ for $1\le j\le m$ and by $-\log(|\langle\,,\,\rangle|(p))$ for $j=0$. On the other hand, since $V(\Sigma)\subset\mathcal{H}om(X,\mathbb{G}_{m,\log})^{(Y)}$ in $\mathcal{H}om(X,\mathbb{G}_{m,\log})$, the image of $X_{\sigma_j}$ in $M_p^{\mathrm{gp}}/\mathcal{O}_p^\times$ under $X\to M^{\mathrm{gp}}_{V(\Sigma)}\to M_p^{\mathrm{gp}}/\mathcal{O}_p^\times$ is contained in $\tau_j^{\mathrm{gp}}$ for any $0\le j\le m$. For $0\le j\le m$, let $\varphi_j:\mathrm{gr}_j(X)\to\mathbb{R}$ be the homomorphism defined as follows. If $1\le j\le m$, we define $\varphi_j$ to be the composite $\mathrm{gr}_j(X)\to\tau_j^{\mathrm{gp}}/\tau_{j-1}^{\mathrm{gp}}\xrightarrow{v_j}\mathbb{R}$. For $j=0$, the image of $X_{\{1\}}$ in $M_p^{\mathrm{gp}}$ is contained in $\mathcal{O}_p^\times$, and we define $\varphi_0$ to be the composition $\mathrm{gr}_0(X)=X_{\{1\}}\to\mathcal{O}_p^\times\xrightarrow{-\log(|\ |(p))}\mathbb{R}$. Let $z\in\mathbb{R}\otimes Y$ be the inverse image of $(\varphi_j)_{0\le j\le m}\in\oplus_{0\le j\le m}\mathrm{Hom}\,(\mathrm{gr}_j(X),\mathbb{R})$ under the isomorphism $e_p$.

It is easily seen that this $z$ is the unique element satisfying the condition in (1) of the proposition.

**5.3.9.** If $p$ belongs to $\mathcal{H}om(X, \mathbb{G}_{m,S}) \subset V(\Sigma)_{\mathrm{val}}$ and if the log structure of $S$ is trivial at the image $s$ of $p$ in $S$, $z = \theta(p)$ is the image of the element of $\mathrm{Hom}\,(X, \mathbb{C}^\times)$ defined by $p$ under $\mathrm{Hom}\,(X, \mathbb{C}^\times) \xrightarrow{\log(|\ \ |)} \mathrm{Hom}\,(X, \mathbb{R}) \xleftarrow{\cong} \mathbb{R} \otimes Y$, where the last arrow is induced by $\log(|\langle\ ,\ \rangle(s)|)$.

**5.3.10.** We prove (2) of 5.3.5. The compatibility with the actions of $Y$ is clear. We prove the continuity of $\theta$. Locally on $S$, $A$ comes from a log complex torus over a log smooth base $S'$ by a base change $S \to S'$ (3.10.3). Hence we may assume that $S$ is log smooth. Let $U$ be the part of $S$ on which the log structure is trivial, which is a dense open set of $S$. Furthermore, $\mathcal{H}om(X, \mathbb{G}_{m,S}) \times_S U$ is dense in $V(\Sigma)_{\mathrm{val}}$. It is sufficient to prove that if $p \in V(\Sigma)_{\mathrm{val}}$ and if $p_\lambda \in \mathcal{H}om(X, \mathbb{G}_{m,S}) \times_S U$ converges to $p$, then the image $z_\lambda$ of $p_\lambda$ in $\mathbb{R} \otimes Y$ converges to the image $z$ of $p$.

For $1 \le j \le m$, fix an element $t_j$ of $M_p$ whose image in $M_p/\mathcal{O}_p^\times$ is contained in $\tau_j$ but not contained in $\tau_{j-1}$. Then, when $p_\lambda$ is sufficiently near to $p$, $|t_j(p_\lambda)| < 1$ for $1 \le j \le m$. Hence we may assume that $|t_j(p_\lambda)| < 1$ for all $\lambda$ and $j$. Consider the $\mathbb{R}$-linear map

$$a_\lambda : \mathbb{R} \otimes Y \to \oplus_{0 \le j \le m} \mathrm{Hom}\,(X_{\sigma_j}, \mathbb{R})$$

defined as follows. For $y \in Y$ and $x \in X_{\sigma_j}$, the $j$-th component of $a_\lambda(y)$ sends $x$ to $\log(|\langle x, y\rangle(p_\lambda)|) v_j(t_j) \log(|t_j(p_\lambda)|)^{-1}$ if $1 \le j \le m$, and to $-\log(|\langle x, y\rangle(p_\lambda)|)$ if $j = 0$. On the other hand, let

$$a = f e_p : \mathbb{R} \otimes Y \to \oplus_{0 \le j \le m} \mathrm{Hom}\,(X_{\sigma_j}, \mathbb{R}),$$

where $e_p$ is the isomorphism $\mathbb{R} \otimes Y \to \oplus_{0 \le j \le m} \mathrm{Hom}\,(\mathrm{gr}_j(X), \mathbb{R})$ constructed in 5.3.8, and $f$ is the canonical injection $\oplus_{0 \le j \le m} \mathrm{Hom}\,(\mathrm{gr}_j(X), \mathbb{R}) \to \oplus_{0 \le j \le m} \mathrm{Hom}\,(X_{\sigma_j}, \mathbb{R})$. That is, for $y \in Y$, the $j$-th component of $a(y)$ sends $x \in X_{\sigma_j}$ to $v_j(\langle x, y\rangle)$ if $1 \le j \le m$, and to $-\log(|\langle x, y\rangle|)$ if $j = 0$. As is easily seen,

(i) $a_\lambda$ converges to $a$.

We have also

(ii) $a_\lambda(z_\lambda)$ converges to $a(z)$.

In fact, for $x \in X_{\sigma_j}$, by 5.3.9, the $j$-th component of $a_\lambda(z_\lambda)$ sends $x$ to $\log(|x(p_\lambda)|)v_j(t_j)\log(|t_j(p_\lambda)|)^{-1}$ if $1 \le j \le m$ (resp. to $-\log(|x(p_\lambda)|)$ if $j = 0$) which converges to $v_j(x)$ (resp. $-\log(|x(p)|)$). But the $j$-th component of $a(z)$ sends $x$ also to $v_j(x)$ (resp. $-\log(|x(p)|)$).

Take a splitting $s : \oplus_{0\le j\le m}\mathrm{Hom}\,(X_{\sigma_j}, \mathbb{R}) \to \mathbb{R}\otimes Y$ of the injection $a$. Then by (i), $sa_\lambda : \mathbb{R}\otimes Y \to \mathbb{R}\otimes Y$ converges to $sa = 1$, and hence $sa_\lambda$ is invertible if $p_\lambda$ is sufficiently near to $p$. Thus we have :

(i$'$) The inverse $(sa_\lambda)^{-1}$ of $sa_\lambda$ converges to the identity map $1 : \mathbb{R}\otimes Y \to \mathbb{R}\otimes Y$.

On the other hand, by (ii),

(ii$'$) $sa_\lambda(z_\lambda)$ converges to $sa(z) = z$.

By applying (i$'$) to (ii$'$), we obtain that $z_\lambda = (sa_\lambda)^{-1}(sa_\lambda(z_\lambda))$ converges to $z$.

We prove (3) of the proposition after some preliminaries.

5.3.11. LEMMA. *Assume that $\Sigma$ is complete. Let $B$ be a bounded set in $\mathbb{R}\otimes Y$. Then the inverse image of $B$ in $V(\Sigma)_{\mathrm{val}}$ is contained in $V(\Delta)_{\mathrm{val}}$ if $\Delta$ is a sufficiently big finitely generated subcone of $C$.*

PROOF. For each face $\sigma$ of $\mathcal{S}$, fix a $\mathbb{Z}$-basis $(x_{\sigma,j})_{1\le j\le r(\sigma)}$ of $X_\sigma$. For each $\sigma$, take an element $\mu_\sigma$ of the interior of $\sigma$. We may assume that there is a finite subset $I$ of $Y$ such that $B$ is the smallest convex set in $\mathbb{R}\otimes Y$ containing $I$. For any $p \in V(\Sigma)_{\mathrm{val}}$, let $V_p$ be the valuative submonoid of $\mathcal{S}^{\mathrm{gp}}_{\mathbb{Q}} \times X_{\mathbb{Q}}$ defined as the inverse image of $(M_p/\mathcal{O}_p^\times)_{\mathbb{Q}}$. Since $p \in V(\Sigma)_{\mathrm{val}}$ is in $V(\Delta)_{\mathrm{val}}$ if and only if $V_p \supset \Delta^\vee$, it is sufficient to find $\Delta$ such that $\bigcap_{\theta(p)=y} V_p \supset \Delta^\vee$ for any $y \in B$. In the following, we regard $\langle\,,\,\rangle$ as the pairing $X_{\mathbb{R}} \times Y_{\mathbb{R}} \longrightarrow \mathcal{S}^{\mathrm{gp}}_{\mathbb{R}}$ by scalar extension. Fix an element $y \in B$. By the definition of $\theta$, the set $\{V_p \,|\, p \in V(\Sigma)_{\mathrm{val}}, \theta(p) = y\}$ coincides with the set of valuative submonoids of $\mathcal{S}^{\mathrm{gp}}_{\mathbb{Q}} \times X_{\mathbb{Q}}$ containing $C^\vee = \mathcal{S} \times X_0$ the boundary (= the set of the points which are neither interior nor exterior) of whose closure in $\mathcal{S}^{\mathrm{gp}}_{\mathbb{R}} \times X_{\mathbb{R}}$ contains $v_{\sigma,j} := (\langle x_{\sigma,j}, y\rangle, x_{\sigma,j}^{-1})$ for all $\sigma$ and $j$. (Here $X_0 := X_{\{1\}}$.) This implies $W_y \supset \bigcap_{\theta(p)=y} V_p = \mathring{W}_y \cup (\mathcal{S} \times X_0)_{\mathbb{Q}}$, where $W_y$ is the $\mathbb{Q}$-cone generated by $\mathcal{S}$ and $v_{\sigma,j}^{\pm 1}$ for all $\sigma$ and $j$, and $\mathring{W}_y$ denotes its interior. Hence it is enough to show that for a sufficiently big $a > 0$, the dual cone $C(a)^\vee$ of the cone

$$\Delta := C(a)$$

in 3.4.9 is contained in $\overset{\circ}{W}_y \cup (\mathcal{S} \times X_0)_\mathbb{Q}$ for any $y \in B$. First, take $a > 0$ such that for any $\sigma$, $j$, and $y \in I$, the $a$-th power $\mu_\sigma^a$ of $\mu_\sigma$ belongs to $\langle x_{\sigma,j}, y\rangle\sigma$. Then any $(N, \ell) \in W_y^\vee$ with $y \in B$ satisfies $\ell(x_{\sigma,j}) = N(\langle x_{\sigma,j}, y\rangle) \leq N(\mu_\sigma)^a$ for all $\sigma$ and $j$. Thus $W_y \supset C(a)^\vee$. To prove furthermore $\overset{\circ}{W}_y \cup (\mathcal{S} \times X_0)_\mathbb{Q} \supset C(a)^\vee$, it is enough to show that $(\langle x, y\rangle, x^{-1})^\vee \not\supset C(a)$ for any $x \in X - X_0$ and $y \in B$. For this, fix a finite set of sharp fs cones $\{\tau\}$ of $X/X_0$ such that $\bigcup \tau = X/X_0$ and fix an $N \in \mathcal{S}^\vee$ whose kernel is trivial. Then for each $\tau$, we can take $\ell_\tau \in \operatorname{Hom}(X/X_0, \mathbb{Z})$ such that $\ell_\tau(x) > N(\langle x, y\rangle)$ for any $y \in B$ and $x \in X_\mathbb{Q}$ lying over $\tau_\mathbb{Q} - \{1\}$. (Take a homomorphism $X/X_0 \to \mathbb{Z}$ which sends $\tau - \{1\}$ into $\mathbb{N} - \{0\}$ and multiply it with a sufficiently big integer to get an $\ell_\tau$.) This means $(N, \ell_\tau) \notin (\langle x, y\rangle, x^{-1})^\vee$. Therefore, $C(a)$ containing $(N, \ell_\tau)$ for all $\tau$ has the desired property. $\square$

**5.3.12.** For a finitely generated subcone $\Delta$ of $C$, a set of generators $(t_1, \cdots, t_n)$ of $\Delta^\vee$, and a real number $a > 0$, let

$$K_a(\Delta) = \{p \in V(\Delta); |t_j| \leq a \text{ at } p \quad (1 \leq j \leq n)\}.$$

Let $K_a(\Delta)_{\text{val}}$ be the inverse image of $K_a(\Delta)$ in $V(\Delta)_{\text{val}}$.

5.3.13. LEMMA. *$K_a(\Delta)$ and $K_a(\Delta)_{\text{val}}$ are proper over $S$.*

PROOF. We may assume that there exists a chart $\mathcal{S} \to M_S$ which lifts $\mathcal{S} \to M_S/\mathcal{O}_S^\times$. For $K_a(\Delta)$, this can be deduced from the fact that the set of all homomorphisms from $\Delta^\vee$ to the multiplicative monoid $\mathbb{C}$ which send every $t_j$ into $\{z \in \mathbb{C}; |z| \leq a\}$ is compact. In fact, $K_a(\Delta) \to S$ is the base change of the proper map from the above subspace of $\operatorname{Hom}(\Delta^\vee, \mathbb{C})$ to $\operatorname{Hom}(\mathcal{S}, \mathbb{C})$ with respect to $S \to \operatorname{Hom}(\mathcal{S}, \mathbb{C})$. For $K_a(\Delta)_{\text{val}}$, use also that $V(\Delta)_{\text{val}} \to V(\Delta)$ is proper. $\square$

**5.3.14.** Now we prove (3) of the proposition. Assume that $\Sigma$ is complete. We may assume that there exists a chart $\mathcal{S} \to M_S$ which lifts $\mathcal{S} \to M_S/\mathcal{O}_S^\times$.

To prove that $\hat{\theta}$ is proper, it is sufficient to prove that the inverse image of any compact subset $B$ of $S \times (\mathbb{R} \otimes Y)$ in $V(\Sigma)_{\mathrm{val}}$ is compact. Let $B_1$ and $B_2$ be the images of $B$ by the projections to $S$ and $\mathbb{R} \otimes Y$ respectively. Let $I \subset Y$ be a finite set satisfying the same condition as in the proof of 5.3.11 for $B_2$. Then 5.3.11 gives a $\Delta$ such that $\theta^{-1}(B_2)$ is contained in $V(\Delta)_{\mathrm{val}}$. Take a set of generators $(t_1, \cdots, t_n)$ of $\Delta^\vee$. Let the notation be as in the proof of 5.3.11. Since $W_y \supset \bigcap_{\theta(p)=y} V_p \supset \Delta^\vee$ for each $y \in B_2$, there are $s_{y,j} \in \mathcal{S}_{\mathbb{Q}_{\geq 0}}$ and $x_j \in X_{\mathbb{Q}}$ such that $t_j = s_{y,j}\langle x_j, y\rangle x_j^{-1}$ for each $j$. Since $B_1$ and $B_2$ are compact, we may assume $\max_{B_1,y,j} |s_{y,j}| < a$ for some positive $a$. On the other hand, $\langle x_j, y\rangle x_j^{-1}$ has the absolute value 1 on $\theta^{-1}(y)$ by definition of $\theta$. This shows that $\hat{\theta}^{-1}(B)$ is a closed subset of $K_a(\Delta)_{\mathrm{val}}$. By 5.3.13, $\hat{\theta}^{-1}(B)$ is proper over $S$ and its image in $S$ is contained in $B_1$. Hence $\hat{\theta}^{-1}(B)$ is compact.

It remains to prove that $\hat{\theta}$ is surjective. For an element $(s, \frac{y}{n}) \in S \times (\mathbb{Q} \otimes Y)$, $n \geq 1$, we will show that it is in the image of $V(\Sigma)_{\mathrm{val}}$. Then the image of $\hat{\theta}$ is dense. Since $\hat{\theta}$ is proper, this shows that $\hat{\theta}$ is surjective. Let $\Delta$ be the finitely generated subcone of $C$ consisting of all elements $(N, \ell)$ of $C$ such that $N(\langle x, y\rangle) = n\ell(x)$ for any $x \in X$. Then $\Delta^\vee = \mathcal{S} \cdot \langle v_x^{\pm 1} \,|\, x \in X\rangle$, where $v_x := (\langle x, y\rangle, x^{-n})$ ($\Delta^\vee$ is the direct product of $\mathcal{S}$ and $\langle v_x^{\pm 1} \,|\, x \in X\rangle$). Hence there is a point of $V(\Delta)$ lying over $s$ at which every $v_x$ has the value 1. Any point of $V(\Delta)_{\mathrm{val}}$ lying over this point maps to $(s, \frac{y}{n})$ by $\hat{\theta}$.

## 5.4. Projective models

Let $S$ be an fs log analytic space. The purpose of this subsection is to prove:

5.4.1. Theorem. *Let $A$ be a locally polarizable log abelian variety over $S$. Then $A$ has a projective model locally on $S$.*

This is essentially known and can be proved algebraically as in [22], [10]. We give here an analytic proof by using theta functions as in [2], [26].

**5.4.2.** Review of the classical theory.

Let $\langle\, , \rangle : X \times Y \to \mathbb{C}^\times$ be a pairing such that $(\mathbb{R} \otimes X) \times (\mathbb{R} \otimes Y) \to \mathbb{R}; (1 \otimes x, 1 \otimes y) \mapsto \log(|\langle x, y\rangle|)$ is a perfect pairing of finite dimensional $\mathbb{R}$-vector spaces, and let $A = T/Y$ with $T = \mathrm{Hom}\,(X, \mathbb{C}^\times)$ being the associated complex torus. We denote the group laws of $X$ and $Y$ multiplicatively.

Let $Q$ be the set of all maps $q : Y \to \mathbb{C}^\times \times X$ satisfying the following condition: If we denote $q$ as $(a, p)$, where $a : Y \to \mathbb{C}^\times$ and $p : Y \to X$, then $p$ is a homomorphism and

$$a(yz) = a(y)a(z)\langle p(y), z\rangle \quad \text{for all } y, z \in Y.$$

Note that for a homomorphism $p : Y \to X$ such that

$$\langle p(y), z\rangle = \langle p(z), y\rangle \quad \text{for all } y, z \in Y,$$

if we denote the map $Y \to \mathbb{C}^\times; y \mapsto \langle p(y), y\rangle$ by $a$, then $(a, p^2) \in Q$.

For $q \in Q$, we define a line bundle $L_q$ on $A$ as follows. For an open set $U$ of $A$, let $\tilde{U}$ be the inverse image of $U$ in $T$, and define

$$L_q(U) = \{f \in \mathcal{O}(\tilde{U}) \,|\, y^*(f) = q(y)^{-1}f \quad \text{for all } y \in Y\}.$$

Here $y^*$ denotes the pull back by the action of $y \in Y$ on $T$, and we regard $q(y)^{-1}$ as a function on $T$ by identifying an element of $X$ with the corresponding function on $T$.

Assume that $p$ is a polarization. Then global sections of $L_q$ have Fourier expansions.

For $x \in X$, define a function $\theta_{q,x}$ on $T$ called a theta function by

$$\theta_{q,x} = \sum_{y \in Y} q(y)\langle x, y\rangle x.$$

Then $\theta_{q,x} \in L_q(A)$. Furthermore, if $R \subset X$ denotes a representative of $\mathrm{Coker}\,(p : Y \to X)$, $(\theta_{q,x})_{x \in R}$ is a basis of the $\mathbb{C}$-vector space $L_q(A)$.

**5.4.3.** Log version.

Let $\langle\ ,\ \rangle : X \times Y \to \mathbb{G}_{m,\log,S}$ be a non-degenerate pairing and let $A = \Psi/Y$ with $\Psi = \mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)}$ being the associated log complex torus.

Let $Q$ be the set of all maps $q : Y \to \mathbb{G}_{m,\log,S} \times X$ satisfying the following condition: If we denote $q$ as $(a, p)$, where $a : Y \to \mathbb{G}_{m,\log,S}$ and $p : Y \to X$, then $p$ is a homomorphism, $a(y) \in M_S$ for all but finitely many $y \in Y$ and

$$(*) \qquad a(yz) = a(y)a(z)\langle p(y), z\rangle \quad \text{for all } y, z \in Y.$$

Note that for a homomorphism $p : Y \to X$ such that

$$\langle p(y), z\rangle = \langle p(z), y\rangle \quad \text{for all } y, z \in Y,$$

if we denote the map $Y \to \mathbb{G}_{m,\log,S}; y \mapsto \langle p(y), y\rangle$ by $a$, then $(a, p^2) \in Q$.

Note that $\langle\, ,\, \rangle$ modulo $\mathbb{G}_m$ lifts locally on $S$ to an $\mathcal{S}$-admissible pairing into $\mathcal{S}^{\mathrm{gp}}$ with an fs chart $\mathcal{S} \to M_S$. We assume that such a lift is given. Define the cone $C$ as in 3.4.

Let $q = (a, p) \in Q$, and assume that $p$ satisfies the conditions in 1.2.5. Note that $a$ modulo $\mathbb{G}_m$ lifts locally on $S$ to a map $Y \to \mathcal{S}^{\mathrm{gp}}$ satisfying $(*)$ in $\mathcal{S}^{\mathrm{gp}}$. We assume that such a lift is given. We assume further that $a(y) \in \mathcal{S}$ for all but finitely many $y \in Y$. We define a cone decomposition $\Sigma = \Sigma_q$ of $C$, which is complete, and a line bundle $L_q$ on $A^{(\Sigma)} = V(\Sigma)/Y$ as follows. These are also essentially known.

First we review how $q$ determines the cone decomposition $\Sigma = \Sigma_q$. For $y \in Y$, let $\sigma_y$ be the set of all elements of $\mathrm{Hom}\,(\mathcal{S}, \mathbb{N}) \oplus \mathrm{Hom}\,(X, \mathbb{Z}) \subset \mathrm{Hom}\,(\mathcal{S}^{\mathrm{gp}} \oplus X, \mathbb{Z})$ which send the images of $q(z)q(y)^{-1}$ in $\mathcal{S}^{\mathrm{gp}} \oplus X$ to non-negative integers for all $z \in Y$. Then $\sigma_y$ is a finitely generated subcone of $C$ (as in [22] 2.4, this is reduced to the statement that for any finitely generated subcone $\tau$ of $C$, $q(z)q(y)^{-1}$ belongs to $(\mathcal{S} - \mathcal{S}^{\times})\tau^{\vee}$ for almost all $z \in Y$). Further, for $m \geq 1$ and $y_1, \cdots, y_m \in Y$, the cone $\sigma_{y_1} \cap \cdots \cap \sigma_{y_m}$ is a face of $\sigma_{y_j}$ for $1 \leq j \leq m$. Let

$$\Sigma_q = \{\sigma_{y_1} \cap \cdots \cap \sigma_{y_m} \,|\, m \geq 1, y_1, \cdots, y_m \in Y\}.$$

Then $\Sigma_q$ is complete.

We show that $\Sigma_q$ is stable under the action of $Y$. Note

$$q(yz) = q(y) \cdot y^*(q(z)) \quad \text{for all } y, z \in Y.$$

This shows $V(\sigma_{yz}) = y^{-1}V(\sigma_z)$. (Here we use only the lifted pairing $X \times Y \to \mathcal{S}^{\mathrm{gp}}$.)

We define $L_q$. First, define a line bundle $\tilde{L}_q$ on $V(\Sigma)$ as follows. Let $[q]$ be the unique global section of $M^{\mathrm{gp}}_{V(\Sigma)}/\mathcal{O}^{\times}_{V(\Sigma)}$ which coincides on $V(\sigma_y)$ $(y \in Y)$ with the image of $q(y)$ in $M^{\mathrm{gp}}_{V(\sigma_y)}/\mathcal{O}^{\times}_{V(\sigma_y)}$. Then $\tilde{L}_q$ is defined to be the line bundle corresponding to the $\mathbb{G}_m$-torsor $[q]$ on $V(\Sigma)$.

For an open set $U$ of $A^{(\Sigma)}$, let $\tilde{U}$ be the inverse image of $U$ in $V(\Sigma)$, and define

$$L_q(U) = \{f \in \tilde{L}_q(\tilde{U}) \,|\, y^*(fq(z)^{-1}) = fq(yz)^{-1} \,\text{in}\, \mathcal{O}_{V(\sigma_{yz})} \,\text{for all}\ y, z \in Y\}.$$

If $S$ is log smooth, by identifying $M_{V(\Sigma)}$ as a subsheaf of $\mathcal{O}_{V(\Sigma)}$, we can write also

$$L_q(U) = \{f \in \tilde{L}_q(\tilde{U}) \,|\, y^*(f) = q(y)^{-1}f \quad \text{for all } y \in Y\}.$$

Here the equality on the right hand side is taken in the space of meromorphic functions on $V(\Sigma)$.

Note that for $n \geq 1$, $\Sigma_{q^n} = \Sigma_q$ and $L_{q^n} = L_q^{\otimes n}$. Here for $q = (a, p)$, $q^n$ denotes $(a^n, p^n)$.

5.4.4. PROPOSITION. *Let $p'$ be a polarization and let $q = (a, p) \in Q$, where $a(y) = \langle p'(y), y \rangle$, $p(y) = p'(y)^2$. Then $L_q$ is ample.*

Theorem follows from this. We will prove this proposition by constructing sufficiently many global sections, which we call the theta functions.

**5.4.5.** Proof of Proposition.

We may assume that $S$ is log smooth by 3.10.3. By replacing $q$ by a power of $q$, we may assume that $q = (q')^n$ for some $q' \in Q$ and $n \geq 3$. (This assumption is used below to have the very ampleness of a line bundle $L_{qp(z)|_{Y'}}$.)

Let $s \in S$. We consider around $s$. It is sufficient to prove that for some $N \geq 1$, the ratios of global sections of $L_{q^N} = L_q^{\otimes N}$ give a finite morphism from $A^{(\Sigma)}$ to a projective space over $S$.

Let $X' = X_{\{1\}}$ and $Y' = Y_{\{1\}}$ be the kernels of $X \times Y \to M_{S,s}^{\mathrm{gp}}/\mathcal{O}_{S,s}^{\times}$.

Define $B = \mathcal{H}om(X', \mathbb{G}_m)/Y'$. This is an abelian variety over $S$. We have a commutative diagram

$$\begin{array}{ccc} V(\Sigma) & \to & \mathcal{H}om(X', \mathbb{G}_m) \\ \downarrow & & \downarrow \\ V(\Sigma)/Y' & \to & \mathcal{H}om(X', \mathbb{G}_m)/Y' \quad = B \\ \downarrow & & \\ V(\Sigma)/Y \quad = A^{(\Sigma)}. & & \end{array}$$

The map $p : Y \to X$ induces $Y' \to X'$.

For $x' \in X'$ and for a homomorphism $\chi : Y \to \mathbb{C}^{\times}$ which kills $Y'$ and $Y^N$ for some integer $N \geq 1$, consider the theta function

$$\theta_{q\chi, x'} = \sum_{y \in Y} q(y)\chi(y)\langle x', y \rangle x' \in L_{q\chi}(A^{(\Sigma)}).$$

(This is well-defined because $S$ is log smooth.) Here $q\chi \in Q$ is defined by $(a\chi, p)$ (write $q = (a, p)$), where $(a\chi)(y) = a(y)\chi(y)$. (Note $\Sigma_{q\chi} = \Sigma_q$.) Take a representative $Z \subset Y$ of $Y/Y'$. For each $z \in Z$, let $qp(z)|_{Y'}\colon Y' \to \mathbb{C}^\times \times X'$ be the map $y' \mapsto q(y')\langle p(z), y'\rangle$. By the classical theory 5.4.2, we have the corresponding line bundle $L_{qp(z)|_{Y'}}$ on $B$. For $x' \in X'$, consider

$$\theta_{qp(z)|_{Y'},x'} = \sum_{y' \in Y'} q(y')\langle p(z), y'\rangle\langle x', y'\rangle x' \in L_{qp(z)|_{Y'}}(B).$$

Here note that $a(y') = \langle p'(y'), y'\rangle \in \mathcal{O}_S^\times$. Let $\hat{q}(z) = q(z)\chi(z)\langle x', z\rangle$. Then we have

$$\theta_{q\chi,x'} = \sum_{z \in Z} \hat{q}(z)\theta_{qp(z)|_{Y'},x'}.$$

Here we explain roughly how the rest goes. By the classical theory, $L_{qp(z)|_{Y'}}$ is very ample. Hence when $x' \in X'$ ranges, the ratios of $\theta_{qp(z)|_{Y'},x'}$ for a fixed $z$ separate points of $B$ in each fiber over $S$. On the other hand, roughly speaking, when $z \in Z$ ranges, the ratios of $\hat{q}(z)$ for a fixed $x'$ give finite-to-one maps from the fibers over $B$ to projective spaces. From these, we deduce that, roughly speaking, when both $x' \in X'$ and $z \in Z$ range, the ratios of $\hat{q}(z)\theta_{qp(z)|_{Y'},x'}$ give finite-to-one maps from the fibers of $A^{(\Sigma)}$ over $S$ to a projective space. Finally, varying $\chi$, we will see that when $x' \in X'$ ranges, the ratios of $\theta_{q\chi,x'}$ give finite-to-one maps from the fibers of $A^{(\Sigma)}$ over $S$ to a projective space.

Now we will explain the details. First note that we have $V(\sigma_{zy'}) = V(\sigma_z)$ and $q(zy') \equiv q(z) \equiv \hat{q}(z) \bmod \mathcal{O}_{V(\Sigma)}^\times$ for all $z \in Z$ and $y' \in Y'$. For each $z \in Z$, by the definition of $\sigma_z$, the dual cone of $\sigma_z$ is generated over $\mathbb{Q}_{\geq 0}$ by $(q(w)q(z)^{-1})_{w \in Z}$, and hence the family of functions $(\hat{q}(w)\hat{q}(z)^{-1})_{w \in Z}$ gives a finite-to-one map from each fiber of $V(\sigma_z) \to \mathrm{Hom}\,(X', \mathbb{G}_m)$ to a projective space over $S$. Furthermore, when $w$ ranges and when $z$ is fixed, almost all $\hat{q}(w)\hat{q}(z)^{-1}$ $(w \in Z)$ are zero as $\mathbb{C}$-valued functions on the fibers of $V(\sigma_z)$ over $S$. Hence, on each fiber of $V(\sigma_z) \to \mathrm{Hom}\,(X', \mathbb{G}_m)$, for any $\chi$, the function $\theta_{q\chi,x'}$ is a finite linear combination of $\hat{q}(z)$ ($x'$ is fixed). Therefore, if we take a sufficiently big $N \geq 1$, each $\hat{q}(z)$ is written as a linear combination of the $\theta_{q\chi,x'}$ for varying $\chi$ for this fixed $N$. Hence if we take $N \geq 1$ sufficiently large, then the ratios of $\theta_{q\chi,x'}$ for varying $\chi$ with fixed $N$ and fixed $x'$ give sufficiently many ratios of $\hat{q}(w)$, $w \in Z$, to give a finite-to-one map from each fiber of $V(\sigma_z) \to B$ to a projective space over $S$.

On the other hand, as we said before, when $x' \in X'$ ranges, the ratios of $\theta_{qp(z)|_{Y'},x'}$ for a fixed $z$ separate points of $B$ in each fiber over $S$.

From these, the $\theta_{q\chi,x'}$ for varying $\chi$ give sufficiently many $\hat{q}(w)\theta_{qp(w)|_{Y'},x'}$, $w \in Z$, for the ratios of these sections for varying $x' \in X'$ to give a finite-to-one map from each fiber of $V(\sigma_z)$ over $S$ to a projective space over $S$. Further, when we replace the $\theta_{q\chi,x'}$ with their $N$-th powers, their ratios for varying $x' \in X'$ still give a finite-to-one map from each fiber of $V(\sigma_z)$ over $S$ to a projective space over $S$.

Hence if $N \geq 1$ is sufficiently large, the global sections $\theta_{q\chi,x'}^N$ of $L_{q^N} = L_q^{\otimes N}$ for varying $\chi$ with this fixed $N$ and for varying $x' \in X'$ give a finite-to-one map from $A^{(\Sigma)}$ to a projective space over $S$.

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