J. Math. Sci. Univ. Tokyo
14 (2007), 531–565.

Push-out of Schemes

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Abstract. We study the existence of a push-out for two morphisms $Z \to X$ and $Z \to Y$ in the category of schemes. Push-out is a generalization of quotient of groupoid. We give a necessary and sufficient condition for the existence of a push-out in the flat projective case. We also give a sufficient condition for the existence of a push-out in the finite normal case, which does not assume any flatness. In particular, this gives a sufficient condition for the existence of a quotient of a finite groupoid on a normal scheme, which does not assume any flatness.

0. Introduction

Let $p : Z \to X$ and $q : Z \to Y$ be morphisms of schemes over a base scheme S. A commutative diagram over S

$$\begin{array}{ccc} Z & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & T \end{array}$$

is called a *push-out* if for any S-scheme T' and any S-morphisms $f' : X \to T'$ and $g' : Y \to T'$ such that $f' \circ p = g' \circ q$, there is a unique morphism $\phi : T \to T'$ such that $f' = \phi \circ f$ and $g' = \phi \circ g$. This is the dual concept of pull-back (i.e. fiber product) in the category of schemes. Unlike pull-back, a push-out may not exist at all (for the given p, q). However, in many interesting cases we can see the existence of a push-out, and push-out is a useful tool.

A special case of push-out is the quotient of a groupoid. Along this line there have been many works and results (cf. e.g. [3], [4]). In many cases we need to use the existence of quotients of groupoids, for example in moduli theory (cf. [6], [8], [11], [13], [14], [15], [16], [17], [19], and see Example 2.4

2000 *Mathematics Subject Classification*. 14A15, 14D22.
*Supported by NSFC of China, grant numbers 19531030, 10531060.

below). We will use quotients of groupoids to construct push-out. A special and important case of groupoid is group scheme action (cf. [1], [7], [10], [12], [18], and see Example 1.1 below).

Assume Z is a closed subscheme of $X \times_S Y$. For the existence of a push-out of $p = \mathrm{pr}_1 : Z \to X$ and $q = \mathrm{pr}_2 : Z \to Y$ which is also a pull-back, a very basic necessary condition is

$$(*) \qquad Z \times_X Z \times_Y Z = Z \times_Y Z \times_X Z \subset X \times_S X \times_S Y \times_S Y$$

In the language of presheaves, this can be expressed as: For any S-scheme U and any $x, x' \in \underline{X}(U)$ and $y, y' \in \underline{Y}(U)$, if $(x, y), (x, y'), (x', y') \in \underline{Z}(U)$, then $(x', y) \in \underline{Z}(U)$.

In §1 we define push-out and some related basic concepts for general categories. We also give some examples for the existence or non-existence of a push-out.

In §2 we give a criteria of push-out. A special case is: for any two faithfully flat morphisms $f : X \to S$ and $g : Y \to S$ of finite type, S is a push-out of $\mathrm{pr}_1 : X \times_S Y \to X$ and $\mathrm{pr}_2 : X \times_S Y \to Y$.

In the following sections we give some sufficient conditions for the existence of push-out in several cases. In §3 we study the flat projective case, the method can be used to study push-out for a general category (see Theorem 1.1). In §4 we study the finite case, here we don't assume any flatness. In particular, this gives a criterion for the existence of quotients of finite groupoids without any assumption of flatness. In §5 we study a partially flat case. In each case we see that condition $(*)$ plays a key role.

We give some examples, which show the connections of push-out with several subjects.

Acknowledgements. I wish to thank A. Ogus, S. Mori, E. Viehweg, J. Yu, G. Faltings, C.-L. Chai, F. Oort, L. Fu and T. Katsura for their very motivative discussions with me. I also wish to thank the Morningside Center of Mathematics (Beijing), the Center of Theoretical Research (Hsinchu), Max-Planck-Institut für Mathematik (Bonn) and the University of Tokyo (Tokyo) for the stimulating environments of mathematical research. Some of the following results were gotten during my visit to these institutions.

1. Push-out in a Category

Let $\mathfrak{C}$ be a category. For any two morphisms $f : X \to T$, $g : Y \to T$ in $\mathfrak{C}$, if there is an object Z of $\mathfrak{C}$ together with two morphisms $p : Z \to X$ and $q : Z \to Y$ satisfying

a) The following diagram is commutative:

$$
\begin{array}{ccc}
Z & \xrightarrow{p} & X \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\
Y & \xrightarrow{g} & T
\end{array} \tag{1}
$$

b) If there is another object Z' together with morphisms $p' : Z' \to X$, $q' : Z' \to Y$ such that $f \circ p' = g \circ q'$, then there is a unique morphism $\phi : Z' \to Z$ such that $p' = p \circ \phi$ and $q' = q \circ \phi$,

then we say Z (or (1), or (Z, p, q)) is a (categorical) *pull-back* of f and g in $\mathfrak{C}$.

The main background of this concept is in geometry: A category of some geometric objects (e.g. the category of topological spaces, the category of differentiable manifolds, the category of complex analytic spaces, the category of schemes, or the category of algebraic stacks, etc.) usually has a pull-back for any two morphisms $f : X \to T$ and $g : Y \to T$, which is the fiber product $X \times_T Y$ in the category. For this reason, we also call the pull-back Z in (1) (in an arbitrary category $\mathfrak{C}$) a *fiber product* of X and Y over T, denoted by $X \times_T Y$ (or more exactly $X\,{}_f\!\times_g Y$), and denote $\mathrm{pr}_1 = p$, $\mathrm{pr}_2 = q$, called the first and second projection respectively.

We often use the "language of presheaves" for the definition of fiber product. For any object $X \in \mathrm{Ob}(\mathfrak{C})$, denote by $\underline{X} = \mathrm{Mor}_{\mathfrak{C}}(\cdot, X) : \mathfrak{C}^{\mathrm{op}} \to$((sets)) the contravariant functor from $\mathfrak{C}$ to ((sets)) sending any $S \in \mathrm{Ob}(\mathfrak{C})$ to $\mathrm{Mor}_{\mathfrak{C}}(S, X)$. (If $\mathfrak{C}$ is a category of some geometric objects, then a morphism $S \to X$ can be understood as an "S-point" of X.) Then conditions a) and b) can be restated as a natural equivalence $\underline{Z} \cong \underline{X}\,{}_{\underline{f}}\!\times_{\underline{g}} \underline{Y}$, i.e. a morphism $Z' \to Z$ is equivalent to two morphisms $p' : Z' \to X$, $q' : Z' \to Y$ such that $f \circ p' = g \circ q'$. Hence we can restate the definition of fiber product as

$$
\underline{X \times_T Y} = \underline{X\,{}_f\!\times_g Y} \cong \underline{X}\,{}_{\underline{f}}\!\times_{\underline{g}} \underline{Y} = \underline{X} \times_{\underline{T}} \underline{Y} \tag{2}
$$

In the following we will often use notation like this, for example, if $x \in \mathrm{Mor}(S, X)$ and $y \in \mathrm{Mor}(S, Y)$ satisfy $f \circ x = g \circ y$, we can denote $(x, y) \in \mathrm{Mor}(S, X \times_T Y)$ the morphism corresponding to $(x, y) \in \mathrm{Mor}(S, X) \times \mathrm{Mor}(S, T)$; and for any morphism $\phi : X \to X'$ and any S-point $x \in \mathrm{Mor}(S, X)$, we can denote $\phi(x) = \phi \circ x$.

Push-out is the dual concept of pull-back.

Definition 1. Let $p : Z \to X$, $q : Z \to Y$ be two morphisms in a category $\mathfrak{C}$. If there is an object T of $\mathfrak{C}$ together with two morphisms $f : X \to T$ and $g : Y \to T$ satisfying

a) The following diagram is commutative:

$$\begin{array}{ccc} Z & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & T \end{array} \tag{3}$$

b) If there is another object T' together with morphisms $f' : X \to T'$, $g' : Y \to T'$ such that $f' \circ p = g' \circ q$, then there is a unique morphism $\psi : T \to T'$ such that $f' = \psi \circ f$ and $g' = \psi \circ g$,

then we say T (or (3), or (T, f, g)) is a (categorical) *push-out* of p and q in $\mathfrak{C}$.

Unlike pull-back, in a category $\mathfrak{C}$ of some geometric objects, a push-out usually may not exist for two general morphisms $p : Z \to X$ and $q : Z \to Y$, as we will see below. But in many cases we are interested to see whether a push-out exists, and push-out is a useful tool.

A special case is the quotient of a groupoid. Let $\mathfrak{C}$ be a category having products and fiber products. By a *groupoid* in $\mathfrak{C}$ we mean morphisms $p_1, p_2 : X_1 \to X_0$ and $q_1, q_2, q_3 : X_2 \to X_1$ satisfying the following conditions:

i) ("reflexivity") The diagonal morphism $\Delta : X_0 \to X_0 \times X_0$ factors through $(p_1, p_2) : X_1 \to X_0 \times X_0$;

ii) ("symmetricity") There is an automorphism ϕ of X_1 such that $(p_1, p_2) \circ \phi = (p_2, p_1) : X_1 \to X_0 \times X_0$;

iii) ("transitivity") $p_1 \circ q_1 = p_1 \circ q_2$, $p_2 \circ q_2 = p_2 \circ q_3$, $(q_1, q_2) : X_2 \to X_1 \times_{X_0} X_1$ and $(q_2, q_3) : X_2 \to X_1 \times_{X_0} X_1$ are epimorphisms, and there is an automorphism ψ of X_2 such that $q_{12} \circ \psi = q_{23} : X_2 \to X_0 \times X_0 \times X_0$, where q_{12} is the composition of (q_1, q_2) and $(p_1, p_2)_{p_1}\times_{p_1} (p_1, p_2) : X_{1p_1}\times_{p_1} X_1 \to (X_0 \times X_0)_{p_1}\times_{p_1} (X_0 \times X_0) \cong X_0 \times X_0 \times X_0$ (where $(X_0 \times X_0)_{p_1}\times_{p_1} (X_0 \times X_0) \cong X_0 \times X_0 \times X_0$ is given by $((x_1, x_2), (x_1, x_3)) \mapsto (x_1, x_2, x_3))$, and q_{23} is the composition of (q_2, q_3) and $(p_1, p_2)_{p_2}\times_{p_1} (p_1, p_2) : X_{1p_2}\times_{p_1} X_1 \to (X_0 \times X_0)_{p_2}\times_{p_1} (X_0 \times X_0) \cong X_0 \times X_0 \times X_0$ (where $(X_0 \times X_0)_{p_2}\times_{p_1} (X_0 \times X_0) \cong X_0 \times X_0 \times X_0$ is given by $((x_1, x_2), (x_2, x_3)) \mapsto (x_1, x_2, x_3))$.

Intuitively, for any object U, these conditions give an eqivalence relation on $\mathrm{Mor}(U, X_0)$. The case of a group object acting on an object is a basic example of groupoid (see Example 1 below). For the groupoid, if there is a morphism $p : X_0 \to Y$ such that

i) $p \circ p_1 = p \circ p_2$;

ii) If $p' : X_0 \to Y'$ is a morphism such that $p' \circ p_1 = p' \circ p_2$, then there is a unique morphism $h : Y \to Y'$ such that $p' = h \circ p$,

then we say Y (or p) is a *quotient* of the groupoid, and denote $Y = X_0/X_1$. Note that for any two morphisms $g, h : X_0 \to Y'$ such that $g \circ p_1 = h \circ p_2 : X_1 \to Y'$, by reflexivity we have $g = h$. Hence a quotient of the groupoid is a push-out of p_1 and p_2. In other words, quotient of a groupoid is a special case of push-out. However, we often use quotients of groupoids to construct push-out (see Theorem 3.1, Theorem 4.2 and Corollary 4.2).

We often consider the case when (q_1, q_2) and $(q_2, q_3) : X_2 \to X_1 \times_{X_0} X_1$ are isomorphisms. In this case condition iii) of groupoid can be simplified as:

> There is an automorphism ψ of $X_1 \times_{X_0} X_1$ such that $q_{12} \circ \psi = q_{23} : X_1 \times_{X_0} X_1 \to X_0 \times X_0 \times X_0$, where $q_{12} = (p_1, p_2)_{p_1}\times_{p_1} (p_1, p_2)$ and $q_{23} = (p_1, p_2)_{p_2}\times_{p_1} (p_1, p_2)$.

Furthermore, we often consider the case when $(p_1, p_2) : X_1 \to X_0 \times X_0$ is a monomorphism, hence q_{12} and q_{23} are monomorphisms (note that for any two morphisms $f : X \to T$ and $g : Y \to T$, if f is a monomorphism, then $\mathrm{pr}_2 : X \times_T Y \to Y$ is a monomorphism). In this case conditions i-iii) can be written as:

i′) $\Delta(X_0) \subset X_1 \subset X_0 \times X_0$;

ii′) $(p_2, p_1)(X_1) = X_1 \subset X_0 \times X_0$;

iii′) $X_{1\,p_1}\!\times_{p_1} X_1 = X_{1\,p_2}\!\times_{p_1} X_1 \subset X_0 \times X_0 \times X_0$.

and we say X_1 is an *equivalence relation* on X_0.

The following two facts are obvious.

Lemma 1. *Let* (3) *be a push-out in a category* $\mathfrak{C}$.

i) *If* $\psi : Z' \to Z$ *is an epimorphism, then*

$$\begin{array}{ccc} Z' & \xrightarrow{p\circ\psi} & X \\ {\scriptstyle q\circ\psi}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & T \end{array}$$

is also a push-out.

ii) *If* f *and* g *have a pull-back in* $\mathfrak{C}$ *(i.e. there is a fiber product* $X \times_T Y$*), then the following diagram*

$$\begin{array}{ccc} X \times_T Y & \xrightarrow{\mathrm{pr}_1} & X \\ {\scriptstyle \mathrm{pr}_2}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & T \end{array}$$

is also a push-out.

In the following we assume $\mathfrak{C}$ has fiber products (i.e. any two morphisms $f : X \to T$ and $g : Y \to T$ in $\mathfrak{C}$ have a pull-back).

Definition 2. Let $\mathfrak{C}$ be a category having fiber products. A push-out (3) is called *geometric* if the induced morphism $Z \to X \times_T Y$ is an epimorphism. Furthermore, a push-out (3) is called *universal* if for any morphism $\tau : T' \to T$, the following diagram is also a push-out:

$$\begin{array}{ccc} Z \times_T T' & \xrightarrow{p\times_T\tau} & X \times_T T' \\ {\scriptstyle q\times_T\tau}\big\downarrow & & \big\downarrow{\scriptstyle f\times_T\tau} \\ Y \times_T T' & \xrightarrow{g\times_T\tau} & T' \end{array}$$

Remark 1. When $\mathfrak{C}$ is the category of schemes, after this section we will replace "epimorphism" by "strong epimorphism" in Definition 2 (see the beginning of §2).

Note that if (3) is a push-out, then

$$\begin{array}{ccc} X \times_T Y & \xrightarrow{\mathrm{pr}_1} & X \\ {\scriptstyle \mathrm{pr}_2}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ Y & \xrightarrow{g} & T \end{array}$$

is a geometric push-out by Lemma 1.

We use the same terminologies for groupoids (this is slightly different from the terminologies in [13]). Intuitively, a quotient $p : X_0 \to X_0/X_1$ of a groupoid is geometric means that for any object U, the fibers of $p_* : \mathrm{Mor}(U, X_0) \to \mathrm{Mor}(U, Y)$ are just the equivalence classes in $\mathrm{Mor}(U, X_0)$.

Example 1. Let $\mathfrak{C}$ be a category having fiber products with a terminal object S (hence $\mathfrak{C}$ has products which are just fiber products over S). A *group object* in $\mathfrak{C}$ is an object G together with morphisms $m : G \times G \to G$ ("multiplication"), $o : S \to G$ ("unit section") and $\iota : G \to G$ ("inverse") such that

i) $m \circ (\mathrm{id}_G \times m) = m \circ (m \times \mathrm{id}_G) : G \times G \times G \to G$ ("associativity");

ii) $m \circ (o \times \mathrm{id}_G) = m \circ (\mathrm{id}_G \times o) = \mathrm{id}_G : G \cong S \times G \to G$;

iii) $m \circ (\iota \times \mathrm{id}_G) \circ \Delta = m \circ (\mathrm{id}_G \times \iota) \circ \Delta = 0 : G \to G$, where $\Delta : G \to G \times G$ is the diagonal morphism, and $0 : G \to G$ is the composition of o and the unique morphism $G \to S$.

Let G be a group object in $\mathfrak{C}$ and X be an object in $\mathfrak{C}$. An *action* of G on X is a morphism $\rho : G \times X \to X$ satisfying

i) $\rho \circ (\mathrm{id}_G \times \rho) = \rho \circ (m \times \mathrm{id}_X) : G \times G \times X \to X$;

ii) $\rho \circ (o \times \mathrm{id}_X) = \mathrm{id}_X : X \cong S \times X \to X$.

In this case, it is easy to check that $\rho, \mathrm{pr}_2 : X_1 = G \times X \to X = X_0$ can be extended to a groupoid. If this groupoid has a quotient Y (i.e. Y is a push-out of ρ and pr_2), we call Y a *quotient of* ρ, and denote $Y = X/\rho$ (or $Y = X/G$ if there is no confusion).

If ρ is the multiplication action of $G = \mathbb{C}^\times$ on $X = \mathbb{C}$, then ρ has two orbits $\mathbb{C}^\times$ and $\{0\}$. In the category of sets, ρ has a quotient which has two elements. But in the category of $\mathbb{C}$-schemes (or in the category of Hausdorff spaces), ρ has a quotient which has just one point, and it is not a geometric quotient.

This shows that a categorical quotient depends on the choice of the category.

Example 2. Let $\mathfrak{C}$ be the category of schemes over an algebraically closed field k. Let $X = \mathbb{A}_k^2 - \{(0,0)\}$. In $X \times_k X$ let x, y (resp. x', y') be the coordinates of the first (resp. second) copy of X. Let $W \subset X \times_k X$ be the union of two subschemes $\{x = x', y = y'\}$ and $\{x = x' = 0\}$. It is not hard to check that W is an equivalence relation on X (i.e. the line $x = 0$ is an equivalence class, while every other equivalence class consists of one single point).

Though there is a set-theoretic quotient of X modulo this equivalence relation, there is no (categorical) quotient X/W in the category of schemes. To show this, first let $R = k[x, xy^r | r = 1, 2, ...]$ (which is not noetherian), $Y' = \mathrm{Spec}R$, $f : X \to Y'$ be the morphism induced by the inclusion $R \hookrightarrow k[x, y]$, and $g : Y' \to \mathbb{A}_k^2$ be the morphism induced by the inclusion $k[x, xy] \subset R$ (note that g induces a topological isomorphism from Y' to $g(Y') = g(f(X))$). Then one checks that the induced morphism $X \to Y'$ equilizes pr_1 and $\mathrm{pr}_2 : W \to X$. Suppose there is a quotient $Y = X/W$. Then f factors through Y, and one can check (locally, point by point, or use Remark 4.2 below) that $Y \to Y'$ is an isomorphism. On the other hand, the morphism $X \to \mathbb{P}_k^1$ given by $(x, y) \mapsto (x : y)$ also equilizes pr_1 and $\mathrm{pr}_2 : W \to X$, hence there is an induced morphism $\phi : Y \to \mathbb{P}_k^1$ sending $f(x, y)$ to $(x : y)$. Therefore for any point $P \in \mathbb{P}_k^1$ we have $(0, 0) \in \phi^{-1}(P)$, this is absurd. In fact we have shown that there is no categorical push-out for pr_1 and $\mathrm{pr}_2 : W \to X$.

If we take $k = \mathbb{C}$, the above argument can also show that the equivalence relation W has no quotient in the category of topological spaces.

Lemma 2. *Suppose the push-out* (3) *is also a pull-back, i.e.* $Z \xrightarrow{\simeq} X \times_T Y$*. Then we have*

$$Z \times_X Z \times_Y Z = Z \times_Y Z \times_X Z \subset X \times X \times Y \times Y \tag{4}$$

where $Z \times_X Z \times_Y Z \to X \times X \times Y \times Y$ *is given by* $((x,y),(x,y'),(x',y')) \mapsto (x,x',y,y')$*, and* $Z \times_Y Z \times_X Z \to X \times X \times Y \times Y$ *is given by* $((x,y),(x',y),(x',y')) \mapsto (x,x',y,y')$*.*

This is because $Z \times_X Z \times_Y Z$ and $Z \times_Y Z \times_X Z$ are both equal to $X \times_T X \times_T Y \times_T Y$ in $X \times X \times Y \times Y$. In the language of presheaves, (4) can be expressed as: For any object U and any $x, x' \in \underline{X}(U)$ and $y, y' \in \underline{Y}(U)$, if $(x,y),(x,y'),(x',y') \in \underline{Z}(U)$, then $(x',y) \in \underline{Z}(U)$.

Conversely, given a monomorphism $Z \to X \times Y$, it is often the case that (4) is a sufficient or almost a sufficient condition for the existence of a (geometric) push-out (3). We now explain this.

Suppose in $\mathfrak{C}$ there are given some morphisms called *fine morphisms*, which satisfy the following conditions:

A) The fine morphisms are epimorphisms; any isomorphism is fine.

B) Let $f : X \to T$ be a fine morphism and $g : Y \to T$ be a morphism. Then $\mathrm{pr}_2 : X \times_T Y \to Y$ is a fine morphism. Furthermore, g is a monomorphism (resp. epimorphism, isomorphism) iff $\mathrm{pr}_1 : X \times_T Y \to X$ is so.

C) For any two morphisms $f : X \to Y$ and $g : Y \to Z$, if two of f, g, $g \circ f$ are fine, then the third is fine.

D) If $f : X \to T$ and $g : Y \to T$ are fine morphisms, then T is a push-out of $\mathrm{pr}_1 : X \times_T Y \to X$ and $\mathrm{pr}_2 : X \times_T Y \to Y$.

E) Let $Z \subset X \times X$ be an equivalence relation. If $\mathrm{pr}_1 : Z \to X$ is fine, then there is a universal geometric quotient X/Z, and the projection $X \to X/Z$ is also fine.

For example, if $\mathfrak{C}$ is the category of quasi-projective schemes over a noetherian base, we can define fine morphisms to be faithfully flat projective morphisms (see Remark 3.2). It is also not hard to show that if $\mathfrak{C}$ is the category of differentiable manifolds, or the category of complex analytic spaces, we can define fine morphisms to be smooth proper epimorphisms.

Theorem 1. *Let $\mathfrak{C}$ be a category having products and fiber products. Suppose fine morphisms are defined in $\mathfrak{C}$ satisfying conditions* A-E). *Let $p : Z \to X$, $q : Z \to Y$ be fine morphisms in $\mathfrak{C}$ such that $(p, q) : Z \to X \times Y$ is a monomorphism. If* (4) *holds, then there is a universal geometric push-out T of p and q. Furthermore, $X \to T$ and $Y \to T$ are also fine.*

Theorem 1 shows that (4) is a fundamental condition for the existence of push-out. For the proof of Theorem 1, one can follow the argument in the proof of Theorem 3.1 step by step. We omit the details here, because in the proof of Theorem 3.1 one can see the key ideas more intuitively.

In the following sections we only consider the category of schemes.

2. A Criterion of Push-out of Schemes

For the category of schemes, we use terminologies "push-out", "groupoid", "equivalence relation" and "quotient" in the same sense as that in §1. For convenience we also use the following terminologies: A morphism of schemes $f : X \to Y$ is called a *strong epimorphism* if f is set-theoretically onto and $f^{\#} : O_Y \to f_* O_X$ is a monomorphism. For example, a faithfully flat morphism is strongly epimorphic. It is easy to see that a strong epimorphism is an epimorphism in the category of schemes (but the converse is not true). A push-out (3) will be called *geometric* if $Z \to X \times_T Y$ is a strong epimorphism (this is different from Definition 1.2, see Remark 1.1), and this terminology is used for quotients of groupoids, equivalence relations or group scheme actions in the same sense. The definition of *universal* push-out is the same as that in Definition 1.2.

The following proposition is a criterion of push-out in the category of schemes.

Proposition 1. *Let S be a noetherian scheme, $f : X \to S$ and $g : Y \to S$ be morphisms of finite type, where g is faithfully flat and f is strongly epimorphic. Then S is a geometric push-out of* $\mathrm{pr}_1 : X \times_S Y \to X$ *and* $\mathrm{pr}_2 : X \times_S Y \to Y$, *which is a universal geometric push-out if f is also flat.*

PROOF. Let $f' : X \to S'$ and $g' : Y \to S'$ be morphisms such that $f' \circ \mathrm{pr}_1 = g' \circ \mathrm{pr}_2$. For any $y \in Y$, let $s = g(y)$, then there exists $x \in X$ such that $f(x) = s$ (since f is set-theoretically onto). For any $y' \in g^{-1}(s)$ there exists $z \in X \times_S Y$ such that $\mathrm{pr}_1(z) = x$ and $\mathrm{pr}_2(z) = y'$, hence all of the points in $g^{-1}(s)$ map to the same point $s' = g'(y) \in S'$ under $f' \circ \mathrm{pr}_1$; similarly all of the points in $f^{-1}(s)$ map to s' under $g' \circ \mathrm{pr}_2$. Therefore there is a (unique) induced map of sets $\phi : S \to S'$ satisfying set-theoretic equations $\phi \circ g = g'$ and $\phi \circ f = f'$. Since g is an open map (of topological spaces), we see ϕ is a continuous map of topological spaces.

Let $U' = \mathrm{Spec}R' \subset S'$ be an affine open subscheme. For any $s \in \phi^{-1}(U')$ and any $y \in Y$ such that $g(y) = s$, there exists an affine open neighborhood $U = \mathrm{Spec}R \subset \phi^{-1}(U')$ of s and an affine open neighborhood $W = \mathrm{Spec}B \subset g^{-1}(U)$ of y such that $g(W) = U$. Note that B is faithfully flat over R, hence $g^* : R \to B$ is monomorphic and $M = B/g^*(R)$ is a flat R-module. Let $V = f^{-1}(U)$ and $A = O_X(V)$, then $f^* : R \to A$ is a monomorphism because f is strongly epimorphic. Since $R \to B$ is flat, we have $O_{X\times_S Y}(V \times_S W) \cong A \otimes_R B$. Now we have a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \to & R & \xrightarrow{g^*} & B & \xrightarrow{\lambda} & M & \to & 0 \\
 & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle \eta} & & \\
0 & \to & A & \xrightarrow{p^*} & A \otimes_R B & \xrightarrow{\mu} & A \otimes_R M & \to & 0
\end{array} \tag{5}
$$

Since η is a monomorphism and $\eta \circ \lambda \circ g'^* = \mu \circ q^* \circ g'^* = \mu \circ p^* \circ f'^* = 0 : R' \to A \otimes_R M$, we have $\lambda \circ g'^* = 0 : R' \to M$, hence there is a (unique) induced homomorphism $\phi^* : R' \to R$ such that $f^* \circ \phi^* = f'^*$ and $g^* \circ \phi^* = g'^*$. Therefore we can define ϕ as a morphism and we have equations of morphisms $\phi \circ f = f'$, $\phi \circ g = g'$.

The last statement holds because faithful flatness is preserved under base change. □

Example 1. Let ρ be the action of $G = O_2(\mathbb{C})$ on $X = \mathbb{A}^2_{\mathbb{C}} = \mathrm{Spec}\mathbb{C}[x, y]$. Let W be the image of $(\rho, \mathrm{pr}_2) : G \times_{\mathbb{C}} X \to X \times_{\mathbb{C}} X$, and $p : X \to Y = \mathbb{A}^1_{\mathbb{C}}$ be induced by $\mathbb{C}[x^2 + y^2] \subset \mathbb{C}[x, y]$. Then one checks that Y is a categorical quotient of W but not a geometric quotient (p maps three ρ-orbits $\{x = \sqrt{-1}y\}$, $\{x = -\sqrt{-1}y\}$ and $\{(0, 0)\}$ to one point). In fact W

has no geometric quotient (because a geometric quotient is isomorphic to the categorical quotient).

Let $X' = X - \{(0,0)\}$ and, by abuse of notation, denote the restriction of W on X' still by W. Let $U_1, U_2 \subset X$ be open subschemes defined by $\{x \neq \sqrt{-1}y\}$, $\{x \neq -\sqrt{-1}y\}$ respectively. Then one checks by Proposition 1 that $U_1/W \cong Y$ and $U_2/W \cong Y$, both being geometric quotients. Hence by Proposition 1, there is a geometric quotient X'/W which is isomorphic to the affine line with original doubled. Note that this is not a separated scheme over $\mathbb{C}$, and W is not a closed subscheme of $X \times_{\mathbb{C}} X$.

Note that for any morphisms $X \to T$, $Y \to T$ and $T \to S$, if $T \to S$ is separated, then $X \times_T Y$ is a closed subscheme of $X \times_S Y$. Hence if $W \subset X \times_S Y$ is not a closed subscheme but $\mathrm{pr}_1 : W \to X$ and $\mathrm{pr}_2 : W \to Y$ have a geometric push-out T, then T is not separated over S.

Example 2. Let S be a noetherian scheme, X be an S-scheme of finite type and G be a group scheme of finite type over S. An action $\rho : G \times_S X \to X$ of G on X is called *transitive* if $(\rho, \mathrm{pr}_2) : G \times_S X \to X \times_S X$ is a strong epimorphism. If ρ has a geometric quotient X/G which is a locally closed subscheme of S, then ρ is transitive by definition; on the other hand, if $X \to S$ is faithfully flat and ρ is transitive, then there is a geometric quotient $X/G \cong S$ by Proposition 1 and Lemma 1.1.

Example 3. For a noetherian scheme X, denote by $O_{X_{\mathrm{et}}}$ the presheaf in the étale topology on X shch that $O_{X_{\mathrm{et}}}(U) = O_U$ for any étale morphism $U \to X$. It is well known that $O_{X_{\mathrm{et}}}$ is an étale sheaf, this means that for any étale morphism $U \to X$ and any two étale covers $q_1 : U_1 \to U$, $q_2 : U_2 \to U$, if $a_1 \in \Gamma(O_{U_1})$, $a_2 \in \Gamma(O_{U_2})$ satisfy $\mathrm{pr}_1^*(a_1) = \mathrm{pr}_2^*(a_2) \in \Gamma(O_{U_1 \times_U U_2})$, then there is a unique section $a \in \Gamma(O_U)$ such that $q_1^*(a) = a_1$, $q_2^*(a) = a_2$.

This fact can be easily shown using Proposition 1. Indeed, it can be reduced to the case when q_1 and q_2 are of finite type, note that a global section of O_U is equivalent to a morphism $U \to \mathbb{A}^1$, hence the problem is equivalent to that for any two morphisms $f_1 : U_1 \to \mathbb{A}^1$, $f_2 : U_2 \to \mathbb{A}^1$ such that $f_1 \circ \mathrm{pr}_1 = f_2 \circ \mathrm{pr}_2 : U_1 \times_U U_2 \to \mathbb{A}^1$, there is a unique morphism $f : U \to \mathbb{A}^1$ such that $f_1 = q_1 \circ f$, $f_2 = q_2 \circ f$. This is clear because U is a push-out of $\mathrm{pr}_1 : U_1 \times_U U_2 \to U_1$ and $\mathrm{pr}_2 : U_1 \times_U U_2 \to U_2$, by Proposition 1.

Example 4. Let $\mathcal{A}_{g,d,n}$ be the moduli scheme of abelian varieties of genus (= dimension) g with a polarization of degree d^2 and a level n-structure. Note that $\mathcal{A}_{g,d,n}$ is defined over $\mathbb{Z}[\frac{1}{n}]$, and is a fine moduli scheme when $n \geq 3$. In particular $\mathcal{A}_{g,d,1} = \mathcal{A}_{g,d}$, the coarse moduli scheme of abelian varieties of genus g with a polarization of degree d^2. If $n|m$, then there is a canonical projection $p_{m,n} : \mathcal{A}_{g,d,m} \to \mathcal{A}_{g,d,n}$ (i.e. a level m-structure gives a level n-structure). Let $m, n > 2$ be coprime integers, then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{A}_{g,d,mn} \otimes \mathbb{Z}[\frac{1}{mn}] & \xrightarrow{p_{mn,n}} & \mathcal{A}_{g,d,n} \otimes \mathbb{Z}[\frac{1}{mn}] \\ \Big\downarrow{\scriptstyle p_{mn,m}} & & \Big\downarrow{\scriptstyle p_{n,1}} \\ \mathcal{A}_{g,d,m} \otimes \mathbb{Z}[\frac{1}{mn}] & \xrightarrow{p_{m,1}} & \mathcal{A}_{g,d} \otimes \mathbb{Z}[\frac{1}{mn}] \end{array} \tag{6}$$

Let S be a noetherian scheme over $\mathbb{Z}[\frac{1}{mn}]$, and X be an abelian scheme of genus g over S with a polarization of degree d^2. Then we can take an étale cover $e_m : S_m \to S$ (resp. $e_n : S_n \to S$) such that $X \times_S S_m$ has a level m-structure (resp. $X \times_S S_n$ has a level n-structure). Let $S_{mn} = S_m \times_S S_n$, then on $X \times_S S_{mn}$ there are an induced level m-structure and an induced level n-structure, which together give a level mn-structure. Denote by $q_1 : S_{mn} \to S_m$ and $q_2 : S_{mn} \to S_n$ the projections respectively. By Proposition 1, S is a push-out of q_1 and q_2.

There are induced morphisms $f_m : S_m \to \mathcal{A}_{g,d,m} \otimes \mathbb{Z}[\frac{1}{mn}]$, $f_n : S_n \to \mathcal{A}_{g,d,n} \otimes \mathbb{Z}[\frac{1}{mn}]$ and $f_{mn} : S_{mn} \to \mathcal{A}_{g,d,mn} \otimes \mathbb{Z}[\frac{1}{mn}]$ satisfying $p_{mn,m} \circ f_{mn} = f_m \circ q_1$ and $p_{mn,n} \circ f_{mn} = f_n \circ q_2$. If T is a scheme and $g_1 : \mathcal{A}_{g,d,m} \otimes \mathbb{Z}[\frac{1}{mn}] \to T$, $g_2 : \mathcal{A}_{g,d,n} \otimes \mathbb{Z}[\frac{1}{mn}] \to T$ are morphisms satisfying $g_1 \circ p_{mn,m} = g_2 \circ p_{mn,n}$, then there is a commutative diagram

$$\begin{array}{ccc} S_{mn} & \xrightarrow{q_2} & S_n \\ \Big\downarrow{\scriptstyle q_1} & & \Big\downarrow{\scriptstyle g_2 \circ f_n} \\ S_m & \xrightarrow{g_1 \circ f_m} & T \end{array} \tag{7}$$

Hence there is a unique induced morphism $S \to T$. This shows that there is a natural transformation from the functor

$$S \mapsto \{\text{abelian schemes of genus } g \text{ over } S \text{ with a polarization of degree } d^2\}$$

to $\underline{T}$. Since $\mathcal{A}_{g,d}$ is a coarse moduli scheme, there is a unique morphism $\phi : \mathcal{A}_{g,d} \otimes \mathbb{Z}[\frac{1}{mn}] \to T$ such that $\phi \circ p_{m,1} = g_1$ and $\phi \circ p_{n,1} = g_2$. This shows that (6) is a push-out.

3. The Existence of Push-out: Flat Projective Case

For convenience we will use the following terminology: A morphism $X \to S$ is called *relatively projective* (resp. *relatively quasi-projective*) if X is a closed (resp. locally closed) subscheme of $\mathbb{P}(\mathcal{E})$ for some coherent sheaf $\mathcal{E}$ on S. In this case we usually fix a tautological invertable sheaf $O_X(1)$, especially when we talk about a Hilbert polynomial.

The following Lemma can be found in [1, p.276], here we give a proof with some different ideas which will be used below.

Lemma 1. *Let S be a noetherian scheme, $X \to S$ be a relatively quasi-projective morphism and $W \subset X \times_S X$ be a locally closed subscheme which is an equivalence relation of X over S. Suppose that $\mathrm{pr}_1 : W \to X$ is proper and flat. Then there is a universal geometric quotient $Y = X/W$ which is relatively quasi-projective (and relatively projective if $X \to S$ is so) over S. Furthermore, $X \to Y$ is faithfully flat and relatively projective.*

Proof. Let X_1 be a connected component of X and $W_1 = \mathrm{pr}_1^{-1}(X_1) \subset W$. Let $X_2 = \mathrm{pr}_2(W_1)$ (this makes sense since pr_2 is proper). Then by the definition of equivalence relation (see §1), we see that $\mathrm{pr}_1^{-1}(X_2) = W \cap (X_2 \times_S X_2)$ which gives a restriction of W on X_2. Hence for simplicity we may assume $X = X_2$.

Take a relatively projective scheme $\bar{X} \to S$ which contains X as an open dense subscheme such that $i^{\#} : O_{\bar{X}} \to i_* O_X$ is a monomorphism (where $i : X \to \bar{X}$ is the inclusion), and fix an $O_{\bar{X}}(1)$. Then W is a closed subscheme of $\bar{X} \times_S X$ since $W \to X$ is proper. The above assumption guarantees that all of the fibers of $\mathrm{pr}_2 : W \to X$ have one and the same Hilbert polynomial, say χ. Let $\mathcal{H} = \mathcal{H}ilb^{\chi}_{\bar{X}/S}$ be the Hilbert scheme representing the following functor:

$$
\begin{aligned}
&((S\text{-schemes})) \to ((\text{sets})) \\
&\qquad T \mapsto \{\text{closed subschemes of } \bar{X} \times_S T, \text{ flat over } T \text{ with } \chi \text{ as} \\
&\qquad\qquad \text{the Hilbert polynomial of each fiber over } T\}
\end{aligned}
$$

and $\mathcal{Z} \subset \bar{X} \times_S \mathcal{H}$ be the universal subscheme. Then W induces an S-morphism $\phi : X \to \mathcal{H}$ such that $W = \mathcal{Z} \times_{\mathcal{H}} X$ (as closed subschemes of $\bar{X} \times_S X$). Let $\bar{Y} \subset \mathcal{H}$ be the closed subscheme defined by the ideal sheaf $\ker(O_{\mathcal{H}} \to \phi_* O_X)$, and $\bar{Z} = \mathcal{Z} \times_{\mathcal{H}} \bar{Y} \subset \bar{X} \times_S \bar{Y}$. Then $W = \bar{Z} \times_{\bar{Y}} X$. Denote by $q : X \to \bar{Y}$, $p_1 : \bar{Z} \to \bar{X}$, $p_2 : \bar{Z} \to \bar{Y}$ and $p : W \to \bar{Z}$ the projections. Then p_2 is faithfully flat, hence p is dominant and $p^{\#} : O_{\bar{Z}} \to p_* O_W$ is a monomorphism. Note that $\mathrm{pr}_1 = p_1 \circ p : W \to \bar{X}$, hence p_1 is onto (because it is dominant and proper), thus strongly epimnorphic.

The pull-back of $\bar{Z} \to \bar{Y}$ and $q \circ \mathrm{pr}_2 : W \to \bar{Y}$ is equal to the pull-back W_{22} of pr_2 and pr_2, while the pull-back of $\bar{Z} \to \bar{Y}$ and $q \circ \mathrm{pr}_1 : W \to \bar{Y}$ is equal to the pull-back W_{12} of pr_1 and pr_2. Since $W_{12} = W_{22}$ (condition iii$'$) in the definition of equivalence relation), by the universality of $\mathcal{H}$ we have $q \circ \mathrm{pr}_1 = q \circ \mathrm{pr}_2 : W \to \bar{Y}$. Hence

$$q \circ p_1 \circ p = q \circ \mathrm{pr}_1 = q \circ \mathrm{pr}_2 = p_2 \circ p \tag{8}$$

Let $Z = p_1^{-1}(X)$. By abuse of notation we still denote by $p_1 : Z \to X$, $p_2 : Z \to \bar{Y}$ and $p : W \to Z$ the projections. Note that $p_1 \circ p : W \to Z \to X$ is proper, hence $p : W \to Z$ is also proper. Since $O_{\bar{Y}} \to q_* O_X$ is a monomorphism by the definition of $\bar{Y}$, and $p_2 : Z \to \bar{Y}$ is flat, we see $O_Z \to \mathrm{pr}_{1*} O_W$ is a monomorphism, hence $p : W \to Z$ is a strong epimorphism, therefore (8) gives

$$q \circ p_1 = p_2 : Z \to \bar{Y} \tag{9}$$

Hence $p : W \cong Z \times_{\bar{Y}} X \to Z$ has a section. Since $p_2 : \bar{Z} \to \bar{Y}$ is an open map we can define $Y = p_2(Z) = q(X)$ which is an open dense subscheme of $\bar{Y}$. Furthermore we have $W \subset X \times_Y X$, hence in $X \times_S X \times_S X$ we have

$$\begin{aligned} Z \times_Y W \subset Z \times_Y (X \times_Y X) &= W \times_Y X = W \times_Z (Z \times_Y X) \\ &= W \times_Z W \subset W \times_X W = Z \times_Y W \end{aligned}$$

i.e. $Z \times_Y W = Z \times_Y (X \times_Y X)$, thus $W = X \times_Y X$ because Z is faithfully flat over Y. Therefore

$$Z \times_Y W = W_{21} = W_{12} = W \times_Y Z = X \times_Y X \times_Y Z = X \times_Y W \tag{10}$$

Now (10)$\times_W Z$ (via the section $Z \to W$ of p) gives

$$Z \times_Y Z \cong X \times_Y Z \tag{11}$$

hence $p_1 : Z \to X$ is an isomorphism because Z is faithfully flat over Y.

Finally, let $Z' = p_2^{-1}(Y)$, then $W = Z' \times_Y X$, hence W is faithfully flat over Z'. This shows $Z' = Z \cong X$, hence $q : X \to Y$ is proper and faithfully flat. We see that Y is a universal geometric quotient X/W by Proposition 2.1. $\square$

Remark 1. In Lemma 1, it is essential to assume that $W \to X$ is both proper and flat. However, without the assumption of flatness, at least W has a "rational quotient" when X is reduced. To explain this, we need to use the following terminologies.

Definition 1. Let $X \to S$ be a scheme and $W \subset X \times_S X$ be a locally closed subscheme. We say W is a *rational equivalence relation* of X over S if

i) ("reflexivity") $\Delta(X) \subset W$;

ii) ("symmetricity") Let $\iota : X \times_S X \to X \times_S X$ be the morphism by switching factors, then $\iota(W) = W$;

iii) ("rational transitivity") Let W_{ij} ($i, j = 1$ or 2) be the pull-back of $\mathrm{pr}_i : W \to X$ and $\mathrm{pr}_j : W \to X$, viewed as a subscheme of $X \times_S X \times_S X$ ($W_{11} = W_{\mathrm{pr}_1} \times_{\mathrm{pr}_1} W \to X \times_S X \times_S X$ is given by $((x_1, x_2), (x_1, x_3)) \mapsto (x_1, x_2, x_3)$, and $W_{21} = W_{\mathrm{pr}_2} \times_{\mathrm{pr}_1} W \to X \times_S X \times_S X$ is given by $((x_1, x_2), (x_2, x_3)) \mapsto (x_1, x_2, x_3)$, etc.), then there is an open dense subscheme $U' \subset X$ such that $W_{11} \cap \mathrm{pr}_1^{-1}(U') = W_{21} \cap \mathrm{pr}_1^{-1}(U')$.

(If $U' = X$, this is just the definition of equivalence relation.) For a rational equivalence relation W, if pr_1 and $\mathrm{pr}_2 : W \to X$ have a push-out Y, then we still call Y a *quotient* of X with respect to W, and denote $Y = X/W$. Furthermore, a rational map $p : X \dashrightarrow Y$ over S is called a *rational quotient* of W if there is an open dense subscheme $U \subset Y$ and an open dense subscheme $V \subset X$ such that p can be defined as a morphism $V \to U$, $W' = W \cap (V \times_S V) \subset V \times_U V$, and U is a push-out of pr_1 and $\mathrm{pr}_2 : W' \to V$ (cf. [5]).

More generally, Let $p : Z \to X$, $q : Z \to Y$ be two morphisms of schemes. If there is a scheme T and open dense subschemes $X' \subset X$, $Y' \subset Y$ together with morphisms $f : X' \to T$, $g : Y' \to T$ such that $p^{-1}(X') = q^{-1}(Y')$ and

T is a push-out of $p|_{p^{-1}(X')} : p^{-1}(X') \to X'$ and $q|_{p^{-1}(X')} : p^{-1}(X') \to Y'$, then we say T is a *rational push-out* of p and q.

COROLLARY 1. *Let S be a noetherian scheme, X be a reduced scheme, $X \to S$ be a relatively quasi-projective morphism and $W \subset X \times_S X$ be a locally closed subscheme defining a rational equivalence relation. Suppose that $\mathrm{pr}_1 : W \to X$ is generically proper. Then there is an open dense subscheme $X' \subset X$ such that the rational equivalence relation of X' given by $W' = W \cap X' \times_S X'$ has a universal geometric quotient. Furthermore, $X'/W' \to S$ is relatively quasi-projective and $X' \to X'/W'$ is faithfully flat and relatively quasi-projective.*

PROOF. For simplicity we may assume $X \to S$ is relatively projective and W is closed. Hence there is an open dense subscheme $X' \subset X$ such that $W \cap (X \times_S X')$ is relatively projective and flat over X'. As in the proof of Lemma 1, we may assume that under a choice of $O_X(1)$, this induces a morphism of finite type $\phi : X' \to \mathcal{H} = \mathcal{H}ilb^{\chi}_{X/S}$ for some Hilbert polynomial χ. By further shrinking X' we may assume ϕ maps X' onto a locally closed subscheme $Y' \subset \mathcal{H}$. Let $W' = W \cap (X' \times_S X')$. Then by the proof of Lemma 1 we can see $W' = X' \times_{Y'} X'$ and $X' \to Y'$ is faithfully flat, hence Y' is a universal geometric quotient X'/W' by Proposition 2.1. $\square$

Therefore under the conditions of Corollary 1, there exists a rational quotient of X with respect to W.

THEOREM 1. *Let S be a noetherian scheme, $X \to S$ and $Y \to S$ be relatively quasi-projective morphisms and $W \subset X \times_S Y$ be a locally closed subscheme satisfying*

$$W \times_X W \times_Y W = W \times_Y W \times_X W \subset X \times_S X \times_S Y \times_S Y \tag{12}$$

(i.e. equation (4) in this case). Suppose that $\mathrm{pr}_1 : W \to X$ and $\mathrm{pr}_2 : W \to Y$ are faithfully flat and proper. Then there is a universal geometric push-out Z of pr_1 and pr_2 which is relatively quasi-projective (and relatively projective if $X \to S$ and $Y \to S$ are so) over S. Furthermore, $X \to Z$ and $Y \to Z$ are faithfully flat and relatively projective.

PROOF. Let $X_1 = W \times_X W$, viewed as a (relatively projective) scheme over $Y_2 = Y \times_S Y$. Then X_1 gives an equivalence relation of W over S and

$W/X_1 \cong X$, by Proposition 2.1. Let $W_1 = X_1 \times_Y W$. Then it is easy to see (by the language of presheaves) that

$$W_1 = X_1 \times_{Y_2} X_1 \tag{13}$$

Indeed, for any S-scheme U, a morphism $U \to W_1$ is equivalent to 3 morphisms $(x,y),(x,y'),(x',y') \in \mathrm{Mor}_S(U,W)$ (where $x,x' \in \mathrm{Mor}_S(U,X)$ and $y,y' \in \mathrm{Mor}_S(U,Y)$), while a morphism $U \to X_1 \times_{Y_2} X_1$ is equivalent to 4 morphisms $(x,y),(x,y'),(x',y'),(x',y) \in \mathrm{Mor}_S(U,W)$ (see §1). Hence W_1 gives an equivalence relation on X_1 over Y_2. Furthermore, $\mathrm{pr}_1 : W_1 \to X_1$ is faithfully flat and relatively projective, hence by Lemma 1 there is a universal geometric quotient $Y_3 = X_1/W_1$, and $X_1 \to Y_3$ is faithfully flat and relatively projective. By (13) we have

$$X_1 \times_{Y_3} X_1 = W_1 = X_1 \times_{Y_3} (Y_3 \times_{Y_2} X_1)$$

hence $X_1 \cong Y_3 \times_{Y_2} X_1$ because X_1 is faithfully flat over Y_3. By the same reason we see $\Delta : Y_3 \to Y_3 \times_{Y_2} Y_3$ is an isomorphism, hence $Y_3 \to Y_2$ is a closed immersion (a proper morphism $T \to T'$ is a closed immersion iff $\Delta : T \to T \times_{T'} T$ is an isomorphism). Furthermore, the following commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\mathrm{pr}_2} & W \\ \downarrow & & \downarrow \\ Y_3 & \xrightarrow{\mathrm{pr}_2} & Y \end{array} \tag{14}$$

is a pull-back, i.e. $X_1 \cong Y_3 \times_Y W$, this comes from

$$X_1 \times_{Y_3} X_1 \cong W_1 = X_1 \times_Y W \cong X_1 \times_{Y_3} (Y_3 \times_Y W) \tag{15}$$

(and that $X_1 \to Y_3$ is faithfully flat). Hence $\mathrm{pr}_2 : Y_3 \to Y$ is faithfully flat and relatively projective (since the composition $Y_3 \hookrightarrow Y_2 \to Y$ is relatively quasi-projective and proper).

We now show that Y_3 gives an equivalence relation of Y over S. The commutativity is obvious. For the reflexivity, note that the composition

$$W \xrightarrow{\Delta} W \times_X W = X_1 \to Y_3 \to Y_2 \tag{16}$$

is equal to the composition $W \to Y \xrightarrow{\Delta} Y_2$, where $W \to Y$ is a strong epimorphism. For the transitivity, we need to show $Y_{311} = Y_{312} \subset Y \times_S Y \times_S Y$, where Y_{3ij} $(i, j = 1$ or $2)$ is the pull-back of $\mathrm{pr}_i : Y_3 \to Y$ and $\mathrm{pr}_j : Y_3 \to Y$. It is enough to show that $X_1 \times_{Y_3} Y_{311} = X_1 \times_{Y_3} Y_{312}$ in $X \times_S Y \times_S Y \times_S Y$. But by (14) we have

$$X_1 \times_{Y_3} Y_{311} \cong X_1 \times_Y Y_3 \cong W \times_X W \times_Y Y_3 \cong W \times_X X_1 \cong W \times_X W \times_X W$$

similarly we have $X_1 \times_{Y_3} Y_{312} \cong W \times_X W \times_X W$, hence we get an isomorphism $X_1 \times_{Y_3} Y_{311} \to X_1 \times_{Y_3} Y_{312}$ compatible with the inclusions to $X \times_S Y \times_S Y \times_S Y$.

By Lemma 1, there is a universal geometric quotient $Z = Y/Y_3$ which is relatively quasi-projective over S, and $Y \to Z$ is faithfully flat and relatively projective. By (14) we see that $W \to Z$ equilizes pr_1 and $\mathrm{pr}_2 : X_1 \to W$, hence $W \to Z$ factors through $W/X_1 \cong X$. Since $W \to X$ and $W \to Z$ are both faithfully flat and relatively projective, we see that $X \to Z$ is faithfully flat and relatively projective. Furthermore,

$$\begin{aligned} W \times_X (X \times_Z Y) &\cong W \times_Z Y \cong W \times_Y (Y \times_Z Y) \\ &\cong W \times_Y Y_3 \cong X_1 \cong W \times_X W \end{aligned} \tag{17}$$

Hence $W \to X \times_Z Y$ is an isomorphism. The statements then follow by Proposition 2.1. □

Remark 2. If we define a fine morphism to be a faithfully flat proper morphism, then the conditions A-E) in §1 obviously hold for the category of relatively quasi-projective schemes over a noetherian base S (with strong epimorphism instead of epimorphism). One checks that these are all of the conditions we need in the proof of Theorem 1, hence it is not hard to change the argument of Theorem 1 to a proof of Theorem 1.1.

4. The Existence of Push-out: Finite Case

In this section we study push-out of finite morphisms. We will not need any assumption of flatness. The following proposition plays the role of Proposition 2.1 for the finite case.

Proposition 1. *Let S be a noetherian normal scheme. Let $f : X \to S$, $g : Y \to S$ be finite strong epimorphisms which map associated points to* *generic points. Then the following diagram is a push-out.*

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{\mathrm{pr}_1} & X \\ {\scriptstyle \mathrm{pr}_2}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ Y & \xrightarrow{g} & S \end{array}$$

Proof. We need only prove the case when S is affine, and then the general case can be shown by the functoriality of pull-back. For simplicity we may assume S is integral. Let $S = \mathrm{Spec}R$, $X = \mathrm{Spec}A$, $Y = \mathrm{Spec}B$, and let $K = \mathrm{q.f.}(R)$. By assumption, the associated primes of A (resp. B) all lie over the zero ideal of R, hence $A \hookrightarrow A \otimes_R K$ (resp. $B \hookrightarrow B \otimes_R K$). This implies that the canonical homomorphisms $A \to A \otimes_R B$ and $B \to A \otimes_R B$ are injective. Thus A and B can be viewed as subrings of $A \otimes_R B$ containig R. Define $\phi : A \oplus B \to A \otimes_R B$ by $\phi(a, b) = a \otimes_R 1 - 1 \otimes_R b$, and let $R' = \ker(\phi)$. Then R' can be viewed as $A \cap B$ in $A \otimes_R B$, hence is an R-subalgebra of A, also a finitely generated R-module of A. Now $A \otimes_R K$ and $B \otimes_R K$ can be viewed as K-subalgebras of $A \otimes_R B \otimes_R K$, and obviously $A \otimes_R K \cap B \otimes_R K = K$ in $A \otimes_R B \otimes_R K$, hence $R' \subset K$. Since R' is integral over R and R is integrally closed, we have $R = R'$, hence S is a push-out of $\mathrm{pr}_1 : X \times_S Y \to X$ and $\mathrm{pr}_2 : X \times_S Y \to Y$. $\Box$

The following theorem gives a sufficient condition of the existence of a quotient in finite case.

Theorem 1. *Let X be a normal scheme which is separated of finite type over a noetherian scheme S, and $W \subset X \times_S X$ be a closed subscheme which is a rational equivalence relation on X such that $\mathrm{pr}_1 : W \to X$ is finite. Suppose that*

A) *$\mathrm{pr}_1 : W \to X$ maps generic points to generic points;*

B) *any point $x \in X$ has an open affine neighborhood $U \subset X$ such that $\mathrm{pr}_2(\mathrm{pr}_1^{-1}(U)) = U$;*

C) *either W is reduced or X is of positive characteristic (i.e. for each irreducible component V of X, it holds that $\mathrm{ch}(K(V)) > 0$),*

then there is a quotient $Y = X/W$ and

i) *Y is of finite type over S and $X \to Y$ is finite;*

ii) *for any open subscheme $U \subset Y$, $W_U = W \times_Y U$ gives a rational equivalence relation on $X_U = X \times_Y U$ which has a quotient $X_U/W_U \cong U$;*

iii) *W gives a set-theoretical equivalence relation on X, under which the fibers of $X \to Y$ are just the equivalence classes in X (as sets);*

iv) *if W has no embedded points, then Y is normal.*

v) *There is an open dense subscheme $Y' \subset Y$ whose inverse image $X' \subset X$ is flat over Y, and $W \times_Y Y' = X' \times_Y X' \subset X \times_S X$ which is an equivalence relation on X' (hence $Y' = X'/(W \times_Y Y')$ is a universal geometric quotient).*

PROOF. By B) and the separatedness of $X \to S$, it is enough to prove the case when X and S are affine (then by ii) we can deduce the general case). Let $S = \mathrm{Spec}R$, $X = \mathrm{Spec}A$ and $W = \mathrm{Spec}B$. For simplicity we can also assume that all of the generic points of X are in one and the same equivalence class (since X is normal). Let $X_1 = \mathrm{Spec}A_1$, ..., $X_n = \mathrm{Spec}A_n$ be the irreducible components of X and $W_1, ..., W_m$ be the irreducible components of W with reduced induced scheme structure. By Corollary 3.1, W has a rational quotient $X \dashrightarrow Y'$. Let $K = K(Y')$.

Case 1. W is reduced. In this case each $K(X_i)$ is separable over K because $K(X_i) \otimes_K K(X_i)$ is reduced. Hence for any W_i, $W_i \to \mathrm{pr}_1(W_i)$ is generically separable, so $K(W_i)$ is separable over K also. Take a finite Galois extension $L \supset K$ such that every $K(W_i)$ is isomorphic to an intermediate field of $L \supset K$ over K. For each i, take a K-homomorphism $K(X_i) \to L$ and let $\tilde{A}_i$ be the integral closure of A_i in L. The induced morphisms $\tilde{X}_i = \mathrm{Spec}(\tilde{A}_i) \to X_i$ $(1 \le i \le n)$ together give a strong epimorphism $q : \tilde{X} \to X$, where $\tilde{X} = \mathrm{Spec}\tilde{A}$ is the disjoint union of all $\tilde{X}_i$'s. Let $\tilde{W} = \mathrm{Spec}\tilde{B}$ be the union of irreducible components of $\tilde{X} \times_X W \times_X \tilde{X}$ whose generic points map to the generic points of $\tilde{X}$ under pr_1, with reduced induced scheme structure. Then $\tilde{W} \subset \tilde{X} \times_S \tilde{X}$ gives a rational equivalence relation on $\tilde{X}$ with rational quotient Y'.

Let $\tilde{W}_1 = \mathrm{Spec}(\tilde{B}_1), ..., \tilde{W}_r = \mathrm{Spec}(\tilde{B}_r)$ be the irreducible components of $\tilde{W}$ with reduced induced scheme structure. Then by our choice of L, we have $K(\tilde{W}_i) \cong L$ $(1 \le i \le r)$. For any i, j $(1 \le i, j \le n)$, there is a $\tilde{W}_t$ such that $\mathrm{pr}_1(\tilde{W}_t) = \tilde{X}_i$ and $\mathrm{pr}_2(\tilde{W}_t) = \tilde{X}_j$. Since $\tilde{A}_i$ is integrally closed and $\tilde{B}_t$ is integral over $\tilde{A}_i$, by $K(\tilde{X}_i) \cong L \cong K(\tilde{W}_t)$ we see $\tilde{A}_i \to \tilde{B}_t$ is an isomorphism, i.e. $\mathrm{pr}_1 : \tilde{W}_t \to \tilde{X}_i$ is an isomorphism. By the same reason $\mathrm{pr}_2 : \tilde{W}_t \to \tilde{X}_j$ is an isomorphism, hence there is an S-isomorphism $\sigma = \mathrm{pr}_2 \circ \mathrm{pr}_1^{-1} : \tilde{X}_i \to \tilde{X}_j$. Let Σ be the set of such isomorphisms (for all i, j). For any $\sigma \in \mathrm{Iso}_S(\tilde{X}_i, \tilde{X}_j) \cap \Sigma$ and any $\sigma' \in \mathrm{Iso}_S(\tilde{X}_j, \tilde{X}_k) \cap \Sigma$, we have $\sigma' \circ \sigma \in \Sigma$ since $\tilde{W}$ gives a rational equivalence relatioin on $\tilde{X}$. Also $\mathrm{id}_{\tilde{X}_i} \in \Sigma$ for each i. Hence $G_i = \mathrm{Aut}_S(X_i) \cap \Sigma$ is a finite group. By checking at the generic fibers of $\tilde{W} \dashrightarrow Y'$, we see that $G_i \cong \mathrm{Gal}(L/K)$. Furthermore, for any j and any $\sigma \in \mathrm{Iso}_S(\tilde{X}_i, \tilde{X}_j) \cap \Sigma$, we have $G_j = \sigma G_i \sigma^{-1}$. Let $C = \tilde{A}_i^{G_i}$ (which is integrally closed in K) and $Y = \mathrm{Spec} C \cong \tilde{X}_i / G_i$. Then $Y \cong \tilde{X}_j / G_j$ for each j, and Y' can be identified with an open subscheme of Y. Since $\mathrm{Gal}(L/K(X_j)) \subset \mathrm{Gal}(L/K)$, each $\tilde{X}_j \to Y$ factors through X_j, hence we have an induced morphism $X \to Y$ which coincides with $X \dashrightarrow Y'$. Note that we have an exact sequence of R-modules

$$0 \to C \to \tilde{A} \xrightarrow{\mathrm{pr}_1^* - \mathrm{pr}_2^*} \tilde{B} \tag{18}$$

This gives an exact sequence of R-modules

$$0 \to C \to A \xrightarrow{\mathrm{pr}_1^* - \mathrm{pr}_2^*} B \tag{19}$$

because $A \to \tilde{A}$ and $B \to \tilde{B}$ are monomorphisms. Since $\tilde{A}$ is integral over C and A is a finitely generated R-algebra, we see X is finite over Y, hence there is a finitely generated R-subalgebra $C' \subset C$ such that X is finite over $\mathrm{Spec} C'$. Since C' is noetherian and A is a finitely generated C'-module, we see C is a finitely generated C'-module, hence $Y \to S$ is of finite type.

We now show that Y is a quotient of W. Let $\phi : X \to Z$ be an S-morphism which equilizes pr_1 and $\mathrm{pr}_2 : W \to X$, then $\psi = \phi \circ q : \tilde{X} \to Z$ equilizes pr_1 and $\mathrm{pr}_2 : \tilde{W} \to \tilde{X}$. Hence for any open affine subscheme $U \subset Z$ and any $x \in \psi^{-1}(U) \cap \tilde{X}_i$, we have $G_i x \subset \psi^{-1}(U)$. Let $I \subset \tilde{A}_i$ be the ideal defining $\tilde{X}_i - \psi^{-1}(U)$. Let $p_1, ..., p_s \subset \tilde{A}_i$ be the prime ideals corresponding to the points of $G_i x$. Then for each j $(1 \le j \le u)$, there is an element $a_j \in I - p_i$. Since $p_j \not\supseteq p_k$ for any $k \ne j$, we can take a_j such that $a_j \in p_k \forall k \ne j$. Then $a = \sum_j a_j \in I - \cup_j p_j$. Let $b = \prod_{\sigma \in G_i} \sigma a$, then $b \in C$ and $G_i x \subset (\tilde{X}_i)_b \subset \psi^{-1}(U)$. By (18) we have an exact sequence

$$0 \to C_b \to \tilde{A}_b \xrightarrow{\mathrm{pr}_1^* - \mathrm{pr}_2^*} \tilde{B}_b \tag{20}$$

hence $(\tilde{X}_i)_b \to U$ factors uniquely through Y_b. From this we see that ϕ factors uniquely through Y. Thus i) is proved. This also shows ii).

When W has no embedded points, the non-zero elements of C are non-zero divisors in B, hence C is integrally closed. This shows iv).

Note that $\tilde{W}$ gives an equivalence relation on $\tilde{X}$, and $\tilde{X}/\tilde{W} \cong Y$ is a geometric (and hence set-theoretic) quotient. Furthermore, the points in each fiber of $\tilde{X} \to X$ are in one and the same equivalence class, hence $\tilde{W}$ induces a set-theoretical equivalence relation $\sim$ on X, and $X/\sim\, \cong Y$ as sets. It is easy to see that $\sim$ is determined by W, i.e. for any $x, x' \in X$, $x \sim x'$ iff there exists $w \in W$ such that $\mathrm{pr}_1(w) = x$, $\mathrm{pr}_2(w) = x'$. This shows iii).

Case 2. X is of positive characteristic, for simplicity we may assume S is an $\mathbb{F}_p$-scheme for some prime number p. Basically we can use the argument in Case 1, hence we mainly write down the points which are different from Case 1. First, in this case $K(W_i)$ may not be separable over K, we can only take a finite normal extension $L \supset K$ such that every $K(W_i)$ is isomorphic to an intermediate field of $L \supset K$ over K. Then we can define $\tilde{X}$ and $\tilde{W}$ as in Case 1, but $\tilde{W}$ may not give a rational equivalence relation on $\tilde{X}$. We can still define the actions of $\mathrm{Gal}(L/K)$ on each $\tilde{A}_i$. Let $C_0 = \tilde{A}_i^{G_i}$, then (18) still holds, i.e. $C_0 = \ker(\mathrm{pr}_1^* - \mathrm{pr}_2^*)$. Here C_0 may not be in A_i, but for large enough N, $C_0^{p^N} \subset A_i$. Again define C by (19). Then $C \subset C_0$ as subrings of $\tilde{A}$. Note that in (19) $\mathrm{pr}_1^* - \mathrm{pr}_2^*$ maps $C_0^{p^N}$ into the nilradical of B, hence for N large enough we have $C_0^{p^N} \subset C$. Let $C_1 = R[C_0^{p^N}] \subset C$, i.e. the R-subalgebra of C generated by $C_0^{p^N}$. Then C_1 is a finitely generated R-algebra and C is a finitely generated C_1-module, hence is also a finitely generated R-algebra.

Let $Y = \mathrm{Spec}(C)$. Note that $Y \to \mathrm{Spec}(C_1)$ is set-theoretically one-to-one. We can use the argument in Case 1 to show that Y is a quotient X/W, and i-iii) hold (or see Remark 2 below).

When W has no embedded point, the associated primes of B all lie over the zero ideal of C. Let $K = \mathrm{q.f.}(C)$, then $B \to B \otimes_C K$ is a monomorphism. Therefore if $a \in C_0 \cap K$ then $a \in C$, hence C is integrally closed. Thus iv) holds also.

Finally, in either case, note that $W \to X$ is generically flat, hence it is easy to take an open dense subscheme $Y' \subset Y$ whose inverse image

$X' \subset X$ is flat over Y', such that $W \times_Y Y' \subset X \times_S X$ is flat over X' and is an equivalence relation on X'. Therefore by ii) and Corollary 3.1, we see $Y' = X'/(W \times_Y Y')$ is a universal geometric quotient. This shows v). $\square$

Remark 1. The conditions in Theorem 1 are almost necessary in the following sense: Let X be a normal scheme which is separated of finite type over a noetherian scheme S, and $W \subset X \times_S X$ be a closed subscheme which gives a rational equivalence relation on X such that $\mathrm{pr}_1 : W \to X$ is finite. Suppose there is a quotient $Y = X/W$ satisfying i) and ii) in Theorem 1. Let $W' = X \times_Y X$. Then by Corollary 3.1, W is a closed subscheme of W' which is generically equal to W', i.e. any irreducible component of W' which is not in W does not dominate any irreducible component of X. Furthermore, W' gives an equivalence relation on X which has a quotient $X/W' \cong Y$ (see Lemma 1.1). From this we see that B) holds. Furthermore, if Y is connected and $\mathrm{ch}(K(Y)) = 0$, then there is an open dense subscheme $U \subset Y$ such that $X \times_Y X \times_Y U$ is reduced, hence W is generically reduced, in other words condition C) holds in the rational sense. By removing the irreducible components of W which do not dominate any irreducible components of X, we get a closed subscheme $W_0 \subset W$ which also gives a rational equivalence relation on X, and there is a quotient $Y' = X/W_0$ which is finite and birational over Y. In particular, if Y is normal, then $Y' \cong Y$, hence in this case we may assume A) also holds.

Corollary 1. *Let* $X_1 \underset{p_2}{\overset{p_1}{\rightrightarrows}} X_0$ *be a groupoid which is separated of finite type over a noetherian scheme* S*, where* X_0 *is normal and* p_1 *is finite. Suppose that*

A) p_1 *maps generic points to generic points;*

B) *any point* $x \in X_0$ *has an open affine neighborhood* $U \subset X_0$ *such that* $p_2(p_1^{-1}(U)) = U$*;*

C) *either* X_1 *is reduced or* X_0 *is of positive characteristic,*

then there is a quotient $Y = X_0/X_1$ *which is of finite type over* S*, and* $X_0 \to Y$ *is finite. Furthermore, if* X_1 *is reduced then* Y *is normal.*

PROOF. Let $X = X_0$ and W be the image of $X_1 \overset{(p_1,p_2)}{\longrightarrow} X \times_S X$. Then W gives a rational equivalence relation on X (because $X_1 \to W$ is generically flat), hence we can apply Theorem 1 to get $Y = X/W$, and by Lemma 1.1 we see that Y is a quotient of X_0 by X_1. The other statements all come from Theorem 1. $\square$

Using Theorem 1 we can set up a criterion of the existence of push-out in the finite normal case, just like Theorem 3.1. Here the key condition is still (12), but in the "rational sense" (i.e. over an open dense subscheme of X).

THEOREM 2. *Let S be a noetherian scheme, X and Y be normal schemes of finite type over S, and $W \subset X \times_S Y$ be a closed subscheme satisfying the following condition:*

($*$) There is an open dense subscheme $U \subset X$ such that

$$W \times_X W \times_Y W \times_X U = W \times_Y W \times_X W \times_X U \subset X \times_S X \times_S Y \times_S Y$$

where $W \times_X W \times_Y W \times_X U \to X \times_S X \times_S Y \times_S Y$ is given by $((x, y), (x, y'), (x', y')) \mapsto (x, x', y, y')$, and $W \times_Y W \times_X W \times_X U \to X \times_S X \times_S Y \times_S Y$ is given by $((x, y), (x', y), (x', y')) \mapsto (x, x', y, y')$ (this can be viewed as a "rational version" of (4)*). Suppose that* $\mathrm{pr}_1 : W \to X$ *and* $\mathrm{pr}_2 : W \to Y$ *are finite strong epimorphisms, and that*

A) $\mathrm{pr}_1 : W \to X$ *and* $\mathrm{pr}_2 : W \to Y$ *map generic points to generic points;*

B) *for any point $x \in X$, there is an open affine neighborhood $U_1 \subset X$ of x and an open affine subscheme $U_2 \subset Y$ such that $\mathrm{pr}_1^{-1}(U_1) = \mathrm{pr}_2^{-1}(U_2) \subset W$;*

C) *either W is reduced or X is of positive characteristic,*

then there is a push-out Z of pr_1 *and* pr_2*. Furthermore,*

i) *Z is of finite type over S, and $X \to Z$, $Y \to Z$ are finite;*

ii) *any open subscheme $U \subset Z$ is a push-out of* $\mathrm{pr}_1 : W \times_Z U \to X \times_Z U$ *and* $\mathrm{pr}_2 : W \times_Z U \to Y \times_Z U$*;*

iii) *Z is a set-theoretic push-out of* pr_1 *and* pr_2*;*

iv) *if W has no embedded point (in particular if W is reduced), then Z is normal.*

v) *There is an open dense subscheme $Z' \subset Z$ whose inverse images $X' \subset X$ and $Y' \subset Y$ are flat over Z, such that $X' \subset U$ and $W \times_Z Z' = X' \times_Z Y' \subset X \times_S Y$ (hence Z' is a universal geometric push-out of* $\mathrm{pr}_1 : W \times_Z Z' \to X'$ *and* $\mathrm{pr}_2 : W \times_Z Z' \to Y'$*).*

Proof. We follow the steps in the proof of Theorem 3.1. For simplicity we only write down the details of the points which are different from that of Theorem 3.1. For convenience we use the following notation: for two closed subschemes V_1, V_2 of an X-scheme T, if there is an open dense subscheme $U \subset X$ such that $V_1 \times_X U = V_2 \times_X U$, then we denote $V_1 \approx V_2$ (for example $(*)$ can be written as $W \times_X W \times_Y W \approx W \times_Y W \times_X W$).

Let $X' = W \times_X W$, viewed as a finite scheme over $Y_2 = Y \times_S Y$. Let $X_1 \subset X'$ be the closed subscheme such that $X_1 \approx X'$ and the associated points of X_1 are all generic. Let $W_1 = X_1 \times_Y W$. Then by the proof of Theorem 3.1 we have $W_1 \approx X_1 \times_{Y_2} X_1$ (which gives a rational equivalence relation on X_1). Let $Y_3' = \mathrm{im}(X_1 \to Y_2)$ (which can be viewed as the rational quotient X_1/W_1). By discarding the irreducible components of Y_3' whose generic points do not lie over the generic points of Y, we get a closed subscheme $Y_3 \subset Y_3'$. By the proof of Theorem 3.1, we see that Y_3 gives a rational equivalence relation on Y over S, hence by Theorem 1 we have a quotient $Z' = Y/Y_3$.

Case 1. W has no embedded point. By Proposition 1 we have $X \cong W/X'$, hence $X \cong W/X_1$ (see Remark 1). Since $W \to Z'$ equilizes pr_1 and $\mathrm{pr}_2 : X_1 \to W$, we see $W \to Z'$ factors through X. Since $X_1 \to W$ is a strong epimorphism, we have the following commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\mathrm{pr}_1} & X \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow \\ Y & \longrightarrow & Z' \end{array} \tag{21}$$

Reducing to the affine case, let $X = \mathrm{Spec}A$, $Y = \mathrm{Spec}B$, $W = \mathrm{Spec}C$ and $Z' = \mathrm{Spec}D'$. For simplicity we may assume D' is a domain. Define

$\phi : A \oplus B \to C$ by $\phi(a,b) = \mathrm{pr}_1^*(a) - \mathrm{pr}_2^*(b)$. Let $D = \ker(\phi)$, then D can be viewed as a subring of C and is a finitely generated D'-module. Furthermore, by $D = A \cap B \subset C \otimes_D \mathrm{q.f.}(D)$ we see D is an integrally closed domain. Let $Z = \mathrm{Spec}D$ and let $f : X \to Z$, $g : Y \to Z$ be the induced morphisms. Then $f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2$. Noting that pr_1 and pr_2 are generically flat, it is easy to see v) holds by Theorem 1.v).

For iii), we use the trick in the proof of Theorem 1. For simplicity assume X, Y, W are affine as above, and $Z = \mathrm{Spec}D$ is integral. Take a finite normal extension $L \supset \mathrm{q.f.}(D)$ such that for each irreducible component $\mathrm{Spec}C \subset W \times_X W \times_Y W$, $\mathrm{q.f.}(C)$ can be embedded into L as an intermediate field of $L \supset \mathrm{q.f.}(D)$. For each irreducible component $\mathrm{Spec}A \subset X$ (resp. $\mathrm{Spec}B \subset Y$), take an integral closure $\tilde{A}$ of A (resp. $\tilde{B}$ of B) in L and replace $\mathrm{Spec}A$ (resp. $\mathrm{Spec}B$) by $\mathrm{Spec}\tilde{A}$ (resp. $\mathrm{Spec}\tilde{B}$), thus we replace X (resp. Y) by some $\tilde{X}$ (resp. $\tilde{Y}$), and the projection $\tilde{X} \to X$ (resp. $\tilde{Y} \to Y$) is a finite strong epimorphism. Replace W by the union $\tilde{W}$ of irreducible components of $\tilde{X} \times_X W \times_Y \tilde{Y}$ which maps onto Z (with reduced induced scheme structure). Then $\tilde{W}$ is isomorphic to a disjoint union of copies of $\mathrm{Spec}\tilde{A} \cong \mathrm{Spec}\tilde{B}$. By v) we see that for any two irreducible components $X_0 \subset \tilde{X}$ and $Y_0 \subset \tilde{Y}$, there is an irreducible component $W_0 \subset \tilde{W}$ lying over both X_0 and Y_0. For any $z \in Z$, $\mathrm{Gal}(L/\mathrm{q.f.}(D))$ acts transitively on the inverse image of z in X_0 (resp. Y_0). From this we see that Z is a set-theoretic push-out of $\mathrm{pr}_1 : \tilde{W} \to \tilde{X}$ and $\mathrm{pr}_2 : \tilde{W} \to \tilde{Y}$. This shows that for any $x \in X$, $y \in Y$ such that $f(x) = g(y)$, there is a point $w \in W$ such that $\mathrm{pr}_1(w) = x$, $\mathrm{pr}_2(w) = y$. Hence Z is a set-theoretic push-out of $\mathrm{pr}_1 : W \to X$ and $\mathrm{pr}_2 : W \to Y$.

We now show that Z is a push-out of pr_1 and pr_2. Let T be an S-scheme and $f' : X \to T$, $g' : Y \to T$ be S-morphisms such that $f' \circ \mathrm{pr}_1 = g' \circ \mathrm{pr}_2$. By iii), for any $z \in Z$, $f' \circ \mathrm{pr}_1$ maps $(f \circ \mathrm{pr}_1)^{-1}(z)$ to one point, say $t \in T$. Take an open affine neighborhood $U = \mathrm{Spec}R \subset T$ of t, then $V = Z - f \circ \mathrm{pr}_1(W - (f' \circ \mathrm{pr}_1)^{-1}(U))$ is an open neighborhood of z in Z. Take $d \in D$ such that $z \in \mathrm{Spec}D_d \subset V$, then f' and g' induce $X_d \to U$ and $Y_d \to U$ respectively. Note that $D_d = A_d \cap B_d$, hence $R \to C_d$ induces $R \to D_d$, i.e. a morphism $Z_d \to U$. Thus there is a unique morphism $\phi : Z \to T$ such that $\phi \circ f = f'$, $\phi \circ g = g'$. Therefore Z is a push-out of pr_1 and pr_2, and i), ii) and iv) hold.

Case 2. X is of positive characteristic. For simplicity we may assume

S is an $\mathbb{F}_p$-scheme for some prime number p. Basically we can use the argument in Case 1, for example we still define $D = \ker(\phi)$. To deal with the points different from that in Case 1, we can use the argument in the proof of Theorem 1, Case 2 (also see Remark 2 below). $\square$

Remark 2. Let $\mathfrak{C}$ be the category of **affine** schemes over S. Then for any $X, Y, W \in \mathrm{Ob}(\mathfrak{C})$, any two S-morphisms $p : W \to X$ and $q : W \to Y$ have a push-out in $\mathfrak{C}$. Indeed, we may assume S is affine, say $S = \mathrm{Spec}R$, $X = \mathrm{Spec}A$, $Y = \mathrm{Spec}B$, $W = \mathrm{Spec}C$, define $\phi : A \times B \to C$ by $\phi(a, b) = p^*(a) - q^*(b)$ and let $D = \ker(\phi)$, then it is easy to see that D is an R-subalgebra of $A \times B$. Let $Z = \mathrm{Spec}D$ and denote $f : X \to Z$, $g : Y \to Z$ the induced morphisms. It is easy to check that Z is a push-out of p and q in $\mathfrak{C}$: for any $Z' = \mathrm{Spec}D' \in \mathrm{Ob}(\mathfrak{C})$ and any S-morphisms $f' : X \to Z'$, $g' : Y \to Z'$ such that $f' \circ p = g' \circ q$, let $\psi : D' \to A \times B$ be the homomorphism $d \mapsto (f'^*(d), g'^*(d))$, then $\phi \circ \psi = 0$, hence there is a unique induced homomorphism $D' \to D$, i.e. an S-morphism $h : Z \to Z'$ such that $h \circ f = f'$, $h \circ g = g'$.

Furthermore, if p, q, f, g are set-theoretically surjective and Z is also a set-theoretic push-out of p and q with quotient topology, then it is a push-out of p and q in the category of S-schemes. Here Z is a set-theoretic push-out means that for any $z \in Z$ and any $x, x' \in f^{-1}(z)$, there are $w_1, ..., w_n \in W$ such that $\mathrm{pr}_1(w_1) = x$, $\mathrm{pr}_1(w_n) = x'$, and for each i $(1 \leq i < n)$, either $\mathrm{pr}_1(w_i) = \mathrm{pr}_1(w_{i+1})$ or $\mathrm{pr}_2(w_i) = \mathrm{pr}_2(w_{i+1})$. Indeed, for any S-scheme Z' and any S-morphisms $f' : X \to Z'$, $g' : Y \to Z'$ such that $f' \circ p = g' \circ q$, there is an induced continuous map $h : Z \to Z'$, and we can define $\psi : O_{Z'} \to f'_* O_X \times g'_* O_Y$ by $\psi(d) = (f'^*(d), g'^*(d))$. Note that ϕ induces $h_*\phi : f'_* O_X \times g'_* O_Y \to (f' \circ p)_* O_W$ and $h_*\phi \circ \psi = 0$, we see there is a unique induced homomorphism $O_{Z'} \to \ker(h_*\phi) = h_* O_Z$. Thus h can be viewed as a morphism of S-schemes, and $h \circ f = f'$, $h \circ g = g'$.

Corollary 2. *Let S be a noetherian scheme, X and Y be normal schemes of finite type over S, and $W \subset X \times_S Y$ be a closed subscheme satisfying the conditions in Theorem 2. Let $W' \subset W$ be a closed subscheme without embedded points such that* $\mathrm{pr}_1 : W' \to X$ *and* $\mathrm{pr}_2 : W' \to Y$ *are both strongly epimorphic and map generic points to generic points. Then there is a push-out Z of* $\mathrm{pr}_1 : W' \to X$ *and* $\mathrm{pr}_2 : W' \to Y$*. Furthermore,*

i) *Z is normal and is of finite type over S, and $X \to Z$, $Y \to Z$ are finite;*

ii) *any open subscheme $U \subset Z$ is a push-out of* $\mathrm{pr}_1 : W' \times_Z U \to X \times_Z U$ *and* $\mathrm{pr}_2 : W' \times_Z U \to Y \times_Z U$*;*

iii) *Z is a set-theoretic push-out of* pr_1 *and* pr_2*;*

iv) *There is an open dense subscheme $Z' \subset Z$ whose inverse images $X' \subset X$ and $Y' \subset Y$ are flat over Y, such that Z' is a universal geometric push-out of* $\mathrm{pr}_1 : W' \times_Z Z' \to X'$ *and* $\mathrm{pr}_2 : W' \times_Z Z' \to Y'$*).*

Proof. By Theorem 2, $\mathrm{pr}_1 : W \to X$ and $\mathrm{pr}_2 : W \to Y$ have a push-out S', and X, Y, W' are all finite S'-schemes. By Remark 2, $\mathrm{pr}_1 : W' \to X$ and $\mathrm{pr}_2 : W' \to Y$ have a push-out Z in the category of affine S'-schemes, and Z is normal, and the projections $f : X \to Z$ and $g : Y \to Z$ are finite morphisms. To show Z is a push-out in the category of S-schemes, it is enough to show that Z is also a set-theoretic push-out of $\mathrm{pr}_1 : W' \to X$ and $\mathrm{pr}_2 : W' \to Y$, by Remark 2.

Let Ξ be the set of generic points of X. For any $x, x' \in \Xi$, define $x \sim x'$ when there are generic points $w_1, ..., w_n \in W'$ such that $\mathrm{pr}_1(w_1) = x$, $\mathrm{pr}_1(w_n) = x'$, and for each i $(1 \leq i < n)$, either $\mathrm{pr}_1(w_i) = \mathrm{pr}_1(w_{i+1})$ or $\mathrm{pr}_2(w_i) = \mathrm{pr}_2(w_{i+1})$. It is easy to see "$\sim$" is an equivalence relation in Ξ. (Similarly we can define an equivalence relation "$\sim$" in the set Θ of generic points of Y.) For any generic point $z \in Z$, $f^{-1}(z) \subset \Xi$ is an equivalence class under $\sim$. Indeed, if $x \sim x' \in \Xi$, it is clear that $f(x) = f(x')$, hence $f^{-1}(z)$ is a union of equivalence classes. On the other hand, if $f^{-1}(z)$ contained an equivalence class Ξ_0 properly, say $f^{-1}(z) = \Xi_0 \coprod \Xi_1$, let $\Theta_0 = \mathrm{pr}_2(\mathrm{pr}_1^{-1}(\Xi_0)) \subset g^{-1}(z)$ and $\Theta_1 = \mathrm{pr}_2(\mathrm{pr}_1^{-1}(\Xi_1)) \subset g^{-1}(z)$, then $g^{-1}(z)$ would be a disjoint union of Θ_0 and Θ_1. Let $Z_0 \subset Z$ (resp. $X_0 \subset X$, $X_1 \subset X$, $Y_0 \subset Y$, $Y_1 \subset Y$) be the component with generic point z (resp. union of components with generic points in Ξ_0, Ξ_1, Θ_0, Θ_1), and let $W_0 = (f \circ \mathrm{pr}_1)^{-1}(Z_0) \subset W'$. Then it would be easy to see that $W_0 = (W_0 \cap (X_0 \times_S Y_0)) \cup (W_0 \cap (X_1 \times_S Y_1))$. Thus $Z_0 \cong (X_0 \cup X_1) \times_S (Y_0 \cup Y_1)/W_0$ would have at least two connected components, a contradiction.

Now we can use the argument in the proof of Theorem 2.iii) to show Z is a set-theoretic push-out of $\mathrm{pr}_1 : W' \to X$ and $\mathrm{pr}_2 : W' \to Y$. $\square$

Example 1. Let $X = \mathbb{A}^2_k$ for a field k, and

$$W = \Delta(X) \cup \{((0,0),(0,1)),((0,1),(0,0))\} \subset X \times_k X$$

Then W is an equivalence relation on X (two points (0,0) and (0,1) form an equivalence class under W, while every other equivalence class consists of a single point). By Remark 2, there is a quotient X/W in the category of affine schemes over k. It is not hard to see that X/W is a topology-theoretic quotient (by "pinching (0,0) and (0,1) together" in X). Hence X/W is a scheme-theoretic quotient by Remark 2. Of course X/W is not normal (in fact $X/W \cong \mathrm{Spec} k[x, xy, y^2 - y, y^3 - y]$).

Example 2. Let $X = Y = \mathbb{A}^2_{\mathbb{C}} - \{(0,0)\}$, and

$$W = \{x_1 = y_1, x_2 = y_2\} \cup \{x_1 = y_1 = 0, x_2 = 2y_2\} \subset X \times_{\mathbb{C}} Y$$

(where x_1, x_2 and y_1, y_2 are the coordinates of X and Y respectively). Then the conditions in Theorem 2 all hold except A). On the other hand, there is no push-out of $\mathrm{pr}_1 : W \to X$ and $\mathrm{pr}_2 : W \to Y$. Indeed, if there were such a push-out Z, then the set $T = \{(0, 2^n) | n \in \mathbb{Z}\} \subset X$ would map to one point in Z, hence the set $V = \{x_1 = 0\} \subset X$ would map to one point since T is Zariski dense in V. Therefore Z would be a quotient X/W', where $W' = \{x_1 = x_1', x_2 = x_2'\} \cup \{x_1 = x_1' = 0\}$ (x_1, x_2 (resp. x_1', x_2') being the coordinates of the first (resp. second) copy of X in $X \times_{\mathbb{C}} X$). This is impossible by Example 1.2.

Remark 3. In the case when $W \subset X \times_S Y$ is finite over X and Y, without the assumption that X, Y are normal, it is usually much harder to give a sufficient condition for the existence of a push-out. However, at least we have the following idea: Take finite strong epimorphisms $X' \to X$ and $Y' \to Y$, where X', Y' are normal. Then take a suitable closed subscheme $W' \subset X' \times_X W \times_Y Y'$, then it might be easier to determine if $W' \to X'$ and $W' \to Y'$ have a push-out, and this might be helpful to determine if $W \to X$ and $W \to Y$ have a push-out.

5. The Existence of Push-out: A Partially Flat Case

The following proposition slightly generalizes Theorem 3.1 (compare Proposition 2.1).

PROPOSITION 1. *Let S be a noetherian scheme, $X \to S$ and $Y \to S$ be relatively quasi-projective morphisms and $W \subset X \times_S Y$ be a locally closed subscheme satisfying* (12). *Suppose that* $\mathrm{pr}_1 : W \to X$ *and* $\mathrm{pr}_2 : W \to Y$ *are proper,* $\mathrm{pr}_1 : W \to X$ *is faithfully flat and* $\mathrm{pr}_2 : W \to Y$ *is a strong epimorphism. Then* $\mathrm{pr}_2 : W \to Y$ *factors through a strong epimorphism $W \to Z$, where $Z \to Y$ is finite and set-theoretically one to one, such that there is a geometric push-out*

$$\begin{array}{ccc} W & \xrightarrow{\mathrm{pr}_1} & X \\ \Big\downarrow{\scriptstyle \mathrm{pr}_2} & & \Big\downarrow \\ Z & \longrightarrow & T \end{array} \tag{24}$$

where T is relatively quasi-projective (and relatively projective if $X \to S$ and $Y \to S$ are so) over S, $X \to T$ and $Z \to T$ are relatively projective, $Z \to T$ is faithfully flat, and $X \to T$ is a strong epimorphism.

PROOF. Take a relatively projective scheme $\bar{Y} \to S$ which contains Y as an open dense subscheme such that $i^{\#} : O_{\bar{Y}} \to i_* O_Y$ is a monomorphism (where $i : Y \to \bar{Y}$ is the inclusion), and fix an $O_{\bar{Y}}(1)$. Then W is a closed subscheme of $X \times_S \bar{Y}$ since $W \to X$ is proper. Let $\mathcal{H} = \mathcal{H}ilb_{\bar{Y}/S}$ be the Hilbert scheme representing the following functor:

$$\begin{aligned} &((S\text{-schemes})) \to ((\text{sets})) \\ &\qquad S' \mapsto \{\text{closed subschemes of } \bar{Y} \times_S S', \text{ flat over } S'\} \end{aligned}$$

and $\mathcal{Z} \subset \bar{Y} \times_S \mathcal{H}$ be the universal subscheme. Then W induces an S-morphism $\phi : X \to \mathcal{H}$ such that $W = X \times_{\mathcal{H}} \mathcal{Z}$ (as closed subschemes of $X \times_S \bar{Y}$). Let $\bar{T} \subset \mathcal{H}$ be the closed subscheme defined by the ideal sheaf $\ker(O_{\mathcal{H}} \to \phi_* O_X)$, and $\bar{Z} = \mathcal{Z} \times_{\mathcal{H}} \bar{T} \subset \bar{Y} \times_S \bar{T}$. Then $W = X \times_{\bar{T}} \bar{Z}$. Denote by $q : X \to \bar{T}$, $p_1 : \bar{Z} \to \bar{Y}$, $p_2 : \bar{Z} \to \bar{T}$ and $p : W \to \bar{Z}$ the projections. Then p_2 is faithfully flat and $q^{\#} : O_{\bar{T}} \to q_* O_X$ is a monomorphism, hence $p^{\#} : O_{\bar{Z}} \to p_* O_W$ is also a monomorphism. Note that $\mathrm{pr}_2 = p_1 \circ p : W \to \bar{Y}$, hence p_1 is proper dominant and $p_1^{\#} : O_{\bar{Y}} \to p_{1*} O_{\bar{Z}}$ is a monomorphism, i.e. p_1 is a strong epimorphism. Let $Z = p_1^{-1}(Y)$, then $Z \to Y$ is relatively projective (because $\bar{Z} \to \bar{Y}$ is so), and $W \to Z$ is proper (because $W \to Y$ is proper), hence is a strong epimorphism. Since p_2 is faithfully flat, we can define the image $T = p_2(Z)$ which is an open subscheme of $\bar{T}$. Then $W \to T$

is a strong epimorphism. Since $W \to X$ is faithfully flat, we see $X \to \bar{T}$ factors through T, and $X \to T$ is onto, hence is a strong epimorphism also. Now $W = X \times_T p_2^{-1}(T)$, hence $W \to p_2^{-1}(T)$ is onto, so $p_2^{-1}(T) \subset p_1^{-1}(Y)$, i.e. $p_2^{-1}(T) = Z$. Therefore T is a geometric push-out of $W = X \times_T Z \to X$ and $W \to Z$, by Proposition 2.1.

The pull-back of $Z \to T$ and $q \circ \mathrm{pr}_1 : W \to \bar{T}$ is equal to $W \times_T Z \cong W \times_X (X \times_T Z) \cong W \times_X W$, hence the pull-back of $Z \to T$ and $q \circ \mathrm{pr}_1 \circ \mathrm{pr}_1 : W \times_Y W \to \bar{T}$ is equal to $W \times_X W \times_Y W$, and the pull-back of $Z \to T$ and $q \circ \mathrm{pr}_1 \circ \mathrm{pr}_2 : W \times_Y W \to \bar{T}$ is equal to $W \times_Y W \times_X W$. By (12) and abstract nonsense, we see the following diagram is commutative:

$$\begin{array}{ccc} W \times_Y W & \xrightarrow{\mathrm{pr}_1} & W \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow \\ W & \longrightarrow & T \end{array} \tag{25}$$

This induces a closed immersion $W \times_Y W \to W \times_T W$.

We now show that $W \times_Z W \to W \times_Y W$ is an isomorphism. Since $W \to X$ is faithfully flat, it is enough to show that $W \times_Z W \times_X W \cong W \times_Y W \times_X W$. Note that $W \times_Z W \times_X W \cong W \times_T W$, hence we have a closed immersion

$$\eta : W \times_Y W \times_X W \hookrightarrow W \times_T W \times_X W \xrightarrow{\cong} W \times_Z W \times_X W \times_X W \tag{26}$$

and $\mathrm{pr}_1 \circ \eta = \mathrm{pr}_1$, $\mathrm{pr}_3 \circ \eta = \mathrm{pr}_2$, $\mathrm{pr}_4 \circ \eta = \mathrm{pr}_3$: $W \times_Y W \times_X W \to W$. Furthermore, we have $\mathrm{pr}_2 \circ \eta = \mathrm{pr}_2$ because $\mathrm{pr}_1 \circ \mathrm{pr}_2 \circ \eta = \mathrm{pr}_1 \circ \mathrm{pr}_2$: $W \times_Y W \times_X W \to X$ and $\mathrm{pr}_2 \circ \mathrm{pr}_2 \circ \eta = \mathrm{pr}_2 \circ \mathrm{pr}_1 = \mathrm{pr}_2 \circ \mathrm{pr}_2$: $W \times_Y W \times_X W \to Y$. Thus we have a commutative diagram

$$\begin{array}{ccc} W \times_Y W \times_X W & \xrightarrow{\eta} & W \times_Z W \times_X W \times_X W \\ \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \mathrm{id}} \\ W \times_Z W \times_X W & \xrightarrow{\mathrm{id}_W \times_Z \Delta \times_X \mathrm{id}_W} & W \times_Z W \times_X W \times_X W \end{array} \tag{27}$$

where $\Delta : W \to W \times_X W$ is the diagonal morphism. Hence ζ is a closed immersion. It is easy to see that the composition of ζ with the closed immersion $W \times_Z W \times_X W \to W \times_Y W \times_X W$ is equal to the identity morphism of $W \times_Z W \times_X W$, hence ζ is an isomorphism.

For a closed point $t : \mathrm{Spec}(K) \hookrightarrow T$ (where K is a field), denote by W_t (resp. X_t, Z_t) the fiber of W (resp. X, Z) over t. Then $W_t \cong X_t \times_K Z_t$, and $W_t \times_Y W_t \subset W \times_Y W_t = W \times_Z W_t = W_t \times_Z W_t$, hence $W_t \times_Y W_t = W_t \times_Z W_t = W_t \times_{Z_t} W_t$. Note that the following commutative diagram

$$\begin{array}{ccc} W_t \times_{Z_t} W_t & \xrightarrow{\simeq} & W_t \times_Y W_t \\ \downarrow & & \downarrow \\ Z_t & \xrightarrow{\Delta} & Z_t \times_Y Z_t \end{array} \tag{28}$$

is a pull-back, and $W_t \times_Y W_t \to Z_t \times_Y Z_t$ is faithfully flat, we see $\Delta : Z_t \to Z_t \times_Y Z_t$ is an isomorphism, hence $Z_t \to Y$ is a closed immersion. This shows that $p_1 : Z \to Y$ is set-theoretically one to one, hence finite.

The other statements hold by Proposition 2.1. $\square$

Corollary 1. *Under the conditions of Proposition 1, if in addition that W is reduced and Y is normal, then $Z \cong Y$.*

Proof. The reducedness of W implies that Z and T are reduced. Let $Y_0 \subset Y$ be a connected component and let $Z_0 = p_1^{-1}(Y_0)$. Then we can take an open dense subscheme $U \subset Y_0$ such that $V = p_1^{-1}(U)$ is flat over U. By (28) we can see that $V \to V \times_U V$ is an isomorphism over U, hence $\deg(V/U) = 1$, i.e. $V \cong U$. This shows that $K(Z_0) \cong K(Y_0)$. But $Z \to Y$ is finite and Y is normal by assumption, we see $Z_0 \cong Y_0$. $\square$

Remark 1. From Corollary 1 we see that for two relatively quasi-projective S-schemes and a closed subscheme $W \subset X \times_S Y$, the following conditions are sufficient for the existence of a geometric push-out T of $\mathrm{pr}_1 : W \to X$ and $\mathrm{pr}_2 : W \to Y$:

i) (12) holds;

ii) $\mathrm{pr}_1 : W \to X$ and $\mathrm{pr}_2 : W \to Y$ are proper;

iii) $\mathrm{pr}_1 : W \to X$ is faithfully flat;

iv) $\mathrm{pr}_2 : W \to Y$ is a strong epimorphism;

v) W is reduced and Y is normal.

In this case the push-out T may not be universal, but any open subscheme $U \subset T$ is a push-out of $\mathrm{pr}_1 : W \times_T U \to X \times_T U$ and $\mathrm{pr}_2 : W \times_T U \to Y \times_T U$.

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(Received October 20, 2004)
(Revised July 26, 2007)

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