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Proper Actions of $SL(2,\mathbb{R})$ *on* $SL(n,\mathbb{R})$ *– Homogeneous Spaces*

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Abstract. This paper gives a necessary and sufficient condition for the homogeneous spaces of $SL(n,\mathbb{R})$ to admit proper actions of $SL(2,\mathbb{R})$, or equivalently, to admit an infinite discontinuous group generated by a unipotent element. The method of our proof is based on Kobayashi's criterion for proper actions on homogeneous spaces of reductive type.

1. Introduction and Statement of Main Results

Let G be a reductive linear Lie group, and H a closed reductive subgroup of G. Then, there exists a pseudo-Riemanian metric on G/H induced from a G-invariant bilinear form on the Lie algebra $\mathfrak{g}$ of G (e.g. the Killing form) such that G acts on G/H as isometries. Semisimple symmetric spaces are a classic example.

Let Γ be a discrete subgroup of G. Then Γ is regarded as a discrete group consisting of isometries of the pseudo–Riemannian manifold G/H. We are interested in the Γ-action on G/H when H is non-compact. Then, the action of a discrete subgroup Γ on G/H is not automatically properly discontinuous. In fact, it may happen that there does not exist an infinite discrete subgroup Γ of G which acts properly discontinuously on G/H. This phenomenon was first discovered by E.Calabi and L.Markus [2] for $(G,H) = (SO(n,1), SO(n-1,1))$, and is called the *Calabi–Markus phenomenon.* T.Kobayashi [4] proved that the Calabi–Markus phenomenon occurs if and only if $\mathrm{rank}_{\mathbb{R}} G = \mathrm{rank}_{\mathbb{R}} H$. The more difficult part is the 'only if' part, because the proof involves a deeper understanding of proper

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actions and the construction of an infinite discrete subgroup Γ acting properly discontinuously on G/H if $\operatorname{rank}_{\mathbb{R}} G > \operatorname{rank}_{\mathbb{R}} H$. In [4], this subgroup Γ is taken to be a free abelian group generated by a *semisimple element* of G.

In this paper, we consider the following question:

QUESTION 1.1. Does there exist a free abelian subgroup Γ generated by a unipotent element γ of G such that Γ acts properly discontinuously on G/H?

It is proved in [14] that Question 1.1 is equivalent to the following:

QUESTION 1.2. Does there exist a subgroup L of G having the following two properties?

(1) L acts properly on G/H.

(2) The Lie algebra of L is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

We also ask:

QUESTION 1.3. Does there exist a discrete subgroup Γ of G having the following two properties?

(1) Γ acts properly discontinuously on G/H.

(2) Γ is isomorphic to a Fucks group.

Since a Fucks group is realized as a discrete subgroup of $SL(2, \mathbb{R})$, we have the following implications:

$$\textit{Question } 1.1 \Leftrightarrow \textit{Question } 1.2 \Rightarrow \textit{Question } 1.3$$

Suppose that Question 1.1 has an affirmative answer for the homogeneous space G/H. Then, the above equivalence implies that there exists also a free abelian subgroup generated by a semisimple element which acts properly discontinuously on G/H. By the criterion of the Calabi–Markus phenomenon, we have $\operatorname{rank}_{\mathbb{R}} G > \operatorname{rank}_{\mathbb{R}} H$ in this case.

The aim of the paper is to examine to which extent the converse statement holds, namely, to which extent Question 1.1 has an affirmative answer for the homogeneous space G/H under the assumption that $\operatorname{rank}_{\mathbb{R}} G > \operatorname{rank}_{\mathbb{R}} H$. We shall focus on the test case $G = SL(n,\mathbb{R})$.

To be more explicit, we shall deal with the following homogeneous spaces:

(1) $SL(n,\mathbb{R})/SL(k,\mathbb{R})$ $(1 \le k \le n-1)$,

(2) $SL(n,\mathbb{R})/SO(p,q)$ $(1 \le p+q \le n)$,

(3) $SL(n,\mathbb{R})/Sp(m,\mathbb{R})$ $(2m \le n)$,

(4) $SL(n,\mathbb{R})/SL(m,\mathbb{C})$ $(2m \le n)$,

(5) $SL(n,\mathbb{R})/\rho(SL(2,\mathbb{R}))$ where ρ is an n-dimensional irreducible representation.

In (1)–(4), we shall consider the action of $SL(2,\mathbb{R})$. On the other hand, in (5) we shall consider the action of more general subgroups of $SL(n,\mathbb{R})$.

The case (1) is a non-symmetric space for which the existence problem of compact Clifford–Klein form was discussed in [5, 6, 10, 11, 15] by various approaches, see also [8, 9] for the survey of different methods. The cases (2), (3), (4) include essentially all the semisimple symmetric spaces of $G = SL(n,\mathbb{R})$ except for $\operatorname{rank}_{\mathbb{R}} G = \operatorname{rank}_{\mathbb{R}} H$ such as $SL(n,\mathbb{R})/S(GL(p,\mathbb{R}) \times GL(n-p,\mathbb{R}))$, where the Calabi–Markus phenomenon occurs. On the other hand, Margulis [12] proved that there does not exist a cocompact lattice for $SL(n,\mathbb{R})/\rho(SL(2,\mathbb{R}))$ if $n > 4$ by using the restriction of unitary representations.

We note that $\operatorname{rank}_{\mathbb{R}} G > \operatorname{rank}_{\mathbb{R}} H$ except for $(p,q,n) = (1,1,2)$ in (2), $(m,n) = (1,2)$ in (3), and $n = 2$ in (5).

In this paper, applying the criterion of proper actions [4], we will prove the following:

Theorem 1.4. *Let (G,H) be $(SL(n,\mathbb{R}), SL(k,\mathbb{R}))$ $(1 \le k \le n-1)$. Question 1.1 (equivalently, Question 1.2) has an affirmative answer if and only if either n is even or n is odd and $k+1 < n$.*

Remark 1.5. It is proved in [5, 6] that $SL(n,\mathbb{R})/SL(k,\mathbb{R})$ does not admit a compact Clifford–Klein form if $n > \frac{3}{2}k$ (k even), or $n > \frac{3}{2}k + \frac{3}{2}$ (k odd) (see [7, Example 5.19]), and in [1] if $n = k+1$ and n odd. The latter condition is precisely when there is no proper action of $SL(2,\mathbb{R})$ by Theorem 1.4. Different approaches such as using the theory of ergodic actions also give similar results to the former condition, see [10, 11, 15]. See also the survey [8, §5.14] or [9] for more details.

THEOREM 1.6. *Let (G,H) be $(SL(n,\mathbb{R}), SO(p,q))$ for $1 \leq p+q \leq n$. Question 1.1 (equivalently, Question 1.2) has an affirmative answer if and only if $n - 2\min(p,q) \geq 2$.*

REMARK 1.7. It is proved in [7, Example 5.18] that $SL(n,\mathbb{R})/SO(p,q)$ does not admit a compact Clifford–Klein form if $p+q<n$ or $p=q$. In particular, $SL(n,\mathbb{R})/SO(p,q)$ $(p+q<n-1)$ does not admit a cocompact discontinuous group, but admits a discontinuous group isomorphic to Fuchs group.

THEOREM 1.8. *Let (G,H) be $(SL(n,\mathbb{R}), Sp(m,R))$ $(2m \leq n)$. Question 1.1 (equivalently, Question 1.2) has an affirmative answer if and only if $n-2m \geq 2$.*

REMARK 1.9. It is proved in [7, Example 5.18] that $SL(n,\mathbb{R})/Sp(m,\mathbb{R})$ does not admit a compact Clifford–Klein form if $2m \leq n-2$. By Theorem 1.8, this is precisely the case where there exists a proper action of $SL(2,\mathbb{R})$ on $SL(n,\mathbb{R})/Sp(m,\mathbb{R})$. Therefore, the pseudo–Riemanian manifolds $SL(n,\mathbb{R})/Sp(m,\mathbb{R})$ $(2m \leq n-2)$ admits a discontinuous group isomorphic to Fuchs groups, but does not admit a cocompact discontinuous group.

THEOREM 1.10. *Let (G,H) be $(SL(n,\mathbb{R}), SL(m,\mathbb{C}))$ $(2m \leq n)$. Question 1.1 (equivalently, Question 1.2) has an affirmative answer for all m.*

Finally, we switch the role of L and H:

THEOREM 1.11. *Let $\rho : SL(2,\mathbb{R}) \rightarrow SL(n,\mathbb{R})$ be an irreducible real representation of $SL(2,\mathbb{R})$, and let L be one of $SL(k,\mathbb{R})$, $SO(p,q)$, $Sp(m,\mathbb{R})$, $SL(m,\mathbb{C})$ which are standard subgroups of G.*

Then, we have the following list of L whether or not L acts properly on $G/\rho(SL(2,\mathbb{R}))$.

L	*proper*	*not proper*
$SL(k,\mathbb{R})$ $(1 \le k \le n-1)$	n : *even*	
	n : *odd*, $k+1<n$	n : *odd*, $k+1=n$
$SO(p,q)$ $(p+q \le n)$	$n-2\min(p,q) \ge 2$	$n-2\min(p,q)=0$ *or* 1
$Sp(m,\mathbb{R})$ $(1 \le 2m \le n)$	$n-2m \ge 2$	$n-2m=0$ *or* 1
$SL(m,\mathbb{C})$ $(1 \le 2m \le n)$	*all cases*	*no case*

2. Review of the Criterion for Proper Actions

Let G be a real reductive linear Lie group. Then, there exists a Cartan involution θ of G. Then, $K = \{g \in G : \theta g = g\}$ is a maximal compact subgroup of G. We write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the $+1$ and -1 eigenspace decomposition of the differential of θ.

Let $\mathfrak{a}$ be any maximal abelian subspace in $\mathfrak{p}$. All such subspaces are mutually conjugate by an element of K. The dimension of $\mathfrak{a}$ is called the *real rank* of G, denoted by $\operatorname{rank}_{\mathbb{R}} G$. A subspace in $\mathfrak{g}$ conjugate to $\mathfrak{a}$ in G is called a *maximally split abelian subspace*. Let M' (resp. M) be the normalizer (resp. centralizer) of $\mathfrak{a}$ in K. Then, the quotient group M'/M is a finite group, to be denoted by W, which is called the Weyl group for the restricted root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$.

Let H be a closed subgroup of G with at most finitely many connected components.

Definition 2.1. We say that H is *reductive* in G if there exists a Cartan involution θ' of G such that $\theta'(H) = H$.

Then, $\theta'|_H$ is a Cartan involution of H, and $K' \cap H$ is a maximal compact subgroup of H, where $K' := \{g \in G : \theta' g = g\}$. Any two Cartan involutions θ and θ' are conjugate to each other, namely, there exists an element $g \in G$ such that

$$\theta = \operatorname{Ad}(g) \circ \theta' \circ \operatorname{Ad}(g^{-1}).$$

This means that gHg^{-1} is θ-stable, i.e. $\theta(gHg^{-1}) = gHg^{-1}$.

We note that the Lie algebra $\mathfrak{h}$ is a reductive Lie algebra, namely, the direct sum of semisimple Lie algebras and abelian Lie algebras. Let $\mathfrak{a}(H)$ be a maximally split abelian subspace for H. Then, $\mathfrak{a}(H)$ is not necessarily contained in $\mathfrak{a}$, however, we can find $g \in G$ such that $\operatorname{Ad}(g)\mathfrak{a}(H) \subset \mathfrak{a}$. Such a subspace $\mathrm{Ad}(g)\mathfrak{a}(H)$ is uniquely determined by H up to the conjugation by the Weyl group W. From now, we set

$$\mathfrak{a}_H := \mathrm{Ad}(g)\mathfrak{a}(H). \tag{1}$$

The criterion for proper actions is discovered in [4, Theorem 4.1] as follows.

FACT 2.2. *Let L, H be subgroups which are reductive in a real reductive linear group G. Let $\mathfrak{a}_L$, $\mathfrak{a}_H \subset \mathfrak{a}$ be as in* (1)*. Then the following three conditions on L and H are equivalent:*

(i) *L acts on G/H properly.*

(ii) *H acts on G/L properly.*

(iii) *For any $w \in W$, $w\mathfrak{a}_L \cap \mathfrak{a}_H = \{0\}$.*

3. Classification of $\mathfrak{a}(\rho(SL(2,\mathbb{R})))$ in $SL(n,\mathbb{R})$

Let $G = SL(n,\mathbb{R})$. We will take a maximally split abelian subspace $\mathfrak{a}$ to be the diagonal subspace in the matrix form, and identify $\mathfrak{a}$ with the vector space

$$\mathfrak{a} := \Big\{(x_1,\dots,x_n) \in \mathbb{R}^n : \sum x_j = 0\Big\}.$$

The Weyl group W is isomorphic to the symmetric group $\mathcal{S}_n$ which acts on $\mathfrak{a}$ as permutations of coodinates.

In this section, we shall determine all possible $\mathfrak{a}_L$ for a subgroup L of G such that L is locally isomorphic to $SL(2,\mathbb{R})$. We note that the condition on L is equivalent to the fact that the Lie algebra $\mathfrak{l}$ of L is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

For a positive integer m, we set

$$(m) := (m-1, m-3, \dots, -(m-1)) \in \mathbb{Z}^m. \tag{2}$$

We shall also use the following notation such as

$$((a),(b)) = (a-1, a-3, \dots, -(a-1), b-1, b-3, \dots, -(b-1)) \in \mathbb{Z}^{a+b}.$$

For subsets $\mathfrak{a}_1$, $\mathfrak{a}_2$ in $\mathfrak{a}$, we write

$$\mathfrak{a}_1 \sim \mathfrak{a}_2 \mod \mathcal{S}_n$$

if there exist $w \in \mathcal{S}_n (= W)$ such that $\mathfrak{a}_1 = w\mathfrak{a}_2$.

LEMMA 3.1. *Suppose L is a subgroup of $SL(n,\mathbb{R})$, which is locally isomorphic to $SL(2,\mathbb{R})$. Then, there exists a partition $n = n_1 + \cdots + n_l$ such that at least one of n_is is greater than one and that*

$$\mathfrak{a}_L \sim \mathbb{R}((n_1),(n_2),\ldots,(n_l)) \mod \mathcal{S}_n.$$

Conversely, for any partition $n = n_1 + \cdots + n_l$ such that at least one of n_is is greater than one, we can find a subgroup L which is locally isomorphic to $SL(2,\mathbb{R})$ such that $\mathfrak{a}_L \sim \mathbb{R}((n_1),\ldots,(n_l)) \mod \mathcal{S}_n$.

PROOF. Suppose the Lie algebra $\mathfrak{l}$ of L is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. This means that we have an injective Lie algebra homomorphism

$$d\rho:\ \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{sl}(n,\mathbb{R}). \tag{3}$$

Let $H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then, $\mathfrak{a}(L) := \mathbb{R}d\rho(H)$ is a maximally split abelian subspace of $\mathfrak{l}$.

We complexify the $\mathbb{R}$–Lie algebra homomorphism (3), and use the same letter $d\rho:\ \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{sl}(n,\mathbb{C})$. Then this is a complex representation, and therefore is decomposed into a direct sum of irreducible representations because any finite dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ is completely reducible. By taking a suitable basis, we will have its irreducible decomposition as $\mathbb{C}^n = \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_l}$ where $n = n_1 + \cdots + n_l$ is a partition.

By changing a basis in each $\mathbb{C}^{n_j} (1 \le j \le l)$ if necessary, we may assume that $d\rho(H)$ is of the following diagonal matrix:

$$A := \begin{pmatrix} D_1 & 0 & \ldots & 0 \\ 0 & D_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & D_k \end{pmatrix} \quad \text{where}$$

$$D_i := \begin{pmatrix} n_i - 1 & 0 & \ldots & 0 \\ 0 & n_i - 3 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 - n_i \end{pmatrix}.$$

Thus, we have found $g \in GL(n, \mathbb{C})$ such that

$$\operatorname{Ad}(g)^{-1}\mathfrak{a}(L) = ((n_1), \ldots, (n_k)). \tag{4}$$

Next, we claim that there exists $g \in SL(n, \mathbb{R})$ satisfying (4). The first step is to show that we can take $g \in GL(n, \mathbb{R})$. To see this, suppose $g = P + iQ \in GL(n, \mathbb{C})$ satisfies $d\rho(H)(P + iQ) = (P + iQ)A$. Since $d\rho(H) \in \mathfrak{sl}(n, \mathbb{R})$, we have $d\rho(H)P = PA$ and $d\rho(H)Q = QA$. On the other hand, $\det(P + tQ)$ is not identically zero as a polynomial of t because $\det(P + iQ) \neq 0$. Therefore, there exists a real number t_0 such that $\det(P + t_0Q) \neq 0$. We set $g_0 := P + t_0Q \in GL(n, \mathbb{R})$. Then, $d\rho(H)g_0 = g_0A$. Hence, we have shown $\operatorname{Ad}(g_0)^{-1}d\rho(H) = A$. We take a diagonal matrix $g_1 := \operatorname{diag}(\det g_0^{-1}, 1, \ldots, 1)$, and set $g := g_0g_1$. Then $\det g = \det g_0 \det g_1 = 1$, and $\operatorname{Ad}(g)^{-1}d\rho(H) = \operatorname{Ad}(g_1)^{-1}\operatorname{Ad}(g_0)^{-1}d\rho(H) = \operatorname{Ad}(g_1)^{-1}A = A$. Therefore $g \in SL(n, \mathbb{R})$ satisfies (4). Thus, we have proved the first statement of Lemma 3.1.

The second statement of Lemma 3.1 follows by taking the direct sum of irreducible representations of $SL(n, \mathbb{R})$ of dimensions $n_1, n_2, \ldots, n_l$. □

4. Proof of Main Theorems

4.1. Proof for $SL(n, \mathbb{R})/SL(k, \mathbb{R})$ $(k < n)$

In this subsection, we complete the proof of of Theorem 1.4. More than this, we shall give its refinement in Lemma 4.2, which provides a criterion for the proper action for each homomorphism ρ from $SL(2, \mathbb{R})$ to $SL(n, \mathbb{R})$.

LEMMA 4.1. *For $H = SL(k, \mathbb{R})$ in $G = SL(n, \mathbb{R})$ $(1 \leq k \leq n - 1)$, we have*

$$\mathfrak{a}_H \sim \{(y_1, \ldots, y_k, \underbrace{0, \ldots, 0}_{n-k}) : \sum y_j = 0,\ y_1, \ldots, y_k \in \mathbb{R}\} \mod \mathcal{S}_n. \tag{5}$$

PROOF. Obvious from the definition (1) of $\mathfrak{a}_H$. □

LEMMA 4.2. *Let $\rho :\ SL(2, \mathbb{R}) \to SL(n, \mathbb{R})$ be a representation, and we set $L := \rho(SL(2, \mathbb{R}))$. Since ρ is completely irreducible, we have a partition $n = n_1 + \cdots + n_l$, according to the dimensions n_j $(1 \leq j \leq l)$ of irreducible summands of ρ.*

Then, $w\mathfrak{a}_L \cap \mathfrak{a}_H = \{0\}$ $(\forall w \in W)$ *if and only if*

$$\sharp\{1 \le i \le k : \ n_i \text{ is odd}\} < n - k. \tag{6}$$

Proof. In light of the definition (2), the number of 0s in $((n_1), (n_2), \ldots, (n_l))$ equals $\sharp\{1 \le i \le k : n_i \text{ is odd}\}$. If (6) holds, then clearly $w\mathfrak{a}_L \cap \mathfrak{a}_H = \{0\}$ for any $w \in W = \mathcal{S}_n$ by Lemma 4.1. Conversely, suppose (6) does not hold. Then, if we sort $((n_1), (n_2), \ldots, (n_l))$ and (5) in descending order, they have a non-trivial intersection. Therefore, Lemma is proved. □

Proof of Theorem 1.4. If n is even, we can take ρ to be irreducible, namely, $n_1 = n$. Then, (6) holds because the left-hand side of (6) is zero. Likewise if n is odd and $n-k > 1$, then we can take again ρ to be irreducible so that $n_1 = n$. Then, (6) holds. On the other hand, if n is odd and $n-k = 1$, there does not exist a partition $n = n_1 + \cdots + n_l$ satisfying (6). In fact, if $n - k = 1$, the left-hand side of (6) must be zero. This holds only if all n_is are even, whence n must be even. This contradicts to the assumption that n is odd. Therefore, Theorem 1.4 is proved. □

4.2. Proof for $SL(n,\mathbb{R})/SO(p,q)$ $(p+q \le n)$

This subsection gives a proof of Theorem 1.6.

Lemma 4.3. *For* $H = SO(p,q)$ *in* $G = SL(n,\mathbb{R})$ $(p+q \le n)$*,*

$$\mathfrak{a}_H \sim \{(y_1, \ldots, y_l, \underbrace{0, \ldots, 0}_{n-2\min(p,q)}, -y_l, \ldots, -y_1) : \ y_1, \ldots, y_l \in \mathbb{R}\} \mod \mathcal{S}_n,$$

where $l = \min(p,q)$.

Proof. The Lie algebra $\mathfrak{h} = \mathfrak{so}(p,q)$ is given in the matrix form as

$$\mathfrak{h} = \{\begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix} : \ A = -{}^tA, \ B \in M(p,q;\mathbb{R}), \ C = -{}^tC\}.$$

Without loss of generality, we may and do assume $p \le q$, that is, $l = p$. Then,

$$\mathfrak{a}(H) := \bigoplus_{j=1}^{l} \mathbb{R}(E_{j,p+j} + E_{p+j,j})$$

is a maximally split abelian subspace for $\mathfrak{h}$, where $E_{i,j} (1 \leq i, j \leq n)$ is the matrix unit. Then, $\mathfrak{a}(H)$ is conjugate to $\oplus_{j=1}^{l} \mathbb{R}(E_{j,j} - E_{n+1-j,n+1-j})$. Thus, Lemma 4.3 is proved. □

LEMMA 4.4. *Let $L = \rho(SL(2,\mathbb{R}))$ where $\rho : SL(2,\mathbb{R}) \to SL(n,\mathbb{R})$ is a representation corresponding to a partition $n = n_1 + \cdots + n_l$.*

Then, $w\mathfrak{a}_L \cap \mathfrak{a}_H = \{0\}$ for any $w \in W$ if and only if

$$\sharp\{1 \leq i \leq l : n_i \textit{ is odd}\} < n - 2\min(p,q). \tag{7}$$

PROOF. The proof parallels to that of Lemma 4.2. By Lemma 4.3, the condition (7) implies $w\mathfrak{a}_L \cap \mathfrak{a}_H = \{0\}$ for any $w \in \mathcal{S}_n$. Conversely, suppose (7) does not hold. Then if we sort $((n_1), (n_2), \ldots, (n_l))$ in descending order, it has a non-trivial intersection with $\mathfrak{a}_H$. Therefore, Lemma is proved. □

PROOF OF THEOREM 1.6. If $n - 2\min(p,q) = 0$, then there does not exist a partition satisfying (7). If $n - 2\min(p,q) = 1$, then (7) holds if and only if all n_is are even. In particular, n must be even. Since $n - 2\min(p,q) = 1$, n never becomes even. Thus, (7) does not hold. If $n - 2\min(p,q) > 1$ and n is even, we can take a partition such that all n_is are even. (e.g.$n_1 = n$) Thus, (7) holds for such a partition. If $n - \min(p,q) > 1$ and n is odd, we can take a partition such that the left hand side of (7) is one. (e.g.$n_1 = n$) Then, (7) holds for such a partition. Therefore, this theorem is proved. □

4.3. Proof for $SL(n,\mathbb{R})/Sp(m,\mathbb{R})$ $(2m \leq n)$

This subsection gives a proof of Theorem 1.8.

LEMMA 4.5. *For $H = Sp(m,\mathbb{R})$ in $G = SL(n,\mathbb{R})$ $(2m \leq n)$,*

$$\mathfrak{a}_H \sim \{(y_1, \ldots, y_m, -y_1, \ldots, -y_m, \underbrace{0, \ldots, 0}_{n-2m}) : y_1, \ldots, y_m \in \mathbb{R}\} \mod \mathcal{S}_n.$$

PROOF. The Lie algebra $\mathfrak{h} = \mathfrak{sp}(m,\mathbb{R})$ is given in the matrix form as

$$\mathfrak{h} = \{\begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} : A, B, C \in M(m,\mathbb{R}),\ {}^tB = B,\ {}^tC = C\}.$$

In this case, $\mathfrak{a} \cap \mathfrak{h}$ is a maximally split abelian subspace of $\mathfrak{h}$. Taking $\mathfrak{a}_H := \mathfrak{a} \cap \mathfrak{h}$, we have seen the lemma. □

PROOF OF THEOREM 1.8. This proof is the same as Theorem 1.6 by comparing Lemma 4.3 ($p = q = m$ case) with Lemma 4.5. □

4.4. Proof for $SL(n,\mathbb{R})/SL(m,\mathbb{C})$ $(2m \leq n)$

This subsection gives a proof of Theorem 1.10.

LEMMA 4.6. *For $H = SL(m,\mathbb{C})$ in $G = SL(n,\mathbb{R})$ $(2m \leq n)$,*

$$\mathfrak{a}_H \sim \{(y_1, \ldots, y_m, y_1, \ldots, y_m, \underbrace{0, \ldots, 0}_{n-2m}) : \sum y_j = 0,\ y_1, \ldots, y_m \in \mathbb{R}\} \quad \text{mod } \mathcal{S}_n.$$

PROOF. The Lie algebra $\mathfrak{h} = \mathfrak{sl}(m,\mathbb{C})$ is given in the matrix form as

$$\mathfrak{h} = \{\begin{pmatrix} A & -B \\ B & A \end{pmatrix} :\ A, B \in M(m,\mathbb{R}), \operatorname{Trace} A = 0,\ \ \operatorname{Trace} B = 0\}.$$

In this case, $\mathfrak{a} \cap \mathfrak{h}$ is a maximally split abelian subspace of $\mathfrak{h}$. Taking $\mathfrak{a}_H := \mathfrak{a} \cap \mathfrak{h}$, we have seen the lemma. □

PROOF OF THEOREM 1.10. We consider $L = \rho(SL(2,\mathbb{R}))$ where $\rho :\ SL(2,\mathbb{R}) \to SL(n,\mathbb{R})$ is an irreducible representation. This corresponds to the partition $n = n_1$. By Lemma 3.1, $\mathfrak{a}_L \sim \mathbb{R}(n-1, n-3, \ldots, -(n-1))$ mod $\mathcal{S}_n$. In particular, no two coordinates are equal to each other. In view of Lemma 4.6, we have clearly

$$w\mathfrak{a}_L \cap \mathfrak{a}_H = \{0\} \ \textit{for any}\ w \in W.$$

Thus, Theorem 1.10 is proved. □

As in the proof of Theorem 1.4, 1.6, 1.8, and 1.10, if Question 1.2 is affirmative for these homogeneous spaces G/H, then $SL(2,\mathbb{R})$ acts properly on G/H through an irreducible n–dimensional representation of $SL(2,\mathbb{R})$. Therefore, the proof of Theorem 1.11 follows from the equivalence of (i) and (ii) in Fact 2.2.

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