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Composition and Decomposition of Multidimensional Polynomial-Normal Distribution

By Monika Maj and Zbigniew Pasternak-Winiarski

Abstract. We show that the characteristic function of a d-dimensional polynomial-normal distribution PND_d has only PND_d factors.

1. Introduction

We say that the distribution of a real-valued random variable X is *polynomial-normal* (PND in short) if the density of X is given by the following formula

$$f_X(x) = p_{2l}(x) \exp\left(-\frac{(x-m)^2}{2c^2}\right),$$

where p_{2l} is a non-negative polynomial of degree $2l$ $(l \geq 0, c > 0)$. It is clear that any normal distribution is polynomial-normal ($l = 0$).

PNDs which are not normal are used in parametric statistical inference to describe the situation when the observed asymptotic behaviour of density f_X is similar to the asymptotic behaviour of a density of a normal distribution (eg. the situation that there exist constants $a > b > 0$ such that $\lim_{|x|\to\infty} f_X(x) \exp\left(\frac{x^2}{2a^2}\right) = 0$ and $\lim_{|x|\to\infty} f_X(x) \exp\left(\frac{x^2}{2b^2}\right) = +\infty$) but f_X is not a density of any normal distribution (eg. f_X has a local minimum – see [3]).

PNDs enjoy several properties of normal distributions (see [1], [3], [8] and [9]). One of the most important properties of the class of normal distributions is the following: *if $X = X_1 + X_2 + \ldots + X_n$, where $X_1, X_2, \ldots, X_n$ are independent random variables and X has a normal distribution, the distribution of every X_i $(i = 1, 2, \ldots, n)$ is normal* (it follows from Cramér's

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theorem – see [6]). E. Lukacs suggested in [7] that the class of PNDs has the same property. The complete proof of this fact in one-dimensional case was given by Plucińska in [9]. In the present paper we will show that this amazing theorem is also true for multidimensional polynomial-normal random variables (Theorem 13).

Following Cramér, Plucińska obtained the result mentioned above as a corollary from the following, more general composition and decomposition theorem.

THEOREM 1. *Let φ, φ_1 and φ_2 be characteristic functions of random variables X, X_1 and X_2 respectively, where X has a polynomial-normal distribution and X_1, X_2 are nondegenerate (the probability $P(X_i = 0) \neq 1$ for $i = 1, 2$). If $\varphi = \varphi_1\varphi_2$ then X_1 and X_2 have polynomial-normal distributions.*

We can express it shortly in the following form: *the characteristic function of a PND has only PND factors.*

In this paper we will consider d-dimensional polynomial-normal distribution (PND_d in short), with the density

$$f(x_1, ..., x_d) = p_{2l}(\mathbf{x}) \exp\left(-\frac{1}{2}\left(\mathbf{x}-\mathbf{b}\right)\mathbf{A}\left(\mathbf{x}-\mathbf{b}\right)'\right)$$

where p_{2l} is a non-negative polynomial which depend on d variables $\mathbf{x} = (x_1, x_2, ..., x_d)$ and $(\mathbf{x}-\mathbf{b})'$ denotes the transpose of a vector (matrix) $\mathbf{x} - \mathbf{b} \in \mathbf{R}^d$. We will prove the following generalization of Plucińska's theorem.

THEOREM 2. *The characteristic function of a PND_d has only PND_d factors.*

The proof of Theorem 1 given in [9] is based on famous Weierstrass and Hadamard decomposition theorems for entire function of one complex variable. In the proof of Theorem 2 we use Theorem 1 and the multidimensional generalization of Weierstrass theorem given by Stoll, Lelong and Ronkin (see [12], [4] or [11]).

2. Preliminaries

The main purpose of this section is to collect some facts related to multidimensional characteristic functions. At the beginning we give the definition of *the analytic characteristic function.*

DEFINITION 3. The characteristic function $\varphi_{\mathbf{Y}}$ of d-dimensional random variable $\mathbf{Y} = (Y_1, ..., Y_d)$ is analytic, if there exist a holomorphic function $\Phi_{\mathbf{Y}}$ on an open set $G \subset \mathbf{C}^d$ and a neighbourhood U of zero in $\mathbf{C}^d$ such that $U \subset G$ and

$$\Phi_{\mathbf{Y}|U\cap\mathbf{R}^d} = \varphi_{\mathbf{Y}|U\cap\mathbf{R}^d}, \tag{1}$$

where $\mathbf{R}^d = \left\{(t_1, ..., t_d) \in \mathbf{C}^d : t_j \in \mathbf{R} \text{ for } j = 1, 2, ..., d\right\}$.

The analytic characteristic function $\varphi_{\mathbf{Y}}$ is an entire characteristic function, when the function $\Phi_{\mathbf{Y}}$ is entire $\left(G = \mathbf{C}^d\right)$.

The function $\Phi_{\mathbf{Y}}$ is said to be a holomorphic extension of the function $\varphi_{\mathbf{Y}}$.

In the above definition we can assume that U is a polydisc

$$U = U(0, \rho) := \left\{(z_1, ..., z_d) \in \mathbf{C}^d : |z_j| < \rho, \quad j = 1, 2, ..., d\right\},$$

where $\rho > 0$.

PROPOSITION 4. *Let us assume that the characteristic function $\varphi_{\mathbf{Y}}$ of d-dimensional random variable $\mathbf{Y}$ is analytic. Then for every $k \in \mathbf{N}$ and every $\mathbf{R}-$linear function $L : \mathbf{R}^d \to \mathbf{R}^k$ the characteristic function $\varphi_{\mathbf{Z}}$ of random variable $\mathbf{Z} = L\mathbf{Y}$ is analytic.*

PROOF. For every $s = (s_1, s_2, ..., s_k) \in \mathbf{R}^k$ we have

$$\begin{aligned}\varphi_{\mathbf{Z}}(s) &= \int_{\Omega} \exp\left(i \langle s\,|\mathbf{Z}(\omega)\rangle\right) dP(\omega) = \int_{\Omega} \exp\left(i \langle s\,|L\mathbf{Y}(\omega)\rangle\right) dP(\omega)\\ &= \int_{\Omega} \exp\left(i \langle L^* s\,|\mathbf{Y}(\omega)\rangle\right) dP(\omega) = \varphi_{\mathbf{Y}}\left(L^* s\right),\end{aligned}$$

where $\langle\cdot|\cdot\rangle$ denotes the standard scalar product in $\mathbf{R}^k$ and $\mathbf{R}^d$ respectively and $L^* : \mathbf{R}^k \to \mathbf{R}^d$ is the adjoint transformation of L (represented by the transpose of the matrix of L with respect to standard bases in $\mathbf{R}^d$ and $\mathbf{R}^k$). Let the function $\Phi_{\mathbf{Y}} : G \to \mathbf{C}$ be a holomorphic extension of the function $\varphi_{\mathbf{Y}}$. Then $\Phi_{\mathbf{Y}} \circ L^*$ is a holomorphic function on $G^* := (L^*)^{-1}(G) \subset \mathbf{C}^k$. Moreover, when U is a neighbourhood of zero in $\mathbf{C}^d$, then $(L^*)^{-1}(U) =: U^*$ is a neighbourhood of zero in $\mathbf{C}^k$ and

$$\varphi_{\mathbf{Z}|U^*\cap\mathbf{R}^k} = (\varphi_{\mathbf{Y}} \circ L^*)_{|(L^*)^{-1}(U)\cap\mathbf{R}^k} = (\Phi_{\mathbf{Y}} \circ L^*)_{|(L^*)^{-1}(U)\cap\mathbf{R}^k} . \ \Box$$

Corollary 5. *Under the assumptions of Proposition 4 for every* $c_1, c_2, ..., c_d \in \mathbf{R}$ *the characteristic function* φ_Z *of the random variable* $Z = \sum_{j=1}^{d} c_j Y_j$ *is analytic and can be written as follows:*

$$\varphi_Z(s) = \varphi_{\mathbf{Y}}(c_1 s, c_2 s, ..., c_d s), \quad s \in \mathbf{R}.$$

The holomorphic extention of this function has the form

$$\Phi_Z(z) = \Phi_{\mathbf{Y}}(c_1 z, c_2 z, ..., c_d z), \quad z \in \mathbf{C}, \quad (c_1 z, c_2 z, ..., c_d z) \in G.$$

In particular the functions φ_{Y_j}, $j = 1, 2, ..., d$, *are analytic, and holomorphic extentions of this functions have the form*

$$\Phi_{Y_j}(z) = \Phi_{\mathbf{Y}}(0, ..., 0, z, 0, ..., 0), \quad |z| < \rho,$$

where z *stays at the* j*-th place.*

It was shown in [6] that, if the characteristic function φ_X of a random variable X is analytic and $\rho > 0$ is a radius of convergence of the Maclaurin series of this function, then there exists a holomorphic extension Φ_X of the function φ_X defined on $B_\rho := \{z \in \mathbf{C} : |\mathrm{Im}\, z| < \rho\}$, and for every $z \in B_\rho$

$$\Phi_X(z) = \int_\Omega \exp\{izX(\omega)\}\, dP(\omega).$$

Moreover for every $y \in (-\rho, \rho)$ the function $\Omega \ni \omega \mapsto \exp\{|y|\,|X(\omega)|\} \in (0, +\infty)$ is integrable with respect to the measure P, and for any $z = t + iy$ we have

$$|\Phi_X(z)| \leq \int_\Omega \exp\{|y|\,|X(\omega)|\}\,dP(\omega) < \infty. \tag{2}$$

Now, let us prove the following theorem:

Theorem 6. *The characteristic function $\varphi_{\mathbf{Y}}$ of d-dimensional random variable $\mathbf{Y} = (Y_1, ..., Y_d)$ is analytic if and only if there exist numbers $y_1, ..., y_d > 0$ such that*

$$\int_\Omega \exp\{y_1\,|Y_1(\omega)| + y_2\,|Y_2(\omega)| + ... + y_d\,|Y_d(\omega)|\}\,dP(\omega) < \infty. \tag{3}$$

In the case when numbers $y_1, ..., y_d$ exist, the function $\Phi_{\mathbf{Y}} : G \to \mathbf{C}$ given by the formula

$$\Phi_{\mathbf{Y}}(\mathbf{z}) := \int_\Omega \exp\{iz_1Y_1(\omega) + ... + iz_dY_d(\omega)\}\,dP(\omega), \quad \mathbf{z} \in G,$$

where

$$G := \Big\{(z_1, ..., z_d) \in \mathbf{C}^d : |\mathrm{Im}\, z_j| < y_j \;\; \textit{for} \;\; j = 1, 2, ..., d\Big\}$$

is holomorphic. Then it is a holomorphic extension of the function $\varphi_{\mathbf{Y}}$.

Proof. ($\Rightarrow$) Let $\rho > 0$ be such that the polydisc $U := U(0, \rho)$ is included in the domain of holomorphic extention $\Phi_{\mathbf{Y}}$ of the function $\varphi_{\mathbf{Y}}$ and the equality (1) holds. From Corollary 5 every function φ_{Y_j} is analytic and the radius of convergence ρ_j of the Maclaurin series of the function φ_{Y_j} satisfies the inequality

$$\rho_j \geq \rho, \quad j = 1, 2, ..., d.$$

Let us take the positive number $y < \rho$ and let $y_j = 2^{-j}y,\ \ j = 1, 2, ..., d$. Then using d times the Schwarz inequality and the inequality (2) we obtain

$$
\begin{aligned}
&\int\limits_\Omega \exp\left\{\sum_{j=1}^d y_j\,|Y_j(\omega)|\right\} dP(\omega) \\
&\leq \left(\int\limits_\Omega \exp\left\{2y_1\,|Y_1(\omega)|\right\} dP(\omega)\right)^{\frac{1}{2}} \left(\int\limits_\Omega \exp\left\{\sum_{j=2}^d 2y_j\,|Y_j(\omega)|\right\} dP(\omega)\right)^{\frac{1}{2}} \\
&\leq \prod_{j=1}^d \left(\int\limits_\Omega \exp\left\{2^j y_j\,|Y_j(\omega)|\right\} dP(\omega)\right)^{\frac{1}{2^j}} \\
&= \prod_{j=1}^d \left(\int\limits_\Omega \exp\left\{y\,|Y_j(\omega)|\right\} dP(\omega)\right)^{\frac{1}{2^j}} < \infty.
\end{aligned}
$$

($\Longleftarrow$)The function $\Phi_{\mathbf{Y}}$ is well defined, because for every $\mathbf{z} = (z_1, ..., z_d) = (t_1 + is_1, ..., t_d + is_d) \in \mathbf{G}$ we have

$$
\left|\exp\left\{i\sum_{j=1}^d z_j Y_j(\omega)\right\}\right| \leq \exp\left(\sum_{j=1}^d |s_j|\,|Y_j(\omega)|\right) \leq \exp\left(\sum_{j=1}^d y_j |Y_j(\omega)|\right).
$$

From the above inequality and from our assumptions we have that the function $\Omega \ni \omega \mapsto \exp\left\{i\sum\limits_{j=1}^d z_j Y_j(\omega)\right\} \in \mathbf{C}$ is integrable on Ω with respect to the measure P.

Let us consider the function

$$
\Omega \ni \omega \mapsto iY_l(\omega)\exp\left\{i\sum_{j=1}^d z_j Y_j(\omega)\right\} := Z_l(\omega) \in \mathbf{C},
$$

where $1 \leq l \leq d$ and $\mathbf{z} = (z_1, ..., z_d) \in \mathbf{G}$ are fixed. We have $z_l = t_l + is_l$, where $|s_l| < y_l$. Let us take $\varepsilon > 0$ such that $|s_l| + \varepsilon < y_l$. Then

$$\begin{aligned}&\left| iY_l(\omega)\exp\left\{ i\sum_{j=1}^{d} z_j Y_j(\omega)\right\}\right| \\ &\quad \leq |Y_l(\omega)|\exp\left\{-\varepsilon\,|Y_l(\omega)|\right\}\exp\left\{\sum_{j=1}^{d} y_j\,|Y_j(\omega)|\right\}.\end{aligned}$$

Let us consider the function

$$g(x) = x\exp\{-\varepsilon x\}, \quad x \in [0, +\infty).$$

It is easy to calculate that g takes its maximal value $M = \frac{1}{e\varepsilon}$. at the point $x_0 = \frac{1}{\varepsilon}$. Then for every $\omega \in \Omega$

$$|Y_l(\omega)|\exp\{-\varepsilon\,|Y_l(\omega)|\} \leq M.$$

The last inequality and (3) prove that the function Z_l is integrable on Ω.

Now we are ready to prove that the function Φ_Y is holomorphic. At first we will show that for any $1 \leq l \leq d$ and any $\mathbf{z} \in \mathbf{G}$ there exists the partial derivative

$$\frac{\partial \Phi_Y}{\partial z_l}(\mathbf{z}) = i\int_{\Omega} Y_l(\omega)\exp\left\{ i\sum_{j=1}^{d} z_j Y_j(\omega)\right\} dP(\omega) = \int_{\Omega} Z_l(\omega) dP(\omega).$$

Let $h \in \mathbf{C}$ be such that $|h| < \frac{\varepsilon}{2}$ (ε like above). Then

$$|\mathrm{Im}\,(z_l + h)| = |s_l + \mathrm{Im}\,h| < |s_l| + \varepsilon < y_l,$$

so the value $\Phi_Y\left(z_1, \dots, z_{l-1}, z_l + h, z_{l+1}, \dots, z_d\right)$ is well defined. We have

$$\left| \frac{\exp\left\{ i\,(z_l + h)\,Y_l(\omega) + i\sum_{j\neq l} z_j Y_j(\omega)\right\} - \exp\left\{ i\sum_{j=1}^{d} z_j Y_j(\omega)\right\}}{h} \right.$$

$$= \left| \frac{1}{h} \exp\left\{ \sum_{j\neq l} z_j Y_j(\omega) \right\} \int_0^1 ihY_l(\omega) \exp\left\{ i\left(z_l + th\right) Y_l(\omega) \right\} dt \right|$$

$$\leq \exp\left\{ -\sum_{j\neq l} s_j Y_j(\omega) \right\} \int_0^1 |Y_l(\omega)| \exp\left\{ -\left(s_l + t \operatorname{Im} h\right) Y_l(\omega) \right\} dt$$

$$\leq |Y_l(\omega)| \exp\left\{ \sum_{j\neq l} y_j \left|Y_j(\omega)\right| \right\} \exp\left\{ \left(|s_l| + \frac{\varepsilon}{2} \right) |Y_l(\omega)| \right\}.$$

Let us consider the function

$$\Omega \ni \omega \mapsto \widetilde{Z}_l(\omega) := |Y_l(\omega)| \exp\left\{ \sum_{j\neq l} y_j \left|Y_j(\omega)\right| \right\} \exp\left\{ \left(|s_l| + \frac{\varepsilon}{2} \right) |Y_l(\omega)| \right\} \in [0, \infty).$$

Using the same arguments as in the case of the function Z_l we obtain that for any $\omega \in \Omega$

$$0 \leq \widetilde{Z}(\omega) \leq M_1 \exp\left\{ \sum_{j=1}^{d} y_j \left|Y_j(\omega)\right| \right\} =: g_1(\omega),$$

where M_1 is the greatest value of the function

$$[0; \infty) \ni x \mapsto x \exp\left\{ -\frac{\varepsilon}{2} x \right\} \in [0, \infty).$$

Then the absolute value of the differential quotient

$$Q_l\left(z, h, \omega\right) = \frac{\exp\left\{ i\left(z_l + h\right) Y_l(\omega) + i \sum_{j\neq l} z_j Y_j(\omega) \right\} - \exp\left\{ i \sum_{j=1}^{d} z_j Y_j(\omega) \right\}}{h}$$

is less then the value of the integrable function g_1 at ω which does not depend on h. Hence we can use Lebesgue theorem for the integral

$$\int_\Omega Q_l\left(z, h, \omega\right) dP(\omega) = \frac{\Phi_{\mathbf{Y}}\left(z_1, ..., z_{l-1}, z_l + h, z_{l+1}, ..., z_d\right) - \Phi_{\mathbf{Y}}\left(z_1, ..., z_d\right)}{h}.$$

We obtain

$$\frac{\partial \Phi_{\mathbf{Y}}}{\partial z_l}(z) = \lim_{h\to 0} \frac{\Phi_{\mathbf{Y}}\left(z_1, ..., z_{l-1}, z_l + h, z_{l+1}, ..., z_d\right) - \Phi_{\mathbf{Y}}\left(z_1, ..., z_d\right)}{h}$$
$$= \int_{\Omega} \lim_{h\to 0} Q_l\left(z, h, \omega\right) dP(\omega) = \int_{\Omega} iY_l(\omega) \exp\left\{\sum_{j=1}^{d} z_j Y_j(\omega)\right\} dP(\omega).$$

which means that the function $\Phi_{\mathbf{Y}}$ is separately holomorphic with respect to each variable z_l on G, where $1 \le l \le d$. From Hartogs theorem on the separate analyticity (see [5] or [13]) the function $\Phi_{\mathbf{Y}}$ is holomorphic on G. □

It is proved in [6] that if the one-dimensional analytic characteristic function is a product of two (a finite number of) characteristic functions then each factor must be analytic, too. Now we generalize this theorem.

Theorem 7. *Let the random variable* $\mathbf{Y} = (Y_1, ..., Y_d)$ *has an analytic characteristic function defined on*

$$B_\rho = \left\{(z_1, ..., z_d) \in \mathbf{C}^d : |\operatorname{Im} z_j| < \rho, j = 1, 2, ..., d\right\} \supset \mathbf{R}^d.$$

Assume that the characteristic function of the random variable $\mathbf{Y}$ *can be represented as*

$$\varphi_{\mathbf{Y}} = \varphi_{\mathbf{X}}\varphi_{\mathbf{Z}},$$

where $\mathbf{X} = (X_1, ..., X_d)\,, \mathbf{Z} = (Z_1, ..., Z_d)$ *are d-dimensional random variables.*

Then $\varphi_{\mathbf{X}}$ *and* $\varphi_{\mathbf{Z}}$ *are analytic characteristic functions.*

Proof. Let us consider one-dimensional characteristic functions. We have

$$\varphi_{Y_j}(t) = \varphi_{\mathbf{Y}}\left(0, ..., 0, t, 0, ..., 0\right) =$$
$$= \varphi_{\mathbf{X}}\left(0, ..., 0, t, 0, ..., 0\right) \varphi_{\mathbf{Z}}\left(0, ..., 0, t, 0, ..., 0\right) = \varphi_{X_j}(t)\varphi_{Z_j}(t),$$

where t stay at the j-th place for $j = 1, 2 \ldots, d$. The function φ_{Y_j} is analytic characteristic function, so the functions φ_{X_j} and φ_{Z_j} are analytic characteristic functions, too. From Theorem 2 there exists $\widetilde{y_j} > 0$, such that

$$\int_\Omega \exp\{\widetilde{y_j}\,|X_j(\omega)|\}\,dP(\omega) < \infty.$$

Let $y_j = 2^{-j}\widetilde{y_j}$, $j = 1, 2, \ldots, d$.

We will prove that

$$\int_\Omega \exp\{y_1\,|X_1(\omega)| + y_2\,|X_2(\omega)| + \ldots + y_d\,|X_d(\omega)|\}\,dP(\omega) < \infty.$$

The Schwarz inequality yields

$$\begin{aligned}\int_\Omega \exp\left\{\sum_{j=1}^d y_j\,|X_j(\omega)|\right\} dP(\omega) &\le \prod_{j=1}^d \left(\int_\Omega \exp\left\{2^j y_j\,|X_j(\omega)|\right\} dP(\omega)\right)^{\frac{1}{2^j}} \\ &= \prod_{j=1}^d \left(\int_\Omega \exp\{\widetilde{y_j}\,|X_j(\omega)|\}\,dP(\omega)\right)^{\frac{1}{2^j}} < \infty\end{aligned}$$

(see the proof of Theorem 6). Then $\varphi_{\mathbf{X}}$ and $\varphi_{\mathbf{Z}}$ are analytic characteristic functions. □

In the next part of this section we will use the Fourier transform. If f is the density of a random vector (a random function) $\mathbf{X}$ then the characteristic function of $\mathbf{X}$ is connected with the Fourier transform

$$\widehat{f}(\mathbf{t}) = \int_{\mathbf{R}^d} f(\mathbf{x}) \exp\left(-2\pi i \langle \mathbf{x}|\mathbf{t}\rangle\right) d\mathbf{x}$$

of f by the equality

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \widehat{f}\left(-\frac{\mathbf{t}}{2\pi}\right).$$

Let us call the function

$$\mathbf{R}^d \ni \mathbf{t} \mapsto \widehat{f}\left(-\frac{\mathbf{t}}{2\pi}\right) \in \mathbf{C}$$

the modified Fourier transform of $f \in L^1(\mathbf{R}^d)$ and let us denote it by $\mathcal{MF}[f]$. Here, $L^1(\mathbf{R}^d)$ denotes the space of all Lebesgue integrable functions on $\mathbf{R}^d$.

If the function $\mathbf{R}^d \ni \mathbf{x} \mapsto \mathbf{x}^\alpha f(\mathbf{x}) \in \mathbf{C}$ belongs to $L^1(\mathbf{R}^d)$ then by a property of Fourier transform we have

$$\frac{\partial^{|\alpha|}\widehat{f}(\mathbf{t})}{\partial \mathbf{t}^\alpha} = (-2\pi i)^{|\alpha|}\,[\widehat{\mathbf{x}^\alpha f(\mathbf{x})}](\mathbf{t}), \tag{4}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d) \in (\mathbf{N} \cup \{0\})^d$ is a multiindex, $|\alpha| = \sum\limits_{k=1}^{d} \alpha_k$, $\frac{\partial^{|\alpha|} f}{\partial \mathbf{t}^\alpha} = \frac{\partial^{|\alpha|} f}{\partial t_1^{\alpha_1} ... \partial t_d^{\alpha_d}}$ and $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdot ... \cdot x_d^{\alpha_d}$ (see [2]).

In the case of modified Fourier transform this equality has a form

$$\frac{\partial^{|\alpha|}\mathcal{MF}\,[f]\,(\mathbf{t})}{\partial \mathbf{t}^\alpha} = i^{|\alpha|}\mathcal{MF}\,[x^\alpha f(x)]\,(\mathbf{t}). \tag{5}$$

As we know the characteristic function of d-dimensional normal distribution (see [6]) with the density

$$f_{b,A}(x_1, ..., x_d) = \frac{\sqrt{\det \mathbf{A}}}{(2\pi)^{\frac{d}{2}}} \exp\left(-\frac{1}{2}\,(\mathbf{x}-\mathbf{b})\,\mathbf{A}\,(\mathbf{x}-\mathbf{b})'\right)$$

($\mathbf{A}$ is a symmetric and positive definite $d \times d$ matrix, $\mathbf{b} \in \mathbf{R}^d$ and y' denotes the transpose of a vector (matrix) $y \in \mathbf{R}^d$) has a form

$$\varphi\,(t_1, ..., t_d) = \exp\left(i\mathbf{b}\mathbf{t}' - \frac{1}{2}\mathbf{t}\mathbf{A}^{-1}\mathbf{t}'\right).$$

For d-dimensional polynomial-normal distribution we have the following proposition:

Proposition 8. *The characteristic function of d-dimensional polynomial-normal distribution (PND_d) with the density*

$$f(x_1, ..., x_d) = p_{2l}(\mathbf{x}) \exp\left(-\frac{1}{2}\,(\mathbf{x}-\mathbf{b})\,\mathbf{A}\,(\mathbf{x}-\mathbf{b})'\right) \tag{6}$$

($\mathbf{A}$ and $\mathbf{b} \in \mathbf{R}^d$ as earlier) has a form

$$\varphi\,(t_1, ..., t_d) = q_{2l}(t) \exp\left(i\mathbf{b}\mathbf{t}' - \frac{1}{2}\mathbf{t}\mathbf{A}^{-1}\mathbf{t}'\right), \tag{7}$$

where q_{2l} is a polynomial of degree 2l.

Proof. Firstly we can write a polynomial p_{2l} in a form

$$p_{2l}(\mathbf{x}) = \sum_{|\alpha|\le 2l} \mathbf{a}_\alpha \mathbf{x}^\alpha.$$

and use equality (4).

Then we have

$$\varphi(t_1, ..., t_d) = \mathcal{MF}\left[p_{2l} f_{b,A}(\mathbf{x})\right](\mathbf{t}) = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det A}} \sum_{|\alpha|\le 2l} \mathbf{a}_\alpha \mathcal{MF}\left[\mathbf{x}^\alpha f_{b,A}(\mathbf{x})\right](\mathbf{t})$$
$$= \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det A}} \sum_{|\alpha|\le 2l} \mathbf{a}_\alpha i^{-|\alpha|} \frac{\partial^{|\alpha|}}{\partial \mathbf{t}^\alpha} \exp\left(i\mathbf{b}\mathbf{t}' - \frac{1}{2}\mathbf{t}\mathbf{A}^{-1}\mathbf{t}'\right).$$

If k_{rs} are the elements of the matrix A^{-1} then

(8) $$\frac{\partial}{\partial t_j} \exp\left(i\mathbf{b}\mathbf{t}' - \frac{1}{2}\mathbf{t}\mathbf{A}^{-1}\mathbf{t}'\right)$$
$$= \left[ib_j - \sum_{r=1}^{d} k_{rj} t_r\right] \exp\left(i\mathbf{b}\mathbf{t}' - \frac{1}{2}\mathbf{t}\mathbf{A}^{-1}\mathbf{t}'\right).$$

From the last equality we have

$$\frac{\partial^{|\alpha|}}{\partial \mathbf{t}^\alpha} \exp\left(i\mathbf{b}\mathbf{t}' - \frac{1}{2}\mathbf{t}\mathbf{A}^{-1}\mathbf{t}'\right) = q_\alpha(\mathbf{t}) \exp\left(i\mathbf{b}\mathbf{t}' - \frac{1}{2}\mathbf{t}\mathbf{A}^{-1}\mathbf{t}'\right),$$

where q_α is a polynomial, which appear when we differentiate the function $\exp\left(i\mathbf{b}\mathbf{t}' - \frac{1}{2}\sum_{r,s=1}^{d} k_{rs} t_r t_s\right)$. This polynomial has a degree $|\alpha|$, because for every j at least one number $k_{rj}, r = 1, 2, ..., d$ is not equal to zero. Then we have

$$\varphi(t_1, ..., t_d) = \sum_{|\alpha|\le 2l} \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det A}} \mathbf{a}_\alpha i^{-|\alpha|} q_\alpha(\mathbf{t}) \exp\left(i\mathbf{b}\mathbf{t}' - \frac{1}{2}\mathbf{t}\mathbf{A}^{-1}\mathbf{t}'\right).$$

Writing

$$\sum_{|\alpha|\le 2l} \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det A}} \mathbf{a}_\alpha \, (-i)^{|\alpha|} \, q_\alpha(\mathbf{t}) = q(\mathbf{t}) \tag{9}$$

$(i^{-|\alpha|} = (-i)^{|\alpha|})$ we obtain

$$\varphi\,(t_1, ..., t_d) = q\,(t_1, ..., t_d)\exp\left(i\mathbf{b}\mathbf{t}' - \frac{1}{2}\mathbf{t}\mathbf{A}^{-1}\mathbf{t}'\right).$$

The polynomial $q(\mathbf{t})$ has a degree $\le 2l$. Taking

$$\tau_j = ib_j - \sum_{r=1}^{d} k_{rj} t_j$$

in (8) and (9) we get a polynomial $\widetilde{q}(\tau),\ \tau = (\tau_1, \tau_2, ..., \tau_d)$ in which the sum of terms of degree $2l$ can be written in the form

$$\sum_{|\alpha|=2l} \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det A}} \mathbf{a}_\alpha \, (-i)^{2l} \, \tau^\alpha = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det A}} \, (-i)^{2l} \sum_{|\alpha|=2l} \mathbf{a}_\alpha \tau^\alpha.$$

The sum on the right hand side of the last equality is the homogeneous part of degree $2l$ of polynomial p_{2l}. Since it is not equal to zero the polynomials $\widetilde{q}$ and q have degree $2l$. We put $q_{2l}(\mathbf{t}) := q(\mathbf{t})$. □

Let us see that the proposition opposite to above is also true i.e.

Proposition 9. *If the characteristic function φ_X of d-dimensional random variable $\mathbf{X}$ has a form (7) then $\mathbf{X}$ is (PND_d) i.e. its density has a form (6).*

Proof. Proof is very similar to the proof of Proposition 8. It follows from (5) and from the fact that Fourier transform is one to one mapping. We left details to the reader. □

The next proposition we are going to prove is about the sum of two independent d-dimensional polynomial-normal distributions.

Proposition 10. *If X_1, X_2 are two independent d-dimensional random vectors with polynomial-normal distributions then the random vector $X_1 + X_2$ has also a polynomial-normal distribution.*

Proof. Let

$$\varphi_{X_j}(t_1, ..., t_d) = q_{2l_j}(t_1, ..., t_d) \exp\left(i\mathbf{b}_j\mathbf{t}' - \frac{1}{2}\mathbf{T}\mathbf{A}_j^{-1}\mathbf{t}'\right),$$

$j = 1, 2$. Then

$$\begin{aligned} &\varphi_{X_1+X_2}(t_1, ..., t_d) \\ &= q_{2l_1}(t_1, ..., t_d) \exp\left(i\mathbf{b}_1\mathbf{t}' - \frac{1}{2}\mathbf{t}\mathbf{A}_1^{-1}\mathbf{t}'\right) \\ &\quad \times q_{2l_2}(t_1, ..., t_d) \exp\left(i\mathbf{b}_2\mathbf{t}' - \frac{1}{2}\mathbf{t}\mathbf{A}_2^{-1}\mathbf{t}'\right) \\ &= q_{2l}(t_1, ..., t_d) \exp\left(i\mathbf{b}\mathbf{t}' - \frac{1}{2}\mathbf{t}\mathbf{A}^{-1}\mathbf{t}'\right), \end{aligned}$$

where $2l := 2l_1 + 2l_2$, $\mathbf{b} := \mathbf{b}_1 + \mathbf{b}_2$, $\mathbf{A}^{-1} := \mathbf{A}_1^{-1} + \mathbf{A}_2^{-1}$ (it is possible since the sum of two symmetric, positive definite matrices is a matrix, which is symmetric and positive definite). It means that the product of two characteristic functions of PND_d is a characteristic function of PND_d with the density

$$f(x_1, ..., x_d) = p_{2l}(\mathbf{x}) \exp\left(-\frac{1}{2}(\mathbf{x} - \mathbf{b})\mathbf{A}(\mathbf{x} - \mathbf{b})'\right). \square$$

3. Decomposition of an Entire Function on C^d

To prove the main result of this paper we should introduce some necessary concepts and results concerning entire functions on $\mathbf{C}^d$. Let Φ be such function. The order of Φ is defined as

$$\rho_\Phi := \limsup_{t\to\infty} \frac{\ln\ln\left\{\max\limits_{|z|\le t} |\Phi(z)|\right\}}{\ln t}.$$

Let

$$m = \left\{\mathbf{z} \in \mathbf{C}^d : \Phi(\mathbf{z}) = \mathbf{0}\right\},$$

and let κ denotes an element from the projective space $P_{d-1} = P_{d-1}(\mathbf{C})$. It means that

$$\kappa = \{\xi \overrightarrow{w} : \xi \in \mathbf{C}\},$$

where $\overrightarrow{w} \in \mathbf{C}^d \setminus \{0\}$ is fixed.

For every $t > 0$ we denote by $n(t, \kappa)$ the number of elements z of the set $\kappa \cap m$ for which $|z| \le t$. Moreover we put

$$n(t) = \frac{1}{\omega_{d-1}} \int\limits_{P_{d-1}} n(t, \kappa) d\omega_{d-1}(\kappa),$$

where $d\omega_{d-1}$ is the standard measure element in the space P_{d-1} and ω_{d-1} is the value of this measure on the set P_{d-1} (see [11]). Finally we define

$$r_\Phi = \limsup_{t \to \infty} \frac{\ln n(t)}{\ln t}.$$

Now we are ready to formulate the Stoll, Lelong and Ronkin result concerning decomposition of entire functions.

THEOREM 11. *Let $\Phi : \mathbf{C}^d \to \mathbf{C}$ be an entire function such that $\Phi(0) \neq 0$. If $r_\Phi < \infty$ then there exists an entire function G on $\mathbf{C}^d$ such that*
1) $\frac{\Phi}{G}$ is an entire function without zeros on $\mathbf{C}^d$;
2) $\rho_G = r_\Phi$.

For more details see [4], [11] or [12].

4. Composition and Decomposition Theorem

We start with the following lemma.

LEMMA 12. *Let f be a holomorfic function in a connected neighbourhood U of a point $z_0 \in \mathbf{C}^d$. Suppose that there exists $n_0 \in \mathbf{N}$ such that*

$$\left. \frac{d^n f(z_0 + t\overrightarrow{w})}{dt^n} \right|_{t=0} = 0$$

for all $n > n_0$ and for all $\overrightarrow{w} \in \mathbf{R}^d \setminus \{0\}$. Then f is a polynomial of degree less than or equal to n_0.

Proof. Choose a polydisc $U(z_0,\rho) \subset U$. Since f is a holomorfic function we can write

$$f(z+z_0) = \sum_{|\alpha|=0}^{+\infty} \frac{\partial^{|\alpha|} f}{\partial z^{\alpha}}(z_0)\frac{1}{\alpha!}z^{\alpha}, \qquad z \in U(0,\rho).$$

It is enough to show that for every α such that $|\alpha| > n_0$ we have

$$\frac{\partial^{|\alpha|} f}{\partial z^{\alpha}}(z_0) = 0.$$

Note that for any $\overrightarrow{w} = (w_1, \dots, w_d) \in \mathbf{R}^d$ and $n = 1, 2, \dots$

$$\left.\frac{d^n f\left(z_0 + \overrightarrow{w}t\right)}{dt^n}\right|_{t=0} = \sum_{j_1,j_2,\dots,j_n=1}^{d} \frac{\partial^n f}{\partial z_{j_1}\dots\partial z_{j_n}}(z_0) w_{j_1} w_{j_2} \dots w_{j_n}.$$

Let us consider the right hand side of the above equality as a polynomial of real variables $w_1, w_2, \dots, w_d$. Since this polynomial equals zero, all its coefficients must equal zero. Then

$$\frac{\partial^n f}{\partial z_{j_1}\dots\partial z_{j_n}}(z_0) = 0$$

for $n > n_0$. and we have

$$f(z+z_0) = \sum_{|\alpha|=0}^{n_0} \frac{\partial^{|\alpha|} f}{\partial z^{\alpha}}(z_0)\frac{1}{\alpha!}z^{\alpha}, \qquad z \in U(0,\rho)$$

By the uniqness theorem for holomorphic functions we conclude that the above formula holds for any $z \in U - z_0$.

Then our function f is a polynomial of degree less than or equal to n_0. □

Now we are in position to prove the composition and decomposition theorem for d-dimensional PND_d.

Proof of Theorem 2. Let φ be the characteristic function of d-dimensional polynomial-normal random variable $\mathbf{X} = (X_1, \dots, X_d)$. Let $\varphi = \varphi_1\varphi_2$, where φ_1 and φ_2 are characteristic functions of some non-degenerate random variables $\mathbf{X}_1 = (X_{1,1}, \dots, X_{1,d})$ and $\mathbf{X}_2 =$

$(X_{2,1}, \ldots, X_{2,d})$. By Proposition 8, φ is an entire characteristic function. Hence ,by Theorem 7, φ_1 and φ_2 are entire characteristic functions too. Let us denote by Φ, Φ_1 and Φ_2 the holomorphic extensions of φ, φ_1 and φ_2 on $\mathbf{C}^d$ respectively. From Proposition 8 we have

$$\Phi(\mathbf{z}) = q_{2l}(\mathbf{z}) \exp\left[h(\mathbf{z})\right], \qquad \mathbf{z} \in \mathbf{C}^d, \tag{10}$$

where q_{2l} is a polynomial of degree equal to $2l$ and $h(\mathbf{z}) = i\mathbf{b}\mathbf{z}' - \frac{1}{2}\mathbf{z}\mathbf{A}^{-1}\mathbf{z}'$.

Let us consider the restriction $\mathbf{C} \ni \mathbf{z} \mapsto \Phi_{\overrightarrow{w}}(z) = \Phi(z\overrightarrow{w}) \in \mathbf{C}$ of the function Φ to a complex plane $\kappa = \{z\overrightarrow{w} : z \in \mathbf{C}\}$, where $\overrightarrow{w} \in \mathbf{R}^d \backslash \{0\} \subset \mathbf{C}^d$ is fixed. If $\overrightarrow{w} = (c_1, \ldots, c_d)$ then

$$\Phi_{\overrightarrow{w}}(z) = \Phi(z\overrightarrow{w}) = \Phi\left(c_1 z, \ldots, c_d z\right), \quad z \in \mathbf{C} \tag{11}$$

and by Corollary 5 function $\Phi_{\overrightarrow{w}}$ is the entire extension of the characteristic function of random variable $\mathbf{Z} = \sum_{j=1}^{d} c_j X_j$. Analogously $\Phi_{k,\overrightarrow{w}}$ is the entire extension of the characteristic function of random variable $\mathbf{Z}_k = \sum_{j=1}^{d} c_j X_{k,j}$ for $k = 1, 2$. It follows from (10), (11) and Proposition 8 that random variable $\mathbf{Z}$ has a polynomial-normal distribution. Moreover we have

$$\Phi_{\overrightarrow{w}}(t) = \Phi_{1,\overrightarrow{w}}(t)\Phi_{2,\overrightarrow{w}}(t), \; t \in \mathbf{R}. \tag{12}$$

Using now the decomposition theorem for one-dimensional polynomial-normal distributions we conclude that $\Phi_{1,\overrightarrow{w}|\mathbf{R}}$ and $\Phi_{2,\overrightarrow{w}|\mathbf{R}}$ are the characteristic functions of one-dimensional polynomial-normal distributions. It means that

$$\Phi_{k,\overrightarrow{w}}(z) = G_{k,\overrightarrow{w}}(z) \exp(h_{k,\overrightarrow{w}}(z)), \qquad z \in \mathbf{C},$$

where $G_{k,\overrightarrow{w}}$ is a polynomial on $\mathbf{C}$ for $k = 1, 2$. By (10), (11) and (12) we obtain

$$q_{2l}(z\overrightarrow{w}) = G_{1,\overrightarrow{w}}(z)G_{2,\overrightarrow{w}}(z), \qquad z \in \mathbf{C}.$$

Hence the sum $2l_1 + 2l_2$ of degrees of $G_{1,\overrightarrow{w}}$ and $G_{2,\overrightarrow{w}}$ is less than or equal $2l$. As a consequence we obtain that $l_k \leq 2l$ for $k = 1, 2$.

It remains to prove that there exist holomorphic polynomials $G_1,\ G_2$ and non-degenerate positive definite quadratic forms $h_1,\ h_2$ on $\mathbf{C}^d$ such that for any $\overrightarrow{w} \in \mathbf{R}^d$ and any $z \in \mathbf{C}$

$$G_{k,\overrightarrow{w}}(z) = G_k(z\overrightarrow{w}),$$

$$h_{k,\overrightarrow{w}}(z) = h_k(z\overrightarrow{w}),$$

where $k = 1, 2$ (see Proposition 8).

Now we can apply Stoll, Lelong and Ronkin's theorem to functions Φ_1 and Φ_2. In our case we have

$$n_k(t, \kappa) \le 2l \quad \text{for} \quad t > 0 \quad \text{and} \quad \kappa \in P_{d-1},$$

$$n_k(t) \le 2l \quad \text{for} \quad t > 0$$

and from Theorem 11, 2)

$$\varrho_{m_k} = 0 \quad \text{for} \quad k = 1, 2.$$

Then there exists function G_k on $\mathbf{C}^d$ with order $\varrho_G = 0$ such that $\frac{\Phi_k}{G_k}$ has no zeros. It is clear that we can compose the function $\frac{\Phi_k}{G_k}$ with some logarithm branch. Then we get entire function h_k and

$$\frac{\Phi_k(z)}{G_k(z)} = \exp(h_k(z)).$$

So we have $\Phi_k = G_k \exp(h_k)$, where G_k is an entire function of order $\varrho_k = 0$. Restricting Φ_k to the plane $\kappa = \{z\overrightarrow{w} : z \in \mathbf{C}\}$, where $\overrightarrow{w} \in \mathbf{R}^d$, we have

$$\Phi_k(z\overrightarrow{w}) = G_k(z\overrightarrow{w}) \exp(h_k(z\overrightarrow{w})) = G_{k,\overrightarrow{w}}(z) \exp(h_{k,\overrightarrow{w}}(z)).$$

Let us assume that $G_k(0) = 1$. By the classical Hadamard's theorem (see [12]) (about canonical representation for holomorphic function of one variable) we have

$$G_k(z\overrightarrow{w}) = \prod_{s=1}^{N_k(\kappa)} \left(1 - \frac{z}{z_s}\right) = G_{k,\overrightarrow{w}}(z) \quad \text{and} \quad h_{k,\overrightarrow{w}}(z) = h_k(z\overrightarrow{w}),$$

where $N_k(\kappa) = \lim_{t\to\infty} n_k(t, \kappa)$ and $\left(z_1, z_2, ..., z_{N_k(\kappa)}\right)$ is a sequence of zeros of function $\Phi_k(z\overrightarrow{w})$ (in our case $N_k(\kappa) \le 2l$).

We know that $n_k(t, \kappa) \le 2l$, so

$$\frac{d^n G_k(z\overrightarrow{w})}{dz^n} = 0 \quad \text{for} \quad n > 2l.$$

Hence from Lemma 12 the function G_k is a polynomial of degree less than or equal to $2l$ on $\mathbf{C}^d$ for k=1,2. It follows by Plucińska theorem (see [8]) that $h_{k,\overrightarrow{w}}(z) = h_k(z\overrightarrow{w})$ is a (non-degenerate) quadratic form (as a function of z) for any $\overrightarrow{w} \in \mathbf{R}^d \setminus \{0\}$. Then

$$\frac{d^n h_k\left(z\overrightarrow{w}\right)}{dz^n} = 0 \qquad \text{for} \quad n > 2$$

and Lemma 12 yields that h_k is a quadratic form on $\mathbf{C}^d$. Finally h_k is non-degenerate, because $h_k\left(z\overrightarrow{w}\right)$ is non-degenerate for any $\overrightarrow{w}$. This completes the proof. □

THEOREM 13. *If the d-dimensional random variable* $\mathbf{X}$ *is* PND_d *and it is the sum of two independent d-dimensional random variables* $\mathbf{X}_{(1)}$ *and* $\mathbf{X}_{(2)}$, *then* $\mathbf{X}_{(1)}$ *and* $\mathbf{X}_{(2)}$ *are* PND_d.

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Monika Maj
Department of Mathematics
Radom University of Technology
Ul. Malczewskiego 20
26-600 Radom, POLAND

Zbigniew Pasternak-Winiarski
Faculty of Mathematics and Information Science
Warsaw University of Technology
Pl. Politechniki 1
00-661 Warsaw, POLAND
E-mail: pastwin@mini.pw.edu.pl