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# *Strong Stability of the Homogeneous Levi Bundle*

By Indranil Biswas

**Abstract.** Let $G$ be a connected semisimple linear algebraic group defined over an algebraically closed field. Let $P \subset G$ be a parabolic subgroup without any simple factor, and let $L(P)$ denote the Levi quotient of $P$. In this continuation of [Bi], we prove that the principal $L(P)$–bundle $(G \times L(P))/P$ over the homogeneous space $G/P$ is stable with respect to any polarization on $G/P$. When the characteristic of the base field is positive, this principal $L(P)$–bundle is shown to be strongly stable with respect to any polarization on $G/P$.

## 1. Introduction

We begin by recalling the main result of [Bi].

Fix a connected semisimple linear algebraic group $G$ defined over an algebraically closed field $k$. Let $P \subset G$ be a reduced parabolic subgroup without any simple factor. This means that the image of $P$ in any simple quotient of $G$ is a reduced proper parabolic subgroup. The principal $P$–bundle over the homogeneous space $G/P$ defined by the quotient morphism $G \longrightarrow G/P$ will be denoted by $E_P$. Let $V$ be a finite dimensional irreducible left $P$–module. Let $E_P(V) := (G \times V)/P$ be the vector bundle over $G/P$ associated to the principal $P$–bundle $E_P$ for the $P$–module $V$. The main result of [Bi] says that $E_P(V)$ is a stable vector bundle with respect to any polarization on $G/P$ (see [Bi, page 135, Theorem 2.1]).

We note that in [Um], Umemura proved that the vector bundle $E_P(V)$ is stable with respect to any polarization on $G/P$ under the assumption that the characteristic of the base field $k$ is zero (see [Um, page 136, Theorem 2.4]). He asked the question in the introduction of [Um] whether $E_P(V)$ is also stable when the characteristic of $k$ is positive. Our earlier paper [Bi] originated from this question of Umemura.

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Let $L(P)$ denote the Levi quotient of $P$. So $L(P)$ is the quotient of $P$ by the unipotent radical of $P$, and $L(P)$ is also the maximal reductive quotient of $P$ (see [Hu, page 125]).

Let

$$E_P(L(P)) := (G \times L(P))/P$$

be the principal $L(P)$–bundle over $G/P$ obtained by extending the structure group of the above defined principal $P$–bundle $E_P$ using the quotient homomorphism $P \longrightarrow L(P)$.

Our aim here is to prove the following theorem (see Theorem 4.1):

Theorem 1.1. *The principal $L(P)$–bundle $E_P(L(P))$ over $G/P$ is stable with respect to any polarization on $G/P$. When the characteristic of the base field $k$ is positive, the principal $L(P)$–bundle $E_P(L(P))$ is strongly stable with respect to any polarization on $G/P$.*

When the characteristic of $k$ is positive, a principal bundle over $G/P$, with a reductive group as the structure group, is called strongly stable (respectively, strongly semistable) if all the iterated pullbacks of it by the Frobenius morphism of $G/P$ are stable (respectively, semistable) principal bundles; the details of these definitions are given in the next section.

The proof of Theorem 1.1 relies heavily on the above mentioned result of [Um] and [Bi]. We first show that $E_P(L(P))$ is strongly semistable with respect to any polarization on $G/P$, and then we show that $E_P(L(P))$ is strongly stable. The above mentioned result of [Um], [Bi] is used in both of these two parts of the proof of Theorem 1.1.

## 2. Preliminaries

Let $k$ be an algebraically closed field of arbitrary characteristic. Henceforth, the characteristic of $k$ will be denoted by $p$. Let $G$ be a connected semisimple linear algebraic group defined over the field $k$. We fix a reduced proper parabolic subgroup

$$P \subsetneq G$$

without any simple factor.

Fix an ample line bundle $\xi$ over $G/P$, which is also called a *polarization* on $G/P$. It is known that any ample line bundle over $G/P$ is very ample.

The *degree* of any torsionfree coherent sheaf on $G/P$ will be defined using $\xi$. If $E$ is a vector bundle defined over a nonempty Zariski open dense subset $U \subseteq G/P$ such that the complement $(G/P)\setminus U$ is of codimension at least two, then the direct image $\iota_* E$ is a torsionfree coherent sheaf on $G/P$, where $\iota : U \longrightarrow G/P$ is the inclusion map. For such a vector bundle $E$, by degree$(E)$ we will mean degree$(\iota_* E)$.

We recall that a torsionfree coherent sheaf $E$ defined over $G/P$ is called *stable* (respectively, *semistable*) if

$$\frac{\mathrm{degree}(E')}{\mathrm{rank}(E')} < \frac{\mathrm{degree}(E)}{\mathrm{rank}(E)}$$

(respectively, $\frac{\mathrm{degree}(E')}{\mathrm{rank}(E')} \leq \frac{\mathrm{degree}(E)}{\mathrm{rank}(E)}$) for every coherent subsheaf $E' \subset E$ with $0 < \mathrm{rank}(E') < \mathrm{rank}(E)$.

If the characteristic $p$ of the base field $k$ positive, then

$$F : G/P \longrightarrow G/P \tag{2.1}$$

will be the Frobenius morphism of the variety $G/P$. For notational convenience, $F$ will denote the identity morphism of $G/P$ when $p = 0$.

A vector bundle $E$ over $G/P$ is called *strongly stable* (respectively, *strongly semistable*) if for each integer $n \geq 1$, the $n$–fold iterated pull back

$$(F^n)^* E := (\overbrace{F \circ \cdots \circ F}^{n\text{-times}})^* E$$

is a stable (respectively, semistable) vector bundle, where $F$ is the map defined above (it is the Frobenius morphism in (2.1) when $p > 0$, and it is the identity morphism of $G/P$ when $p = 0$).

We note that a strongly stable (respectively, strongly semistable) vector bundle is stable (respectively, semistable). Indeed, if $E$ is not stable (respectively, not semistable) then the pullback $F^*E$ is not stable (respectively, not semistable). We also note that by our convention, when $p = 0$, a strongly stable (respectively, strongly semistable) vector bundle is simply a stable (respectively, semistable) vector bundle.

We will now recall the definition of a (semi)stable principal bundle. Let $H$ be a connected reductive linear algebraic group defined over the field $k$. A principal $H$–bundle $E_H$ over $G/P$ is called *stable* (respectively, *semistable*) if for every triple of the form $(Q\,, U\,, \sigma)$, where

- $Q \subset H$ is a reduced maximal proper parabolic subgroup,
- $U \subseteq G/P$ is a Zariski open dense subset such that the codimension of the complement $(G/P) \setminus U$ is at least two, and
- $\sigma : U \longrightarrow (E_H/Q)|_U$ is a reduction of structure group to the subgroup $Q$, over $U$, of the principal $H$–bundle $E_H$,

the following inequality holds:

$$\text{degree}(\sigma^* T_{\text{rel}}) > 0$$

(respectively, $\text{degree}(\sigma^* T_{\text{rel}}) \geq 0$), where $T_{\text{rel}} \longrightarrow E_H/Q$ is the relative tangent bundle for the natural projection $E_H/Q \longrightarrow G/P$ (see [Ra, page 129, Definition 1.1] and [Ra, page 131, Lemma 2.1]); as before, the degree is defined using the polarization $\xi$ on $G/P$.

A principal $H$–bundle $E_H$ over $G/P$ is called *strongly stable* (respectively, *strongly semistable*) if for each integer $n \geq 1$, the iterated $n$–fold pullback $(F^n)^* E_H$ is a stable (respectively, semistable) principal $H$–bundle, where the map $F$, as before, is the Frobenius morphism in (2.1) when $p > 0$ and it is the identity morphism of $G/P$ when $p = 0$.

So, by our convention, when $p = 0$, a strongly stable (respectively, strongly semistable) principal bundle is just a stable (respectively, semistable) principal bundle. Also, a strongly stable (respectively, strongly semistable) principal bundle is automatically stable (respectively, semistable).

Remark 2.1. For any vector $E$ of rank $r$ over $G/P$, there is a corresponding principal $\text{GL}(r, k)$–bundle over $G/P$ defined by the space of all linear isomorphisms of $k^{\oplus r}$ with the fibers of $E$. It is straight–forward to check that the vector bundle $E$ is stable (respectively, semistable) if and only if the corresponding principal $\text{GL}(r, k)$–bundle over $G/P$ is stable (respectively, semistable). Similarly, $E$ is strongly stable (respectively, strongly semistable) if and only if the corresponding principal $\text{GL}(r, k)$–bundle over $G/P$ is strongly stable (respectively, strongly semistable).

Let

$$R_u(P) \subset P$$

be the *unipotent radical* of the parabolic subgroup $P$ of $G$. So, in particular, $R_u(P)$ is a normal subgroup of $P$. The quotient

$$L(P) := P/R_u(P)\,, \tag{2.2}$$

which is called the *Levi quotient* of $P$, is a connected reductive linear algebraic group defined over $k$. Let

$$q : P \longrightarrow L(P) \tag{2.3}$$

be the quotient map.

The natural projection $G \longrightarrow G/P$ defines a principal $P$–bundle over the projective variety $G/P$. This principal $P$–bundle over $G/P$ will be denoted by $E_P$. Let

$$E_P(L(P)) := (G \times L(P))/P \tag{2.4}$$

be the principal $L(P)$–bundle over $G/P$ obtained by extending the structure group of the principal $P$–bundle $E_P$ using the homomorphism $q$ in (2.3). We recall that in the construction of the quotient in (2.4), the action of any point $z \in P$ sends any point

$$(g\,,h) \in G \times L(P)$$

to $(gz\,,q(z^{-1})h) \in G \times L(P)$.

## 3. Strong Semistability of Associated Vector Bundles

Let

$$Z(L(P)) \subset L(P) \tag{3.1}$$

denote the subgroup–scheme of $L(P)$ defined by the center of $L(P)$. It is straight–forward to see that $Z(L(P))$ is a normal subgroup–scheme of $L(P)$. Since the quotient group $L(P)/Z(L(P))$ is semisimple, it does not admit any nontrivial character.

Let $V$ be a finite dimensional left $L(P)$–module satisfying the following condition: the action of $Z(L(P))$ on $V$ is the trivial action, that is, $Z(L(P))$ is contained in the kernel of the homomorphism $L(P) \longrightarrow \mathrm{GL}(V)$ defined by the action of $L(P)$ on $V$.

Consequently, the action of $L(P)$ on $V$ factors through the quotient $L(P)/Z(L(P))$. Hence $V$ is also a left $L(P)$–module.

Lemma 3.1. *The vector bundle $E_{L(P)}(V)$ over $G/P$ associated to the principal $L(P)$–bundle $E_P(L(P))$ in* (2.4) *for the above left $L(P)$–module $V$ is semistable of degree zero with respect to any polarization on $G/P$.*

Proof. Fix a filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell-1} \subset V_\ell = V \tag{3.2}$$

of the left $L(P)$–module $V$ such that each successive quotient $V_i/V_{i-1}$, $1 \leq i \leq \ell$, is an irreducible left $L(P)$–module.

For any integer $0 \leq i \leq \ell$, let $E_{L(P)}(V_i)$ denote the vector bundle over $G/P$ associated to the principal $L(P)$–bundle $E_P(L(P))$ (defined in (2.4)) for the left $L(P)$–module $V_i$ in (3.2). As $V_\ell = V$, the vector bundle $E_{L(P)}(V_\ell)$ will also be denoted by $E_{L(P)}(V)$. So $E_{L(P)}(V)$ is the vector bundle associated to the principal $L(P)$–bundle $E_P(L(P))$ for the left $L(P)$–module $V$. The filtration of $L(P)$–modules in (3.2) gives a filtration of subbundles

$$\begin{aligned} 0 &= E_{L(P)}(V_0) \subset E_{L(P)}(V_1) \subset \cdots \subset E_{L(P)}(V_{\ell-1}) \subset E_{L(P)}(V_\ell) \\ &= E_{L(P)}(V) \end{aligned} \tag{3.3}$$

of the vector bundle $E_{L(P)}(V)$.

For any integer $1 \leq i \leq \ell$, the quotient vector bundle $E_{L(P)}(V_i)/E_{L(P)}(V_{i-1})$ (for the filtration in (3.3)) is identified with the vector bundle over $G/P$ associated to the principal $L(P)$–bundle $E_P(L(P))$ for the $L(P)$–module $V_i/V_{i-1}$ in (3.2). Since each successive quotient $V_i/V_{i-1}$, where $1 \leq i \leq \ell$, is an irreducible $L(P)$–module, from [Bi, page 135, Theorem 2.1] and [Um, page 136, Theorem 2.4] we conclude the following:

For each integer $1 \leq i \leq \ell$, the associated vector bundle $E_{L(P)}(V_i)/E_{L(P)}(V_{i-1})$ is stable with respect to any polarization on $G/P$.

We will next show that

$$\text{degree}(E_{L(P)}(V_i)/E_{L(P)}(V_{i-1})) = 0$$

for each $i \in [1, \ell]$.

Since $Z(L(P))$ (defined in (3.1)) acts trivially on $V$, we conclude that $Z(L(P))$ acts trivially on each quotient $L(P)$–module $V_i/V_{i-1}$, where $1 \leq i \leq \ell$. In other words, the action of $L(P)$ on $V_i/V_{i-1}$ factors through the quotient group $L(P)/Z(L(P))$. We noted earlier that $L(P)/Z(L(P))$ does not admit any nontrivial character. Hence the one–dimensional $L(P)$–module $\bigwedge^{\rm top}(V_i/V_{i-1})$ is isomorphic to the trivial $L(P)$–module of dimension one. This immediately implies that the associated line bundle

$$L_i := \bigwedge^{\rm top}(E_{L(P)}(V_i)/E_{L(P)}(V_{i-1}))$$

is isomorphic to the trivial line bundle over $G/P$, where $1 \leq i \leq \ell$. Note that $L_i$ is the line bundle over $G/P$ associated to the principal $L(P)$–bundle $E_P(L(P))$ for the $L(P)$–module $\bigwedge^{\rm top}(V_i/V_{i-1})$. In particular, we have

$$\text{degree}(E_{L(P)}(V_i)/E_{L(P)}(V_{i-1})) = \text{degree}(L_i) = 0$$

for all $1 \leq i \leq \ell$ and with respect to every polarization on $G/P$.

We have already shown that the vector bundle $E_{L(P)}(V_i)/E_{L(P)}(V_{i-1})$ is stable with respect to any polarization on $G/P$. Therefore, we conclude that (3.3) is a filtration of subbundles of the vector bundle $E_{L(P)}(V)$ such that each successive quotient is a stable vector bundle of degree zero (with respect to any polarization on $G/P$). This immediately implies that $E_{L(P)}(V)$ is a semistable vector bundle of degree zero (with respect to any polarization on $G/P$). This completes the proof of the lemma. □

Using Lemma 3.1, we will prove the following stronger version of it.

PROPOSITION 3.2. *Let $V$ be a finite dimensional left $L(P)$–module on which $Z(L(P))$ acts trivially. Then the associated vector bundle $E_{L(P)}(V)$ in Lemma 3.1 is strongly semistable.*

PROOF. Let

$$F_{L(P)} : L(P) \longrightarrow L(P) \tag{3.4}$$

be the Frobenius morphism of the algebraic group $L(P)$, if the characteristic of the base field $k$ is positive; if $p = 0$, then $F_{L(P)}$ will denote the identity morphism of $L(P)$.

Let

$$\delta : L(P) \longrightarrow \mathrm{GL}(V)$$

be the homomorphism giving the action of $L(P)$ on $V$. For any integer $n \geq 1$, let $V(n)$ denote the left $L(P)$–module constructed using the following composition homomorphism

$$L(P) \stackrel{F^n_{L(P)}}{\longrightarrow} L(P) \stackrel{\delta}{\longrightarrow} \mathrm{GL}(V)\,, \tag{3.5}$$

where

$$F^n_{L(P)} = \overbrace{F_{L(P)} \circ \cdots \circ F_{L(P)}}^{n\text{-times}}$$

with $F_{L(P)}$ being the self–map of $L(P)$ in (3.4). Note that we have

$$F^n_{L(P)}(Z(L(P))) \subset Z(L(P))\,,$$

where $Z(L(P))$ is defined in (3.1). In view of this and the fact that $Z(L(P))$ acts trivially on $V$, from the above definition of the $L(P)$–module $V(n)$ it follows immediately that $Z(L(P))$ also acts trivially on $V(n)$.

Let $E_{L(P)}(V(n))$ denote the vector bundle over $G/P$ associated to the principal $L(P)$–bundle $E_P(L(P))$ for the left $L(P)$–module $V(n)$ constructed in (3.5). We noted above that $Z(L(P))$ acts trivially on $V(n)$. Substituting $V(n)$ in place of $V$ in Lemma 3.1 we conclude that for each integer $n \geq 1$, the vector bundle $E_{L(P)}(V(n))$ is semistable with respect to any polarization on $G/P$.

From the definition of $E_{L(P)}(V(n))$ it follows that the vector bundle $E_{L(P)}(V(1))$ over $G/P$ is identified with the pullback $F^*E_{L(P)}(V)$, where $F$, as in (2.1), is the Frobenius morphism of $G/P$ when $p > 0$ and it is the identity morphism of $G/P$ when $p = 0$. Consequently, using induction on $n$, for any integer $n \geq 1$, the vector bundle $E_{L(P)}(V(n))$ is identified with the $n$–fold iterated pullback $(F^n)^*E_{L(P)}(V)$.

We already noted above that the vector bundle $E_{L(P)}(V(n))$ is semistable with respect to any polarization on $G/P$. Hence $(F^n)^*E_{L(P)}(V)$ is semistable with respect to any polarization on $G/P$. In other words, the vector bundle $E_{L(P)}(V)$ is strongly semistable with respect to any polarization on $G/P$. This completes the proof of the proposition. $\square$

## 4. Strong Stability of the Levi Bundle

Our aim in this section is to prove the following theorem.

THEOREM 4.1. *The principal $L(P)$–bundle $E_P(L(P))$ over $G/P$, defined in* (2.4)*, is stable with respect to any polarization on $G/P$. When the characteristic of the base field $k$ is positive, the principal $L(P)$–bundle $E_P(L(P))$ is strongly stable with respect to any polarization on $G/P$.*

PROOF. As the first step in the proof of the theorem, we will prove the following lemma.

LEMMA 4.2. *The principal $L(P)$–bundle $E_P(L(P))$ is strongly semistable with respect to any polarization on $G/P$.*

PROOF. Let $\mathfrak{l}(\mathfrak{p})$ denote the Lie algebra of $L(P)$. The adjoint action of $L(P)$ on $\mathfrak{l}(\mathfrak{p})$ makes it a left $L(P)$–module. The subgroup–scheme $Z(L(P))$ defined in (3.1) clearly acts trivially on $\mathfrak{l}(\mathfrak{p})$. The vector bundle associated to the principal $L(P)$–bundle $E_P(L(P))$ for the $L(P)$–module $\mathfrak{l}(\mathfrak{p})$ is, by definition, the adjoint vector bundle $\mathrm{ad}(E_P(L(P)))$.

Setting $V = \mathfrak{l}(\mathfrak{p})$ in Proposition 3.2 we conclude that the adjoint vector bundle $\mathrm{ad}(E_P(L(P)))$ over $G/P$ is strongly semistable with respect to any polarization on $G/P$. Using this we will show that the principal $L(P)$–bundle $E_P(L(P))$ is semistable with respect to any polarization on $G/P$.

Take any reduction of structure group

$$E_Q \subset E_P(L(P))|_U \tag{4.1}$$

of the principal $L(P)$–bundle $E_P(L(P))$, to a maximal reduced proper parabolic subgroup $Q \subset L(P)$, over a Zariski open dense subset $U \subseteq G/P$ such that the complement $(G/P) \setminus U$ is of codimension at least two. The dimension of this variety $Q$ will be denoted by $m$. Let $\mathrm{Gr}(\mathfrak{l}(\mathfrak{p}), m)$ be the Grassmann variety that parametrizes linear subspaces of $\mathfrak{l}(\mathfrak{p})$ of dimension $m$.

We have an embedding

$$f_0 \, : \, L(P)/Q \longrightarrow \mathrm{Gr}(\mathfrak{l}(\mathfrak{p}), m) \tag{4.2}$$

that sends any $g \in L(P)/Q$ to $\overline{g}\mathfrak{q}\overline{g}^{-1} \subset \mathfrak{l}(\mathfrak{p})$, where $\mathfrak{q}$ is the Lie algebra of $Q$, and $\overline{g} \in L(P)$ projects to $g$. We note that $f_0$ is equivariant for the left translation actions of $L(P)$ on $L(P)/Q$ and $\mathrm{Gr}(\mathfrak{l}(\mathfrak{p}), m)$.

Since both $L(P)/Q$ and $\mathrm{Gr}(\mathfrak{l}(\mathfrak{p}), m)$ are Fano varieties, and

$$\mathrm{Pic}(L(P)/Q) = \mathbb{Z} = \mathrm{Pic}(\mathrm{Gr}(\mathfrak{l}(\mathfrak{p}), m)),$$

there are positive integers $a$ and $b$ such that

$$(f_0^* K^{-1}_{\mathrm{Gr}(\mathfrak{l}(\mathfrak{p}),m)})^{\otimes a} = (K^{-1}_{L(P)/Q})^{\otimes b}, \tag{4.3}$$

where $K^{-1}_{\mathrm{Gr}(\mathfrak{l}(\mathfrak{p}),m)}$ and $K^{-1}_{L(P)/Q}$ are the anticanonical bundles of $\mathrm{Gr}(\mathfrak{l}(\mathfrak{p}), m)$ and $L(P)/Q$ respectively, and $f_0$ is the embedding in (4.2).

Let $\mathrm{Gr}(\mathrm{ad}(E_P(L(P))), m)$ be the Grassmann bundle over $G/P$ parametrizing all linear subspaces of dimension $m$ in the fibers of the vector bundle $\mathrm{ad}(E_P(L(P)))$. The reduction of structure group $E_Q$ in (4.1) and the map $f_0$ together define an embedding

$$f : (E_P(L(P))/Q)|_U \longrightarrow \mathrm{Gr}(\mathrm{ad}(E_P(L(P))), m))|_U \tag{4.4}$$

that commutes with the projections to $U$.

Let $\mathcal{L}_1 \longrightarrow \mathrm{Gr}(\mathrm{ad}(E_P(L(P))), m)|_U$ be the relative anticanonical line bundle for the natural projection

$$\mathrm{Gr}(\mathrm{ad}(E_P(L(P))), m)|_U \longrightarrow U. \tag{4.5}$$

Similarly, let $\mathcal{L}_2 \longrightarrow (E_P(L(P))/Q)|_U$ be the relative anticanonical line bundle for the projection

$$(E_P(L(P))/Q)|_U \longrightarrow U. \tag{4.6}$$

From (4.3) it follows that

$$(f^*\mathcal{L}_1)^{\otimes a} = \mathcal{L}_2^{\otimes b}, \tag{4.7}$$

where $f$ is the morphism in (4.4). We note that both the line bundles $\mathcal{L}_2$ and $f^*\mathcal{L}_1$ are associated to characters of $Q$, and the character group of $Q$ is isomorphic to $\mathbb{Z}$. Hence (4.7) follows from (4.3).

Let

$$\sigma' := f \circ \sigma : U \longrightarrow (E_P(L(P))/Q)|_U$$

be the section of the projection in (4.5), where $\sigma$ is the section of the projection in (4.6) defined by the reduction in (4.1) and $f$ is constructed in (4.4). From (4.7) we conclude that

$$((\sigma')^*\mathcal{L}_1)^{\otimes a} = \sigma^*\mathcal{L}_2^{\otimes b}. \tag{4.8}$$

We have shown earlier that the adjoint vector bundle $\mathrm{ad}(E_P(L(P)))$ is semistable. Hence,

$$\mathrm{degree}((\sigma')^*\mathcal{L}_1) \geq 0 .$$

Therefore, from (4.8) we conclude that

$$\mathrm{degree}(\sigma^*\mathcal{L}_2) \geq 0 .$$

Consequently, the principal $L(P)$–bundle $E_P(L(P))$ is semistable with respect to any polarization of $G/P$.

For each integer $n \geq 1$, the adjoint vector bundle $\mathrm{ad}((F^n)^*E_P(L(P)))$ is clearly identified with the pullback $(F^n)^*\mathrm{ad}(E_P(L(P)))$, where $F$ is the Frobenius morphism in (2.1) (as before, it is the identity morphism of $G/P$ when $p = 0$). Indeed, this follows immediately from the general fact that taking adjoint bundle commutes with pullback. We noted earlier that from Proposition 3.2 it follows that adjoint vector bundle $\mathrm{ad}(E_P(L(P)))$ is strongly semistable with respect to any polarization on $G/P$. Therefore, using following the above argument for the semistability of $E_P(L(P))$ we now conclude that the principal $L(P)$–bundle $E_P(L(P))$ is strongly semistable with respect to any polarization on $G/P$. This completes the proof of the lemma. □

To prove the theorem using contradiction, assume that the principal $L(P)$–bundle $E_P(L(P))$ is not strongly stable with respect to some polarization $\xi$ on $G/P$. Fix an integer $n_0$ such that the principal $L(P)$–bundle

$$(F^{n_0})^*E_P(L(P)) \longrightarrow G/P \tag{4.9}$$

is not stable. Therefore, there exists a triple $(Q\,, U\,, \sigma)$, where

(i) $Q \subset L(P)$ is a reduced maximal proper parabolic subgroup,

(ii) $U \subseteq G/P$ is a Zariski open dense subset such that the codimension of the complement $(G/P) \setminus U$ is at least two, and

$$\sigma : U \longrightarrow ((F^{n_0})^*E_P(L(P))/Q)|_U \tag{4.10}$$

is a reduction of structure group, to the subgroup $Q$, over the open subset $U$, of the principal $L(P)$–bundle $(F^{n_0})^*E_P(L(P))$, with the property that the following inequality holds:

$$\text{degree}(\sigma^* T_{\text{rel}}) \leq 0\,, \tag{4.11}$$

where $T_{\text{rel}} \longrightarrow (F^{n_0})^* E_P(L(P))/Q$ is the relative tangent bundle for the natural projection $(F^{n_0})^* E_P(L(P))/Q \longrightarrow G/P$.

The principal $L(P)$–bundle $(F^{n_0})^* E_P(L(P))$ is semistable by Lemma 4.2. Therefore, we have

$$\text{degree}(\sigma^* T_{\text{rel}}) \geq 0\,.$$

Combining this with (4.11) we conclude that

$$\text{degree}(\sigma^* T_{\text{rel}}) = 0\,. \tag{4.12}$$

We will need the following proposition.

Proposition 4.3. *There is a finite dimensional irreducible nontrivial left $L(P)$–module*

$$\rho : L(P) \longrightarrow \mathrm{GL}(W) \tag{4.13}$$

*such that the image $\rho(Q)$ is contained in a proper parabolic subgroup of* $\mathrm{GL}(W)$*, where $Q$ is the parabolic subgroup in* (4.10).

Proof. First consider the quotient group $L(P)/Z(L(P))$, where $Z(L(P)) \subset L(P)$ is the subgroup–scheme in (3.1) defined by the center of $L(P)$. Since $L(P)$ is reductive, the group $L(P)/Z(L(P))$ is a product of simple groups. In other words, we have

$$L(P)/Z(L(P)) = \prod_{i=1}^{d} H_i\,, \tag{4.14}$$

where each $H_i$ is a simple linear algebraic group defined over $k$. Any parabolic subgroup of $L(P)/Z(L(P))$ is of the form $\prod_{i=1}^{d} P_i$ where $P_i$ is a parabolic subgroup of $H_i$. We note that a parabolic subgroup need not be a proper subgroup, hence some $P_i$ may coincide with $H_i$.

Since $Q$ is a reduced maximal proper parabolic subgroup of $L(P)$, the image of $Q$ in $L(P)/Z(L(P))$ is a parabolic subgroup of the form

$$P_{j_0} \times (\prod_{i \neq j_0} H_i) \subset \prod_{i=1}^{d} H_i$$

(see (4.14)), where $P_{j_0}$ is a reduced maximal proper parabolic subgroup of $H_{j_0}$.

Take any finite dimensional irreducible nontrivial left $H_{j_0}$–module $W'$. Let

$$\rho_0 \,:\, H_{j_0} \longrightarrow \mathrm{GL}(W') \tag{4.15}$$

be the corresponding homomorphism. Since $P_{j_0}$ is a proper parabolic subgroup of the simple group $H_{j_0}$, and $W'$ is a nontrivial irreducible $H_{j_0}$–module, it can be shown that the image $\rho_0(P_{j_0})$ (see (4.15)) is contained in some proper parabolic subgroup $Q_0$ of $\mathrm{GL}(W')$. To prove this, let

$$R_u(P_{j_0}) \subset P_{j_0}$$

be the unipotent radical. Let

$$0 =: W'_0 \subset W'_1 \subset \cdots \subset W'_{b-1} \subset W'_b = W' \tag{4.16}$$

be the unique filtration of subspaces of $W'$ satisfying the following two conditions:

- $\rho_0(R_u(P_{j_0}))(W'_i) \subset W'_i$ for all $0 \leq i \leq b$, where $\rho_0$ is the homomorphism in (4.15), and
- $W'_i/W'_{i-1} = (W'/W'_{i-1})^{\rho_0(R_u(P_{j_0}))}$ for all $1 \leq i \leq b$.

Since $R_u(P_{j_0})$ is a normal subgroup of $P_{j_0}$, the filtration in (4.16) is preserved by the action of $P_{j_0}$ on $W'$. Therefore,

$$\rho_0(P_{j_0}) \subset Q_0 \subset \mathrm{GL}(W'), \tag{4.17}$$

where $Q_0$ is the parabolic subgroup of $\mathrm{GL}(W')$ that preserves the filtration in (4.16) by its standard action.

Let

$$\rho \,:\, L(P) \longrightarrow \mathrm{GL}(W)$$

be the composition of the homomorphism $\rho_0$ in (4.15) with the natural projection of $L(P)$ to $P_{j_0}$. So from (4.17) it follows that

$$\rho(Q) \subset Q_0\,.$$

This completes the proof of the proposition. □

Continuing with the proof of the theorem, fix any $L(P)$–module $W$ satisfying the condition in Proposition 4.3. Let $(F^{n_0})^*E_P(L(P))(W)$ denote the vector bundle over $G/P$ associated to the principal $L(P)$–bundle $(F^{n_0})^*E_P(L(P))$ for the $L(P)$–module $W$, where $n_0$ is the integer in (4.9).

The proof of the theorem will be completed using the following lemma.

Lemma 4.4. *The above vector bundle $(F^{n_0})^*E_P(L(P))(W)$ over $G/P$ is not stable with respect to the polarization $\xi$ (the same polarization with respect to which the principal bundle $(F^{n_0})^*E_P(L(P))$ in* (4.9) *is not stable).*

Proof. Fix a reduced maximal proper parabolic subgroup $Q' \subset \mathrm{GL}(W)$ such that

$$\rho(Q) \subset Q',$$

where $\rho$ is the homomorphism in (4.13), and $Q$ is the parabolic subgroup in (4.10). Since the image $\rho(Q)$ is contained in a proper parabolic subgroup of $\mathrm{GL}(W)$ (see Proposition 4.3), such a maximal parabolic subgroup $Q' \subset \mathrm{GL}(W)$ exists. The homomorphism $\rho$ in (4.13) induces an embedding

$$\widehat{\rho} : L(P)/Q \longrightarrow \mathrm{GL}(W)/Q'. \tag{4.18}$$

The morphism $\widehat{\rho}$ is clearly equivariant for the left translation actions of $L(P)$ on $L(P)/Q$ and $\mathrm{GL}(W)/Q'$.

We note that

$$\mathrm{Pic}(L(P)/Q) = \mathbb{Z} = \mathrm{Pic}(\mathrm{GL}(W)/Q'),$$

and also both $L(P)/Q$ and $\mathrm{GL}(W)/Q'$ are Fano varieties. Therefore, there are positive integers $a$ and $a'$ such that

$$(\widehat{\rho}^* K^{-1}_{\mathrm{GL}(W)/Q'})^{\otimes a'} = (K^{-1}_{L(P)/Q})^{\otimes a}, \tag{4.19}$$

where $\widehat{\rho}$ is the morphism in (4.18).

Let $m$ be the dimension of $Q'$. Let $\mathrm{Gr}((F^{n_0})^*E_P(L(P))(W), m)$ be the Grassmann bundle over $G/P$ parametrizing all linear subspaces of dimension $m$ in the fibers of the vector bundle $(F^{n_0})^*E_P(L(P))(W)$.

We now note that the reduction $\sigma$ in (4.10) and the morphism $\widehat{\rho}$ in (4.18) together give an embedding

$$\gamma : ((F^{n_0})^* E_P(L(P))/Q)|_U \longrightarrow \mathrm{Gr}((F^{n_0})^* E_P(L(P))(W), m)|_U \tag{4.20}$$

which commutes with the projections to $U$. Let

$$\mathcal{L}' \longrightarrow \mathrm{Gr}((F^{n_0})^* E_P(L(P))(W), m)|_U$$

be the relative anticanonical line bundle for the natural projection

$$\mathrm{Gr}((F^{n_0})^* E_P(L(P))(W), m)|_U \longrightarrow U\,. \tag{4.21}$$

Let

$$\mathcal{L} \longrightarrow ((F^{n_0})^* E_P(L(P))/Q)|_U$$

be the relative anticanonical line bundle for the projection

$$((F^{n_0})^* E_P(L(P))/Q)|_U \longrightarrow U\,. \tag{4.22}$$

From (4.19) it follows that

$$(\gamma^* \mathcal{L}')^{\otimes a'} = \mathcal{L}^{\otimes a}\,. \tag{4.23}$$

We note that both the line bundles $\mathcal{L}$ and $\gamma^* \mathcal{L}'$ are associated to characters of $Q$. Also, the character group of $Q$ is isomorphic to $\mathbb{Z}$. Hence (4.23) follows from (4.19).

Let

$$\sigma' := \gamma \circ \sigma : U \longrightarrow \mathrm{Gr}((F^{n_0})^* E_P(L(P))(W), m)|_U$$

be the section of the projection in (4.21), where $\sigma$ is the section in (4.10) of the projection in (4.22) and $\gamma$ is the map in (4.20). From (4.19) we conclude that

$$(\sigma')^* (\mathcal{L}')^{\otimes a'} = \sigma^* \mathcal{L}^{\otimes a}\,.$$

Hence from (4.12) it follows immediately that

$$\mathrm{degree}((\sigma')^* \mathcal{L}') = 0\,.$$

Consequently, the vector bundle $(F^{n_0})^* E_P(L(P))(W)$ is not stable with respect to the polarization $\xi$ on $G/P$. This completes the proof of the lemma. □

Now we are in a position to complete the proof of the theorem.

Since $W$ in Lemma 4.4 is an irreducible left $L(P)$–module, the composition

$$L(P) \xrightarrow{F^{n_0}_{L(P)}} L(P) \xrightarrow{\rho} \mathrm{GL}(W) \tag{4.24}$$

defines an irreducible left $L(P)$–module, where the homomorphisms $F^{n_0}_{L(P)}$ and $\rho$ are defined in (3.5) and (4.13) respectively. The vector bundle associated to the principal $L(P)$–bundle $E_P(L(P))$ for this left $L(P)$–module constructed in (4.24) is identified with the vector bundle $(F^{n_0})^* E_P(L(P))(W)$. Therefore, using [Bi, page 135, Theorem 2.1] and [Um, page 136, Theorem 2.4] we conclude that the vector bundle $(F^{n_0})^* E_P(L(P))(W)$ is stable with respect to any polarization on $G/P$. This contradicts Lemma 4.4. Hence we conclude that the principal $L(P)$–bundle $E_P(L(P))$ is strongly stable. This completes the proof of the theorem. □

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School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay 400005, India
E-mail: indranil@math.tifr.res.in