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Classification of Log del Pezzo Surfaces of Index Two

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Abstract. In this article, a log del Pezzo surface of index two means a projective normal non-Gorenstein surface S such that $(S, 0)$ is a log-terminal pair, the anti-canonical divisor $-K_S$ is ample and that $2K_S$ is Cartier. The log del Pezzo surfaces of index two are shown to be constructed from data (X, E, Δ) called fundamental triplets consisting of a non-singular rational surface X, a simple normal crossing divisor E of X, and an effective Cartier divisor Δ of E satisfying a suitable condition. A geometric classification of the fundamental triplets gives a classification of the log del Pezzo surfaces of index two. As a result, any log del Pezzo surface of index two can be described explicitly as a subvariety of a weighted projective space or of the product of two weighted projective spaces. This classification does not use the theory of K3 lattices, which is essential for the classification by Alexeev–Nikulin [4]. The comparison between two classifications is also discussed.

Contents

1. Introduction

In this article, we classify a certain class of generalized del Pezzo surfaces. Since 19th century, study of del Pezzo surfaces (non-singular projective surfaces with ample anti-canonical divisor) has been one of the principal topics

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in the theory of algebraic surfaces (for instance, see [10]). From the view point of logarithmic birational geometry, the classical notion of del Pezzo surface is naturally generalized to the notion of *del Pezzo pair* (S, B), where S is a normal projective surface (or a normal complete algebraic space of dimension two) and B is an effective $\mathbb{Q}$-divisor on S with $-(K_S + B)$ being ample in some sense (a precise definition of del Pezzo pairs will be given in Section 3.1 below). A log del Pezzo surface in the sense of Alexeev and Nikulin is, by definition, a normal projective surface S such that $(S, 0)$ is a del Pezzo pair with only log-terminal singularities.

An important invariant of a given log del Pezzo surface is the *index*, which is defined to be the smallest positive integer i such that iK_S is Cartier. Log del Pezzo surfaces of index one, i.e., projective surfaces with only rational double points and with ample anti-canonical divisor, have been intensively studied in many papers ([8], [10], [13], [14], [31], [32], [33]). The subject of the present paper is the next case: log del Pezzo surfaces of index two.

Log del Pezzo surfaces S of index two defined over the complex number field $\mathbb{C}$ were studied by Alexeev and Nikulin [4] (cf. [1], [2], [3]) through the theory of $K3$ lattice. Roughly speaking, their argument is as follows:

(1) (Smooth Divisor Theorem) A general member $C \in |-2K_S|$ is a non-singular curve of genus ≥ 2.

(2) Fix the general member $C \in |-2K_S|$ and construct a surface $\overline{\mathcal{X}}$ as the double-covering of S branching along $C \cup \operatorname{Sing} S$.

(3) The minimal desingularization $\mathcal{X}$ of $\overline{\mathcal{X}}$ is a K3 surface with a non-symplectic involution θ and $\mathcal{Y} = \mathcal{X}/\langle\theta\rangle$ is non-singular. The birational morphism $\mathcal{Y} \to S$ is called the *right resolution* in [4] or the canonical resolution by Horikawa, and the connected components of the exceptional locus are completely determined.

(4) Via the correspondence above, the classification of (S, C) is reduced to that of $(\mathcal{X}, \theta)$, the pair of a $K3$ surface $\mathcal{X}$ and a non-symplectic involution θ which fixes a non-singular curve of genus ≥ 2 in $\mathcal{X}$.

(5) By the Torelli Theorem, the classification of $(\mathcal{X}, \theta)$, up to deformation, is described in terms of the invariant sublattice $\mathbb{S}$ of the K3

lattice by the action of θ. Moreover, $\mathbb{S}$ is classified by certain numerical data of the lattice, which are called the *main invariants*.

(6) The nef cone of $\mathcal{X}$ gives another information on $\mathbb{S}$, which is called the *root invariant*. The main and root invariants determine a finer deformation equivalence class of the pair $(\mathcal{X}, \theta)$. The root invariants are all constructed from extremal ones by suitable combinatorial methods, and the extremal root invariants are completely classified.

Depending on the period map for $K3$ surfaces, the argument of Alexeev and Nikulin [4] requires many notions of the lattice theory, is far from being geometric, and does not give the classification of the isomorphism classes of log del Pezzo surfaces of index two. It would make sense to seek for a more geometric and direct classification. In this direction, Kojima has succeeded in classifying such surfaces with Picard number one by using the theory of open surfaces.

In this paper, we present a complete geometric classification of log del Pezzo surfaces of index two, over an algebraically closed field $\Bbbk$ of any characteristic. Our main idea, which stems from a technique used in [15], enables us to classify all the isomorphism classes of log del Pezzo pairs of index one or two. In the most essential part of the classification, we consider the following three objects:

- A del Pezzo pair (S, B) of index at most two of a certain class discussed from Section 3.2.
- A *basic pair* (M, E_M) consisting of a non-singular projective rational surface M and an effective divisor E_M satisfying the condition $\mathcal{C}$ in Definition 3.13.
- A *fundamental triplet* (X, E, Δ) consisting of a rational surface X isomorphic to a Hirzebruch surface $\mathbb{F}_n$ or $\mathbb{P}^2$, of an effective divisor E of X, and of a zero-dimensional subscheme $\Delta \subset E$ which satisfy the conditions in Section 4.1.

These objects are related as follows: From a del Pezzo pair (S, B) of index two in the class above, we have a basic pair (M, E_M) by the minimal desingularization $\alpha\colon M \to S$ and by the formula $-2K_M = \alpha^*(-2(K_S + B)) + E_M$. For a basic pair (M, E_M), the linear system $|L_M|$ is base point free for

$L_M = -2K_M - E_M$ by Theorem 3.18, which gives another proof of the Smooth Divisor Theorem in [4] when $\operatorname{char} \Bbbk = 0$. The linear system $|L_M|$ defines the minimal desingularization $\alpha\colon M \to S$ of a normal projective surface S in which (S, B) is a del Pezzo pair of the class above for $B = (1/2)\alpha_* E_M$. By the cone and the contraction theorems (cf. [23]) in the minimal model theory, from a basic pair (M, E_M), we have a *minimal* basic pair (X, E) (cf. Section 3.2) and a birational morphism $\phi\colon M \to X$ with $K_M + L_M = \phi^*(K_X + L)$ for $L = -2K_X - E$. Here, X is a Hirzebruch surface $\mathbb{F}_n$ or $\mathbb{P}^2$. There exists a zero-dimensional subscheme $\Delta \subset E$ such that $\nu_P(\Delta) = 1$ for any $P \in \Delta$ (cf. Definition 2.2) and that ϕ is expressed as the *elimination* of Δ (cf. Definition 2.5, Proposition 2.9). The triplet (X, E, Δ) is a quasi-fundamental triplet (cf. Definition 4.1), but we can replace the birational morphism $\phi\colon M \to X$ so that (X, E, Δ) to satisfy the additional condition required for fundamental triplets. The fundamental triplet (X, E, Δ) is determined uniquely by the basic pair (M, E_M) with the exception mentioned in Theorem 4.9 (cf. Example 4.12). The minimal basic pairs are classified by an elementary calculation (cf. Section 3.3). The fundamental triplets are classified also by an information of Δ, which is done in Theorem 4.6. The type of the fundamental triplet (X, E, Δ) defined in Theorem 4.6 depends only on the associated del Pezzo pair (S, B) (cf. Theorem 4.9). The list of types gives essentially the geometric classification of del Pezzo pairs of the class.

The information on fundamental triplets enables us to study the structure of del Pezzo pairs in detail. For example, we can determine the dual graph of exceptional divisors of the minimal desingularization of S for any the rational del Pezzo pairs (S, B) of index two (cf. Section 4.3), and also we can study several deformation types on (S, B), (M, E_M), and on (X, E, Δ) (cf. Section 5). For a log del Pezzo surface S of index two, we shall show in Theorem 5.16 that S is deformed to a non-singular del Pezzo surface of the same genus $g = K_S^2 + 1$ under a $\mathbb{Q}$-Gorenstein deformation. The author was informed the result from Yongnam Lee in the case of $\operatorname{char} \Bbbk = 0$. For the positive characteristic case, we need a local $\mathbb{Q}$-Gorenstein smoothing of the singularity of type K_n, which is prepared in Section 4.4.

There are exactly 41 types for the log del Pezzo surfaces S of index two, which are listed in Table 6. The list of types corresponds to the list of equi-singular deformation types of (M, E_M) with one exception: basic pairs of type $[2;1,2]_0$ and of type $[0;1,1]_0$ are connected by *equi-singular deformation* (cf. Theorem 6.1, Proposition 5.10). We can show in Theorem 6.28 below that if $\operatorname{char} \Bbbk \neq 2$, then the equi-singular deformation type of a log del Pezzo surface S of index two is determined by the type of S and by the dual graph of curves on M with negative self-intersection number.

By TABLE 6, we infer that the list of equi-singular deformation type of (M, E_M) corresponds to the list of the main invariants (r, a, δ) of $\mathbb{S}$ given in [4]. The numerical information of Δ for a given E seems to correspond to the root invariant of $\mathbb{S}$. It is interesting to define a root invariant directly from the data of fundamental triplet for the comparison between the classification of [4] and our classification by fundamental triplets. By Theorem 6.28, it is almost true that Alexeev and Nikulin have classified in [4] not the isomorphism classes but the equi-singular deformation types of log del Pezzo surfaces of index two.

We can describe a log del Pezzo surface of index two as a subvariety of a weighted projective space or of the product of two weighted projective spaces with explicit defining equations (cf. Section 7, TABLE 14). The idea of description follows from a description of the blowing up of X along Δ as a divisor of a $\mathbb{P}^1$-bundle over X (cf. Section 2.3). We have a morphism from S into a toric variety W by a certain linear system on the $\mathbb{P}^1$-bundle. If the nef divisor $K_X + L = -(K_X + E)$ is big, then the morphism is an embedding, and if $K_X + L$ is not big, then it is a double-covering. In some cases, W is a weighted projective space or is realized as a subvariety of a weighted projective space. In the case where E is a minimal section of $X \simeq \mathbb{F}_n$, the description of S and W seems to be complicated, and we consider another method of description. In this case, S is obtained as the blowing up of $\mathbb{P}(1,1,4)$ along a zero-dimensional subscheme of degree $4-n$ (cf. Proposition 7.1). In particular, $S \simeq \mathbb{P}(1,1,4)$ is case $n = 4$. For other n, S is realized as a subvariety of the product $\mathbb{P}(1,1,n) \times \mathbb{P}(1,1,4)$ in case $n > 0$, and of the product $\mathbb{P}^1 \times \mathbb{P}(1,1,4)$ in case $n = 0$. In the case where $S \to W$ is a double-covering, W is $\mathbb{P}(1,1,4)$ or $\mathbb{P}(1,1,2)$. Using some ad hoc method, we can describe S as a divisor of a weighted projective space of dimension three. In the recent paper [16], we find another method of describing the defining equations of S in a weighted projective space when $\operatorname{char} \Bbbk = 0$ and the genus is small.

In many arguments in our study, the case of type $[1;2,2]_0$ and the case of char $\mathbb{k} = 2$ appear as exceptional cases. The log del Pezzo surfaces in the cases seem to have interesting and complicated structure.

This article is organized as follows: The notion of elimination is introduced in Section 2. The notions of del Pezzo pair and basic pair are introduced in Section 3, where the minimal basic pairs are classified, and the anti log-canonical rings of del Pezzo pairs of index at most two are studied. The notion of fundamental triplet is introduced and the fundamental triplets are classified by types in Section 4.2. Here, in Tables 3 and 4, the list of the dual graphs of exceptional divisors for the minimal desingularization of non-Gorenstein singular points of S is given. Section 5 is devoted to the study of deformation. Especially, deformations of fundamental triplets, and equi-singular deformations of (M, E_M) and of (S, B) are studied. In Sections 6 and 7, we consider only the log del Pezzo surfaces of index two. The structure of the minimal desingularization M is studied in Section 6. Here, we determine all the curves on M with negative self-intersection number. Using it, we study the equi-singular deformations of (M, E_M) and of S. The comparison with the classification by Alexeev–Nikulin [4] is explained in Section 6.6. Section 7 is devoted to giving an explicit description of the log del Pezzo surface from the data of fundamental triplet.

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Notation and terminology. We work in the category of algebraic schemes (or algebraic spaces) over a fixed algebraically closed field $\Bbbk$. A scheme (or an algebraic space) proper over $\Bbbk$ is called *complete*. If $\Bbbk$ is the complex number field $\mathbb{C}$, then the completeness is equivalent to the compactness of the associated analytic space.

First, we explain things on divisors on a normal variety. Let X be a normal variety.

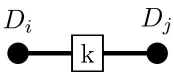
- A divisor on X means a *Weil* divisor. Thus a $\mathbb{Q}$-divisor is a linear combination $D = \sum a_i\Gamma_i$ of prime divisors Γ_i with rational coefficients a_i. The $\mathbb{Q}$-divisor D is called *effective* and we write $D \geq 0$ if all $a_i \geq 0$. A $\mathbb{Q}$-divisor D is called $\mathbb{Q}$-*Cartier* if some positive multiple mD is a Cartier divisor.

- For a reflexive sheaf $\mathcal{L}$ of rank one, a global section ξ of $\mathcal{L}$ defines a homomorphism $\mathcal{O}_X \to \mathcal{L}$. If $\xi \neq 0$, then the image of the dual homomorphism $\mathcal{L}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \to \mathcal{O}_X$ is the ideal sheaf of an effective divisor. The divisor is denoted by $\operatorname{div}(\xi) = \operatorname{div}(\xi)_{\mathcal{L}}$. If $D = \operatorname{div}(\xi)_{\mathcal{L}}$, then ξ is called a *defining equation* of D in $\mathcal{L}$. In this case, there is an injection from $\mathcal{L}$ into the sheaf of germs of rational functions of X sending ξ to 1. The image is just the sheaf $\mathcal{O}_X(D)$ of germs of rational functions f with $\mathrm{div}(f) + D \geq 0$. The cohomology group $\mathrm{H}^i(X, \mathcal{O}_X(D))$ is denoted by $\mathrm{H}^i(X, D)$, for short.

- Suppose that X is complete. A Cartier divisor D is called *nef* if $DC \geq 0$ for any irreducible curve C, where DC denotes the intersection number of D and C. A Cartier divisor D is called *big* if some positive multiple mD is linearly equivalent to $A + E$ for an ample divisor A and an effective divisor E. Note that a nef Cartier divisor D is big if and only if $D^n > 0$ for $n = \dim X$. The intersection theory is generalized to divisors on normal surfaces by the Mumford pullback (cf. Section 3.1).

Second, we explain things related to surfaces. Let S be a non-singular surface.

- An irreducible complete curve γ on S is called a *negative curve* if the self-intersection number γ^2 is negative. If $\gamma \simeq \mathbb{P}^1$ in addition, then γ is called a $(-d)$-*curve* for $d = -\gamma^2$.
- The *dual graph* of a reduced divisor $D = \sum D_j$ on S is defined as follows in the case where irreducible components D_j are all non-singular: A vertex corresponds to an irreducible component D_j. Let v_j be the vertex corresponding to D_j. If $D_iD_j = 0$ for two irreducible components D_i, D_j, then there is no edge joining v_i and v_j. If $D_iD_j = 1$, then v_i and v_j are joined by a (simple) line. If $D_iD_j = k > 1$, then v_i and v_j are joined by a thick line with the numbered box $\boxed{\mathrm{k}}$: If the vertices v_j are written as black circles labelled by D_j, then

in case $D_iD_j = 1$,


in case $D_iD_j = k > 1$.

The set of vertices of such a dual graph Γ is denoted by $\mathrm{Ver}(\Gamma)$.

- In the dual graphs of divisors, a vertex corresponding to a $(-d)$-curve is expressed as follows:

(-1)-curve	(-2)-curve	(-3)-curve	(-4)-curve	$(-d)$-curve
○	●	⊙	◎	ⓓ

On the other hand, an arbitrary irreducible curve is expressed by the symbol $\oslash$ when it is not necessarily a $(-d)$-curve.

- A *straight chain* of non-singular curves of length n on a non-singular surface means a divisor $D = D_1 + D_2 + \cdots + D_n$ such that

  (1) any irreducible component D_i of D is a non-singular projective curve,

  (2) $D_i \cap D_j = \emptyset$ for $|i - j| > 1$,

  (3) $D_1 D_2 = D_2 D_3 = \cdots = D_{n-1} D_n = 1$.

  The dual graph of D is written as:

- Let $\mathbb{F}_n \to \mathbb{P}^1$ denote the $\mathbb{P}^1$-bundle associated with the locally free sheaf $\mathcal{O} \oplus \mathcal{O}(n)$ of $\mathbb{P}^1$ for $n \geq 0$. The surface $\mathbb{F}_n$ is called the *Hirzebruch surface* of degree n. A section $\sigma \subset \mathbb{F}_n$ with $\sigma^2 = -n$ is called a *minimal section*. If $n > 0$, then the minimal section is called the *negative section* since it is a unique negative curve on $\mathbb{F}_n$. The contraction of the negative section is denoted by $\mathbb{F}_n \to \overline{\mathbb{F}}_n$. Here, $\overline{\mathbb{F}}_n$ is isomorphic to the weighted projective space $\mathbb{P}(1, 1, n)$. A section σ_∞ with $\sigma \cap \sigma_\infty = \emptyset$, which is necessarily linearly equivalent to $\sigma + n\ell$ for a fiber ℓ, is called a *section at infinity*.

Finally, we explain additional things.

- A weighted projective space $\mathbb{P}(a_0, a_1, \ldots, a_l)$ over $\Bbbk$ is defined as $\operatorname{Proj} R$ for the graded polynomial ring $R = \Bbbk[\mathtt{X}_0, \mathtt{X}_1, \ldots, \mathtt{X}_l]$ where $\mathtt{X}_i$ is a homogeneous element of degree a_i for $1 \leq i \leq l$. The tautological sheaf $\mathcal{O}(n)$ for $n \in \mathbb{Z}$ is defined as $R(n)^\sim$. If $a_i \mid n$ for any i, then $\mathcal{O}(n)$ is invertible. A *homogeneous coordinate* $(\mathtt{Y}_0, \ldots, \mathtt{Y}_l)$ of $\mathbb{P}(a_0, \ldots, a_l)$ means that $\mathtt{Y}_i$ is a global section of $\mathcal{O}(a_i)$ for any i and $\mathbb{P}(a_0, \ldots, a_l) \simeq \operatorname{Proj} \Bbbk[\mathtt{Y}_0, \ldots, \mathtt{Y}_l]$.

- A lattice $\mathbb{S}$ means a free abelian group $\mathbb{S}$ of finite rank together with a non-degenerate symmetric integral bilinear form $(.\,,\ .)\colon \mathbb{S} \times \mathbb{S} \to \mathbb{Z}$.

- The intersection $C \cap E$ of subschemes C, $E \subset X$ means the scheme-theoretic intersection.

2. Elimination of Zero-Dimensional Subschemes

We introduce the notion of elimination for a zero-dimensional subscheme of a non-singular surface satisfying a suitable condition. A typical example of such a subscheme is the scheme-theoretic intersection $C \cap E$ of a non-singular curve C and an effective divisor E with $C \not\subset E$. The notion of elimination is a generalization of the notion of separation introduced in [15].

2.1. Succession of blowups

Let X be a non-singular surface and let Δ be a zero-dimensional subscheme of X. The defining ideal sheaf of Δ is denoted by $\mathcal{I}_\Delta$.

Definition 2.1 (weak transform). Let $f\colon Z \to X$ be a proper birational morphism from a non-singular surface.

(1) Then the image $\mathcal{I}_\Delta \mathcal{O}_Z$ of $f^*\mathcal{I}_\Delta \to \mathcal{O}_Z$ is written as $\mathcal{O}_Z(-G)\mathcal{J}$ for an effective f-exceptional divisor G of Z and an $\mathcal{O}_Z$-ideal $\mathcal{J}$ defining a subscheme of Z of dimension ≤ 0. The ideal $\mathcal{J}$ is called the *weak transform* of $\mathcal{I}_\Delta$. Similarly, the subscheme Δ_Z defined by $\mathcal{J}$ is called the *weak transform* of Δ.

(2) Let E be an effective divisor on X. We define E_Z^Δ to be the effective divisor $f^*E - f^*E \wedge G$, where G is the f-exceptional divisor in (1) and

$$f^*E \wedge G := \sum\nolimits_\Gamma \min\{\mathrm{mult}_\Gamma(f^*E), \mathrm{mult}_\Gamma(G)\}\Gamma.$$

Remark. If Δ is a subscheme of an effective divisor E, i.e., $\mathcal{O}_X(-E) \subset \mathcal{I}_\Delta \subset \mathcal{O}_X$, then the weak transform Δ_Z is a subscheme of E_Z^Δ. In fact, the inclusion $\mathcal{O}_Z(-f^*E) \subset \mathcal{I}_\Delta\mathcal{O}_Z = \mathcal{J}\mathcal{O}_Z(-G)$ implies that $E_Z^\Delta = f^*E - G \geq 0$ and $\mathcal{O}_Z(-E_Z^\Delta) \subset \mathcal{J} = \mathcal{I}_{\Delta_Z}$.

The following is related to the notion of multiplicity of Δ at a point:

Definition 2.2. Let P be a point of the zero-dimensional subscheme Δ.

(1) The *multiplicity* $\mathrm{mult}_P(\Delta)$ at P is defined as the length of the Artinian local ring $\mathcal{O}_{\Delta,P}$.

(2) The *degree* $\deg\Delta$ coincides with $\mathrm{h}^0(\mathcal{O}_\Delta) = \sum_{P\in\Delta}\mathrm{mult}_P(\Delta)$.

(3) Let us define another invariant $\nu_P(\Delta)$ by

$$\nu_P(\Delta) = \max\{\nu \in \mathbb{N} \mid \mathcal{I}_\Delta \subset \mathfrak{m}_P^\nu\},$$

where $\mathfrak{m}_P \subset \mathcal{O}_X$ is the maximal ideal at P.

Remark. For an effective divisor D and for a point P, we have

$$\begin{aligned}&\max\{\nu \in \mathbb{N} \mid \mathcal{O}_X(-D) \subset \mathfrak{m}_P^\nu\}\\ &= \min\{\mathrm{mult}_P(C\cap D) \mid \text{ a non-singular curve } C \not\subset D \text{ passing through } P\}.\end{aligned}$$

This number is called the multiplicity of D at P and is denoted by $\mathrm{mult}_P(D)$. For two effective divisors D_1, D_2 with no common irreducible components, the local intersection number $(D_1, D_2)_P$ at a point P is defined by $\mathrm{mult}_P(D_1 \cap D_2)$.

Remark. $\nu_P(\Delta) = 1$ if and only if Δ is an effective divisor of a non-singular curve over a neighborhood of P. In fact, if $\nu_P(\Delta) = 1$, then $\mathcal{I}_{\Delta,P} = (x, y^k)$ for a system of parameters (x, y) of the regular local ring $\mathcal{O}_{X,P}$ and for $k = \mathrm{mult}_P(\Delta)$.

Lemma 2.3. *Assume that* $\mathrm{Supp}\,\Delta$ *is a point* P *with* $\nu_P(\Delta) = 1$ *and* $k = \mathrm{mult}_P(\Delta) \geq 2$. *Let* $V \to X$ *be the blowing-up along* Δ. *Then* V *is normal and has a unique singular point* $Q \in V$, *which is an* A_{k-1}*-singularity.*

Proof. We may assume that $X = \mathrm{Spec}\,\Bbbk[\mathtt{x},\mathtt{y}]$ and $\mathcal{I}_\Delta = (\mathtt{x}, \mathtt{y}^k)$. Then $V = V_0 \cup V_1$ for

$$V_0 \simeq \mathrm{Spec}\,\Bbbk[\mathtt{x},\mathtt{y},\mathtt{z}]/(\mathtt{xz} - \mathtt{y}^k) \quad \text{and} \quad V_1 \simeq \mathrm{Spec}\,\Bbbk[\mathtt{x},\mathtt{y},\mathtt{w}]/(\mathtt{x} - \mathtt{wy}^k).$$

Here, V_1 is non-singular and V_0 has the unique singular point $(0, 0, 0)$ of type A_{k-1}. $\square$

In what follows in Sections 2.1–2.3, we assume that $\nu_P(\Delta) = 1$ for any $P \in \Delta$.

We shall investigate the weak transform of Δ by blowups. Let $\mu\colon Y \to X$ be the blowing-up at a point $P \in \Delta$. If $\mathrm{mult}_P(\Delta) = 1$, then $\mathcal{I}_\Delta \mathcal{O}_Y = \mathcal{O}_Y(-l)$ for the exceptional curve $l = \mu^{-1}(P)$ and hence the weak transform Δ_Y is empty. If $\mathrm{mult}_P(\Delta) > 1$, then $\mathcal{I}_\Delta \mathcal{O}_Y = \mathcal{O}_Y(-l) \otimes \mathcal{I}_{\Delta_Y}$ and $\ell \cap \Delta_Y = \{P'\}$ for a point P', where $\nu_{P'}(\Delta_Y) = 1$ and $\mathrm{mult}_{P'}(\Delta_Y) = \mathrm{mult}_P(\Delta) - 1$. In fact, if $\mathcal{I}_{\Delta,P} = (x, y^k)$ for a local coordinate (x, y), then $\mathcal{I}_{\Delta_Y,P'} = (x', y'^{k-1})$ and $(x, y) = (x'y', y')$ for a local coordinate (x', y') around P'. For an effective divisor E on X, we have $E_Y^\Delta = \mu^*E - l$ in case $P \in E$ and $E_Y^\Delta = \mu^*E$ in case $P \notin E$.

By the argument above on the blowing-up at a point, we infer that if $\deg(\Delta) = n < \infty$, then there exists a succession of blowups

$$\phi\colon M = Y_n \to Y_{n-1} \to \cdots \to Y_1 \to Y_0 = X \tag{2–1}$$

such that

(1) the weak transform Δ_{Y_i} of Δ in Y_i is not empty for $i < n$ and $\Delta_{Y_n} = \emptyset$,

(2) $Y_{i+1} \to Y_i$ is the blowing-up at a point $P_i \in \Delta_{Y_i}$ for $i < n$.

In particular, the weak transform of Δ is eliminated by the succession of blowups (2–1).

LEMMA 2.4. *The non-singular surface M in* (2–1) *is isomorphic over X to the minimal desingularization of the blowup V of X along Δ.*

PROOF. By construction, $\mathcal{I}_\Delta \mathcal{O}_M = \mathcal{O}_M(-G)$ for the ϕ-exceptional effective divisor $G \sim K_M - \phi^*K_X$. By the universality of blowing up, there is a morphism $\lambda\colon M \to V$ over X such that $\lambda^*\mathcal{O}_V(1) \simeq \mathcal{O}_M(-G)$, where $\mathcal{O}_V(1)$ denotes the tautological invertible sheaf associated to the graded $\mathcal{O}_X$-algebra $\bigoplus_{m\geq 0} \mathcal{I}_\Delta^m$. In particular, $K_M \sim \lambda^*K_V$. Hence, $\lambda\colon M \to V$ is the minimal desingularization. $\square$

DEFINITION 2.5 (elimination). Let $M \to V$ be the minimal desingularization for the blowing up V along Δ. The composite $\phi\colon M \to X$ is called the *elimination* of Δ.

Even though the definition of elimination can be applied to arbitrary zero-dimensional subscheme Δ, we consider only the case where $\nu_P(\Delta) = 1$ for any $P \in \Delta$.

REMARK 2.6. The elimination $\phi\colon M \to X$ of Δ is characterized by the following two conditions:

(1) The weak transform of Δ is empty;

(2) $K_M \sim \phi^* K_X + G$ for the effective divisor G determined by $\mathcal{I}_\Delta \mathcal{O}_M = \mathcal{O}_M(-G)$.

In fact, there is a birational morphism $\lambda\colon M \to V$ by (1) and λ is the minimal desingularization by (2). Conversely, the elimination $\phi\colon M \to X$ satisfies these two conditions by Lemma 2.4.

LEMMA 2.7. *Let $\phi\colon M \to X$ be the elimination of Δ.*

(1) *Let Δ' be a subscheme of Δ. Then ϕ factors through the elimination of Δ'.*

(2) *Let E be an effective divisor on X containing Δ as a subscheme. Then E_M^Δ is a unique effective divisor of M such that $\phi_* E_M^\Delta = E$ and $K_M + E_M^\Delta \sim \phi^*(K_X + E)$.*

(3) *For an effective divisor E on X, let $M' \to X$ be the elimination of $\Delta \cap E$. Then E_M^Δ is the total transform of $E_{M'}^\Delta$. In particular, if E is non-singular at $\Delta \cap E$, then E_M^Δ is the proper transform of E in M.*

(4) *Let E be an effective divisor on X such that $\Delta \cap E$ consists of finitely many points. Then the difference $\Theta = \phi^* E - E_M^\Delta$ is a complete ϕ-exceptional effective divisor satisfying*

$$-\Theta^2 = -\Theta K_M = \Theta E_M^\Delta = \deg(\Delta \cap E).$$

(5) *For two complete effective divisors D and E on X,*

$$D_M^\Delta E_M^\Delta = DE - \deg(\Delta \cap D \cap E).$$

PROOF. (1): In the expression (2–1) of the elimination ϕ of Δ as a succession of blowups at points, we can choose the center of blowing-up $Y_{i+1} \to Y_i$ from points of the weak transform of Δ' whenever the weak transform is not empty. Hence ϕ factors through the elimination of Δ'.

(2): Let G be the effective divisor on M such that $\mathcal{O}_M(-G) = \mathcal{I}_\Delta \mathcal{O}_M$. Then $G \le \phi^* E$ by $\mathcal{O}_X(-E) \subset \mathcal{I}_\Delta$. Since $G \sim K_M - \phi^* K_X$, we have $E_M^\Delta = \phi^* E - G \sim \phi^*(K_X + E) - K_M$.

(3): $\mathcal{I}_{\Delta \cap E}\mathcal{O}_{M'} = \mathcal{O}_{M'}(-G')$ for the ϕ'-exceptional effective divisor G' on M' with $K_{M'} \sim \phi'^* K_X + G'$. The equality $\mathcal{I}_\Delta \mathcal{O}_{M'} + \mathcal{O}_{M'}(-\phi'^* E) = \mathcal{I}_{\Delta \cap E}\mathcal{O}_{M'}$ implies $\Delta_{M'} \cap E_{M'}^\Delta = \emptyset$. For the induced morphism $\phi''\colon M \to M'$, there is an effective divisor G'' such that $\mathcal{I}_{\Delta_{M'}}\mathcal{O}_M = \mathcal{O}_M(-G'')$ and $G = \phi''^* G' + G''$. Hence, $E_M^\Delta = \phi''^* E_{M'}^\Delta$.

(4): Θ is complete by the assumption and it coincides with $\phi''^* G'$ in the proof of (3). Thus $-\Theta^2 = -G'^2 = \deg(\Delta \cap E)$, and

$$-\Theta K_M = -\Theta G = -\Theta^2 = \Theta(-\phi^* E + E_M^\Delta) = \Theta E_M^\Delta$$

by the equality $G = \phi''^* G' + G''$.

(5): We may assume $\Delta \subset D$ by (3). Thus $\phi^* D - D_M^\Delta = G$. Hence, by (4), we have

$$D_M^\Delta E_M^\Delta = (\phi^* D - G)(\phi^* E - \Theta) = DE + G\Theta = DE - \deg(\Delta \cap D \cap E). \ \square$$

Remark. Let C be a non-singular curve and let E be a non-zero effective divisor with $C \not\subset E$. Then the scheme-theoretic intersection $\Delta = C \cap E$ satisfies $\nu_P(\Delta) = 1$ for any $P \in \Delta$. The separation of C and E defined in [15] is nothing but the elimination of Δ.

The following well-known result is important for showing some vanishing of cohomologies and for showing the base point freeness of some linear systems, especially in characteristic $p > 0$ (cf. [5], [6]):

Lemma 2.8. *Let E be a one-dimensional projective scheme satisfying $\mathrm{H}^1(E, \mathcal{O}_E) = 0$. If $\mathcal{L}$ is a nef invertible sheaf of E, then $\mathcal{L}$ is generated by global sections and $\mathrm{H}^1(E, \mathcal{L}) = 0$.*

Proof. Let $E_1, E_2, \dots, E_l$ be the one-dimensional irreducible components of E. We may assume that E is connected, and hence $E = \bigcup_{i=1}^l E_i$. Let $J_i \subset \mathcal{O}_E$ be the ideal sheaf defining E_i. Then J_i^n is a skyscraper sheaf for $n \gg 0$. We set

$$a(E) := \sum\nolimits_{i=1}^l \sum\nolimits_{n \ge 0} \operatorname{rank}_{\mathcal{O}_{E_i}} J_i^n / J_i^{n+1}.$$

Note that $a(E)$ is an invariant for any one-dimensional algebraic scheme E. We also set $d_i = \deg(\mathcal{L}|_{E_i}) \geq 0$.

We first consider the case where $\mathcal{L}$ is numerically trivial; we shall show that if $d_i = 0$ for any i, then $\mathcal{L} \simeq \mathcal{O}_E$. There is an exact sequence

$$0 \to \mathcal{L} \otimes J_i \to \mathcal{L} \to \mathcal{O}_{E_i} \to 0$$

for any E_i, since $E_i \simeq \mathbb{P}^1$. Note that J_i is regarded as an $\mathcal{O}_{D_i}$-module for a subscheme $D_i \subset E$ such that $\dim D_i \leq 0$ or that $\dim D_i = 1$ with $a(D_i) = a(E) - 1$. By using the induction on $a(E)$, we may assume $\mathcal{L} \otimes J_i \simeq J_i$. The surjection $\mathrm{H}^0(E, \mathcal{O}_E) \to \mathrm{H}^0(E_i, \mathcal{O}_{E_i}) \simeq \Bbbk$ and the vanishing $\mathrm{H}^1(E, \mathcal{O}_E) = 0$ induce $\mathrm{H}^1(J_i) = 0$. Therefore, the restriction map

$$\pi_i \colon \mathrm{H}^0(E, \mathcal{L}) \to \mathrm{H}^0(E_i, \mathcal{O}_{E_i})$$

is surjective for any i. There is a section $s \in \mathrm{H}^0(E, \mathcal{L})$ such that $\pi_i(s) \neq 0$ for any i. Let $\mathcal{F}$ be the cokernel of the homomorphism $\mathcal{O}_E \to \mathcal{L}$ sending 1 to s. Then $\mathcal{F} \otimes \mathcal{O}_{E_i} = 0$ for any i. Thus $\mathcal{O}_E \to \mathcal{L}$ is surjective, and is isomorphic.

Next, we consider the general case. For any i, let us take an arbitrary point $P_i \in E_i$ not contained in other irreducible components E_j. Then there is an effective Cartier divisor B_i of E with $\operatorname{Supp} B_i = \{P_i\}$ and $B_i|_{E_i} = P_i$. In fact, an open neighborhood U of P_i can be regarded as a subscheme of an affine space A and there is a regular function f on A with $\operatorname{div}(f) \cap E_i \cap U = P_i$. Therefore the invertible sheaf $\mathcal{L} \otimes \mathcal{O}_E(-B)$ is numerically trivial for the effective Cartier divisor $B = \sum d_i B_i$. Hence, $\mathcal{L} \simeq \mathcal{O}_E(B)$. Thus $\mathcal{L}$ is generated by global sections by the freeness of the choice of $\{P_i\}$. Since $0 \to \mathcal{O}_E \to \mathcal{L} \simeq \mathcal{O}_E(B) \to \mathcal{O}_B \to 0$ is exact, we have $\mathrm{H}^1(E, \mathcal{L}) = 0$. □

Remark. In Lemma 2.8, we have $\mathrm{H}^1(E', \mathcal{O}_{E'}) = 0$ for any subscheme $E' \subset E$. In particular, if E is an effective divisor of a non-singular surface, then $E_{\mathrm{red}} = \sum E_i$ is a simple normal crossing divisor consisting of rational curves whose dual graph is a tree.

Proposition 2.9. *Let $\phi \colon M \to X$ be a non-isomorphic proper birational morphism of non-singular surfaces such that $-K_M$ is ϕ-nef. Let G be the ϕ-exceptional effective divisor with $G \sim K_M - \phi^* K_X$ and let $\Delta \subset X$ be the zero-dimensional scheme defined by the ideal $\mathcal{I}_\Delta = \phi_* \mathcal{O}_M(-G)$. Then*

$\nu_P(\Delta) = 1$ for any $P \in \Delta$, and ϕ is the elimination of Δ. If E_M be an effective divisor of M such that $K_M + E_M$ is ϕ-numerically trivial, then Δ is a subscheme of the non-zero effective divisor $E = \phi_* E_M$ and $E_M = E_M^{\Delta}$.

Proof. First, we shall show the following two properties to be satisfied for any ϕ-nef divisor D:

(1) $\mathrm{R}^1 \phi_* \mathcal{O}_M(D) = 0$;

(2) $\mathcal{O}_M(D)$ is ϕ-generated, i.e., $\phi^* \phi_* \mathcal{O}_M(D) \to \mathcal{O}_M(D)$ is surjective.

Let B be a ϕ-exceptional effective divisor of M. Then $\mathrm{H}^1(\mathcal{O}_B) = 0$ by $\mathrm{R}^1 \phi_* \mathcal{O}_M = 0$. Thus $\mathrm{H}^1(\mathcal{O}_B \otimes \mathcal{O}_M(D)) = 0$ by Lemma 2.8. Hence, we have the vanishing $\mathrm{R}^1 \phi_* \mathcal{O}_M(D) = 0$ by the theorem of holomorphic functions:

$$\left(\mathrm{R}^1 \phi_* \mathcal{O}_M(D)\right)_x^{\wedge} \simeq \varprojlim{}_m \mathrm{H}^1(\mathcal{O}_{mB} \otimes \mathcal{O}_M(D)),$$

where x is an arbitrary point of X and B is an effective divisor of M with $\operatorname{Supp} B = \phi^{-1}(x)$. Since $D - G \sim D - K_M + \phi^* K_X$ is ϕ-nef, $\mathrm{R}^1 \phi_* \mathcal{O}_M(D - G) = 0$, $\phi_* \mathcal{O}_M(D) \to \phi_* \mathcal{O}_G(D|_G)$ is surjective, and $\mathcal{O}_G(D|_G)$ is generated by global sections (cf. Lemma 2.8). Hence, $\mathcal{O}_M(D)$ is ϕ-generated, since $\operatorname{Supp} G$ is the exceptional locus of ϕ.

Second, we shall show that ϕ is the elimination of Δ by the characterization in Remark 2.6. Since $\mathcal{O}_M(-G)$ is ϕ-generated, $\mathcal{I}_\Delta \mathcal{O}_M = \mathcal{O}_M(-G)$. In particular, the weak transform of Δ in M is empty. Since $K_M \sim \phi^* K_X + G$, ϕ is just the elimination of Δ.

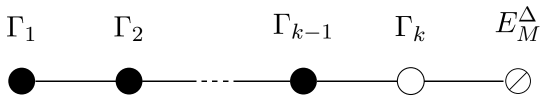
Finally, we shall show the remaining thing. It is derived from $0 \le E_M = \phi^* E - G$. In fact, it induces $\mathcal{O}_X(-E) \subset \mathcal{I}_\Delta$; hence Δ is a subscheme of E and $E_M = E_M^{\Delta}$ by Lemma 2.7, (2). □

2.2. Transformation of an effective divisor

Let E be a non-zero effective divisor of X containing Δ as a subscheme, i.e., $\mathcal{O}_X(-E) \subset \mathcal{I}_\Delta$. Note that Δ is a Cartier divisor of E if and only if $\mathcal{I}_\Delta / \mathcal{O}_X(-E)$ is a locally free $\mathcal{O}_E$-module. We shall study the divisor E_M^{Δ} for the elimination $\phi\colon M \to X$ of Δ.

The following is easily derived from Lemma 2.7:

Lemma 2.10. *Suppose that E is non-singular and Δ is supported on a point P of E. Then, for the elimination $\phi\colon M \to X$ of Δ, the set-theoretic* *inverse image* $\phi^{-1}(P)$ *is a straight chain* $\sum_{j=1}^{k} \Gamma_j$ *of non-singular rational curves,* E_M^Δ *is the proper transform of* E *in* M*, and the dual graph of* $\phi^{-1}(E)$ *is as follows* (cf. Notation and terminology):



LEMMA 2.11. *If* Δ *is supported on a singular point* P *of* E*, then there exists a non-singular curve* C *on an open neighborhood of* P *in* X *such that* $\Delta \subset C \cap E$*. If furthermore* Δ *is a Cartier divisor of* E*, then one can choose the non-singular curve* C *so that* $\Delta = C \cap E$*.*

PROOF. For a local defining equation η of E around P, we have $\eta \in \mathfrak{m}_P^2$ for the maximal ideal $\mathfrak{m}_P$ at P. Thus the ideal $\mathcal{I}_\Delta$ contains η and another function $\xi \in \mathfrak{m}_P \setminus \mathfrak{m}_P^2$, since $\nu_P(\Delta) = 1$. Hence the divisor $C = \mathrm{div}(\xi)$ is non-singular at P and $\Delta \subset C \cap E$. If Δ is a Cartier divisor of E, then we can choose ξ so that $\mathcal{I}_\Delta$ is generated by η and ξ; thus $\Delta = C \cap E$. $\square$

LEMMA 2.12. *Suppose that* $E = E_1 + E_2$ *for non-singular divisors* E_1*,* E_2 *and that* E_1 *and* E_2 *intersect transversally at a unique point* $P = E_1 \cap E_2$*. Suppose also that the zero-dimensional subscheme* Δ *is supported on* P*. Then* Δ *is contained in an effective Cartier divisor* $\widehat{\Delta}$ *of* E *supported on* P *with* $\nu_P(\widehat{\Delta}) = 1$*. In particular,* $\min\{\mathrm{mult}_P(\Delta \cap E_1), \mathrm{mult}_P(\Delta \cap E_2)\} = 1$*. Furthermore, the following conditions are mutually equivalent*:

(1) Δ *is a Cartier divisor of* E;

(2) Δ *is neither a subscheme of* E_1 *nor* E_2;

(3) $\mathrm{mult}_P(\Delta) = \mathrm{mult}_P(\Delta \cap E_1) + \mathrm{mult}_P(\Delta \cap E_2)$.

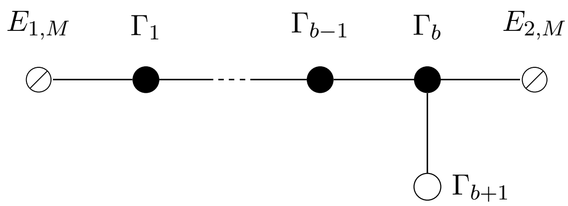
PROOF. We may assume that $\mathrm{div}(x_i) = E_i$ for a regular function x_i of X for $i = 1, 2$. Since $\nu_P(\Delta) = 1$, $\mathcal{I}_{\Delta,P}$ contains a function $\xi \in \mathfrak{m}_P \setminus \mathfrak{m}_P^2$. We may assume that

$$\xi = \lambda_1 x_1^{m_2} + \lambda_2 x_2^{m_1}$$

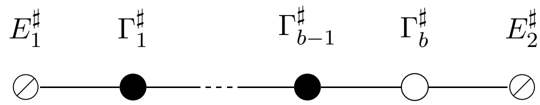
for unit functions λ_1, λ_2 at P and for positive integers m_1, m_2 with $\min\{m_1, m_2\} = 1$. Let $\widehat{\Delta}$ be the subscheme $\mathrm{div}(\xi) \cap E$, i.e., the subscheme defined by the ideal (ξ, x_1x_2). Then $\widehat{\Delta}$ satisfies the required property. Moreover, $\mathrm{mult}_P(\widehat{\Delta}) = m_1 + m_2$, and $\mathrm{mult}_P(\widehat{\Delta} \cap E_i) = m_i$ for $i = 1, 2$. Suppose that $m_1 = 1$ and $\Delta \neq \widehat{\Delta}$. Then $\mathcal{I}_{\Delta,P} = (\xi, x_1x_2, x_1^k) = (x_1^k, x_2)$ for some $1 \leq k \leq m_2$. Hence, $\Delta \subset E_2$, $\mathrm{mult}_P(\Delta) = k$, $\mathrm{mult}_P(\Delta \cap E_1) = 1$, and $\mathrm{mult}_P(\Delta \cap E_2) = k$. Thus the condition: $\widehat{\Delta} = \Delta$, is equivalent to all the conditions (1)–(3) above. □

Corollary 2.13. *In the situation of* Lemma 2.12, *suppose that* Δ *is a Cartier divisor of* E. *If* $\Delta' \subset \Delta$ *is a Cartier divisor of* E, *then* $\Delta' = \emptyset$ *or* $\Delta' = \Delta$.

Lemma 2.14. *Suppose that* $E = E_1 + E_2$ *satisfies the same assumption as in* Lemma 2.12. *Suppose furthermore that* Δ *is an effective Cartier divisor of* E *supported on* P *with* $\mathrm{mult}_P(\Delta \cap E_1) = 1$ *and* $\mathrm{mult}_P(\Delta \cap E_2) = b \geq 1$. *Then, for the elimination* ϕ *of* Δ, *the set-theoretic inverse image* $\phi^{-1}(P)$ *is a straight chain* $\sum_{j=1}^{b+1} \Gamma_j$ *of non-singular rational curves,* $E_M^{\Delta} = E_{1,M} + E_{2,M} + \sum_{j=1}^{b} \Gamma_j$ *for the proper transform* $E_{i,M}$ *of* E_i *for* $i = 1, 2$, *and the dual graph of* $\phi^{-1}(E)$ *is as follows*:



Proof. Let $\phi^\sharp \colon M^\sharp \to X$ be the elimination of $\Delta \cap E_2$. By Lemma 2.10, $(\phi^\sharp)^{-1}(P)$ is a straight chain $\sum_{j=1}^{b} \Gamma_j^\sharp$ of non-singular rational curves. For the proper transform $E_i^\sharp$ of E_i for $i = 1, 2$, the dual graph of the union $(\phi^\sharp)^{-1}(P) \cup E_1^\sharp \cup E_2^\sharp$ is written as follows:



The weak transform $\Delta_{M^\sharp}$ of Δ in $M^\sharp$ is just a point $P^\sharp \in \Gamma_b^\sharp \setminus (\Gamma_{b-1}^\sharp \cup E_2^\sharp)$, where $\Gamma_0^\sharp = E_1^\sharp$ in case $b = 1$. The elimination M of Δ is obtained as the blowing-up $M \to M^\sharp$ at $P^\sharp$. Therefore, the expected dual graph of $\phi^{-1}(E)$ is obtained. Here, Γ_{b+1} is the exceptional curve for $M \to M^\sharp$, and $E_{1,M}$, $E_{2,M}$, Γ_j for $j \leq b$ are the proper transforms in M of $E_1^\sharp$, $E_2^\sharp$, $\Gamma_j^\sharp$, respectively. The divisor E_M^Δ is just $E_{1,M} + E_{2,M} + \sum_{j=1}^{b} \Gamma_j$. $\square$

Remark 2.15. In the situation of Lemma 2.14, the ideal $\mathcal{I}_\Delta$ is expressed as

$$\mathcal{I}_\Delta = (\phi^\sharp)_* \left(\mathfrak{m}_{P^\sharp} \otimes \mathcal{O}_{M^\sharp} \left(-\sum\nolimits_{j=1}^{b} j\Gamma_j^\sharp \right) \right).$$

Therefore, Δ is determined by a point $P^\sharp$ lying on $\Gamma_b^\sharp \setminus (\Gamma_{b-1}^\sharp \cup E_2^\sharp)$. The point $P^\sharp \in \Gamma_b^\sharp$ corresponds to the point $(\lambda_1(P) : \lambda_2(P)) \in \mathbb{P}^1$ for λ_1, λ_2 appearing in the proof of Lemma 2.12.

Lemma 2.16. *Suppose that Δ is supported on a point P of E and that $E = mE_0$ for a non-singular divisor E_0 and for a positive integer m. Then* $\operatorname{mult}_P(\Delta) \leq m \operatorname{mult}_P(\Delta \cap E_0)$*, where the equality holds if and only if Δ is a Cartier divisor of E.*

Proof. We may assume that $m \geq 2$ and that $E_0 = \operatorname{div}(x)$ for a regular function x. Then $x^m \in \mathcal{I}_\Delta$. By using the induction on m, we may assume that $x^{m-1} \notin \mathcal{I}_\Delta$. There is another regular function ξ such that $(\xi, x^m) \subset \mathcal{I}_\Delta$ and $\xi \in \mathfrak{m}_P \setminus \mathfrak{m}_P^2$. If Δ is a Cartier divisor of E, then we can choose ξ so that $\mathcal{I}_\Delta = (\xi, x^m)$ by Lemma 2.11.

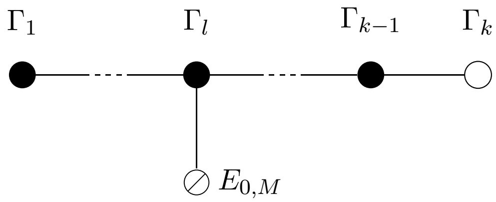
Suppose that $\operatorname{mult}_P(\Delta \cap E_0) = 1$. Then we may assume that $\xi = y$ for a local coordinate system (x, y) around P. Then $\mathcal{I}_\Delta = (x^m, y)$ since $x^{m-1} \notin \mathcal{I}_\Delta$. Thus Δ is a Cartier divisor of E with $\operatorname{mult}_P(\Delta) = m$, $\operatorname{mult}_P(\Delta \cap E_0) = 1$.

Suppose that $\operatorname{mult}_P(\Delta \cap E_0) = l \geq 2$. Then we may assume that $\xi = x + \varepsilon y^l$ for a local coordinate system (x, y) around P and a unit function ε at P. Here, $(x + \varepsilon y^l, x^m) = (x + \varepsilon y^l, y^{ml})$. Thus $\mathcal{I}_\Delta = (x + \varepsilon y^l, y^k)$ for a positive integer k with $(m-1)l < k \leq ml$, since $(x + \varepsilon y^l, x^{m-1}) = (x + \varepsilon y^l, y^{(m-1)l})$. Hence, the required inequality follows from $\operatorname{mult}_P(\Delta) = k$. Moreover if $k = ml$, then Δ is a Cartier divisor of E. $\square$

LEMMA 2.17. *In the situation of* Lemma 2.16, *let* $E_{0,M} \subset M$ *be the proper transform of* E_0 *for the elimination* $\phi\colon M \to X$ *of* Δ. *Then*

$$E_M^{\Delta} = mE_{0,M} + \sum\nolimits_{i=1}^{l} i(m-1)\Gamma_i + \sum\nolimits_{i=l+1}^{k} (ml-i)\Gamma_i$$

for the straight chain $\phi^{-1}(P) = \sum_{i=1}^{k} \Gamma_i$ *of non-singular rational curves, where* $k = \mathrm{mult}_P(\Delta)$ *and* $l = \mathrm{mult}_P(\Delta \cap E_0)$. *If* $k = l$, *then the dual graph of* $\phi^{-1}(E_0)$ *is the same graph as in* Lemma 2.10. *If* $k > l$, *then the dual graph of* $\phi^{-1}(E_0)$ *is written as follows*:



PROOF. The inverse image $\phi^{-1}(P)$ is a straight chain $\sum_{i=1}^{k} \Gamma_i$ of non-singular rational curves where an end curve Γ_k is the unique (-1)-curve of the chain. Let $\phi^{\sharp}\colon M^{\sharp} \to X$ be the elimination of $\Delta \cap E_0$ and let $\phi'\colon M \to M^{\sharp}$ be the induced morphism. Then the curves Γ_i for $i > l$ are ϕ'-exceptional and the images $\Gamma_i^{\sharp} = \phi'(\Gamma_i)$ for $i \le l$ form the straight chain $(\phi^{\sharp})^{-1}(P) = \sum_{i=1}^{l} \Gamma_i^{\sharp}$ of rational curves. The proper transform $E_0^{\sharp} \subset M^{\sharp}$ of E_0 intersects only the unique (-1)-curve $\Gamma_l^{\sharp}$ in the chain $(\phi^{\sharp})^{-1}(P)$. Here, we have

$$\begin{aligned}(\phi^{\sharp})^*E_0 &= E_0^{\sharp} + \sum\nolimits_{i=1}^{l} i\Gamma_i^{\sharp},\\ E_{M^{\sharp}}^{\Delta} &= (\phi^{\sharp})^*(mE_0) - (K_{M^{\sharp}} - (\phi^{\sharp})^*K_X) = mE_0^{\sharp} + \sum\nolimits_{i=1}^{l}(m-1)i\Gamma_i^{\sharp}.\end{aligned}$$

Thus we are done in the case where $k = l$, since $\Delta \subset E_0$ and $\phi = \phi^{\sharp}$. Hence, we may assume $k > l$. Then the morphism ϕ' is the elimination of the weak transform $\Delta^{\sharp} \subset M^{\sharp}$ of Δ. The weak transform $\Delta^{\sharp}$ is supported on a point $P^{\sharp}$ of $\Gamma_l^{\sharp}$ which is not contained in other components of $(\phi^{\sharp})^*E_0$. Thus

$$E_M^{\Delta} = \left((m-1)l\Gamma_l^{\sharp}\right)_M^{\Delta^{\sharp}} + (\phi')^*\left(E_{M^{\sharp}}^{\Delta} - (m-1)l\Gamma_l^{\sharp}\right).$$

Let us consider the special case where $k = m$ and $l = 1$. Then $\mathcal{I}_\Delta = (x^m, y)$ and $E_0 = \operatorname{div}(x)$ for a local coordinate system (x, y) around P. Thus $\phi^\sharp \colon M^\sharp \to M$ is nothing but the blowing up at P. Thus there is a local coordinate $(x^\sharp, y^\sharp)$ around $P^\sharp$ such that $\Gamma_1^\sharp = \operatorname{div}(x^\sharp)$ and $\mathcal{I}_{\Delta^\sharp} = ((x^\sharp)^{m-1}, y^\sharp)$. Thus we have

$$E_M^\Delta = mE_{0,M} + \sum\nolimits_{i=1}^{m-1}(m-i)\Gamma_i$$

by induction on m.

For a general case, by the proof of Lemma 2.16, we may assume $\mathcal{I}_\Delta = (x, y^k)$ and $E_0 = \operatorname{div}(x - \varepsilon y^l)$ for a local coordinate system (x, y) around P and for a unit function ε at P. Then there is a local coordinate system $(x^\sharp, y^\sharp)$ around $P^\sharp$ such that $\Gamma_l^\sharp = \operatorname{div}(x^\sharp)$ and $\mathcal{I}_{\Delta^\sharp} = ((x^\sharp)^{k-l}, y^\sharp)$ around $P^\sharp$. Thus the situation $\Delta^\sharp \subset (k-l)\Gamma_l^\sharp$ belongs to the special case above. Hence,

$$\begin{aligned}
\left((m-1)l\Gamma_l^\sharp\right)_M^{\Delta^\sharp} &= (ml-k)(\phi')^*(\Gamma_l^\sharp) + \left((k-l)\Gamma_l^\sharp\right)_M^{\Delta^\sharp} \\
&= (ml-k)\left(\Gamma_l + \sum\nolimits_{j=1}^{k-l}\Gamma_{l+j}\right) \\
&\quad + (k-l)\Gamma_l + \sum\nolimits_{j=1}^{k-l-1}(k-l-j)\Gamma_{l+j} \\
&= (m-1)l\Gamma_l + \sum\nolimits_{j=1}^{k-l}((m-1)l-j)\Gamma_{l+j}.
\end{aligned}$$

Thus we are done. □

2.3. Global description

Assume that Δ is an effective Cartier divisor of a non-zero effective divisor E of X and that there is a divisor L of X with $L|_E \sim \Delta$, i.e., $\mathcal{O}_X(L)|_E \simeq \mathcal{O}_E(\Delta)$. We shall describe the blowup $V \to X$ along Δ explicitly under the assumption.

We have an extension

$$0 \to \mathcal{O}_X(L-E) \to \mathcal{E} \to \mathcal{O}_X \to 0 \tag{2–2}$$

of locally free sheaves which makes the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X(L-E) & \longrightarrow & \mathcal{I}_\Delta\mathcal{O}_X(L) & \longrightarrow & \mathcal{O}_E & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_X(L-E) & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0
\end{array}$$

of exact sequences, where the top sequence is derived from $\mathcal{O}_X(-E) \subset \mathcal{I}_\Delta$ and from the isomorphism $\mathcal{O}_X(L)|_E \simeq \mathcal{O}_E(\Delta)$. The diagram induces another exact sequence

$$0 \to \mathcal{O}_X(-E) \to \mathcal{E} \to \mathcal{I}_\Delta \mathcal{O}_X(L) \to 0. \tag{2–3}$$

Let $p\colon \mathbb{P} := \mathbb{P}_X(\mathcal{E}) \to X$ be the $\mathbb{P}^1$-bundle associated with $\mathcal{E}$ and let $\mathcal{O}_{\mathcal{E}}(1)$ denote the tautological line bundle of $\mathbb{P}$ with respect to $\mathcal{E}$.

Lemma 2.18. *The blowing up V of X along Δ is realized as a Cartier divisor of $\mathbb{P}$ with $\mathcal{O}_{\mathbb{P}}(V) \simeq \mathcal{O}_{\mathcal{E}}(1) \otimes p^*\mathcal{O}_X(E)$.*

Proof. By the exact sequence (2–3), we infer that $\bigoplus_{d \ge 0} \mathcal{I}_\Delta^d$ is a quotient algebra of the symmetric algebra of the locally free sheaf $\mathcal{E} \otimes \mathcal{O}_X(-L)$. Hence, V is isomorphic to a closed subspace of $\mathbb{P}$. The inclusion $\mathcal{O}_X(-E) \subset \mathcal{E}$ of (2–3) defines an irreducible Cartier divisor $D \subset \mathbb{P}$ with $\mathcal{O}_{\mathbb{P}}(D) \simeq \mathcal{O}_{\mathbb{P}}(1) \otimes p^*\mathcal{O}_X(E)$ and $V \subset D$. Thus $V = D$. $\square$

Proposition 2.19. *The extension* (2–2) *is split if and only if* $\operatorname{div}(\xi) \cap E = \Delta$ *for a global section ξ of $\mathcal{O}_X(L)$. In the split case, V is isomorphic to the divisor*

$$V(\xi, \eta) := \operatorname{div}(p^*(\xi)\mathtt{v} - p^*(\eta)\mathtt{u}) \subset \mathbb{P}$$

for a defining equation η of E, where the section $\mathtt{v} \in \mathrm{H}^0(\mathbb{P}, \mathcal{O}_{\mathcal{E}}(1) \otimes p^*\mathcal{O}_X(E - L))$ corresponds to the injection $\mathcal{O}_X(L - E) \to \mathcal{E}$ of (2–2) *and the section $\mathtt{u} \in \mathrm{H}^0(\mathbb{P}, \mathcal{O}_{\mathcal{E}}(1))$ corresponds to a splitting $\mathcal{O}_X \to \mathcal{E}$.*

Proof. If such a section ξ of $\mathcal{O}_X(L)$ exists, then ξ gives an injection $\mathcal{O}_X \to \mathcal{I}_\Delta \mathcal{O}_X(L)$ inducing a splitting $\mathcal{O}_X \to \mathcal{E}$ of (2–2).

Next, suppose that (2–2) is split. Then we have $\mathcal{E} = \mathcal{O}_X(L-E)\mathtt{v} \oplus \mathcal{O}_X\mathtt{u}$. For the injection $\mathcal{O}_X(-E) \to \mathcal{E}$ of (2–3) and for the surjection $\mathcal{E} \to \mathcal{O}_X$ of (2–2), the composite $\eta\colon \mathcal{O}_X(-E) \to \mathcal{E} \to \mathcal{O}_X$ is an injection defining E. Thus η is regarded as a defining equation of E. For the other projection $\mathcal{E} \to \mathcal{O}_X(L - E)\mathtt{v}$, the composite $\mathcal{O}_X(-E) \to \mathcal{E} \to \mathcal{O}_X(L - E)\mathtt{v}$ defines a section ξ of $\mathcal{O}_X(L)$. Replacing ξ with $-\xi$, we infer that

- the twist $\mathcal{O}_X \to \mathcal{E} \otimes \mathcal{O}_X(E)$ of the injection $\mathcal{O}_X(-E) \to \mathcal{E}$ of (2–3) is given by $1 \mapsto \eta\mathtt{u} - \xi\mathtt{v}$, and

- the surjection $\mathcal{E} \to \mathcal{I}_\Delta \mathcal{O}_X(L)$ in (2–3) is given by

$$\mathcal{O}_X(L-E) \oplus \mathcal{O}_X \ni (s_1, s_2) \mapsto (s_1\eta + s_2\xi).$$

Therefore, $V = V(\xi, \eta)$ and $\operatorname{div}(\xi) \cap E = \Delta$. □

REMARK. If $\mathrm{H}^1(X, L-E) = 0$ and if $\operatorname{Bs}|L-E| = \emptyset$, then $\mathcal{I}_\Delta \mathcal{O}_X(L)$ is generated by global sections. In fact, (2–2) is split by $\mathrm{H}^1(X, L-E) = 0$, and thus $\mathcal{E}$ is generated by global sections by $\operatorname{Bs}|L-E| = \emptyset$. Hence, $\mathcal{I}_\Delta \mathcal{O}_X(L)$ is so by the exact sequence (2–3).

2.4. Simultaneous elimination

LEMMA 2.20. *Let $\widetilde{X} \to T$ be a smooth family of surfaces over a non-singular curve T and let $\widetilde{\Delta} \subset \widetilde{X}$ be a subscheme such that $\widetilde{\Delta} \to T$ is finite and flat and that the fiber $\Delta_t = \widetilde{\Delta} \times_T \{t\}$ satisfies $\nu_P(\Delta_t) = 1$ for any point $P \in \Delta_t$ as a zero-dimensional subscheme of the fiber $X_t = \widetilde{X} \times_T \{t\}$ over any $t \in T$. Then there exist a finite ramified covering $\tau\colon T' \to T$ from another non-singular curve T' and a simultaneous elimination $\widetilde{M} \to \widetilde{X} \times_T T'$ of $\widetilde{\Delta} \times_T T'$ in the following sense: $\widetilde{M}$ is smooth over T' and the fiber of $\widetilde{M} \times_{T'} \{t'\}$ over any point $t' \in T'$ is the elimination of $\Delta_t \subset X_t$ for $t = \tau(t')$.*

PROOF. Taking a succession of base changes $\widetilde{\Gamma} \to T$ from the normalizations $\widetilde{\Gamma}$ of irreducible components Γ of $\operatorname{Supp} \widetilde{\Delta}$, we may assume that any irreducible component of $\widetilde{\Delta}$ is a section of $\widetilde{X} \to T$. For a point $P \in \widetilde{\Delta}$, we have a local coordinate system $(\mathtt{x}, \mathtt{y}, \mathtt{t})$ of $\widetilde{X}$ such that $X \to T$ is given by $(\mathtt{x}, \mathtt{y}, \mathtt{t}) \mapsto \mathtt{t}$ and that the defining ideal $\mathcal{I}_{\widetilde{\Delta},P}$ of $\widetilde{\Delta}$ at P contains $\mathtt{y}$. Thus, locally on T, $\widetilde{\Delta}$ is a subscheme of a divisor $\widetilde{E} \subset \widetilde{X}$ which is smooth over T. Then $\widetilde{\Delta}$ is regarded as an effective divisor $\sum n_i \Gamma_i$ of $\widetilde{E}$ for sections Γ_i of $\widetilde{E} \to T$. Hence, we may write

$$\mathcal{I}_{\widetilde{\Delta},P} = (\mathtt{y}, \mathtt{x}^{n_1} \varphi)$$

for a regular function φ at P, where $\{\mathtt{x} = \mathtt{y} = 0\} = \Gamma_1$ and $\sum_{i \geq 2} n_i \Gamma_i$ is defined by $\varphi = \mathtt{y} = 0$. Let $\mu\colon \widetilde{Y} \to \widetilde{X}$ be the blowing-up along the section Γ_1. Then $\widetilde{Y} \to T$ is smooth and the weak transform $\widetilde{\Delta}_{\widetilde{Y}}$ of $\widetilde{\Delta}$ is defined by

$$\left(\mathtt{y}', \mathtt{x}'^{n_1 - 1} \mu^* \varphi\right)$$

for a coordinate system $(\mathtt{x}', \mathtt{y}', \mathtt{t})$ of $\widetilde{Y}$ satisfying $\mu^*\mathtt{x} = \mathtt{x}'$, $\mu^*\mathtt{y} = \mathtt{x}'\mathtt{y}'$, $\mu^*\mathtt{t} = \mathtt{t}$. Thus $\widetilde{\Delta}_{\widetilde{Y}} \to T$ is finite and flat, and the degree of $\widetilde{\Delta}_{\widetilde{Y}} \to T$ is less than the degree of $\widetilde{\Delta} \to T$ by one. Hence, we have a simultaneous elimination by taking a succession of blowups along sections. □

PROPOSITION 2.21. *Suppose that E is a complete simple normal crossing divisor of a non-singular surface X. Let Δ_1 and Δ_2 be zero-dimensional subschemes of E such that*

(a) $\deg(\Delta_1 \cap E_j) = \deg(\Delta_2 \cap E_j)$ *for any irreducible component E_j of E,*

(b) $\mathrm{mult}_P(\Delta_1) = \mathrm{mult}_P(\Delta_2)$ *and* $\mathrm{mult}_P(\Delta_1 \cap E_j) = \mathrm{mult}_P(\Delta_2 \cap E_j)$ *for any node P of E and for any E_j,*

(c) $\nu_{P_i}(\Delta_i) = 1$ *for any $P_i \in \Delta_i$ for $i = 1, 2$.*

Then there exist a connected curve T, a subscheme $\widetilde{\Delta}$ of $E \times T$ flat and finite over T, and two points $t_1, t_2 \in T$ satisfying the following properties where Δ_t is the restriction $\widetilde{\Delta} \cap (E \times \{t\})$ for $t \in T$:

(1) $\Delta_{t_1} = \Delta_1$ *and* $\Delta_{t_2} = \Delta_2$.

(2) $\deg(\Delta_t \cap E_j) = \deg(\Delta_1 \cap E_j)$ *for any $t \in T$ and E_j.*

(3) $\mathrm{mult}_P(\Delta_t) = \mathrm{mult}_P(\Delta_1)$ *and* $\mathrm{mult}_P(\Delta_t \cap E_j) = \mathrm{mult}_P(\Delta_1 \cap E_j)$ *for any $t \in T$, E_j, and for any node P of E.*

(4) $\nu_{P_t}(\Delta_t) = 1$ *for any $t \in T$ and $P_t \in \Delta_t$.*

In particular, there is a birational morphism $\widetilde{\phi}\colon \widetilde{M} \to X \times T$ such that $\widetilde{M}$ is smooth over T and the fiber

$$\widetilde{\phi}|_{t_i}\colon \widetilde{M} \times_T \{t_i\} \to X \times \{t_i\} = X$$

is the elimination of Δ_i for $i = 1, 2$.

PROOF. Let $\Delta_3 \subset \Delta_1 \cap \Delta_2$ be the subscheme supported on nodes of E such that

$$\mathrm{mult}_P(\Delta_3) = \max\{\mathrm{mult}_P(\Delta_1 \cap E_j) \mid P \in E_j\}$$

for any node P of E. Note that if $P \in E_1 \cap E_2$ and $\mathrm{mult}_P(\Delta_1 \cap E_1) = 1$, then $\Delta_3 = \Delta_1 \cap E_2$ near the point P. Let $\phi^\sharp \colon M^\sharp \to X$ be the elimination of Δ_3. Let $\Delta_i^\sharp$ be the weak transform of Δ_i in $M^\sharp$ for $i = 1$, 2, and set

$$E^\sharp := E_{M^\sharp}^{\Delta_3} \sim \phi^{\sharp *}(K_X + E) - K_{M^\sharp}.$$

Then $\Delta_i^\sharp$ is empty or an effective divisor supported on the non-singular part $E^\sharp \setminus \mathrm{Sing}\, E^\sharp$ by Lemma 2.12 (cf. Remark 2.15). Since the degrees of $\Delta_1^\sharp$ and $\Delta_2^\sharp$ on an irreducible component of $E^\sharp$ coincide, the divisors $\Delta_1^\sharp$ and $\Delta_2^\sharp$ of $E^\sharp$ are algebraically equivalent to each other. Therefore, we have a connected non-singular curve T and a relative effective Cartier divisor $\widetilde{\Delta^\sharp} \subset E^\sharp \times T$ such that $\Delta_i^\sharp = \widetilde{\Delta^\sharp} \cap (E^\sharp \times \{t_i\})$ for a suitable point $t_i \in T$ for $i = 1$, 2. By Lemma 2.20, we have a simultaneous elimination $\widetilde{M} \to M^\sharp \times T$ of $\widetilde{\Delta^\sharp}$ by replacing T with a finite ramified covering of T. The subscheme $\widetilde{\Delta} \subset X \times T$ defined by the ideal

$$(\phi^\sharp \times \mathrm{id}_T)_* \mathcal{I}_{\widetilde{\Delta^\sharp}} \mathcal{O}_{M^\sharp \times T}(-E^\sharp \times T) \subset \mathcal{O}_{X \times T}$$

satisfies the required conditions and $\widetilde{\phi} \colon \widetilde{M} \to M^\sharp \times T \to X \times T$ is the simultaneous elimination. □

Lemma 2.22. *Let E_1, E_2 be non-singular prime divisors of a non-singular surface X which intersect transversely at one point P. Let Δ be a zero-dimensional subscheme of $E = E_1 + E_2$ supported at $\{P\}$ with $\nu_P(\Delta) = 1$, $\mathrm{mult}_P(\Delta \cap E_1) = 1$, and $\mathrm{mult}_P(\Delta \cap E_2) = k \geq 1$. Then there exist a connected non-singular curve T, a point $0 \in T$, a subscheme $\widetilde{\Delta} \subset E \times T$ satisfying the following conditions*:

(1) *$\widetilde{\Delta} \to T$ is flat and finite*;

(2) *Δ is isomorphic to the fiber $\Delta_0 = \widetilde{\Delta} \times_T \{0\}$ over the point $0 \in T$*;

(3) *$P \notin \Delta_t$ for the fiber $\Delta_t = \widetilde{\Delta} \times_T \{t\}$ over any point $t \neq 0$.*

Proof. Let $(\mathtt{x}, \mathtt{y})$ be a local coordinate system of X around P such that $E_1 = \mathrm{div}(\mathtt{x})$ and $E_2 = \mathrm{div}(\mathtt{y})$. We may assume that the defining ideal $\mathcal{I}_\Delta$ is one of the following two ideals by the proof of Lemma 2.12:

$$(1) \quad \mathcal{I}_\Delta = (\mathtt{y}, \mathtt{x}^k); \qquad (2) \quad \mathcal{I}_\Delta = (\mathtt{xy}, \mathtt{y} + \varepsilon \mathtt{x}^k),$$

where ε is a unit function at P. Let T be the affine line $\mathbb{A}^1 = \operatorname{Spec} \Bbbk[\mathtt{t}]$. We choose mutually distinct non-zero constants $a_1, a_2, \dots, a_k \in \Bbbk$. In case (1), the subscheme $\widetilde{\Delta}$ of $X \times T$ defined by the ideal

$$\left(\mathtt{y}, \prod_{j=1}^{k} (\mathtt{x} - a_j \mathtt{t})\right)$$

satisfies the required conditions. In case (2), the subscheme $\widetilde{\Delta}$ of $X \times T$ defined by the ideal

$$\left(\mathtt{xy}, \mathtt{y} + \varepsilon \prod_{j=1}^{k} (\mathtt{x} - a_j \mathtt{t})\right)$$

satisfies the required conditions. □

LEMMA 2.23. *Let E_0 be a non-singular prime divisor of a non-singular surface X and let Δ be a zero-dimensional subscheme of $E = mE_0$ for some $m \geq 1$ such that Δ is supported at one point $P \in E_0$. Then there exist a connected non-singular curve T, a point $0 \in T$, a subscheme $\widetilde{\Delta} \subset E \times T$ satisfying the following conditions*:

(1) *$\widetilde{\Delta} \to T$ is flat and finite*;

(2) *Δ is isomorphic to the fiber $\Delta_0 = \widetilde{\Delta} \times_T \{0\}$ over the point $0 \in T$*;

(3) *$\Delta_t \cap E_0$ is reduced for the fiber $\Delta_t = \widetilde{\Delta} \times_T \{t\}$ over any point $t \neq 0$.*

PROOF. Let $(\mathtt{x}, \mathtt{y})$ be a local coordinate system of X around P such that $E_0 = \operatorname{div}(\mathtt{x})$. We may assume that $\Delta \not\subset (m-1)E_0$ and $\operatorname{mult}_P(\Delta \cap E_0) = l \geq 2$. If $m = 1$, then the defining ideal $\mathcal{I}_{\Delta,P}$ at P can be written as $(\mathtt{x}, \mathtt{y}^l)$. If $m \geq 2$, then, by the proof of Lemma 2.16, we may assume that the defining ideal $\mathcal{I}_{\Delta,P}$ at P is written as $(\mathtt{x} + \varepsilon \mathtt{y}^l, \mathtt{y}^k)$ for an integer $(m-1)l < k \leq ml$ and for a unit function ε at P. Let T be the affine line $\mathbb{A}^1 = \operatorname{Spec} \Bbbk[\mathtt{t}]$. We choose mutually distinct non-zero constants $a_1, a_2, \dots, a_l \in \Bbbk$. In case $m = 1$, the subscheme $\widetilde{\Delta}$ of $X \times T$ defined by the ideal

$$\left(\mathtt{x}, \prod_{i=1}^{l} (\mathtt{y} - a_i \mathtt{t})\right)$$

satisfies the required condition. In case $m \geq 2$, the subscheme $\widetilde{\Delta}$ of $X \times T$ defined by the ideal

$$\left(\mathtt{x} + \varepsilon \prod\nolimits_{i=1}^{l} (\mathtt{y} - a_i \mathtt{t}),\ \left(\prod\nolimits_{i=1}^{l} (\mathtt{y} - a_i \mathtt{t})\right)^{m-1} \prod\nolimits_{j=1}^{k-(m-1)l} (\mathtt{y} - a_j \mathtt{t})\right)$$

satisfies the required condition. □

3. Del Pezzo Pairs and Basic Pairs

We introduce the notions of del Pezzo pair and of basic pair in this section. The first one is a generalization of the notion of del Pezzo surface to pairs (S, B) of surfaces S and $\mathbb{Q}$-divisors B, where the del Pezzo property for (S, B) are considered in the most general situation. If $(S, 0)$ is a log-terminal del Pezzo pair, then S is called a log del Pezzo surface. The notion of basic pair naturally comes from studying the minimal desingularization of S for del Pezzo pairs (S, B) of index at most two. The set of isomorphism classes of basic pairs is in one-to-one correspondence with the set of isomorphism classes of rational del Pezzo pairs (S, B) of index at most two and of genus at least two which are not $(S, 0)$ of index one. Applying a kind of minimal model program to a basic pair, we have a birational morphism to a minimal basic pair, which is expressed as the elimination of a zero-dimensional subscheme. The minimal basic pairs are classified by some numerical data.

3.1. Definition of del Pezzo pairs

Let S be an irreducible normal algebraic space of dimension two proper over $\operatorname{Spec} \Bbbk$. There is a birational morphism $\alpha \colon M \to S$ from a non-singular algebraic surface projective over $\operatorname{Spec} \Bbbk$, by Chow's lemma and by the resolution of singularities of algebraic surfaces. We may assume that there is no (-1)-curve of M contracted to a point by α. Then α is uniquely determined up to isomorphism and is called the minimal resolution of singularities (or the minimal desingularization) of S.

Let Θ be a $\mathbb{Q}$-divisor of S. The Mumford pullback $\alpha^*\Theta$ (cf. [24]) is defined to be a $\mathbb{Q}$-divisor of the form

$$\Theta_M + \sum a_i E_i,$$

where Θ_M is the proper transform of Θ in M, E_i is an irreducible component of the exceptional locus of α, and the coefficients a_i are rational numbers determined by the condition: $\Theta_M E_i = 0$ for any i. We say that Θ is *numerically Cartier* if $\alpha^*\Theta$ is Cartier. For another $\mathbb{Q}$-divisor Θ' of S, the intersection number $\Theta\Theta'$ is well-defined to be $(\alpha^*\Theta)(\alpha^*\Theta')$. We say that Θ is *nef* if $\Theta\Gamma \geq 0$ for any irreducible curve Γ on S. Similarly, we say that Θ is *numerically ample* if $\Theta\Gamma > 0$ for any irreducible curve Γ on S and if the self-intersection number Θ^2 is positive.

We recall the following results related to rational singularities (cf. [5, Theorem (2.3)]):

THEOREM 3.1. *If S has only rational singularities, i.e., $\mathrm{R}^1\,\alpha_*\mathcal{O}_M = 0$, then S is a projective scheme over* $\operatorname{Spec}\Bbbk$. *For the minimal desingularization $\alpha\colon M \to S$ and for any α-nef divisor L of M, $\mathrm{R}^1\,\alpha_*\mathcal{O}_M(L) = 0$ and $\alpha^*\alpha_*\mathcal{O}_M(L) \to \mathcal{O}_M(L)$ is surjective.*

PROOF. First, we shall show the latter half assertion. Let Z be the fundamental cycle, i.e., the smallest non-zero effective divisor supported on the α-exceptional locus $\bigcup E_i$ such that $-ZE_i \geq 0$ for any i. Note that $\operatorname{Supp} Z = \bigcup E_i$ and $L - nZ$ is α-nef for any $n \geq 0$. Thus $\mathrm{H}^1(\mathcal{O}_{nZ}(L)) = 0$ by Lemma 2.8. Hence, the vanishing $\mathrm{R}^1\,\alpha_*\mathcal{O}_M(L) = 0$ follows from the theorem of holomorphic functions for algebraic spaces (cf. [19]). Applying the vanishing for $L - Z$ to the exact sequence

$$0 \to \mathcal{O}_M(L - Z) \to \mathcal{O}_M(L) \to \mathcal{O}_Z(L) \to 0,$$

we infer that $\alpha_*\mathcal{O}_M(L) \to \alpha_*\mathcal{O}_Z(L)$ is surjective. Let $\mathcal{G}(L)$ be the image of $\alpha^*\alpha_*\mathcal{O}_M(L) \to \mathcal{O}_M(L)$. By Lemma 2.8, $\mathcal{O}_Z(L)$ is generated by global sections. Thus $\mathcal{G}(L) \subset \mathcal{O}_M(L) \to \mathcal{O}_Z(L)$ is surjective. Since $\mathcal{O}_M(L)/\mathcal{G}(L)$ is supported in $\bigcup E_i$, we have $\mathcal{G}(L) = \mathcal{O}_M(L)$.

Next, we shall prove the projectivity of S. Let A be a very ample divisor of M with $\mathrm{H}^1(M, A) = 0$ and let H be the pushforward α_*A. Then the Mumford pullback of H is written by

$$\alpha^*H = A + \sum a_i E_i$$

for positive rational numbers a_i. By multiplying A, we may assume a_i are all integral; thus α^*H is Cartier. By the previous argument, we infer that $\mathcal{O}_Z(\alpha^*H) \simeq \mathcal{O}_Z$ and $\alpha_*\mathcal{O}_M(\alpha^*H) \to \alpha_*\mathcal{O}_Z$ is surjective. In particular, there is an effective divisor D on a Zariski-open neighborhood U of a connected component of Z such that $D \sim \alpha^* H|_U$ and $D \cap Z = \emptyset$. This implies that H is Cartier and $\alpha^* H$ coincides with the pullback as a Cartier divisor. We shall show that H is an ample divisor of S. Let E be the effective divisor $\sum a_i E_i$. From the exact sequence

$$0 \to \mathcal{O}_M(A) \to \mathcal{O}_M(\alpha^* H) \to \mathcal{O}_E(\alpha^* H) \simeq \mathcal{O}_E \to 0$$

and the vanishing $\mathrm{H}^1(M, A) = 0$, we infer that $|H|$ is base point free. If $C\alpha^* H = 0$ for an irreducible curve $C \subset M$, then $C \subset E$. Hence, $|H|$ defines a finite morphism from S into a projective space. Therefore, H is ample and S is projective. □

Definition 3.2. Let B be an effective $\mathbb{Q}$-divisor of S.

(1) The *index* of (S, B) is defined to be the minimum positive integer a with $a(K_S + B)$ being numerically Cartier.

(2) Let $f\colon Z \to S$ be a birational morphism from a non-singular projective surface Z such that the union of $f^{-1}(B)$ and the f-exceptional locus is a normal crossing divisor $\sum E_i$. The pair (S, B) is called log-terminal (resp. log-canonical) if $\delta_i > -1$ (resp. $\delta_i \geq -1$) for any δ_i for the formula

$$K_Z = f^*(K_S + B) + \sum \delta_i E_i.$$

Note that the condition does not depend on the choice of $f\colon Z \to S$.

(3) (S, B) is called a *del Pezzo pair* if $-(K_S + B)$ is numerically ample.

(4) A del Pezzo pair (S, B) is called *rational* if S is a rational surface.

(5) If $(S, 0)$ is a log-terminal del Pezzo pair, then S is called a *log del Pezzo surface.*

Note that a del Pezzo surface is a non-singular projective surface with ample anti-canonical divisor, which is always rational.

Proposition 3.3 (cf. [27, Proposition 4.4]). *Let M be a non-singular projective surface with $\kappa(-K_M) = 2$. Then M has only finitely many negative curves. If $\rho(M) > 2$ in addition, then the cone* $\mathrm{NE}(M)$ *of numerical* *classes of effective* 1*-cycles on* M (cf. [23]) *is generated by the numerical classes of negative curves.*

PROOF. $-K_M$ is $\mathbb{Q}$-linearly equivalent to $A+D$ for an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor D. Let Γ be a negative curve. If $K_M\Gamma < 0$, then Γ is a (-1)-curve. If $K_M\Gamma \geq 0$, then Γ is an irreducible component of D.

Assume that there are infinitely many (-1)-curves C_i on M. By the cone theorem [23], we may assume that the limit

$$\zeta = \lim_{i\to\infty} \frac{1}{AC_i}[C_i]$$

exists in $\overline{\mathrm{NE}}(M)$ with $K_M\zeta = 0$. Since $A\zeta = 1$, $DC_i < 0$ for infinitely many i. This is a contradiction, since $C_i \subset \operatorname{Supp} D$. Therefore, M has only finitely many negative curves.

Suppose that $\rho(M) \geq 3$. Then any extremal ray $R \subset \overline{\mathrm{NE}}(M)$ with $K_M R < 0$ is generated by the class of a (-1)-curve by [23]. Let

$$\Lambda := \sum \mathbb{R}_{\geq 0}[\Gamma_j] \subset \mathrm{NE}(M) \subset \overline{\mathrm{NE}}(M)$$

be the polyhedral cone generated by the set $\{\Gamma_j\}$ of negative curves on M. Assume that there is an element $z \in \overline{\mathrm{NE}}(M) \setminus \Lambda$. By the cone theorem [23], there exists an element $\zeta_1 \in \Lambda$ satisfying $z-\zeta_1 \in \overline{\mathrm{NE}}(M)$ and $K_M(z-\zeta_1) \geq 0$. Since $z \neq \zeta_1$, we have $A(z-\zeta_1) > 0$ and $D(z-\zeta_1) < 0$. Thus the negative part of the Zariski-decomposition of $z-\zeta_1$ is not zero. Hence $z-\zeta_1-\zeta_2 \in \overline{\mathrm{NE}}(M)$ for some $\zeta_2 \in \Lambda \setminus \{0\}$. Therefore, $0 < c(z) \leq Az$ for the number

$$c(z) = \sup\{Ay \mid y \in \Lambda, z-y \in \overline{\mathrm{NE}}(M)\}.$$

Let $\{y_i\}$ be a sequence of elements of Λ such that $z-y_i \in \overline{\mathrm{NE}}(M)$ and $\lim_{i\to\infty} Ay_i = c(z)$. Then we have an accumulation point $y_\infty \in \Lambda$ of $\{y_i\}$. Since $z-y_\infty \in \overline{\mathrm{NE}}(M) \setminus \Lambda$, we have a contradiction by $0 < c(z-y_\infty) \leq c(z) - Ay_\infty = 0$. Hence $\overline{\mathrm{NE}}(M) = \Lambda$. $\square$

COROLLARY 3.4. *Let* (S,B) *be a del Pezzo pair and let* $\alpha\colon M \to S$ *be the minimal desingularization. Then* M *has only finitely many negative* *curves. If a negative curve* Γ *is not* α*-exceptional, then* Γ *is a* (-1)*-curve or* $\alpha(\Gamma) \subset \operatorname{Supp} B$*.*

PROOF. For the nef and big $\mathbb{Q}$-divisor $L = -\alpha^*(K_S + B)$, there is an effective $\mathbb{Q}$-divisor E with $-K_M \sim_{\mathbb{Q}} L + E$. Thus $\kappa(-K_M) = 2$. Hence, M has only finitely many negative curves by Proposition 3.3. Suppose that Γ is neither an α-exceptional curve nor a (-1)-curve. Then $K_M\Gamma \geq 0$ and $L\Gamma > 0$. Hence, $E\Gamma < 0$ and $\alpha(\Gamma) \subset \operatorname{Supp} B$. □

PROPOSITION 3.5. *Let* (S, B) *be a rational del Pezzo pair of index* a *and let* $\alpha\colon M \to S$ *be the minimal desingularization. Then* S *is a projective surface with only rational singularities,* $-a(K_S+B)$ *is an ample Cartier divisor, and the* α*-exceptional locus is a simple normal crossing divisor whose dual graph is a tree.*

PROOF. Let B_M be the proper transform of B. Let b be a positive integer such that abK_S and abB are numerically Cartier. For the α-exceptional locus $\bigcup E_i$, we define effective divisors $E_{(1)}$, $E_{(2)}$ supported on the locus by

$$abK_M = \alpha^*(abK_S) - E_{(1)}, \quad abB_M = \alpha^*(abB) - E_{(2)}.$$

Then $-E_{(1)}$ and $-E_{(2)}$ are both α-nef. We set $L := -a\alpha^*(K_S + B)$ and $E := E_{(1)} + E_{(2)}$. Then L is nef and big, $LE = 0$, and $-ab(K_M + B_M) = bL + E$. Moreover,

$$(K_M + E)L = K_M L \leq (K_M + B_M)L = -a^{-1}L^2 < 0.$$

In particular $\mathrm{H}^0(M, K_M + E) = 0$. By duality, we have $\mathrm{H}^2(M, -E) = 0$ and thus $\mathrm{H}^1(E, \mathcal{O}_E) = 0$ from the exact sequence $0 \to \mathcal{O}_M(-E) \to \mathcal{O}_M \to \mathcal{O}_E \to 0$. Thus $\operatorname{Supp} E$ is a simple normal crossing divisor whose dual graph is a tree. Since $-E$ is α-nef, $\operatorname{Supp} E$ is the inverse image of a finite set of S and $\mathrm{H}^1(E, -jE|_E) = 0$ for any $j \geq 0$ by Lemma 2.8. Hence $\mathrm{H}^1(mE, \mathcal{O}_{mE}) = 0$ for any $m \geq 1$ by the exact sequences

$$0 \to \mathcal{O}_E(-(m-1)E) \to \mathcal{O}_{mE} \to \mathcal{O}_{(m-1)E} \to 0,$$

and we infer that S has only rational singularities by applying the theorem of holomorphic functions to $\mathrm{R}^1\,\alpha_*\mathcal{O}_M$. In particular, S is projective by

Theorem 3.1 and $\mathcal{O}_M(L)$ is the pullback of an invertible sheaf of S. Hence $-a(K_S+B)$ is Cartier. □

Proposition 3.6. *Let (S,B) be a del Pezzo pair.*

(1) *If (S,B) is log-terminal, then S is rational.*

(2) *Assume that S is not rational. Let $\phi\colon M\to X$ be a birational morphism from the minimal desingularization M of S into a $\mathbb{P}^1$-bundle X over a non-singular curve C of genus $g\geq 1$. Then $X\to C$ has a negative section Γ with $-\Gamma^2>2g-2$. If (S,B) is log-canonical in addition, then C is an elliptic curve and the proper transform Γ_M of Γ in M is exceptional for $M\to S$.*

Proof. Suppose that S is not rational. Let $\alpha\colon M\to S$ be the minimal desingularization and $\pi\colon X\to C$ be the $\mathbb{P}^1$-bundle. Then $-K_M\sim_{\mathbb{Q}} L_M+E_M$ for the nef and big $\mathbb{Q}$-divisor $L_M=-\alpha^*(K_S+B)$ and for an effective $\mathbb{Q}$-divisor E_M. Thus $-K_X\sim_{\mathbb{Q}} L+E$ for the nef and big $\mathbb{Q}$-divisor $L=\phi_*L_M$ and the effective $\mathbb{Q}$-divisor $E=\phi_*E_M$. Since $(-K_X)^2=-8(g-1)$, E is not nef. Hence, there is a negative curve Γ on X with $E\Gamma<0$. Moreover, Γ is a unique negative curve of X since the cone $\overline{\mathrm{NE}}(X)$ is spanned by Γ and a fiber ℓ of π. Since Γ dominates C, we have $(K_X+\Gamma)\Gamma=2p_a(\Gamma)-2\geq 2g-2\geq 0$. We set $c=\mathrm{mult}_\Gamma(E)\in\mathbb{Q}$. Then

$$\begin{aligned}
&0<L\ell=(-K_X-E)\ell\leq(-K_X-c\Gamma)\ell=2-c\Gamma\ell,\\
&0\leq L\Gamma=(-K_X-E)\Gamma\leq(-K_X-c\Gamma)\Gamma=-(K_X+\Gamma)\Gamma+(1-c)\Gamma^2.
\end{aligned}$$

Hence, $1\leq c<2$ and Γ is a section of π. In particular, (S,B) is not log-terminal, and (S,B) is log-canonical only when $c=1$.

Suppose that $c=1$. Then $g=1$, $L\Gamma=0$, and $E=\Gamma+D$ for an effective $\mathbb{Q}$-divisor D with $D\cap\Gamma=0$, $D\ell<1$. In particular, $0\leq L_M\Gamma_M\leq L\Gamma=0$ and $\alpha(\Gamma_M)$ is a point.

If $c>1$, then $2g-2\leq(c-1)(-\Gamma^2)\leq-\Gamma^2$. If $c=1$, then $0=2g-2<-\Gamma^2$. Thus we are done. □

Remark 3.7. Let $X\to C$ be a $\mathbb{P}^1$-bundle over a non-singular curve C of genus $g\geq 1$ admitting a negative section σ with $-\sigma^2>2g-2$. Then $X\simeq\mathbb{P}_C(\mathcal{O}_C\oplus\mathcal{O}_C(A))$ for an ample divisor A with $\mathcal{O}_C(A)\simeq\mathcal{O}_\sigma(-\sigma)$.

Thus there is a section σ_∞ with $\sigma \cap \sigma_\infty = \emptyset$, i.e., a section at infinity. Here $\mathrm{Bs}\,|m\sigma_\infty| = \emptyset$ for $m \geq 2$, since $\deg(mA - K_C) \geq 2$. For the contraction morphism $\mu\colon X \to V$ of σ, V is a projective surface of Picard number one, and $\mathcal{O}_V(\mu_*\sigma_\infty)$ is an ample generator.

In what follows, we consider only del Pezzo pairs (S, B) of index at most two.

Convention 3.8. For a del Pezzo pair (S, B) of index at most two, let $\alpha\colon M \to S$ denote the minimal desingularization of S. Then we can write

$$K_M = \alpha^*(K_S + B) - \sum \delta_i E_i$$

for $\delta_i \in (1/2)\mathbb{Z}_{\geq 0}$, where $\bigcup E_i$ is the union of $\alpha^{-1}(B)$ and the α-exceptional locus. We introduce two Cartier divisors on M by

$$E_M := 2\sum \delta_i E_i, \qquad L_M := -2K_M - E_M.$$

Note that E_M is effective, $K_M + L_M = -K_M - E_M$, and $2(K_M + L_M) = L_M - E_M$. The *genus* $g = g(S, B)$ is defined by $2g - 2 = (K_M + L_M)L_M$. In other words, $g = (K_S + B)(K_S + 2B) + 1$. If $-2(K_S + B)$ is Cartier and $|-2(K_S + B)|$ contains an irreducible and reduced curve C, then the arithmetic genus $p_a(C)$ equals $g(S, B)$.

Remark 3.9. Suppose that $E_M = 0$. Then $B = 0$ and $K_M \sim \alpha^* K_S$. Thus $-K_S$ is ample and S has only rational double points as singularities; in other words, S is a log del Pezzo surface of index one. If $(S, 0)$ is a rational del Pezzo pair of index one, then S has only rational double points by Proposition 3.5, and hence S is a log del Pezzo surface of index one. The log del Pezzo surfaces S of index one have been studied by many people as a degenerate case of del Pezzo surfaces (cf. [8], [10], [13], [14], [31], [32], [33]). Here, $2 \leq g = K_S^2 + 1 \leq 10$ and the minimal desingularization M is obtained as the blowing up of $\mathbb{P}^2$ at $10 - g$ points in a general position in certain sense.

Lemma 3.10. *Let (S, B) be a del Pezzo pair of index at most two. Assume that the minimal desingularization M is a $\mathbb{P}^1$-bundle over a non-singular projective curve C of genus $g \geq 1$. Then S is projective, M has a negative section σ, and one of the following cases occurs*:

(1) *C is an elliptic curve, $E_M = 2\sigma$, $L_M \sim 2\sigma_\infty$ for a section σ_∞ at infinity, and α is the contraction morphism of σ. In particular, $B = 0$ and $(S, 0)$ is log-canonical of index one with $g(S, 0) = K_S^2 + 1 = 2$.*

(2) *C is an elliptic curve, $E_M = 2\sigma + \sigma_\infty$, $L_M \sim \sigma_\infty$ for a section σ_∞ at infinity, and α is the contraction morphism of σ. In particular, $B = (1/2)\alpha_*\sigma_\infty$ and (S, B) is log-canonical of index two with $g(S, B) = 1$.*

(3) *$E_M = 3\sigma + \pi^*\Delta$ for the projection $\pi\colon M \to C$ and for an effective divisor Δ on C with $-\sigma^2 \geq 4g - 4 + \deg(\Delta)$. In particular, (S, B) is of index two but not log-canonical, and $g(S, B) = g(C)$. Here, α contracts σ if and only if $-\sigma^2 = 4g - 4 + \deg(\Delta)$.*

Proof. By the proof of Proposition 3.6, we infer that M admits a negative section σ with $m := \mathrm{mult}_\sigma(E_M) \in \{2, 3\}$ and admits a section σ_∞ at infinity (cf. Remark 3.7). In particular, S is always projective. Let D be the effective divisor $E_M - m\sigma$. By the calculation of $(1/2)L_M\gamma = (-K_M - (1/2)E_M)\gamma$ for $\gamma = \ell$ and $\gamma = \sigma$ in Proposition 3.6, we have

$$\begin{aligned} &0 < 2 - (m/2) - (1/2)D\ell \quad \text{and} \\ &0 \leq -(2g - 2) + (1 - (m/2))\sigma^2 - (1/2)D\sigma. \end{aligned}$$

If $m = 2$, then $g = 1$, $D\sigma = 0$, and $D\ell \leq 1$; hence, $D = 0$ or $D = \sigma_\infty$ for a section σ_∞ with $\sigma \cap \sigma_\infty = \emptyset$. If $m = 2$ and $D = 0$, then $L_M \sim 2\sigma_\infty$ for a section σ_∞ at infinity; this is in the case (1). If $m = 2$ and $D = \sigma_\infty$, then $L_M \sim \sigma_\infty$; this is in the case (2). If $m = 3$, then $D\ell = 0$ and $-\sigma^2 \geq 4g - 4 + D\sigma$; thus $D = \pi^*\Delta$ for an effective divisor Δ on C, and $L_M \sim \sigma_\infty + \pi^*(A - 2K_C - \Delta)$ for a divisor A of C with $\mathcal{O}_C(A) \simeq \mathcal{O}_\sigma(-\sigma)$. Thus the case (3) occurs. Since $-\sigma^2 > 2g - 2$, S is projective (cf. Remark 3.7). $\square$

Remark. In the case (3) of Lemma 3.10, suppose that α contracts σ. Then $K_S + B$ is $\mathbb{Q}$-Cartier if and only if $A \sim_{\mathbb{Q}} 2K_C + \Delta$. Here, the Cartier index of $K_S + B$ is the double of the order of $A - 2K_C - \Delta$ in $\mathrm{Pic}^0(C)$.

Proposition 3.11. *If $K_M + L_M$ is not nef, then (S, B) is one of the following*:

(1) *$S \simeq \mathbb{P}^2$ and* $\deg(2B) \in \{4, 5\}$.

(2) $S \simeq \mathbb{F}_n$ *and* $2B \in |3\sigma + (2n+4-b)\ell|$ *for* $n < b \leq 2n+4$.

(3) $S \simeq \mathbb{P}(1,1,n)$ *for* $n \geq 2$ *and* $2B \in |(n+4)\overline{\ell}|$ *for a generating line* $\overline{\ell}$.

(4) *The case* (2) *of* Lemma 3.10.

(5) *The case* (3) *of* Lemma 3.10.

In any case above, the genus $g(S,B)$ *coincides with the irregularity of* M.

Proof. There exists an extremal ray $R \subset \overline{\mathrm{NE}}(M)$ with $(K_M + L_M)R < 0$ by [23]. If R contains the class of a (-1)-curve γ, then $K_M\gamma = -1$ and $L_M\gamma = 0$. This contradicts the minimality of α. Hence, either $M \simeq \mathbb{P}^2$ with $\deg(K_M + L_M) < 0$ or X is a $\mathbb{P}^1$-bundle over a non-singular curve C with $(K_M + L_M)\ell < 0$ for a fiber ℓ.

Suppose that $M \simeq \mathbb{P}^2$. Then $(M, (1/2)E_M) \simeq (S,B)$ and $K_M + L_M$ corresponds to $-K_S - 2B$. Thus $\deg(K_S + B) < 0$ and $\deg(K_S + 2B) > 0$. Hence, $3 < 2\deg B < 6$. Since $2B$ is Cartier, $\deg(2B) \in \{4,5\}$; equivalently, $\deg L_M = 1$ or 2. Thus $g = 0$.

Suppose that $M \simeq \mathbb{F}_n$ for $n \geq 0$. Then $L_M\ell = 1$ for a fiber ℓ. Hence, $L_M \sim \sigma + b\ell$ for a minimal section σ and $b \geq n$. In particular, $g = 0$. If $n = 0$, then $b > 0$. Here, $E_M = -2K_M - L_M \sim 3\sigma + (2n+4-b)\ell$. Thus $n \leq b \leq 2n+4$. If $b > n$, then L_M is ample and $\alpha\colon M \to S$ is isomorphic. If $b = n$, then $n > 0$ and S is isomorphic to the cone $\overline{\mathbb{F}}_n \simeq \mathbb{P}(1,1,n)$ and $2B \sim (n+4)\overline{\ell}$. Here, the case $n = 1$ does not occur since $(K_M + L_M)\sigma = -1$ for the negative section σ.

Suppose that M is a $\mathbb{P}^1$-bundle over C of genus $q \geq 1$. Then (M, E_M) is in one of the three cases in Lemma 3.10. Here, $(K_M + L_M)\ell = 0$ in the case (1), $(K_M + L_M)\ell = -1$ in the cases (2) and (3). We have $g(S,B) = q$ by Lemma 3.10. □

Lemma 3.12. *If* $K_M + L_M$ *is nef and* $g(S,B) = 1$, *then* S *is a log del Pezzo surface of index one and* $2B \sim -K_S$.

Proof. By the Hodge index theorem, we infer that $K_M + L_M$ is numerically trivial. In particular, $-K_M$ is nef and big, which implies that M is rational. Thus S is a log del Pezzo surface of index one. Since $E_M \sim L_M \sim -K_M$, we have the assertion. □

3.2. Definition of basic pairs

For the classification of del Pezzo pairs of index at most two, there remains the case where $E_M \neq 0$, $K_M + L_M$ is nef, and $g(S, B) \geq 2$. In order to study the case, we introduce the following notion of basic pairs:

Definition 3.13. Let X be a non-singular projective surface and let E be a non-zero effective divisor of X satisfying the following three conditions $(\mathcal{C}1)$–$(\mathcal{C}3)$ for the divisor $L = -2K_X - E$:

$(\mathcal{C}1)$ $K_X + L$ is nef;

$(\mathcal{C}2)$ $(K_X + L)L > 0$;

$(\mathcal{C}3)$ $LE_i \geq 0$ for any irreducible component E_i of E.

If X is rational, then (X, E) is called a *basic pair*. The positive integer $g \geq 2$ defined by $2g - 2 = (K_X + L)L$ is called the *genus* of (X, E).

For a del Pezzo pair (S, B) of index at most two of the remaining case, the pair (M, E_M) satisfies $(\mathcal{C}1)$–$(\mathcal{C}3)$ and $g(S, B)$ coincides with the genus of (M, E_M).

Lemma 3.14. *Let (X, E) be a pair satisfying $(\mathcal{C}1)$–$(\mathcal{C}3)$. Then the following two conditions are also satisfied*:

$(\mathcal{C}3')$ $L = -2K_X - E$ *is nef and big*;

$(\mathcal{C}4)$ $K_X^2 \geq 0$.

If X is rational, then the following condition is also satisfied:

$(\mathcal{C}5)$ $\mathrm{H}^1(E, \mathcal{O}_E) = 0$.

Proof. We have $L^2 > 0$ by $0 < 2(K_X + L)L = L^2 - LE \leq L^2$. Thus either L or $-L$ is big by the Riemann–Roch formula for $\chi(X, mL)$. Now $(K_X + L)L > 0$ for the nef divisor $K_X + L$. Thus L is big. If L is not nef, then $L\gamma = (L - E)\gamma + E\gamma < 0$ for an irreducible curve γ. Since $L - E = 2(K_X + L)$ is nef, γ is an irreducible component of E, which contradicts the condition $(\mathcal{C}3)$. Hence, L is nef and $(\mathcal{C}3')$ is satisfied. The condition $(\mathcal{C}4)$ is satisfied by

$$K_X^2 = (K_X + L)^2 + LE \geq LE \geq 0. \tag{3–1}$$

Suppose that X is rational. We have $\mathrm{H}^0(X, K_X + E) \simeq \mathrm{H}^0(X, -K_X - L) = 0$ by ($\mathcal{C}1$), ($\mathcal{C}2$), and ($\mathcal{C}3'$). The Serre duality, the exact sequence $0 \to \mathcal{O}_X(-E) \to \mathcal{O}_X \to \mathcal{O}_E \to 0$, and the rationality of X imply the vanishing $\mathrm{H}^2(X, -E) \simeq \mathrm{H}^1(E, \mathcal{O}_E) = 0$. Thus ($\mathcal{C}5$) is satisfied. □

Corollary 3.15. *Let (X, E) be a pair satisfying* ($\mathcal{C}1$)–($\mathcal{C}3$). *Suppose that X is irrational. Then X is a $\mathbb{P}^1$-bundle over an elliptic curve and $E = 2\sigma$ for a negative section σ.*

Proof. It follows from ($\mathcal{C}4$) and Lemma 3.10. □

Corollary 3.16. *An irrational del Pezzo pair (S, B) of index at most two is one of the three cases in* Lemma 3.10. *In particular, S is projective.*

The rational del Pezzo pairs (S, B) of index at most two are classified by genus g as follows (cf. Remark 3.9):

- If $g = 0$, then (S, B) is a pair in (1)–(3) of Proposition 3.11;
- If $g = 1$, then (S, B) is a pair in Lemma 3.12;
- If $g \geq 2$, then (S, B) is either the pair $(S, 0)$ for a log del Pezzo surface S of index one or has a basic pair as the minimal desingularization.

Therefore, the classification of del Pezzo pairs of index at most two is reduced to the classifications of log del Pezzo surfaces of index one, and of basic pairs.

Let (X, E) be a basic pair and set $L = -2K_X - E$. Suppose that $-E\gamma = (2K_X + L)\gamma < 0$ for a (-1)-curve γ. Then $(K_X + L)\gamma = 0$ and $L\gamma = E\gamma = 1$. Let $\tau\colon X \to Z$ be the blow-down of γ to a point $P \in Z$. Then $E_Z := \tau_*(E)$ is not zero and $K_X + L = \tau^*(K_Z + L_Z)$ for the divisor $L_Z = -2K_Z - E_Z$. Therefore, (Z, E_Z) is a basic pair. Here, the genus of (X, E) equals the genus of (Z, E_Z) since $(K_X + L)L = (K_Z + L_Z)L_Z$.

A basic pair (X, E) is called *minimal* if $-E\gamma = (2K_X + L)\gamma \geq 0$ for any (-1)-curve γ of X. By the theory of extremal rays [23], if (X, E) is minimal, then there is an extremal ray $R \subset \overline{\mathrm{NE}}(X)$ with $(2K_X + L)R < 0$ such that the contraction morphism of R is either the structure morphism of a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ or the trivial morphism from $X \simeq \mathbb{P}^2$ to a point.

LEMMA 3.17. *Any basic pair* (X,E) *satisfies the following stronger condition than* $(\mathcal{C}1)$ *for* $L=-2K_X-E$:

$(\mathcal{C}1')$ $\mathrm{Bs}\,|K_X+L|=\emptyset$.

Moreover, $\mathrm{H}^1(X,m(K_X+L))=0$ *for any* $m\geq 0$.

PROOF. By successive contractions of (-1)-curves γ with $(2K_X+L)\gamma<0$, we may assume that (X,E) is minimal. Then $X\simeq\mathbb{P}^2$ or $X\simeq\mathbb{F}_n$. It is well known that $\mathrm{H}^1(X,D)=0$ and $\mathcal{O}_X(D)$ is generated by global sections for any nef divisor D of X. Thus we are done. □

THEOREM 3.18. $\mathrm{Bs}\,|L|=\emptyset$ *for* $L=-2K_X-E$ *for any basic pair* (X,E)*. Moreover* $\mathrm{H}^1(X,mL-jE)=0$ *for any* $m\geq j\geq 0$.

PROOF. Since $2(K_X+L)=L-E$, we have $\mathrm{Bs}\,|L-E|=\emptyset$ and $\mathrm{H}^1(X,L-E)=0$ by Lemma 3.17. Hence the base point freeness follows from the exact sequence $0\to\mathcal{O}_X(L-E)\to\mathcal{O}_X(L)\to\mathcal{O}_E(L|_E)\to 0$ and from Lemma 2.8. By the exact sequences

$$0\to\mathcal{O}_X(mL-(j+1)E)\to\mathcal{O}_X(mL-jE)\to\mathcal{O}_E((mL-jE)|_E)\to 0$$

and by Lemma 3.17, the vanishing of $\mathrm{H}^1(X,mL-jE)$ is reduced to the vanishing of $\mathrm{H}^1(E,(mL-jE)|_E)$, which follows from Lemma 2.8 since $mL-jE=(m-j)L+j(L-E)$ is nef. □

PROPOSITION 3.19. *Let* (M,E_M) *be a basic pair. Then there exist a rational del Pezzo pair* (S,B) *of index at most two with* $g(S,B)\geq 2$ *such that* (M,E_M) *is obtained as the minimal desingularization* $\alpha\colon M\to S$*. Here,* (S,B) *is log-terminal if and only if* E_M *is reduced*; (S,B) *is log-canonical if and only if* $\llcorner(1/2)E_M\lrcorner$ *is reduced.*

PROOF. Let $\Phi\colon M\to|L_M|^\vee=\mathbb{P}(\mathrm{H}^0(M,L_M))$ be the morphism associated to the linear system $|L_M|$. Let $\alpha\colon M\to S$ be the Stein factorization of Φ. Then S is a normal projective surface and $L_M\sim\alpha^*L_0$ for an ample divisor L_0 of S. Since $L_M=-2K_M-E_M$, we have $L_0\sim-2(K_S+B)$ for $B=(1/2)\alpha_*E_M$. Then $-(K_S+B)$ is ample and

$$K_M\sim_{\mathbb{Q}}\alpha^*(K_S+B)-(1/2)E_M. \tag{3–2}$$

Hence, (S,B) is a rational del Pezzo pair of index at most two. If $B=0$, then the index of (S,B) is two by $E_M \neq 0$. Since K_M+L_M is nef, α is the minimal desingularization. The log-terminal and log-canonical properties follow from (3–2) and ($\mathcal{C}5$). $\square$

Corollary 3.20. *Let (S,B) be a del Pezzo pair of index at most two. Suppose either that S is rational or that (S,B) is log-canonical. Then the index of (S,B) coincides with the Cartier index of K_S+B and* $\mathrm{Bs}\,|-2m(K_S+B)|=\emptyset$ *for $m \geq 2$. If* $\mathrm{Bs}\,|-2(K_S+B)| \neq \emptyset$*, then (S,B) is one of the following*:

(1) *S is a log del Pezzo surface of index one with $K_S^2=1$ and $2B \sim -K_S$;*

(2) *M is a $\mathbb{P}^1$-bundle over an elliptic curve with a negative section σ and a section σ_∞ at infinity such that $\sigma^2=-1$, $B=(1/2)\alpha_*\sigma_\infty$, where $\alpha\colon M \to S$ is the contraction of σ.*

In particular, $|-2(K_S+B)|$ contains a non-singular member if $\mathrm{char}\,\Bbbk = 0$.

Proof. If M is irrational, then $\mathrm{Bs}\,|L_M|$ can be analyzed by Lemma 3.10. Here, we have the exceptional case (2) above, where $\alpha_*\sigma_\infty$ is a non-singular member of $|-2(K_S+B)|$. Thus, we may assume M to be rational. If $E_M=0$, then the property $\mathrm{Bs}\,|-2K_M|=\emptyset$ is well-known. If K_M+L_M is not nef, then $M \simeq \mathbb{P}^2$ or $M \simeq \mathbb{F}_n$ by Proposition 3.11, and hence $\mathrm{Bs}\,|L_M|=\emptyset$ for the nef divisor L_M. If K_M+L_M is nef and $g(S,B)=1$, then $L_M \sim -K_M$ by Lemma 3.12. In this case, it is well known that $\mathrm{Bs}\,|-K_M|=\emptyset$ for $K_M^2>1$ and that, in $\mathrm{char}\,\Bbbk=0$, $|-K_M|$ contains a non-singular member even if $\mathrm{Bs}\,|-K_M| \neq \emptyset$. The assertion for the remaining case follows from Theorem 3.18. $\square$

Remark. A similar result to Corollary 3.20 has been proved as Smooth Divisor Theorem in [4] in the case where $B=0$, E_M is reduced, and $\mathrm{char}\,\Bbbk=0$, by the use of Kawamata–Viehweg's vanishing theorem ([17], [29]). The Smooth Divisor Theorem asserts that a general member of $|-2K_S|$ is non-singular for a log del Pezzo surface S of index ≤ 2. Even if $\mathrm{char}\,\Bbbk>0$, it holds for S with $K_S^2 \geq 2$ by Theorem 3.32 below. However, it does not hold for certain S with $K_S^2=1$ in case $\mathrm{char}\,\Bbbk=2$ as in Example 7.22 below.

3.3. Minimal basic pairs

We shall classify all the minimal basic pairs. Let (X, E) be a minimal basic pair and set $L = -2K_X - E$. Then, either $X \simeq \mathbb{P}^2$ or X is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. In the latter case, $(2K_X + L)\ell = -E\ell < 0$ for a fiber ℓ of the $\mathbb{P}^1$-bundle structure $X \to \mathbb{P}^1$.

LEMMA 3.21. *Let (X, E) be a minimal basic pair with $X \simeq \mathbb{P}^2$. Then* $\deg E = 1$ *or* 2.

PROOF. This follows from $\deg L + \deg E = \deg(-2K_X) = 6$ and $(K_X + L)L > 0$. □

LEMMA 3.22. *Let (X, E) be a minimal basic pair with $X \simeq \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Let ℓ_i be a fiber of the i-th projection $p_i \colon X \to \mathbb{P}^1$ for $i = 1$, 2. Let (e_1, e_2) be the pair of non-negative integers determined by $E \sim e_1\ell_1 + e_2\ell_2$. Assume that $e_1 \geq e_2$. Then*

$$(e_1, e_2) \in \{(1, 0), (1, 1), (2, 0), (2, 1)\}.$$

In particular, E has at most three irreducible components.

PROOF. Since $K_X \sim -2\ell_1 - 2\ell_2$ and $L \sim (4 - e_1)\ell_1 + (4 - e_2)\ell_2$, we have $4 \geq e_1 \geq e_2$, and

$$0 < (K_X + L)L = 2(e_1 - 3)(e_2 - 3) - 2.$$

Hence, $e_1 \leq 2$ and $e_2 \leq 1$. Thus we are done. □

CONVENTION 3.23. In what follows, for a minimal basic pair (X, E) with $X \simeq \mathbb{F}_0$, we fix a $\mathbb{P}^1$-bundle structure $\pi \colon X \to \mathbb{P}^1$ such that $E \sim e_1\sigma + e_2\ell$ with $e_1 \geq e_2$ for a fiber ℓ and for a minimal section σ of π. Here, we express a fiber of π as ℓ and a fiber of another projection to $\mathbb{P}^1$ as σ. The projection π is uniquely determined except for the case $(e_1, e_2) = (1, 1)$.

LEMMA 3.24. *Let (X, E) be a minimal basic pair with $X \simeq \mathbb{F}_n$ for $n \geq 1$. Let $\sigma \subset X$ be the negative section and let ℓ be a fiber of the $\mathbb{P}^1$-bundle structure $\pi \colon X \to \mathbb{P}^1$. Let (e_1, e_2) be the pair of non-negative integers* *determined by* $E \sim e_1\sigma + e_2\ell$*. If* $E \not\geq 2\sigma$*, then* $n \leq 4$ *and* (e_1, e_2) *is one of the following*:

$$\begin{aligned} &\textit{Case } n = 1: && (1,0), (1,1), (2,1), (2,2). \\ &\textit{Case } n = 2: && (1,0), (1,1), (1,2), (2,2), (2,3). \\ &\textit{Case } n = 3: && (1,0), (1,1), (2,3), (2,4). \\ &\textit{Case } n = 4: && (1,0), (2,4). \end{aligned}$$

If $E \geq 2\sigma$*, then* $e_1 = 2$ *and* $0 \leq e_2 \leq \min\{n+1, 4\}$*. The number of irreducible components of* E *is at most* 3 *in case* $E \not\geq 2\sigma$*, and is at most* 5 *in case* $E \geq 2\sigma$*.*

PROOF. The formula $-K_X \sim 2\sigma + (n+2)\ell$ implies $L \sim (4-e_1)\sigma + (2n+4-e_2)\ell$ and $K_X + L = (2-e_1)\sigma + (n+2-e_2)\ell$. Here, $2 - e_1 \geq 0$ by $(K_X + L)\ell \geq 0$, and $e_1 = E\ell > 0$ by $(2K_X + L)\ell < 0$. Hence $e_1 \in \{1, 2\}$. The condition $(\mathcal{C}1)$ is equivalent to: $n + 2 - e_2 \geq n(2 - e_1)$. Similarly, $(\mathcal{C}3')$ is equivalent to: $2n + 4 - e_2 \geq n(4 - e_1)$. Therefore

$$\begin{aligned} e_2 &\leq \min\{n(e_1-1)+2, n(e_1-2)+4\} \\ &= \begin{cases} \min\{2, 4-n\}, & \text{in case } e_1 = 1; \\ \min\{n+2, 4\}, & \text{in case } e_1 = 2. \end{cases} \end{aligned} \tag{3–3}$$

The genus g of (X, E) is calculated as follows:

$$\begin{aligned} 2g - 2 &= (K_X + L)L \\ &= -n(2-e_1)(4-e_1) + (2-e_1)(2n+4-e_2) + (4-e_1)(n+2-e_2) \\ &= (2-e_1)(n(e_1-1)+2-e_2) + (2-e_1)(4-e_2) + 2(n+2-e_2). \end{aligned}$$

Therefore, we have

$$2 \leq g = \begin{cases} n + 3 - e_2, & \text{in case } e_1 = 2; \\ n + 6 - 2e_2, & \text{in case } e_1 = 1. \end{cases} \tag{3–4}$$

Comparing with the inequality (3–3), we have a new inequality $e_2 \leq n + 1$ in case $e_1 = 2$, but no new inequalities in case $e_1 = 1$.

If $E \not\supset \sigma$, then $\sigma E = e_2 - ne_1 \geq 0$. If $E \supset \sigma$ but $E \not\geq 2\sigma$, then $E = \sigma + D$ for a divisor $D \not\supset \sigma$; thus $\sigma D = e_2 - n(e_1 - 1) \geq 0$. Combining with (3–3) and (3–4), we have

$$n \leq e_2 \leq \min\{2, 4-n\}, \quad \text{in case } e_1 = 1, E \not\supset \sigma;$$

$$\begin{aligned} 0 \le\, & e_2 \le \min\{2, 4-n\}, && \text{in case } e_1 = 1, E \supset \sigma; \\ 2n \le\, & e_2 \le \min\{n+1, 4\}, && \text{in case } e_1 = 2, E \not\supset \sigma; \\ n \le\, & e_2 \le \min\{n+1, 4\}, && \text{in case } e_1 = 2, E \not\geq 2\sigma; \\ 0 \le\, & e_2 \le \min\{n+1, 4\}, && \text{in case } e_1 = 2, E \ge 2\sigma. \end{aligned}$$

Therefore, $n \le 4$ in case $E \not\geq 2\sigma$, and the list of (e_1, e_2) is obtained for $n \ge 2$. In case $n = 1$, the minimality of (X, E) requires another condition: $0 \le (2K_X + L)\sigma = -E\sigma = e_1 - e_2$. Hence the case $(e_1, e_2) = (1, 2)$ is erased and the list is obtained.

Finally, we bound the number k_E of irreducible components of E. If $E \ge 2\sigma$, then $E = 2\sigma + \sum a_i \ell_i$ for fibers ℓ_i with $\sum a_i = e_2 \le 4$; thus $k_E \le 5$. Suppose that $E \not\geq 2\sigma$. If $e_1 = 1$, then E is a section of π or the union of σ and at most two fibers, since $e_2 \le 2$; thus $k_E \le 3$. The remaining case satisfies $E \not\geq 2\sigma$, $e_1 = 2$, and $4 \ge e_2 \in \{n, n+1\}$. If $e_2 = n$, then $E = \sigma + \sigma_\infty$ for a section σ_∞ at infinity; thus E is the disjoint union of two copies of $\mathbb{P}^1$ and $k_E = 2$. If $e_2 = n + 1$, then we have the following three possibilities:

(A) $E \not\supset \sigma$.

(B) $E = \sigma + D$ for a section $D \sim \sigma + (n+1)\ell$ of π.

(C) $E = \sigma + \sigma_\infty + \ell_0$ for a section σ_∞ at infinity and for a fiber ℓ_0 of π.

Then $k_E = 2$ in case (B), and $k_E = 3$ in case (C). In case (A), we have $n = 1$ and $k_E \le 2$. In fact, if $E \sim 2\sigma + 2\ell$ is reducible, then $E = D_1 + D_2$ for two sections D_1, D_2 at infinity, where $D_1 D_2 = 1$. □

We can classify the minimal basic pairs (X, E) by the following types:

$[e]$: $X \simeq \mathbb{P}^2$ and $\deg E = e \in \{1, 2\}$.

$[n; e_1, e_2]$: $X \simeq \mathbb{F}_n$ with $E \sim e_1\sigma + e_2\ell$. Here, σ is a minimal section and ℓ is a fiber for the $\mathbb{P}^1$-bundle structure $\pi\colon X \to \mathbb{P}^1$ (cf. Convention 3.23).

The types of minimal basic pairs are listed in TABLE 1 with the invariants g, LE, and $(K_X + L)^2$ by the results in Lemmas 3.21, 3.22, 3.24. We note that $K_X^2 = (K_X + L)^2 + LE$ (cf. (3–1)).

TABLE 1. The types of minimal basic pairs (X, E)

Type	g	LE	$(K_X + L)^2$	Type	g	LE	$(K_X + L)^2$
[1]	6	5	4	[3; 1, 0]	9	1	7
[2]	3	8	1	[3; 1, 1]	7	3	5
[0; 1, 0]	6	4	4	[3; 2, 0]	6	8	0
[0; 1, 1]	4	6	2	[3; 2, 1]	5	8	0
[0; 2, 0]	3	8	0	[3; 2, 2]	4	8	0
[0; 2, 1]	2	8	0	[3; 2, 3]	3	8	0
[1; 1, 0]	7	3	5	[3; 2, 4]	2	8	0
[1; 1, 1]	5	5	3	[4; 1, 0]	10	0	8
[1; 2, 0]	4	8	0	[4; 2, 0]	7	8	0
[1; 2, 1]	3	8	0	[4; 2, 1]	6	8	0
[1; 2, 2]	2	8	0	[4; 2, 2]	5	8	0
[2; 1, 0]	8	2	6	[4; 2, 3]	4	8	0
[2; 1, 1]	6	4	4	[4; 2, 4]	3	8	0
[2; 1, 2]	4	6	2	[n $\geq$ 5; 2, 0]	n + 3	8	0
[2; 2, 0]	5	8	0	[n $\geq$ 5; 2, 1]	n + 2	8	0
[2; 2, 1]	4	8	0	[n $\geq$ 5; 2, 2]	n + 1	8	0
[2; 2, 2]	3	8	0	[n $\geq$ 5; 2, 3]	n	8	0
[2; 2, 3]	2	8	0	[n $\geq$ 5; 2, 4]	n - 1	8	0

COROLLARY 3.25. *Let (X, E) be a minimal basic pair and set $L = -2K_X - E$. If $K_X + L$ is ample, then it is very ample. If $K_X + L$ is not ample but big, then (X, E) is of type $[2; 1, 2]$. If $K_X + L$ is not big, then (X, E) is of type $[n; 2, e_2]$ with $0 \leq e_2 \leq \min\{n + 1, 4\}$.*

PROOF. An ample divisor on X is always very ample for $X = \mathbb{P}^2$ or $X = \mathbb{F}_n$. If $X = \mathbb{P}^2$, then $K_X + L$ is ample. Thus we have only to determine when $K_X + L$ is ample for $X = \mathbb{F}_n$. If we write $K_X + L \sim d_1\sigma + d_2\ell$, then $d_1 = 2 - e_1$, $d_2 = n + 2 - e_2$. Here, $K_X + L$ is ample if and only if $d_2 > nd_1$ and $d_1 > 0$. Thus $K_X + L$ is not big if $e_1 = 2$. If $e_1 = 1$ and $K_X + L$ is not ample, then (X, E) is of type $[2; 1, 2]$. $\square$

3.4. Anti log-canonical rings

For a graded $\Bbbk$-algebra $R = \bigoplus_{m \geq 0} R_m$, the m-th piece R_m denotes the module of homogeneous elements of degree m. The n-th truncation $R^{(n)}$ for $n > 0$ is defined by $R^{(n)} = \bigoplus_{m \geq 0} R_{nm}$, i.e., $(R^{(n)})_m = R_{nm}$.

For a normal complete variety Z and a $\mathbb{Q}$-divisor D, we define a graded

$\Bbbk$-algebra by

$$R(Z,D) = \bigoplus\nolimits_{m\geq 0} R(Z,D)_m = \bigoplus\nolimits_{m\geq 0} \mathrm{H}^0(Z, \llcorner mD \lrcorner)$$

(cf. [11]), where $\llcorner\cdot\lrcorner$ stands for the *round-down*. Here, $R(Z,D)^{(n)} \simeq R(Z,nD)$ for $n>0$.

Let (S,B) be a del Pezzo pair of index at most two. We consider the anti log-canonical ring $R[S,B] := R(S, -K_S - B)$ and its second truncation $R[S,B]^{(2)} = R(S, -2(K_S+B))$. The latter is isomorphic to $R(M, L_M)$. We set $E_M^\circ = \llcorner (1/2)E_M \lrcorner$. Then $E_M - 2E_M^\circ$ is a reduced divisor or zero. Note that $E_M^\circ = 0$ if (S,B) is log-terminal.

LEMMA 3.26. *There is an isomorphism*

$$R[S,B]_{2k-1} = \mathrm{H}^0\left(S, \llcorner -(2k-1)(K_S+B) \lrcorner\right) \simeq \mathrm{H}^0(M, K_M + E_M^\circ + kL_M)$$

for any positive integer k.

PROOF. Let D be a $\mathbb{Q}$-divisor on M which is relatively numerically trivial with respect to the minimal desingularization $\alpha\colon M \to S$. Then $\alpha_*\mathcal{O}_M(\llcorner D \lrcorner)$ is a reflexive sheaf. This is shown as follows: We may replace S with an open subset freely since the property is local. If it is not reflexive, then $\alpha_*\mathcal{O}_M(\llcorner D \lrcorner) \subsetneq \alpha_*\mathcal{O}_M(\llcorner D \lrcorner + E')$ for an α-exceptional effective divisor E'. A section of $\alpha_*\mathcal{O}_M(\llcorner D \lrcorner + E')$ defines an effective $\mathbb{Q}$-divisor D' on M such that $\langle D' \rangle = \langle D \rangle$ and $D' - (D + E') = \llcorner D' \lrcorner - (\llcorner D \lrcorner + E')$ is linearly equivalent to 0. Then $D' \geq E'$, since $DE'_i = E'E'_i$ for any irreducible component E'_i of E'. This argument says essentially that the negative part of the relative Zariski-decomposition of $D+E'$ is E'. Therefore, the section defining D' comes from a section of $\alpha_*\mathcal{O}_M(\llcorner D \lrcorner)$. Thus, $\alpha_*\mathcal{O}_M(\llcorner D \lrcorner)$ is reflexive.

We can apply the reflexive property to the $\mathbb{Q}$-divisor $K_M + (1/2)E_M + kL_M$, since $K_M + (1/2)E_M + kL_M = (k-(1/2))L_M$ is α-numerically trivial. Hence,

$$\alpha_*\mathcal{O}_M\left(\llcorner K_M + (1/2)E_M + kL_M \lrcorner\right) \simeq \mathcal{O}_S\left(\llcorner -(2k-1)(K_S+B) \lrcorner\right),$$

since $\alpha_* L_M \sim -2(K_S+B)$ and $(1/2)\alpha_* E_M = B$. $\square$

Therefore, $R[S,B]$ is isomorphic to the graded ring

$$\bigoplus\nolimits_{m=2k,\,k\geq 0}\mathrm{H}^0(M,kL_M)\oplus\bigoplus\nolimits_{m=2k-1,\,k\geq 1}\mathrm{H}^0(M,K_M+E_M^\circ+kL_M),$$

where $R[S,B]_{2k-1}\otimes R[S,B]_{2l-1}\to R[S,B]_{2(k+l-1)}$ is induced from

$$\begin{aligned}&\mathrm{H}^0(M,2K_M+2E_M^\circ+(k+l)L_M)\\&\qquad=\mathrm{H}^0(M,-E_M+2E_M^\circ+(k+l-1)L_M)\subset\mathrm{H}^0(M,(k+l-1)L_M).\end{aligned}$$

Suppose that K_M+L_M is nef. For a positive integer k with $R[S,B]_{2k-1}\neq 0$, equivalently, $|K_M+E_M^\circ+kL|\neq 0$, let us consider the set $\mathcal{S}_k$ of effective divisors $N\leq E_M^\circ$ such that $K_M+E_M^\circ+kL-N$ is nef. Then $E_M^\circ\in\mathcal{S}_k$. We define

$$N^{(k)}:=\sum\nolimits_\Gamma\min\{\mathrm{mult}_\Gamma(N)\mid N\in\mathcal{S}_k\}\Gamma.$$

Then $N^{(k)}\in\mathcal{S}_k$. In fact, for an irreducible curve γ on M, there is an effective divisor $N\in\mathcal{S}_k$ with $\mathrm{mult}_\gamma(N)=\mathrm{mult}_\gamma(N^{(k)})$ and $(N-N^{(k)})\gamma\geq 0$; hence

$$(K_M+E_M^\circ+kL-N^{(k)})\gamma=(K_M+E_M^\circ+kL-N)\gamma+(N-N^{(k)})\gamma\geq 0.$$

We define $E_M^{(k)}:=E_M^\circ-N^{(k)}$ if $R[S,B]_{2k-1}\neq 0$; and $E_M^{(k)}:=0$ if $R[S,B]_{2k-1}=0$. Then $E_M^{(k)}\leq E_M^{(k+1)}$ and $K_M+E_M^{(k)}+kL_M$ is nef for any $k>0$. We also define $E_M^{(\infty)}$ to be $E_M^{(k)}$ for $k\gg 0$. Then $K_M+E_M^{(\infty)}$ is α-nef with an isomorphism

$$\alpha_*\mathcal{O}_M(K_M+E_M^{(\infty)})\simeq\alpha_*\mathcal{O}_M(K_M+E_M^\circ)\simeq\mathcal{O}_S\left(K_S+\llcorner(1/2)B\lrcorner\right).$$

Lemma 3.27. *If K_M+L_M is nef, then there is an isomorphism*

$$R[S,B]_{2k-1}=\mathrm{H}^0\left(S,\llcorner-(2k-1)(K_S+B)\lrcorner\right)\simeq\mathrm{H}^0(M,K_M+E_M^{(k)}+kL_M)$$

for any positive integer $k>0$.

Proof. Assume the contrary. Then $R[S,B]_{2k-1}\neq 0$ and $E_M^\circ\neq E_M^{(k)}$ by Lemma 3.26. Let $D'\leq E_M^\circ-E_M^{(k)}$ be any non-zero effective divisor. Then $(K_M+E_M^{(k)}+D'+kL_M)\gamma<0$ for an irreducible curve γ. Here, $D'\geq\gamma$ and

$$\mathrm{H}^0(M,K_M+E_M^{(k)}+(D'-\gamma)+kL_M)\simeq\mathrm{H}^0(M,K_M+E_M^{(k)}+D'+kL_M).$$

By induction on $\deg D'$, we have a contradiction. $\square$

LEMMA 3.28 (cf. [12, Lemma 1.8]). *Let Z be a scheme and D an effective Cartier divisor. For two invertible sheaves $\mathcal{L}$ and $\mathcal{M}$ on Z, the multiplication map $\mathrm{H}^0(Z,\mathcal{L})\otimes\mathrm{H}^0(Z,\mathcal{M})\to\mathrm{H}^0(Z,\mathcal{L}\otimes\mathcal{M})$ is surjective provided that the following three conditions are satisfied*:

(S1) $\mathrm{H}^1(Z,\mathcal{L}(-D))=0$;

(S2) $\mathrm{H}^0(D,\mathcal{L}|_D)\otimes\mathrm{H}^0(Z,\mathcal{M})\to\mathrm{H}^0(D,\mathcal{L}\otimes\mathcal{M}|_D)$ *is surjective*;

(S3) $\mathrm{H}^0(Z,\mathcal{L}(-D))\otimes\mathrm{H}^0(Z,\mathcal{M})\to\mathrm{H}^0(Z,\mathcal{L}\otimes\mathcal{M}(-D))$ *is surjective.*

PROOF. By the three conditions, we have a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{H}^0(\mathcal{L}(-D))\otimes\mathrm{H}^0(\mathcal{M}) & \longrightarrow & \mathrm{H}^0(\mathcal{L})\otimes\mathrm{H}^0(\mathcal{M}) & \longrightarrow & \mathrm{H}^0(\mathcal{L}|_D)\otimes\mathrm{H}^0(\mathcal{M}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{H}^0(\mathcal{L}\otimes\mathcal{M}(-D)) & \longrightarrow & \mathrm{H}^0(\mathcal{L}\otimes\mathcal{M}) & \longrightarrow & \mathrm{H}^0((\mathcal{L}\otimes\mathcal{M})|_D) & &
\end{array}$$

of exact sequences in which the left and right vertical arrows are surjective. Thus the middle vertical arrow is also surjective. $\square$

LEMMA 3.29. *Let Z be a one-dimensional projective scheme with $\mathrm{H}^1(Z,\mathcal{O}_Z)=0$, $\mathcal{L}$ a nef invertible sheaf, and let $\mathcal{F}$ be a coherent sheaf on Z generated by global sections. Then the multiplication map $\mathrm{H}^0(Z,\mathcal{L})\otimes\mathrm{H}^0(Z,\mathcal{F})\to\mathrm{H}^0(Z,\mathcal{L}\otimes\mathcal{F})$ is surjective.*

PROOF. By the proof of Lemma 2.8, there is an effective Cartier divisor D of Z such that $\mathcal{L}\simeq\mathcal{O}_Z(D)$ and $\mathcal{F}\to\mathcal{F}\otimes\mathcal{O}_Z(D)$ is injective outside a closed subset of dimension ≤ 0. Let $\mathcal{F}'$ be the image of $\mathcal{F}\to\mathcal{F}\otimes\mathcal{O}_Z(D)$. Then $\mathrm{H}^0(Z,\mathcal{F})\to\mathrm{H}^0(Z,\mathcal{F}')$ is surjective. As in the proof of Lemma 3.28, we have a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{H}^0(\mathcal{O}_Z)\otimes\mathrm{H}^0(\mathcal{F}) & \longrightarrow & \mathrm{H}^0(\mathcal{O}_Z(D))\otimes\mathrm{H}^0(\mathcal{F}) & \longrightarrow & \mathrm{H}^0(\mathcal{O}_Z(D)|_D)\otimes\mathrm{H}^0(\mathcal{F}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{H}^0(\mathcal{F}') & \longrightarrow & \mathrm{H}^0(\mathcal{O}_Z(D)\otimes\mathcal{F}) & \longrightarrow & \mathrm{H}^0(\mathcal{O}_Z(D)\otimes\mathcal{F}\otimes\mathcal{O}_D) & &
\end{array}$$

of exact sequences, where the left vertical arrow is surjective, and the right vertical arrow is surjective, since $\dim D=0$ and $\mathcal{F}$ is generated by global sections. Thus the middle one is also surjective. $\square$

LEMMA 3.30. *For a basic pair (M,E_M), the following properties hold*:

(1) $\mathrm{H}^1(M, mL_M + j(K_M + L_M)) = 0$ *for any* m, $j \geq 0$.

(2) $\mathrm{H}^1(M, K_M + mL_M - jE_M) = 0$ *and* $\mathrm{H}^1(M, K_M + E_M^{(m)} + mL_M) = 0$ *for any* $m > j \geq 0$.

(3) *If* $K_M + L_M$ *is big, then* $\mathrm{H}^1(M, j(K_M + L_M) - E_M) = 0$ *for* $j \geq 0$.

(4) $\mathrm{H}^0(M, K_M+L_M)^{\otimes m} \to \mathrm{H}^0(M, m(K_M+L_M))$ *is surjective for* $m \geq 1$.

(5) *If* $K_M + L_M$ *is not big, then*

$$\begin{aligned}\mathrm{H}^0(M, j(K_M + L_M) + E_M) \otimes \mathrm{H}^0(M, K_M + L_M)&\\ \to \mathrm{H}^0(M, (j+1)(K_M + L_M) + E_M)&\end{aligned}$$

is surjective for $j \geq 3$.

(6) *If* $K_M + L_M$ *is not big with* $(K_M + L_M)L_M > 2$, *then*

$$\begin{aligned}\mathrm{H}^0(M, j(K_M + L_M)) \otimes \mathrm{H}^0(E_M, \mathcal{O}_{E_M})&\\ \to \mathrm{H}^0(E_M, j(K_M + L_M)|_{E_M}),&\end{aligned} \tag{3–5}$$

$$\begin{aligned}\mathrm{H}^0(M, j(K_M + L_M)) \otimes \mathrm{H}^0(E_M, L_M|_{E_M})&\\ \to \mathrm{H}^0(E_M, (jK_M + (j+1)L_M)|_{E_M})&\end{aligned} \tag{3–6}$$

are surjective for $j \geq 0$.

(7) *If* $K_M + L_M$ *is not big with* $(K_M + L_M)L_M = 2$, *then*

$$\begin{aligned}\mathrm{H}^0(M, j(K_M + L_M)) \otimes \mathrm{H}^0(E_M, (K_M + L_M)|_{E_M})&\\ \to \mathrm{H}^0(E_M, (j+1)(K_M + L_M)|_{E_M}),&\end{aligned} \tag{3–7}$$

$$\begin{aligned}\mathrm{H}^0(M, j(K_M + L_M)) \otimes \mathrm{H}^0(E_M, (K_M + 2L_M)|_{E_M})&\\ \to \mathrm{H}^0(E_M, ((j+1)K_M + (j+2)L_M)|_{E_M})&\end{aligned} \tag{3–8}$$

are surjective for $j \geq 0$.

Proof. Let $\phi\colon M \to X$ be a birational map such that (X, E) is a minimal basic triplet for $E = \phi_*(E_M)$ and that $K_M + L_M \sim \phi^*(K_X + L)$. Since $X \simeq \mathbb{P}^1$ or $\mathbb{F}_n$, we have a non-singular member $C \in |K_M + L_M|$. If

$K_M + L_M$ is big, then $C \simeq \mathbb{P}^1$. If $K_M + L_M$ is not big, then C is a union of copies of $\mathbb{P}^1$, which are fibers of $\pi \circ \phi\colon M \to X \to \mathbb{P}^1$.

(1): The vanishing follows from

$$\begin{aligned}0 &\to \mathrm{H}^1(M, mL_M + (i-1)C) \to \mathrm{H}^1(M, mL_M + iC)\\ &\to \mathrm{H}^1(C, (mL_M + iC)|_C) = 0\end{aligned}$$

for $1 \le i \le j$ and the vanishing $\mathrm{H}^1(M, mL_M) = 0$ by Theorem 3.18.

(2): The first vanishing follows from (1), since

$$K_M + mL_M - jE_M = (m - j - 1)L_M + (2j + 1)(K_M + L_M).$$

For the second, we may assume $E_M^{(m)} \neq 0$. Then $\mathrm{H}^1(E_M^{(m)}, \mathcal{O}_{E_M^{(m)}}) = 0$ by $E_M^{(m)} \le E_M$, and $K_M + E_M^{(m)} + mL_M$ is nef. Therefore,

$$\mathrm{H}^1(E_M^{(m)}, (K_M + E_M^{(m)} + mL_M)|_{E_M^{(m)}}) = 0$$

by Lemma 2.8. Combing with the first vanishing for $j = 0$, we have the second vanishing.

(3): We have

$$\begin{aligned}\mathrm{H}^1(M, j(K_M + L_M) - E_M) &\simeq \mathrm{H}^1(M, (j + 1)(K_M + L_M) + K_M)\\ &\simeq \mathrm{H}^1(X, (j + 1)(K_X + L) + K_X).\end{aligned}$$

Since $K_X + L$ is nef and big, this cohomology group vanishes for $j \ge 0$ if $\operatorname{char} \Bbbk = 0$. Since $X \simeq \mathbb{P}^2$ or $\mathbb{F}_n$, X is a toric variety and thus this cohomology group is described by combinatorial data which do not depend on $\operatorname{char} \Bbbk$. Thus we have the vanishing.

(4): The homomorphism is isomorphic to

$$\mathrm{H}^0(X, K_X + L)^{\otimes m} \to \mathrm{H}^0(X, m(K_X + L))$$

If $X \simeq \mathbb{P}^2$, then this is surjective. If $X \simeq \mathbb{F}_n$, then $K_X + L \sim d_1\sigma + d_2\ell$ for $d_1 \in \{0, 1\}$ and $d_2 \ge nd_1$. If $d_1 = 0$, then the surjectivity follows from that of

$$\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(1))^{\otimes m} \to \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(m)).$$

If $d_1 = 1$, then it also follows from the surjectivity of

$$\operatorname{Sym}^m \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(d_2) \oplus \mathcal{O}(d_2 - n)) \to \mathrm{H}^0(\mathbb{P}^1, \operatorname{Sym}^m(\mathcal{O}(d_2) \oplus \mathcal{O}(d_2 - n))).$$

(5): We have $K_X + L \sim d_2\ell$ for an integer $d_2 > 0$. Since $\mathrm{H}^1(M, L_M - E_M + E_M) = 0$, we have

$$\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(2d_2) \otimes \pi_*\phi_*\mathcal{O}_M(E_M)) = 0.$$

Hence, $\pi_*\phi_*\mathcal{O}_M(E_M) \simeq \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \mathcal{O}(a_3)$ for integers $a_i \geq -2d_2 - 1$. If $jd_2 + a_i \geq 0$ for any i, (this is satisfied for $j \geq 3$), then the multiplication map in question is surjective since so is

$$\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(jd_2 + a_i)) \otimes \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(d_2)) \to \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}((j+1)d_2 + a_i)).$$

(6): As in (5), we have $K_M + L_M \sim d_2\phi^*\ell$. Then $d_2 > 1$ by $(K_M + L_M)L_M > 2$. For the commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^0(j(K_M + L_M)) \otimes \mathrm{H}^0(\mathcal{O}_{E_M}) & \longrightarrow & \mathrm{H}^0(j(K_M + L_M)) \otimes \mathrm{H}^1(-E_M) \\
\downarrow & & \downarrow \\
\mathrm{H}^0(j(K_M + L_M)|_{E_M}) & \longrightarrow & \mathrm{H}^1(j(K_M + L_M) - E_M),
\end{array}$$

the horizontal arrows are surjective with the isomorphic kernels. The surjectivity of (3–5) follows from that of the right vertical arrow, which is just the H^1 of the surjection

$$\mathrm{H}^0(M, j(K_M + L_M)) \otimes_{\Bbbk} \mathcal{O}_M(-E_M) \to \mathcal{O}_M(j(K_M + L_M) - E_M).$$

Since we have an exact sequence

$$0 \to \mathcal{O}(-1)^{\oplus m} \to \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(m)) \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}(m) \to 0 \tag{3–9}$$

for $m \geq 1$, the expected surjectivity follows from

$$\begin{aligned}
\mathrm{H}^2(M, -\phi^*\ell - E_M) &\simeq \mathrm{H}^0(M, K_M + E_M + \phi^*\ell)^\vee \\
&\simeq \mathrm{H}^0(M, (1 - d_2)\phi^*\ell)^\vee = 0.
\end{aligned}$$

For the homomorphism (3–6), it is enough to prove that the composite

$$\begin{aligned}
&\mathrm{H}^0(M, j(K_M + L_M)) \otimes \mathrm{H}^0(M, L_M) \otimes \mathrm{H}^0(E_M, \mathcal{O}_{E_M}) \\
&\qquad \to \mathrm{H}^0(M, j(K_M + L_M)) \otimes \mathrm{H}^0(E_M, L_M|_{E_M}) \\
&\qquad\qquad \to \mathrm{H}^0(E_M, (jK_M + (j+1)L_M)|_{E_M})
\end{aligned}$$

is surjective. This is also written as the composite

$$\begin{aligned}&\mathrm{H}^0(M, j(K_M+L_M)) \otimes \mathrm{H}^0(M, L_M) \otimes \mathrm{H}^0(E_M, \mathcal{O}_{E_M})\\ &\qquad\to \mathrm{H}^0(E_M, j(K_M+L_M)|_{E_M}) \otimes \mathrm{H}^0(M, L_M)\\ &\qquad\to \mathrm{H}^0(E_M, j(K_M+L_M)|_{E_M}) \otimes \mathrm{H}^0(E_M, L_M|_{E_M})\\ &\qquad\to \mathrm{H}^0(E_M, (jK_M+(j+1)L_M)|_{E_M}).\end{aligned}$$

This is surjective by the surjectivity of (3–5), $\mathrm{H}^1(L_M - E_M) = 0$, and by Lemma 3.29.

(7): We have $K_M + L_M \sim \phi^*\ell$ by assumption. For the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^0(j(K_M+L_M)) \otimes \mathrm{H}^0((K_M+L_M)|_{E_M}) & \longrightarrow & \mathrm{H}^0(j(K_M+L_M)) \otimes \mathrm{H}^1(K_M+L_M-E_M) \\ \downarrow & & \downarrow \\ \mathrm{H}^0((j+1)(K_M+L_M)|_{E_M}) & \longrightarrow & \mathrm{H}^1((j+1)(K_M+L_M)-E_M), \end{array}$$

the horizontal arrows are surjective, and a surjection is induced between the kernels by (4). Hence, the surjectivity of (3–7) follows from that of the right vertical arrow, which is just the H^1 of the surjection

$$\mathrm{H}^0(M, j(K_M+L_M)) \otimes_{\Bbbk} \mathcal{O}_M(K_M+L_M-E_M) \to \mathcal{O}_M((j+1)(K_M+L_M)-E_M).$$

The kernel of the sheaf homomorphism is isomorphic to the direct sum of some copies of $\mathcal{O}_M(-\phi^*\ell) \otimes \mathcal{O}_M(K_M+L_M-E_M) \simeq \mathcal{O}_M(-E_M)$ by the exact sequence (3–9) for $m = j$. Since

$$\mathrm{H}^2(M, -E_M) \simeq \mathrm{H}^0(M, K_M+E_M)^\vee \simeq \mathrm{H}^0(M, -\phi^*\ell)^\vee = 0,$$

the expected surjectivity follows. For the homomorphism (3–8), it is enough to show the composite

$$\begin{aligned}&\mathrm{H}^0(M, j(K_M+L_M)) \otimes \mathrm{H}^0(M, L_M) \otimes \mathrm{H}^0(E_M, (K_M+L_M)|_{E_M})\\ &\qquad\to \mathrm{H}^0(M, j(K_M+L_M)) \otimes \mathrm{H}^0(E_M, (K_M+2L_M)|_{E_M})\\ &\qquad\to \mathrm{H}^0(E_M, ((j+1)K_M+(j+2)L_M)|_{E_M})\end{aligned}$$

is surjective. This is written also as the composite

$$\begin{aligned}\mathrm{H}^0(M, j(K_M+L_M)) \otimes \mathrm{H}^0(M, L_M) \otimes \mathrm{H}^0(E_M, (K_M+L_M)|_{E_M}) \\ \to \mathrm{H}^0(E_M, (j+1)(K_M+L_M)|_{E_M}) \otimes \mathrm{H}^0(M, L_M) \\ \to \mathrm{H}^0(E_M, ((j+1)K_M+(j+2)L_M)|_{E_M}).\end{aligned}$$

This is surjective by the surjectivity of (3–7), $\mathrm{H}^1(L_M - E_M) = 0$, and by Lemma 3.29. □

Proposition 3.31. *Let (M, E_M) be a basic pair. Then the multiplication maps*

$$\begin{aligned}\mu_m &\colon \mathrm{H}^0(M, mL_M) \otimes \mathrm{H}^0(M, L_M) \to \mathrm{H}^0(M, (m+1)L_M), \\ \mu'_m &\colon \mathrm{H}^0(M, K_M + mL_M) \otimes \mathrm{H}^0(M, L_M) \to \mathrm{H}^0(M, K_M + (m+1)L_M) \\ \mu''_m &\colon \mathrm{H}^0(M, K_M + E_M^{(m)} + mL_M) \otimes \mathrm{H}^0(M, L_M) \\ &\qquad \to \mathrm{H}^0(M, K_M + E_M^{(m)} + (m+1)L_M)\end{aligned}$$

are surjective for $m \geq 2$. If $(K_M+L_M)L_M > 2$, then these are surjective for $m \geq 1$. If $(K_M+L_M)L_M = 2$, then the following homomorphism is also surjective:

$$\begin{aligned}\mu''' \colon \big(\mathrm{H}^0(M, K_M+L_M) \otimes \mathrm{H}^0(K_M+2L_M)\big) \\ \oplus \mathrm{H}^0(M, L_M)^{\otimes 2} \to \mathrm{H}^0(M, 2L_M).\end{aligned}$$

Proof. We have the following three cases of (M, E_M):

(i) $K_M + L_M$ is big;

(ii) $K_M + L_M$ is not big and $(K_M+L_M)L_M > 2$;

(iii) $(K_M+L_M)L_M = 2$.

Note that $(K_M+L_M)L_M > 2$ if K_M+L_M is big (cf. Table 1). In the proof below, *Step* 1 gives a reduction for the proof related to μ_m and μ'_m. We shall show the surjectivity of μ_m and μ'_m in the cases (i) and (ii) in *Step* 2. The same thing in the case (iii) is shown in *Step* 3. The surjectivity of μ''_m is shown in *Step* 4, and that of μ''' in *Step* 5.

Step 1. Let us consider the following multiplication maps:

$$\begin{aligned}\mu_{m,j}\colon\ &\mathrm{H}^0(M, mL_M - jE_M) \otimes \mathrm{H}^0(M, L_M) \to \mathrm{H}^0(M, (m+1)L_M - jE_M),\\ \mu'_{m,j}\colon\ &\mathrm{H}^0(M, K_M + mL_M - jE_M) \otimes \mathrm{H}^0(M, L_M)\\ &\qquad\qquad\to \mathrm{H}^0(M, K_M + (m+1)L_M - jE_M)\end{aligned}$$

for $0 \le j \le m$. We have $\mathrm{H}^1(M, mL_M - jE_M) = 0$ for $m \ge j \ge 0$ and $\mathrm{H}^1(M, K_M + mL_M - jE_M) = 0$ for $m > j \ge 0$ by Lemma 3.30, (1), (2). We infer that the natural homomorphisms

$$\begin{aligned}&\mathrm{H}^0(E_M, (mL_M - jE_M)|_{E_M}) \otimes \mathrm{H}^0(M, L_M)\\ &\qquad\to \mathrm{H}^0(E_M, ((m+1)L_M - jE_M)|_{E_M}),\\ &\mathrm{H}^0(E_M, (K_M + mL_M - jE_M)|_{E_M}) \otimes \mathrm{H}^0(M, L_M)\\ &\qquad\to \mathrm{H}^0(E_M, (K_M + (m+1)L_M - jE_M)|_{E_M})\end{aligned}$$

are both surjective by Lemma 3.29 and by $\mathrm{H}^1(M, L_M - E_M) = 0$. Applying Lemma 3.28 to the case $Z = M$, $D = E_M$, $\mathcal{L} = \mathcal{O}(mL_M - jE_M)$, $\mathcal{M} = \mathcal{O}_M(L_M)$, for $0 \le j \le m$, we infer that the surjectivity of μ_m is reduced to that of $\mu_{m,j}$ for $j \le m$. Similarly, applying Lemma 3.28 to the case $Z = M$, $D = E_M$, $\mathcal{L} = \mathcal{O}(K_M + mL_M - jE_M)$, $\mathcal{M} = \mathcal{O}_M(L_M)$, for $0 \le j < m$, we infer that the surjectivity of μ'_m is reduced to that of $\mu'_{m,j}$ for $j < m$.

Step 2. We consider the cases (i) and (ii). We shall check the surjectivity of $\mu_{m,m}$ for $m \ge 1$ by applying Lemma 3.28 to the case $Z = M$, $D = E_M$, $\mathcal{L} = \mathcal{O}_M(L_M)$, $\mathcal{M} = \mathcal{O}_M(m(L_M - E_M))$. Here, the condition (S1) is satisfied by $\mathrm{H}^1(L_M - E_M) = 0$. The homomorphism of (S2) is

$$\begin{aligned}&\mathrm{H}^0(M, m(L_M - E_M)) \otimes \mathrm{H}^0(E_M, L|_{E_M})\\ &\qquad\qquad\qquad\to \mathrm{H}^0(E_M, ((m+1)L_M - mE_M)|_{E_M}),\end{aligned}$$

which is surjective by Lemma 3.30, (3), and Lemma 3.29 for the case (i), and by the surjectivity of (3–6) for the case (ii). The homomorphism of (S3) is

$$\mathrm{H}^0(M, m(L_M - E_M)) \otimes \mathrm{H}^0(M, L_M - E_M) \to \mathrm{H}^0(M, (m+1)(L_M - E_M)),$$

which is also surjective by Lemma 3.30, (4). Thus $\mu_{m,m}$ and μ_m are surjective.

Still in the cases (i) and (ii), we shall check the surjectivity of $\mu'_{m,m-1}$ for $m \geq 1$ by applying Lemma 3.28 to the case $Z = M$, $D = E_M$, $\mathcal{L} = \mathcal{O}_M(L_M)$, $\mathcal{M} = \mathcal{O}_M(K_M + mL_M - (m-1)E_M))$. Here, (S1) is satisfied by $\mathrm{H}^1(L_M - E_M) = 0$. The homomorphism of (S2) is written as

$$\begin{aligned}\mathrm{H}^0(K_M + mL_M - (m-1)E_M)) \otimes \mathrm{H}^0(E_M, L|_{E_M}) \\ \to \mathrm{H}^0((K_M + (m+1)L_M - (m-1)E_M)|_{E_M})\end{aligned}$$

and it is surjective. In fact, in the case (i), it follows from the vanishing

$$\mathrm{H}^1(M, K_M + m(L_M - E_M)) \simeq \mathrm{H}^1(X, (2m-1)(K_M + L_M) - E_M) = 0$$

shown in Lemma 3.30, (3), and from Lemma 3.29; in the case (ii), it is just the homomorphism (3–6) for $j = 2m - 1$. The homomorphism (S3) is

$$\begin{aligned}\mathrm{H}^0(K_M + mL_M - (m-1)E_M)) \otimes \mathrm{H}^0(L_M - E_M) \\ \to \mathrm{H}^0(K_M + (m+1)L_M - mE_M),\end{aligned}$$

which is surjective by Lemma 3.30, (4). Thus, $\mu'_{m,m-1}$ and μ'_m are surjective. Hence, we are done for μ_m and μ'_m in the cases (i) and (ii).

Step 3. We consider the case (iii). We shall check the surjectivity of $\mu_{m,m-1}$ for $m \geq 2$ by applying Lemma 3.28 to the case $Z = M$, $D = E_M$, $\mathcal{L} = \mathcal{O}_M(L_M)$, $\mathcal{M} = \mathcal{O}_M(mL_M - (m-1)E_M)$. Here, (S1) is satisfied by $\mathrm{H}^1(L_M - E_M) = 0$. The homomorphism of (S2) is

$$\begin{aligned}\mathrm{H}^0(mL_M - (m-1)E_M)) \otimes \mathrm{H}^0(E_M, L|_{E_M}) \\ \to \mathrm{H}^0(((m+1)L_M - (m-1)E_M)|_{E_M}),\end{aligned}$$

which is surjective by Theorem 3.18 and Lemma 3.29. The homomorphism of (S3) is

$$\mathrm{H}^0(mL_M - (m-1)E_M)) \otimes \mathrm{H}^0(L_M - E_M) \to \mathrm{H}^0((m+1)L_M - mE_M),$$

which is surjective for $2m \geq 3$ by Lemma 3.30, (5). Thus, $\mu_{m,m-1}$ and μ_m are surjective for $m \geq 2$.

We shall check the surjectivity of $\mu'_{m,m-2}$ for $m \geq 2$ by applying Lemma 3.28 to the case $Z = M$, $D = E_M$, $\mathcal{L} = \mathcal{O}_M(L_M)$, $\mathcal{M} = \mathcal{O}_M(K_M +$

$mL_M - (m-2)E_M)$. Here, (S1) is satisfied by $\mathrm{H}^1(L_M - E_M) = 0$. The homomorphism of (S2) is

$$\mathrm{H}^0(K_M + mL_M - (m-2)E_M)) \otimes \mathrm{H}^0(E_M, L|_{E_M}) \\ \to \mathrm{H}^0\left(((m+1)L_M - (m-2)E_M)|_{E_M}\right),$$

which is surjective by Lemma 3.30, (1) and Lemma 3.29. The homomorphism of (S3) is

$$\mathrm{H}^0(K_M + mL_M - (m-2)E_M)) \otimes \mathrm{H}^0(L_M - E_M) \\ \to \mathrm{H}^0(K_M + (m+1)L_M - (m-1)E_M),$$

which is surjective by (4), (5) of Lemma 3.30, since $K_M + mL_M - (m-2)E_M = (2m-1)(K_M + L_M) + E_M$. Hence, $\mu'_{m,m-2}$ and μ'_m are surjective for $m \geq 2$. Therefore, we are done for μ_m and μ'_m.

Step 4. We shall show the surjectivity of μ''_m for $m \geq 1$ in the cases (i), (ii), and for $m \geq 2$ in the case (iii). We apply Lemma 3.28 to the case $Z = M$, $D = E_M^{(m)}$, $\mathcal{L} = \mathcal{O}_M(K_M + E_M^{(m)} + mL_M)$, $\mathcal{M} = \mathcal{O}_M(L_M)$. Here, (S1) is satisfied by Lemma 3.30, (1). The homomorphism of (S3) is nothing but the surjection $\mu'_{m,m-1}$. By $\mathrm{H}^1(L_M - E_M) = 0$, (S2) is derived from the surjectivity of

$$\mathrm{H}^0(E_M^{(m)}, (K_M + E_M^{(m)} + mL_M)|_{E_M^{(m)}}) \otimes \mathrm{H}^0(E_M, L_M|_{E_M}) \\ \to \mathrm{H}^0(E_M^{(m)}, (K_M + E_M^{(m)} + (m+1)L_M)|_{E_M^{(m)}})$$

Here, $\mathcal{F} = \mathcal{O}_M(K_M + E_M^{(m)} + mL_M)|_{E_M^{(m)}}$ is generated by global sections, since $(K_M + E_M^{(m)} + mL_M)|_{E_M^{(m)}}$ is nef and $\mathrm{H}^1(\mathcal{O}_{E_M^{(m)}}) = 0$ (cf. Lemma 2.8). Since $L_M|_{E_M}$ is nef, the homomorphism above is surjective by Lemma 3.29. Therefore, μ''_m is surjective.

Step 5. Since the composite $\mathrm{H}^0(M, L_M)^{\otimes 2} \to \mathrm{H}^0(M, 2L_M) \to \mathrm{H}^0(E_M, L|_{E_M})$ is surjective, it is enough to show the surjectivity of

$$\mathrm{H}^0(M, K_M + L_M) \otimes \mathrm{H}^0(M, K_M + 2L_M) \\ \to \mathrm{H}^0(M, 2K_M + 3L_M) \simeq \mathrm{H}^0(M, 2L_M - E_M).$$

By Lemma 3.28 applied to the case $Z = M$, $D = E_M$, $\mathcal{L} = \mathcal{O}_M(K_M+2L_M)$, $\mathcal{M} = \mathcal{O}_M(K_M+L_M)$ and by $\mathrm{H}^1(K_M+2L_M-E_M) = 0$, this is also reduced to showing the surjectivity of

$$\mathrm{H}^0(K_M+L_M) \otimes \mathrm{H}^0(K_M+2L_M-E_M) \to \mathrm{H}^0(2K_M+3L_M-E_M) \quad \text{and}$$
$$\mathrm{H}^0(M, K_M+L_M) \otimes \mathrm{H}^0(E_M, (K_M+2L_M)|_{E_M}) \to \mathrm{H}^0(E_M, (2K_M+3L_M)|_{E_M}).$$

The first one is surjective by Lemma 3.30, (4), and the second one is just the surjection (3–8) for $j = 1$. Thus we are done. □

THEOREM 3.32. *Let (S, B) be a del Pezzo pair of index at most two obtained from a basic pair (M, E_M). Let $m^\star$ be the minimum positive integer m such that $K_M + E_M^{(\infty)} + mL_M$ is nef.*

(1) *If $\llcorner B \lrcorner = 0$, then $m^\star = 1$. If $\llcorner B \lrcorner$ is reduced, then $m^\star \le 2$.*

(2) *If $g(S, B) > 2$, then $R[S, B]^{(2)}$ is simply generated. In particular, $-2(K_S+B)$ is very ample and $|L_M|$ contains a non-singular member.*

(3) *Suppose that $g(S, B) > 2$. Then $R[S, B]_{2k-1}R[S, B]_2 = R[S, B]_{2k+1}$ for $k \ge m^\star$. In particular, $R[S, B]$ is generated by homogeneous elements of degree at most* $\max\{2, 2m^\star - 1\}$.

(4) *If $g(S, B) = 2$, then $R[S, B]^{(2)}$ is generated by homogeneous elements of degree at most 2. If $B = 0$ in addition, then $-2(K_S + B)$ is not very ample and $R[S, B]^{(2)}$ is not simply generated.*

(5) *Suppose that $g(S, B) = 2$. Then $R[S, B]_4 = (R[S, B]_2)^2 + R[S, B]_1R[S, B]_3$ and $R[S, B]_{2k-1}R[S, B]_2 = R[S, B]_{2k+1}$ for $k \ge$* $\max\{2, m^\star\}$. *In particular, $R[S, B]$ is generated by homogeneous elements of degree at most* $\max\{2, 2m^\star - 1\}$.

PROOF. (1): Suppose that $\llcorner B \lrcorner = 0$. Then $E_M^{(\infty)}$ is α-exceptional and hence $(K_M + E_M^{(\infty)} + L_M)\gamma \ge 0$ for any irreducible component γ of $E_M^{(\infty)}$. Thus $K_M + E_M^{(\infty)} + L_M$ is nef, and $m^\star = 1$. Suppose next that $\llcorner B \lrcorner$ is reduced. If γ is an irreducible component of $E_M^{(\infty)}$ with $L_M\gamma > 0$, then $\mathrm{mult}_\gamma(E_M^{(\infty)}) = 1$ and

$$(K_M + E_M^{(\infty)} + 2L_M)\gamma \ge -2 + 2L_M\gamma \ge 0.$$

Thus $K_M + E_M^{(\infty)} + 2L_M$ is nef, and $m^\star \leq 2$.

(2) follows from the surjectivity of μ_m for $m \geq 1$ shown in Proposition 3.31. Here, the existence of non-singular member of $|L_M|$ follows from the Bertini Theorem applied to the very ample divisor $-2(K_S + B)$ of a variety S with only isolated singularities.

(3): By the surjectivity of μ''_m for $m \geq 1$ shown in Proposition 3.31, we infer that $R[S,B]_{2k-1}R[S,B]_2 = R[S,B]_{2k+1}$ if and only if $E_M^{(k)} = E_M^{(k+1)}$. Thus the assertion holds.

(4): The first assertion also follows from Proposition 3.31. If $B = 0$, then $L_M^2 = 4$ and $\dim \mathrm{H}^0(M, L_M) = \chi(M, L_M) = 4$. If $-2(K_S + B)$ is very ample, then S is realized as a quartic surface in $\mathbb{P}^3$, contradicting that S has a non-Gorenstein singular point.

(5) follows from the surjectivity of μ''' and μ''_m shown in Proposition 3.31 and by the same argument as in the proof of (3) above. □

Example 3.33. There is an example (M, E_M) of basic pairs such that $\llcorner B \lrcorner$ is reduced and $m^\star = 2$. We use results in Section 4 in order to describe the example: Let (X, E, Δ) be a fundamental triplet of type $[n; 2, 3]_2$ for $n \geq 2$ in which $\Delta = 0$ and $E = 2\sigma + F$ for the union F of three fibers of $\pi\colon X \to \mathbb{P}^1$. Then $M = X = S$, $L\sigma = 1$, and $E_M^\circ = E_M^{(\infty)} = \llcorner B \lrcorner = \sigma$. Thus $K_M + E_M^{(\infty)} + kL_M$ is nef if and only if $k \geq 2$. Hence, $m^\star = 2$.

By using the classification of fundamental triplets in Section 4.2 below, we have:

Proposition 3.34. *$m^\star \leq 2$ for any basic pair (M, E_M).*

Proof. A basic pair (M, E_M) is obtained from a fundamental triplet (X, E, Δ) by the elimination of Δ. We may assume that $\llcorner B \lrcorner$ is not reduced. Let $\Gamma \subset M$ be the proper transform of an irreducible component of $\llcorner B \lrcorner$ with multiplicity > 1. We set $m_\Gamma = \mathrm{mult}_\Gamma(E_M)$. Then $m_\Gamma \geq 4$ and

$$\mathrm{mult}_\Gamma(\llcorner B \lrcorner) = \mathrm{mult}_\Gamma(E_M^\circ) = \mathrm{mult}_\Gamma(E_M^{(\infty)}) = \llcorner (1/2)m_\Gamma \lrcorner > 1.$$

Let k_Γ be the minimum positive integer k with $(K_M + E_M^{(\infty)} + kL_M)\Gamma \geq 0$. It is enough to show that $k_\Gamma \leq 2$ for any such Γ.

Case 1. $\underline{\Gamma \text{ is not } \phi\text{-exceptional}}$: Then $\phi(\Gamma)$ an irreducible component of E with multiplicity $m_\Gamma \geq 4$. By Theorem 4.6, the type of (X, E, Δ) is

$[n; 2, 4]_2$ for $n \geq 3$, $m_\Gamma = 4$, $E = 2\sigma + 4\phi(\Gamma)$, $\phi(\Gamma)$ is a fiber of $X = \mathbb{F}_n \to \mathbb{P}^1$, and $2 = L\phi(\Gamma) > \deg(\Delta \cap \phi(\Gamma))$. Thus $-1 \leq \Gamma^2 \leq 0$. If $\Gamma^2 = 0$, then $L_M\Gamma = 2$ and

$$(K_M + E_M^{(\infty)} + L_M)\Gamma \geq (K_M + 2\Gamma + L_M)\Gamma = 0.$$

Hence, $k_\Gamma \leq 1$. Suppose that $\Gamma^2 = -1$. Then $E_M = 2\sigma_M + 3\Gamma_1 + 4\Gamma$ for the proper transform $\sigma_M \subset M$ of σ and a ϕ-exceptional curve Γ_1 by Lemma 2.17. Here, $L_M\sigma_M = 0$, $\Gamma_1^2 = -1$, and $L_M\Gamma = L_M\Gamma_1 = 1$. Thus $E_M^\circ = E_M^{(\infty)} = \sigma_M + \Gamma_1 + 2\Gamma$. In particular, $(K_M + E_M^{(\infty)} + L_M)\Gamma = 0$, and hence $k_\Gamma = 1$.

Case 2. $\underline{\Gamma \text{ is } \phi\text{-exceptional}}$: Let $E_0 \subset E$ be the irreducible component containing the point $P = \phi(\Gamma)$. Note that E_0 is unique and $m_0 := \operatorname{mult}_{E_0}(E) \geq 2$ and that $m_0 \leq 4$ by Theorem 4.6. Let $E_{0,M} \subset M$ be the proper transform of E_0. Since $(K_M + L_M)\Gamma = 0$, Γ is a (-1)-curve and $L_M\Gamma = 1$. Since $(K_M + E_M^{(\infty)} + kL_M)\Gamma = (k-1) + E_M^{(\infty)}\Gamma$, it is enough to show $E_M^{(\infty)}\Gamma \geq -1$.

We set $k_P = \operatorname{mult}_P(\Delta)$ and $l_P = \operatorname{mult}_P(\Delta \cap E_0)$. Over an open neighborhood of $\phi^{-1}(P)$, $\phi^{-1}(E_0)$ is a the union of $E_{0,M}$ and a straight chain $\Gamma_1 + \Gamma_2 + \cdots + \Gamma_{k_P}$ of non-singular rational curves where the dual graph of $\phi^{-1}(E_0)$ is the same as that of $\phi^{-1}(E_0)$ in Lemma 2.17. Here, $L_M\Gamma_i = 0$ except for $i = k_P$. Thus $\Gamma = \Gamma_{k_P}$. Therefore, $m_\Gamma = \operatorname{mult}_\Gamma(E_M) = l_Pm_0 - k_P$ by Lemma 2.17.

Subcase 2A. $\underline{m_0 = 2}$: Then $l_P \geq 4$. In particular, $\deg(\Delta \cap E_0) \geq 4$. Thus, (X, E, Δ) is of type $[2]_2$ and $\operatorname{Supp}(\Delta) = \{P\}$ with $l_P = 4$, by Theorem 4.6. Thus $k_P = 4$ and

$$E_M = 2E_{0,M} + \Gamma_1 + 2\Gamma_2 + 3\Gamma_3 + 4\Gamma$$

by Lemma 2.17. Here, $L_ME_{0,M} = L_M\Gamma_i = 0$ for $1 \leq i \leq 3$. It implies that $E_M^\circ = E_{0,M} + \Gamma_2 + \Gamma_3 + 2\Gamma$ and $E_M^{(\infty)} = E_{0,M} + \Gamma_3 + 2\Gamma$. Therefore, $E_M^{(\infty)}\Gamma = 0$.

Subcase 2B. $\underline{m_0 = 3}$: Then (X, E, Δ) is of type $[n; 2, e]_2$ with $e \in \{3, 4\}$ and $n \geq 2$, E_0 is a fiber of $\pi\colon X \to \mathbb{P}^1$, and $E = 2\sigma + 3E_0 + F$ for an effective divisor $F \sim (e-3)E_0$ by Theorem 4.6. Since $m_\Gamma \geq 4$ and $\deg(\Delta \cap E_0) \leq 2$, we have $k_P = l_P = 2$ and $m_\Gamma = 4$. Thus $E_M = 2\sigma_M + 3E_{0,M} + 2\Gamma_1 + 4\Gamma + F'$

for the proper transform $\sigma_M \subset M$ of σ and for an effective divisor F' with $\phi_* F' = F$. Then $E_M^\circ = \sigma_M + E_{0,M} + \Gamma_1 + 2\Gamma$ and $E_M^{(\infty)}\Gamma = E_M^\circ\Gamma = 0$.

Subcase 2C. $\underline{m_0 = 4}$: Then (X, E, Δ) is of type $[n; 2, 4]_2$ for $n \geq 3$, and E_0 is a fiber of $\pi\colon X \to \mathbb{P}^1$ with $E \geq 2\sigma + 4E_0$ by Theorem 4.6. Since $m_\Gamma \geq 4$ and $\deg(\Delta \cap E_0) \leq 2$, we have $l_P = 2$ and $m_\Gamma \leq 6$. Note that the proper transform $\sigma_M \subset M$ of σ, and $E_{0,M}$ are α-exceptional.

Suppose that $m_\Gamma = 6$. Then $k_P = 2$ and $E_M = 2\sigma_M + 4E_{0,M} + 3\Gamma_1 + 6\Gamma$. Thus $E_M^\circ = \sigma_M + 2E_{0,M} + \Gamma_1 + 3\Gamma$. Hence, $E_M^{(\infty)} = E_M^\circ$ and $E_M^{(\infty)}\Gamma = 0$.

Suppose that $m_\Gamma = 5$. Then $k_P = 3$ and $E_M = 2\sigma_M + 4E_{0,M} + 3\Gamma_1 + 6\Gamma_2 + 5\Gamma$. Thus $E_M^\circ = \sigma_M + 2E_{0,M} + \Gamma_1 + 3\Gamma_2 + 2\Gamma$ and $E_M^{(\infty)} = \sigma_M + E_{0,M} + \Gamma_1 + 2\Gamma_2 + 2\Gamma$. Therefore, $E_M^{(\infty)}\Gamma = 0$.

Suppose that $m_\Gamma = 4$. Then $k_P = 4$ and $E_M = 2\sigma_M + 4E_{0,M} + 3\Gamma_1 + 6\Gamma_2 + 5\Gamma_3 + 4\Gamma$. Thus $E_M^\circ = \sigma_M + 2E_{0,M} + \Gamma_1 + 3\Gamma_2 + 2\Gamma_3 + 2\Gamma$ and $E_M^{(\infty)} = \sigma_M + E_{0,M} + \Gamma_1 + 2\Gamma_2 + 2\Gamma_3 + 2\Gamma$. Therefore, $E_M^{(\infty)}\Gamma = 0$.

Thus, we are done. □

Hence, we have the following by Theorem 3.32 and Proposition 3.34:

Theorem 3.35. *If (S, B) is a del Pezzo pair obtained from a basic pair (M, E_M), then $R[S, B]$ is generated by homogeneous elements of degree at most* 3*, and $R[S, B]^{(2)}$ is generated by homogeneous elements of degree at most* 2*.*

Next, we consider the rings $R[S, B]$ and $R[S, B]^{(2)}$ for a del Pezzo pair (S, B) of index at most two which is not obtained from any basic pair.

Proposition 3.36. *Let (S, B) be an irrational del Pezzo pair of index ≤ 2. If (S, B) is log-canonical, then $R[S, B]$ is generated by homogeneous elements of degree at most* 6*, and $R[S, B]^{(2)}$ is generated by homogeneous elements of degree at most* 3*. However, in the non-log-canonical case, $R[S, B]$ is not always finitely generated. Furthermore, there is no bound of degrees of minimal generators of $R[S, B]$ even if $R[S, B]$ is finitely generated.*

Proof. (S, B) is in one of the cases in Lemma 3.10. For the minimal desingularization $\alpha\colon M \to S$, M has a $\mathbb{P}^1$-bundle structure $\pi\colon M = \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(A)) \to C$ over a non-singular projective curve C of genus ≥ 1 for an ample divisor A.

Let σ be the negative section and let σ_∞ be a section at infinity on M. We can calculate $R[S, B]$ in each case of Lemma 3.10 as follows:

Case (1) *of* Lemma 3.10. Then, C is an elliptic curve, $E_M = 2\sigma$, $L_M \sim 2\sigma_\infty$, and $B = 0$. Thus,

$$R[S, B] \simeq R(M, \sigma_\infty) \simeq R(C, A)[\mathtt{t}]$$

for a variable $\mathtt{t}$ of degree one. Thus $R[S, B]$ is generated by homogeneous elements of degree at most 3 by the following well-known result for an elliptic curve C and an ample divisor A:

- If $\deg A \geq 3$, then $R(C, A)$ is simply generated.
- If $\deg A = 2$, then $R(C, A)$ is generated by homogeneous elements of degree ≤ 2.
- If $\deg A = 1$, then $R(C, A)$ is generated by homogeneous elements of degree ≤ 3.

Case (2) *of* Lemma 3.10. Then, C is an elliptic curve, $E_M = 2\sigma + \sigma_\infty$ for a section σ_∞ at infinity, $L_M \sim \sigma_\infty$, $B = (1/2)\alpha_*\sigma_\infty$, and $E_M^\circ = \sigma$. Since $K_M + E_M^\circ \sim -\sigma_\infty$, we have

$$R[S, B] \simeq R(M, (1/2)\sigma_\infty) \simeq R(C, A)[\theta, \mathtt{t}]/(\theta^2 - \mathtt{f})$$

for two variables θ, $\mathtt{t}$, where $\mathtt{f} \in R[S, B]_2 = \mathrm{H}^0(M, \sigma_\infty)$ is a defining equation of σ_∞ and

$$(R(C, A)[\theta, \mathtt{t}])_m = \bigoplus\nolimits_{2k+i+j=m} R(C, A)_k \theta^i \mathtt{t}^j.$$

Thus $R[S, B]$ (resp. $R[S, B]^{(2)}$) is generated by homogeneous elements of degree at most 6 (resp. 3).

Case (3) *of* Lemma 3.10. Then, $E_M = 3\sigma + \pi^*\Delta$ for an effective divisor Δ on C with $\deg(A - 2K_C - \Delta) \geq 0$ and $L_M \sim \sigma + \pi^*(2A - 2K_C - \Delta)$. We can choose the effective divisor Δ so that $\mathcal{O}_C(A - 2K_C - \Delta)$ is a non-torsion element of $\mathrm{Pic}^0(C)$. In this case, α is the contraction morphism of σ, but $-(K_S + B)$ is not $\mathbb{Q}$-Cartier; hence $R[S, B]^{(2)}$ and $R[S, B]$ are not finitely generated. On the other hand, we can take Δ so that $\mathcal{O}_C(A - 2K_C - \Delta)$ is a torsion element of $\mathrm{Pic}^0(C)$ with sufficiently large order. Thus we can not bound the degree of homogeneous generators of $R[S,B]$, even if $R[S,B]$ is finitely generated. $\square$

Proposition 3.37. *Let (S,B) be a del Pezzo pair of index at most two with $g(S,B)=0$. Then $R[S,B]^{(2)}$ is simply generated, and $R[S,B]$ is generated by homogeneous elements of degree at most* 5.

Proof. (S,B) is described as one of the cases (1), (2), (3) of Proposition 3.11. We first consider the case (1). Then $M \simeq S \simeq \mathbb{P}^2$ and $(\deg(L_M), \deg(E_M)) \in \{(1,5),(2,4)\}$. Thus $R[S,B]^{(2)} \simeq R(M, L_M)$ is simply generated. Since $\deg(K_M + E_M^\circ + kL_M) \geq k-3$,

$$\mathrm{H}^0(K_M + E_M^\circ + kL_M) \otimes \mathrm{H}^0(L_M) \to \mathrm{H}^0(K_M + E_M^\circ + (k+1)L_M)$$

is surjective for $k \geq 3$. Thus $R[S,B]$ is generated by homogeneous elements of degree at most 5.

Next, we consider the cases (2) and (3). Then $M \simeq \mathbb{F}_n$, $E_M \sim 3\sigma + (2n+4-b)\ell$, $L_M \sim \sigma + b\ell$ for a minimal section σ and a fiber ℓ of $\pi\colon X \to \mathbb{P}^1$, and for a positive integer b with $n \leq b \leq 2n+4$. Thus

$$R[S,B]^{(2)} \simeq R(M, L_M) \simeq \bigoplus\nolimits_{m \geq 0} \mathrm{H}^0(\mathbb{P}^1, \mathrm{Sym}^m(\mathcal{O}(b) \oplus \mathcal{O}(b-n)))$$

is simply generated. If we write $E_M^\circ \sim e_1^\circ \sigma + e_2^\circ \ell$, then

$$K_M + E_M^\circ + kL_M \sim (k-2+e_1^\circ)\sigma + (kb-(n+2)+e_2^\circ)\ell,$$

and hence

$$\begin{aligned}&\mathrm{H}^0(M, K_M + E_M^\circ + kL_M)\\ &\quad \simeq \mathrm{H}^0\left(\mathbb{P}^1, \mathrm{Sym}^{k-2+e_1^\circ}\big(\mathcal{O}(kb-2n-2+e_2^\circ) \oplus \mathcal{O}(kb-n-2+e_2^\circ)\big)\right)\end{aligned}$$

for $k \geq 2$. Since $b > n$ for the case $0 \leq n \leq 1$, we have $kb - 2n - 2 + e_2^\circ \geq 0$ for $k \geq 3$. Thus $R[S,B]$ is generated by homogeneous elements of degree at most 5. $\square$

Proposition 3.38. *Let (S,B) be a rational del Pezzo pair of index at most two with $g(S,B)=1$. Then $R[S,B]^{(2)}$ is generated by homogeneous elements of degree at most* 3, *and $R[S,B]$ is generated by homogeneous elements of degree at most* 6.

PROOF. S is a log del Pezzo surface of index one and $2B \in |-K_S|$ (cf. Lemma 3.12). Hence, $R[S,B]^{(2)} \simeq R(S,-K_S)$, which is known to be generated by homogeneous elements of degree at most 3 (cf. [10, Chapter V, Proposition 2]). Since $K_M + L_M \sim 0$ is nef, we can define $E_M^{(m)}$ for $m \geq 1$ as above, i.e., $E_M^{(m)}$ is the maximum divisor $\leq E_M^\circ$ with $E_M^{(m)} - (m-1)K_M$ being nef.

Suppose that $\alpha_* E_M^\circ = \llcorner B \lrcorner = 0$. Then $E_M^{(m)} = E_M^{(\infty)} = 0$ for any $m \geq 1$, and $R[S,B]_{2k-1} \simeq \mathrm{H}^0(M, -(k-1)K_M)$ for $k \geq 1$. Since $R[S,B]_1 \otimes R[S,B]_{2k} \to R[S,B]_{2k+1}$ is just the isomorphism $\mathrm{H}^0(M, \mathcal{O}_M) \otimes \mathrm{H}^0(M, -kK_M) \simeq \mathrm{H}^0(M, -kK_M)$ for $k \geq 1$, $R[S,B]$ is generated by homogeneous elements of degree at most 6.

Next, suppose $\alpha_* E_M^\circ \neq 0$. Then $E_M^{(\infty)} \neq 0$. The dualizing sheaf ω_{E_M} is isomorphic to $\mathcal{O}_{E_M}$, since $E_M \sim -K_M$. Furthermore, $\mathrm{H}^1(\mathcal{O}_{E_M}) \simeq \mathrm{H}^2(M, K_M) \simeq \Bbbk$. From the exact sequence

$$0 \to \omega_{E_M^{(\infty)}} \to \omega_{E_M} \simeq \mathcal{O}_{E_M} \to \mathcal{O}_{E_M - E_M^{(\infty)}} \to 0,$$

we have the vanishing

$$\mathrm{H}^1\big(\mathcal{O}_{E_M^{(\infty)}}\big) \simeq \mathrm{H}^0\big(\omega_{E_M^{(\infty)}}\big)^\vee = 0.$$

An inequality $K_M^2 = (-K_M)E_M \geq 2L_M E_M^{(\infty)} \geq 2$ follows from $E_M \geq 2E_M^{(\infty)}$. Hence, $R[S,B]^{(2)} = R(S,-K_S)$ is generated by homogeneous elements of degree at most 2.

Let γ be an irreducible curve with $E_M^{(\infty)}\gamma < 0$. Then γ is a (-1)-curve, since any (-2)-curve is α-exceptional. We set $b = b_\gamma = \mathrm{mult}_\gamma(E_M^\circ)$. Since $-K_M \sim E_M \geq 2E_M^{(\infty)}$ and M has a (-1)-curve, we have $8 \geq K_M^2 \geq 2b$.

We shall show $b \leq 2$. First, we consider the case where $K_M^2 = 8$. Then $M \simeq \mathbb{F}_1$ and $\gamma = \sigma$. Since $-K_M - 2b\gamma$ is linearly equivalent to an effective divisor, we have $(-K_M - 2b\sigma)\ell = 2 - 2b \geq 0$ for a fiber ℓ of $\pi\colon M \to \mathbb{P}^1$. Hence, $b \leq 1$. Next, we consider the case where $K_M^2 \leq 7$. Then there is a birational morphism $M \to \mathbb{F}_n$ for $0 \leq n \leq 2$. Here, we may assume that γ is contained in a fiber of the composite $M \to \mathbb{F}_n \to \mathbb{P}^1$. Thus, by replacing the birational morphism $M \to \mathbb{F}_n$ if necessary, we may also assume that γ is the proper transform of a fiber ℓ of $X = \mathbb{F}_n \to \mathbb{P}^1$. Since $-K_X - 2b\ell \sim 2\sigma + (n+2-2b)\ell$ is linearly equivalent to an effective divisor, we have $2b \leq n+2 \leq 4$. Hence, $b \leq 2$.

Therefore, $K_M + E_M^{(\infty)} + kL_M \sim E_M^{(\infty)} - (k-1)K_M$ is nef and $E_M^{(k)} = E_M^{(\infty)}$ for $k \geq 3$. In order to show $R[S,B]_{2k-1}R[S,B]_2 = R[S,B]_{2k+1}$ for $k \geq 3$, we shall apply Lemma 3.28 to the case $Z = M$, $D = E_M^{(\infty)}$, $\mathcal{L} = \mathcal{O}_M(-(k-1)K_M + E_M^{(\infty)})$, $\mathcal{M} = \mathcal{O}_M(-K_M)$. Here, (S1) follows from $\mathrm{H}^1(M, -(k-1)K_M) = 0$ for $k \geq 1$. The homomorphism of (S3) is nothing but $\mathrm{H}^0(M, -(k-1)K_M) \otimes \mathrm{H}^0(M, -K_M) \to \mathrm{H}^0(M, -kK_M)$, which is surjective for $k \geq 3$, since $K_S^2 \geq 2$. The restriction map $\mathrm{H}^0(M, -K_M) \to \mathrm{H}^0(E_M, -K_M|_{E_M})$ is surjective by $\mathrm{H}^1(M, -K_M - E_M) = \mathrm{H}^1(-2K_M) = 0$. Thus (S2) is derived from the surjectivity of

$$\mathrm{H}^0\big(E_M^{(\infty)}, (-(k-1)K_M + E_M^{(\infty)})|_{E_M^{(\infty)}}\big) \otimes \mathrm{H}^0(E_M, -K_M|_{E_M}) \\ \to \mathrm{H}^0\big(E_M^{(\infty)}, (-kK_M + E_M^{(\infty)})|_{E_M^{(\infty)}}\big),$$

which is shown by Lemma 3.29. Therefore, $R[S,B]_{2k-1}R[S,B]_2 = R[S,B]_{2k+1}$ for $k \geq 3$, and $R[S,B]$ is generated by homogeneous elements of degree at most 6. □

Finally, we consider a rational del Pezzo pair (S,B) of index at most two of genus $g(S,B) \geq 2$ which is not obtained from any basic pair. Then S is a log del Pezzo surface of index one and $B = 0$. Thus, $R[S,B] = R(S, -K_S)$. Hence, by [10, Chapter V, Proposition 2], $R[S,B]^{(2)}$ (resp. $R[S,B]$) is generated by homogeneous elements of degree at most 2 (resp. 3), respectively.

Therefore, we have proved the following:

Theorem 3.39. *Let (S,B) be a del Pezzo pair of index at most two. Suppose either that S is rational or that (S,B) is log-canonical. Then $R[S,B]$ is generated by homogeneous elements of degree at most* 6*, and $R[S,B]^{(2)}$ is generated by homogeneous elements of degree at most* 3*.*

4. Fundamental Triplets

In this section, the notion of fundamental triplet is introduced. Any basic pair is shown to be obtained as the elimination of a fundamental triplet. The fundamental triplets are classified by their types. The uniqueness of fundamental triplet for a given basic pair does not hold in general but the type is uniquely determined. By the list of types, we can classify all the non-Gorenstein singularities on S for rational del Pezzo pairs (S, B) of index at most two.

4.1. Definition of fundamental triplet

DEFINITION 4.1. A triplet (X, E, Δ) is called a *quasi-fundamental triplet* if the following conditions $(\mathcal{F}1)$–$(\mathcal{F}3)$ are satisfied:

($\mathcal{F}1$) (X, E) is a minimal basic pair;

($\mathcal{F}2$) Δ is empty or a zero-dimensional subscheme of X with $\nu_P(\Delta) = 1$ for any $P \in \Delta$;

($\mathcal{F}3$) Δ is a subscheme of E such that $LE_i \geq \deg(\Delta \cap E_i)$ for any irreducible component E_i of E, where $L = -2K_X - E$.

LEMMA 4.2.

(1) *Let (X, E, Δ) be a quasi-fundamental triplet and let $\phi\colon M \to X$ be the elimination of Δ. Then (M, E_M^Δ) is a basic pair.*

(2) *If (M, E_M) is a basic pair, then there exist a quasi-fundamental triplet (X, E, Δ) and a birational morphism $\phi\colon M \to X$ such that ϕ is the elimination of Δ and $E_M = E_M^\Delta$.*

PROOF. (1): We set $E_M = E_M^\Delta$. By Lemma 2.7, (2), $K_M + E_M \sim \phi^*(K_X + E)$. Hence, $K_M + L_M \sim \phi^*(K_X + L)$ for $L_M = -2K_M - E_M$. Let G be the ϕ-exceptional effective divisor determined by $\mathcal{I}_\Delta \mathcal{O}_M = \mathcal{O}_M(-G)$. Then $L_M = \phi^*L - G$ and $\phi_*\mathcal{O}_M(-G) \simeq \mathcal{I}_\Delta$. If $E_{i,M}$ is the proper transform of an irreducible component E_i of E, then $E_{i,M} = (E_i)_M^\Delta$ and $GE_{i,M} = \deg(\Delta \cap E_i)$ by Lemma 2.7; thus

$$L_M E_{i,M} = LE_i - \deg(\Delta \cap E_i) \geq 0.$$

Since $-K_M$ is ϕ-nef, $L_M\Gamma = -K_M\Gamma \geq 0$ for any ϕ-exceptional irreducible component Γ of E_M. Therefore, the conditions $(\mathcal{C}1)$–$(\mathcal{C}3)$ are all satisfied for (M, E_M).

(2): If (M, E_M) is minimal, then (M, E_M, Δ) is the expected quasi-fundamental triplet for $\Delta = \emptyset$. If (M, E_M) is not minimal, then by successive contractions of (-1)-curves, we have a minimal basic pair (X, E) and a birational morphism $\phi\colon M \to X$ such that $E = \phi_* E_M$ and $K_M + E_M \sim \phi^*(K_X + E)$. Hence $K_M + L_M \sim \phi^*(K_X + L)$ for nef divisors $L_M = -2K_M - E_M$ and $L = -2K_X - E$. Thus ϕ is the elimination of a zero-dimensional subscheme $\Delta \subset E$ with $\nu_P(\Delta) = 1$ for any P and $E_M = E_M^{\Delta}$ by Proposition 2.9. For an irreducible component E_i of E and for the proper transform $E_{i,M}$ in M, we have

$$0 \leq L_M E_{i,M} = (\phi^* L - G) E_{i,M} = L E_i - \deg(\Delta \cap E_i).$$

Hence, (X, E, Δ) is a quasi-fundamental triplet. $\square$

For a quasi-fundamental triplet (X, E, Δ), the basic pair (M, E_M) obtained as above by the elimination of Δ is called the *elimination* of (X, E, Δ).

Let (M, E_M) be a basic pair and set $L_M = -2K_M - E_M$.

Suppose that $K_M + L_M$ is big. Then the quasi-fundamental triplet (X, E, Δ) whose elimination is (M, E_M) is unique up to isomorphism. In fact, if the type of (M, E_M) is not $[2; 1, 2]$, then elimination $\phi\colon M \to X$ of Δ is associated to the complete linear system $|K_M + L_M|$, since $K_M + L_M \sim \phi^*(K_X + L)$ for the very ample divisor $K_X + L$ (cf. Corollary 3.25). If the type is $[2; 1, 2]$, then $|K_M + L_M|$ gives a birational morphism into $\overline{\mathbb{F}}_2 \simeq \mathbb{P}(1, 1, 2)$; thus the morphism ϕ into the minimal desingularization X of $\overline{\mathbb{F}}_2$ is uniquely determined.

On the other hand, if $K_M + L_M$ is not big, then the quasi-fundamental triplet (X, E, Δ) whose elimination is (M, E_M) is not necessarily unique as in the proof of Proposition 4.4 below. In this case, $X \simeq \mathbb{F}_n$ and $K_X + L$ is linearly equivalent to a multiple of fiber of π. Thus the linear system $|K_M + L_M|$ defines only the composition $M \to X \to \mathbb{P}^1$.

The notion of fundamental triplet below is introduced for establishing similar uniqueness also for the non-big case; However, the uniqueness does not hold in general even for the artificial notion (cf. Theorem 4.9, Example 4.12).

Definition 4.3. A quasi-fundamental triplet (X, E, Δ) is called a *fundamental triplet* either if $K_X + L$ is big or if $K_X + L$ is not big and the following three conditions $(\mathcal{F}4)$–$(\mathcal{F}6)$ are satisfied:

($\mathcal{F}4$) $\Delta \cap \sigma = \emptyset$ for a minimal section σ; In particular, $\Delta = \emptyset$ if $X \simeq \mathbb{F}_0$.

($\mathcal{F}5$) If $E \geq \sigma + D$ for a minimal section σ and a section $D \neq \sigma$, then $D^2 + n \geq \deg(\Delta \cap D)$, where $X \simeq \mathbb{F}_n$.

($\mathcal{F}6$) If E does not contain a minimal section σ and if E is either reducible or non-reduced, then $\Delta = \emptyset$.

PROPOSITION 4.4. *Any basic pair is obtained as the elimination of a fundamental triplet.*

For the proof, we need the following:

LEMMA 4.5. *Let $f\colon Y \to T$ be a proper surjective morphism from a non-singular surface Y into a non-singular curve T such that a general fiber is isomorphic to $\mathbb{P}^1$. Let $E \subset Y$ be a section of f. Then $\mathcal{O}_Y(E)$ is f-generated and $\mathcal{F} = f_*\mathcal{O}_Y(E)$ is a locally free sheaf of rank two. In particular, there is a birational morphism $\mu\colon Y \to \mathbb{P}_T(\mathcal{F})$ over T such that $E = \mu^* D$ for a section D of $\mathbb{P}_T(\mathcal{F}) \to T$.*

PROOF. Y is a blowup of a $\mathbb{P}^1$-bundle over T. Hence, $f_*\mathcal{O}_Y \simeq \mathcal{O}_T$ and $\mathrm{R}^1 f_*\mathcal{O}_Y = 0$. Thus, from the exact sequence $0 \to \mathcal{O}_Y \to \mathcal{O}_Y(E) \to \mathcal{O}_E(E) \to 0$, we have an exact sequence

$$0 \to \mathcal{O}_T \to \mathcal{F} = f_*\mathcal{O}_Y(E) \to f_*\mathcal{O}_E(E) \to 0.$$

Since E is a section, $\mathcal{F}$ is locally free of rank two. The surjectivity of $f^*\mathcal{F} \to \mathcal{O}_Y(E)$ follows from the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & f^*\mathcal{O}_T & \longrightarrow & f^*\mathcal{F} & \longrightarrow & f^*f_*\mathcal{O}_E(E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_Y & \longrightarrow & \mathcal{O}_Y(E) & \longrightarrow & \mathcal{O}_E(E) & \longrightarrow & 0
\end{array}$$

of exact sequences. The surjection defines the birational morphism μ and the injection $\mathcal{O}_T \to \mathcal{F}$ defines the section D with $\mu^* D = E$. $\square$

We shall prove Proposition 4.4.

PROOF. Let (M, E_M) be a basic pair and let (X, E, Δ) be a quasi-fundamental triplet whose elimination is (M, E_M). We may assume that $K_X + L_X$ is not big, i.e., the type of (X, E) is $[n, 2, e_2]$. Applying Lemma 4.5, we want to replace (X, E, Δ) with another quasi-fundamental triplet (X', E', Δ') which satisfies some conditions on fundamental triplet.

Step 1. *We can find a quasi-fundamental triplet* (X, E, Δ) *satisfying* $(\mathcal{F}4)$.

Let $\sigma_M \subset M$ be the proper transform of a minimal section of σ with $\sigma \cap \Delta \neq \emptyset$. By Lemma 4.5, there is a birational morphism $\phi' \colon M \to X' = \mathbb{F}_{n'}$ over $\mathbb{P}^1$ with $n' = -(\sigma_M)^2 = n + \deg(\Delta \cap \sigma) > n$ such that σ_M is the total transform of the negative section σ' of $X' \to \mathbb{P}^1$. Since $K_M + E_M$ is linearly equivalent to a multiple of a fiber of $M \to \mathbb{P}^1$, $K_M + E_M \sim \phi'^*(K_{X'} + E')$ for the effective divisor $E' = \phi'_* E_M$. By Proposition 2.9, we infer that ϕ' is the elimination of a zero-dimensional subscheme $\Delta' \subset E'$. We infer also that (X', E', Δ') is a quasi-fundamental triplet whose elimination is (M, E_M). Here, $\sigma' \cap \Delta' = \emptyset$ since ϕ' is an isomorphism around σ'. Thus $(\mathcal{F}4)$ is satisfied.

Step 2. *The case where E contains a minimal section.*

We may assume $n > 0$, $\Delta \neq \emptyset$, $\sigma \cap \Delta = \emptyset$ for the negative section σ. Suppose that $E \geq \sigma + D$ for a section $D \neq \sigma$ with $D^2 + n < \deg(\Delta \cap D)$. Then $n' := -D_M^2 = -D^2 + \deg(\Delta \cap D) > n$ for the proper transform $D_M \subset M$ of D. By Lemma 4.5, there is a birational morphism $\phi' \colon M \to X' = \mathbb{F}_{n'}$ over $\mathbb{P}^1$ such that D_M is the total transform of the negative section σ' of X'. By the same argument as in *Step 1*, (M, E_M) is the elimination of a quasi-fundamental triplet (X', E', Δ') satisfying $(\mathcal{F}4)$, where $E' = \phi'_* E_M$. For the proper transform $\sigma_M \subset M$ of σ, $D' = \phi'_* \sigma_M$ is a section with $E' \geq \sigma' + D'$. Since $\sigma_M^2 = \sigma^2 = -n$, we have $-n = D'^2 - \deg(\Delta' \cap D')$. Thus (X', E', Δ') satisfies also $(\mathcal{F}5)$. Since E' contains σ', (X', E', Δ') is a fundamental triplet.

Final step. The case where E does not contain a minimal section.

We may assume that $n > 0$, $\Delta \neq \emptyset$, and that (X, E, Δ) satisfies the condition $(\mathcal{F}4)$. Then $E \geq D_1 + D_2$ for sections $D_1 \neq \sigma$, $D_2 \neq \sigma$. Then $2n \leq e_2 \leq \min\{n+1, 4\}$ by the proof of Lemma 3.24. Hence, $n = 1$, $e_2 = 2$, and $E = D_1 + D_2$ for the sections D_1, D_2 at infinity. We may assume $D_1 \cap \Delta \neq \emptyset$. Let $D_{1,M} \subset M$ be the proper transform of D_1. Then $-n' := D_{1,M}^2 = D_1^2 - \deg(\Delta \cap D_1) \leq 0$. Let $\phi' \colon M \to X' \simeq \mathbb{F}_{n'}$ be the birational morphism such that $D_{1,M}$ is the total transform of a minimal section σ' of $X' \to \mathbb{P}^1$. Then (M, E_M) is the elimination of a quasi-fundamental triplet (X', E', Δ'). Let $D_2' \subset X'$ be the proper transform of D_2. Then $E' \geq \sigma' + D_2'$. By *Step 1*, *Step 2*, we have a fundamental triplet (X'', E'', Δ'') whose elimination is (M, E_M). □

4.2. Classification of fundamental triplets

Let (X, E, Δ) be a fundamental triplet and let $\phi\colon (M, E_M) \to (X, E, \Delta)$ be the elimination. We set $E_M = E_M^{\Delta}$, $L = -2K_X - E$, and $L_M = -2K_M - E_M$. Let (S, B) be the del Pezzo pair associated to (M, E_M) (cf. Proposition 3.19). Here, the birational morphism $\alpha\colon M \to S$ given by the linear system $|L_M|$ is the minimal desingularization of S, and $B = (1/2)\alpha_* E_M$.

THEOREM 4.6. *The fundamental triplets* (X, E, Δ) *are classified by the types defined as follows*:

The case $X = \mathbb{P}^2$:

$[1]_0$: *E is a line and* $\deg \Delta \leq LE = 5$.

$[2]_0$: *E is a non-singular conic and* $\deg \Delta \leq LE = 8$.

$[2]_+(b)$: *$E = \ell_1 + \ell_2$ for two lines ℓ_1, ℓ_2, and* $\deg(\Delta \cap \ell_i) \leq L\ell_i = 4$ *for $i = 1, 2$. For $P = \ell_1 \cap \ell_2$,*

$$b = \max\{\mathrm{mult}_P(\Delta \cap \ell_1), \mathrm{mult}_P(\Delta \cap \ell_2)\} \in \{0, 1, 2, 3, 4\}.$$

$[2]_2$: *$E = 2\ell$ for a line ℓ and* $\deg(\Delta \cap \ell) \leq L\ell = 4$.

For $X = \mathbb{F}_n$, let $\pi\colon X \to \mathbb{P}^1$ be the $\mathbb{P}^1$-bundle structure, σ a minimal section, σ_∞ a section at infinity, and ℓ a fiber of π (cf. Convention 3.23).

The case $X = \mathbb{F}_0$:

$[0; 1, 0]_0$: *$E = \sigma$ and* $\deg \Delta \leq LE = 4$.

$[0; 1, 1]_0$: *$E \sim \sigma + \ell$ is non-singular and* $\deg \Delta \leq LE = 6$.

$[0; 1, 1]_+(b)$: *$E = \sigma + \ell$,* $\deg(\Delta \cap \sigma) \leq L\sigma = 3$, *and* $\deg(\Delta \cap \ell) \leq L\ell = 3$. *For $P = \sigma \cap \ell_2$,*

$$b = \max\{\mathrm{mult}_P(\Delta \cap \sigma), \mathrm{mult}_P(\Delta \cap \ell)\} \in \{0, 1, 2, 3\}.$$

$[0;2,0]_{00}$: *$E = \sigma_1 + \sigma_2$ for two distinct minimal sections σ_1 and σ_2, and $\Delta = \emptyset$, where $L\sigma_1 = L\sigma_2 = 4$.*

$[0;2,0]_2$: *$E = 2\sigma$ and $\Delta = \emptyset$, where $L\sigma = 4$.*

$[0;2,1]_0$: *$E \sim 2\sigma + \ell$ is non-singular and $\Delta = \emptyset$, where $LE = 8$.*

$[0;2,1]_+$: *$E = \sigma + D$ for a section $D \sim \sigma + \ell$, and $\Delta = \emptyset$, where $L\sigma = 3$ and $LD = 5$.*

$[0;2,1]_{++}$: *$E = \sigma_1 + \sigma_2 + \ell$ for two distinct minimal sections σ_1, σ_2, and $\Delta = \emptyset$, where $L\sigma_1 = L\sigma_2 = 3$ and $L\ell = 2$.*

$[0;2,1]_2$: *$E = 2\sigma + \ell$ and $\Delta = \emptyset$, where $L\sigma = 3$ and $L\ell = 2$.*

The case $X = \mathbb{F}_1$:

$[1;1,0]_0$: *$E = \sigma$ and $\deg\Delta \le LE = 3$.*

$[1;1,1]_0$: *$E \sim \sigma + \ell$ is non-singular and $\deg\Delta \le LE = 5$.*

$[1;1,1]_+(a,b)$: *$E = \sigma + \ell$, $\deg(\Delta \cap \sigma) \le L\sigma = 2$, and $\deg(\Delta \cap \ell) \le L\ell = 3$. For $P = \sigma \cap \ell$,*

$$(a,b) = (\mathrm{mult}_P(\Delta \cap \sigma), \mathrm{mult}_P(\Delta \cap \ell)) \in \{(0,0),(1,1),(2,1),(1,2),(1,3)\}.$$

$[1;2,e]_2$: *$0 \le e \le 2$, $E = 2\sigma + F$ for an effective divisor $F \sim e\ell$, $\Delta \cap \sigma = \emptyset$, and $\deg(\Delta \cap \ell) \le L\ell = 2$ for any fiber $\ell \le F$, where $L\sigma = 4 - e$.*

$[1;2,1]_{00}$: *$E = \sigma + \sigma_\infty$ and $\Delta \subset \sigma_\infty$ with $\deg\Delta \le 2$, where $L\sigma = 3$ and $L\sigma_\infty = 5$.*

$[1;2,2]_0$: *$E \sim 2\sigma + 2\ell$ is non-singular and $\deg\Delta \le LE = 8$.*

$[1;2,2]_\times$: *$E = \sigma_\infty + \sigma'_\infty$ for two distinct sections σ_∞, σ'_∞ at infinity, and $\Delta = \emptyset$, where $L\sigma_\infty = L\sigma'_\infty = 4$.*

$[1;2,2]_{2\infty}$: *$E = 2\sigma_\infty$ and $\Delta = \emptyset$, where $L\sigma_\infty = 4$.*

$[1;2,2]_+$: *$E = \sigma + D$ for a section $D \sim \sigma + 2\ell$ and $\Delta \subset D \setminus \sigma$ with $\deg\Delta \le 4$, where $L\sigma = 2$ and $LD = 6$.*

$[1;2,2]_{++}(a,b)$**:** *$E = \sigma + \sigma_\infty + \ell$, $\Delta \cap \sigma = \emptyset$, $\deg(\Delta \cap \sigma_\infty) \leq 2$, and $\deg(\Delta \cap \ell) \leq 2$, where $L\sigma = 2$, $L\sigma_\infty = 4$, and $L\ell = 2$. For $P = \sigma_\infty \cap \ell$,*

$$\begin{aligned}(a,b) &= (\mathrm{mult}_P(\Delta \cap \ell), \mathrm{mult}_P(\Delta \cap \sigma_\infty)) \\ &\in \{(0,0),(1,1),(2,1),(1,2)\}.\end{aligned}$$

The case $X = \mathbb{F}_2$:

$[2;1,0]_0$**:** *$E = \sigma$ and $\deg \Delta \leq LE = 2$.*

$[2;1,1]_+(a,b)$**:** *$E = \sigma + \ell$, $\deg(\Delta \cap \sigma) \leq L\sigma = 1$, and $\deg(\Delta \cap \ell) \leq L\ell = 3$. For $P = \sigma \cap \ell$,*

$$(a,b) = (\mathrm{mult}_P(\Delta \cap \sigma), \mathrm{mult}_P(\Delta \cap \ell)) \in \{(0,0),(1,1),(1,2),(1,3)\}.$$

$[2;1,2]_0$**:** *$E = \sigma_\infty$ and $\deg \Delta \leq LE = 6$.*

$[2;1,2]_{++}$**:** *$E = \sigma + \ell_1 + \ell_2$ for two distinct fibers ℓ_1 and ℓ_2, $\Delta \cap \sigma = \emptyset$ and $\deg(\Delta \cap \ell_i) \leq L\ell_i = 3$ for $i = 1$, 2, where $L\sigma = 0$.*

$[2;1,2]_{2+}$**:** *$E = \sigma + 2\ell$ for a fiber ℓ, and $\Delta \cap \sigma = \emptyset$ and $\deg(\Delta \cap \ell) \leq L\ell = 3$, where $L\sigma = 0$.*

$[2;2,e]_2$**:** *$0 \leq e \leq 3$, $E = 2\sigma + F$ for an effective divisor $F \sim e\ell$, $\Delta \cap \sigma = \emptyset$, and $\deg(\Delta \cap \ell) \leq L\ell = 2$ for any fiber $\ell \leq F$, where $L\sigma = 4 - e$.*

$[2;2,2]_{00}$**:** *$E = \sigma + \sigma_\infty$ and $\Delta \subset \sigma_\infty$ with $\deg \Delta \leq 4$, where $L\sigma = 2$ and $L\sigma_\infty = 6$.*

$[2;2,3]_+$**:** *$E = \sigma + D$ for a section $D \sim \sigma + 3\ell$ and $\Delta \subset D \backslash \sigma$ with $\deg \Delta \leq 6$, where $L\sigma = 1$ and $LD = 7$.*

$[2;2,3]_{++}(a,b)$**:** *$E = \sigma + \sigma_\infty + \ell$, $\Delta \cap \sigma = \emptyset$, $\deg(\Delta \cap \sigma_\infty) \leq 4$, and $\deg(\Delta \cap \ell) \leq L\ell = 2$, where $L\sigma = 1$ and $L\sigma_\infty = 5$. For $P = \sigma_\infty \cap \ell$,*

$$\begin{aligned}(a,b) &= (\mathrm{mult}_P(\Delta \cap \ell), \mathrm{mult}_P(\Delta \cap \sigma_\infty)) \\ &\in \{(0,0),(1,1),(2,1),(1,2),(1,3),(1,4)\}.\end{aligned}$$

The case $X = \mathbb{F}_3$:

$[3;1,0]_0$**:** *$E=\sigma$ and* $\deg\Delta \leq LE = 1$.

$[3;1,1]_+$**:** *$E=\sigma+\ell$, $\Delta\cap\sigma=\emptyset$ and* $\deg(\Delta)\leq L\ell=3$, *where $L\sigma=0$.*

$[3;2,e]_2$**:** $0\leq e\leq 4$, *$E=2\sigma+F$ for an effective divisor $F\sim e\ell$, $\Delta\cap\sigma=\emptyset$, and* $\deg(\Delta\cap\ell)\leq L\ell=2$ *for any fiber $\ell\leq F$, where $L\sigma=4-e$.*

$[3;2,3]_{00}$**:** *$E=\sigma+\sigma_\infty$ and $\Delta\subset\sigma_\infty$ with* $\deg(\Delta)\leq 6$, *where $L\sigma=0$ and $L\sigma_\infty=7$.*

$[3;2,4]_+$**:** *$E=\sigma+D$ for a section $D\sim\sigma+4\ell$, $\Delta\cap\sigma=\emptyset$, and* $\deg(\Delta\cap D)\leq LD=8$, *where $L\sigma=0$.*

$[3;2,4]_{++}(a,b)$**:** *$E=\sigma+\sigma_\infty+\ell$, $\Delta\cap\sigma=\emptyset$,* $\deg(\Delta\cap\sigma_\infty)\leq L\sigma_\infty=6$, *and* $\deg(\Delta\cap\ell)\leq L\ell=2$, *where $L\sigma=0$. For $P=\sigma_\infty\cap\ell$,*

$$(a,b)=(\operatorname{mult}_P(\Delta\cap\ell),\operatorname{mult}_P(\Delta\cap\sigma_\infty))\\ \in\{(0,0),(1,1),(2,1),(1,2),(1,3),(1,4),(1,5),(1,6)\}.$$

The case $X=\mathbb{F}_4$:

$[4;1,0]_0$**:** *$E=\sigma$ and $\Delta=\emptyset$, where $LE=0$.*

$[4;2,e]_2$**:** $0\leq e\leq 4$, *$E=2\sigma+F$ for an effective divisor $F\sim e\ell$, $\Delta\cap\sigma=\emptyset$, and* $\deg(\Delta\cap\ell)\leq L\ell=2$ *for any fiber $\ell\leq F$, where $L\sigma=4-e$.*

$[4;2,4]_{00}$**:** *$E=\sigma+\sigma_\infty$, $\Delta\subset\sigma_\infty$, and* $\deg\Delta\leq L\sigma_\infty=8$, *where $L\sigma=0$.*

The case $X=\mathbb{F}_n$ *for* $n\geq 5$:

$[n;2;e]_2$**:** $0\leq e\leq 4$, *$E=2\sigma+F$ for an effective divisor $F\sim e\ell$, $\Delta\cap\sigma=\emptyset$, and* $\deg(\Delta\cap\ell)\leq L\ell=2$ *for any fiber $\ell\leq F$, where $L\sigma=4-e$.*

Here, $[e]$ indicates that $X\simeq\mathbb{P}^2$ and $\deg E=e$; *$[n;e_1,e_2]$ indicates that $X\simeq\mathbb{F}_n$ and $E\sim e_1\sigma+e_2\ell$. The subscripts $_0$, $_{00}$, $_+$, $_{++}$, $_2$, $_\times$ have the following meaning*:

$_0$: *E is non-singular and irreducible*

$_{00}$: *E is non-singular with two components*

$_+$: *E has exactly one node*

$_{++}$: *E has exactly two nodes*

$_2$: *E is not reduced*

$_\times$: *E has exactly one node .*

The subscript $_\times$ *is used for distinguishing the type* $[1;2,2]_\times$ *from* $[1;2,2]_+$.

Proof. We consider the structure of fundamental triplet (X, E, Δ) from properties of (X, E).

We first consider the case $X = \mathbb{P}^2$. If (X, E) is of type [1], then $\deg \Delta \le LE = 5$; thus (X, E, Δ) is of type $[1]_0$. Suppose that (X, E) is of type [2]. If E is irreducible and reduced, then E is a non-singular conic (even if $\operatorname{char} \Bbbk = 2$), and $\deg \Delta \le LE = 8$; this case is of type $[2]_0$. If E is not reduced, then $E = 2\ell$ for a line ℓ and $\deg(\Delta \cap \ell) \le L\ell = 4$; this case is of type $[2]_2$. Suppose E is reducible and reduced, then $E = \ell_1 + \ell_2$ for two lines ℓ_i with $\deg(\Delta \cap \ell_i) \le L\ell_i = 4$ for $i = 1, 2$. Since $\min\{\operatorname{mult}_P(\Delta \cap \ell_1), \operatorname{mult}_P(\Delta \cap \ell_2)\} \le 1$ by Lemma 2.12, the type is $[2]_+(b)$ for $0 \le b \le 4$.

Next, we consider the case $X = \mathbb{F}_n$. Then one of the following subcases occurs:

(1) $E = \sigma + F$ for an effective divisor F supported on fibers of π;

(2) $E = \sigma + D + F$ for a section $D \ne \sigma$ and an effective divisor F supported on fibers;

(3) $E = 2\sigma + F$ for an effective divisor F supported on fibers;

(4) E is irreducible and reduced with $E \ne \sigma$;

(5) $E \not\ge \sigma$ and E is either non-reduced or reducible.

Case (1). (X, E) is of type $[n; 1, e]$ for $e = F\sigma$ with $0 \le e \le \min\{2, 4-n\}$; if $n = 0$, then $e \le 1$ by Convention 3.23. If $e = 0$, then $E = \sigma$ and $\deg \Delta \le L\sigma = 4 - n$; this case is of type $[n; 1, 0]_0$ for $0 \le n \le 4$.

Suppose that $e = 1$. Then $n \le 3$ and $E = \sigma + \ell$ for a fiber ℓ with $\deg(\Delta \cap \sigma) \le L\sigma = 3 - n$, $\deg(\Delta \cap \ell) \le L\ell = 3$. This case is one of types $[0; 1, 1]_+(b)$, $[1; 1, 1]_+(a, b)$, $[2; 1, 1]_+(a, b)$, and $[3; 1, 1]_+$. Note that $(a, b) = (0, 0)$ or $\min\{a, b\} = 1$ by Lemma 2.12.

Suppose that $e = 2$. Then $n = 2$, since $[1; 1, 2]$ is not a type of (X, E) (cf. Lemma 3.24). Note that $\sigma \cap \Delta = \emptyset$ by $L\sigma = 0$. Thus this case is of type $[2; 1, 2]_{++}$ or $[2; 1, 2]_{2+}$.

Case (2). (X, E) is of type $[n; 2, e]$ for $n \le e \le \min\{n + 1, 4\}$, where $D \sim \sigma + m\ell$ for $n \le m \le e$.

Suppose that $m = n + 1$. Then $e = n + 1$, $n \leq 3$, and $E = \sigma + D$, where $D\sigma = 1$. Here, $\Delta \subset D$ by $(\mathcal{F}4)$, $\Delta = \emptyset$ for $n = 0$ by $(\mathcal{F}4)$, and $\deg \Delta \leq D^2 + n = 2n + 2$ by $(\mathcal{F}5)$. This case is of type $[n; 2, n + 1]_+$ for $0 \leq n \leq 3$.

Suppose that $m = e = n$. Then $E = \sigma + \sigma_\infty$ for a section $D = \sigma_\infty$ at infinity. Here $\Delta \subset \sigma_\infty$ by $(\mathcal{F}4)$ and $\deg \Delta \leq 2n$ by $(\mathcal{F}5)$. This case is of type $[n; 2, n]_{00}$ for $0 \leq n \leq 4$.

Suppose that $m = n$ and $e = n + 1$. Then $n \leq 3$ and $E = \sigma + \sigma_\infty + \ell$ for a section $D = \sigma_\infty$ at infinity and a fiber ℓ. Here, $\Delta \cap \sigma = \emptyset$ by $(\mathcal{F}4)$, $\Delta = \emptyset$ for $n = 0$ by $(\mathcal{F}4)$, $\deg(\Delta \cap \sigma_\infty) \leq 2n$ by $(\mathcal{F}5)$, and $\deg(\Delta \cap \ell) \leq L\ell = 2$. Thus the case is one of types $[0; 2, 1]_{++}$, $[1; 2, 2]_{++}(a, b)$, $[2; 2, 3]_{++}(a, b)$, $[3; 2, 4]_{++}(a, b)$.

Case (3). (X, E) is of type $[n; 2, e]$ for $e = F\sigma$ with $e \leq \min\{n + 1, 4\}$. Here $\Delta \cap \sigma = \emptyset$ by $(\mathcal{F}4)$ and $\deg(\Delta \cap \ell) \leq L\ell = 2$ for any fiber $\ell \leq F$. This case is of type $[n; 2, e]_2$ for $0 \leq e \leq \min\{n + 1, 4\}$, $n \geq 0$.

Case (4). Suppose that (X, E) is of type $[n; 1, e]$. Then $[n; 1, e]$ is one of $[0; 1, 1]$, $[1; 1, 1]$, and $[2; 1, 2]$ by Lemma 3.24. Here E is non-singular. Thus the type is one of $[0; 1, 1]_0$, $[1; 1, 1]_0$, and $[2; 1, 2]_0$.

Suppose that (X, E) is of type $[n, 2, e]$. Then $2n \leq e \leq \min\{n + 1, 4\}$ by the proof of Lemma 3.24. Hence $[n; 2, e]$ is $[0; 2, 1]$ or $[1; 2, 2]$, where E is non-singular. Thus the type is $[0; 2, 1]_0$ or $[1; 2, 2]_0$.

Case (5). This case is treated essentially in *Final step* of the proof of Proposition 4.4. By the proof of Lemma 3.24, the case is of type $[1; 2, 2]_\times$ or $[1; 2, 2]_{2\infty}$.

Thus we are done. □

Corollary 4.7.

(1) *For a fundamental triplet, the associated del Pezzo pair is log-terminal if and only if the type is one of the followings*:

$[1]_0, [2]_0, [2]_+(b),$

$[0; 1, 0]_0, [0; 1, 1]_0, [0; 1, 1]_+(b), [0; 2, 0]_{00}, [0; 2, 1]_0, [0; 2, 1]_+, [0; 2, 1]_{++},$

$[1; 1, 0]_0, [1; 1, 1]_0, [1; 1, 1]_+(a, b), [1; 2, 1]_{00},$

$[1; 2, 2]_0, [1; 2, 2]_\times, [1; 2, 2]_+, [1; 2, 2]_{++}(a, b),$

$[2; 1, 0]_0, [2; 1, 1]_+(a, b), [2; 1, 2]_0, [2; 1, 2]_{++},$

TABLE 2. The fundamental triplets with $LE = \deg\Delta$

Type	$\deg\Delta$	Type	$\deg\Delta$	Type	$\deg\Delta$
$[1]_0$	5	$[1;1,0]_0$	3	$[2;1,2]_{++}$	6
$[2]_0$	8	$[1;1,1]_0$	5	$[3;1,0]_0$	1
$[2]_+(b)$	8	$[1;1,1]_+(a,b)$	5	$[3;1,1]_+$	3
$[2]_2$	8	$[1;2,2]_0$	8	$[3;2,4]_+$	8
$[0;1,0]_0$	4	$[2;1,0]_0$	2	$[3;2,4]_{++}(a,b)$	8
$[0;1,1]_0$	6	$[2;1,1]_+(a,b)$	2	$[4;1,0]_0$	0
$[0;1,1]_+(b)$	6	$[2;1,2]_0$	6	$[4;2,4]_{00}$	8
				$[n;2,4]_2$ $(n \geq 3)$	8

$$[2;2,2]_{00}, [2;2,3]_+, [2;2,3]_{++}(a,b),$$
$$[3;1,0]_0, [3;1,1]_+, [3;2,3]_{00}, [3;2,4]_+, [3;2,4]_{++}(a,b),$$
$$[4;1,0]_0, [4;2,4]_{00}.$$

(2) *For a fundamental triplet, the associated del Pezzo pair is log-canonical but not log-terminal if and only if it has one of the following types with extra condition*:

$$[2]_2 \text{ with } \operatorname{mult}_P(\Delta \cap \ell) \leq 2 \text{ for any } P \in \ell,$$
$$[0;2,0]_2,\ [0;2,1]_0,\ [0;2,1]_2,\ [1;2,e]_2 \text{ for } 0 \leq e \leq 2,\ [1;2,2]_{2\infty},$$
$$[2;1,2]_{2+} \text{ with } \operatorname{mult}_P(\Delta \cap \ell) \leq 2 \text{ for any } P \in \ell,$$
$$[n;2,e]_2 \text{ for } n \geq 2, e \leq 2,$$
$$[n;2,e]_2 \text{ for } n \geq 2, e \geq 3 \text{ with } \operatorname{mult}_\ell F \leq 2 \text{ for any } \ell \leq F.$$

(3) *For a fundamental triplet (X, E, Δ), the associated del Pezzo pair (S, B) has $B = 0$ if and only if it belongs to one of the types with extra condition on* $\deg\Delta$ *listed in* TABLE 2. *Here, if the type is not $[2]_2$ nor $[n;2,4]_2$, then the fundamental triplet is log-terminal, i.e., defining a log del Pezzo surface of index two.*

PROOF. For a fundamental triplet (X, E, Δ) and its elimination (M, E_M), the log-terminal condition is equivalent to that E_M is reduced. This also equivalent to that E is reduced by Lemmas 2.10 and 2.14. Thus the list of (1) is obtained from Theorem 4.6. The log-canonical condition is equivalent to that the multiplicity of E_M along any irreducible component is at most two. If (X, E, Δ) is not log-terminal but log-canonical, then $\max\{\mathrm{mult}_{E_i}(E)\} = 2$ for the irreducible components $E_i \subset E$. In this case, by Theorem 4.6, Δ does not contain any node of E_{red}. By Lemma 2.17, we infer that (X, E, Δ) is log-canonical if and only if $\max\{\mathrm{mult}_{E_i}(E)\} = 2$ and $\mathrm{mult}_P(\Delta \cap E_i) \leq 2$ for any irreducible component $E_i \subset E$ with $\mathrm{mult}_{E_i}(E) = 2$. Thus we have the list of (2). For (3), we note that the three conditions: $B = 0$, $L_M E_M = 0$, and $LE = \deg \Delta$ are mutually equivalent. Thus we have TABLE 2. $\square$

THEOREM 4.8. *A del Pezzo pair (S, B) of index one with $B \neq 0$ is one of the following*:

(1) $S = \mathbb{P}^2$ *and* $\deg B \in \{1, 2\}$.

(2) $S = \mathbb{F}_n$ *and B is a minimal section of* $\mathbb{F}_n \to \mathbb{P}^1$ *for* $n \geq 0$.

(3) $S = \mathbb{F}_n$ *and* $B \sim \sigma + \ell$ *for a minimal section σ and a fiber ℓ of* $\mathbb{F}_n \to \mathbb{P}^1$ *for* $n \geq 0$.

(4) $S = \mathbb{P}(1, 1, n)$ *and* $B \sim 2\overline{\ell}$ *for a generating line $\overline{\ell}$ for* $n \geq 2$.

PROOF. We infer that S is rational by Lemma 3.10, Proposition 3.11, and Corollary 3.16. Moreover, if $g(S, B) = 0$, then $S = \mathbb{P}^2$ with $\deg B = 2$ by Proposition 3.11.

Suppose that $g(S, B) = 1$. Then S is a log del Pezzo surface of index one and $-K_S \sim 2B$ by Lemma 3.12. For the minimal desingularization $\alpha\colon M \to S$, $K_M \sim \alpha^* K_S$ is divisible by two; hence M has no (-1)-curve. Thus $M = \mathbb{F}_m$ for $m \in \{0, 2\}$. If $m = 0$, then (S, B) belongs to the case (3) with $n = 0$. If $m = 2$, then (S, B) belongs to the case (4) with $n = 2$.

Therefore, we may assume that (S, B) is obtained from a fundamental triplet (X, E, Δ), where $(1/2)E_M$ is Cartier for the elimination (M, E_M) of (X, E, Δ). Then Δ does not contain any nodes of E_{red} by Theorem 4.6. Furthermore, $\Delta = \emptyset$ by Lemma 2.17. By Theorem 4.6, we have only the following types of possible $(X, E, \Delta = \emptyset)$:

(a) $[2]_2$.

(b) $[n; 2, 0]_2$ for $n \geq 0$.

(c) $[n; 2, 2]_2$ for $n \geq 1$, where $E = 2\sigma + 2\ell$,

(d) $[1; 2, 2]_{2\infty}$,

(e) $[n; 2, 4]_2$ for $n \geq 3$, where $E = 2\sigma + 2F'$ for an effective divisor $F' \sim 2\ell$.

According to the cases (a), (b), (c), (d), (e), the associated del Pezzo pair (S, B) belongs to (1), (2), (3), (3), (4). Hence, we have the list of $(S, B \neq \emptyset)$ of index one. □

THEOREM 4.9. *Let (X, E, Δ) be a fundamental triplet and let (M, E_M) be the elimination. Then the type of the fundamental triplet (X, E, Δ) and* $\deg(\Delta)$ *depend only on (M, E_M). Moreover, the isomorphism class of (X, E, Δ) depends only on (M, E_M) except for the following two cases*:

- *(X, E, Δ) is of type $[1; 2, 2]_0$.*
- *(X, E, Δ) is of type $[n; 2, n+1]_{++}(1, b)$ for $1 \leq n \leq 3$, where*

$$\deg(\Delta \cap \sigma_\infty) = 2n \quad \textit{and} \quad \mathrm{mult}_P(\Delta) + \deg(\Delta \cap \ell) = 2 + b$$

for the irreducible decomposition $E = \sigma + \sigma_\infty + \ell$ and for the node $P = \sigma_\infty \cap \ell$.

The proof needs the following:

PROPOSITION 4.10. *Let $f: Y \to T$ be a proper surjective morphism from a non-singular surface Y into a non-singular curve T such that a general fiber is isomorphic to $\mathbb{P}^1$. Let E_1 and E_2 be two sections of f such that $E_1 \cap E_2 = \emptyset$ and $K_Y + E_1 + E_2$ is f-numerically trivial. Let $\phi: Y \to X = \mathbb{P}_T(f_*\mathcal{O}_Y(E_1))$ be the morphism defined in* Lemma 4.5 *for E_1. Then $E_{i,X} := \phi(E_i)$ is a section of $X \to \mathbb{P}^1$ for $i = 1, 2$ with $E_{1,X} \cap E_{2,X} = \emptyset$ and ϕ is the elimination of a zero-dimensional subscheme $\Delta \subset E_{2,X}$. In particular, there is an action of the algebraic group $\mathbb{G}_\mathtt{m} = \operatorname{Spec} \Bbbk[\mathtt{t}, \mathtt{t}^{-1}]$ on Y such that it fixes every point of $E_1 \cup E_2$ and that it acts non-trivially on every irreducible component of any fiber of f. Moreover, if $f_*\mathcal{O}_{E_1}(E_1) \simeq f_*\mathcal{O}_{E_2}(E_2)$, then the following assertions hold*:

(1) *Let ℓ be a non-singular fiber of f and let P_1, P_2 be any points of $\ell \setminus (E_1 \cup E_2)$ including the case $P_1 = P_2$. Then there exists an involution ι of Y over T such that $\iota(E_1) = E_2$ and $\iota(P_1) = P_2$.*

(2) *Let Γ_1 and Γ_2 be irreducible components of a reducible fiber F of f with $E_1\Gamma_1 = E_2\Gamma_2 = 1$. Then, for any points $P_1 \in \Gamma_1 \setminus (E_1 \cup \operatorname{Sing} F)$ and $P_2 \in \Gamma_2 \setminus (E_2 \cup \operatorname{Sing} F)$, there is an involution ι of Y over T such that $\iota(E_1) = E_2$ and $\iota(P_1) = P_2$.*

(3) *Let $\Gamma_1 + \Gamma_2$ be a fiber of f, $\widehat{Y} \to Y$ the blowing up along the intersection point $\Gamma_1 \cap \Gamma_2$, G the exceptional curve for the blowing up, $\widehat{\Gamma}_i$ the proper transform of Γ_i in $\widehat{Y}$ for $i = 1$, 2, and let P_1, P_2 be any points of $G \setminus (\widehat{\Gamma}_1 \cup \widehat{\Gamma}_2)$. Then there is an involution $\hat{\iota}$ of $\widehat{Y}$ over T such that $\hat{\iota}(\widehat{\Gamma}_1) = \widehat{\Gamma}_2$ and $\hat{\iota}(P_1) = P_2$.*

Proof. $E_1 = \phi^* E_{1,X}$ by Lemma 4.5. Thus ϕ is the elimination of a subscheme $\Delta \subset E_{2,X}$ by Proposition 2.9. We have a natural action of $\mathbb{G}_{\mathtt{m}}$ on the $\mathbb{P}^1$-bundle X which fixes every point of $E_{1,X} \cup E_{2,X}$. Since $\mathbb{G}_{\mathtt{m}}$ fixes the subscheme Δ, the action lifts to Y, by the following observation:

Let $\mathbb{A}^2 = \operatorname{Spec} \Bbbk[\mathtt{u}, \mathtt{v}]$ be an affine plane with an action of $\mathbb{G}_{\mathtt{m}} = \operatorname{Spec} \Bbbk[\mathtt{t}, \mathtt{t}^{-1}]$ given by $(\mathtt{u}, \mathtt{v}) \mapsto (\mathtt{tu}, \mathtt{v})$. Then every point of $\{\mathtt{u} = 0\}$ is fixed by the action. Let $U \to \mathbb{A}^2$ be the blowing up at the origin. Then $U = U_1 \cup U_2$ for two affine open subsets $U_1 = \operatorname{Spec} \Bbbk[\mathtt{u}_1, \mathtt{v}_1]$, $U_2 = \operatorname{Spec} \Bbbk[\mathtt{u}_2, \mathtt{v}_2]$, where the morphism to $\mathbb{A}^2$ is described as

$$(\mathtt{u}_1, \mathtt{v}_1) \mapsto (\mathtt{u}, \mathtt{v}) = (\mathtt{u}_1, \mathtt{u}_1\mathtt{v}_1) \quad \text{and} \quad (\mathtt{u}_2, \mathtt{v}_2) \mapsto (\mathtt{u}, \mathtt{v}) = (\mathtt{u}_2\mathtt{v}_2, \mathtt{v}_2).$$

Here, $\{\mathtt{u}_1 = 0\} \cup \{\mathtt{v}_2 = 0\}$ is the exceptional divisor. Then the action of $\mathbb{G}_{\mathtt{m}}$ lifts to U as

$$(\mathtt{u}_1, \mathtt{v}_1) \mapsto (\mathtt{tu}_1, \mathtt{t}^{-1}\mathtt{v}_1) \quad \text{and} \quad (\mathtt{u}_2, \mathtt{v}_2) \mapsto (\mathtt{tu}_2, \mathtt{v}_2).$$

If we consider the blowing up of U at the point $(\mathtt{u}_2, \mathtt{v}_2) = (0, 0) \in U_2$, then the action also lifts to the blowing up in the same way.

Therefore, $\mathbb{G}_{\mathtt{m}}$ acts on Y, and acts non-trivially on every irreducible component of a fiber of f. Let $\Phi_t \colon Y \to Y$ be the action of $t \in \mathbb{G}_{\mathtt{m}}(\Bbbk) = \Bbbk \setminus \{0\}$. Let $(\mathtt{x} : \mathtt{y})$ be a coordinate of a non-singular fiber $Y_o = f^{-1}(o) \simeq \mathbb{P}^1$ of f such that $E_1 \cap Y_o = \operatorname{div}(\mathtt{x})$ and $E_2 \cap Y_o = \operatorname{div}(\mathtt{y})$. Then we may assume that Φ_t induces the automorphism $(\mathtt{x} : \mathtt{y}) \mapsto (t\mathtt{x} : \mathtt{y})$ on Y_o.

Let $\mathcal{L}$ be an invertible sheaf on T and suppose that $f_*\mathcal{O}_{E_i}(E_i) \simeq \mathcal{L}$ for $i = 1, 2$. Then we have an isomorphism

$$\chi\colon f_*\mathcal{O}_{E_1+E_2}(E_1+E_2) \simeq f_*\mathcal{O}_{E_1}(E_1+E_2) \oplus f_*\mathcal{O}_{E_2}(E_1+E_2) \xrightarrow{\simeq} \mathcal{L}^{\oplus 2}.$$

For $\lambda \in \Bbbk \setminus \{0\}$, let $\mathcal{L}^{\oplus 2} \to \mathcal{L}$ be the homomorphism given by $(x, y) \mapsto \lambda x - y$ and let $\mathcal{M}_\lambda \subset f_*\mathcal{O}_{E_1+E_2}(E_1+E_2)$ be the subsheaf isomorphic via χ to the kernel of $\mathcal{L}^{\oplus 2} \to \mathcal{L}$. Then we have a locally free subsheaf $\mathcal{E}_\lambda$ of $f_*\mathcal{O}_Y(E_1+E_2)$ and a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_T & \longrightarrow & \mathcal{E}_\lambda & \longrightarrow & \mathcal{M}_\lambda & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_T & \longrightarrow & f_*\mathcal{O}_Y(E_1+E_2) & \longrightarrow & f_*\mathcal{O}_{E_1+E_2}(E_1+E_2) & \longrightarrow & 0
\end{array}$$

of exact sequences. Note that, under the isomorphism

$$f_*\mathcal{O}_Y(E_1+E_2) \otimes \Bbbk(o) \simeq \Bbbk\mathtt{x}^2 + \Bbbk\mathtt{xy} + \Bbbk\mathtt{y}^2,$$

the fiber $\mathcal{E}_\lambda \otimes \Bbbk(o)$ corresponds to the subspace $\Bbbk(\lambda\mathtt{x}^2 + \mathtt{y}^2) + \Bbbk\mathtt{xy}$. Hence, $\Phi_t^*\mathcal{E}_\lambda = \mathcal{E}_{t^2\lambda}$. The natural homomorphism $f^*\mathcal{E}_\lambda \to \mathcal{O}_Y(E_1+E_2)$ is surjective since the projection $\mathcal{M}_\lambda \to f_*\mathcal{O}_{E_i}(E_1+E_2)$ is surjective for $i = 1, 2$. Hence, we have a morphism $h_\lambda\colon Y \to \mathbb{P} = \mathbb{P}_T(\mathcal{E}_\lambda)$ over T and a section Σ of $\mathbb{P} \to T$ such that $h_\lambda^*\Sigma = E_1 + E_2$. We may assume that the restriction of h_λ to Y_o is described as $(\mathtt{x}:\mathtt{y}) \mapsto (\lambda\mathtt{x}^2 + \mathtt{y}^2 : \mathtt{xy})$. Let $Y \to Y' \to \mathbb{P}$ be the Stein factorization. Then $Y' \to \mathbb{P}$ is a separable double-covering and Y is the minimal desingularization of Y'. Thus the Galois involution ι_λ acts on Y as an automorphism, where $\iota_\lambda(E_1) = E_2$. Moreover the restriction of ι_λ to Y_o is described as $(\mathtt{x}:\mathtt{y}) \mapsto (\mathtt{y}:\lambda\mathtt{x})$. Hence,

$$\iota_\lambda \circ \Phi_t = \Phi_t \circ \iota_{t^2\lambda} = \iota_{t\lambda}.$$

For the assertions (1)–(3), it is enough to find an involution ι_λ with $\iota_\lambda(P_1) = P_2$. The existence of λ is shown as follows:

(1): Since the action of $\mathbb{G}_\mathtt{m}$ on the fiber ℓ is non-trivial, $\Phi_t(P_1) = P_2$ for some t. Hence, $\iota_\lambda(P_1) = P_2$ for some λ.

(2): Since the action of $\mathbb{G}_\mathtt{m}$ on Γ_2 is non-trivial, $\Phi_t \circ \iota_\lambda(P_1) = P_2$ for some λ and t. Thus $\iota_{t^{-1}\lambda}(P_1) = P_2$.

(3): The involution ι_λ lifts to an involution $\hat{\iota}_\lambda$ of $\widehat{Y}$, since ι_λ fixes the intersection point $\Gamma_1 \cap \Gamma_2$. Similarly, $\mathbb{G}_\mathtt{m}$ acts on $\widehat{Y}$. We infer that $\mathbb{G}_\mathtt{m}$ acts non-trivially also on the exceptional divisor G by the observation above. Hence, $\hat{\iota}_\lambda(P_1) = P_2$ for some λ. $\square$

We shall prove Theorem 4.9.

Proof. We may assume that $K_M + L_M$ is not big and $\Delta \neq \emptyset$. Then (X, E) is of type $[n; 2, e]$ for $n > 0$ and $e \leq n + 1$. Let T be the type of the fundamental triplet (X, E, Δ). Let (X', E', Δ') be another fundamental triplet of type T' whose elimination is (M, E_M). Let $[n'; 2, e']$ be the type of (X', E'). We may assume that $\pi \circ \phi = \pi' \circ \phi'$ for the elimination morphisms $\phi\colon M \to X$, $\phi'\colon M \to X'$, and the $\mathbb{P}^1$-bundle structures $\pi\colon X \to \mathbb{P}^1$, $\pi'\colon X' \to \mathbb{P}^1$, since $\pi \circ \phi$ is just the morphism $M \to \mathbb{P}^1$ associated with the linear system $|K_M + L_M|$. Let σ and σ' be the negative sections of X and X', respectively.

By Theorem 4.6, one of the following three cases occurs:

(1) $E \geq 2\sigma$; (2) $E \geq \sigma + D$ for a section $D \neq \sigma$; (3) $\mathsf{T} = [1; 2; 2]_0$.

Case (1). $\mathsf{T} = [n, 2, e]_2$ by Theorem 4.6, and $E_M \geq 2\sigma_M$ for the total transform σ_M of σ in M. Thus $E' = \phi'_* E_M \geq 2\phi'_*\sigma_M$ for the section $\phi'_*\sigma_M$. Then $\sigma' = \phi'_*\sigma_M$ and $\mathsf{T}' = [n'; 2, e']_2$ by Theorem 4.6. In particular, σ_M is also the total transform of σ' and $n = n'$. By Lemma 4.5, $\phi \simeq \phi'$ over $\mathbb{P}^1$, and hence $(X, E, \Delta) \simeq (X', E', \Delta')$.

Case (2). $D^2 + n \geq \deg(\Delta \cap D)$ by ($\mathcal{F}5$). Hence $E_M \geq \sigma_M + D_M$ for the total transform $\sigma_M \subset M$ of σ and the proper transform $D_M \subset M$ of D, where $D_M^2 \geq -n$. Moreover, T is one of

$$[n; 2, n]_{00}\ (1 \leq n \leq 4), \quad [n; 2, n+1]_+\ (1 \leq n \leq 3),$$
$$[n; 2, n+1]_{++}(a, b)\ (1 \leq n \leq 3),$$

by the proof of Theorem 4.6. Since $E' = \phi'_* E_M$ is also reducible and $\Delta' \neq \emptyset$, $E' \geq \sigma' + D'$ for a section $D' \neq \sigma'$ by ($\mathcal{F}6$). In particular, $E_M \geq \sigma'_M + D'_M$ for the total transform $\sigma'_M \subset M$ of σ' and the proper transform $D'_M \subset M$ of D', where $D'^2_M \geq -n'$. If $\sigma_M = \sigma'_M$, then $\phi \simeq \phi'$ and $(X, E, \Delta) \simeq (X', E', \Delta')$ by Lemma 4.5. Thus we may assume that $\sigma_M \neq \sigma'_M$. Therefore, $n = n' = -D_M^2 = -D'^2_M$, $\sigma_M = D'_M$, and $\sigma_{M'} = D_M$. In particular, one of the following cases occurs:

(2-i) $\mathsf{T} = [n; 2, n]_{00}$ and $\deg \Delta = 2n$;

(2-ii) $\mathsf{T} = [n; 2, n+1]_{+}$ and $\deg \Delta = 2n + 2$;

(2-iii) $\mathsf{T} = [n; 2, n+1]_{++}(a, b)$ and $\deg(\sigma_\infty \cap \Delta) = 2n$ for $D = \sigma_\infty$.

Subcase (2-i). Applying Proposition 4.10 to $\pi \circ \phi \colon M \to T = \mathbb{P}^1$ and two sections σ_M, D_M, we infer that $\iota(\sigma_M) = D_M$ for an involution of M over $\mathbb{P}^1$. Hence, $\phi' \simeq \phi \circ \iota$ and $(X, E, \Delta) \simeq (X', E', \Delta')$.

Subcase (2-ii). Let $\mathcal{Y} \to M$ be the blowing up at the point $P = \sigma_M \cap D_M$ and let $\mathcal{Y} \to \widehat{M}$ be the contraction of the proper transform $\ell_{\mathcal{Y}} \subset \mathcal{Y}$ of the fiber ℓ of $M \to \mathbb{P}^1$ passing through P. Let $\hat{\sigma}$ and $\hat{D}$ be the proper transforms of σ_M and D_M in $\widehat{M}$, respectively. Then $\hat{\sigma} \cap \hat{D} = \emptyset$ and $K_{\widehat{M}} + \hat{\sigma} + \hat{D}$ is relatively numerically trivial over $\mathbb{P}^1$. Let $\hat{\ell}$ be the fiber of $\widehat{M} \to \mathbb{P}^1$ over the point $\pi \circ \phi(P)$ and let $Q \in \hat{\ell}$ be the image of $\ell_{\mathcal{Y}}$. Applying Proposition 4.10 to $\widehat{M} \to \mathbb{P}^1$, two sections $\hat{\sigma}$, $\hat{D}$, and to the point Q, we have an involution $\hat{\iota}$ of $\widehat{M}$ over $\mathbb{P}^1$ such that $\hat{\iota}(\hat{\sigma}) = \hat{D}$ and $\hat{\iota}(Q) = Q$. Thus $\hat{\iota}$ induces an involution ι of M over $\mathbb{P}^1$ with $\iota(\sigma_M) = D_M$. Hence, $\phi' \simeq \phi \circ \iota$ and $(X, E, \Delta) \simeq (X', E', \Delta')$.

Subcase (2-iii). Then $E = \sigma + \sigma_\infty + \ell$ for $D = \sigma_\infty$ and for a fiber ℓ of π. Let P be the node $\sigma_\infty \cap \ell$. We write $D_M = \sigma_{\infty, M}$.

If $(a, b) = (0, 0)$, then we have an involution ι of M over T with $\iota(\sigma_M) = \sigma_{\infty, M}$ by Proposition 4.10 as above. Thus we may assume that $(a, b) \neq (0, 0)$.

Suppose that $(a, b) = (2, 1)$, i.e., $\mathrm{mult}_P(\Delta \cap \ell) = 2$. Then $\Delta \cap \ell$ is supported on P. Let $\phi^\sharp \colon M^\sharp \to X$ be the elimination of the subscheme $(\Delta \setminus P) \cup (\Delta \cap \ell)$. Then $\phi^{\sharp *} \ell = \ell^\sharp + 2\Gamma_1^\sharp + \Gamma_2^\sharp$ for the proper transform $\ell^\sharp \subset M^\sharp$ of ℓ, a (-1)-curve $\Gamma_1^\sharp$, and for a (-2)-curve $\Gamma_2^\sharp$ such that $\ell^\sharp + \Gamma_1^\sharp + \Gamma_2^\sharp$ is a chain of rational curves and that $\Gamma_2^\sharp$ only intersects the proper transform of σ_∞ in $M^\sharp$. Suppose that Δ is not a Cartier divisor of E at P. Then $M = M^\sharp$, and by Proposition 4.10, (3), there is an involution ι of M over $\mathbb{P}^1$ satisfying $\iota(\sigma_M) = \sigma_{\infty, M}$. Thus $\phi' \simeq \phi \circ \iota$ and $(X, E, \Delta) \simeq (X', E', \Delta')$. Suppose next that Δ is a Cartier divisor of E at P. Then $M \to M^\sharp$ is given as the blowing up along a point $P_1 \in \Gamma_1^\sharp \setminus (\ell^\sharp \cup \Gamma_2^\sharp)$. Thus by Proposition 4.10, (3), there is an involution ι of M over $\mathbb{P}^1$ satisfying $\iota(\sigma_M) = \sigma_{\infty, M}$. Thus $\phi' \simeq \phi \circ \iota$ and $(X, E, \Delta) \simeq (X', E', \Delta')$.

Suppose that $a = 1$, i.e., $\mathrm{mult}_P(\Delta \cap \ell) = 1$. Let $\phi^\sharp \colon M^\sharp \to X$ be the elimination of $\Delta \cap \sigma_\infty$. Then $\phi^{\sharp *}\ell = \ell^\sharp + \Gamma_1^\sharp + \cdots + \Gamma_b^\sharp$ is a chain of rational curves for the proper transform $\ell^\sharp \subset M^\sharp$ of ℓ, (-2)-curves $\Gamma_i^\sharp$ for $i < b$, and for a (-1)-curve $\Gamma_b^\sharp$, such that $\Gamma_b^\sharp$ only intersects the proper transform of σ_∞ in $M^\sharp$.

If Δ is not a Cartier divisor of E at P and if $\deg(\Delta \cap \ell) = 1$, then $M \simeq M^\sharp$ and $\iota(\sigma_M) = \sigma_{\infty,M}$ for an involution of M by Proposition 4.10, (2). Thus, $\phi' \simeq \phi \circ \iota$ and $(X, E, \Delta) \simeq (X', E', \Delta')$.

If Δ is a Cartier divisor of E at P and if $\deg(\Delta \cap \ell) = 2$, then $M \to M^\sharp$ is the blowing up at certain two points $P_1^\sharp \in \ell^\sharp$ and $P_b^\sharp \in \Gamma_b^\sharp$, and hence $\iota(\sigma_M) = \sigma_{\infty,M}$ for an involution of M by Proposition 4.10, (2). Thus, $\phi' \simeq \phi \circ \iota$ and $(X, E, \Delta) \simeq (X', E', \Delta')$.

Therefore, it remains only the case where $\mathrm{mult}_P(\Delta) + \deg(\Delta \cap \ell) = b + 2$. This is divided into the following two cases:

(A) Δ is a Cartier divisor of E at P and $\deg(\Delta \cap \ell) = 1$;

(B) Δ is not a Cartier divisor of E at P and $\deg(\Delta \cap \ell) = 2$.

We shall show that if (X, E, Δ) belongs to the case (A), then (X', E', Δ') is also of type $[n; 2, n+1]_{++}(1, b)$ belonging to the case (B), and vice versa.

Suppose that (X, E, Δ) belongs to the case (A). Then $M \to M^\sharp$ is the blowing-up at a certain point $P_b^\sharp \in \Gamma_b^\sharp$. By Proposition 4.10, (2), there is an involution $\iota^\sharp$ of $M^\sharp$ which interchanges the proper transforms of σ and σ_∞ in $M^\sharp$. Thus $\phi' \colon M \to X'$ is the composite of $M \to M^\sharp$ and $\phi^\sharp \circ \iota^\sharp$. Hence, $(X', E', \Delta' \setminus \ell') \simeq (X, E, \Delta \setminus \ell)$ for the fiber ℓ' over $\phi(\ell)$, and (X', E', Δ') is of type $[n; 2, n+1]_{++}(1, b)$ belonging to (B).

Similarly, if (X, E, Δ) belongs to (B), then (X', E', Δ') is of type $[n; 2, n+1]_{++}(1, b)$ belonging to (A).

Case (3). We have $\mathsf{T}' = [1; 2, 2]_0$ by the results in the cases (1) and (2). Thus, we are done. $\square$

There are some ideas of dividing the type $[1; 2, 2]_0$ into suitable subtypes by properties related to the double-covering $\pi|_E \colon E \subset X \to \mathbb{P}^1$. For example, $\pi|_E$ is not necessarily separable if $\mathrm{char}\, \Bbbk = 2$. For the type $[1; 2, 2]_0$, (X, E) has the following explicit description:

Lemma 4.11. *For the ruled surface $\pi \colon X = \mathbb{F}_1 \to \mathbb{P}^1$, let $E \subset X$ be a non-singular curve linearly equivalent to $2\sigma + 2\ell$ for the negative section*

σ and a fiber ℓ of π. Then there exist a homogeneous coordinate $(\mathtt{X}:\mathtt{Y}:\mathtt{Z})$ of $\mathbb{P}^2$ and an isomorphism from X to the blowing up of $\mathbb{P}^2$ at the point $(0:0:1)$ such that π is induced from the projection $(\mathtt{X}:\mathtt{Y}:\mathtt{Z}) \mapsto (\mathtt{X}:\mathtt{Y})$ and E corresponds to the total transform of the one of following curves:

(1) $\{\mathtt{Z}^2 = \mathtt{XY}\}$;

(2) $\{\mathtt{Z}^2 + \mathtt{XZ} + \mathtt{Y}^2 = 0\}$.

If $\operatorname{char} \Bbbk = 2$, *then* $\pi|_E \colon E \to \mathbb{P}^1$ *is inseparable in case* (1), *and separable in case* (2). *If* $\operatorname{char} \Bbbk \neq 2$, *then* (1) *and* (2) *define the same* (X, E) *up to isomorphism.*

Proof. Let $\mathtt{g}$ be a defining equation of σ and $\mathtt{f}$ be a defining equation of a section σ_∞ at infinity. Let $(\mathtt{s}, \mathtt{t})$ denote a homogeneous coordinate of $\mathbb{P}^1$. A defining equation $\eta \in \mathrm{H}^0(X, 2\sigma + 2\ell)$ of E is written by

$$\eta = \mathtt{f}^2 + a(\mathtt{s}, \mathtt{t})\mathtt{fg} + b(\mathtt{s}, \mathtt{t})\mathtt{g}^2$$

for homogeneous polynomials $a(\mathtt{s}, \mathtt{t})$ and $b(\mathtt{s}, \mathtt{t})$ of degree 1 and 2, respectively. We can replace $\mathtt{f}$ with $\mathtt{f} + c(\mathtt{s}, \mathtt{t})\mathtt{g}$ for a linear form $c = c(\mathtt{s}, \mathtt{t})$. By the replacement, (a, b) is changed to $(a + 2c, b + ac + c^2)$. Thus we may assume one of the following two cases occurs:

$$\text{(i)} \quad a = 0; \qquad \text{(ii)} \quad b = b_1^2 \ \text{ for a linear form } b_1.$$

In fact, this is shown as follows: If $\operatorname{char} \Bbbk \neq 2$, then the case (i) can be occur since $a + 2c = 0$ for some c; If $\operatorname{char} \Bbbk = 2$ and $a \neq 0$, then we can take $(a, b) = (\mathtt{s}, \lambda \mathtt{t}^2)$ for a non-zero constant $\lambda \in \Bbbk$. If (i) and (ii) occur at the same time, then we have

$$\mathtt{f}^2 + a\mathtt{fg} + b\mathtt{g}^2 = (\mathtt{f} + \sqrt{-1}b_1\mathtt{g})(\mathtt{f} - \sqrt{-1}b_1\mathtt{g}),$$

which contradicts the irreducibility of E. In case (i), we may assume $b = \mathtt{st}$ by a suitable coordinate change of $(\mathtt{s}, \mathtt{t})$, and thus we have the case (1). In case (ii), we may assume similarly $a = \mathtt{s}$ and $b = \mathtt{t}^2$, and thus we have the case (2). If $\operatorname{char} \Bbbk \neq 2$, then $(a, b) = (\mathtt{s}, \mathtt{t}^2)$ is changed to

$$(a + 2c, b + ac + c^2) = (0, (\mathtt{t} + (1/2)\mathtt{s})(\mathtt{t} - (1/2)\mathtt{s}))$$

by $c = -(1/2)a$; thus (1) and (2) define the same (X, E) up to isomorphism. $\square$

Even if $\operatorname{char} \Bbbk \neq 2$, the uniqueness of fundamental triplet (cf. Theorem 4.9) does not hold in general for the type $[1; 2, 2]_0$ as follows:

Example 4.12. Let (X, E, Δ) be a fundamental triplet of type $[1; 2, 2]_0$ with a fiber ℓ of $\pi \colon X \to \mathbb{P}^1$ such that $\ell \cap E$ consists of two points P_1, P_2. We set $\operatorname{mult}_{P_i}(\Delta) = m_i$ for $i = 1$, 2, and assume that $m_1 \geq 2$, $m_2 \geq 0$.

We shall show the existence of a section σ_∞ at infinity with $\operatorname{mult}_{P_1}(\sigma_\infty \cap E) = 2$. In fact, from the exact sequence

$$\begin{aligned} 0 \to \mathrm{H}^0(X, -\sigma - \ell) \to \mathrm{H}^0(X, \sigma + \ell) \to \mathrm{H}^0(E, (\sigma + \ell)|_E) \\ \simeq \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(2)) \to 0, \end{aligned}$$

there is an effective divisor $D \sim \sigma + \ell$ with $D|_E = 2P_1$ on E. If D is reducible, then $D = \sigma + \ell$ but $\ell \cap E \neq 2P_1$; this is a contradiction. Thus D is a section at infinity.

Let $\phi^{-1}(P_i) = \sum_{j=1}^{m_i} \Gamma_j^{(i)}$ be the chain of ϕ-exceptional curves over P_i for $i = 1$, 2; however we do not consider $\phi^{-1}(P_2)$ in case $m_2 = 0$. Here, $\Gamma_{m_i}^{(i)}$ is an end (-1)-curve and others are (-2)-curves. For the proper transform $\ell_M \subset M$ of ℓ, the inverse image $\phi^{-1}(\ell)$ is a straight chain of rational curves written as

$$\begin{cases} \sum_{i=1}^{m_1} \Gamma_i^{(1)} + \ell_M + \sum_{j=1}^{m_2} \Gamma_j^{(2)}, & \text{if } m_2 > 0; \\ \sum_{i=1}^{m_1} \Gamma_i^{(1)} + \ell_M, & \text{if } m_2 = 0, \end{cases}$$

where ℓ_M intersects only $\Gamma_1^{(1)}$ and $\Gamma_1^{(2)}$ in the chain $\phi^{-1}(\ell)$ when $m_2 > 0$, and intersects only $\Gamma_1^{(1)}$ when $m_2 = 0$. The proper transform $\sigma_{\infty,M}$ of σ_∞ in M intersects only $\Gamma_2^{(1)}$ in the chain $\phi^{-1}(\ell)$. Note that the section $\sigma_{\infty,M}$ of $M \to \mathbb{P}^1$ is a (-1)-curve with $\sigma_{\infty,M} \cap E_M = \emptyset$. Let $\phi' \colon M \to X'$ be the morphism of Lemma 4.5 defined for the section $\sigma_{\infty,M}$, and let $\sigma' \subset X'$ be the image $\phi'(\sigma_{\infty,M})$. Then $\sigma_{\infty,M} = \phi'^*(\sigma')$. Therefore, $X' \simeq \mathbb{F}_1$, ϕ' contracts any irreducible component of $\phi^{-1}(\ell)$ except for $\Gamma_2^{(1)}$, and $\sigma' \cap \phi'(E_M) = \emptyset$. Thus ϕ' is the elimination of a fundamental triplet (X', E', Δ') of type $[1; 2, 2]_0$ which is isomorphic to (X, E, Δ) over $\mathbb{P}^1 \setminus \pi(\ell)$. Furthermore, for the fiber ℓ' of $X' \to \mathbb{P}^1$ over $\pi(\ell)$, we have $\ell' \cap E' = \{P'_1, P'_2\}$ with $\operatorname{mult}_{P'_1}(\Delta') = m_1 - 2$ and $\operatorname{mult}_{P'_2}(\Delta') = m_2 + 2$. Thus (X', E', Δ') is not isomorphic to (X, E, Δ).

4.3. Non-Gorenstein exceptional graphs

LEMMA 4.13. *Let* (X, E, Δ) *be a fundamental triplet,* (M, E_M) *the elimination of* Δ*, and let* (S, B) *be the associated del Pezzo pair of index two. An irreducible curve* $\Gamma \subset M$ *is exceptional for* $\alpha\colon M \to S$ *if and only if one of the following conditions is satisfied*:

(1) Γ *is a* (-2)*-curve contracted by the elimination* $\phi\colon M \to X$ *of* Δ;

(2) Γ *is the proper transform in* M *of an irreducible component* $E_i \subset E$ *with* $LE_i = \deg(\Delta \cap E_i)$;

(3) Γ *is the total transform in* M *of* σ *in the case of type* $[2; 1, 2]_0$;

(4) Γ *is the proper transform in* M *of a fiber* ℓ *of* $\pi\colon X \to \mathbb{P}^1$ *with* $\deg(\ell \cap \Delta) = 2$ *in the case of type* $[1; 2, 2]_0$.

Moreover, if an irreducible component Γ *of* E_M *is* α*-exceptional, then* $m = \operatorname{mult}_\Gamma E_M \leq 4$ *and the following properties hold*:

(i) *If* $m = 1$*, then* $\Gamma^2 \geq -4$*, where the equality holds if and only if* Γ *is a connected component of* E_M.

(ii) *In case* $m = 1$*,* $\Gamma^2 = -3$ *if and only if* $(E_M - \Gamma)\Gamma = 1$.

(iii) *In case* $m = 2$*,* $\Gamma^2 = -n \geq -4$ *if and only if* Γ *is the proper transform of* σ *in the case of type* $[n; 2, 4]_2$.

(iv) *If* $m = 2$ *and* $\Gamma^2 = -3$*, then* Γ *is one of the following curves*:

  (a) *The proper transform of* ℓ *in the case of type* $[2]_2$;
  (b) *The proper transform of* σ *in the case of type* $[3; 2, 4]_2$;
  (c) *The proper transform of* ℓ *in the case of type* $[2; 1, 2]_{2+}$.

(v) *If* $m \geq 3$*, then* $\Gamma^2 = -2$.

PROOF. We fix an irreducible curve $\Gamma \subset M$ with $\Gamma^2 < 0$. Note that Γ is α-exceptional if and only if $L_M\Gamma = 0$. Since $-2K_M = L_M + E_M$, it is also equivalent to $-2K_M\Gamma = E_M\Gamma$. If Γ is α-exceptional and ϕ-exceptional, then Γ is not a (-1)-curve by the minimality of α, hence it is a (-2)-curve. Conversely, if Γ is a ϕ-exceptional (-2)-curve, then $L_M\Gamma = 0$ by

$K_M + L_M \sim \phi^*(K_X + L)$. Therefore, it is enough to consider only the case where Γ is the proper transform in M of an irreducible curve γ of X. Then, by Lemma 2.7, we have

$$\Gamma^2 = \gamma^2 - \deg(\gamma \cap \Delta), \quad L_M\Gamma = L\gamma - \deg(\gamma \cap \Delta), \quad \text{and}$$
$$E_M\Gamma = E\gamma - \deg(\gamma \cap \Delta).$$

Suppose that $\gamma \subset E$. Then $m = \operatorname{mult}_\Gamma E_M = \operatorname{mult}_\gamma E \leq 4$ by Theorem 4.6. If $m = 4$, then γ is a fiber of $\pi\colon X \to \mathbb{P}^1$ in the case of type $[n; 2, 4]_2$ for $n \geq 3$, and $\Gamma^2 \geq -2$. If $m = 3$, then γ is also a fiber in the case of type $[n; 2, e]_2$ for $n \geq 2$, $e \geq 3$, and $\Gamma^2 \geq -2$. In particular, the property (v) holds. If $m = 2$ and Γ is α-exceptional, then one of the following cases occurs:

- $\gamma = \ell$ in the case of type $[2]_2$ and $\deg(\Delta \cap \ell) = 4$.
- $\gamma = \sigma$ in the case of type $[2; 1, 2]_{2+}$
- $\gamma = \ell$ in the case of type $[2; 1, 2]_{2+}$ with $\deg(\Delta \cap \ell) = 3$.
- $\gamma = \ell$ in the case of type $[n; 2, e]_2$ for $n \geq 1$, $e \geq 2$ with $\deg(\Delta \cap \ell) = 2$
- $\gamma = \sigma$ in the case of type $[n; 2, 4]_2$ for $n \geq 3$.

Thus the properties (iii) and (iv) hold. If $m = 1$ and Γ is α-exceptional, then $L_M\Gamma = 0$ induces

$$\begin{aligned} -2 &= (K_M + \Gamma)\Gamma = -(1/2)E_M\Gamma + \Gamma^2 \\ &= -(1/2)(E_M - \Gamma)\Gamma + (1/2)\Gamma^2 \leq (1/2)\Gamma^2. \end{aligned}$$

Thus the properties (i) and (ii) hold.

Then there remains only the case: $\gamma \not\subset E$. Assume that Γ is α-exceptional. Then $K_M\Gamma \geq 0$ and $E_M\Gamma \geq 0$ imply that Γ is a (-2)-curve and $L\gamma = E\gamma = \gamma^2 + 2 = \deg(\gamma \cap \Delta)$. In particular, $K_X + L$ is not ample, since $2(K_X + L) = L - E$. If (X, E) is of type $[2; 1, 2]$, then (X, E, Δ) is of type $[2; 1, 2]_0$ and $\gamma = \sigma$. If $K_X + L$ is not big, then γ is a fiber ℓ of $\pi\colon X \to \mathbb{P}^1$ with $\deg(\Delta \cap \ell) = 2$; such a fiber ℓ exists only in the case of type $[1; 2, 2]_0$ by Theorem 4.6.

Conversely, assume that γ is the curve σ in (3) or the curve ℓ in (4). Then

$$\begin{aligned}L_M\Gamma &= L\gamma - \deg(\gamma \cap \Delta) = (K_X + L)\gamma - K_X\gamma - \deg(\gamma \cap \Delta)\\ &= 2 + \gamma^2 - \deg(\gamma \cap \Delta) = 0.\end{aligned}$$

Hence, Γ is α-exceptional. Thus, we are done. $\square$

THEOREM 4.14. *For a rational del Pezzo pair (S, B) of index at most two, the dual graph of the exceptional divisors for the minimal desingularization of a non-Gorenstein singular point of S is one of the graphs listed in* TABLES 3 *and* 4.

The singularities having the graph K_l are discussed in Section 4.4 below.

PROOF. We may assume that (S, B) is constructed from a fundamental triplet (X, E, Δ) by Proposition 3.11 and Lemma 3.12. Let $\phi\colon (M, E_M) \to (X, E, \Delta)$ be the elimination and let $\alpha\colon M \to S$ be the minimal desingularization. Let $\Xi = \Xi_Q$ be the reduced divisor $\alpha^{-1}(Q)$ for a non-Gorenstein point $Q \in S$. Then $\Xi \leq E_M$ by the equality $K_M = \alpha^*(K_S + B) - (1/2)E_M$. Hence, Ξ is a connected component of the reduced divisor ${}_{\alpha}E_M$ consisting of the irreducible components of E_M exceptional for α. Conversely, a connected component of ${}_{\alpha}E_M$ is the exceptional divisor Ξ_Q for a non-Gorenstein point $Q \in S$.

Since Ξ defines a non-Gorenstein point, there is an irreducible component $E_1 \subset E$ such that the proper transform $E_{1,M}$ in M is contained in Ξ and $E_{1,M}^2 \leq -3$. By Theorem 4.6, we can divide the argument into the following seven cases of (X, E, Δ):

(1) $E = E_1$.

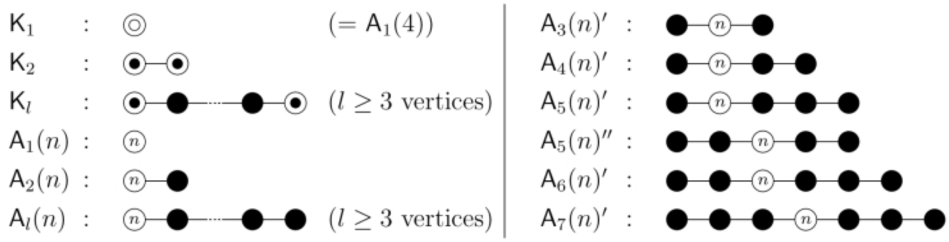
(2) $E = E_1 + E_2$ for another irreducible component E_2.

(3) The type $[2; 1, 2]_{++}$ with $\deg(\Delta \cap \ell_i) = 3$ for $i = 1$ or 2.

(4) The type $[3; 2, 4]_{++}(a, b)$.

(5) The type $[2]_2$ with $\deg(\Delta \cap \ell) = 4$.

TABLE 3. Exceptional graphs of types K, A, D and $\widetilde{\mathsf{D}}$ ($n \geq 3$)



(The bounds of l: K_l for $l \leq 9$; $\mathsf{A}_l(n)$ for $l \leq 5$ in case $n \geq 4$; $\mathsf{A}_l(3)$ for $l \leq 7$)

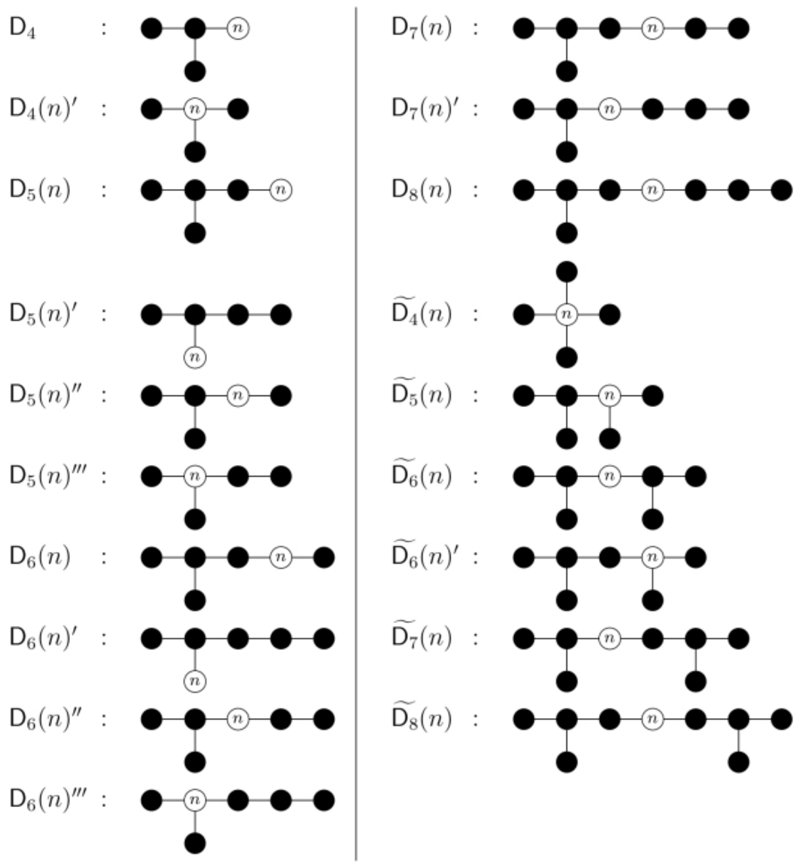
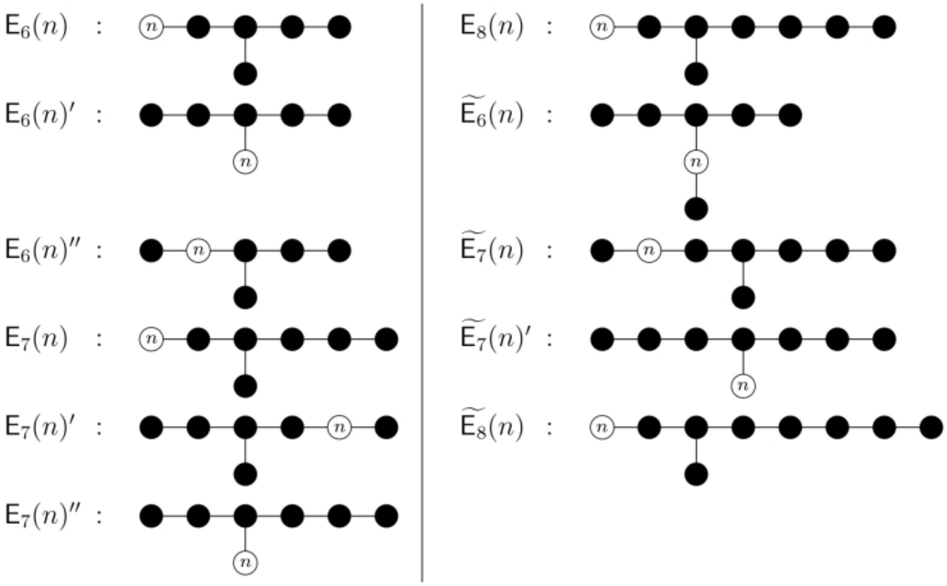


TABLE 4. Exceptional graphs of types E and $\widetilde{\mathsf{E}}$ ($n \geq 3$)



(6) The type $[2;1,2]_{2+}$ with $\deg(\Delta \cap \ell) = 3$.

(7) The type $[n;2,4]_2$ for $n \geq 3$.

Case (1). E_M is a (-4)-curve by Lemma 4.13. Hence the dual graph of $\Xi = E_M$ is K_1.

Case (2). Let $E_{2,M} \subset M$ be the proper transform of E_2.

Subcase (2-1) $\underline{E_1 \cap E_2 = \emptyset}$. Then $\Xi = E_{i,M}$ for $i = 1$ or 2 and the dual graph of Ξ is K_1 by Lemma 4.13.

In case $E_1 \cap E_2 \neq \emptyset$, let P denote the intersection point $E_1 \cap E_2$.

Subcase (2-2) $\underline{P \notin \Delta}$. Then ${}_\alpha E_M = E_{1,M} + E_{2,M}$ or $E_{1,M}$. Hence, the dual graph of $\Xi = {}_\alpha E_M$ is K_2 or $\mathsf{A}_1(3)$.

In case $P \in \Delta$, we may assume that $b = \mathrm{mult}_P(\Delta \cap E_1) \geq \mathrm{mult}_P(\Delta \cap E_2) = 1$. Here $b \leq 4$ and the maximum is attained when the type is $[2]_+(4)$ by Theorem 4.6.

Subcase (2-3) $\underline{\Delta \text{ is a Cartier divisor of } E \text{ at } P}$. If $E_{2,M}$ is also α-exceptional, then the dual graph of ${}_\alpha E_M$ is of type K_{b+2}, since ${}_\alpha E_M$ consists of $E_{1,M}$, $E_{2,M}$, and of the (-2)-curves contained in $\phi^{-1}(P)$. If $E_{2,M}$ is not α-exceptional, then the dual graph of ${}_\alpha E_M$ is $\mathsf{A}_{b+1}(3)$.

Subcase (2-4) $\underline{\Delta \text{ is not a Cartier divisor of } E \text{ at } P}$. Then $\mathrm{mult}_P(\Delta) = b$. Hence E_M has two connected components; one is $E_{1,M}$ and the other component consists of $E_{2,M}$ and of the (-2)-curves contained in $\phi^{-1}(P)$. Hence the dual graph of Ξ is $\mathsf{A}_1(3)$ or $\mathsf{A}_b(3)$.

Case (3). We may assume $E_1 = \ell_1$ and $\deg(\Delta \cap \ell_1) = 3$. If $\deg(\Delta \cap \ell_2) = 3$, then the dual graph of ${}_\alpha E_M$ is K_3. If $\deg(\Delta \cap \ell_2) < 3$, then the dual graph of ${}_\alpha E_M$ is $\mathsf{A}_2(3)$.

Case (4). We may assume $E_1 = \sigma$. We set $E_2 = \ell$, $E_3 = \sigma_\infty$, and $P = E_2 \cap E_3$. Let $E_{i,M}$ be the proper transform of E_i in M for $1 \le i \le 3$.

Subcase (4-1) $\underline{E_{2,M} \text{ and } E_{3,M} \text{ are } \alpha\text{-exceptional}}$.

Subcase (4-1-1) $\underline{\Delta \text{ is a Cartier divisor of } E}$. Then E_M is α-exceptional and connected. If $(a, b) = (0, 0)$, i.e., $P \notin \Delta$, then the dual graph of E_M is K_3. If $(a, b) \neq (0, 0)$, then the dual graph is K_{a+b+2}. Hence, we have K_l for $l \le 9$.

Subcase (4-1-2) $\underline{\Delta \text{ is not a Cartier divisor of } E}$. Then $(a, b) \neq (0, 0)$ and $\mathrm{mult}_P(\Delta) = a + b - 1$. Hence, E_M has two connected components; one contains $E_{1,M} + E_{2,M}$ and the other contains $E_{3,M}$. Thus the dual graph of Ξ is $\mathsf{A}_l(3)$ for $l \le 7$, where the maximum $l = 7$ is attained in the case $(a, b) = (1, 6)$.

Subcase (4-2) $\underline{E_{2,M} \text{ is } \alpha\text{-exceptional but } E_{3,M} \text{ is not}}$. Then $b \le \deg(\Delta \cap \sigma_\infty) < 6$.

Subcase (4-2-1) $\underline{\Delta \text{ is a Cartier divisor of } E}$. Then ${}_\alpha E_M$ is connected and the dual graph is $\mathsf{A}_2(3)$ if $(a, b) = (0, 0)$, and $\mathsf{A}_{1+a+b}(3)$ if $(a, b) \neq (0, 0)$. Thus we have $\mathsf{A}_l(3)$ for $l \le 7$, where the maximum $l = 7$ is attained in the case $(a, b) = (1, 5)$.

Subcase (4-2-2) $\underline{\Delta \text{ is not a Cartier divisor of } E}$. Then $(a, b) \neq (0, 0)$ and $\mathrm{mult}_P(\Delta) = a+b-1$. Thus ${}_\alpha E_M$ is connected and its dual graph is $\mathsf{A}_l(3)$ for $l \le 6$, where the maximum $l = 6$ is attained in the case $(a, b) = (1, 5)$.

Subcase (4-3) $\underline{E_{3,M} \text{ is } \alpha\text{-exceptional but } E_{2,M} \text{ is not}}$. Then $\Xi = E_{1,M}$ or Ξ contains $E_{3,M}$. Thus the dual graph of Ξ is $\mathsf{A}_l(3)$ for $1 \le l \le 7$, where the maximum $l = 7$ is attained in the case $(a, b) = (1, 6)$.

Subcase (4-4) $\underline{E_{2,M} \text{ and } E_{3,M} \text{ are not } \alpha\text{-exceptional}}$. Then $\Xi = E_{1,M}$ and the dual graph is $\mathsf{A}_1(3)$.

Case (5). Now $E_1 = \ell$. In fact, the proper transform of ℓ is an α-exceptional (-3)-curve contained in Ξ. The dual graph of Ξ is obtained by using Lemma 2.17 as follows.

Subcase (5-1) $\underline{\Delta \cap \ell = 4P \text{ for a point } P}$. Then $4 \leq k = \mathrm{mult}_P(\Delta) \leq 8$ and the dual graph of Ξ is as follows:

k	4	5	6	7	8
Graph	$\mathsf{A}_1(3)$	$\mathsf{A}_5(3)$	$\mathsf{D}_6(3)'$	$\mathsf{E}_7(3)''$	$\widetilde{\mathsf{E}_7}(3)'$

Subcase (5-2) $\underline{\Delta \cap \ell = 3P + P' \text{ for points } P \neq P'}$. Then $3 \leq k \leq 6$ and $1 \leq k' \leq 2$ for $k = \mathrm{mult}_P(\Delta)$ and $k' = \mathrm{mult}_P(\Delta')$. The dual graph of Ξ is as follows:

(k,k')	$(3,1)$	$(3,2)$	$(4,1)$	$(4,2)$	$(5,1)$	$(5,2)$	$(6,1)$	$(6,2)$
Graph	$\mathsf{A}_1(3)$	$\mathsf{A}_2(3)$	$\mathsf{A}_4(3)$	$\mathsf{A}_5(3)'$	$\mathsf{D}_5(3)'$	$\mathsf{E}_6(3)''$	$\mathsf{E}_6(3)'$	$\widetilde{\mathsf{E}_6}(3)$

Subcase (5-3) $\underline{\Delta \cap \ell = 2P + 2P' \text{ for points } P \neq P'}$. Then $2 \leq k,\ k' \leq 4$ for $k = \mathrm{mult}_P(\Delta)$ and $k' = \mathrm{mult}_P(\Delta')$. We may assume $k \geq k'$. Then the dual graph of Ξ is as follows:

(k,k')	$(2,2)$	$(3,2)$	$(3,3)$	$(4,2)$	$(4,3)$	$(4,4)$
Graph	$\mathsf{A}_1(3)$	$\mathsf{A}_3(3)$	$\mathsf{A}_5(3)''$	$\mathsf{D}_4(3)$	$\mathsf{D}_6(3)''$	$\widetilde{\mathsf{D}_6}(3)$

Subcase (5-4) $\underline{\Delta \cap \ell = 2P + P' + P'' \text{ for three points } P,\ P',\ P''}$. Then $2 \leq k \leq 4$ and $1 \leq k',\ k'' \leq 2$ for $k = \mathrm{mult}_P(\Delta)$, $k' = \mathrm{mult}_{P'}(\Delta)$, $k'' = \mathrm{mult}_{P''}(\Delta)$. We set $l = k' + k'' - 2$. Then the dual graph of Ξ is as follows:

(k,l)	$(2,0)$	$(2,1)$	$(2,2)$	$(3,0)$	$(3,1)$	$(3,2)$	$(4,0)$	$(4,1)$	$(4,2)$
Graph	$\mathsf{A}_1(3)$	$\mathsf{A}_2(3)$	$\mathsf{A}_3(3)'$	$\mathsf{A}_3(3)$	$\mathsf{A}_4(3)'$	$\mathsf{D}_5(3)'''$	$\mathsf{D}_4(3)$	$\mathsf{D}_5(3)''$	$\widetilde{\mathsf{D}_5}(3)$

Subcase (5-5) $\underline{\Delta \cap \ell \text{ consists of 4 points}}$. Then $1 \leq \mathrm{mult}_P(\Delta) \leq 2$ for $P \in \Delta \cap \ell$. Let l be the number of points $P \in \Delta \cap \ell$ with $\mathrm{mult}_P(\Delta) = 2$. Then the dual graph of Ξ is as follows:

l	0	1	2	3	4
Graph	$\mathsf{A}_1(3)$	$\mathsf{A}_2(3)$	$\mathsf{A}_3(3)'$	$\mathsf{D}_4(3)'$	$\widetilde{\mathsf{D}_4}(3)$

Case (6). Now $E_1 = \ell$. The proper transform of E is M is α-exceptional whose dual graph is $\mathsf{A}_2(3)$. It is contained in Ξ and the dual graph of Ξ is obtained by using Lemma 2.17 as follows.

Subcase (6-1) $\underline{\Delta \cap \ell = 3P}$. Then $3 \leq k = \mathrm{mult}_P(\Delta) \leq 6$ and the dual graph is as follows:

k	3	4	5	6
Graph	$\mathsf{A}_2(3)$	$\mathsf{A}_5(3)'$	$\mathsf{E}_6(3)''$	$\widetilde{\mathsf{E}_6}(3)$

Subcase (6-2) $\Delta \cap \ell = 2P + P'$ for two points P, $P' \in \ell$. Then $2 \leq k = \mathrm{mult}_P(\Delta) \leq 4$ and $1 \leq k' = \mathrm{mult}_{P'}(\Delta) \leq 2$. The dual graph is as follows:

(k, k')	$(2,1)$	$(2,2)$	$(3,1)$	$(3,2)$	$(4,1)$	$(4,2)$
Graph	$\mathsf{A}_2(3)$	$\mathsf{A}_3(3)'$	$\mathsf{A}_4(3)'$	$\mathsf{D}_5(3)'''$	$\mathsf{D}_5(3)''$	$\widetilde{\mathsf{D}_5}(3)$

Subcase (6-3) $\Delta \cap \ell$ consists of three points. Then $1 \leq \mathrm{mult}_P(\Delta) \leq 2$ for any $P \in \Delta \cap \ell$. Let l be the number of points P with $\mathrm{mult}_P(\Delta) = 2$. Then the dual graph is as follows:

l	0	1	2	3
Graph	$\mathsf{A}_2(3)$	$\mathsf{A}_3(3)'$	$\mathsf{D}_4(3)'$	$\widetilde{\mathsf{D}_4}(3)$

Case (7). We may assume $E_1 = \sigma$. The proper transform $E_{1,M} \subset M$ is a $(-n)$-curve.

Subcase (7-1) $F = 4\ell$ for a fiber ℓ of π. Then $\deg(\Delta \cap \ell) = 2$, and the proper transform in M of E is α-exceptional which is contained in Ξ.

Subcase (7-1-1) $\Delta \cap \ell = 2P$ for a point $P \in \ell$. Then $2 \leq k = \mathrm{mult}_P(\Delta) \leq 8$ and the dual graph of Ξ is as follows:

k	2	3	4	5	6	7	8
Graph	$\mathsf{A}_2(n)$	$\mathsf{A}_4(n)$	$\mathsf{D}_5(n)$	$\mathsf{E}_6(n)$	$\mathsf{E}_7(n)$	$\mathsf{E}_8(n)$	$\widetilde{\mathsf{E}_8}(n)$

Subcase (7-1-2) $\Delta \cap \ell = P + P'$ for two points P, $P' \in \ell$. Then $1 \leq k, k' \leq 4$ for $k = \mathrm{mult}_P(\Delta)$ and $k' = \mathrm{mult}_{P'}(\Delta)$. We may assume $k \geq k'$. The dual graph of Ξ is as follows:

(k, k')	$(1,1)$	$(2,1)$	$(2,2)$	$(3,1)$	$(3,2)$	$(3,3)$	$(4,1)$	$(4,2)$	$(4,3)$	$(4,4)$
Graph	$\mathsf{A}_2(n)$	$\mathsf{A}_3(n)$	$\mathsf{D}_4(n)$	$\mathsf{A}_4(n)$	$\mathsf{D}_5(n)'$	$\mathsf{E}_6(n)'$	$\mathsf{A}_5(n)$	$\mathsf{D}_6(n)'$	$\mathsf{E}_7(n)''$	$\widetilde{\mathsf{E}_7}(n)'$

Subcase (7-2) $F = 3\ell_1 + \ell_2$ for two fibers ℓ_1, ℓ_2 of π.

Subcase (7-2-1) $\Delta \cap \ell_1 = 2P$ for a point $P \in \ell_1$ and $\deg(\Delta \cap \ell_2) = 2$. Then $2 \leq k = \mathrm{mult}_P(\Delta) \leq 6$ and the dual graph of Ξ is as follows:

k	2	3	4	5	6
Graph	$\mathsf{A}_3(n)'$	$\mathsf{A}_5(n)'$	$\mathsf{D}_6(n)$	$\mathsf{E}_7(n)'$	$\widetilde{\mathsf{E}_7}(n)$

Subcase (7-2-2) $\Delta \cap \ell = 2P$ for a point $P \in \ell_1$ and $\deg(\Delta \cap \ell_2) < 2$. Then $2 \le k = \mathrm{mult}_P(\Delta) \le 6$ and the dual graph of Ξ is as follows:

k	2	3	4	5	6
Graph	$\mathsf{A}_2(n)$	$\mathsf{A}_4(n)$	$\mathsf{D}_5(n)$	$\mathsf{E}_6(n)$	$\mathsf{E}_7(n)$

Subcase (7-2-3) $\Delta \cap \ell_1 = P + P'$ for two points P, $P' \in \ell_1$ and $\deg(\Delta \cap \ell_2) = 2$. We may assume $3 \ge k \ge k' \ge 1$ for $k = \mathrm{mult}_P(\Delta)$ and $k' = \mathrm{mult}_{P'}(\Delta)$. Then the dual graph of Ξ is as follows:

(k, k')	$(1,1)$	$(2,1)$	$(2,2)$	$(3,1)$	$(3,2)$	$(3,3)$
Graph	$\mathsf{A}_3(n)'$	$\mathsf{A}_4(n)'$	$\mathsf{D}_5(n)''$	$\mathsf{A}_5(n)'$	$\mathsf{E}_6(n)''$	$\widetilde{\mathsf{E}_6}(n)$

Subcase (7-2-4) $\Delta \cap \ell_1 = P + P'$ for two points P, $P' \in \ell_1$ and $\deg(\Delta \cap \ell_2) < 2$. We may assume $3 \ge k \ge k' \ge 1$ for $k = \mathrm{mult}_P(\Delta)$ and $k' = \mathrm{mult}_{P'}(\Delta)$. Then the dual graph of Ξ is as follows:

(k, k')	$(1,1)$	$(2,1)$	$(2,2)$	$(3,1)$	$(3,2)$	$(3,3)$
Graph	$\mathsf{A}_2(n)$	$\mathsf{A}_3(n)$	$\mathsf{D}_4(n)$	$\mathsf{A}_4(n)$	$\mathsf{D}_5(n)'$	$\mathsf{E}_6(n)'$

Subcase (7-2-5) $\deg(\Delta \cap \ell_1) < 2$. If $\deg(\Delta \cap \ell_2) = 2$, then the dual graph of Ξ is $\mathsf{A}_2(n)$. If $\deg(\Delta \cap \ell_2) < 2$, then it is $\mathsf{A}_1(n)$.

Subcase (7-3) $F = 2\ell_1 + 2\ell_2$ for two fibers ℓ_1, ℓ_2 of π.

Subcase (7-3-1) $\Delta \cap \ell_1 = 2P_1$ and $\Delta \cap \ell_2 = 2P_2$ for points $P_1 \in \ell_1$, $P_2 \in \ell_2$. Then $2 \le k_i = \mathrm{mult}_{P_i}(\Delta) \le 4$ for $i = 1, 2$. We may assume $k_1 \ge k_2$. Then the dual graph is as follows:

(k_1, k_2)	$(2,2)$	$(3,2)$	$(3,3)$	$(4,2)$	$(4,3)$	$(4,4)$
Graph	$\mathsf{A}_3(n)'$	$\mathsf{A}_5(n)'$	$\mathsf{A}_7(n)'$	$\mathsf{D}_6(n)$	$\mathsf{D}_8(n)$	$\widetilde{\mathsf{D}_8}(n)$

Subcase (7-3-2) $\Delta \cap \ell_1 = 2P_1$ and $\Delta \cap \ell_2 = P_2 + P_2'$ for a point $P_1 \in \ell_1$ and for two points P_2, $P_2' \in \ell_2$. Then $2 \le k_1 = \mathrm{mult}_{P_1}(\Delta) \le 4$ and $1 \le k_2, k_2' \le 2$ for $k_2 = \mathrm{mult}_{P_2}(\Delta)$ and $k_2' = \mathrm{mult}_{P_2'}(\Delta)$. Let $l = k_2 + k_2' - 2$. Then the dual graph is as follows:

(k_1, l)	$(2,0)$	$(2,1)$	$(2,2)$	$(3,0)$	$(3,1)$	$(3,2)$	$(4,0)$	$(4,1)$	$(4,2)$
Graph	$\mathsf{A}_3(n)'$	$\mathsf{A}_4(n)'$	$\mathsf{D}_5(n)''$	$\mathsf{A}_5(n)'$	$\mathsf{A}_6(n)'$	$\mathsf{D}_7(n)'$	$\mathsf{D}_6(n)$	$\mathsf{D}_7(n)$	$\widetilde{\mathsf{D}_7}(n)$

Subcase (7-3-3) $\Delta \cap \ell_1$ consists of two points and $\Delta \cap \ell_2$ consists of two points. For $i = 1, 2$, let l_i be the number of points $P \in \Delta \cap \ell_i$ with $\mathrm{mult}_P(\Delta) = 2$. We may assume $l_1 \geq l_2$. Then the dual graph is as follows:

(l_1, l_2)	$(0,0)$	$(1,0)$	$(1,1)$	$(2,0)$	$(2,1)$	$(2,2)$
Graph	$\mathsf{A}_3(n)'$	$\mathsf{A}_4(n)'$	$\mathsf{A}_5(n)''$	$\mathsf{D}_5(n)''$	$\mathsf{D}_6(n)''$	$\widetilde{\mathsf{D}_6}(n)$

Subcase (7-3-4) $\Delta \cap \ell_1 = 2P$ for a point $P \in \ell_1$ and $\deg(\Delta \cap \ell_2) < 2$. Then $2 \leq k = \mathrm{mult}_P(\Delta) \leq 4$ and the dual graph is as follows:

k	2	3	4
Graph	$\mathsf{A}_2(n)$	$\mathsf{A}_4(n)$	$\mathsf{D}_5(n)$

Subcase (7-3-5) $\Delta \cap \ell_1$ consists of two points and $\deg(\Delta \cap \ell_2) < 2$. For the number l of points $P \in \Delta \cap \ell_1$ with $\mathrm{mult}_P(\Delta) = 2$, the dual graph is as follows:

l	0	1	2
Graph	$\mathsf{A}_2(n)$	$\mathsf{A}_3(n)$	$\mathsf{D}_4(n)$

Subcase (7-3-6) $\deg(\Delta \cap \ell_1) < 2$ and $\deg(\Delta \cap \ell_2) < 2$. Then $\Xi = E_{1,M}$ and the dual graph is $\mathsf{A}_1(n)$.

Subcase (7-4) $F = 2\ell_1 + \ell_2 + \ell_3$ for three fibers ℓ_1, ℓ_2, ℓ_3 of π.

Subcase (7-4-1) $\Delta \cap \ell_1 = 2P$ for a point $P \in \ell$. Then $2 \leq k = \mathrm{mult}_P(\Delta) \leq 4$. Let l be the number of fibers ℓ_i for $i = 2, 3$ with $\deg(\Delta \cap \ell_i) = 2$. Then the dual graph is as follows:

(k, l)	$(2,0)$	$(3,0)$	$(4,0)$	$(2,1)$	$(3,1)$	$(4,1)$	$(2,2)$	$(3,2)$	$(4,2)$
Graph	$\mathsf{A}_2(n)$	$\mathsf{A}_4(n)$	$\mathsf{D}_5(n)$	$\mathsf{A}_3(n)'$	$\mathsf{A}_5(n)'$	$\mathsf{D}_6(n)$	$\mathsf{D}_4(n)'$	$\mathsf{D}_6(n)'''$	$\widetilde{\mathsf{D}_6}(n)'$

Subcase (7-4-2) $\Delta \cap \ell_1$ consists of two points. Let l be the number of points $P \in \ell_1$ with $\mathrm{mult}_P(\Delta) = 2$ and l' be the number of fibers ℓ_i for $i = 2$, 3 with $\deg(\Delta \cap \ell_i) = 2$. Then $0 \leq l, l' \leq 2$ and the dual graph is as follows:

(l, l')	$(0,0)$	$(1,0)$	$(2,0)$	$(0,1)$	$(1,1)$	$(2,1)$	$(0,2)$	$(1,2)$	$(2,2)$
Graph	$\mathsf{A}_2(n)$	$\mathsf{A}_3(n)$	$\mathsf{D}_4(n)$	$\mathsf{A}_3(n)'$	$\mathsf{A}_4(n)'$	$\mathsf{D}_5(n)''$	$\mathsf{D}_4(n)'$	$\mathsf{D}_5(n)'''$	$\widetilde{\mathsf{D}_5}(n)$

Subcase (7-4-3) $\underline{\deg(\Delta \cap \ell_1) < 2}$. Let l be the number of fibers ℓ_i for $i = 2, 3$ with $\deg(\Delta \cap \ell_i) = 2$. Then the dual graph is as follows:

l	0	1	2
Graph	$\mathsf{A}_1(n)$	$\mathsf{A}_2(n)$	$\mathsf{A}_3(n)'$

Subcase (7-5) $\underline{F = \ell_1 + \ell_2 + \ell_3 + \ell_4 \text{ for 4 fibers } \ell_i\ (1 \le i \le 4) \text{ of } \pi}$. Let l be the number of fibers ℓ_i with $\deg(\Delta \cap \ell_i) = 2$. Then the dual graph is as follows:

l	0	1	2	3	4
Graph	$\mathsf{A}_1(n)$	$\mathsf{A}_2(n)$	$\mathsf{A}_3(n)'$	$\mathsf{D}_4(n)'$	$\widetilde{\mathsf{D}_4}(n)$

Thus we are done. □

4.4. Remarks on two-dimensional log-terminal singularity of index two

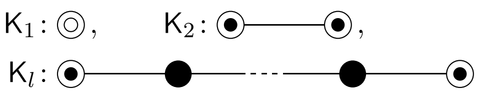
We note on two-dimensional log-terminal singularities in arbitrary characteristics. Let S be a germ of normal surface at a point Q and let $\alpha\colon M \to S$ be the minimal desingularization. Suppose that $2K_S$ is numerically Cartier and let E_M be the effective divisor supported in $\alpha^{-1}(Q)$ determined by $2K_M \sim \alpha^*(2K_S) - E_M$.

Lemma 4.15. *Under the situation, the following conditions are mutually equivalent*:

(1) $(S, 0)$ *is log-terminal of index two*;

(2) E_M *is a non-zero reduced divisor*;

(3) E_M *is a straight chain of non-singular rational curves whose dual graph is* K_n *defined below* (cf. Notation (1.)):



(*consisting of* $l \ge 3$ *vertices*).

If the conditions above are satisfied, then S has only rational singularities.

The same symbol K_n is used in Table 3.

PROOF. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3): Any irreducible component $E_{i,M}$ of E_M is isomorphic to $\mathbb{P}^1$ by

$$\begin{aligned}(K_M + E_{i,M})E_{i,M} &= -(1/2)E_M E_{i,M} + E_{i,M}^2 \\ &= -(1/2)(E_M - E_{i,M})E_{i,M} + (1/2)E_{i,M}^2 < 0.\end{aligned}$$

Moreover, we have

$$E_{i,M}^2 = -4 + (E_M - E_{i,M})E_{i,M} \geq -4. \tag{4–1}$$

If E_M is irreducible, then E_M is a (-4)-curve, thus the dual graph is K_1. Hence we may assume that E_M is reducible.

If there are two irreducible components $E_{1,M}$, $E_{2,M}$ with $E_{1,M}E_{2,M} \geq 2$, then $E_{1,M}^2 = E_{2,M}^2 = -2$, $E_{1,M}E_{2,M} = 2$ by (4–1); this induces $(E_{1,M} + E_{2,M})^2 = 0$ contradicting that the intersection matrix $(E_{i,M}E_{j,M})$ is negative definite. Thus $E_{i,M}E_{j,M} \leq 1$ for any i, j.

Suppose that there are three irreducible components $E_{1,M}$, $E_{2,M}$, $E_{3,M}$ which contain the same point P. Then $E_{1,M} \cap E_{2,M} = E_{2,M} \cap E_{3,M} = E_{3,M} \cap E_{1,M} = \{P\}$ and $E_{i,M}^2 = -2$ for $1 \leq i \leq 3$ by (4–1). Thus we have a contradiction by $(E_{1,M} + E_{2,M} + E_{3,M})^2 = 0$. Therefore, E_M is a simple normal crossing divisor consisting of non-singular rational curves $E_{i,M}$ such that $E_{i,M}E_{j,M} \leq 1$ for any i, j.

Suppose that $E_{i,M}^2 = -2$ for any i. Then $(E_M - E_{i,M})E_{i,M} = 2$ and the dual graph of E_M is a circle. Thus we have a contradiction by $E_M^2 = 0$.

Hence, there is an irreducible component $E_{1,M}$ with $E_{1,M}^2 = -3$. Let $E_{2,M}$ be the unique irreducible component with $E_{1,M}E_{2,M} = 1$. If $E_{2,M}^2 = -3$, then $E_M = E_{1,M} + E_{2,M}$ and the dual graph is K_2. If $E_{2,M}^2 = -2$, then there is a unique irreducible component $E_{3,M}$ with $E_{1,M}E_{3,M} = 0$ and $E_{2,M}E_{3,M} = 1$. In this way, we can show that the dual graph of E_M is K_n.

(3) $\Rightarrow$ (1): The fundamental cycle of S is E_M since $E_M E_{i,M} = 0$ if $E_{i,M}^2 = -2$, and $E_M E_{i,M} = -2$ if $E_{i,M}^2 = -3$. Since $(K_M + E_M)E_M = (1/2)E_M^2 = -2$, S has only rational singularities. Furthermore, $(2K_M + E_M)E_{i,M} = 0$ for any i. Thus $2K_M + E_M \sim \alpha^* L$ for a Cartier divisor by Theorem 3.1. Hence $2K_S \sim L$ is Cartier and $(S, 0)$ is log-terminal of index two. $\square$

Definition 4.16. If the conditions in Lemma 4.15 are satisfied and if the number of irreducible components of E_M is n, then the singularity of S is called of type K_n.

Example 4.17. Let N be a free abelian group of rank two with a basis (e_1, e_2) and let M be the dual $\mathrm{Hom}(\mathsf{N}, \mathbb{Z})$. For a positive integer n, we set

$$\mathsf{N}' = \mathsf{N} + \mathbb{Z}\frac{1}{4n}(e_1 + (2n-1)e_2) \subset \mathsf{N} \otimes \mathbb{Q} \quad \text{and} \quad \mathsf{M}' = \mathrm{Hom}(\mathsf{N}', \mathbb{Z}).$$

For the first quadrant $\boldsymbol{\sigma} = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}e_2$, let $X = X(\mathsf{N}', \boldsymbol{\sigma})$ be the affine toric variety $\operatorname{Spec} \Bbbk[\boldsymbol{\sigma}^\vee \cap \mathsf{M}']$ associated with $(\mathsf{N}', \boldsymbol{\sigma})$. Let $\mathtt{x}$, $\mathtt{y}$ be the generators of the polynomial ring $\Bbbk[\boldsymbol{\sigma}^\vee \cap \mathsf{M}]$ in which $(\mathtt{x}, \mathtt{y})$ corresponds to the basis of M dual to (e_1, e_2). Then the toric variety $X(\mathsf{N}, \boldsymbol{\sigma})$ is isomorphic to $\mathbb{A}^2$ and the natural morphism $\mathbb{A}^2 \simeq X(\mathsf{N}, \boldsymbol{\sigma}) \to X$ is regarded as the quotient map for the following action of the algebraic subgroup $\boldsymbol{\mu}_{4n} = \operatorname{Spec} \Bbbk[\zeta]/(\zeta^{4n} - 1)$ of $\mathbb{G}_{\mathtt{m}} = \operatorname{Spec} \Bbbk[\zeta, \zeta^{-1}]$ on $\mathbb{A}^2$:

$$(\mathtt{x}, \mathtt{y}) \mapsto (\zeta\mathtt{x}, \zeta^{2n-1}\mathtt{y}).$$

In fact, $\Bbbk[\boldsymbol{\sigma}^\vee \cap \mathsf{M}']$ is isomorphic to the invariant ring $\Bbbk[\mathtt{x}, \mathtt{y}]^{\boldsymbol{\mu}_{4n}}$, which is generated by five monomials

$$\mathtt{x}^{4n}, \quad \mathtt{y}^{4n}, \quad \mathtt{x}^2\mathtt{y}^2, \quad \mathtt{x}^{2n+1}\mathtt{y}, \quad \mathtt{x}\mathtt{y}^{2n+1},$$

over $\Bbbk$. Note that $\zeta^{2n} \neq -1$ if $\operatorname{char} \Bbbk = 2$. We write $X = X((1, 2n-1)/(4n))$ and $\Bbbk[\mathtt{x}, \mathtt{y}]^{\boldsymbol{\mu}_{4n}} = R((1, 2n-1)/(4n))$. Actually, X is a cyclic quotient singularity of type $(1, 2n-1)/(4n)$ if $4n$ and $\operatorname{char} \Bbbk$ are coprime. We define $v_0 = e_2$, $v_{n+1} = e_1$, and

$$v_j = \frac{2j-1}{4n}e_1 + \left(\frac{1}{2} - \frac{2j-1}{4n}\right)e_2 \in \mathsf{N}'$$

for $1 \leq j \leq n$. Furthermore, we set $\boldsymbol{\sigma}_j = \mathbb{R}_{\geq 0}v_{j-1} + \mathbb{R}_{\geq 0}v_j$ for $1 \leq j \leq n+1$. Since $\mathbb{Z}v_{j-1} + \mathbb{Z}v_j = \mathsf{N}'$ for any $1 \leq j \leq n+1$, $X(\mathsf{N}', \boldsymbol{\sigma}_j)$ is non-singular and the toric variety $\widetilde{X} = X(\mathsf{N}', \{\boldsymbol{\sigma}_j\}) = \bigcup X(\mathsf{N}', \boldsymbol{\sigma}_j)$ is a desingularization of X. Let Γ_j be the prime divisor of $\widetilde{X}$ corresponding to the ray $\mathbb{R}_{\geq 0}v_j$. Then $\Gamma_j \simeq \mathbb{P}^1$ and $\sum \Gamma_j$ is a simple normal crossing divisor whose dual graph is K_n. Thus $\widetilde{X} \to X$ is the minimal desingularization and the singularity of X at the origin is K_n.

PROPOSITION 4.18. *For a singularity S of type K_n and for the minimal desingularization $\alpha\colon M \to S$, suppose that* $\mathrm{Pic}(M) \to \mathrm{Pic}(E_M) \simeq \mathbb{Z}^{\oplus n}$ *is surjective. Then there is an étale morphism from S into $X((1, 2n-1)/(4n))$ in* Example 4.17. *In particular, the Henselization of a singularity of type K_n is unique.*

PROOF. We may assume that $S = \operatorname{Spec} R$ for a two-dimensional local ring R essentially of finite type over $\Bbbk$.

First, we treat the case: $n = 1$. Then $\mathcal{O}_M(-E_M) \simeq \mathcal{L}^{\otimes 4}$ for an invertible sheaf $\mathcal{L}$, by assumption. Then $|\mathcal{L}|$ is base point free by Theorem 3.1. Hence, we can choose two sections s_1 and s_2 of $\mathcal{L}$ such that $\mathrm{div}(s_1) \cap \mathrm{div}(s_2) \cap E_M = \emptyset$. Let y be a defining equation of E_M, i.e., y is a section of $\mathcal{O}_M(E_M)$ with $\mathrm{div}(y) = E_M$. Then we have the following five regular functions

$$\xi_1 = s_1^4 y, \quad \xi_2 = s_2^4 y, \quad \theta = s_1^2 s_2^2 y, \quad \eta_1 = s_1^3 s_2 y, \quad \eta_2 = s_1 s_2^3 y$$

over S. Since these five functions satisfy the same relation as the five generators of $R((1,1)/4)$, there is a ring homomorphism $R((1,1)/4) \to R$, and equivalently a morphism $S \to X((1,1)/4)$. Since E_M is the fundamental cycle, the maximal ideal $\mathfrak{m}$ of R is regarded as $\alpha_*\mathcal{O}_M(-E_M)$ and $\mathfrak{m}/\mathfrak{m}^2$ is identified with $\mathrm{H}^0(E_M, \mathcal{O}_{E_M}(-E_M))$ (cf. [6, Theorem 4]). Therefore, the five regular functions above form a basis of $\mathfrak{m}/\mathfrak{m}^2$, which implies that $R((1,1)/4) \to R$ is étale.

Next, we treat the case $n > 1$. By assumption, there exist invertible sheaves $\mathcal{L}_0$ and $\mathcal{L}_{n+1}$ on M with $\deg \mathcal{L}_0|_{E_{j,M}} = \delta_{1,j}$ and $\deg \mathcal{L}_{n+1}|_{E_{j,M}} = \delta_{n,j}$ for $1 \le j \le n$. For $i = 0$, $n+1$, $|\mathcal{L}_i|$ has no base points by Theorem 3.1. Thus there exist a section s_0 of $\mathcal{L}_0$ and a section s_{n+1} of $\mathcal{L}_{n+1}$ such that $\mathrm{div}(s_0)$ intersects $E_{1,M}$ transversely, $\mathrm{div}(s_{n+1})$ intersects $E_{n,M}$ transversely, $\mathrm{div}(s_0) \cap E_{j,M} = \emptyset$ for $j \ne 1$, and $\mathrm{div}(s_{n+1}) \cap E_{j,M} = \emptyset$ for $j \ne n$. Note that

$$\mathcal{L}_0 + \sum\nolimits_{j=1}^{n} \left(\frac{1}{2} - \frac{2j-1}{4n}\right) E_{j,M} \quad \text{and} \quad \mathcal{L}_{n+1} + \sum\nolimits_{j=1}^{n} \frac{2j-1}{4n} E_{j,M}$$

are numerically trivial. Let y_j be a defining equation of $E_{j,M}$. Then we have five regular functions

$$\xi_1 = s_0^{4n} \prod\nolimits_{j=1}^{n} y_j^{2n-2j+1}, \quad \xi_2 = s_{n+1}^{4n} \prod\nolimits_{j=1}^{n} y_j^{2j-1}, \quad \theta = s_0^2 s_{n+1}^2 \prod\nolimits_{j=1}^{n} y_j,$$
$$\eta_1 = s_0^{2n+1} s_{n+1} \prod\nolimits_{j=1}^{n} y_j^{n-j+1}, \quad \eta_2 = s_0 s_{n+1}^{2n+1} \prod\nolimits_{j=1}^{n} y_j^{j}$$

over S. Hence, by the same argument as in the case of $n = 1$, there is an étale morphism $S \to X((1, 2n-1)/(4n))$. The remaining assertion on Henselization follows from [21, Lemma 14.3]. $\square$

PROPOSITION 4.19. *There exists a $\mathbb{Q}$-Gorenstein smoothing (of index two) of the singularity K_n at the origin of $X((1, 2n-1)/(4n))$.*

PROOF. In Example 4.17, we can consider another subgroup

$$\mathsf{N}'' = \mathsf{N} + \mathbb{Z}\frac{1}{2n}(1, 2n-1) \subset \mathsf{N}'$$

and the associated toric variety $Y = X(\mathsf{N}'', \boldsymbol{\sigma})$. Then $\Bbbk[\boldsymbol{\sigma}^\vee \cap \mathsf{M}'']$ for the dual $\mathsf{M}'' = \mathrm{Hom}(\mathsf{N}'', \mathbb{Z})$ is the invariant subring of $\Bbbk[\boldsymbol{\sigma}^\vee \cap \mathsf{M}] = \Bbbk[\mathtt{x}, \mathtt{y}]$ by the action $(\mathtt{x}, \mathtt{y}) \mapsto (\zeta^2\mathtt{x}, \zeta^{-2}\mathtt{y})$ of ζ^2, which is generated by three monomials $\mathtt{xy}$, $\mathtt{x}^{2n}$, $\mathtt{y}^{2n}$. Thus the invariant subring may be written as $R((1, (2n-1))/(2n))$ and is isomorphic to

$$\Bbbk[\mathtt{z}, \mathtt{u}, \mathtt{v}]/(\mathtt{z}^{2n} - \mathtt{uv}),$$

by $\mathtt{z} \mapsto \mathtt{xy}$, $\mathtt{u} \mapsto \mathtt{x}^{2n}$, and $\mathtt{v} \mapsto \mathtt{y}^{2n}$. In particular, Y has a singularity of type A_{2n} at the origin. The action of ζ on $\Bbbk[\mathtt{x}, \mathtt{y}]$ induces an action on $R((1, (2n-1))/(2n))$, which is expressed as

$$(\mathtt{z}, \mathtt{u}, \mathtt{v}) \mapsto (\zeta^{2n}\mathtt{z}, \zeta^{2n}\mathtt{u}, \zeta^{2n}\mathtt{v}).$$

Thus the quotient group $\boldsymbol{\mu}_2 = \mathrm{Spec}\,\Bbbk[\xi]/(\xi^2 - 1)$ of $\boldsymbol{\mu}_{4n}$ acts on the polynomial ring $\Bbbk[\mathtt{z}, \mathtt{u}, \mathtt{v}]$ by the same way, where ζ^{2n} is replaced with ξ. Note that $X = X((1, 2n-1)/(4n))$ is the quotient of Y by the action of $\boldsymbol{\mu}_2$. The invariant ring $A = \Bbbk[\mathtt{z}, \mathtt{u}, \mathtt{v}]^{\boldsymbol{\mu}_2}$ has a singularity only at the origin and it is a toric terminal singularity of index two. We define a $\Bbbk$-algebra homomorphism $\Bbbk[\mathtt{t}] \to A$ by $\mathtt{t} \mapsto \mathtt{z}^{2n} - \mathtt{uv}$. For a constant $c \in \Bbbk$, let $\Bbbk[\mathtt{t}] \to \Bbbk$ be the $\Bbbk$-algebra homomorphism given by $\mathtt{t} \mapsto c$ and let A_c be the tensor product $A \otimes_{\Bbbk[\mathtt{t}]} \Bbbk$. Then $A_0 \simeq R((1, 2n-1)/(4n))$. It is enough to show that $\mathrm{Spec}\, A_c$ is nonsingular for any $c \neq 0$. Note that $\mathrm{Spec}\, A_c$ is covered by three open subsets $\{\mathtt{z}^2 \neq 0\}$, $\{\mathtt{u}^2 \neq 0\}$, and $\{\mathtt{v}^2 \neq 0\}$, since $c \neq 0$.

The localization $A_c[\mathtt{z}^{-2}]$ contains $\mathtt{u}/\mathtt{z}$ and $\mathtt{v}/\mathtt{z}$. Thus it is isomorphic to

$$\Bbbk[z, z^{-1}, u, v]/(z^n - uvz - c)$$

by $z \mapsto \mathtt{z}^2$, $u \mapsto \mathtt{u}/\mathtt{z}$, $v \mapsto \mathtt{v}/\mathtt{z}$. If the ring is not regular, then, by the Jacobian criterion, $u = v = nz^{n-1} = z^n - c = 0$ has a solution, but it is impossible. Hence, $A_c[\mathtt{z}^{-2}]$ is regular.

The localization $A_c[\mathtt{u}^{-2}]$ contains $\mathtt{z}/\mathtt{u}$ and $\mathtt{v}/\mathtt{u}$. Thus it is isomorphic to

$$\Bbbk[u, u^{-1}, z, v]/(z^{2n}u^n - uv - c)$$

by $u \mapsto \mathtt{u}^2$, $z \mapsto \mathtt{z}/\mathtt{u}$, $v \mapsto \mathtt{v}/\mathtt{u}$. By the Jacobian criterion, the ring is regular since

$$\frac{\partial}{\partial v}(z^{2n}u^n - uv - c) = -u \neq 0.$$

Similarly, the localization $A_c[\mathtt{v}^{-2}]$ is also regular. Thus we are done. $\square$

5. Deformations

We shall study deformation of fundamental triplets, of basic pairs, and of del Pezzo pairs of index at most two. The notion of equi-singular deformation is introduced.

5.1. Deformation of several objects

Definition 5.1.

(1) Let $\tau\colon \widetilde{X} \to T$ be a proper surjective smooth morphism into a connected curve T, $\widetilde{E} \subset \widetilde{X}$ an effective divisor flat over T, and let $\widetilde{\Delta} \subset \widetilde{X}$ be a subscheme finite and flat over T. If (X_t, E_t, Δ_t) is a fundamental triplet for the fibers $X_t = \tau^{-1}(t)$, $E_t = \widetilde{E} \cap X_t$, and $\Delta_t = \widetilde{\Delta} \cap X_t$ over any closed point $t \in T$, then $\tau\colon (\widetilde{X}, \widetilde{E}, \widetilde{\Delta}) \to T$ is called a *family of fundamental triplets*. If two fundamental triplets appear as fibers of a family of fundamental triplets over a connected curve, then the fundamental triplets are called deformation equivalent to each other.

(2) Let $h\colon \widetilde{M} \to T$ be a proper surjective smooth morphism into a connected curve T and let $\widetilde{E} \subset \widetilde{M}$ be an effective divisor flat over T. If (M_t, E_t) is a basic pair for the fibers $M_t = h^{-1}(t)$ and $E_t = \widetilde{E} \cap M_t$ over any closed point $t \in T$, then $h\colon (\widetilde{M}, \widetilde{E}) \to T$ is called a *family of basic pairs*. If two basic pairs appear as fibers of a family of basic pairs over a connected curve, then the basic pairs are called deformation equivalent to each other.

(3) Let $f\colon \widetilde{S} \to T$ be a proper surjective flat morphism from a normal variety $\widetilde{S}$ into a connected non-singular curve T and let $\widetilde{B}$ be an effective $\mathbb{Q}$-divisor of $\widetilde{S}$ such that $K_{\widetilde{S}} + \widetilde{B}$ is $\mathbb{Q}$-Cartier and $\operatorname{Supp} \widetilde{B}$ is flat over T. If, for any closed point $t \in T$, (S_t, B_t) is a del Pezzo pair for the fiber $S_t = f^{-1}(t)$ and for the $\mathbb{Q}$-divisor B_t defined by

$$(K_{\widetilde{S}} + \widetilde{B})_{|S_t} = K_{S_t} + B_t,$$

then $f\colon (\widetilde{S}, \widetilde{B}) \to T$ is called a *family of del Pezzo pairs*.

- The *index* of the family $(\widetilde{S}, \widetilde{B}) \to T$ is defined to be the $\mathbb{Q}$-Cartier index of $K_{\widetilde{S}} + \widetilde{B}$.
- If the index of $K_{S_t} + B_t$ for any closed point $t \in T$ is equal to the index k of $K_{\widetilde{S}} + \widetilde{B}$, then $(\widetilde{S}, \widetilde{B}) \to T$ is called to have the *constant index* k.

Two del Pezzo pairs (S_1, B_1) and (S_2, B_2) are called deformation equivalent to each other if there exist finitely many families $(\widetilde{S}_{(j)}, \widetilde{B}_{(j)}) \to T_{(j)}$ of del Pezzo pairs over connected non-singular curves $T_{(j)}$ $(1 \le j \le l)$ and points $t^a_{(j)}$, $t^b_{(j)} \in T_{(j)}$ such that

$$(S_1, B_1) \simeq (S_{(1),t^a_1}, B_{(1),t^a_1}), \quad (S_2, B_2) \simeq (S_{(l),t^b_l}, B_{(l),t^b_l}), \quad \text{and}$$
$$(S_{(j),t^b_j}, B_{(j),t^b_j}) \simeq (S_{(j+1),t^a_{j+1}}, B_{(j+1),t^a_{j+1}})$$

for $1 \le j \le l-1$. If any $(\widetilde{S}_{(j)}, \widetilde{B}_{(j)}) \to T_{(j)}$ has the same index (resp. constant index) equal to k, then (S_1, B_1) and (S_2, B_2) are called to be connected by deformations of index (resp. constant index) k.

Remark. The genus g is a deformation invariant for fundamental triplets (X, E, Δ), basic pairs (M, E_M), and for del Pezzo pairs (S, B) of index two, where

$$2g - 2 = (K_X + L)L = (K_M + L_M)L_M = 2(K_S + 2B)(K_S + B)$$

for $L = -2K_X - E$ and $L_M = -2K_M - E_M$. Moreover, LE and L^2 are deformation invariants for fundamental triplets (X, E, Δ); and $L_M E_M$ and L_M^2 are deformation invariants for basic pairs (M, E_M).

LEMMA 5.2.

(1) *If two fundamental triplets are deformation equivalent to each other, then their eliminations are also deformation equivalent to each other as basic pairs.*

(2) *For a family $h\colon (\widetilde{M}, \widetilde{E}) \to T$ of basic pairs over a smooth connected curve T, there exist a family $f\colon (\widetilde{S}, \widetilde{B}) \to T$ of del Pezzo pairs of index at most two and a birational morphism $\widetilde{\alpha}\colon \widetilde{M} \to \widetilde{S}$ over T such that*

$$-2K_{\widetilde{M}} = \alpha^*(-2(K_{\widetilde{S}} + \widetilde{B})) + \widetilde{E}$$

and that, for any closed point $t \in T$, the restriction $\alpha_t = \widetilde{\alpha}|_{M_t}\colon M_t \to S_t$ of $\widetilde{\alpha}$ to the fibers $M_t = h^{-1}(t)$ and $S_t = f^{-1}(t)$ is the minimal desingularization.

PROOF. (1) follows from Lemma 2.20.

(2): We set $\widetilde{L} := -2K_{\widetilde{M}} - \widetilde{E}$ and $L_t := -2K_{M_t} - E_t$. By Theorem 3.18 and by the upper semi-continuity theorem, we have an isomorphism

$$h_*\mathcal{O}_{\widetilde{M}}(m\widetilde{L}) \otimes \Bbbk(t) \simeq \mathrm{H}^0(M_t, mL_t) \tag{5–1}$$

for any closed point $t \in T$ and for any $m \geq 0$. Hence the natural homomorphism

$$h^*h_*\mathcal{O}_{\widetilde{M}}(m\widetilde{L}) \to \mathcal{O}_{\widetilde{M}}(m\widetilde{L}) \tag{5–2}$$

is surjective for any $m \geq 0$ by Theorem 3.18. Since L_t is big, there exist a proper surjective morphism $f\colon \widetilde{S} \to T$ from a normal variety $\widetilde{S}$, and a birational morphism $\widetilde{\alpha}\colon \widetilde{M} \to \widetilde{S}$ over T such that $\widetilde{L}$ is linearly equivalent to the pullback of an f-ample divisor of $\widetilde{S}$. Then $-2K_{\widetilde{S}} - \widetilde{\alpha}_*\widetilde{E}$ is the f-ample divisor. The morphism $\widetilde{\alpha}$ is induced from the surjection (5–2) for sufficiently large m. Hence, by the base change property (5–1), any fiber $S_t = f^{-1}(t)$ is a normal variety, and $\alpha_t = \widetilde{\alpha}|_{M_t}\colon M_t \to S_t$ is isomorphic to the birational morphism into the del Pezzo pair constructed in Proposition 3.19. Thus $f\colon (\widetilde{S}, \widetilde{B}) \to T$ is a family of del Pezzo pairs of index at most two for $\widetilde{B} = (1/2)\widetilde{\alpha}_*\widetilde{E}$. $\square$

LEMMA 5.3.

(1) *A fundamental triplet (X, E, Δ) is deformation equivalent to the fundamental triplet (X, E, Δ') for a zero-dimensional subscheme $\Delta' \subset E$ such that Δ' contains no nodes of E and that $\Delta' \cap E_{\mathrm{red}}$ is reduced.*

(2) *A fundamental triplet (X, E, Δ) is deformation equivalent to the fundamental triplet (X, E', Δ') for an effective divisor E' linearly equivalent to E and for a reduced zero-dimensional subscheme $\Delta' \subset E'$ such that Δ' contains no nodes of E' and that E' is reduced along Δ'.*

(3) *For a fundamental triplet (X, E, Δ), suppose that $E = E^{(1)} + E^{(2)}$ for effective divisors $E^{(1)}$ and $E^{(2)}$ such that $\Delta \cap E^{(2)} = \emptyset$ and that $E^{(1)}$ is linearly equivalent to a non-singular divisor. Then (X, E, Δ) is deformation equivalent to the fundamental triplet $(X, E' + E^{(2)}, \Delta')$ for a non-singular divisor E' and a reduced subscheme $\Delta' \subset E'$.*

PROOF. (1): If Δ contains a node of E, then E is reduced by Theorem 4.6. Thus the assertion follows from Lemmas 2.22 and 2.23.

(2): By (1) and Theorem 4.6, we may assume that $\Delta \cap E_{\mathrm{red}}$ is reduced and that the type of (X, E, Δ) is one of $[2]_2$, $[2; 1, 2]_{2+}$, and $[n; 2, e]_2$ for $n \geq 1$, $2 \leq e \leq \min\{n+1, 4\}$. Let Γ be an irreducible component Γ with $\Delta \cap \Gamma \neq \emptyset$ such that $\mathrm{mult}_\Gamma(E) = m \geq 2$ and that $\Gamma \neq \sigma$ if the type is $[n; 2, e]_2$. Thus Γ is a line of $\mathbb{P}^2$ or a fiber of the Hirzebruch surface $\mathbb{F}_n$. There exists an effective divisor $\widetilde{D} \subset X \times T$ for an open neighborhood T of 0 of the affine line $\mathbb{A}^1 = \operatorname{Spec} \Bbbk[\mathtt{t}]$ such that $D_t = \widetilde{D} \cap (X \times \{t\})$ is a non-singular divisor for $t \neq 0$ and that $D_0 = m\Gamma$. We may assume that $\Delta \cap \Gamma$ is reduced by (1). For a point $P \in \Delta \cap \Gamma$, Δ is locally defined by the ideal $(\mathtt{x}^m, \mathtt{y})$ for a local coordinate system $(\mathtt{x}, \mathtt{y})$ of X at P, where Γ is defined by $\mathtt{x} = 0$. Thus, for a suitable choice of $\widetilde{D}$, we infer that the divisor $\mathrm{div}(\mathtt{y})$ intersects transversely with D_t for any $t \neq 0$ on a neighborhood of P. By replacing Δ with $\mathrm{div}(\mathtt{y}) \cap D_t$ for $t \neq 0$ around P, we have a deformation to a fundamental triplet (X, E', Δ') satisfying the required condition.

(3): We may assume that Δ is reduced and is supported on the non-singular part of $E^{(1)}$. There exist a non-singular connected curve T with a point 0 and an effective divisor $\widetilde{D}$ of $X \times T$ such that $\widetilde{D} \to T$ is flat, the fiber $D_t = \widetilde{D} \cap (X \times \{t\})$ over $t \in T$ is non-singular for $t \neq 0$, and that $D_0 = E^{(1)}$.

Table 5. A list of types of fundamental triplets

genus g	Type	genus g	Type
2	$[0;2,1]_0$, $[1;2,2]_0$, $[2;2,3]_+$	7	$[1;1,0]_0$
3	$[2]_0$, $[0;2,0]_{00}$, $[1,2,1]_{00}$, $[2;2,2]_{00}$, $[3;2,3]_{00}$, $[4;2,4]_{00}$	8	$[2;1,0]_0$
4	$[0;1,1]_0$	9	$[3;1,0]_0$
5	$[1;1,1]_0$	10	$[4;1,0]_0$
6	$[1]_0$, $[0;1,0]_0$	$n+3-e$	$[n;2,e]_2$ for $n \geq 1$, $e \leq \min\{n-1,4\}$

Since $\widetilde{D} \to T$ is smooth along $\Delta \times \{0\}$, there exists a non-singular curve $\widetilde{\Delta} \subset \widetilde{D}$ smooth over T such that the fiber of $\widetilde{\Delta} \to T$ over 0 is Δ. Thus (X, E, Δ) is deformed to $(X, D_t + E^{(2)}, \Delta_t)$ for $t \neq 0$. $\square$

We introduce a relation $\lessdot$ for the types of fundamental triplets, as follows: $\mathsf{T}_1 \lessdot \mathsf{T}_2$ means that any fundamental triplet of type T_1 is deformation equivalent to a fundamental triplet of type T_2.

Proposition 5.4. *A fundamental triplet is deformation equivalent to a fundamental triplet of one of the types listed in* Table 5.

Proof. Let (X, E, Δ) be a fundamental triplet. By Lemma 5.3, we may assume that Δ is reduced and that either E is non-singular or $E = E^{(1)} + E^{(2)}$ for a non-singular divisor $E^{(1)}$ and an effective divisor $E^{(2)}$ with $\Delta \cap E^{(2)} = \emptyset$. More explicitly, we have the following relations by Lemma 5.3:

$[1;1,1]_+(a,b) \lessdot [1;1,1]_0$;

$[n;2,n+1]_{++}(a,b) \lessdot [n;2,n+1]_+$ for $1 \leq n \leq 3$;

$[n;2,n]_2 \lessdot [n;2,n]_{00}$ for $0 \leq n \leq 4$;

$[n;2,n+1]_2 \lessdot [n;2,n+1]_+$ for $1 \leq n \leq 3$;

$[2]_+(b) \lessdot [2]_0$; $[2]_2 \lessdot [2]_0$; $[0;1,1]_+(b) \lessdot [0;1,1]_0$; $[1;2,2]_+ \lessdot [1;2,2]_0$;

$[1;2,2]_{2\infty} \lessdot [1;2,2]_\times \lessdot [1;2,2]_0$;

$[0;2,1]_2 \lessdot [0;2,1]_{++} \lessdot [0;2,1]_+ \lessdot [0;2,1]_0$;

$[2;1,1]_+(a,b) \lessdot [2;1,1]_+(0,0)$; $[2;1,2]_{2+} \lessdot [2;1,2]_{++} \lessdot [2;1,2]_0$.

In order to obtain Table 5, it is enough to show the following relations in addition:

$$[3;1,1]_+ \lessdot [1;1,0]_0; \quad [2;1,1]_+(0,0) \lessdot [0;1,0]_0;$$
$$[2;1,2]_0 \lessdot [0;1,1]_0; \quad [3;2,4]_+ \lessdot [1;2,2]_0.$$

These are shown in Proposition 5.10, (1) below, in which Lemma 5.5 and Corollary 5.6 are required. □

In order to construct some interesting deformations, we note the following well-known:

Lemma 5.5. *For positive integers n, a, b with $a+b=n$, there exists an exact sequence*

$$0 \to p_1^*\mathcal{O}_{\mathbb{P}^1} \to \widetilde{\mathcal{E}} \to p_1^*\mathcal{O}_{\mathbb{P}^1}(n) \to 0$$

on the product $\mathbb{P}^1 \times \mathbb{A}^1$, where p_1 denotes the projection $\mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{P}^1$, such that $\widetilde{\mathcal{E}}$ is isomorphic to $p_1^*(\mathcal{O}(a) \oplus \mathcal{O}(b))$ over $\mathbb{P}^1 \times (\mathbb{A}^1 \setminus \{0\})$ and that the restriction of $\widetilde{\mathcal{E}}$ to $\mathbb{P}^1 \times \{0\}$ is isomorphic to $\mathcal{O} \oplus \mathcal{O}(n)$.

Proof. Let us take global sections $\zeta_1 \in \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(a))$ and $\zeta_2 \in \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(b))$ so that $\mathrm{div}(\zeta_1) \cap \mathrm{div}(\zeta_2) = \emptyset$. Then we have a short exact sequence $0 \to \mathcal{O} \to \mathcal{O}(a) \oplus \mathcal{O}(b) \to \mathcal{O}(n) \to 0$ over $\mathbb{P}^1$, where the surjection $\mathcal{O}(a) \oplus \mathcal{O}(b) \to \mathcal{O}(n)$ is given by $(x,y) \mapsto x\zeta_2 - y\zeta_1$ and the injection $\mathcal{O} \to \mathcal{O}(a) \oplus \mathcal{O}(b)$ is given by $z \mapsto (z\zeta_1, z\zeta_2)$. Let $\eta \in \mathrm{Ext}^1(\mathbb{P}^1; \mathcal{O}(n), \mathcal{O})$ be the extension class associated with the exact sequence above and let $\widetilde{\mathcal{E}}$ be the locally free sheaf of rank two given by the extension class

$$\eta \otimes \mathtt{t} \in \mathrm{Ext}^1(\mathbb{P}^1; \mathcal{O}(n), \mathcal{O}) \otimes \mathrm{H}^0(\mathbb{A}^1, \mathcal{O}) \simeq \mathrm{Ext}^1(\mathbb{P}^1 \times \mathbb{A}^1; p_1^*\mathcal{O}(n), p_1^*\mathcal{O}),$$

where $\mathbb{A}^1 = \operatorname{Spec} \Bbbk[\mathtt{t}]$. Then $\widetilde{\mathcal{E}}$ restricted to $\mathbb{P}^1 \times \{0\}$ is $\mathcal{O} \oplus \mathcal{O}(n)$. The extensions defined by $\eta \otimes \mathtt{t}$ and by $\eta \otimes 1$ are mutually isomorphic over $\mathbb{P}^1 \times (\mathbb{A}^1 \setminus \{0\})$. Thus $\widetilde{\mathcal{E}}$ restricted to $\mathbb{P}^1 \times (\mathbb{A}^1 \setminus \{0\})$ is isomorphic to $p_1^*(\mathcal{O}(a) \oplus \mathcal{O}(b))$. □

Corollary 5.6. *Let n and a be positive integers with $n \geq 2a$. Then there is a $\mathbb{P}^1$-bundle $\widetilde{X} \to \mathbb{P}^1 \times \mathbb{A}^1$ such that the fiber X_t of $\widetilde{X} \to \mathbb{A}^1$ over $t \in \mathbb{A}^1$ is isomorphic to $\mathbb{F}_{n-2a}$ if $t \neq 0$ and to $\mathbb{F}_n$ if $t = 0$. Moreover, there exist a section $\Sigma(1,n)$ and rational sections $\Sigma(1,a)$, $\Sigma(1,n-a)_\infty$ of the $\mathbb{P}^1$-bundle $\widetilde{X} \to \mathbb{P}^1 \times \mathbb{A}^1$ satisfying the following conditions*:

(1) $\Sigma(1,a) \sim \Sigma(1,n) + p_1^*\mathcal{O}(a-n)$, $\Sigma(1,n-a)_\infty \sim \Sigma(1,n) + p_1^*\mathcal{O}(-a)$, *and* $\Sigma(1,n)$ *is a tautological divisor with respect to* $\widetilde{\mathcal{E}}$.

(2) *Suppose that* $t \neq 0$. *Then* $\Sigma(1,a)|_{X_t}$ *is a minimal section* $\sigma^{(n-2a)}$ *of* $X_t = \mathbb{F}_{n-2a}$, $\Sigma(1,n)|_{X_t} \sim \sigma^{(n-2a)} + (n-a)\ell$ *for a fiber* ℓ *of* $X_t \to \mathbb{P}^1$, *and* $\Sigma(1,n-a)_\infty|_{X_t}$ *is a section at infinity.*

(3) $\Sigma(1,n)|_{X_0}$ *is a section at infinity of* $X_0 = \mathbb{F}_n$, $\Sigma(1,a)|_{X_0} = \sigma^{(n)} + F_1$, *and* $\Sigma(1,n-a)_\infty|_{X_0} = \sigma^{(n)} + F_2$ *for a negative section* $\sigma^{(n)}$ *and effective divisors* $F_1 \sim a\ell$, $F_2 \sim (n-a)\ell$ *with* $F_1 \cap F_2 = \emptyset$ *for a fiber* ℓ *of* $X_0 \to \mathbb{P}^1$.

PROOF. The $\mathbb{P}^1$-bundle defined by $\widetilde{X} = \mathbb{P}(\widetilde{\mathcal{E}})$ for the locally free sheaf $\widetilde{\mathcal{E}}$ of Lemma 5.5 for $b = n-a$ satisfies the first required condition. The section defined by the surjection $\widetilde{\mathcal{E}} \to p_1^*\mathcal{O}(n)$ satisfies the condition of $\Sigma(1,n)$. In order to find other rational sections, we look at the isomorphism between $\widetilde{\mathcal{E}}$ and $p_1^*(\mathcal{O}(a)\oplus\mathcal{O}(n-a))$ over $\mathbb{P}^1\times(\mathbb{A}^1\setminus\{0\})$ shown in Lemma 5.5. Let $\Sigma(1,a)^\star$ and $\Sigma(1,n-a)_\infty^\star$ be the sections over $\mathbb{P}^1 \times (\mathbb{A}^1 \setminus \{0\})$ corresponding to the surjections to $p_1^*\mathcal{O}(a)$ and to $p_1^*\mathcal{O}(n-a)$, respectively. Here $\Sigma(1,a)^\star\cap\Sigma(1,n)$ is isomorphic to $\operatorname{div}(\zeta_1) \times (\mathbb{A}^1 \setminus \{0\})$ for the section $\zeta_1 \in \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(a))$ in Lemma 5.5. Similarly, $\Sigma(1,n-a)_\infty^\star \cap \Sigma(1,n)$ is isomorphic to $\operatorname{div}(\zeta_2) \times (\mathbb{A}^1 \setminus \{0\})$ for the section $\zeta_2 \in \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(n-a))$. Let $\Sigma(1,a)$ and $\Sigma(1,n-a)_\infty$ be the closures of $\Sigma(1,a)^\star$ and $\Sigma(1,n-a)_\infty^\star$ in $\widetilde{X}$, respectively. Then $\Sigma(1,a)|_{X_0} = \sigma^{(n)} + \pi^*\operatorname{div}(\zeta_1)$ and $\Sigma(1,n-a)_\infty|_{X_0} = \sigma^{(n)} + \pi^*\operatorname{div}(\zeta_2)$ for the projection $\pi\colon X_0 = \mathbb{F}_n \to \mathbb{P}^1$. Thus we are done. $\square$

Example 5.7. Applying Corollary 5.6 to the case $n = 4$, $a = 2$, we have a $\mathbb{P}^1$-bundle $M \to \mathbb{P}^1 \times \mathbb{A}^1$ and a tautological divisor $\Sigma = \Sigma(1,4)$ such that $M_t \simeq \mathbb{F}_0$ and $\Sigma|_{M_t}$ is ample for the fiber M_t of $M \to \mathbb{A}^1$ over $t \neq 0$ and that $M_0 \simeq \mathbb{F}_4$ and $\Sigma|_{M_0}$ is a section at infinity. There is a birational morphism $M \to V$ into a normal variety V over $\mathbb{A}^1$ such that Σ is linearly equivalent to the pullback of a relatively ample divisor of V over $\mathbb{A}^1$. Thus we have a flat surjective morphism $V \to \mathbb{A}^1$ whose fiber V_t over $t \in \mathbb{A}^1$ is isomorphic to $X_t \simeq \mathbb{F}_0$ if $t \neq 0$ and to $\overline{\mathbb{F}}_4 \simeq \mathbb{P}(1,1,4)$ if $t = 0$. Note that $\mathbb{P}(1,1,4)$ is a log del Pezzo surface of index two defined by the fundamental triplet $(\mathbb{F}_4, \sigma, \emptyset)$ of type $[4;1,0]_0$. However, V is not $\mathbb{Q}$-Gorenstein since the exceptional locus of $M \to V$ is just the negative curve $\sigma^{(4)}$ of $M_0 \simeq \mathbb{F}_4$ and

$K_M\sigma^{(4)} > 0$. Therefore, $(V, 0) \to \mathbb{A}^1$ is not a deformation of del Pezzo pairs in the sense of Definition 5.1. Indeed, $K_{V_t}^2 = 8 \neq K_{V_0}^2 = 9$ for $t \neq 0$.

The following generalizes the construction called *sweeping out the cone with hyperplane sections* due to Pinkham [26, Remarks (7.6), iii)]:

LEMMA 5.8. *Let S be a non-singular projective variety and let $A \subset S$ be an effective ample divisor. Then there exist a proper flat morphism $\pi\colon \widetilde{S} \to \mathbb{P}^1$ and a point $0 \in \mathbb{P}^1$ such that $\pi^{-1}(t) \simeq S$ for $t \neq 0$ and that $\pi^{-1}(0) \simeq \operatorname{Proj} R$ for the image R of the restriction homomorphism*

$$\bigoplus\nolimits_{k\geq 0} \mathrm{H}^0\left(S, \mathrm{Sym}^k(\mathcal{O}_S \oplus \mathcal{O}_S(A))\right) \\ \to \bigoplus\nolimits_{k\geq 0} \mathrm{H}^0\left(A, \mathrm{Sym}^k(\mathcal{O}_A \oplus \mathcal{O}_A(A))\right).$$

In particular, if A is a non-singular variety and if $\mathrm{H}^1(S, \mathcal{O}_S(mA)) = 0$ for $m \geq 0$, then $\pi^{-1}(0)$ is normal and is a cone over A.

PROOF. Let $p\colon Z \to S$ be the $\mathbb{P}^1$-bundle associated with $\mathcal{V} = \mathcal{O}_S \oplus \mathcal{O}_S(A)$ and let H be a tautological divisor with respect to $\mathcal{V}$. Let Σ and $W \subset Z$ be the sections of p corresponding to the first projection $\mathcal{V} \to \mathcal{O}_S$ and the second projection $\mathcal{V} \to \mathcal{O}_S(A)$, respectively. Let Λ be the linear system consisting of the members of $|H|$ containing $B := p^{-1}(A)\cap W$. Then $\Lambda \simeq \mathbb{P}^1$ and $\operatorname{Bs}\Lambda = B$. Let $0 \in \Lambda$ correspond to $p^*A+\Sigma$. Then any another member of Λ corresponds to a section of p. The complete linear system $|mH|$ for suitable $m > 0$ defines a birational morphism $\mu\colon Z \to Z'$ into the normal variety $Z' = \operatorname{Proj}\bigoplus_{k\geq 0}\mathrm{H}^0(S, \mathrm{Sym}^k(\mathcal{V}))$ such that $\mu(\Sigma)$ is a point, $\Sigma = \mu^{-1}(\mu(\Sigma))$, and that μ is an isomorphism outside Σ. Thus Λ can be regarded as a linear system on Z'. Let $\widetilde{S} \to Z'$ be the blowing up along $\mu(B)$. Then the induced morphism $\pi\colon \widetilde{S} \to \Lambda$ is flat, and the fiber over a point $t \in \Lambda$ is isomorphic to the corresponding member of Λ as a divisor of Z'. In particular, $\pi^{-1}(t) \simeq S$ for $t \neq 0$ and $\pi^{-1}(0)$ is isomorphic to the image of $p^{-1}(A) = \mathbb{P}_A(\mathcal{V}|_A)$ under the morphism μ. Thus $\pi^{-1}(0) \simeq \operatorname{Proj} R$. If A is a non-singular variety and $\mathrm{H}^1(S, \mathcal{O}_S(mA)) = 0$ for $m \geq 0$, then $R \simeq \bigoplus_{k\geq 0}\mathrm{H}^0(A, \mathrm{Sym}^k(\mathcal{V})|_A)$. Thus we are done. $\square$

Example 5.9. Applying Lemma 5.8 to $S = \mathbb{P}^2$ and a non-singular conic A, we have a proper flat morphism $\pi\colon \widetilde{S} \to \mathbb{P}^1$ such that $\pi^{-1}(0) \simeq \mathbb{P}(1, 1, 4)$

and $\pi^{-1}(t) \simeq \mathbb{P}^2$ for $t \neq 0$. Here, $\widetilde{S}$ has a unique singular point, which is obtained by contracting a divisor isomorphic to $\mathbb{P}^2$ with the normal bundle $\mathcal{O}(-2)$. Hence, the singularity of $\widetilde{S}$ is terminal of index two and $(\widetilde{S}, 0) \to \mathbb{P}^1$ is a family of del Pezzo pairs of index two. The morphism π gives a $\mathbb{Q}$-Gorenstein smoothing of the rational singularity of type K_1.

Remark. The formal moduli space of the cone $\mathbb{P}(1,1,4)$ has been shown to be reduced with two components, of dimension 3 and 1 meeting transversely, by Pinkham [25], [26, (8. 6)]. Here, the 3-dimensional component corresponds to the deformation in Example 5.7 and the 1-dimensional component to the deformation in Example 5.9.

Proposition 5.10.

(1) *The following relations hold*:

$$[3;1,1]_+ \lessdot [1;1,0]_0, \quad [2;1,1]_+(0,0) \lessdot [0;1,0]_0,$$
$$[2;1,2]_0 \lessdot [0;1,1]_0, \quad [3;2,4]_+ \lessdot [1;2,2]_0.$$

(2) *If (X, E, Δ) is a fundamental triplet of type $[2;2,3]_+$ with $\Delta = \emptyset$, then it is deformation equivalent to a fundamental triplet of type $[0;2,1]_0$.*

(3) *If (X, E, Δ) is a fundamental triplet of type $[2;2,3]_+$ with $\Delta \neq \emptyset$, then its elimination is deformation equivalent to the elimination of a fundamental triplet of type $[1;2,2]_0$.*

(4) *The del Pezzo pair associated with a fundamental triplet of type $[4;2,4]_{00}$ is deformation equivalent to the del Pezzo pair associated with a fundamental triplet of type $[2]_0$.*

Proof. (1): For $[3;1,1]_+$, applying Corollary 5.6 to $n=3$ and $a=1$, we have a family $\widetilde{X} \to T$ of ruled surfaces and a rational section $\Sigma = \Sigma(1,1)$ such that $\Sigma|_{X_0} = \sigma^{(3)} + \ell$ and $\Sigma|_{X_t} = \sigma^{(1)}$ for $t \neq 0$ for the fiber X_t over $t \in T$; moreover the zero-dimensional subscheme Δ of a fundamental triplet of type $[3;1,1]_+$ on the central fiber X_0 extends to a subscheme $\widetilde{\Delta}$ of $\widetilde{X}$ which is finite and flat over T. Therefore $[3;1,1]_+ \lessdot [1;1,0]_0$. For $[2;1,1]_+(0,0) \lessdot [0;1,0]_0$, it is similarly proved by applying Corollary 5.6 to $n=2$ and $a=1$, and by considering $\Sigma = \Sigma(1,1)$. For $[2;1,2]_0 \lessdot [0;1,1]_0$, it is similarly proved by applying Corollary 5.6 to $n=2$ and $a=1$, and by considering $\Sigma=\Sigma(1,2)$.

For $[3;2,4]_+ \lessdot [1;2,2]_0$, we need more complicated argument. Let (X,E,Δ) be a fundamental triplet of type $[3;2,4]_+$. Then $X\simeq\mathbb{F}_3$, $E=\sigma^{(3)}+D$ for the negative section $\sigma^{(3)}$ and a section $D\sim\sigma^{(3)}+4\ell$ for a fiber ℓ of $\pi\colon X\to\mathbb{P}^1$, and $\Delta\subset D\setminus\sigma^{(3)}$. Let ℓ_1 be the fiber passing through $D\cap\sigma^{(3)}$ and let ℓ_2 be another fiber with $\ell_2\cap\Delta=\emptyset$. We set $P_i=\pi(\ell_i)$ for $i=1$, 2. Then there exists a member $\Theta\in|\sigma^{(3)}+3\ell|$ such that $\Theta|_D=4\ell_2\cap D$ as divisors on D by the exact sequence

$$0=\mathrm{H}^0(X,-\ell)\to\mathrm{H}^0(X,\sigma^{(3)}+3\ell)\to\mathrm{H}^0(D,\mathcal{O}(4))\to\mathrm{H}^1(X,-\ell)=0.$$

Note that $\sigma^{(3)}\not\leq\Theta$ since $\sigma^{(3)}\cap D\not\subset\Theta\cap D$. Thus Θ is a section at infinity. The exact sequence above shows that D is a member of the pencil spanned by $\sigma^{(3)}+4\ell_2$ and $\Theta+\ell_1$. Let $\widetilde{X}\to\mathbb{P}^1\times\mathbb{A}^1$ be the $\mathbb{P}^1$-bundle obtained by applying Corollary 5.6 to $n=3$ and $a=1$. Let $\mathtt{h}$, $\mathtt{g}$, and $\mathtt{f}$ be defining equations of the rational sections $\Sigma(1,3)$, $\Sigma(1,1)$, and $\Sigma(1,2)_\infty$ of the $\mathbb{P}^1$-bundle, respectively. We may assume that $\Sigma(1,3)|_{X_0}=\Theta$, $\Sigma(1,1)|_{X_0}=\sigma^{(3)}+\ell_1$, and $\Sigma(1,2)_\infty|_{X_0}=\sigma^{(3)}+2\ell_2$. Thus $E=\mathrm{div}(\mathtt{f}^2+c\mathtt{gh})|_{X_0}$ for a non-zero constant $c\in\Bbbk$. For $t\neq0$, $\Sigma(1,3)|_{X_t}\sim\sigma^{(1)}+2\ell$ is a section, $\Sigma(1,1)|_{X_t}=\sigma^{(1)}$, and $\Sigma(1,2)_\infty|_{X_t}$ is a section at infinity, where the point $\Sigma(1,3)\cap\Sigma(1,1)\cap X_t$ lies on the fiber of $X_t\to\mathbb{P}^1$ over P_1, and $\Sigma(1,3)\cap\Sigma(1,2)_\infty\cap X_t$ is a zero-dimensional subscheme of multiplicity two supported on the fiber of $X_t\to\mathbb{P}^1$ over P_2. If we consider X_t as the blowing up at a point P of $\mathbb{P}^2$, then $\mathrm{div}(\mathtt{f})|_{X_t}$ is the pullback of a line γ not containing the center P, and $\mathrm{div}(\mathtt{gh})|_{X_t}$ is the total transform of a non-singular conic C containing P, where γ is a tangent line of C. Hence, $\mathrm{div}(\mathtt{f}^2+c\mathtt{gh})|_{X_t}$ is isomorphic to $\mathrm{div}(\mathtt{z}^2+c(\mathtt{x}^2+\mathtt{yz}))$ for a suitable homogeneous coordinate $(\mathtt{x},\mathtt{y},\mathtt{z})$ of $\mathbb{P}^2$. Therefore, the divisor $\widetilde{E}:=\mathrm{div}(\mathtt{f}^2+c\mathtt{gh})$ of $\widetilde{X}$ is smooth over $\mathbb{A}^1\setminus\{0\}$. Moreover, Δ is a fiber of a subscheme $\widetilde{\Delta}\subset\widetilde{E}$ which is finite and flat over $\mathbb{A}^1$ by Lemma 2.23. Thus we have a family $(\widetilde{X},\widetilde{E},\widetilde{\Delta})\to\mathbb{A}^1$ of fundamental triplets, and hence $[3;2,4]_+\lessdot[1;2,2]_0$.

(2) and (3): Let (X,E,Δ) be a fundamental triplet of type $[2;2,3]_+$. Then $X\simeq\mathbb{F}_2$, $E=\sigma^{(2)}+D$ for a section $D\sim\sigma^{(2)}+3\ell$ for a fiber ℓ of $\pi\colon X\to\mathbb{P}^1$, and $\Delta\subset D\setminus\sigma^{(2)}$. Let ℓ_1 be the fiber passing through $D\cap\sigma^{(2)}$ and let ℓ_2 be another fiber with $\ell_2\cap\Delta=\emptyset$. We set $P_i=\pi(\ell_i)$ for $i=1$, 2. Then there exists a member $\Theta\in|\sigma^{(2)}+2\ell|$ such that $\Theta|_D=3\ell_2\cap D$ as

divisors on D by the exact sequence

$$0 = \mathrm{H}^0(X, -\ell) \to \mathrm{H}^0(X, \sigma^{(2)} + 2\ell) \to \mathrm{H}^0(D, \mathcal{O}(3)) \to \mathrm{H}^1(X, -\ell) = 0.$$

Note that $\sigma^{(2)} \not\leq \Theta$ since $\sigma^{(2)} \cap D \not\subset \Theta \cap D$. Thus Θ is a section at infinity. The exact sequence above shows that D is a member of the pencil spanned by $\sigma^{(2)} + 3\ell_2$ and $\Theta + \ell_1$. Let $\widetilde{X} \to \mathbb{P}^1 \times \mathbb{A}^1$ be the $\mathbb{P}^1$-bundle obtained by applying Corollary 5.6 to $n = 2$ and $a = 1$. Then $X_t \simeq \mathbb{F}_0$ for $t \in \mathbb{A}^1 \setminus \{0\}$. Let $\mathtt{h}$, $\mathtt{g}$, and $\mathtt{f}$ be defining equations of the rational sections $\Sigma(1,2)$, $\Sigma(1,1)$, and $\Sigma(1,1)_\infty$ of the $\mathbb{P}^1$-bundle, respectively. We may assume that $\Sigma(1,2)|_{X_0} = \Theta$, $\Sigma(1,1)|_{X_0} = \sigma^{(2)} + \ell_1$, and $\Sigma(1,1)_\infty|_{X_0} = \sigma^{(2)} + \ell_2$. Let $\mathtt{s}$ be a defining equation of ℓ_2, in other words, $P_2 \in \mathbb{P}^1$ is defined by $\mathtt{s} = 0$. Then $E = \operatorname{div}(\mathtt{s}\mathtt{f}^2 + c\mathtt{g}\mathtt{h})|_{X_0}$ for a non-zero constant $c \in \Bbbk$. For $t \neq 0$, $\Sigma(1,2)|_{X_t} \sim \sigma^{(0)} + \ell$ is a section, $\Sigma(1,1)|_{X_t} = \sigma^{(0)}$, and $\Sigma(1,1)_\infty|_{X_t}$ is a section at infinity, where the point $\Sigma(1,2) \cap \Sigma(1,1) \cap X_t$ lies on the fiber of $X_t \to \mathbb{P}^1$ over P_1, and the point $\Sigma(1,2) \cap \Sigma(1,1)_\infty \cap X_t$ lies on the fiber over P_2. Let Λ be the pencil on X_t generated by $2\Sigma(1,1)_\infty|_{X_t} + \ell_{2,t}$ and $\Sigma(1,2)|_{X_t} + \Sigma(1,1)|_{X_t}$, where $\ell_{2,t}$ is the fiber of $X_t \to \mathbb{P}^1$ over P_2. Then Λ is a sublinear system of $|2\sigma^{(0)} + \ell|$ having no fixed components. We infer that a member of Λ is a section for the other projection $\pi' \colon X_t \simeq \mathbb{F}_0 \to \mathbb{P}^1$ except for $2\Sigma(1,1)_\infty|_{X_t} + \ell_{2,t}$ and $\Sigma(1,2)|_{X_t} + \Sigma(1,1)|_{X_t}$. Thus $\operatorname{div}(\mathtt{s}\mathtt{f}^2 + c\mathtt{g}\mathtt{h})|_{X_t}$ is a section of π'. Therefore, the divisor $\widetilde{E} = \operatorname{div}(\mathtt{s}\mathtt{f}^2 + c\mathtt{g}\mathtt{h})$ of $\widetilde{X}$ is smooth over $\mathbb{A}^1 \setminus \{0\}$. Moreover, Δ is a fiber of a subscheme $\widetilde{\Delta} \subset \widetilde{E}$ which is finite and flat over $\mathbb{A}^1$, by Lemma 2.23. Thus we have a family $(\widetilde{X}, \widetilde{E}, \widetilde{\Delta}) \to \mathbb{A}^1$ of quasi-fundamental triplets, and is a family of fundamental triplets of type $[0; 2, 1]_0$ when $\Delta = \emptyset$. When $\Delta \neq \emptyset$, for the family of quasi-fundamental triplets, we also have a family of basic pairs by taking the simultaneous eliminations as in Lemma 5.2, (1) (cf. Lemma 2.20). If (X', E', Δ') is a quasi-fundamental triplet such that $X' \simeq \mathbb{F}_0$, $E' \sim \sigma^{(0)} + 2\ell$ is a non-singular divisor, and $\Delta' \neq \emptyset$, then its elimination is the basic pair obtained from a fundamental triplet of type $[1; 2, 2]_0$ by Proposition 4.4. Thus the assertion is proved.

(4): For a fundamental triplet (X, E, Δ) of type $[4; 2, 4]_{00}$, $E = \sigma + \sigma_\infty$ and $\Delta \subset \sigma_\infty$ for the negative section σ and a section σ_∞ at infinity of $X \simeq \mathbb{F}_4$. Hence, the del Pezzo pair associated with (X, E, Δ) is constructed from the elimination of $q(\Delta)$ for the contraction morphism $q \colon X \to \mathbb{P}(1, 1, 4)$ of σ. Here, $q(\sigma_\infty)$ is a cross section of the cone $\mathbb{P}(1, 1, 4)$. We consider the deformation $V = \widetilde{S} \to \mathbb{P}^1$ in Example 5.9. Here, we may assume that there is an effective divisor $Q \subset V$ such that $Q \to \mathbb{P}^1$ is smooth and that the fiber Q_t over $t \in \mathbb{P}^1$ is a non-singular conic of $V_t \simeq \mathbb{P}^2$ for $t \neq 0$ and that $Q_0 \simeq q(\sigma_\infty)$ for an isomorphism $V_0 \simeq \mathbb{P}(1,1,4)$. Since Δ can be assumed to be reduced, it extends to an effective divisor $\widetilde{\Delta}$ of an open neighborhood of V_0 in Q which is smooth over $\mathbb{P}^1$ (cf. Lemma 2.23). Hence, $(V_0, q(\sigma_\infty), q(\Delta))$ is deformed to a fundamental triplet of type $[2]_0$. Thus the associated del Pezzo pair with (X, E, Δ) is deformation equivalent to the del Pezzo pair associated with a fundamental triplet of type $[2]_0$. $\square$

5.2. Equi-singular deformations

We shall consider the equi-singular deformation types of del Pezzo pairs of index two.

DEFINITION 5.11.

(1) A family $h\colon (\widetilde{M}, \widetilde{E}) \to T$ of basic pairs over a connected non-singular curve T is called *equi-singular* if $\widetilde{E}$ is a relative simple normal crossing divisor over T, i.e., any irreducible component $\widetilde{E}_j$ of $\widetilde{E}$ is smooth over T, any non-empty intersection $\widetilde{E}_i \cap \widetilde{E}_j$ of two irreducible components is smooth over T, and any intersection $\widetilde{E}_i \cap \widetilde{E}_j \cap \widetilde{E}_k$ of three irreducible components is an empty set.

(2) A family $f\colon (\widetilde{S}, \widetilde{B}) \to T$ of del Pezzo pairs over a connected non-singular curve T is called *equi-singular* if there exist a proper smooth morphism $h\colon \widetilde{M} \to T$ and a birational morphism $\widetilde{\alpha}\colon \widetilde{M} \to \widetilde{S}$ with $h = f \circ \widetilde{\alpha}$ such that

   (a) $M_t = h^{-1}(t) \to S_t = f^{-1}(t)$ is the minimal desingularization for any closed point $t \in T$,

   (b) the union of the exceptional locus of $\widetilde{\alpha}$ and $\widetilde{\alpha}^{-1}(\operatorname{Supp} \widetilde{B})$ is a relative simple normal crossing divisor over T.

If $f\colon (\widetilde{S}, \widetilde{B}) \to T$ is an equi-singular family of del Pezzo pairs whose fibers (S_t, B_t) are constructed from basic pairs, then f is constructed from an equi-singular family $h\colon (\widetilde{M}, \widetilde{E}) \to T$ of basic pairs by Lemma 5.2, (2). However, the family of del Pezzo pairs constructed from an equi-singular family of basic pairs by Lemma 5.2, (2) is not necessarily equi-singular.

Two basic pairs are called *equi-singular deformation equivalent* to each other if they are connected by equi-singular families of basic pairs. Similarly, two del Pezzo pairs are called *equi-singular deformation equivalent* to each other if they are connected by equi-singular families of del Pezzo pairs.

Remark. Let (S, B) be a del Pezzo pair of index at most two associated with a basic pair (M, E_M). Then the number k of irreducible components of E_M is an equi-singular deformation invariant both for (M, E_M) and for (S, B).

Definition 5.12. Let $\tau\colon (\widetilde{X}, \widetilde{E}, \widetilde{\Delta}) \to T$ be a family of fundamental triplets over a non-singular connected curve T. The family is called *equi-singular* if the following conditions are satisfied:

(1) $\widetilde{E}$ is a relative simple normal crossing divisor over T;

(2) $\widetilde{\Delta} \cap \widetilde{E}_j$ is flat over T for any irreducible component $\widetilde{E}_j$ of $\widetilde{E}$;

(3) $\widetilde{\Delta} \cap \widetilde{E}_i \cap \widetilde{E}_j$ are flat over T for any two irreducible components $\widetilde{E}_i$ and $\widetilde{E}_j$.

If the following conditions are also satisfied, then the family τ is called *strongly equi-singular*:

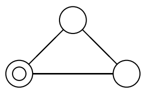
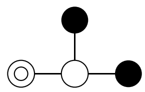
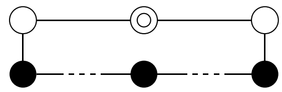
(4) Any two fibers of $\widetilde{\Delta} \cap \widetilde{E}_j \to T$ are isomorphic to each other for any j;

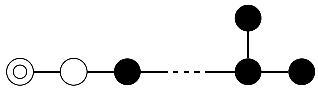
(5) If a fiber (X_t, E_t, Δ_t) of τ is of type $[2; 1, 2]_0$, then any fiber is of type $[2; 1, 2]_0$;

(6) Suppose that a fiber (X_t, E_t, Δ_t) of τ is of type $[1; 2, 2]_0$. Then there is an effective divisor $\widetilde{L} \subset \widetilde{X}$ smooth over T such that $\widetilde{L} \cap \widetilde{E}$ is flat over T and that $\widetilde{L} \cap X_t$ is the union of fibers ℓ of $X_t \to \mathbb{P}^1$ with $\deg(\Delta_t \cap \ell) = 2$.

Two fundamental triplets are called *equi-singular* (resp. *strongly equi-singular*) *deformation equivalent* to each other if they are connected by equi-singular (resp. strongly equi-singular) families of fundamental triplets.

LEMMA 5.13. *Let (X, E, Δ) be a fundamental triplet of type $[1; 2, 2]_0$ and let $\phi\colon (M, E_M) \to (X, E, \Delta)$ be the elimination. For a reducible fiber F of $M \to X \to \mathbb{P}^1$, the dual graph of $E_M + F$ is one of the following, where the number of black vertices is at most* 7 *in* (3), *and is at most* 8 *in* (4):

(1)  (2)  (3) 

(4) 

PROOF. The image $\ell = \phi(F)$ is a fiber of $\pi\colon X \to \mathbb{P}^1$ with $\ell \cap \Delta \neq \emptyset$, and $F = \phi^{-1}(\ell)$. If $\ell \cap E$ consists of two points Q_1, Q_2, then the dual graph of $F + E_M$ is either (1) or (3) above, and the number of black vertices is $\mathrm{mult}_{Q_1}(\Delta) + \mathrm{mult}_{Q_2}(\Delta) - 1$. If ℓ intersects E tangentially at a point P, then $P \in \Delta$ and the dual graph of $F + E_M$ is one of (1), (2), and (4) above. Here the number of black vertices equals $\mathrm{mult}_P(\Delta)$ if $\mathrm{mult}_P(\Delta) \geq 2$, and equals 0 if $\mathrm{mult}_P(\Delta) = 1$. Thus, we are done. □

LEMMA 5.14. *Let (X, E) be a minimal basic pair and Δ_1, Δ_2 be two zero-dimensional subschemes of X such that*

(1) *(X, E, Δ_1) and (X, E, Δ_2) are fundamental triplets of the same type,*

(2) $\deg(\Delta_1 \cap E_j) = \deg(\Delta_2 \cap E_j)$ *for any irreducible component E_j of E,*

(3) $\mathrm{mult}_P(\Delta_1) = \mathrm{mult}_P(\Delta_2)$ *and* $\mathrm{mult}_P(\Delta_1 \cap E_j) = \mathrm{mult}_P(\Delta_2 \cap E_j)$ *for any node P of E and for any irreducible component $E_j \ni P$.*

Then (X, E, Δ_1) and (X, E, Δ_2) are equi-singular deformation equivalent to each other. They are strongly equi-singular deformation equivalent if the following conditions are satisfied in addition:

(4) $\Delta_1 \cap (E_j \setminus \{\textit{node of } E\}) \simeq \Delta_2 \cap (E_j \setminus \{\textit{node of } E\})$ *as schemes for any E_j;*

(5) *Suppose that* (X, E) *is of type* $[1; 2, 2]$, E *is non-singular, and* $\pi|_E \colon E \to \mathbb{P}^1$ *is separable. Let* $L_i \subset X$ *be the union of fibers* ℓ *of* π *with* $\deg(\ell \cap \Delta_i) = 2$ *for* $i = 1, 2$. *Then there is an isomorphism* $\Delta_1 \simeq \Delta_2$ *inducing* $\Delta_1 \cap L_1 = E \cap L_1 \simeq \Delta_2 \cap L_2 = E \cap L_2$.

Proof. By Proposition 2.21, we have an equi-singular family $(X \times T, E \times T, \widetilde{\Delta}) \to T$ of fundamental triplets over a connected non-singular curve T. Thus the first assertion follows. Suppose that the latter two conditions are satisfied. Then, by (4), the subschemes $\Delta_1^\sharp$ and $\Delta_2^\sharp$ in the proof of Proposition 2.21 are isomorphic to each other on any irreducible components of $E^\sharp$. Thus $\Delta_t = \widetilde{\Delta} \cap (X \times \{t\})$ is isomorphic to Δ_1 for any t, and the condition (4) of Definition 5.12 is satisfied. Since the condition (5) of Definition 5.12 is automatically satisfied, we may assume that (X, E) is of type $[1; 2, 2]$ and E is non-singular.

Suppose that $\pi|_E$ is inseparable. For $i = 1, 2$, and $m \geq 1$, let $\Delta_i^{[m]}$ be the set of points P with $\mathrm{mult}_P(\Delta_i) = m$. Then we can write

$$\Delta_1 = \sum\nolimits_{m \geq 1} m\Delta_1^{[m]}, \quad \Delta_2 = \sum\nolimits_{m \geq 1} m\Delta_2^{[m]}.$$

Here $\Delta_1^{[m]}$ is linearly equivalent to $\Delta_2^{[m]}$ for any $m \geq 1$, since $\Delta_1 \simeq \Delta_2$. Hence, we have a smooth family $\widetilde{\Delta}^{[m]} \subset X \times T$ of reduced effective divisors for $m \geq 1$ over a non-singular connected curve T such that $\Delta_i^{[m]} = \widetilde{\Delta}^{[m]} \cap (X \times \{t_i\})$ for suitable point $t_i \in T$ for $i = 1, 2$. We set $\widetilde{\Delta} = \sum_{m \geq 1} m\widetilde{\Delta}^{[m]}$. For the union L_i of fibers ℓ of π with $\deg(\ell \cap \Delta_i) = 2$ for $i = 1, 2$, we have $L_i \cap E = L_i \cap \Delta = 2\sum_{m \geq 2} \Delta_i^{[m]}$. Thus, for the family $(X \times T, E \times T, \widetilde{\Delta}) \to T$ of fundamental triplets, we have an effective divisor $\widetilde{L} \subset X \times T$ satisfying the condition (6) of Definition 5.12.

Suppose that $\pi|_E$ is separable. Then the set R of the ramification points of $\pi|_E$ consists of one point if $\mathrm{char}\, \Bbbk = 2$, and two points if $\mathrm{char}\, \Bbbk \neq 2$, by Lemma 4.11. If $\mathrm{mult}_Q(\Delta_1) = m \geq 2$ for a point $Q \in R$, then $\ell \cap E = 2Q$ for the fiber ℓ of π containing Q, and hence, by (5), $\mathrm{mult}_Q(\Delta_2) = m$ or $\mathrm{mult}_{Q'}(\Delta_2) = m$ for the other point $Q' \in R$. If $\mathrm{char}\, \Bbbk \neq 2$, then, by Lemma 4.11, we have an involution of X preserving E and π, and interchanging Q and Q'. Thus we may assume that if $\mathrm{mult}_Q(\Delta_1) = m \geq 2$ for a point $Q \in R$, then $\mathrm{mult}_Q(\Delta_2) = m$. Let L_i^R be the union of fibers ℓ of π passing through a point $Q \in R$ with $\mathrm{mult}_Q(\Delta) \geq 2$ for $i = 1, 2$. Then

$L_1^R = L_2^R$ and $L_1^R \cap E = L_1^R \cap \Delta_1 = L_2^R \cap E = L_2^R \cap \Delta_2$ by the assumption. We set Δ_0 to be the divisor

$$\sum\nolimits_{P \in L_1^R} \mathrm{mult}_P(\Delta_1)P = \sum\nolimits_{P \in L_2^R} \mathrm{mult}_P(\Delta_2)P.$$

In order to construct a divisor $\widetilde{L} \subset X \times T$ satisfying the condition (6) of Definition 5.12, it is enough to consider the restrictions of Δ_1 and Δ_2 to $E \setminus \mathrm{Supp}\,\Delta_0$. Note that the Galois involution ι associated with the double-covering $\pi|_E \colon E \to \mathbb{P}^1$ acts on $E \setminus R$ freely. We have a finite number of morphisms $P_j \colon T \to E \setminus (L_1^R \cap E)$ from a connected non-singular curve T with fixed points t_1, t_2, and natural numbers $m_j \geq 1$ such that $\Delta_i = \sum m_j P_j(t_i) + \Delta_0$ for $i = 1, 2$. By the condition (5) and by replacing T with an open subset, we may assume that, for a natural number k and for any $t \in T$,

- $P_j(t) \neq P_{j'}(t)$ for any $j \neq j'$,
- $P_j(t) \notin R$ for $1 \leq j \leq 2k$,
- $\iota \circ P_j(t) = P_{j+k}(t)$ for $1 \leq j \leq k$,
- $\iota \circ P_j(t) \neq P_{j'}(t)$ for j, $j' > 2k$, except for the case where $j = j'$ and $P_j(t) \in R$.

Let $\widetilde{\Delta} \subset X \times T$ be the effective divisor $\sum m_j \Gamma_j + (\Delta_0 \times T)$, where Γ_j is the graph of P_j. Then, for the family $(X \times T, E \times T, \widetilde{\Delta}) \to T$, we can find an expected divisor $\widetilde{L} \subset X \times E$. $\square$

THEOREM 5.15. *Let $\tau \colon (\widetilde{X}, \widetilde{E}, \widetilde{\Delta}) \to T$ be an equi-singular family of fundamental triplets over a connected non-singular curve T. Then there is a simultaneous elimination $\widetilde{M} \to \widetilde{X}$ of $\widetilde{\Delta}$ over T if T is replaced with a finite covering over T. Moreover the induced family $h \colon (\widetilde{M}, \widetilde{E}_{\widetilde{M}}) \to T$ of basic pairs is equi-singular. If τ is strongly equi-singular, then h induces an equi-singular family $f \colon (\widetilde{S}, \widetilde{B}) \to T$ of del Pezzo pairs.*

PROOF. The existence of the simultaneous elimination is shown by Lemma 2.20 and by a similar argument to the proof of Proposition 2.21. By (1)–(3) of Definition 5.12, we infer that $\widetilde{E}_{\widetilde{M}}$ is a relative simple normal crossing divisor over T. In order to show the equi-singularity of f, we apply

Lemma 4.13. The exceptional curves for the eliminations and $\widetilde{E}_{\widetilde{M}}$ form a relative simple normal crossing divisor over T by (4) of Definition 5.12. In case (X_t, E_t, Δ_t) is of type $[2; 1, 2]_0$, then the negative section σ on X_t forms a divisor of $\widetilde{M}$ smooth over T which does not intersect $\widetilde{E}_{\widetilde{M}}$. If (X_t, E_t, Δ_t) is of type $[1; 2, 2]_0$, then the proper transform of the divisor $\widetilde{L}$ in $\widetilde{M}$ is smooth over T and is away from $\widetilde{E}_{\widetilde{M}}$. Thus the induced family $(\widetilde{S}, \widetilde{B}) \to T$ is equi-singular. $\square$

5.3. Deformation of log del Pezzo surfaces of index two

Recall that S is called a log del Pezzo surface if $(S, 0)$ is a log-terminal del Pezzo pair. By a deformation of a log del Pezzo surface S, we mean a deformation of the del Pezzo pair $(S, 0)$ in the sense of Definition 5.1, (3). If the index of S is at most two, then the genus g is a deformation invariant, since $2g-2 = (K_M + L_M)L_M = 2K_S^2$. The author has learned the following result in the case of characteristic zero from Yongnam Lee.

Theorem 5.16. *A log del Pezzo surface of index two is deformation equivalent to a* (*non-singular*) *del Pezzo surface by a deformation of index two of log del Pezzo surfaces in the sense of* Definition 5.1. *In particular, a log del Pezzo surface of index at most two admits a* $\mathbb{Q}$*-Gorenstein smoothing.*

Proof. A non-Gorenstein singular point of a log del Pezzo surface S of index two is of type K_n for $n \leq 9$ by Theorem 4.14. Moreover, the local ring of the singularity is isomorphic to the local ring at the origin of $X((1, 2n-1)/(4n))$ of Example 4.17. In fact, the morphism to $X((1, 2n-1)/(4n))$ in Proposition 4.18 is birational by construction of the minimal desingularization M. Thus, the singularity admits a $\mathbb{Q}$-Gorenstein smoothing (of index two) by Proposition 4.19.

In order to show that the smoothing extends to a global deformation of S, it is enough to prove that $\mathrm{H}^2(S, T_S) = 0$ for $T_S = \mathcal{H}om(\Omega^1_S, \mathcal{O}_S)$ (cf. [30, Proposition 6.4], [22, Lemma 1]). In fact, we have a formal global deformation by the vanishing, which is algebraizable by $\mathrm{H}^2(S, \mathcal{O}_S) = 0$. Note that $\mathrm{H}^2(S, T_S)$ is dual to $\mathrm{Hom}_S(T_S, \omega_S)$ for the dualizing sheaf $\omega_S \simeq \mathcal{O}_S(K_S)$ and that a member of $|-K_S|$ induces an injection $\omega_S \hookrightarrow \mathcal{O}_S$. Thus $\mathrm{H}^2(S, T_S) = 0$ follows from another vanishing $\mathrm{Hom}_S(T_S, \mathcal{O}_S) \simeq \mathrm{H}^0(S, (\Omega^1_S)^{\vee\vee}) = 0$. Since S has only toric singularities, the double-dual $(\Omega^1_S)^{\vee\vee}$ is isomorphic to $\alpha_*\Omega^1_M$

(cf. [9]). Thus the vanishing is established by $\mathrm{H}^0(M, \Omega^1_M) = 0$. Hence, S admits a $\mathbb{Q}$-Gorenstein smoothing.

Let S_t be a smooth surface obtained as a smooth fiber of the $\mathbb{Q}$-Gorenstein smoothing. Since $-2K_S$ is an ample Cartier divisor, $-K_{S_t}$ is also ample. Thus S_t is a del Pezzo surface. □

Since the genus g can be taken between 2 and 10, any del Pezzo surface degenerates into a log del Pezzo surface of index two by a $\mathbb{Q}$-Gorenstein deformation.

For deformations of constant index two (cf. Definition 5.1, (3)), we have the following result by Proposition 5.4 and Proposition 5.10.

LEMMA 5.17. *If two log del Pezzo surfaces of index two have the same genus $g \neq 6$, then they are connected by deformations of constant index two. A log del Pezzo surface of index two and of genus $g = 6$ is connected to a log del Pezzo surface of type $[1]_0$ or $[0;1,0]_0$ by deformations of constant index two.*

In the case of $g = 6$, we have exactly two deformation equivalence classes for deformations of constant index two by:

LEMMA 5.18. *Let $f : \widetilde{S} \to T$ be a flat family of normal surfaces over a non-singular connected curve T such that $2K_{\widetilde{S}}$ is Cartier and that any fiber $S_t = f^{-1}(t)$ is a log del Pezzo surface of index two. If a fiber S_o is of type $[1]_0$, then so is any fiber S_t.*

PROOF. The type of a fiber S_t is one of $[1]_0$, $[0;1,0]_0$, $[2;1,1]_+(a,b)$, since these are the types with genus 6. We have isomorphisms

$$\omega_{S_t} \simeq \mathcal{E}xt^1_{\mathcal{O}_{\widetilde{S}}}(\mathcal{O}_{S_t}, \omega_{\widetilde{S}}) \simeq \omega_{\widetilde{S}} \otimes \mathcal{O}_{S_t} \quad \text{and} \quad \mathcal{O}_{S_t}(2K_{S_t}) \simeq \mathcal{O}_{\widetilde{S}}(2K_{\widetilde{S}}) \otimes \mathcal{O}_{S_t}$$

for any $t \in T$. Since $-K_{S_t} = K_{S_t} + (-2K_{S_t})$, we have

$$\mathcal{O}_{S_t}(-K_{S_t}) \simeq \mathcal{O}_{\widetilde{S}}(-K_{\widetilde{S}}) \otimes \mathcal{O}_{S_t}.$$

We also have the base change isomorphism

$$f_*\mathcal{O}_{\widetilde{S}}(-K_{\widetilde{S}}) \otimes_{\mathcal{O}_T} \Bbbk(t) \simeq \mathrm{H}^0(S_t, -K_{S_t})$$

by $\mathrm{H}^1(S_t, -K_{S_t}) = 0$. Let $\mathbb{P}_T(\mathcal{E}) \to T$ be the projective bundle associated with the locally free sheaf $\mathcal{E} = f_*\mathcal{O}_{\widetilde{S}}(-K_{\widetilde{S}})$ and let $\Phi\colon \widetilde{S} \dashrightarrow \mathbb{P}_T(\mathcal{E})$ be the rational map over T associated with the homomorphism $f^*f_*\mathcal{E} \to \mathcal{O}_{\widetilde{S}}(-K_{\widetilde{S}})$. Then the restriction of Φ to S_t coincides with the rational map associated with the linear system $|-K_{S_t}|$. Thus $\Phi(S_o) \simeq \mathbb{P}^2$, and

$$\Phi(S_t) \simeq \begin{cases} \mathbb{F}_0, & \text{if } S_t \text{ is of type } [0;1,0]_0; \\ \mathbb{F}_2, & \text{if } S_t \text{ is of type } [2;1,1]_+(a,b). \end{cases}$$

Let $V \subset \mathbb{P}_T(\mathcal{E})$ be the image of the rational map Φ. Then a general fiber V_t of $V \to T$ is just the image $\Phi(S_t)$. For a tautological divisor H of $\mathbb{P}_T(\mathcal{E})$ with respect to $\mathcal{E}$, we have $\Phi(S_t)H^2 = 6$, since $\Phi|_{S_t}$ is birational to the morphism associated with $|K_{M_t} + L_t|$ for the minimal desingularization $\alpha_t\colon M_t \to S_t$ and for $L_t = \alpha_t^*(-2K_{S_t})$. Therefore, $V_t = \Phi(S_t)$ for any t. Moreover, $V_t \simeq \mathbb{P}^2$, since $V_o \simeq \mathbb{P}^2$. Hence, S_t is of type $[1]_0$ for any t. $\square$

Therefore, the number of the deformation types of log del Pezzo surfaces of index two with respect to the deformations of constant index two is 10.

6. The Structure of Log del Pezzo Surfaces of Index Two

In the remaining part of this paper, we consider only log del Pezzo surfaces S of index two. In this section, the negative curves on the minimal desingularization M are studied. We shall show that the dual graph of negative curves on M and the type of S almost determine the equi-singular deformation equivalence class of S. We shall also compare the classification of log del Pezzo surfaces of index two by the types of fundamental triplet with the classification by Alexeev–Nikulin [4].

6.1. Types of log del Pezzo surfaces of index two

For a log del Pezzo surface S of index two, let $\alpha\colon M \to S$ be the minimal resolution of singularities. Then $-2K_M \sim \alpha^*(-2K_S) + E_M$ for a non-zero α-exceptional simple normal crossing divisor E_M, and (M, E_M) is a basic pair with $L_M E_M = 0$ for $L_M = -2K_M - E_M$. Conversely, S is determined by M since $|-2K_M| = |L_M| + E_M$ and since α is given as the Stein factorization of the morphism associated with the base point free linear system $|L_M|$.

Let (X, E, Δ) be a fundamental triplet whose elimination $\phi\colon M \to X$ of Δ defines the basic pair (M, E_M) by $E_M = E_M^{\Delta}$. Here, E is also a non-zero simple normal crossing divisor and $LE = \deg(\Delta)$ for $L = -2K_X - E$.

There is an isomorphism $\alpha_* \mathcal{O}_M(K_M + L_M) \simeq \mathcal{O}_S(-K_S)$ by $K_M + L_M \sim K_M + \alpha^*(-2K_S)$. Thus the morphism $M \to \mathbb{P}|K_M + L_M|$ associated with the base point free linear system $|K_M + L_M|$ is birational to the rational map $\Phi_{|-K_S|}\colon S \dashrightarrow \mathbb{P}|-K_S|$ associated with the anti-canonical linear system $|-K_S|$, even though $-K_S$ is not Cartier.

If $K_X + L$ is ample, then X is the image of $\Phi_{|-K_S|}$ and E is the image of the non-Gorenstein locus of S. If $K_X + L$ is not ample but big, then the rational map $\Phi_{|-K_S|}$ induces the contraction morphism $X \simeq \mathbb{F}_2 \to \overline{\mathbb{F}}_2 \simeq \mathbb{P}(1,1,2)$ of the negative section $\sigma \subset X$. If $K_M + L_M$ is not big, then the morphism $\pi \circ \phi\colon M \to X \to \mathbb{P}^1$ is obtained as the Stein factorization of the composite $\Phi_{|-K_S|} \circ \alpha$.

A log del Pezzo surface S of index two determines the isomorphism class of the basic pair (M, E_M), and moreover, the isomorphism class of the fundamental triplet (X, E, Δ) except for the case where (X, E, Δ) is of type $[1;2,2]_0$, by Theorem 4.9 (cf. Example 4.12). In particular, the type of (X, E, Δ) depends only on S. Thus we define the type of S to be the type of (X, E, Δ). Let T be the type of S. Then the genus g_{T} is defined as the genus of the minimal basic pair (X, E), but it equals the genus of the basic pair (M, E_M) and also the genus of the del Pezzo pair $(S, 0)$. In particular, $g_{\mathsf{T}} = K_S^2 + 1$.

The number of irreducible components of E_M also depends on the type T, which is denoted by k_{T}. In Section 6.3 below, we shall introduce another invariant δ_{T}, which is calculated in Proposition 6.14. We have TABLE 6 of the list of types T of log del Pezzo surfaces of index two together with the invariants g_{T}, k_{T}, and δ_{T}.

By TABLE 6, we shall show in Lemma 6.15 below that δ_{T} depends on the equi-singular deformation equivalence class of basic pairs (M, E_M) with $L_M E_M = 0$. In particular, we have:

THEOREM 6.1. *The list of types of fundamental triplets coincides with the list of equi-singular deformation equivalence classes of basic pairs defining log del Pezzo surfaces of index two with one exception*; *The two types* $[0;1,1]_0$ *and* $[2;1,2]_0$ *define the same equi-singular deformation equivalence class.*

Table 6. The types of log del Pezzo surfaces of index two

Type T	g_{T}	k_{T}	δ_{T}	Type T	g_{T}	k_{T}	δ_{T}	Type T	g_{T}	k_{T}	δ_{T}
$[1]_0$	6	1	0	$[1;1,1]_0$	5	1	1	$[3;1,0]_0$	9	1	1
$[2]_0$	3	1	1	$[1;1,1]_+(0,0)$	5	2	1	$[3;1,1]_+$	7	2	0
$[2]_+(0)$	3	2	1	$[1;1,1]_+(1,1)$	5	3	1	$[3;2,4]_+$	2	2	1
$[2]_+(1)$	3	3	1	$[1;1,1]_+(2,1)$	5	4	0	$[3;2,4]_{++}(0,0)$	2	3	1
$[2]_+(2)$	3	4	1	$[1;1,1]_+(1,2)$	5	4	1	$[3;2,4]_{++}(1,1)$	2	4	1
$[2]_+(3)$	3	5	1	$[1;1,1]_+(1,3)$	5	5	1	$[3;2,4]_{++}(2,1)$	2	5	0
$[2]_+(4)$	3	6	0	$[1;2,2]_0$	2	1	1	$[3;2,4]_{++}(1,2)$	2	5	1
$[0;1,0]_0$	6	1	1	$[2;1,0]_0$	8	1	1	$[3;2,4]_{++}(1,3)$	2	6	1
$[0;1,1]_0$	4	1	1	$[2;1,1]_+(0,0)$	6	2	1	$[3;2,4]_{++}(1,4)$	2	7	1
$[0;1,1]_+(0)$	4	2	1	$[2;1,1]_+(1,1)$	6	3	1	$[3;2,4]_{++}(1,5)$	2	8	1
$[0;1,1]_+(1)$	4	3	1	$[2;1,1]_+(1,2)$	6	4	1	$[3;2,4]_{++}(1,6)$	2	9	0
$[0;1,1]_+(2)$	4	4	1	$[2;1,1]_+(1,3)$	6	5	0	$[4;1,0]_0$	10	1	0
$[0;1,1]_+(3)$	4	5	1	$[2;1,2]_0$	4	1	1	$[4;2,4]_{00}$	3	2	0
$[1;1,0]_0$	7	1	1	$[2;1,2]_{++}$	4	3	0				

6.2. The negative curves on M

Proposition 6.2. *A negative curve γ on M is a $(-d)$-curve for $1 \le d \le 4$. Moreover, the $(-d)$-curves are classified as follows*:

(1) *A (-4)-curve is a connected component of E_M and is the proper transform of an irreducible connected component of E. A (-4)-curve exists if and only if E is non-singular.*

(2) *A (-3)-curve γ is the proper transform of an irreducible component E_1 of E with $(E - E_1)E_1 = 1$. Here, $(E_M - \gamma)\gamma = 1$.*

(3) *A ϕ-exceptional (-2)-curve is a ϕ-exceptional irreducible curve γ satisfying $\gamma \cap E_M = \emptyset$ or $\gamma \subset E_M$. If $\gamma \cap E_M = \emptyset$, then $\phi(\gamma)$ is a non-singular point of E. If $\gamma \subset E_M$, then $\phi(\gamma)$ is a node of E.*

(4) *A (-2)-curve which is not ϕ-exceptional is the proper transform of one of the following curves on $X \simeq \mathbb{F}_n$:*

   (a) *The section σ when the type is $[2;1,2]_0$ or $[2;1,2]_{++}$;*

   (b) *A fiber ℓ of π with $\ell \cap E \subset \Delta$ when the type is $[1;2,2]_0$.*

   (c) *The fiber ℓ of π contained in E when the type is $[3;2,4]_{++}(a,b)$.*

(5) *A ϕ-exceptional (-1)-curve is either the curve Γ_k in the situation of* Lemma 2.10 *or the curve Γ_{b+1} in the situation of* Lemma 2.14.

(6) *A (-1)-curve γ with $L_M\gamma = 1$ which is not ϕ-exceptional is the proper transform of a fiber ℓ of $\pi\colon X \simeq \mathbb{F}_n \to \mathbb{P}^1$ such that $E\ell = 2$, $\ell \not\subset E$, and $\deg(\Delta \cap \ell) = 1$. Here, $\gamma \cap E_M$ is a non-singular point of E_M.*

(7) *A (-1) curve γ with $L_M\gamma \neq 1$ satisfies $L_M\gamma = 2$ and $E_M \cap \gamma = \emptyset$, and is the proper transform of one of the following curves:*

  (a) *A line ℓ of $\mathbb{P}^2$ with $\deg(\Delta \cap \ell) = 2$ when $\deg E = 2$;*

  (b) *A fiber ℓ of π with $\deg(\Delta \cap \ell) = 1$ when $X \simeq \mathbb{F}_n$ and $E\ell = 1$;*

  (c) *A minimal section σ with $\sigma \cap E \subset \Delta$ when the type is $[0,1,1]_0$ or $[0;1,1]_+(b)$;*

  (d) *The negative section σ when the type is $[1;1,1]_0$;*

  (e) *A section Θ at infinity with $\Theta \cap E \subset \Delta$ in the case where the type is one of $[3;2,4]_+$, $[3;2,4]_{++}(0,0)$, $[3;2,4]_{++}(1,1)$, $[3;2,4]_{++}(1,2)$, $[3;2,4]_{++}(1,3)$. Here, for a given Cartier divisor $\Delta' \subset \Delta$ of E with $\Delta' \sim (\sigma + 3\ell)|_E$, there exists uniquely the section Θ at infinity with $\Theta \cap E = \Delta'$;*

  (f) *The negative section σ when the type is $[1;2,2]_0$;*

  (g) *A section $\Theta \sim \sigma + m\ell$ of π with $\Theta \cap E \subset \Delta$ for $1 \leq m \leq 4$ when the type is $[1;2,2]_0$. Here, for a given Cartier divisor $\Delta' \subset \Delta$ of E with $\deg \Delta' = 2m$ such that $E \cap \ell \not\subset \Delta'$ for any fiber ℓ of π, there exists uniquely the section Θ with $\Theta \cap E = \Delta'$.*

Note that the α-exceptional curves are classified in Lemma 4.13 for any basic pairs (M, E_M). However, here, we consider only the basic pairs with $L_M E_M = 0$. A part of the proof below overlaps with the proof of Lemma 4.13.

Proof. If γ is ϕ-exceptional, then γ is a (-1)-curve or a (-2)-curve, and the assertions (3) and (5) have been shown in Lemmas 2.10 and 2.14.

We have the following properties (i)–(iv) of a negative curve γ on M:

(i) If γ is not ϕ-exceptional, then the equality

$$\gamma^2 = \phi(\gamma)^2 - \deg(\Delta \cap \phi(\gamma))$$

holds, by Lemma 2.7.

(ii) If $\gamma \not\subset E_M$, then γ is a (-1)-curve or a (-2)-curve, by $-2K_M\gamma = L_M\gamma + E_M\gamma \geq 0$.

(iii) Suppose that $\phi(\gamma)$ is an irreducible component E_1 of E in M. Then

$$-4 \leq \gamma^2 = -4 + (E - E_1)E_1 \leq -3,$$

which is derived from

$$\begin{aligned}\gamma^2 &= E_1^2 - \deg(\Delta \cap E_1) = E_1^2 - LE_1 = E_1^2 - (-2K_X - E)E_1 \\ &= 2(K_X + E_1)E_1 + (E - E_1)E_1 = -4 + (E - E_1)E_1.\end{aligned}$$

In particular, γ is a (-3)-curve or a (-4)-curve.

(iv) If $\gamma \subset E_M$, then $L_M\gamma = 0$ and

$$-4 = 2(K_M + \gamma)\gamma = -E_M\gamma - L_M\gamma + 2\gamma^2 = \gamma^2 - (E_M - \gamma)\gamma \leq \gamma^2.$$

The properties above show that $\gamma \simeq \mathbb{P}^1$ with $\gamma^2 \geq -4$. The assertions (1) and (2) follow from (iii), (iv). Note that if E has an irreducible connected component, then E is non-singular by Theorem 4.6.

In the proof of (4), (6), (7) below, let e_1, e_2 be the integers with $E \sim e_1\sigma + e_2\ell$ when $X \simeq \mathbb{F}_n$.

(4): Let γ be the (-2)-curve. Then $L_M\gamma = E_M\gamma = 0$ by $-2K_M = L_M + E_M$ and $L_ME_M = 0$. In particular, $(K_M + L_M)\gamma = (K_X + L)\phi(\gamma) = 0$. Hence, $K_X + L$ is not ample. If $K_X + L$ is big, then the type of (X, E, Δ) is $[2; 1, 2]_0$ or $[2; 1, 2]_{++}$, and $\phi(\gamma) = \sigma$. Conversely, the proper transform of σ in the case $[2; 1, 2]_0$ or $[2; 1, 2]_{++}$ is a (-2)-curve since $\Delta \cap \sigma = \emptyset$. This is the case of (4a).

Suppose that $K_X + L$ is not big. Then $e_1 = 2$. Since $K_X + L \sim (n + 2 - e_2)\ell$, $\phi(\gamma)$ is a fiber ℓ of π. Conversely, if γ is the proper transform of ℓ in the case $e_1 = 2$, then γ is a (-2)-curve if and only if $\deg(\Delta \cap \ell) = 2$, by (i). Here, if $\ell \not\subset E$, then the type is $[1; 2, 2]_0$ by $2 = \deg(\Delta \cap \ell) \leq E\ell$, and we have $\ell \cap E \subset \Delta$ by $E\ell \leq 2$. This is the case of (4b). If $\ell \subset E$, then the type is $[3; 2, 4]_{++}(a, b)$ and the fiber ℓ is unique, where $\deg(\Delta \cap \ell) = 2$. This is the case of (4c).

(6): Now $\phi(\gamma) \not\subset E$ by (iii) and $K_M\gamma = L_M\gamma = E_M\gamma = 1$. Hence, $(K_X + L)\phi(\gamma) = 0$. Thus $K_X + L$ is not ample. If $K_X + L$ is big, then

$X \simeq \mathbb{F}_2$ and $\phi(\gamma) = \sigma$, which contradicts (i). Hence, $K_X + L$ is not big. Thus $e_1 = 2$ and $\phi(\gamma)$ is a fiber ℓ of π. Conversely, if γ is the proper transform of a fiber $\ell \not\subset E$ in the case $e_1 = 2$, then $\gamma^2 = -\deg(\Delta \cap \ell)$ by (i). Thus γ is a (-1)-curve if and only if $\deg(\Delta \cap \ell) = 1$. If E has a node, then the type is $[3; 2, 4]_+$ or $[3; 2, 4]_{++}(a, b)$, but a fiber $\ell \not\subset E$ with $\Delta \cap \ell \neq \emptyset$ does not contain the nodes of E.

(7): The curve γ is not ϕ-exceptional by (5), and $\phi(\gamma) \not\subset E$ by (iii). The equality $2 = -2K_M\gamma = L_M\gamma + E_M\gamma$ implies that $L_M\gamma = 2$ and $E_M \cap \gamma = \emptyset$. In particular, $(K_X + L)\phi(\gamma) = 1$. We consider the proof in the following cases:

(A) $X \simeq \mathbb{P}^2$; (B) $X \simeq \mathbb{F}_n$ and $e_1 = 1$; (C) $X \simeq \mathbb{F}_n$ and $e_1 = 2$.

Case (A). $\deg E = 2$ and $\phi(\gamma)$ is a line ℓ by $\deg(K_X + L) = 3 - \deg E$. Conversely, if γ is the proper transform of a line ℓ and if $\deg E = 2$, then γ is a (-1)-curve if and only if $\deg(\Delta \cap \ell) = 2$, by (i). This is the case of (7a).

Case (B). $K_X + L \sim \sigma + (n + 2 - e_2)\ell$ with $e_2 \leq 2$. Note that $e_2 = 2$ only in the case $[2; 1, 2]_{++}$.

If $\phi(\gamma)$ is a fiber ℓ of π, then $\deg(\Delta \cap \ell) = \ell^2 - \gamma^2 = 1$. Conversely, the proper transform of a fiber ℓ with $\deg(\Delta \cap \ell) = 1$ is a (-1)-curve. This is the case of (7b).

If $\phi(\gamma)$ is a minimal section σ', then $e_2 = 1$, $\sigma' \not\subset E$, and $\deg(\sigma' \cap \Delta) = -n + 1$; hence, the type is $[0; 1, 1]_0$, $[0; 1, 1]_+(b)$, or $[1; 1, 1]_0$. For the types $[0; 1, 1]_0$ and $[0; 1, 1]_+(b)$, we have $\sigma' \cap E \subset \Delta$. This is the case of (7c). For the type $[1; 1, 1]_0$, σ' is the negative section σ and $\sigma \cap E = \sigma \cap \Delta = \emptyset$. This is the case of (7d).

Assume that $\phi(\gamma)$ is neither a fiber nor a minimal section. Then $\phi(\gamma)^2 > 0$. By the Hodge index theorem, we have

$$1 = ((K_X + L)\phi(\gamma))^2 \geq (K_X + L)^2\phi(\gamma)^2 = (n + 4 - 2e_2)\phi(\gamma)^2 > 0.$$

Thus $2e_2 = n + 3$ and $\phi(\gamma)^2 = 1$. Then $n = 1$ and $e_2 = 2$, which is a contradiction since $e_2 \leq 1$ for $n \neq 2$.

Case (C). Then $K_X + L \sim (n + 2 - e_2)\ell$, where $0 \leq e_2 \leq n + 1$. Since $(K_X + L)\phi(\gamma) = 1$, $\phi(\gamma)$ is a section $\Theta \sim \sigma + m\ell$ for some m and $e_2 = n + 1$.

Then the type is $[1;2,2]_0$, $[3;2,4]_+$, or $[3;2,4]_{++}(a,b)$. We treat the case of types $[3;2,4]_+$ and $[3;2,4]_{++}(a,b)$ in *Subcase* (C1), and the case of type $[1;2,2]_0$ in *Subcase* (C2) below.

Subcase (C1). This will corresponds to the case (7e). Here $n = 3$. Then $m \geq 3$, since $\sigma \subset E$ and $\Theta \not\subset E$. Here,

$$2m - 2 = \Theta^2 - \gamma^2 = \deg(\Delta \cap \Theta) \leq \Theta(E - \sigma) = m + 1.$$

Hence, $m = 3$ and $\Theta \cap E \subset \Delta$. Conversely, let $\Delta' \subset \Delta$ be a Cartier divisor such that $\Delta' \sim (\sigma + 3\ell)|_E$. Since $\mathrm{H}^p(X, \sigma + 3\ell - E) = \mathrm{H}^p(X, -\sigma - \ell) = 0$ for any p, we have an isomorphism

$$\mathrm{H}^0(X, \sigma + 3\ell) \xrightarrow{\simeq} \mathrm{H}^0(E, \mathcal{O}_E(\sigma + 3\ell)).$$

Here the subspace $\mathrm{H}^0(X, 3\ell)$ of the left hand side is isomorphic to the kernel of

$$\mathrm{H}^0(E, \mathcal{O}_E(\sigma + 3\ell)) \to \mathrm{H}^0(\sigma, \mathcal{O}_\sigma).$$

Since $\Delta \cap \sigma = \emptyset$, there exists a unique section $\Theta \sim \sigma + 3\ell$ at infinity with $\Theta \cap E = \Delta'$. Furthermore, the proper transform of Θ in M is a (-1)-curve.

We have to consider the existence of $\Delta' \sim (\sigma + 3\ell)|_E$ with $\Delta' \subset \Delta$. If the type is $[3;2,4]_+$, then Δ does not contain the node of E and hence any subscheme $\Delta' \subset \Delta$ with $\deg \Delta' = 4$ is linearly equivalent to $(\sigma + 3\ell)|_E$.

Suppose that the type is $[3;2,4]_{++}(a,b)$. Then $E = \sigma + \ell + \sigma_\infty$ for a section σ_∞ at infinity and for a fiber ℓ of π, where $\Delta \cap \sigma = \emptyset$. If $(a,b) \neq (0,0)$, then Δ contains the node $P = \ell \cap \sigma_\infty$ and hence $\mathrm{mult}_P(\Delta' \cap \ell) = a$, $\mathrm{mult}_P(\Delta' \cap \sigma_\infty) = b$ for any Cartier divisor $\Delta' \subset \Delta$ of E containing P by Corollary 2.13. If $\Delta' \sim (\sigma + 3\ell)|_E$, then $\deg(\Delta' \cap \sigma_\infty) = 3 \leq \deg(\Delta \cap \sigma_\infty) = 6$ and $\deg(\Delta' \cap \ell) = 1 \leq \deg(\Delta \cap \ell) = 2$. Therefore, the Cartier divisor $\Delta' \subset \Delta$ with $\Delta' \sim (\sigma + 3\ell)|_E$ exists if and only if the type is one of $[3;2,4]_{++}(0,0)$, $[3;2,4]_{++}(1,1)$, $[3;2,4]_{++}(1,2)$, $[3;2,4]_{++}(1,3)$.

Subcase (C2). This will corresponds to the cases (7f), (7g). Here, $E \sim 2\sigma + 2\ell$ is non-singular and $\sigma \cap E = \emptyset$. We have

$$2m = \Theta^2 - \gamma^2 = \deg(\Theta \cap \Delta) \leq \Theta E = 2m.$$

Hence, $\Theta \cap E \subset \Delta$ and $0 \leq m \leq 4$ by $2m = \Theta E \leq \deg \Delta = 8$. If $m = 0$, then $\Theta = \sigma$. In the case $m > 0$, Θ is determined by $\Theta \cap E$. In fact, the vanishings $\mathrm{H}^p(X, -\sigma + j\ell) = 0$ for p, $j \in \mathbb{Z}$ induce an isomorphism

$$\mathrm{H}^0(X, \sigma + m\ell) \xrightarrow{\simeq} \mathrm{H}^0(E, \mathcal{O}(2m)).$$

Hence, for a given subscheme $\Delta' \subset \Delta$ of $\deg \Delta' = 2m$ such that $\ell \cap E \not\subset \Delta'$ for any fiber ℓ, the section $\Theta \sim \sigma + m\ell$ with $\Theta \cap E = \Delta'$ exists uniquely. Thus we are done. $\square$

Let $\psi \colon \mathcal{Y} \to M$ be the blowing-up at all the nodes of E_M. Then the proper transform $E_{\mathcal{Y}}$ of E_M in $\mathcal{Y}$ is a disjoint union of (-4)-curves. Let G_q be the ψ-exceptional curve over a node q of E_M. Then $E_{\mathcal{Y}} = \psi^*(E_M) - 2\sum G_q$ and

$$\begin{aligned} -2K_{\mathcal{Y}} &= \psi^*(-2K_M) - 2\sum G_q \sim \psi^*(L_M + E_M) - 2\sum G_q \\ &= \psi^*(L_M) + E_{\mathcal{Y}}. \end{aligned} \tag{6–1}$$

Definition 6.3 (cf. [4]). The birational morphism $\beta = \alpha \circ \psi \colon \mathcal{Y} \to S$ is called the *right resolution* of S. If a non-singular projective surface $\mathcal{Y}$ is the right resolution of a log del Pezzo surface of index two, then $\mathcal{Y}$ is called a *DPN surface*, for short.

In $\operatorname{char} \Bbbk = 0$, the notion of DPN surface above coincides with that of right DPN surface of elliptic type in [4].

Lemma 6.4. *For a DPN surface $\mathcal{Y}$, suppose that there exists a negative curve $\gamma \subset \mathcal{Y}$ such that γ is not ψ-exceptional and $\psi(\gamma)^2 \geq 0$. Then the type of (X, E, Δ) is $[3; 2, 4]_+$, γ is a (-1)-curve, and $\phi \circ \psi(\gamma)$ is the unique fiber of $\pi \colon X \to \mathbb{P}^1$ passing through the node of E.*

Proof. We have $-2K_{\mathcal{Y}}\gamma \geq 0$ by (6–1), since $\psi(\gamma) \not\subset E_M$. Since $L_M\psi(\gamma) > 0$ by the Hodge index theorem, γ is a (-1)-curve and $L_M\psi(\gamma) + E_{\mathcal{Y}}\gamma = 2$. Then

$$L_M\psi(\gamma) \geq E_{\mathcal{Y}}\gamma + 2\sum G_q\gamma > E_{\mathcal{Y}}\gamma,$$

since $L_M - E_M$ is nef. Hence, $L_M\gamma = 2$, $E_{\mathcal{Y}}\gamma = 0$, and $\sum G_q\gamma = 1$. It follows that $\psi(\gamma)^2 = 0$ and $\phi^*(K_X + L)\psi(\gamma) = 0$ by $2(K_M + L_M) \sim$

$L_M - E_M$. Therefore, $X \simeq \mathbb{F}_n$, $K_X + L$ is not big, and $\phi \circ \psi(\gamma)$ is a fiber ℓ_0 of π. Here, $\ell_0 \cap \Delta = \emptyset$ and ℓ_0 contains a node of E. Hence, the type of (X, E, Δ) is $[3; 2, 4]_+$ and ℓ_0 is the unique fiber passing through the node of E. Conversely, the proper transform of the fiber ℓ_0 in $\mathcal{Y}$ is a (-1)-curve. □

Corollary 6.5. *A negative curve on a DPN surface $\mathcal{Y}$ is a $(-d)$-curve for $d = 1$, 2, 4.*

(1) *The set of (-4)-curves on $\mathcal{Y}$ coincides with the set of the proper transforms of irreducible components of E_M.*

(2) *The set of (-2)-curves on $\mathcal{Y}$ coincides with the set of the total transforms of (-2)-curves on M not contained in E_M.*

(3) *The set of (-1)-curves on $\mathcal{Y}$ consists of the following curves*:

  (a) *The ψ-exceptional curves*;
  (b) *The total transforms of (-1)-curves on M*;
  (c) *The proper transform of the fiber containing the node of E when the type is $[3; 2, 4]_+$.*

Proof. By Lemma 6.4, it is enough to consider the proper transforms of negative curves on M. Then the proper transform of any irreducible component of E_M is a (-4)-curve by (1), (2), and (4) of Proposition 6.2. The proper transform in $\mathcal{Y}$ of a (-2)-curve not contained in E_M is a (-2)-curve by (3) and (4) of Proposition 6.2. The proper transform in $\mathcal{Y}$ of a (-1)-curve is a (-1)-curve by (5), (6), and (7) of Proposition 6.2. Thus we are done. □

Corollary 6.6. *The Picard number $r = \rho(\mathcal{Y})$ of $\mathcal{Y}$ equals $11 - g_{\mathsf{T}} + k_{\mathsf{T}}$ for the type T of S.*

Proof. $E_{\mathcal{Y}}$ is non-singular with k_{T} components where any component is a (-4)-curve. Hence, $4K_{\mathcal{Y}}^2 = L_M^2 - 4k_{\mathsf{T}}$ by (6–1). Since $(K_M + L_M)L_M = 2g_{\mathsf{T}} - 2$ induces $L_M^2 = 4g_{\mathsf{T}} - 4$, we have $r = 10 - K_{\mathcal{Y}}^2 = 11 - g_{\mathsf{T}} + k_{\mathsf{T}}$. □

Let $n(E_M)$ be the number of nodes of E_M. Then $n(E_M) = k_{\mathsf{T}} - 1$ when the type is not $[4; 2, 4]_{00}$, and $n(E_M) = 0$ when the type is $[4; 2, 4]_{00}$.

COROLLARY 6.7. *The Picard number $\rho(M)$ equals $10-(K_X+E)^2$. It is also calculated as follows*:

$$\begin{aligned}\rho(M) &= 11 - g_{\mathsf{T}} + k_{\mathsf{T}} - n(E_M)\\ &= \begin{cases} 12 - g_{\mathsf{T}}, & \text{if the type is not } [4;2,4]_{00};\\ 13 - g_{\mathsf{T}} = 10, & \text{if the type is } [4;2,4]_{00}.\end{cases}\end{aligned}$$

PROOF. The first equality follows from $K_M^2 = (K_M + L_M)^2 = (K_X + L)^2 = (K_X + E)^2$ by (3–1). The second follows from Corollary 6.6. □

We have the following characterization for a rational projective surface to be a DPN surface:

LEMMA 6.8. *A non-singular projective rational surface $\mathcal{Y}$ is a DPN surface if and only if there is a non-zero non-singular divisor $E_{\mathcal{Y}}$ such that $L_{\mathcal{Y}} = -2K_{\mathcal{Y}} - E_{\mathcal{Y}}$ is nef and big, and $L_{\mathcal{Y}}E_{\mathcal{Y}} = 0$.*

PROOF. It is enough to show the 'if' part. Let $\beta\colon \mathcal{Y} \to S$ be the birational morphism into a normal complete algebraic space S of dimension two such that β-exceptional curves are the curves γ with $L_{\mathcal{Y}}\gamma = 0$. Then S is a log del Pezzo surface of index two (cf. Definition 3.2, Proposition 3.5). Let $\alpha\colon M \to S$ be the minimal desingularization. Then $\beta = \alpha \circ \psi$ for a birational morphism $\psi\colon \mathcal{Y} \to M$ and $\psi^*E_M = E_{\mathcal{Y}} + 2G$ for the ψ-exceptional divisor $G \sim K_{\mathcal{Y}} - \psi^*K_M$. Let $\mathcal{Y} = Y_m \to Y_{m-1} \to \cdots \to Y_1 \to Y_0 = M$ be the succession of blowups at points representing ψ. For $0 \le i \le m-1$, let $\psi_i\colon Y_{i+1} \to Y_i$ be the blowing up, $G_{i+1} \subset Y_{i+1}$ the ψ_i-exceptional divisor, and let $E_i \subset Y_i$ be the pushforward of $E_{\mathcal{Y}}$. Then $\psi_i^*E_i = E_{i+1} + 2G_{i+1}$ for any i. In particular, the center of $\psi_i\colon Y_{i+1} \to Y_i$ is a node of E_i. Hence, $\psi\colon \mathcal{Y} \to M$ is the blowing up at all the nodes of E_M. Therefore, $\beta\colon \mathcal{Y} \to S$ is the right resolution. □

6.3. Another invariant δ

Let $\beta\colon \mathcal{Y} \to S$ be the right resolution and let $\psi\colon \mathcal{Y} \to M$ be the blowing up at all the nodes of M, as before. For an irreducible component $E_{i,M}$ of E_M, let $E_{i,\mathcal{Y}}$ be the proper transform in $\mathcal{Y}$, which is a (-4)-curve. The proper transform $E_{\mathcal{Y}} = \sum E_{i,\mathcal{Y}}$ of E_M in $\mathcal{Y}$ is a disjoint union of the (-4)-curves. Moreover, $E_{\mathcal{Y}}$ is the union of all the (-4)-curves on $\mathcal{Y}$ by Corollary 6.5. We infer that $E_{\mathcal{Y}}$ coincides with the fixed part of the linear system $|-2K_{\mathcal{Y}}|$ by the relation (6–1). Since $L_{\mathcal{Y}} = -2K_{\mathcal{Y}} - E_{\mathcal{Y}} \sim \psi^* L_M \sim \beta^*(-2K_S)$, $\beta\colon \mathcal{Y} \to S$ is induced from the morphism associated with $|-2K_{\mathcal{Y}}|$. We have

$$
\begin{aligned}
(6\text{–}2) \qquad \dim \mathrm{H}^0(\mathcal{Y}, -2K_{\mathcal{Y}}) &= \dim \mathrm{H}^0(S, -2K_S) = 3K_S^2 + 1 = 3g_{\mathsf{T}} - 2, \\
\dim \mathrm{H}^1(\mathcal{Y}, -2K_{\mathcal{Y}}) &= \dim \mathrm{H}^0(\mathcal{Y}, -2K_{\mathcal{Y}}) - \chi(\mathcal{Y}, -2K_{\mathcal{Y}}) \\
&= 3(K_S^2 - K_{\mathcal{Y}}^2) = 3k_{\mathsf{T}},
\end{aligned}
$$

by Theorem 3.18, $\mathrm{H}^2(\mathcal{Y}, -2K_{\mathcal{Y}}) = 0$, $E_{\mathcal{Y}}^2 = -4k_{\mathsf{T}}$, and by (6–1).

DEFINITION 6.9. We introduce an invariant $\delta \in \{0, 1\}$ for a DPN surface $\mathcal{Y}$ as follows: For the number k of irreducible components of $E_{\mathcal{Y}}$ and for a vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ with $\varepsilon_i \in \{1, -1\}$, we set

$$
B_{\mathcal{Y}}^{\varepsilon} := L_{\mathcal{Y}} + \sum\nolimits_{i=1}^{k} \varepsilon_i E_{i,\mathcal{Y}}.
$$

Then we define $\delta = 0$ if there exists a vector $\varepsilon \in \{1, -1\}^k$ such that the numerical class $\mathrm{cl}(B_{\mathcal{Y}}^{\varepsilon}) \in \mathrm{NS}(\mathcal{Y})$ is divisible by 4, i.e., $\mathrm{cl}(B_{\mathcal{Y}}^{\varepsilon}) \in 4\,\mathrm{NS}(\mathcal{Y})$. If $\delta \neq 0$, then we define $\delta = 1$. Note that δ can be considered as an invariant of S which depends only on the type of S.

REMARK. The invariant δ above is nothing but the geometric interpretation of δ of the main invariants (r, a, δ) for the invariant lattice $\mathbb{S}$ (cf. Section 6.6, [4, Section 2.3]).

PROPOSITION 6.10. *Let $\varpi\colon \widetilde{\mathcal{Y}} \to T$ be a proper smooth morphism over a non-singular connected curve T whose fibers $\mathcal{Y}_t = \varpi^{-1}(t)$ are DPN surfaces. Then the invariant $\delta(\mathcal{Y}_t)$ is constant on T.*

PROOF. We may replace T with another curve étale over T, since T is connected. The rationality of $\mathcal{Y}_t$ implies that the relative Picard scheme $\mathrm{Pic}_{\widetilde{\mathcal{Y}}/T}$ is étale over T. Hence, we may assume that the restriction map $\mathrm{Pic}(\widetilde{\mathcal{Y}}) \to \mathrm{Pic}(\mathcal{Y}_o)$ is surjective for a given point $o \in T$. The kernel of the restriction map is just the image of $\varpi^*\colon \mathrm{Pic}(T) \to \mathrm{Pic}(\widetilde{\mathcal{Y}})$. In fact, it is shown as follows: Suppose that $\mathcal{M}|_{\mathcal{Y}_o} \simeq \mathcal{O}_{\mathcal{Y}_o}$ for an invertible sheaf

$\mathcal{M} \in \mathrm{Pic}(\widetilde{\mathcal{Y}})$. Then $(\mathcal{M}|_{\mathcal{Y}_t})^2 = (\mathcal{M}|_{\mathcal{Y}_t}) \cdot (\mathcal{A}|_{\mathcal{Y}_t}) = 0$ for a ϖ-ample invertible sheaf $\mathcal{A}$ on $\widetilde{\mathcal{Y}}$ and for any point $t \in T$. It implies that $\mathcal{M}|_{\mathcal{Y}_t} \simeq \mathcal{O}_{\mathcal{Y}_t}$ by the Hodge index theorem and by the rationality of $\mathcal{Y}_t$. Hence, $\varpi_*\mathcal{M}$ is an invertible sheaf and $\varpi^*\varpi_*\mathcal{M} \simeq \mathcal{M}$.

By (6–2), we have the base change isomorphism

$$\varpi_*\mathcal{O}_{\widetilde{\mathcal{Y}}}(-2K_{\widetilde{\mathcal{Y}}}) \otimes \Bbbk(t) \simeq \mathrm{H}^0(\mathcal{Y}_t, -2K_{\mathcal{Y}_t}).$$

Hence there exist a family $f\colon \widetilde{S} \to T$ of log del Pezzo surfaces of index two, a birational morphism $\tilde{\beta}\colon \widetilde{\mathcal{Y}} \to \widetilde{S}$ over T, and an effective divisor $E_{\widetilde{\mathcal{Y}}} \subset \widetilde{\mathcal{Y}}$ such that

(1) $\tilde{\beta}|_{\mathcal{Y}_t}\colon \mathcal{Y}_t \to S_t = f^{-1}(t)$ is the right resolution of S_t,

(2) $E_{\widetilde{\mathcal{Y}}}|_{\mathcal{Y}_t} = E_{\mathcal{Y}_t}$,

(3) $-2K_{\widetilde{\mathcal{Y}}} - E_{\widetilde{\mathcal{Y}}} \sim \tilde{\beta}^*(-2K_{\widetilde{S}})$.

Here, $E_{\widetilde{\mathcal{Y}}} \to T$ is smooth. Replacing T with a curve étale over T, we may assume that any irreducible component $E_{i,\widetilde{\mathcal{Y}}}$ of $E_{\widetilde{\mathcal{Y}}}$ is a $\mathbb{P}^1$-bundle over T. Thus $E_{i,\mathcal{Y}_t} = E_{i,\widetilde{\mathcal{Y}}}|_{\mathcal{Y}_t}$ is an irreducible component of $E_{\mathcal{Y}_t}$ for $t \in T$.

For a vector $\varepsilon = (\varepsilon_i)$, we consider a divisor

$$\tilde{B}^\varepsilon = B^\varepsilon_{\widetilde{\mathcal{Y}}} = \tilde{\beta}^*(-2K_{\widetilde{S}}) + \sum \varepsilon_i E_{i,\widetilde{\mathcal{Y}}}.$$

Then $\tilde{B}^\varepsilon|_{\mathcal{Y}_t} = B^\varepsilon_{\mathcal{Y}_t}$ for any $t \in T$. Suppose that $B^\varepsilon_{\mathcal{Y}_o} \sim 4L_o$ for a divisor L_o of $\mathcal{Y}_o$. Then $\mathcal{O}_{\mathcal{Y}_o}(L_o) \simeq \mathcal{L}|_{\mathcal{Y}_o}$ for an invertible sheaf $\mathcal{L}$ of $\widetilde{\mathcal{Y}}$. Thus the invertible sheaf $\mathcal{M} = \mathcal{L}^{\otimes 4} \otimes \mathcal{O}_{\widetilde{\mathcal{Y}}}(-\tilde{B}^\varepsilon)$ of $\widetilde{\mathcal{Y}}$ comes from T. Therefore, $B^\varepsilon_{\mathcal{Y}_t}$ is divisible by 4 in $\mathrm{Pic}(\mathcal{Y}_t)$ for any $t \in T$. Thus δ is constant. □

The following result is useful for calculating δ:

Lemma 6.11. *Let $f\colon S_1 \to S_2$ be a birational morphism between non-singular projective varieties and let D be a divisor of S_1. Then* $\mathrm{cl}(D) \in 4\,\mathrm{NS}(S_1)$ *if and only if*

(1) *$D\gamma \in 4\mathbb{Z}$ for any f-exceptional curve γ and,*

(2) $\mathrm{cl}(f_*D) \in 4\,\mathrm{NS}(S_2)$.

Proof. Since f is a succession of blowups at points, we may assume that f is the blowing-up at a point. Let Γ be the exceptional divisor. It is enough to prove the 'if' part. If the two conditions are satisfied, then $f_*D - 4L$ is numerically trivial for a divisor L, and $f^*(f_*D) - D = 4n\Gamma$ for some $n \in \mathbb{Z}$; hence, $D - 4(f^*L - n\Gamma)$ is numerically trivial. $\square$

Applying Lemma 6.11 to $\phi \circ \psi \colon \mathcal{Y} \to X$, we have:

Lemma 6.12. *$\delta = 0$ if and only if there exists a vector $\varepsilon = (\varepsilon_i) \in \{1, -1\}^k$ such that,*

(1) *$\varepsilon_i + \varepsilon_j = 0$ for $i \neq j$ if $E_{i,M} \cap E_{j,M} \neq \emptyset$,*

(2) *$1 + \varepsilon_i = 0$ if there is a (-1)-curve γ with $\gamma \cap E_{i,M} \neq \emptyset$,*

(3) *$\mathrm{cl}\left(\phi_*\left(L_M + \sum_{i=1}^k \varepsilon_i E_{i,M}\right)\right) \in 4\,\mathrm{NS}(X)$.*

Proof. An exceptional curve Γ for $\phi \circ \psi$ is either a ψ-exceptional curve or the proper transform of a ϕ-exceptional curve. In the former case, $B^{\varepsilon}_{\mathcal{Y}}\Gamma = \varepsilon_i + \varepsilon_j$ if $\psi(\Gamma) = E_{i,M} \cap E_{j,M}$. In the second case, if $\psi(\Gamma)$ is a (-2)-curve, then $L_M\psi(\Gamma) = E_M\psi(\Gamma) = 0$ and $B^{\varepsilon}_{\mathcal{Y}}\Gamma \in 4\mathbb{Z}$. If $\psi(\Gamma)$ is a (-1)-curve, then $L_M\psi(\Gamma) = E_M\psi(\Gamma) = 1$ and $B^{\varepsilon}_{\mathcal{Y}}\Gamma = 1 + \varepsilon_i$ for the unique irreducible component $E_{i,M}$ of E_M intersecting $\psi(\Gamma)$. Thus, we are done. $\square$

Corollary 6.13. *Suppose that $\mathrm{cl}(B^{\varepsilon}_{\mathcal{Y}}) \in 4\,\mathrm{NS}(\mathcal{Y})$ for a vector $\varepsilon \in \{1, -1\}^k$.*

(1) *If $E_{1,\mathcal{Y}}$ is the proper transform of an irreducible component E_1 of E with $\Delta \cap (E_1 \setminus \mathrm{Sing}\, E) \neq \emptyset$, then $\varepsilon_1 = -1$.*

(2) *Let E_1 and E_2 be irreducible components of E intersecting with each other at a point P such that $\mathrm{mult}_P(\Delta \cap E_1) = 1$ and $\mathrm{mult}_P(\Delta \cap E_2) = b$. Let $E_{i,\mathcal{Y}}$ be the proper transform of E_i in $\mathcal{Y}$ for $i = 1, 2$. Then $\varepsilon_1 = (-1)^{b+1}$ and $\varepsilon_2 = 1$.*

Proof. (1): By Lemma 2.10, there is a (-1)-curve $\Gamma_k \not\subset E_M$ such that $\Gamma_k E_M = \Gamma_k E_{1,M} = 1$ and $\Gamma_k \cap E_{1,M}$ is a non-singular point of E_M. Thus $B^{\varepsilon}_{\mathcal{Y}}\psi^*(\Gamma_k) = L_M\Gamma_k + \varepsilon_1 = 1 + \varepsilon_1 \in 4\mathbb{Z}$.

(2): By Lemma 2.14, there is a straight chain $\sum_{j=1}^{b+1}\Gamma_j$ of non-singular rational curves on M such that

- $E_{1,M}+\sum_{j=1}^{b}\Gamma_j+E_{2,M}$ is a straight chain of rational curves contained in E_M,
- the end Γ_{b+1} is a (-1)-curve with $\Gamma_{b+1}\cap E_M=\Gamma_b\cap\Gamma_{b+1}$.

Let $\Gamma_{j,\mathcal{Y}}$ be the proper transform of Γ_j in $\mathcal{Y}$ and let $\varepsilon[j]$ be the coefficient of ε at $\Gamma_{j,\mathcal{Y}}$ for $1\le j\le b$. Then $B^{\varepsilon}_{\mathcal{Y}}\Gamma_{b+1,\mathcal{Y}}=\varepsilon[b]+1\in 4\mathbb{Z}$. Thus $\varepsilon[b]=-1$. By (1) of Lemma 6.12, we have $\varepsilon[j]=(-1)^{b+1-j}$ for $1\le j\le b$, $\varepsilon_1=(-1)^{b+1}$, and $\varepsilon_2=1$. $\square$

Proposition 6.14.

(1) *Suppose that E is irreducible. Then $\delta=1$ except for the types $[1]_0$ and $[4;1,0]_0$.*

(2) *Suppose that E is non-singular and reducible. Then the type is $[4;2,4]_{00}$ and $\delta=0$.*

(3) *Suppose that E is reducible and singular, and has no nodes P with $P\in\Delta$. Then $\delta=1$ except for the types $[2;1,2]_{++}$ and $[3;1,1]_+$.*

(4) *Suppose that E has exactly one node P and that* $\mathrm{mult}_P(\Delta\cap E_1)=1$, $\mathrm{mult}_P(\Delta\cap E_2)=b$ *for the irreducible components E_1, E_2 of E. Then $\delta=1$ except for the types $[2]_+(4)$, $[1;1,1]_+(2,1)$, and $[2;1,1]_+(1,3)$.*

(5) *Suppose that E has two nodes P and P' and that* $\mathrm{mult}_P(\Delta\cap E_1)=1$, $\mathrm{mult}_P(\Delta\cap E_2)=b$ *for the irreducible components E_1, E_2 of E. Then $\delta=1$ except for the types $[3;2,4]_{++}(2,1)$ and $[3;2,4]_{++}(1,6)$.*

Proof. (1): If $\Delta=\emptyset$, then (X,E,Δ) is of type $[4;1,0]_0$. In this case, $X=M$ and $L+E\sim 4(\sigma+3\ell)$. Hence, $\delta=0$. Suppose that $\Delta\ne\emptyset$. Then there is a (-1)-curve $\gamma\subset M$ contracted by $\phi\colon M\to X$. By Lemma 6.12, Corollary 6.13, and by $L_M-E_M\sim -2\phi^*(K_X+E)$, we infer that $\delta=0$ if and only if $\mathrm{cl}(K_X+E)\in 2\,\mathrm{NS}(X)$. Here, $\mathrm{cl}(K_X+E)\notin 2\,\mathrm{NS}(X)$ except for the type $[1]_0$.

(2) follows from $L_M-E_M\sim -2\phi^*(K_X+E)\sim 4\phi^*(\ell)$.

(3): Let E_1, E_2 be irreducible components of E with $E_1\cap E_2\ne 0$. Let $E_{i,\mathcal{Y}}$ be the proper transform of E_i in $\mathcal{Y}$ for $i=1$, 2. Suppose that $\mathrm{cl}(B^{\varepsilon}_{\mathcal{Y}})\in 4\,\mathrm{NS}(\mathcal{Y})$ for some ε. If $\deg(\Delta\cap E_1)>0$ and $\deg(\Delta\cap E_2)>0$, then

$\varepsilon_1 = \varepsilon_2 = -1$ by Corollary 6.13. But it contradicts Lemma 6.12. Hence, it is enough to consider the types $[2; 1, 2]_{++}$, $[3; 2, 4]_{+}$, and $[3; 1, 1]_{+}$.

Case $[2; 1, 2]_{++}$. $E = \ell_1 + \ell_2 + \sigma$ for two fibers ℓ_1, ℓ_2 of π and for the negative section σ. Then $L - \ell_1 - \ell_2 + \sigma \sim 4(\sigma + \ell)$. Here,

$$\phi^*\psi^*(L - \ell_1 - \ell_2 + \sigma) = \psi^* L_M - \ell_{1,\mathcal{Y}} - \ell_{2,\mathcal{Y}} + \sigma_{\mathcal{Y}} = B^{\varepsilon}_{\mathcal{Y}}$$

for a suitable $\varepsilon \in \{1, -1\}^k$, where $\ell_{1,\mathcal{Y}}$, $\ell_{2,\mathcal{Y}}$, and $\sigma_{\mathcal{Y}}$ are the proper transforms in $\mathcal{Y}$. Thus $\delta = 0$.

Case $[3; 2, 4]_{+}$. $E = \sigma + D$ for the negative section σ and for a section $D \sim \sigma + 4\ell$. Let $\sigma_{\mathcal{Y}}$ and $D_{\mathcal{Y}}$ be the proper transforms in $\mathcal{Y}$. Then $\mathrm{cl}(B^{\varepsilon}_{\mathcal{Y}}) \in 4\,\mathrm{NS}(\mathcal{Y})$ implies that $B^{\varepsilon}_{\mathcal{Y}} = \psi^*(L_M) - D_{\mathcal{Y}} + \sigma_{\mathcal{Y}}$ and hence $\mathrm{cl}(L - D + \sigma) \in 4\,\mathrm{NS}(X)$ by Lemma 6.12. However, $\mathrm{cl}(L - D + \sigma) = \mathrm{cl}(2\sigma + 2\ell) \not\in 4\,\mathrm{NS}(X)$. Hence, $\delta = 1$.

Case $[3; 1, 1]_{+}$. $E = \sigma + \ell$ for a fiber ℓ of π and for the negative section σ. Then $L - \ell + \sigma \sim 4(\sigma + 2\ell)$. Here,

$$\phi^*\psi^*(L - \ell + \sigma) = \psi^* L_M - \ell_{\mathcal{Y}} + \sigma_{\mathcal{Y}} = B^{\varepsilon}_{\mathcal{Y}}$$

for a suitable $\varepsilon \in \{1, -1\}^k$, where $\ell_{\mathcal{Y}}$ and $\sigma_{\mathcal{Y}}$ are the proper transforms in $\mathcal{Y}$. Thus $\delta = 0$.

(4): Suppose that $B^{\varepsilon}_{\mathcal{Y}} \in 4\,\mathrm{NS}(\mathcal{Y})$ and let ε_i be the coefficient of ε at the proper transform $E_{i,\mathcal{Y}}$ of E_i for $i = 1$, 2. Then $\varepsilon_2 = 1$ by Corollary 6.13. Thus $\deg(\Delta \cap E_2) = b$ also by Corollary 6.13. If $\deg(\Delta \cap E_1) > 1$, then b is even since $\varepsilon_1 = -1 = (-1)^{b+1}$ by Corollary 6.13.

Suppose that $\deg(\Delta \cap E_1) = 1$ and $\deg(\Delta \cap E_2) = b$. Then the type is $[2; 1, 1]_{+}(1, 3)$. Here, E_1 is the negative section σ, E_2 is a fiber of π, and $L \sim -2K_X - E \sim 3\sigma + 7\ell$. Then $\mathrm{cl}(L + E_1 + E_2) \in 4\,\mathrm{NS}(X)$ by $L + E_1 + E_2 \sim 4\sigma + 8\ell$. Thus

$$\begin{aligned}\mathrm{cl}(\psi^*\phi^*(L + E_1 + E_2)) - \mathrm{cl}(L_{\mathcal{Y}} + E_{1,\mathcal{Y}} + E_{2,\mathcal{Y}} - \Gamma_{1,\mathcal{Y}} + \Gamma_{2,\mathcal{Y}} - \Gamma_{3,\mathcal{Y}}) \\ \in 4\,\mathrm{NS}(\mathcal{Y})\end{aligned}$$

for the curves $\Gamma_{j,\mathcal{Y}}$ in the proof of Corollary 6.13, (2). Hence, $\delta = 0$.

Suppose that $\deg(\Delta \cap E_1) > 1$ and $\deg(\Delta \cap E_2) = b$. Then b is even and the following types remain: $[2]_{+}(4)$, $[0; 1, 1]_{+}(3)$, $[1; 1, 1]_{+}(2, 1)$, $[1; 1, 1]_{+}(1, 3)$. We can write

$$B^{\varepsilon}_{\mathcal{Y}} = \psi^*(L_M) - E_{1,\mathcal{Y}} + E_{2,\mathcal{Y}} + \sum\nolimits_{j=1}^{b} (-1)^{b+1-j}\Gamma_{j,\mathcal{Y}}$$

for the curves $\Gamma_{j,\mathcal{Y}}$ is the proof of Corollary 6.13, (2). Thus $\delta = 0$ if and only if $\mathrm{cl}(L - E_1 + E_2) \in 4\,\mathrm{NS}(X)$.

Case $[2]_{+}(4)$. E_1 and E_2 are lines of $\mathbb{P}^2$. Here $\deg L = \deg(L - E_1 + E_2) = 4$. Hence $\delta = 0$.

Case $[0; 1, 1]_{+}(3)$. We may assume that E_1 is a minimal section σ and E_2 is a fiber. Here $L \sim 3\sigma + 3\ell$ and $L - E_1 + E_2 \sim 2\sigma + 4\ell$. Hence $\delta = 1$.

Case $[1; 1, 1]_{+}(2, 1)$. E_1 is a fiber ℓ of π and E_2 is the negative section σ. Here, $L \sim 3\sigma + 5\ell$ and $L - E_1 + E_2 \sim 4\sigma + 4\ell$. Hence $\delta = 0$.

Case $[1; 1, 1]_{+}(1, 3)$. E_1 is the negative section σ and E_2 is a fiber ℓ of π. Here $L \sim 3\sigma + 5\ell$ and $L - E_1 + E_2 \sim 2\sigma + 6\ell$. Hence $\delta = 1$.

(5): The types in this case are $[3; 2, 4]_{++}(a, b)$. Here, $E = \sigma + \ell + \sigma_\infty$ for the negative section σ, a fiber ℓ, and a section σ_∞ at infinity, and furthermore $P = \ell \cap \sigma_\infty$. If $\delta = 0$, then $(a, b) = (2, 1)$ or $(1, 6)$ by the same argument as in the proof of (4) above.

Case $(a, b) = (2, 1)$. Then $E_1 = \sigma_\infty$ and $E_2 = \ell$. We set $E_3 = \sigma$. As in the proof of (4), we infer that $\delta = 0$ if and only if $\mathrm{cl}(L - E_1 + E_2 - E_3) \in 4\,\mathrm{NS}(X)$. Now $L \sim 2\sigma + 6\ell$ and $L - E_1 + E_2 - E_3 \sim 4\ell$. Hence $\delta = 0$.

Case $(a, b) = (1, 6)$. Then $E_1 = \ell$ and $E_2 = \sigma_\infty$. We set $E_3 = \sigma$. As in the proof of (4), we infer that $\delta = 0$ if and only if $\mathrm{cl}(L - E_1 + E_2 + E_3) \in 4\,\mathrm{NS}(X)$. Now $L - E_1 + E_2 + E_3 \sim 4\sigma + 8\ell$. Hence $\delta = 0$. □

As a result, the invariant δ depends only on the type T of (X, E, Δ) and is calculated as in TABLE 6.

LEMMA 6.15. *For a log del Pezzo surface S of index two, the deformation type of the right resolution $\mathcal{Y}$ depends only on the equi-singular deformation type of the basic pair (M, E_M), and vice versa. The invariant δ depends only on the equi-singular deformation type of the basic pair.*

PROOF. Let $h\colon (\widetilde{M}, \widetilde{E}_M) \to T$ be an equi-singular family of basic pairs over a connected non-singular curve T whose fibers define log del Pezzo surfaces of index two. Then there exist a family $f\colon \widetilde{S} \to T$ of log del Pezzo surfaces of index two and a birational morphism $\tilde{\alpha}\colon \widetilde{M} \to \widetilde{S}$ over T by Lemma 5.2. Let $\tilde{\psi}\colon \widetilde{\mathcal{Y}} \to \widetilde{M}$ be the blowing up along the double locus $\bigcup(\widetilde{E}_i \cap \widetilde{E}_j)$ of $\widetilde{E} = \sum \widetilde{E}_i$. Then the induced smooth family $\varpi\colon \widetilde{\mathcal{Y}} \to T$ is a simultaneous right resolution of f. Thus, if two such basic pairs are equi-singular deformation equivalent, then the associated right resolutions are deformation equivalent, and they have the same δ by Proposition 6.10. Conversely, if two basic pairs have the same invariants g, k, δ, then by TABLE 6, we infer that either they have the same type or they are of types $[0;1,1]_0$ and $[2;1,2]_0$. In both cases, the basic pairs are equi-singular deformation equivalent by results in Section 5.2 and by Proposition 5.10, (1). □

6.4. The singular points of S

We consider the singular points on S. A connected component of the exceptional locus for $\alpha\colon M \to S$ is written as $\alpha^{-1}(Q)$ for a singular point Q of S. If $\alpha^{-1}(Q) \subset E_M$, then $Q \in S$ is a singular point of type K_n. If $\alpha^{-1}(Q) \not\subset E_M$, then $Q \in S$ is a rational double point, and an irreducible component of $\alpha^{-1}(Q)$ is one of following (-2)-curves by Proposition 6.2:

- A ϕ-exceptional (-2)-curve such that $\phi(\gamma)$ is a non-singular point of E;
- The proper transform of the negative section σ when the type is $[2;1,2]_0$;
- The proper transform of a fiber ℓ of π with $\ell \cap E \subset \Delta$ when the type is $[1;2,2]_0$.

LEMMA 6.16.

(1) *If the type is not* $[4;2,4]_{00}$, *then S has a unique non-Gorenstein singular point, which is of type K_k for the number k of irreducible components of E_M. If the type is* $[4;2,4]_{00}$, *then S has two singular points, which are of type K_1.*

(2) *Suppose that the type is neither* $[1;2,2]_0$ *nor* $[2;1,2]_0$. *Then a rational double point $Q \in S$ is of type A_{l-1} where $\alpha^{-1}(Q)$ is the maximal straight chain of (-2)-curves in $\phi^{-1}(P)$ for a non-singular point P of E with* $\mathrm{mult}_P(\Delta) = l \geq 2$. *In particular, $l \leq \deg\Delta$.*

(3) *Suppose that the type is* $[2;1,2]_0$. *Then the total transform of the negative section σ in M is a (-2)-curve defining an A_1-singularity on S. The other rational double points $Q \in S$ are of type A_{l-1}, where $\alpha^{-1}(Q)$ is the maximal straight chain of (-2)-curves in $\phi^{-1}(P)$ for a point $P \in E$ with* $\mathrm{mult}_P(\Delta) = l \geq 2$.

(4) *Suppose that the type is* $[1;2,2]_0$ *and that* $\pi|_E\colon E \to \mathbb{P}^1$ *is separable. Then a rational double point* $Q \in S$ *is of type* A_l *for* $1 \le l \le 7$ *or of type* D_l *for* $4 \le l \le 8$*.*

(5) *Suppose that the type is* $[1;2,2]_0$ *and that* $\pi|_E\colon E \to \mathbb{P}^1$ *is inseparable. Then a rational double point* $Q \in S$ *is of type* A_l *for* $l \in \{1,3\}$ *or of type* D_l *for* $4 \le l \le 8$*.*

PROOF. (1): E_M is connected if and only if the type is not $[4;2,4]_{00}$. If the type is $[4;2,4]_{00}$, then E_M is a disjoint union of two (-4)-curves. Thus (1) follows.

(2) and (3): If the type is neither $[1;2,2]_0$ nor $[2;1,2]_0$, then any (-2)-curve is contained in $\phi^{-1}(P)$ for a non-singular point P of E with $\mathrm{mult}_P(\Delta) \ge 2$. If the type is $[2;1,2]_0$, then there is one more (-2)-curve which is the total transform of σ.

(4) and (5): Any (-2)-curve is contained in a fiber of $M \to \mathbb{P}^1$. Thus the assertion follows from Lemmas 5.13 and 4.11. □

Let $\Gamma = \Gamma[M] = \Gamma(S) = \Gamma(X,E,\Delta)$ be the dual graph of the negative curves on M. The part Γ_{K} is defined to be the subgraph consisting of the irreducible components of E_M. Another part Γ_{RDP} is defined to be the subgraph consisting of the (-2)-curves not contained in E_M. Then a connected component of Γ_{K} corresponds to a non-Gorenstein point on S, and a connected component of Γ_{RDP} corresponds to a rational double point on S. Thus $\Gamma_{\mathsf{K}} \sqcup \Gamma_{\mathrm{RDP}}$ is the dual graph of the minimal resolution of singularities of S. By Lemma 6.16, (1), if S is not of type $[4;2,4]_{00}$, then $\Gamma_{\mathsf{K}} = \mathsf{K}_k$ for $k = k_{\mathsf{T}}$; If S is of type $[4;2,4]_{00}$, then Γ_{K} is the disjoint union of two K_1. Thus Γ_{K} depends on the type T of S.

Let $a(i)$ be the number of singular points on S of type A_i for $i \ge 1$. Similarly, let $d(i)$ be the number of singular points of type D_i for $i \ge 4$. The formal linear combination

$$\mathcal{D}(S) = \mathcal{D}(X,E,\Delta) = \sum a(i)\mathsf{A}_i + \sum d(j)\mathsf{D}_j$$

of Dynkin diagrams is called the *distribution* (of rational double points) of S. Then $\Gamma(S)_{\mathrm{RDP}}$ is identified with $\mathcal{D}(S)$. We define $\sigma(S) = \sigma(X,E,\Delta) = \sum ia(i) + \sum jd(j)$. Note that $\sigma(S)$ is not determined by the type T, in general.

The birational morphism $\alpha\colon M \to S$ contracts $k_{\mathsf{T}}+\sigma(S)$ rational curves. Hence, the Picard number $\rho(S)$ equals $\rho(M) - k_{\mathsf{T}} - \sigma(S)$, since S is $\mathbb{Q}$-factorial. Therefore,

$$\begin{aligned}\rho(S) &= 10 - (K_X + E)^2 - k_{\mathsf{T}} - \sigma(S)\\ &= \begin{cases} 12 - g_{\mathsf{T}} - k_{\mathsf{T}} - \sigma(S), & \text{if the type is not } [4;2,4]_{00};\\ 8 - \sigma(S), & \text{if the type is } [4;2,4]_{00}.\end{cases}\end{aligned}$$

DEFINITION 6.17. For a type T of fundamental triplet, we define $\sigma_{\mathsf{T}}^{\max}$ (resp. $\sigma_{\mathsf{T}}^{\min}$) to be the maximum (resp. the minimum) of $\sigma(S)$ for the log del Pezzo surfaces S of index two of type T. For a log del Pezzo surface S of type T, if $\sigma(S) = \sigma_{\mathsf{T}}^{\max}$, then S is called *extremal*. If $\sigma(S) = \sigma_{\mathsf{T}}^{\min}$, then S is called *generic*. A fundamental triplet (X, E, Δ) is called extremal (resp. generic) if the associated log del Pezzo surface S is so. We also define $\rho_{\mathsf{T}}^{\min}$ (resp. $\rho_{\mathsf{T}}^{\max}$) to be the minimum (resp. the maximum) of $\rho(S)$ for the log del Pezzo surfaces S of index two of type T.

REMARK. The notion of extremal in Definition 6.17 is slightly different from that used in [4]; this is related to the equi-singular deformation equivalence between types $[0;1,1]_0$ and $[2;1,2]_0$ in Theorem 6.1.

By Lemma 6.16, (X, E, Δ) is generic if and only if

- Δ is reduced on $E \setminus \operatorname{Sing} E$ when $\mathsf{T} \neq [1;2,2]_0$, and
- Δ is reduced and $\deg(\Delta \cap \ell) \leq 1$ for any fiber ℓ of π when $\mathsf{T} = [1;2,2]_0$.

In particular, $\sigma_{\mathsf{T}}^{\min} = 0$ for any T. Thus $\sigma_{\mathsf{T}}^{\max} = \rho_{\mathsf{T}}^{\max} - \rho_{\mathsf{T}}^{\min}$. If $\mathsf{T} \neq [4;2,4]_{00}$, then $\rho_{\mathsf{T}}^{\min} = 12 - g_{\mathsf{T}} - k_{\mathsf{T}} - \sigma_{\mathsf{T}}^{\max}$ and $\rho_{\mathsf{T}}^{\max} = 12 - g_{\mathsf{T}} - k_{\mathsf{T}}$. If $\mathsf{T} = [4;2,4]_{00}$, then $\rho_{\mathsf{T}}^{\min} = 8 - \sigma_{\mathsf{T}}^{\max}$ and $\rho_{\mathsf{T}}^{\max} = 8$. The numbers $\rho_{\mathsf{T}}^{\max}$ and $\rho_{\mathsf{T}}^{\min}$ are calculated as in TABLE 7, by:

PROPOSITION 6.18.

(1) *Suppose that (X, E, Δ) is not of type $[1;2,2]_0$. Then (X, E, Δ) is extremal if and only if any irreducible component of $E \setminus \operatorname{Sing} E$ has at most one point contained in Δ.*

TABLE 7. The maximum and minimum Picard numbers

Type T	$\rho_{\mathsf{T}}^{\max}$	$\rho_{\mathsf{T}}^{\min}$	Type T	$\rho_{\mathsf{T}}^{\max}$	$\rho_{\mathsf{T}}^{\min}$	Type T	$\rho_{\mathsf{T}}^{\max}$	$\rho_{\mathsf{T}}^{\min}$
$[1]_0$	5	1	$[1;1,1]_0$	6	2	$[3;1,0]_0$	2	2
$[2]_0$	8	1	$[1;1,1]_+(0,0)$	5	2	$[3;1,1]_+$	3	1
$[2]_+(0)$	7	1	$[1;1,1]_+(1,1)$	4	3	$[3;2,4]_+$	8	1
$[2]_+(1)$	6	2	$[1;1,1]_+(2,1)$	3	2	$[3;2,4]_{++}(0,0)$	7	1
$[2]_+(2)$	5	2	$[1;1,1]_+(1,2)$	3	3	$[3;2,4]_{++}(1,1)$	6	2
$[2]_+(3)$	4	2	$[1;1,1]_+(1,3)$	2	2	$[3;2,4]_{++}(2,1)$	5	1
$[2]_+(4)$	3	1	$[1;2,2]_0$	9	1	$[3;2,4]_{++}(1,2)$	5	2
$[0;1,0]_0$	5	2	$[2;1,0]_0$	3	2	$[3;2,4]_{++}(1,3)$	4	2
$[0;1,1]_0$	7	2	$[2;1,1]_+(0,0)$	4	2	$[3;2,4]_{++}(1,4)$	3	2
$[0;1,1]_+(0)$	6	2	$[2;1,1]_+(1,1)$	3	2	$[3;2,4]_{++}(1,5)$	2	2
$[0;1,1]_+(1)$	5	3	$[2;1,1]_+(1,2)$	2	2	$[3;2,4]_{++}(1,6)$	1	1
$[0;1,1]_+(2)$	4	3	$[2;1,1]_+(1,3)$	1	1	$[4;1,0]_0$	1	1
$[0;1,1]_+(3)$	3	2	$[2;1,2]_0$	7	1	$[4;2,4]_{00}$	8	1
$[1;1,0]_0$	4	2	$[2;1,2]_{++}$	5	1			

(2) *Suppose that* $\operatorname{char}\Bbbk \neq 2$ *and that* (X,E,Δ) *is of type* $[1;2,2]_0$. *Then* (X,E,Δ) *is extremal if and only if* $\Delta = n_1P_1+n_2P_2$ *for the ramification points* P_1, P_2 *of* $\pi|_E\colon E \to \mathbb{P}^1$ *where* $(\max\{n_1,n_2\}, \min\{n_1,n_2\}) = (8,0)$, $(6,2)$, $(5,3)$, *or* $(4,4)$.

(3) *Suppose that* $\operatorname{char}\Bbbk = 2$, (X,E,Δ) *is of type* $[1;2,2]_0$, *and that* $\pi|_E\colon E \to \mathbb{P}^1$ *is separable. Then* (X,E,Δ) *is extremal if and only if* $\Delta = 8P$ *for the unique ramification points* P *of* $\pi|_E$.

(4) *Suppose that* $\operatorname{char}\Bbbk = 2$, (X,E,Δ) *is of type* $[1;2,2]_0$, *and that* $\pi|_E\colon E \to \mathbb{P}^1$ *is inseparable. Then* (X,E,Δ) *is extremal if and only if* $\operatorname{mult}_P(\Delta) \geq 2$ *for any point* $P \in \Delta$.

PROOF. (1): Suppose that $\operatorname{Supp}\Delta \cap (E_i \setminus \operatorname{Sing} E)$ contains two points P_1, P_2 for an irreducible component $E_i \subset E$. We set $m_i = \operatorname{mult}_{P_i}(\Delta)$ for $i = 1$, 2 and set $\Delta' = \Delta + m_2(P_1 - P_2)$ which is an effective Cartier divisor of E. Then $\operatorname{mult}_{P_1}(\Delta') = m_1 + m_2$ and $P_2 \notin \Delta'$. Since the Dynkin diagram $\mathsf{A}_{m_1+m_2-1}$ contains the disjoint union of A_{m_1-1} and A_{m_2-1}, $\mathit{\Gamma}(X,E,\Delta)_{\mathrm{RDP}}$ is regarded as a subgraph of $\mathit{\Gamma}(X,E,\Delta')_{\mathrm{RDP}}$. In particular, (X,E,Δ) is not extremal.

Next suppose that $\operatorname{Supp}\Delta \cap (E_i \setminus \operatorname{Sing} E)$ consists of at most one point for any irreducible component $E_i \subset E$, then $\mathit{\Gamma}(X,E,\Delta)_{\mathrm{RDP}}$ is uniquely determined by Lemma 6.16. Thus (X, E, Δ) is extremal.

(2): We define

$$\Delta' = \Delta + \sum\nolimits_{P \in \Delta, P \neq P_1, P_2} \mathrm{mult}_P(\Delta)(P_1 - P).$$

Since $\mathsf{A}_{m-1} \subset \mathsf{D}_m$, $\mathit{\Gamma}(X, E, \Delta)_{\mathrm{RDP}}$ is a subgraph of $\mathit{\Gamma}(X, E, \Delta')_{\mathrm{RDP}}$ by Lemma 6.16. In particular, if $\mathrm{Supp}(\Delta) \not\subset \{P_1, P_2\}$, then (X, E, Δ) is not extremal.

Suppose that $\Delta = n_1 P_1 + n_2 P_2$ for $n_1 \geq n_2$. Then $n_1 + n_2 = 8$. Then $\mathcal{D}(X, E, \Delta)$ is calculated as follows:

n_2	0	1	2	3	4
$\mathcal{D}(X, E, \Delta)$	D_8	D_7	$\mathsf{D}_6 + 2\mathsf{A}_1$	$\mathsf{D}_5 + \mathsf{A}_3$	$2\mathsf{D}_4$

Since $\mathcal{D}_7 \subset \mathcal{D}_8$, the case $n_1 = 1$ is not extremal. The other cases are extremal.

(3) and (4) follow from a similar argument to (2) above and Lemma 6.16. $\square$

We define an *extremal distribution* of type T to be $\mathcal{D}(S)$ for an extremal log del Pezzo surface S of type T.

If $\mathsf{T} \neq [1; 2, 2]_0$, then an extremal distribution $\mathcal{D}_{\mathsf{T}}$ of type T is uniquely determined. In fact, for an extremal fundamental triplet (X, E, Δ) of type T, $\Delta \cap (E_i \setminus \mathrm{Sing}\, E)$ consists at most one point for any irreducible component $E_i \subset E$, and hence $\mathcal{D}_{\mathsf{T}}$ is the direct sum $\sum_{d_i \geq 2} \mathsf{A}_{d_i - 1}$ for the degree $d_i = \deg(\Delta \cap (E_i \setminus \mathrm{Sing}\, E))$, where the numbers d_i depend only on T.

The extremal distributions of type $[1; 2, 2]_0$ has been classified in Lemma 6.18, (2), (3), when $\pi|_E \colon E \subset X \to \mathbb{P}^1$ is separable. Let (X, E, Δ) be an extremal fundamental triplet of type $[1; 2, 2]_0$ such that $\pi|_E \colon E \to \mathbb{P}^1$ is inseparable. Then Δ can be written as a divisor $\sum_{i=1}^{l} m_i P_i$ of E for $m_i \geq 2$ with $\sum m_i = 8$. We may assume that $m_1 \geq m_2 \geq \cdots \geq m_l$. Then $(m_1, \ldots, m_l)$ is one of

$$(8), \quad (6, 2), \quad (5, 3), \quad (4, 4), \quad (4, 2, 2), \quad (3, 3, 2), \quad (2, 2, 2, 2).$$

Therefore, the extremal distributions are classified as in Table 8, where the case $[1; 2, 2]_0$ is treated in *).

TABLE 8. Extremal distributions

Type T	$\mathcal{D}_{\mathsf{T}}$	Type T	$\mathcal{D}_{\mathsf{T}}$	Type T	$\mathcal{D}_{\mathsf{T}}$
$[1]_0$	A_4	$[1;1,1]_0$	A_4	$[3;1,0]_0$	$\emptyset$
$[2]_0$	A_7	$[1;1,1]_+(0,0)$	$\mathsf{A}_2+\mathsf{A}_1$	$[3;1,1]_+$	A_2
$[2]_+(0)$	$2\mathsf{A}_3$	$[1;1,1]_+(1,1)$	A_1	$[3;2,4]_+$	A_7
$[2]_+(1)$	$2\mathsf{A}_2$	$[1;1,1]_+(2,1)$	A_1	$[3;2,4]_{++}(0,0)$	$\mathsf{A}_5+\mathsf{A}_1$
$[2]_+(2)$	$\mathsf{A}_2+\mathsf{A}_1$	$[1;1,1]_+(1,2)$	$\emptyset$	$[3;2,4]_{++}(1,1)$	A_4
$[2]_+(3)$	A_2	$[1;1,1]_+(1,3)$	$\emptyset$	$[3;2,4]_{++}(2,1)$	A_4
$[2]_+(4)$	A_2	$[1;2,2]_0$	see *) below	$[3;2,4]_{++}(1,2)$	A_3
$[0;1,0]_0$	A_3	$[2;1,0]_0$	A_2	$[3;2,4]_{++}(1,3)$	A_2
$[0;1,1]_0$	A_5	$[2;1,1]_+(0,0)$	A_2	$[3;2,4]_{++}(1,4)$	A_1
$[0;1,1]_+(0)$	$2\mathsf{A}_2$	$[2;1,1]_+(1,1)$	A_1	$[3;2,4]_{++}(1,5)$	$\emptyset$
$[0;1,1]_+(1)$	$2\mathsf{A}_1$	$[2;1,1]_+(1,2)$	$\emptyset$	$[3;2,4]_{++}(1,6)$	$\emptyset$
$[0;1,1]_+(2)$	A_1	$[2;1,1]_+(1,3)$	$\emptyset$	$[4;1,0]_0$	$\emptyset$
$[0;1,1]_+(3)$	A_1	$[2;1,2]_0$	$\mathsf{A}_5+\mathsf{A}_1$	$[4;2,4]_{00}$	A_7
$[1;1,0]_0$	A_2	$[2;1,2]_{++}$	$2\mathsf{A}_2$		

*) Extremal distributions of type $[1;2,2]_0$:

$\operatorname{char}\Bbbk \neq 2$	D_8, $\mathsf{D}_6+2\mathsf{A}_1$, $\mathsf{D}_5+\mathsf{A}_3$, $2\mathsf{D}_4$
$\operatorname{char}\Bbbk = 2$	D_8, $\mathsf{D}_6+2\mathsf{A}_1$, $\mathsf{D}_5+\mathsf{A}_3$, $2\mathsf{D}_4$, $\mathsf{D}_4+4\mathsf{A}_1$, $2\mathsf{A}_3+2\mathsf{A}_1$, $8\mathsf{A}_1$

COROLLARY 6.19. *The distribution $\mathcal{D}(S)$ of rational double points of a log del Pezzo surface S of type T is realized as a subdiagram of an extremal distribution of type T. Conversely, any subdiagram of an extremal distribution of type T is realized as $\mathcal{D}(S)$ for a log del Pezzo surface S of type T, provided that $\mathsf{T} \neq [2;1,2]_0$. An extremal distribution of type $[2;1,2]_0$ is $\mathsf{K}_1+\mathsf{A}_5+\mathsf{A}_1$ and any subdiagram containing the part $\mathsf{K}_1+\mathsf{A}_1$ is realized as $\mathcal{D}(S)$ for a log del Pezzo surface S of type $[2;1,2]_0$.*

PROOF. The first assertion follows from Proposition 6.18. A subdiagram of A_{m-1} is also a direct sum of A_{m_i-1} with $m \geq \sum m_i$. Similarly, a subdiagram of D_m is the sum of D_n and A_{m_j-1} with $m \geq n + \sum m_j$. If (X, E, Δ) is of type $[2;1,2]_0$, then $\mathcal{D}(X, E, \Delta)$ always contains A_1 which corresponds to the total transform of the negative section $\sigma \subset X$. Thus, we have the converse assertion. □

THEOREM 6.20. *For a given type T, an extremal fundamental triplet of type T is unique up to isomorphism if $\mathsf{T} \neq [1;2,2]_0$. In case $\mathsf{T} = [1;2,2]_0$,* *the isomorphism class of extremal fundamental triplet is determined by the extremal distribution $\mathcal{D}$ either if* $\operatorname{char}\Bbbk \neq 2$ *or if* $\mathcal{D} \notin \{\mathsf{D}_8, 8\mathsf{A}_1\}$.

PROOF. Suppose that the type T is not $[2]_0$, $[0;1,1]_0$, $[1;2,2]_0$, $[3;2,4]_+$, nor $[3;2,4]_{++}(a,b)$ with $(a,b) \neq (0,0)$. Then for two extremal fundamental triplets (X,E,Δ_1), (X,E,Δ_2) of type T, there exists an effective divisor E' such that $\Delta_1 \cap E' = \Delta_2 \cap E' = \emptyset$ and that $X \setminus (E+E') \subset X$ is a torus embedding. Since every irreducible component is an orbit of the torus, we have an automorphism f of X such that $f(E_i) = E_i$ for any irreducible component $E_i \subset E$ and $f(\Delta_1) = \Delta_2$ outside the nodes of E. Suppose that E has a node P contained in Δ_1. Then $P = E_1 \cap E_2$ and $E = E_1 + E_2$ for two irreducible components E_1 and E_2. We may assume the following properties to be satisfied:

- There is an effective divisor E' such that $\operatorname{Supp}(\Delta_1) \setminus P \subset E'$, $\operatorname{Supp}(\Delta_2) \setminus P \subset E'$, and $X \setminus (E+E') \subset X$ is a torus embedding.
- $\operatorname{mult}_P(\Delta_1 \cap E_1) = \operatorname{mult}_P(\Delta_2 \cap E_1) = 1$ and $\operatorname{mult}_P(\Delta_1 \cap E_2) = \operatorname{mult}_P(\Delta_2 \cap E_2) = b$.

Let $\phi^\sharp \colon M^\sharp \to X$ be the elimination of $\Delta_1 \cap (E_2 \setminus E') = \Delta_2 \cap (E_2 \setminus E')$. Then $\phi^\sharp$ is a toric blowing-up defined by a subdivision of the fan corresponding to $X \setminus (E+E') \subset X$. The weak transform of Δ_i is supported on a non-singular point P_i of an exceptional curve $\Gamma \subset (\phi^\sharp)^{-1}(P)$ and on nodes of $(\phi^\sharp)^{-1}(E+E')$ for $i = 1$, 2. The open torus acts transitively on $\Gamma \setminus \operatorname{Sing}(\phi^\sharp)^{-1}(E+E')$. Therefore, we have an automorphism f of X with $f(\Delta_1) = \Delta_2$.

Next, we consider the exceptional types.

Case $[2]_0$. $E \simeq \mathbb{P}^1 \subset X \simeq \mathbb{P}^2$ is considered as the Veronese embedding by $|\mathcal{O}(2)|$. Thus an automorphism of E lifts to an automorphism of X. An extremal fundamental triplet (X,E,Δ) is determined by a point $P \in E$ by $\Delta = 8P$. Thus the isomorphism class of the extremal fundamental triplet is unique.

Case $[0;1,1]_0$. We may assume that E is the diagonal locus of $X = \mathbb{P}^1 \times \mathbb{P}^1$. Thus an automorphism of E lifts to an automorphism of X. Thus the isomorphism class of extremal fundamental triplet is unique by the same reason above.

Case $[1;2,2]_0$. The extremal distributions are classified as in *) of TABLE 8 by Proposition 6.18, (2)–(4). For an extremal fundamental triplet

(X, E, Δ), if $\operatorname{char} \Bbbk \neq 2$, then Δ is supported on the two ramification points of $\pi|_E \colon E \to \mathbb{P}^1$. If $\operatorname{char} \Bbbk = 2$ and $\mathcal{D}(X, E, \Delta) \notin \{\mathcal{D}_8, 8\mathsf{A}_1\}$, then $\pi|_E \colon E \to \mathbb{P}^1$ is inseparable and Δ is supported on at most three points. Thus the isomorphism class of the extremal fundamental triplet (X, E, Δ) is determined by the distribution.

Case $[3; 2, 4]_+$. $E = \sigma + D$ for a section $D \sim \sigma + 4\ell$ and an extremal fundamental triplet (X, E, Δ) is given by $\Delta = 8P$ for a point $P \in D \setminus \sigma$. For given two points P_1, $P_2 \in D \setminus \sigma$, we take another point $Q \in D \setminus (\sigma \cup \{P_1, P_2\})$ and consider the elementary transformation at Q: $X \dashrightarrow X_2 \simeq \mathbb{F}_2$. Let $Q_2 \in X_2$ be the intersection point of the proper transform $D_2 \subset X_2$ of D and the fiber over $\pi(Q)$ and let $X_2 \dashrightarrow X_1 \simeq \mathbb{F}_1$ be the elementary transformation at Q_2. Let $Q_1 \in X_1$ be the intersection point of the proper transform $D_1 \subset X_1$ of D and the fiber over $\pi(P)$ and let $X_1 \dashrightarrow X_0 \simeq \mathbb{F}_0$ be the elementary transformation at Q_1. Let $\sigma_0 \subset X_0$ be the proper transform of σ and let $Q_0 \in X_0$ be the intersection point of the proper transform $D_0 \subset X_0$ of D and the fiber over $\pi(P)$. Note that D_0 is regarded as the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$. There is an automorphism φ of D_0 such that $\varphi(D_0 \cap \sigma_0) = D_0 \cap \sigma_0$, $\varphi(Q_0) = Q_0$, and $\varphi(f(P_1)) = \varphi(f(P_2))$ for the rational map $f \colon X \dashrightarrow X_0$. Then φ lists to an automorphism $\widetilde{\varphi}$ of X_0 which preserves the section σ_0, the fiber over $\pi(P)$, and D_0. Hence $\widetilde{\varphi}$ induces an automorphism $\widehat{\varphi}$ of X such that $\widehat{\varphi}(D) = D$, $\widehat{\varphi}(\sigma) = \sigma$, and $\widehat{\varphi}(P_1) = P_2$. Hence, the isomorphism class of extremal fundamental triplet is unique.

Case $[3; 2, 4]_{++}(a, b)$ with $(a, b) \neq (0, 0)$. $E = \sigma + \ell + \sigma_\infty$ for a fiber ℓ and a section σ_∞ at infinity. Let P be the point $\sigma_\infty \cap \ell$. Let Δ_1 and Δ_2 be effective Cartier divisors of E giving extremal fundamental triplet of this type. By the argument above, we may assume that $\operatorname{Supp}(\Delta_1 \cap \sigma_\infty) = \operatorname{Supp}(\Delta_2 \cap \sigma_\infty) = \{P\} \cup (\sigma_\infty \cap \ell')$ for another fiber ℓ' and that $\operatorname{Supp}(\Delta_1 \cap \ell) = \operatorname{Supp}(\Delta_2 \cap \ell)$. Let $\phi^\sharp \colon M^\sharp \to X$ be the elimination of $\Delta_1 \cap \sigma_\infty$ in case $\operatorname{mult}_P(\Delta_1 \cap \ell) = 1$, and the the elimination of $\Delta_1 \cap \ell$ in case $\operatorname{mult}_P(\Delta_1 \cap \sigma_\infty) = 1$. Then the weak transform $\Delta_i^\sharp$ for $i = 1$, 2 is supported on a non-singular point P_i of a $\phi^\sharp$-exceptional curve Γ, on a point $Q \in \ell \setminus \{P\}$, and on the inverse image of the intersection point $\sigma_\infty \cap \ell'$. Since Γ and the proper transform of ℓ are two irreducible component of the boundary of the torus imbedding into $M^\sharp$, an element of the open torus acts trivially on the proper transform of ℓ and moves $P_1^\sharp$ to $P_2^\sharp$. Thus $f(\Delta_1) = \Delta_2$ for an automorphism f of X. Hence, the isomorphism class of extremal fundamental triplet is unique. □

Remark 6.21. In case $\operatorname{char} \Bbbk = 2$, the isomorphism class of an extremal fundamental triplet of type $[1; 2, 2]_0$ with the extremal distribution $\mathcal{D}$ is not unique if $\mathcal{D} = \mathsf{D}_8$ or $8\mathsf{A}_1$. In fact, if $\mathcal{D} = \mathsf{D}_8$, then there are two fundamental triplets $(X, E, 8P)$ and $(X, E', 8P')$ for $X = \mathbb{F}_1$ such that

- $\pi|_E \colon E \to \mathbb{P}^1$ is separable and P is the unique ramification point of $\pi|_E$,
- $\pi|_E \colon E \to \mathbb{P}^1$ is inseparable and P is any point of E.

If (X, E, Δ) is an extremal fundamental triplet with the distribution $\mathcal{D} = 8\mathsf{A}_1$, then $\pi|_E \colon E \to \mathbb{P}^1$ is inseparable and $\operatorname{Supp} \Delta$ consists of four points. Thus Δ is not unique up to isomorphism of E. Moreover, there are infinitely many isomorphism classes of (X, E, Δ) with $\mathcal{D}(X, E, \Delta) = 8\mathsf{A}_1$; This fact was pointed out by Ohashi.

Corollary 6.22 (cf. [4], [20]). *There is a one-to-one correspondence between the set of isomorphism classes of log del Pezzo surfaces of index two with Picard number one and the set of isomorphism classes of extremal fundamental triplets of the following types*:

$$[1]_0,\ [2]_0,\ [2]_+(0),\ [2]_+(4),\ [1;2,2]_0,\ [2;1,1]_+(1,3),\ [2;1,2]_0,\ [2;1,2]_{++},$$
$$[3;1,1]_+,\ [3;2,4]_+,\ [3;2,4]_+(0,0),\ [3;2,4]_{++}(2,1),\ [3;2,4]_{++}(1,6),$$
$$[4;1,0]_0,\ [4;2,4]_{00}.$$

In particular, if $\operatorname{char} \Bbbk \neq 2$*, then there are exactly* 18 *isomorphism classes of log del Pezzo surfaces of index two with Picard number one, in which* 4 *isomorphism classes are of type* $[1; 2, 2]_0$*. If* $\operatorname{char} \Bbbk = 2$*, then there are exactly* 14 *isomorphism classes of log del Pezzo surfaces of index two with Picard number one not of type* $[1; 2, 2]_0$*, and there are infinitely many isomorphism classes of log del Pezzo surfaces of index two with Picard number one of type* $[1; 2, 2]_0$.

6.5. Dual graph of the negative curves

We consider the dual graph $\varGamma = \varGamma(S) = \varGamma[M]$ of negative curves on M. The proper transform of an irreducible component E_j of E in M is represented by a vertex in $\mathit{\Gamma}_{\mathsf{K}}$. Thus we have a natural injection $\nu\colon \mathcal{J}(E) \to \mathrm{Ver}(\mathit{\Gamma}_{\mathsf{K}})$ from the set $\mathcal{J}(E)$ of irreducible components of E to the set $\mathrm{Ver}(\mathit{\Gamma}_{\mathsf{K}})$ of vertices of $\mathit{\Gamma}_{\mathsf{K}}$.

Let $\mathcal{C}(\mathit{\Gamma}_{\mathrm{RDP}})$ be the set of connected components of $\mathit{\Gamma}_{\mathrm{RDP}}$. Let $\mathcal{C}(\mathsf{A}_m)$ and $\mathcal{C}(\mathsf{D}_n)$ be the sets of connected components of $\mathit{\Gamma}_{\mathrm{RDP}}$ which are Dynkin diagrams of types A_m and D_n, respectively.

Let $\mathcal{V}$ be the subset of white vertices joined to $\mathit{\Gamma}_{\mathsf{K}}$. A vertex $v \in \mathcal{V}$ represents a (-1)-curve γ on M with $E_M \cap \gamma \neq \emptyset$, equivalently a (-1)-curve belonging to the case (5) or (6) of Proposition 6.2.

Let $\mathit{\Gamma}^\flat$ be the subgraph of $\mathit{\Gamma}$ consisting of vertices of $\mathcal{V} \cup \mathit{\Gamma}_{\mathsf{K}} \cup \mathit{\Gamma}_{\mathrm{RDP}}$. Let $\mathcal{W}$ be the set of white vertices of $\mathit{\Gamma}$ which is not joined to $\mathit{\Gamma}_{\mathsf{K}}$. Then a vertex in $\mathcal{W}$ represents a (-1)-curve γ with $E_M \cap \gamma = \emptyset$. Thus

$$\mathrm{Ver}(\mathit{\Gamma}) = \mathrm{Ver}(\mathit{\Gamma}^\flat) \sqcup \mathcal{W} = \mathrm{Ver}(\mathit{\Gamma}_{\mathsf{K}}) \sqcup \mathrm{Ver}(\mathit{\Gamma}_{\mathrm{RDP}}) \sqcup \mathcal{V} \sqcup \mathcal{W}.$$

Note that $\mathit{\Gamma}_{\mathsf{K}}$ and $\mathit{\Gamma}_{\mathrm{RDP}}$ are uniquely determined as the subgraphs of $\mathit{\Gamma}^\flat$. In fact, $\mathit{\Gamma}_{\mathsf{K}} \sqcup \mathit{\Gamma}_{\mathrm{RDP}}$ is the subgraph consisting of non-white vertices, and a connected component of $\mathit{\Gamma}_{\mathsf{K}}$ contains a non-black vertex.

Lemma 6.23. *Suppose that S is not of type $[1;2,2]_0$. Then, for any irreducible component $E_j \in \mathcal{J}(E)$, the scheme*

$$\Delta \cap (E_j \setminus \mathrm{Sing}\, E)$$

is uniquely determined up to isomorphism by the type T, the graph $\mathit{\Gamma}^\flat$, and by $\nu(E_j) \in \mathit{\Gamma}_{\mathsf{K}}$. Moreover, the number $\sharp\mathcal{V}$ of the finite set $\mathcal{V}$ is calculated as follows:

- *If $K_M + L_M$ is big, then $\sharp\mathcal{V} = \deg(\Delta) - \sigma(S) - b_{\mathsf{T}}$ for the number b_{T} of black vertices in $\mathit{\Gamma}_{\mathsf{K}}$.*
- *If $\mathsf{T} = [3;2,4]_+$ or $[4;2,4]_{00}$, then $\sharp\mathcal{V} = 16 - 2\sigma(S)$.*
- *If $\mathsf{T} = [3;2,4]_{++}(1,b)$, then $\sharp\mathcal{V} = 15 - 2\sigma(S) - 2b$.*
- *If $\mathsf{T} = [3;2,4]_{++}(2,1)$, then $\sharp\mathcal{V} = 12 - 2\sigma(S)$.*
- *Suppose that $\mathsf{T} = [3;2,4]_{++}(0,0)$. If a vertex in $\mathit{\Gamma}_{\mathrm{RDP}}$ joined to a vertex $v \in \mathcal{V}$ and v is joined to a black vertex of $\mathit{\Gamma}_{\mathsf{K}}$, then $\sharp\mathcal{V} = 15 - 2\sigma(S)$. If there is no such a vertex in $\mathit{\Gamma}_{\mathrm{RDP}}$ above, then $\sharp\mathcal{V} = 14 - 2\sigma(S)$.*

PROOF. We have $\mathcal{C}(\varGamma_{\mathrm{RDP}}) = \bigcup \mathcal{C}(\mathsf{A}_m)$. In case $\mathsf{T} = [2;1,2]_0$, we set $\mathcal{C}' \subset \mathcal{C}(\varGamma_{\mathrm{RDP}})$ to be the complement of a unique element of $\mathcal{C}(\varGamma_{\mathrm{RDP}})$ representing the total transform of the negative section of $X \simeq \mathbb{F}_2$. In case $\mathsf{T} \neq [2;1,2]_0$, we set $\mathcal{C}' = \mathcal{C}(\varGamma_{\mathrm{RDP}})$. In the both cases, we set $\mathcal{C}'(\mathsf{A}_m) = \mathcal{C}' \cap \mathcal{C}(\mathsf{A}_m)$.

Let $\mathcal{V}_\phi \subset \mathcal{V}$ be the subset of vertices representing a ϕ-exceptional (-1)-curve. Let $P_v \in X$ denote the point to which the (-1)-curve is contracted. Note that $\mathcal{V}_\phi = \mathcal{V}$ if $K_M + L_M$ is big. Let $\mathcal{V}_+$ be the set of vertices $v \in \mathcal{V}_\phi$ such that P_v is a node of E. Let $\mathcal{V}_m$ be the set of vertices $v \in \mathcal{V}_\phi$ such that $P_v \notin \operatorname{Sing} E$ and $\operatorname{mult}_{P_v}(\Delta) = m \geq 1$. The number $\sharp\mathcal{V}_+$ is 0 or 1, which depends on the type T. There is a one to one correspondence between $\mathcal{C}'(\mathsf{A}_l)$ and $\mathcal{V}_{l+1}$ for $l \geq 1$ as follows (cf. Lemma 6.16): A connected component $\varGamma_\lambda \in \mathcal{C}'(\mathsf{A}_l)$ represents the set of (-2)-curves in the fiber $\phi^{-1}(P)$ over a point $P \in \Delta \setminus \operatorname{Sing} E$ with $\operatorname{mult}_P(\Delta) = l+1$, where the end (-1)-curve of $\phi^{-1}(P)$ is represented by a vertex $v_\lambda \in \mathcal{V}_{l+1}$. Here, $\nu(E_j) \in \operatorname{Ver}(\varGamma_{\mathsf{K}})$ is the unique vertex of $\varGamma_{\mathsf{K}}$ joined to v_λ for the irreducible component $E_j = E_{j(\lambda)}$ of E containing P. Conversely, for a vertex $v \in \mathcal{V}_{l+1}$, the set of (-2)-curves in $\phi^{-1}(P_v)$ is represented by a connected component $\varGamma_{\lambda(v)} \in \mathcal{C}'(\mathsf{A}_l)$. Therefore, we have

$$\{P \in E_j \setminus \operatorname{Sing} E \mid \operatorname{mult}_P(\Delta) = l+1\} = \{P_\lambda \mid \varGamma_\lambda \in \mathcal{C}'(\mathsf{A}_l),\ j = j(\lambda)\}$$

for any $l \geq 1$ and for any irreducible component E_j of E. Since $\deg(\Delta \cap (E_j \setminus \operatorname{Sing} E))$ is determined by T, the scheme $\Delta \cap (E_j \setminus \operatorname{Sing} E)$ is determined up to isomorphism by T, $\varGamma^\flat$ and $\nu(E_j)$. We have

$$\deg(\Delta \setminus \operatorname{Sing} E) = \sum\nolimits_{l \geq 1} l \,\sharp\mathcal{V}_l \quad \text{and} \quad \sigma(S) = \sum\nolimits_{l \geq 2} (l-1)\,\sharp\mathcal{V}_l.$$

If $\Delta \cap \operatorname{Sing} E \neq \emptyset$, then $\deg(\Delta) - \deg(\Delta \setminus \operatorname{Sing} E) = 1 + b_{\mathsf{T}}$. Therefore,

$$\deg \Delta - \sigma(S) = b_{\mathsf{T}} + \sharp\mathcal{V}_\phi.$$

Hence, we may assume that $K_M + L_M$ is not big, i.e., T is one of $[3;2,4]_+$, $[3;2,4]_{++}(a,b)$, or $[4;2,4]_{00}$. Here, $\deg \Delta = 8$. A vertex $v \in \mathcal{V} \setminus \mathcal{V}_\phi$ represents the proper transform in M of a fiber of π passing through a point of $\Delta \setminus \operatorname{Sing} E$. Let $E_1 \subset E$ be the horizontal component which is not the negative section. Then

$$\sharp(\mathcal{V} \setminus \mathcal{V}_\phi) = \sharp\{v \in \mathcal{V}_\phi \mid 1 = j(v)\}.$$

Thus $\sharp\mathcal{V} = 2\sharp\mathcal{V}_\phi - \varepsilon$ for $\varepsilon = \sharp\{v \in \mathcal{V}_\phi \mid j(v) \neq 1\}$. Here, $\varepsilon = 0$ for $\mathsf{T} = [3;2,4]_+$, $[3;2,4]_{++}(2,1)$, $[4;2,4]_{00}$; and $\varepsilon = 1$ for $[3;2,4]_{++}(1,b)$. If $\mathsf{T} = [3;2,4]_{++}(0,0)$, then

$$\varepsilon = \begin{cases} 1, & \text{if } j(\lambda) \neq 1 \text{ for some } \Gamma_\lambda \in \mathcal{C}(\mathsf{A}_1), \\ 2, & \text{otherwise.} \end{cases}$$

Thus we are done. □

Corollary 6.24. *If $\mathsf{T} \neq [1;2,2]_0$, then the graph $\Gamma(S)$ depends only on the subgraph $\Gamma(S)^\flat$.*

Proof. It is enough to show the set $\mathcal{W}$ and the lines joining $\mathcal{W}$ and $\Gamma^\flat \cup \mathcal{W}$ are all determined. A vertex of $\mathcal{W}$ represents a (-1)-curve belonging to one of the cases (7a)–(7e) of Proposition 6.2.

Case. $\underline{X \simeq \mathbb{P}^2}$. Then $\mathcal{W} = \emptyset$ if $\mathsf{T} = [1]_0$. Hence, we may assume that $\mathsf{T} = [2]_0$ or $[2]_+(b)$. Then a vertex of $\mathcal{W}$ represents the proper transform in M of a line $\ell \subset X$ with $\deg(\ell \cap \Delta) = 2$, by Proposition 6.2. The line ℓ is not a component of E and is one of the following:

- A line joining two distinct points of Δ.
- The tangent line of E at a point $P \in \Delta \setminus \operatorname{Sing} E$ with $\operatorname{mult}_P(\Delta) \geq 2$.
- The line ℓ passing through the node P of E with $\operatorname{mult}_P(\Delta \cap \ell) = 2$ in the case $\mathsf{T} = [2]_+(1)$.

Therefore, the set $\mathcal{W}$ is determined by the graph $\Gamma(S)^\flat$. Let ℓ_1 and ℓ_2 be two such lines above. Then the proper transforms in M intersects if and only if the intersection point $\ell_1 \cap \ell_2$ is not contained in Δ. Therefore, the graph $\Gamma(S)$ is also determined by $\Gamma(S)^\flat$.

Case. $\underline{X \simeq \mathbb{F}_n,\ K_X + L \text{ is big, and } \mathsf{T} \text{ is not of type } [0;1,1]_0, [0;1,1]_+(b), \text{ nor } [1;1,1]_0}$.

Then a vertex in $\mathcal{W}$ represents the proper transform of a fiber ℓ of $\pi\colon X \to \mathbb{P}^1$ with $\deg(\Delta \cap \ell) = 1$ by Proposition 6.2. Since $\Delta \cap \ell$ is not a node of E, the set $\mathcal{W}$ is determined by $\Delta \setminus \operatorname{Sing} E$. For $P \in \Delta \cap \ell$, the proper transform $\ell_M \subset M$ of ℓ intersects the (-1)-curve $\phi^{-1}(P)$ if $\operatorname{mult}_P(\Delta) = 1$, and intersects the end (-2)-curve of the straight chain

$\phi^{-1}(P)$ if $\operatorname{mult}_P(\Delta) \geq 2$. There are no other negative curves intersecting ℓ_M. Therefore, $\varGamma(S)$ is also determined by $\varGamma(S)^\flat$.

Case. $\underline{\mathsf{T} = [0;1,1]_0 \text{ or } [0;1,1]_+(b)}$. A vertex in $\mathcal{W}$ represents the proper transform of a fiber ℓ of $\pi\colon X \to \mathbb{P}^1$ with $\deg(\Delta \cap \ell) = 1$ or the proper transform of a minimal section σ with $\deg(\Delta \cap \sigma) = 1$ by Proposition 6.2. Since $\Delta \cap \ell$ is not a node of E, the set $\mathcal{W}$ is determined by $\Delta \setminus \operatorname{Sing} E$. Let ℓ be such a fiber. Then a negative curve intersecting the proper transform ℓ_M is either an end curve of the chain $\phi^{-1}(P)$ for $P = \ell \cap \Delta$ or the proper transform σ_M of a minimal section σ with $\deg(\Delta \cap \sigma) = 1$, $\sigma \cap \ell \cap \Delta = \emptyset$. We have a similar assertion for a minimal section σ above. Therefore, $\varGamma(S)$ is also determined by $\varGamma(S)^\flat$.

Case. $\underline{\mathsf{T} = [1;1,1]_0}$. A vertex in $\mathcal{W}$ represents the proper transform of a fiber ℓ of $\pi\colon X \to \mathbb{P}^1$ with $\deg(\Delta \cap \ell) = 1$ or the total transform of the negative section σ by Proposition 6.2. By a similar argument to the cases above, we infer that $\varGamma(S)$ is determined by $\varGamma(S)^\flat$.

In the remaining case, $K_M + L_M$ is not big. The set $\mathcal{W}$ is empty for $\mathsf{T} = [4;2,4]_{00}$, $\mathsf{T} = [3;2,4]_{++}(2,1)$, $\mathsf{T} = [3;2,4]_{++}(1,b)$ with $4 \leq b \leq 6$ by Proposition 6.2. Thus the remaining types we must consider are $\mathsf{T} = [3;2,4]_+$, $[3;2,4]_{++}(0,0)$, and $[3;2,4]_{++}(1,b)$ with $1 \leq b \leq 3$.

Case. $\underline{\mathsf{T} = [3;2,4]_+}$. $E = \sigma + D$ and $D \sim \sigma + 4\ell$ for a fiber ℓ. A vertex in $\mathcal{W}$ represents the proper transform Θ_M of a section Θ at infinity with $\Theta \cap D \subset \Delta$ by Proposition 6.2. Moreover, the section Θ at infinity is uniquely determined by a subscheme $\Delta' \subset \Delta$ of degree 4 by $\Delta' = \Theta \cap D$. The (-2)-curves on M intersecting Θ_M are determined from the divisor Δ'. For $i = 1, 2$, let Θ_i be a section at infinity with $\Delta_i = \Theta_i \cap D \subset \Delta$, and let $\Theta_{i,M}$ be the proper transform in M. Then

$$\Theta_{1,M}\Theta_{2,M} = \Theta_1\Theta_2 - \deg(\Delta_1 \cap \Delta_2) = 3 - \deg(\Delta_1 \cap \Delta_2).$$

Therefore, $\mathcal{W}$ and $\varGamma(S)$ are determined by $\varGamma(S)^\flat$.

Case. $\underline{\mathsf{T} = [3;2,4]_{++}(a,b) \text{ for } (a,b) \in \{(0,0),(1,1),(1,2),(1,3)\}}$. $E = \sigma + \ell + \sigma_\infty$ for a fiber ℓ and for a section σ_∞ at infinity. A vertex in $\mathcal{W}$ represents the proper transform Θ_M of a section Θ at infinity with $\Theta \cap E \subset \Delta$ by Proposition 6.2. Moreover, the section Θ at infinity is uniquely determined by subschemes $\Delta' \subset \Delta \cap \sigma_\infty$ of degree $3 - b$ and $\Delta'' \subset \Delta \cap \ell$ of degree $1 - a$ by $\Delta' \cup \Delta'' = \Theta \cap (E \setminus \operatorname{Sing} E)$. Thus, by the same argument as above, we infer that $\mathcal{W}$ and $\varGamma(S)$ are determined by $\varGamma(S)^\flat$. □

The same assertion as Lemma 6.23 does not hold for type $\mathsf{T} = [1;2,2]_0$.

Example 6.25. Suppose that $\operatorname{char} \Bbbk \neq 2$. Let $X = \mathbb{F}_1$ and let $E \sim 2\sigma + 2\ell$ be a non-singular divisor. Let $P \in E$ be a non-ramification point with respect to $\pi|_E \colon E \to \mathbb{P}^1$, ℓ_P the fiber of π passing through P, and let P' be the other point of $\ell_P \cap E$. We consider two divisors $\Delta_1 := 8P$ and $\Delta_2 := 7P + P'$ on E. Then (X, E, Δ_1) and (X, E, Δ_2) are fundamental triplets of type $[1;2,2]_0$, and $\mathit{\Gamma}(X, E, \Delta_1)^\flat \simeq \mathit{\Gamma}(X, E, \Delta_2)^\flat$, which is written as the graph (3) of Lemma 5.13 with 7 black vertices. However, the number of white vertices of $\mathit{\Gamma}(X, E, \Delta_1)$ is 7 and the number for $\mathit{\Gamma}(X, E, \Delta_2)$ is 6, by Proposition 6.2, (7f), (7g). In TABLE 12 below, we have the graphs $\mathit{\Gamma}(X, E, \Delta_1)$ and $\mathit{\Gamma}(X, E, \Delta_2)$.

LEMMA 6.26. *Suppose that S is of type $[1;2,2]_0$. Let $w \in \mathcal{W}$ be the vertex representing the total transform of the negative section σ of $X \simeq \mathbb{F}_1$. Let L be the union of the fibers ℓ of π with* $\deg(\ell \cap \Delta) = 2$. *Then,* $(\Delta, \Delta \cap L = E \cap L)$ (cf. Lemma 5.14) *is uniquely determined up to isomorphism by the graph $\mathit{\Gamma}^\flat$ and w. Moreover, the dual graph $\mathit{\Gamma}(S)$ is determined by the subgraph consisting of $\mathit{\Gamma}^\flat$ and w.*

PROOF. A reducible fiber F of $M \to \mathbb{P}^1$ corresponds to a connected component of the graph consisting of $\mathcal{V} \cup \mathit{\Gamma}_{\mathrm{RDP}}$, by Proposition 6.2 and Lemma 5.13. The image $\ell = \phi(F)$ a fiber of $\pi \colon X \to \mathbb{P}^1$ and $F = \phi^*\ell$. Let ℓ_M be the proper transform of ℓ in M. Then ℓ_M be the irreducible component of F which intersects the total transform σ_M of σ.

Suppose that the dual graph of $F + E_M$ is either (1) or (3) of Lemma 5.13. Then F is written as the straight chain $F_0 + F_1 + \cdots + F_m$ of rational curves for $m \geq 1$ such that the end curves F_0 and F_m are represented by vertices in $\mathcal{V}$ and that $\sum_{i=1}^{m-1} F_i$ corresponds to a connected component of $\mathit{\Gamma}_{\mathrm{RDP}}$. If $\ell_M = F_i$ for $0 < i < m$, then $\ell \cap E = \ell \cap \Delta$ and it consists of two points Q_1, Q_2 with $\operatorname{mult}_{Q_1}(\Delta) = i$, $\operatorname{mult}_{Q_2}(\Delta) = m - i$. If $\ell_M = F_0$ or F_m, then $\ell \cap \Delta$ consists of one point Q with $\operatorname{mult}_Q(\Delta) = m$. If $m = 1$, then either that ℓ intersects E transversely, or that $\ell \cap E$ consists of two points.

Next, suppose that the dual graph of $F + E_M$ is either (2) or (4) of Lemma 5.13. Then ℓ intersects E tangentially at a point P, and the number of (-2)-curves in F is $\operatorname{mult}_P(\Delta) \geq 2$. The vertex representing ℓ_M is a black vertex joined to the unique white vertex.

Hence, w and $\Gamma^{\flat}$ determine the scheme structures of Δ and $\Delta \cap L = E \cap L$.

By Proposition 6.2, a vertex $w_i \in \mathcal{W} \setminus \{w\}$ represents the proper transform $\Theta_{i,M}$ in M of a section $\Theta_i \sim \sigma + n_i \ell$ of π with $\Theta_i \cap E \subset \Delta$ for $1 \leq n_i \leq 4$. Furthermore, Θ_i corresponds to a subscheme $\Delta_i \subset \Delta$ with $\deg \Delta_i = 2n_i$, $E \cap \ell \not\subset \Delta_i \cap \ell$. The unique component of a reducible fiber F intersecting $\Theta_{i,M}$ is determined by the information on $\ell \cap \Delta_i$. We have $\Theta_{i,M} \sigma_M = n_i - 1$. The intersection number $\Theta_{i,M} \Theta_{j,M}$ for w_i, $w_j \in \mathcal{W} \setminus \{w\}$ is calculated as

$$(\sigma + n_i \ell)(\sigma + n_j \ell) - \deg(\Delta_i \cap \Delta_j) = n_i + n_j - 1 - \deg(\Delta_i \cap \Delta_j).$$

Thus the full graph $\Gamma = \Gamma(S)$ is also determined by w and $\Gamma^{\flat}$. $\square$

LEMMA 6.27. *Suppose that S is of type $[1; 2, 2]_0$. Let $\mathcal{V}_i$ for $0 \leq i \leq 4$ be the following subsets of $\mathcal{V}$:*

- *$v \in \mathcal{V}_0$ if and only if v is not joined to any black vertex.*
- *$v \in \mathcal{V}_1$ if and only if v is joined to exactly one black vertex and the black vertex is an end of a connected component of Γ_{RDP} of type A_l for $l \geq 1$.*
- *$v \in \mathcal{V}_2$ if and only if v is joined to two black vertices.*
- *$v \in \mathcal{V}_3$ if and only if v is joined to exactly one black vertices and the black vertex is the middle vertex of a connected component of Γ_{RDP} of type A_3.*
- *$v \in \mathcal{V}_4$ if and only if v is joined to exactly one black vertices and the black vertex is an end of a connected component of Γ_{RDP} of type D_l for $l \geq 4$.*

Then $\mathcal{V} = \bigsqcup_{i=0}^{4} \mathcal{V}_i$. Let $\mathcal{V}_{1,l} \subset \mathcal{V}_1$ be the subset of vertices v such that the connected component joined to v is of type A_l. Let $\mathcal{V}_{4,l} \subset \mathcal{V}_4$ be the subset of vertices v such that the connected component joined to v is of type D_l. Then

$$\sigma(S) = 2\,\sharp\mathcal{V}_2 + 3\,\sharp\mathcal{V}_3 + (1/2)\sum\nolimits_{l\geq 1} l\,\sharp\mathcal{V}_{1,l} + \sum\nolimits_{l \geq 4} l\,\sharp\mathcal{V}_{4,l},$$

$$\begin{aligned}\deg \Delta = 8 = {} & (1/2)\sharp\mathcal{V}_0 + 2\,\sharp\mathcal{V}_2 + 3\,\sharp\mathcal{V}_3 \\ & + (1/2)\sum\nolimits_{l \geq 1}(l+1)\,\sharp\mathcal{V}_{1,l} + \sum\nolimits_{l\geq 4} l\,\sharp\mathcal{V}_{4,l}.\end{aligned}$$

PROOF. The subsets $\mathcal{V}_i$ are related to the graphs of Lemma 5.13 as follows: If $v \in \mathcal{V}_0$, then v is one of the two white vertices of the graph (1). If $v \in \mathcal{V}_{1,l}$, then v is one of the two white vertices of the graph (3) with l black vertices. If $v \in \mathcal{V}_2$, then v is the white vertex of the graph (2). If $v \in \mathcal{V}_3$, then v is the white vertex of the graph (4) with three black vertices. If $v \in \mathcal{V}_{4,l}$, then v is the white vertex of the graph (4) with l black vertices. Thus $\mathcal{V} = \bigsqcup \mathcal{V}_i$. Since any (-2)-curve of M is contained in a fiber of $M \to \mathbb{P}^1$, $\sigma(S)$ is calculated as above. For a point $P \in E$, let ℓ_P be the fiber of π passing through P, and $m_P := \mathrm{mult}_P(\Delta)$. If P is not a ramification point of $\pi|_E \colon E \to \mathbb{P}^1$, then $\ell \cap E = \{P, P'\}$ for another point $P' \in E$. In this case, if $m_P + m_{P'} = 1$, then the dual graph of $\phi^{-1}(\ell_P) + E_M$ is the graph (1); If $m_P + m_{P'} = m > 1$, then the dual graph is the graph (3) with $m-1$ black vertices. If $m_P = 2$ and $\ell_P \cap E = 2P$, then the dual graph of $\phi^{-1}(\ell_P) + E_M$ is the graph (2). If $m_P = 3$ and $\ell_P \cap E = 2P$, then the dual graph of $\phi^{-1}(\ell_P) + E_M$ is the graph (4) with three black vertices. If $m_P \geq 4$ and $\ell_P \cap E = 2P$, then the dual graph of $\phi^{-1}(\ell_P) + E_M$ is the graph (4) with m_P black vertices. Thus $\deg \Delta = 8 = \sum_{P \in \Delta} m_P$ is calculated as above. $\square$

THEOREM 6.28. *Let S_1 and S_2 be log del Pezzo surfaces of index two. For $i = 1$, 2, let (M_i, E_{M_i}) be the basic pair associated with S_i and let $\mathit{\Gamma}(S_i)$ be the dual graph of negative curves on M_i. If* $\operatorname{char} \Bbbk \neq 2$, *then the following conditions are mutually equivalent*:

(1) *(M_1, E_{M_1}) and (M_2, E_{M_2}) are equi-singular deformation equivalent, and $\mathit{\Gamma}(S_1)$ and $\mathit{\Gamma}(S_2)$ are isomorphic*;

(2) *S_1 and S_2 have the same type, and $\mathit{\Gamma}(S_1)$ and $\mathit{\Gamma}(S_2)$ are isomorphic*;

(3) *There exist fundamental triplets (X_1, E_1, Δ_1) and (X_2, E_2, Δ_2) defining S_1 and S_2, respectively, such that (X_1, E_1, Δ_1) and (X_2, E_2, Δ_2) are strongly equi-singular deformation equivalent*;

(4) *S_1 and S_2 are equi-singular deformation equivalent.*

PROOF. (1) $\Rightarrow$ (2): Let T_i be the type of S_i for $i = 1$, 2. Then $\mathsf{T}_1 = \mathsf{T}_2$, $(\mathsf{T}_1, \mathsf{T}_2) = ([0;1,1]_0, [2;1,2]_0)$, or $(\mathsf{T}_1, \mathsf{T}_2) = ([2;1,2]_0, [0;1,1]_0)$ by Theorem 6.1. Under the isomorphism $\mathit{\Gamma}(S_1) \simeq \mathit{\Gamma}(S_2)$, we have isomorphisms $\mathit{\Gamma}(S_1)_{\mathsf{K}} \simeq \mathit{\Gamma}(S_2)_{\mathsf{K}}$ and $\mathit{\Gamma}(S_1)_{\mathrm{RDP}} \simeq \mathit{\Gamma}(S_2)_{\mathrm{RDP}}$. If $\mathsf{T}_1 = [2;1,2]_0$, then there is an isolated black vertex in $\Gamma(S_1)_{\mathrm{RDP}}$. If $\mathsf{T}_2 = [0; 1, 1]_0$, then there is no isolated black vertex in $\Gamma(S_2)_{\mathrm{RDP}}$. Hence, $\mathsf{T}_1 = \mathsf{T}_2$.

(2) $\Rightarrow$ (1) follows from Theorem 6.1.

(2) $\Rightarrow$ (3) Since $\mathsf{T}_1 = \mathsf{T}_2$ and $\operatorname{char} \Bbbk \neq 2$, there exist a minimal basic pair (X, E) and zero-dimensional subschemes Δ_1 and Δ_2 of E such that (M_i, E_{M_i}) is obtained as the elimination of the fundamental triplet (X, E, Δ_i) for $i = 1$, 2. Thus the assertion (3) follows from Lemmas 5.14, 6.23, and 6.26.

(3) $\Rightarrow$ (4) is shown in Theorem 5.15.

(4) $\Rightarrow$ (2): Let $f \colon \widetilde{S} \to T$ be an equi-singular deformation of log del Pezzo surfaces of index two over a non-singular connected curve T. Let $\widetilde{M} \to \widetilde{S}$ be the simultaneous minimal resolution and $h \colon (\widetilde{M}, E_{\widetilde{M}}) \to T$ be the induced equi-singular deformation of basic pairs. Then, $\Gamma(S_t)_{\mathsf{K}} \sqcup \Gamma(S_t)_{\mathrm{RDP}}$ is independent for any fiber $S_t = f^{-1}(t)$. In particular, all the fibers S_t have the same type T by the argument in (1) $\Rightarrow$ (2) above. If γ is a (-1)-curve on the fiber $M_0 = h^{-1}(o)$ over a point $o \in T$, then γ is the fiber over o of a divisor $\widetilde{\Gamma}$ of $h^{-1}(U)$ for a Zariski open neighborhood U of o such that any fiber of $\widetilde{\Gamma} \to U$ is a (-1)-curve. In particular, the number of (-1)-curves on M_t for $t \in T$ defines a lower semi-continuous function. Let $\mathcal{V}(t)$ be the set of white vertices in $\Gamma(S(t))$ which are joined to $\Gamma(S(t))_{\mathrm{RDP}}$. Then $t \mapsto \sharp\mathcal{V}(t)$ is also lower semi-continuous. If $\sharp\mathcal{V}(t)$ is constant, then $\Gamma(S(t))^{\flat}$ is uniquely determined, and hence $\Gamma(S(t))$ is also constant by Corollary 6.24 and Lemma 6.26. Thus, it is enough to show the function $\sharp\mathcal{V}(t)$ is constant. If $\mathsf{T} \neq [3; 2, 4]_{++}(0, 0)$ and $\mathsf{T} \neq [1; 2, 2]_0$, then $\sharp\mathcal{V}(t)$ is constant, since it is determined by T and $\Gamma(S(t))$ by Lemma 6.23.

Suppose that $\mathsf{T} = [3; 2, 4]_{++}(0, 0)$. Let $\mathcal{V}'(t) \subset \mathcal{V}(t)$ be the subset of vertices v which is joined to a black vertex in $\Gamma(S(t))_{\mathsf{K}}$. Then $\sharp\mathcal{V}'(t) = 1$ or 2, and $t \mapsto \sharp\mathcal{V}'(t)$ is lower semi-continuous. On the other hand, $\sharp\mathcal{V}(t) = 16 - 2\sigma(S(t)) - \sharp\mathcal{V}'(t)$ by Lemma 6.23. Hence, $\mathcal{V}(t)$ and $\mathcal{V}'(t)$ are constant.

Suppose that $\mathsf{T} = [1; 2, 2]_0$. Let $\mathcal{V}_i(t)$ be the set $\mathcal{V}_i$ for $S(t)$ in Lemma 6.27. Similarly, we define $\mathcal{V}_{1,l}(t)$ and $\mathcal{V}_{4,l}(t)$. Then $\sharp\mathcal{V}_i(t)$, $\sharp\mathcal{V}_{1,l}(t)$, and $\sharp\mathcal{V}_{4,l}(t)$ are all lower semi-continuous functions. Let $a(l)$ be the number of connected components of $\Gamma(S(t))_{\mathrm{RDP}}$ of type A_l for $l \geq 1$ and let $d(l)$ be the number of connected components of $\Gamma(S(t))_{\mathrm{RDP}}$ of type D_l for $l \geq 4$.

Then

$$a(1) = (1/2)\sharp\mathcal{V}_{1,1}(t) + 2\sharp\mathcal{V}_2(t), \qquad a(2) = \sharp(1/2)\mathcal{V}_{1,2}(t),$$
$$a(3) = (3/2)\sharp\mathcal{V}_{1,3}(t) + 3\sharp\mathcal{V}_3(t), \qquad a(l) = (1/2)\sharp\mathcal{V}_{1,l}(t) \quad \text{for} \quad l \geq 4,$$
$$d(l) = \sharp\mathcal{V}_{4,l}(t) \quad \text{for} \quad l \geq 4.$$

By the formula for $\deg(\Delta)$ in Lemma 6.27, we infer that all the $\sharp\mathcal{V}_i(t)$ are constant. In particular, $\sharp\mathcal{V}(t)$ is constant. □

6.6. Comparison with the classification by Alexeev–Nikulin

The right resolution plays an important role in the classification theory of log del Pezzo surfaces of index two by Alexeev–Nikulin [4]. We assume $\operatorname{char} \Bbbk = 0$ in Section 6.6.

A general member $C_S \in |-2K_S|$ is non-singular, by Bertini's theorem. Let $C_{\mathcal{Y}}$ be the total transform in $\mathcal{Y}$. Then the divisor $C_{\mathcal{Y}} + E_{\mathcal{Y}}$ is non-singular and linearly equivalent to $-2K_{\mathcal{Y}}$. The pair $(\mathcal{Y}, C_{\mathcal{Y}} + E_{\mathcal{Y}})$ is called a *right DPN pair of elliptic type* in [4]. Let $\tau\colon \mathcal{X} \to \mathcal{Y}$ be the double-covering branched along $C_{\mathcal{Y}} + E_{\mathcal{Y}}$. Then $\mathcal{X}$ is non-singular and is a $K3$ surface. Note that $\mathcal{X}$ does depend on the choice of C_S. Let θ be the covering involution of $\mathcal{X}$ with respect to τ. Then θ does not preserve a nowhere vanishing holomorphic 2-form on $\mathcal{X}$, i.e., θ is non-symplectic. The θ-fixed locus $\mathcal{X}^\theta$ is non-singular and is isomorphic to $\tau(\mathcal{X}^\theta) = C_{\mathcal{Y}} + E_{\mathcal{Y}}$. We call $\mathcal{X}$ the $K3$ surface associated with (S, C_S).

Remark. Let $\mathcal{X} \to \mathcal{X}' \to S$ be the Stein factorization of the composite $\beta \circ \tau\colon \mathcal{X} \to S$. Then $\mathcal{X}' \to S$ is a double-covering étale outside $\operatorname{Sing} C_S \cup \operatorname{Sing} S$ and $\mathcal{O}_{\mathcal{X}'} \simeq \mathcal{O}_S \oplus \mathcal{O}_S(K_S)$. Moreover, $\mathcal{X}'$ has only rational double points as singularities and has a trivial dualizing sheaf. Thus the notion of right resolution of S is just the notion of *canonical resolution* in the sense of Horikawa with respect to the double-covering $\mathcal{X}' \to S$.

Remark. Giving a non-singular member $C_S \in |-2K_S|$ is equivalent to giving a non-singular member $C_M \in |L_M|$ for the associated basic pair (M, E_M). Let (X, E, Δ) be a fundamental triplet defining the log del Pezzo surface S. Then a non-singular member $C_S \in |-2K_S|$ is the proper transform of a non-singular member $C \in |L|$ with $C \cap E = \Delta$.

Conversely, let us consider a K3 surface $\mathcal{X}$ with a non-symplectic involution θ. Then the θ-fixed locus $\mathcal{X}^\theta$ is a non-singular divisor. Let $\mathcal{Y}$ be the quotient surface of $\mathcal{X}$ by the action of θ and let $\tau\colon \mathcal{X} \to \mathcal{Y}$ be the quotient map. Since $K_\mathcal{X} \sim \tau^* K_\mathcal{Y} + \mathcal{X}^\theta$, $\tau(\mathcal{X}^\theta)$ is a non-singular divisor linearly equivalent to $-2K_\mathcal{Y}$.

LEMMA 6.29. *Suppose that $\mathcal{X}^\theta$ is reducible and contains an irreducible curve of genus $g \geq 2$. Then $(\mathcal{X}, \theta)$ is constructed from a log del Pezzo surface S of index two and a non-singular member $C_S \in |-2K_S|$ as above.*

PROOF. Let $C_\mathcal{Y} \subset \mathcal{Y}$ be the image of the curve of genus g and let $E_\mathcal{Y}$ be the rest of $\tau(\mathcal{X}^\theta)$. Then $K_\mathcal{Y} C_\mathcal{Y} + C_\mathcal{Y}^2 = (1/2)C_\mathcal{Y}^2 = 2g-2 > 0$. By the Hodge index theorem, $E_{i,\mathcal{Y}}^2 < 0$ for any irreducible component $E_{i,\mathcal{Y}}$ of $E_\mathcal{Y}$. Thus, $E_{i,\mathcal{Y}}$ is a (-4)-curve by $-2K_\mathcal{Y} E_{i,\mathcal{Y}} = E_{i,\mathcal{Y}}^2$. Hence, $\mathcal{Y}$ is the right resolution of a log del Pezzo surface of index two by Lemma 6.8. Moreover, $C_\mathcal{Y}$ is the total transform of a non-singular member C_S of $|-2K_S|$. Thus, we are done. □

Therefore, the classification problem of log del Pezzo surfaces of index two is reduced in some sense to the classification of K3 surfaces with non-symplectic involutions, if $\operatorname{char} \Bbbk = 0$.

Let S_1 and S_2 be two log del Pezzo surfaces of index two whose right resolutions $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are deformation equivalent. For $i = 1, 2$, let $\mathcal{X}_i$ be the K3 surface associated with (S_i, C_i) for a non-singular member $C_i \in |-2K_{S_i}|$, and let $\theta_i \in \operatorname{Aut}(S_i)$ be the associated non-symplectic involution. Then $(\mathcal{X}_1, \theta_1)$ and $(\mathcal{X}_2, \theta_2)$ are deformation equivalent by an argument in Proposition 6.10. In fact, $\mathcal{X}_1$ and $\mathcal{X}_2$ appear as fibers of a smooth family $\widetilde{\mathcal{X}} \to T$ of K3 surfaces over a connected curve T where $\widetilde{\mathcal{X}}$ admits an involution $\tilde{\theta}$ over T and the restriction of $\tilde{\theta}$ to S_i is θ_i for $i = 1, 2$. Therefore, the deformation type of $(\mathcal{X}, \theta)$ depends on the deformation type of the DPN surface $\mathcal{Y}$, and vice versa.

Assume further that $\Bbbk$ is the complex number field $\mathbb{C}$. In order to study $(\mathcal{X}, \theta)$, Alexeev and Nikulin have considered the invariant part $\mathbb{S} = \mathrm{H}^2(\mathcal{X}^{\mathrm{an}}, \mathbb{Z})^{\theta^*}$ of the $K3$ lattice $\mathrm{H}^2(\mathcal{X}^{\mathrm{an}}, \mathbb{Z})$ by the induced involution θ^*. Then $\mathbb{S}$ is an even hyperbolic 2-elementary lattice contained in $\mathrm{NS}(\mathcal{X})$ in the following sense:

Let Λ be a non-degenerate lattice and let $Q(x, y) \in \mathbb{Z}$ denote the intersection pairing for x, $y \in \Lambda$. Then Λ is called *even* if $Q(x, x) \in 2\mathbb{Z}$ for any $x \in \Lambda$. It is called *hyperbolic* if the signature of $Q(\cdot,\cdot)$ is $(1, r-1)$ for $r = \operatorname{rank}\Lambda$. It is called 2-*elementary* if $\Lambda^*/\Lambda \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus a}$ for the dual lattice $\Lambda^* = \operatorname{Hom}(\Lambda, \mathbb{Z}) \subset \Lambda \otimes \mathbb{Q}$.

For an even hyperbolic 2-elementary lattice Λ, the main invariants are defined to be (r, a, δ), where the remaining invariant $\delta \in \{0, 1\}$ is determined as follows: $\delta = 0$ if and only if $Q(x^*, x^*) \in \mathbb{Z}$ for any $x^* \in \Lambda^*$. It is shown that the isomorphism classes of even hyperbolic 2-elementary lattices are determined by the main invariants (cf. [4, Appendix A.2]). Furthermore, the main invariants for even hyperbolic 2-elementary lattices Λ admitting primitive embeddings into a K3 lattice are classified in [4, Appendix A.2] by an algebraic argument of the lattice theory.

The main invariants of $\mathbb{S}$ have the following geometric interpretation (cf. [4, Section 2.3, Appendix A.2]): Let g be the genus of C_S and let k be the number of irreducible components of E_M. Note that $L_M^2 = 4g - 4$, $K_S^2 = g - 1 \geq 1$, and k equals the number of (-4)-curves on $\mathcal{Y}$. Then (g, k) and (r, a) are related by

$$k = (r-a)/2, \quad g = (22 - r - a)/2; \quad r = 11 - g + k, \quad a = 11 - g - k.$$

The invariant δ coincides with the δ of Definition 6.9 (cf. [4, Section 2.3]).

By the geometric interpretation and by Table 6, we have the list of the main invariants for all the types of log del Pezzo surfaces of index two in Table 9. Here, the number N in Table 9 is the entry number N used in [4, Table 1], which is given by the lexicographic order with respect to (k, r, δ). Note that Alexeev and Nikulin [4] has treated also log del Pezzo surfaces of index one and that the list with $N \leq 10$ in [4, Table 1] corresponds to the case of index one.

By the Torelli type theorem for K3 surfaces, Alexeev–Nikulin proved that the set of the pairs $(\mathcal{X}, \theta)$ of K3 surfaces $\mathcal{X}$ and non-symplectic involutions θ having fixed main invariants (r, a, δ) forms a connected family.

In [4], the log del Pezzo surfaces of index at most two are classified not only by the main invariants but also by another invariant called the root invariant. We omit the explanation of the root invariant here, but it almost corresponds to an information on the set of negative curves on the DPN surface $\mathcal{Y}$. They classified the root invariants for any $(\mathcal{X}, \theta)$ by an algebraic argument of lattices and by the Torelli type theorem for K3 surfaces. The method of calculating the dual graph $\Gamma[\mathcal{Y}]$ of the negative curves on $\mathcal{Y}$ from

Table 9. The main invariants of fundamental triplets

Type T	r	a	δ	N	Type T	r	a	δ	N
$[1]_0$	6	4	0	15	$[2;1,0]_0$	4	2	1	13
$[2]_0$	9	7	1	19	$[2;1,1]_+(0,0)$	7	3	1	22
$[2]_+(0)$	10	6	1	26	$[2;1,1]_+(1,1)$	8	2	1	28
$[2]_+(1)$	11	5	1	32	$[2;1,1]_+(1,2)$	9	1	1	34
$[2]_+(2)$	12	4	1	38	$[2;1,1]_+(1,3)$	10	0	0	40
$[2]_+(3)$	13	3	1	43	$[2;1,2]_0$	8	6	1	18
$[2]_+(4)$	14	2	0	46	$[2;1,2]_{++}$	10	4	0	30
$[0;1,0]_0$	6	4	1	16	$[3;1,0]_0$	3	1	1	12
$[0;1,1]_0$	8	6	1	18	$[3;1,1]_+$	6	2	0	21
$[0;1,1]_+(0)$	9	5	1	24	$[3;2,4]_+$	11	7	1	27
$[0;1,1]_+(1)$	10	4	1	31	$[3;2,4]_{++}(0,0)$	12	6	1	33
$[0;1,1]_+(2)$	11	3	1	37	$[3;2,4]_{++}(1,1)$	13	5	1	39
$[0;1,1]_+(3)$	12	2	1	42	$[3;2,4]_{++}(2,1)$	14	4	0	44
$[1;1,0]_0$	5	3	1	14	$[3;2,4]_{++}(1,2)$	14	4	1	45
$[1;1,1]_0$	7	5	1	17	$[3;2,4]_{++}(1,3)$	15	3	1	47
$[1;1,1]_+(0,0)$	8	4	1	23	$[3;2,4]_{++}(1,4)$	16	2	1	48
$[1;1,1]_+(1,1)$	9	3	1	29	$[3;2,4]_{++}(1,5)$	17	1	1	49
$[1;1,1]_+(2,1)$	10	2	0	35	$[3;2,4]_{++}(1,6)$	18	0	0	50
$[1;1,1]_+(1,2)$	10	2	1	36	$[4;1,0]_0$	2	0	0	11
$[1;1,1]_+(1,3)$	11	1	1	41	$[4;2,4]_{00}$	10	6	0	25
$[1;2,2]_0$	10	8	1	20					

the main invariants and a root invariant is explained in detail in [4]. The nef cone of $\mathcal{Y}$ is determined by $\Gamma[\mathcal{Y}]$ up to the action of certain Weyl group defined by the root invariant. The nef cone is used for the Torelli type theorem.

Let $\widehat{\Gamma}(S)$ be the dual graph $\Gamma[\mathcal{Y}]$. Then we have a natural map $\mathrm{Ver}(\Gamma(S)) \to \mathrm{Ver}(\widehat{\Gamma}(S))$ by taking proper transforms in $\mathcal{Y}$. Let $\widehat{\Gamma}(S)_{\mathsf{K}}$ be the subgraph of $\widehat{\Gamma}(S)$ consisting of the vertices representing irreducible components of $\psi^* E_M$. This is called the logarithmic part of $\Gamma[\mathcal{Y}]$ in [4]. If a connected component of $\widehat{\Gamma}(S)_{\mathsf{K}}$ corresponds to a singular point of S of type K_n for $n \geq 2$, then the component is written as

$$\circledcirc - \bigcirc - \circledcirc - \bigcirc - \cdots - \bigcirc - \circledcirc - \bigcirc - \circledcirc$$

where the total number of the vertices is $2n+1$. The subgraph $\widehat{\Gamma}(S)_{\mathrm{RDP}} \subset \widehat{\Gamma}(S)$ consisting of the (-2)-curves on $\mathcal{Y}$ is called the Du Val part of $\Gamma[\mathcal{Y}]$ in [4], and is canonically isomorphic to $\Gamma(S)_{\mathrm{RDP}}$. The union $\widehat{\Gamma}(S)_{\mathsf{K}} \sqcup \widehat{\Gamma}(S)_{\mathrm{RDP}}$ is just the dual graph of the β-exceptional curves. Note that $\widehat{\Gamma}(S)$ is determined by $\Gamma(S)$ by Corollary 6.5.

Therefore, the classification of the main invariants and the root invariants seems to correspond to the classification of equi-singular deformation types by Theorems 6.1, 6.28.

6.7. Dual graph of the negative curves for extremal cases

We shall write the graph $\widehat{\Gamma}(S)$ for an extremal log del Pezzo surface S of index two. The notion of extremal in [4] is the same as our notion in Definition 6.17 if we erase the case of type $[2;1,2]_0$. Then we have the list of dual graphs for $\operatorname{char} \Bbbk = 0$ in [4, Table 3]. We can calculate the graph by a geometric way by using results in Section 6.2. This method is completely different from that in [4].

Let us fix an extremal fundamental triplet (X, E, Δ) defining S. A negative curve on $\mathcal{Y}$ is one of the following curves:

(1) An exceptional curve for the composite $\mathcal{Y} \to M \to X$.

(2) The proper transform of an irreducible component of E; in other words, an irreducible component of $E_{\mathcal{Y}}$.

(3) The proper transform of an irreducible curve of X not contained in E.

By Proposition 6.2 and Corollary 6.5, we can classify the negative curves in the case (3) as follows.

Proposition 6.30. *Let $\mathfrak{S}$ be the set of irreducible curves γ of X with $\gamma \not\subset E$ whose proper transform in $\mathcal{Y}$ is negative. Then $\mathfrak{S}$ is described as follows according to the type T of the extremal fundamental triplet (X, E, Δ):*

(1) $\mathfrak{S} = \emptyset$ *if* T *is one of*

$$[1]_0, \quad [2]_+(4), \quad [1;1,1]_+(2,1), \quad [2;1,1]_+(1,1), \quad [2;1,1]_+(1,2),$$
$$[2;1,1]_+(1,3), \quad [2;1,2]_{++}, \quad [3;1,1]_+, \quad [3;2,4]_{++}(1,6), \quad [4;1,0]_0.$$

(2) *If* $\mathsf{T} = [2]_0$, *then* $\Delta = 8P$ *for a point* P *of the non-singular conic* E, *and* $\mathfrak{S}$ *consists of the tangent line at* P.

(3) *Suppose that* $\mathsf{T} = [2]_+(b)$. *Then* $E = E_1 + E_2$ *for two lines* E_1, E_2. *Let* P *be the node* $E_1 \cap E_2$. *If* $b = 0$, *then* $\Delta = 4Q_1 + 4Q_2$ *for points* $Q_1 \in E_1 \setminus \{P\}$, $Q_2 \in E_2 \setminus \{P\}$. *If* $b > 0$, *then* $\Delta = 3Q_1 + (4 - b)Q_2 + \Delta_P$ *for points* $Q_1 \in E_1 \setminus \{P\}$, $Q_2 \in E_2 \setminus \{P\}$ *and for an effective Cartier divisor* Δ_P *of* E *supported on* P *with* $\mathrm{mult}_P(\Delta_P \cap E_1) = 1$, $\mathrm{mult}_P(\Delta_P \cap E_2) = b$.

   (a) *If* $b \neq 1, 4$, *then* $\mathfrak{S}$ *consists of the line passing through* Q_1, Q_2.

   (b) *If* $b = 1$, *then* $\mathfrak{S}$ *consists of the line passing through* Q_1, Q_2 *and the unique line* ℓ *with* $\ell \cap E = \Delta_P$.

(4) $\mathfrak{S}$ *consists of one fiber of the* $\mathbb{P}^1$*-bundle* $\pi \colon X \to \mathbb{P}^1$ *if* T *is one of*

$$\begin{array}{llll}
[0;1,0]_0, & [0;1,1]_+(3), & & \\
[1;1,0]_0, & [1;1,1]_+(0,0), & [1;1,1]_+(1,1), & [1;1,1]_+(1,2), \\
[1;1,1]_+(1,3) & & & \\
[2;1,0]_0, & [2;1,1]_+(0,0), & & \\
[3;1,0]_0, & [3;2,4]_{++}(2,1), & [3;2,4]_{++}(1,4), & [3;2,4]_{++}(1,5), \\
[4;2,4]_{00}. & & &
\end{array}$$

(5) $\mathfrak{S}$ *consists of a fiber and a minimal section of* $\pi \colon X \to \mathbb{P}^1$ *if* T *is one of*

$$\begin{array}{llll}
[0;1,1]_0, & [0;1,1]_+(0), & [0;1,1]_+(1), & [0;1,1]_+(2), \\
[1;1,1]_0, & [2;1,2]_0. & &
\end{array}$$

(6) *Suppose that* $\mathsf{T} = [3;2,4]_+$. *Then* $E = \sigma + D$ *for a section* $D \sim \sigma + 4\ell$. *Let* P *be the node* $\sigma \cap D$. *Then* $\Delta = 8Q$ *for a point* $Q \in D \setminus \{P\}$. *Let* ℓ_P *and* ℓ_Q *be the fibers of* π *passing through* P *and* Q, *respectively. Then* $\mathfrak{S}$ *consists of* ℓ_P, ℓ_Q, *and the section* Θ *at infinity with* $\Theta \cap E = 4Q$.

(7) *Suppose that* $\mathsf{T} = [3;2,4]_{++}(a,b)$ *for* $(a,b) \in \{(0,0),(1,1),(1,2),(1,3)\}$. *Then* $E = \sigma + \sigma_\infty + \ell$ *for a section* σ_∞ *at infinity and a fiber* ℓ

of π. Let P be the node $\sigma_\infty \cap \ell$. Then $\Delta = (6-b)Q+(2-a)Q'+\Delta_P$ for points $Q \in \sigma_\infty \setminus \{P\}$, $Q' \in \ell \setminus \{P\}$ and for an effective Cartier divisor Δ_P of E supported on P with $\mathrm{mult}_P(\Delta_P \cap \sigma_\infty) = b$, $\mathrm{mult}_P(\Delta_P \cap \ell) = a$. Let ℓ_Q be the fiber of π passing through Q.

(a) *If $(a,b) = (0,0)$, then $\mathfrak{S}$ consists of ℓ_Q and the section Θ with $\Theta \cap E = 3Q + Q'$.*

(b) *If $(a,b) = (1,1)$, then $\mathfrak{S}$ consists of ℓ_Q and two sections Θ_1, Θ_2 at infinity such that $\Theta_1 \cap E = 3Q+Q'$ and $\Theta_2 \cap E = 2Q+\Delta_P$.*

(c) *If $(a,b) = (1,2)$, then $\mathfrak{S}$ consists of ℓ_Q and two sections Θ_1, Θ_2 at infinity such that $\Theta_1 \cap E = 3Q+Q'$ and $\Theta_2 \cap E = Q+\Delta_P$.*

(d) *If $(a,b) = (1,3)$, then $\mathfrak{S}$ consists of ℓ_Q and two sections Θ_1, Θ_2 at infinity such that $\Theta_1 \cap E = 3Q + Q'$ and $\Theta_2 \cap E = \Delta_P$.*

(8) *Suppose that $\mathsf{T} = [1;2,2]_0$ and $\mathrm{char}\,\Bbbk \neq 2$. Then $\Delta = n_1P_1+n_2P_2$ for the ramification points P_1, $P_2 \in E$ of the double-covering $\pi|_E \colon E \to \mathbb{P}^1$ and for $(n_1,n_2) \in \{(8,0),(6,2),(5,3),(4,4)\}$. Let ℓ_i be the fiber of π passing through P_i for $i = 1$, 2.*

(a) *If $(n_1,n_2) = (8,0)$, then $\mathfrak{S} = \{\sigma, \ell_1\}$.*

(b) *If $(n_1,n_2) \neq (8,0)$, then $\mathfrak{S}$ consists of σ, the fibers ℓ_1, ℓ_2, and the section Θ at infinity passing through P_1 and P_2.*

(9) *Suppose that $\mathsf{T} = [1;2;2]_0$ and $\mathrm{char}\,\Bbbk = 2$. If $\pi|_E \colon E \to \mathbb{P}^1$ is separable, then $\Delta = 8P$ for the unique ramification point $P \in E$, and $\mathfrak{S}$ consists of the fiber ℓ_P passing through P and σ. Suppose that $\pi|_E$ is inseparable. Then $\Delta = \sum_{i=1}^{l} m_iP_i$ for l distinct points $P_1, \dots, P_l$ for $l \leq 4$, and $m_1 \geq m_2 \geq \cdots \geq m_l \geq 2$ with $\sum_{i=1}^{l} m_i = 8$. Let ℓ_i be the fiber of π passing through P_i.*

(a) *If $l = 1$, then $\mathfrak{S} = \{\sigma, \ell_1\}$.*

(b) *If $2 \leq l \leq 3$, then $\mathfrak{S}$ consists of σ, the fibers ℓ_i for $1 \leq i \leq l$, and the sections $\Theta_{i,j}$ at infinity with $\Theta_{i,j}|_E = P_i + P_j$ for $1 \leq i < j \leq l$.*

(c) *If $l = 4$, then $\mathfrak{S}$ consists of σ, the fibers ℓ_i for $1 \leq i \leq 4$, the sections $\Theta_{i,j}$ at infinity with $\Theta_{i,j}|_E = P_i+P_j$ for $1 \leq i < j \leq l$, and the section $\Upsilon \sim \sigma + 2\ell$ with $\Upsilon|_E = \sum_{i=1}^{4} P_i$.*

Proof. (1), (2), (4), (5) are shown directly from Proposition 6.2 and Corollary 6.5.

(3): If $b > 1$, then there is no line ℓ with $\ell \cap E \subset \Delta_P$ by Corollary 2.13. If $b = 1$, then there exists uniquely the line ℓ with $\ell \cap E = \Delta_P$. Thus $\mathfrak{S}$ is described as above by Proposition 6.2, (7a).

(6): The proper transform of ℓ_Q in M is a (-1)-curve in Proposition 6.2, (6). The proper transform of ℓ_P in $\mathcal{Y}$ is the (-1)-curve appearing at Lemma 6.4. Since $\Delta = 8P$ and $(\sigma + 3\ell)D = 4$, the section Θ at infinity with $\Theta \cap E \subset \Delta$ is unique. Thus $\mathfrak{S}$ consists of these three curves.

(7): It is enough to determine the sections Θ at infinity satisfying $\Theta \cap E \subset \Delta$. Since $\Theta\sigma_\infty = 3$, $\Theta\ell = 1$, we have the unique section Θ in case $(a, b) = (0, 0)$ and the two sections Θ_1, Θ_2 in other cases by Corollary 2.13.

(8) and (9): It is enough to determine the sections $\Theta \sim \sigma + m\ell$ for $1 \le m \le 4$ with $\Theta \cap E \subset \Delta$. For the fiber ℓ_P passing through a point $P \in \Delta$, we have $\ell_P|_E = 2P$. Hence, the sections Θ are determined by Proposition 6.2, (7g). Thus we are done. □

Using Proposition 6.30, we can calculate the graph $\widehat{\Gamma}(S)$ for any extremal log del Pezzo surface S of index two. If the type T is not $[1; 2, 2]_0$, then the extremal fundamental triplet (X, E, Δ) of type T is unique up to isomorphism by Theorem 6.20, so the graph $\widehat{\Gamma}(S)$ for the extremal log del Pezzo surface S is denoted by $\widehat{\Gamma}_{\mathsf{T}}$ for $\mathsf{T} \ne [1; 2, 2]_0$. We shall explain how to calculate $\widehat{\Gamma}(S)$ for some types in each case of Proposition 6.30, and have the list of graphs for some types in cases (1)–(7) in Table 10. In the cases (8)–(9), we list the graph $\widehat{\Gamma}(S)$ for two extremal cases in Table 11. We can obtain the same graphs as in [4, Table 3] for all the types if $\operatorname{char} \Bbbk \ne 2$, but we omit the calculation in the remaining types.

In the graphs in Table 10, a vertex labeled with an irreducible curve γ of X represents the proper transform of γ in $\mathcal{Y}$.

Case (1). $\mathfrak{S} = \emptyset$. If $\mathsf{T} = [4; 1, 0]_0$, then ◎ is the graph $\widehat{\Gamma}_{\mathsf{T}}$ since $\Delta = 0$ and $\mathcal{Y} \simeq M \simeq X$. If $\mathsf{T} = [1]_0$, then $\Delta = 5P$ for a point P of a line E of $\mathbb{P}^2$, $\mathcal{Y} \simeq M$, and hence $\widehat{\Gamma}_{\mathsf{T}}$ is written as in Table 10. For other types with $\mathfrak{S} = \emptyset$, E is reducible and $\mathcal{Y} \to M \to X$ is a succession of blowups whose centers lie on the proper transform of E or on the inverse image of the nodes of E. Thus $\widehat{\Gamma}_{\mathsf{T}}$ is naturally obtained. For example, we consider the case $\mathsf{T} = [2; 1, 1]_+(1, 2)$. Then $E = \sigma + \ell$ and $\Delta = Q + \Delta_P$ for a point $Q \in \ell \setminus \sigma$ and for an effective Cartier divisor Δ_P supported on the node

$P = \sigma \cap \ell$ such that $\mathrm{mult}_P(\Delta_P \cap \sigma) = 1$ and $\mathrm{mult}_P(\Delta_P \cap \ell) = 2$. Thus we have the graph $\widehat{\Gamma}_{\mathsf{T}}$ as in TABLE 10.

Case (2). $\mathsf{T} = [2]_0$. E is a non-singular conic of $\mathbb{P}^2$ and $\Delta = 8P$. For the tangent line ℓ_P of E at P, we have the graph $\widehat{\Gamma}_{\mathsf{T}}$ in TABLE 10.

Case (3). $\mathsf{T} = [2]_+(b)$. $E = E_1 + E_2$ for two lines E_1, E_2 of $\mathbb{P}^1$. Suppose that $b = 0$. Then $\Delta = 4Q_1 + 4Q_2$. For the line ℓ_0 passing through Q_1 and Q_2, we have the graph $\widehat{\Gamma}_{\mathsf{T}}$ in TABLE 10. For the case $b \neq 1$, $\widehat{\Gamma}_{\mathsf{T}}$ is similarly obtained. Suppose that $b = 1$. Let $\ell_0 \in \mathfrak{S}$ be the line passing through Q_1, Q_2 and let $\ell_1 \in \mathfrak{S}$ be the other line. Then the point $\ell_0 \cap \ell_1$ is not lying on E. Thus $\widehat{\Gamma}_{\mathsf{T}}$ is as in TABLE 10.

Case (4). Here, we pick up three types $[2;1,0]_0$, $[1;1,1]_+(1,1)$, and $[3;2,4]_{++}(2,1)$. Suppose that $\mathsf{T} = [2;1,0]_0$. Then $E = \sigma$ and $\Delta = 2P$. Thus we have the graph $\widehat{\Gamma}_{\mathsf{T}}$ in TABLE 10 for the fiber ℓ_P of π passing through P.

Suppose that $\mathsf{T} = [1;1,1]_+(1,1)$. Then $E = \sigma + \ell$ and $\Delta = Q_1 + 2Q_2 + \Delta_P$ for $Q_1 \in \sigma \setminus \ell$, $Q_2 \in \ell \setminus \sigma$, and for an effective Cartier divisor Δ_P supported on $P = \sigma \cap \ell$ with $\mathrm{mult}_P(\Delta_P \cap \sigma) = \mathrm{mult}_P(\Delta_P \cap \ell) = 1$. Thus we have the graph $\widehat{\Gamma}_{\mathsf{T}}$ in TABLE 10 for the fiber ℓ_1 passing through Q_1.

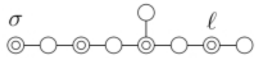
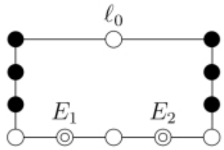
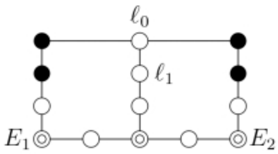
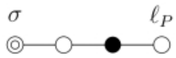
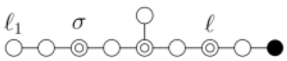
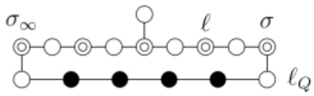
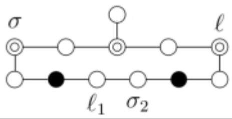
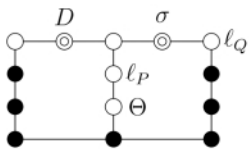
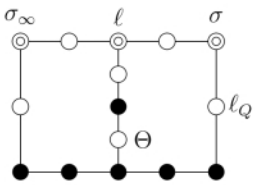
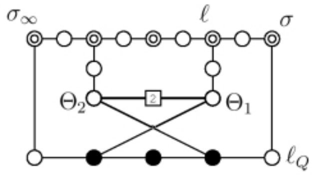
Suppose that $\mathsf{T} = [3;2,4]_{++}(2,1)$. Then $E = \sigma + \sigma_\infty + \ell$ and $\Delta = 5Q + \Delta_P$ for $Q \in \sigma_\infty \setminus \ell$ and for an effective Cartier divisor Δ_P supported on $P = \sigma_\infty \cap \ell$ with $\mathrm{mult}_P(\Delta_P \cap \sigma_\infty) = 1$, $\mathrm{mult}_P(\Delta_P \cap \ell) = 2$. Thus we have the graph $\widehat{\Gamma}_{\mathsf{T}}$ in TABLE 10 for the fiber ℓ_Q passing through Q.

Case (5). Here, we pick up three types $[0;1,1]_0$, $[0;1,1]_+(1)$, and $[2;1,2]_0$. Suppose that $\mathsf{T} = [0;1,1]_0$. Then E is regarded as the diagonal locus of $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $\Delta = 6P$ for a point $P \in E$. Let ℓ_i be the fiber passing through P of the i-th projection $X \to \mathbb{P}^1$ for $i = 1$, 2. Then $\widehat{\Gamma}_{\mathsf{T}}$ is as in TABLE 10. Note that this graph is not included in [4] since this is not extremal in the sense of [4]. In fact, the extremal distribution $\mathcal{D}_{[0;1,1]_0}$ is a subdiagram of $\mathcal{D}_{[2;1,2]_0}$.

Suppose that $\mathsf{T} = [0;1,1]_+(1)$. Then $E = \sigma + \ell$ and $\Delta = 2Q_1 + 2Q_2 + \Delta_P$ for $Q_1 \in \sigma \setminus \ell$, $Q_2 \in \ell \setminus \sigma$, and for an effective Cartier divisor Δ_P supported on the node $P = \sigma \cap \ell$ with $\mathrm{mult}_P(\Delta_P \cap \sigma) = \mathrm{mult}_P(\Delta_P \cap \ell) = 1$. Let ℓ_1 be the fiber passing through Q_1 and let σ_2 be the minimal section passing through Q_2. Then $\widehat{\Gamma}_{\mathsf{T}}$ is as in TABLE 10.

Suppose that $\mathsf{T} = [2;1,2]_0$. Then E is a section at infinity and $\Delta = 6P$ for $P \in E$. Let ℓ_P be the fiber passing through P. Then $E \cap \sigma = \emptyset$ and

TABLE 10. Some graphs $\widehat{\Gamma}_{\mathsf{T}}$

T	$[1]_0$	$[2;1,1]_+(1,2)$
$\widehat{\Gamma}_{\mathsf{T}}$		
T	$[2]_0$	$[2]_+(0)$
$\widehat{\Gamma}_{\mathsf{T}}$		
T	$[2]_+(1)$	$[2;1,0]_0$
$\widehat{\Gamma}_{\mathsf{T}}$		
T	$[1;1,1]_+(1,1)$	$[3;2,4]_{++}(2,1)$
$\widehat{\Gamma}_{\mathsf{T}}$		
T	$[0;1,1]_0$	$[0;1,1]_+(1)$
$\widehat{\Gamma}_{\mathsf{T}}$		
T	$[2;1,2]_0$	$[3;2,4]_+$
$\widehat{\Gamma}_{\mathsf{T}}$		
T	$[3;2,4]_{++}(0,0)$	$[3;2,4]_{++}(1,2)$
$\widehat{\Gamma}_{\mathsf{T}}$		

$E\ell_P = \sigma\ell_P = 1$. Hence $\widehat{\Gamma}_{\mathsf{T}}$ is as in TABLE 10.

Case (6). $\mathsf{T} = [3; 2, 4]_+$. Then $E = \sigma + D$ for a section $D \sim \sigma + 4\ell$. For the node $P = \sigma \cap D$, we have $\Delta = 8Q$ for $Q \in D \setminus \{P\}$. Let ℓ_P, ℓ_Q, and Θ be the same divisors as in Proposition 6.30, (6). Then $\widehat{\Gamma}_{\mathsf{T}}$ is as in TABLE 10.

Case (7). We pick up two types $[3; 2, 4]_{++}(0, 0)$ and $[3; 2, 4]_{++}(1, 2)$. Let $E = \sigma + \sigma_\infty + \ell$, $P = \sigma_\infty \cap P$, Q, Q', Δ_P, ℓ_Q, Θ, Θ_1, Θ_2 be the same as in Proposition 6.30, (7). Then $\widehat{\Gamma}_{\mathsf{T}}$ is as in TABLE 10 by the description of $\mathfrak{S}$.

Case (8). $\mathsf{T} = [1; 2, 2]_0$ and $\operatorname{char} \Bbbk \neq 2$. We pick up the case where $\mathcal{D}(S) = \mathsf{D}_8$. Then $\Delta = 8P_1$ for a ramification P_1 point of $\pi|_E : E \to \mathbb{P}^1$. Then $\widehat{\Gamma}(S) = \Gamma(S)$ is as in TABLE 11 for the fiber ℓ_1 of $\pi : X \to \mathbb{P}^1$ passing through P_1.

Case (9). $\mathsf{T} = [1; 2, 2]_0$ and $\operatorname{char} \Bbbk = 2$. We pick up the case where Δ consists of four points $P_1, \dots, P_4$. This is just the case where $\mathcal{D}(S) = 8\mathsf{A}_1$. Then $\Delta = 2(P_1 + \cdots + P_4)$. Let $\Theta_{i,j,M}$ be the proper transform in M of the section $\Theta_{i,j}$ at infinity with $\Theta_{i,j}|_E = P_i + P_j$ for $1 \leq i < j \leq 4$. Let σ_M be the proper transform in M of the negative section σ and let Υ_M be the proper transform in M of the section $\Upsilon \sim \sigma + 2\ell$ with $\Upsilon|_E \sim \sum_{i=1}^4 P_i$. Then $\Upsilon_M \cap \Theta_{i,j,M} = \sigma_M \cap \Theta_{i,j,M} = \emptyset$ for any $i < j$, $\Upsilon_M \sigma_M = 1$, and

$$\Theta_{i_1,j_1,M}\Theta_{i_2,j_2,M} = \begin{cases} 1, & \text{if } \{i_1, j_1\} \cap \{i_2, j_2\} = \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $\widehat{\Gamma}(S) = \Gamma(S)$ is as in TABLE 11.

REMARK 6.31. Suppose that $\operatorname{char} \Bbbk \neq 2$. We have two isomorphism classes of log del Pezzo surfaces S of index two of type $[1; 2, 2]_0$ with $\mathcal{D}(S) = \mathsf{A}_7$. These are constructed from the fundamental triplets (X, E, Δ_1) and (X, E, Δ_2) for the two zero dimensional subschemes $\Delta_1 = 8P$ and $\Delta_2 = 7P + P'$ defined in Example 6.25. Let ℓ_P be the fiber of $\pi : X \to \mathbb{P}^1$ passing through P. Then $\ell_P \cap E = \{P, P'\}$. Let $\gamma_j \sim \sigma + j\ell$ be the unique section of π with $\gamma_j|_E = 2jP$ for $j \geq 1$ (cf. Proposition 6.2, (7g)). Then the dual graph $\Gamma(X, E, \Delta_i)$ for $i = 1, 2$ is written as in TABLE 12.

For a ramification point $P_1 \in E$ of $\pi|_E$, the fundamental triplet $(X, E, 8P_1)$ is extremal and the dual graph $\Gamma := \Gamma(X, E, 8P_1)$ is given in

Table 11. Graphs $\Gamma(S)$ for two extremal cases of type $[1;2,2]_0$

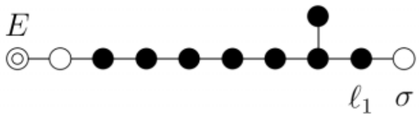
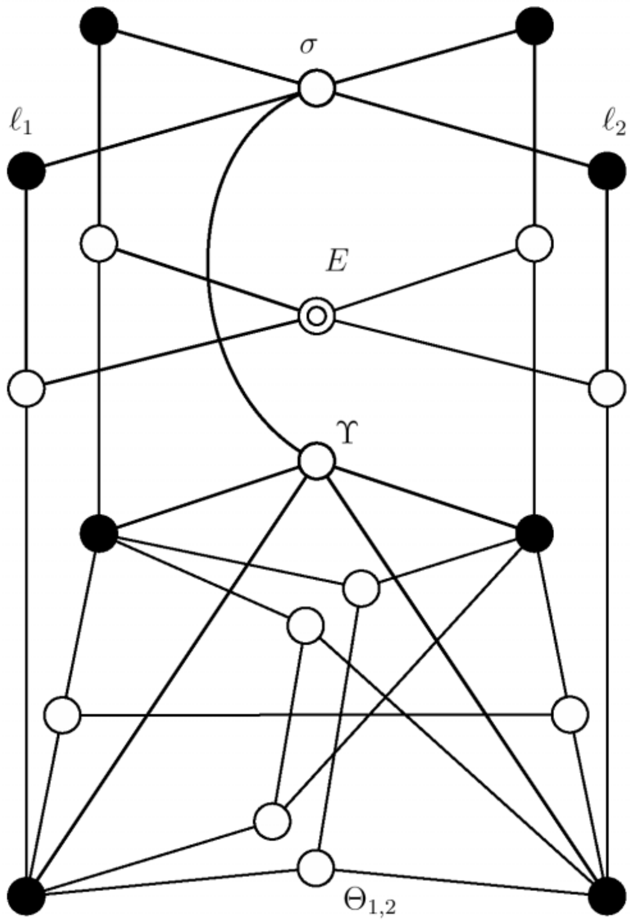
$\mathcal{D}(S)$	$\widehat{\Gamma}(S)$
D_8	
$8\mathsf{A}_1$	

TABLE 12. Graphs $\Gamma(S)$ for two non-extremal cases of type $[1;2,2]_0$ with $\mathcal{D}(S) = \mathsf{A}_7$

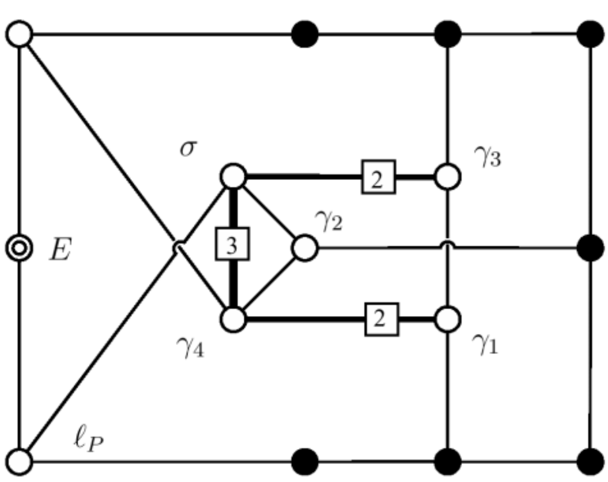
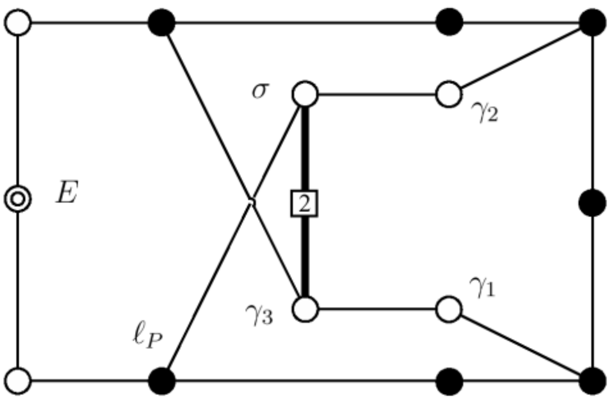
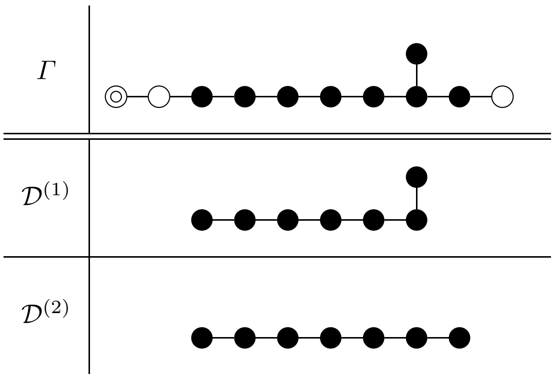
	$\Gamma(S)$
$\Delta_1 = 8P$	
$\Delta_2 = 7P + P'$	

TABLE 11. According to Alexeev–Nikulin [4], we have a non-extremal root invariant from a subgraph $\mathcal{D}^\sharp$ of the Dynkin diagram $\Gamma_{\mathrm{RDP}} = \mathsf{D}_8$ and we can calculate the dual graph $\widehat{\Gamma}(S^\sharp) = \widehat{\Gamma}(\mathcal{D}^\sharp)$ for a log del Pezzo surface $S^\sharp$ of type $[1;2,2]_0$ having the same non-extremal root invariant determined by $\mathcal{D}^\sharp$. Ohashi has calculated the graph $\widehat{\Gamma}(\mathcal{D}^\sharp)$ for the subgraph $\mathcal{D}^\sharp = \mathcal{D}^{(1)}$

TABLE 13. Two subgraphs $\mathsf{A}_7 \subset \mathsf{D}_8 = \Gamma_{\mathrm{RDP}}$ defining non-extremal root invariants



or $\mathcal{D}^{(2)}$ in TABLE 13. As a result, we infer that $\widehat{\Gamma}(\mathcal{D}^{(i)})$ coincides with $\Gamma(X, E, \Delta_i)$ for $i = 1, 2$.

7. Description of Log del Pezzo Surfaces of Index Two

A log del Pezzo surface S of index two is determined by a fundamental triplet (X, E, Δ) with E reduced and with $LE = \deg(\Delta)$. The classification of fundamental triplets gives the geometric description of S. From the information of the fundamental triplet, we shall describe the surface S explicitly as a subvariety of a weighted projective space or of the product of two weighted projective spaces (cf. TABLE 14).

7.1. Description by blowing up

Let (X, E, Δ) be a fundamental triplet such that $X \simeq \mathbb{F}_n$ and E is a section of the $\mathbb{P}^1$-bundle structure $\pi\colon X \to \mathbb{P}^1$. For the elimination $\phi\colon M \to X$ of Δ, the proper transform $E_M \subset M$ of E is a section of $\pi \circ \phi\colon M \to \mathbb{P}^1$ with $E_M^2 = -4$. By Lemma 4.5, there is a birational morphism $\mu\colon M \to \mathbb{F}_4$ over $\mathbb{P}^1$ such that E_M is the total transform of the negative section $\sigma^{(4)}$ of $\mathbb{F}_4$. For an irreducible curve $\gamma \subset M$, it is μ-exceptional if and only if γ is an irreducible component of a fiber of $M \to \mathbb{P}^1$ with $E_M \cap \gamma = \emptyset$. In particular, $K_M\gamma \leq 0$ for any μ-exceptional curve γ. Thus μ is isomorphic to the elimination of a zero-dimensional subscheme $\mathbb{D}' \subset \mathbb{F}_4$ such that $\nu_P(\mathbb{D}') = 1$ for any $P \in \mathbb{D}'$ and $\mathbb{D}' \cap \sigma^{(4)} = \emptyset$, by Proposition 2.9.

The birational morphism $\alpha\colon M \to S$ contracts E_M to a singular point of type K_1 and ϕ-exceptional (-2)-curves to rational double points. In the case $[2;1,2]_0$, α contracts also the proper transform of σ to a singular point of type A_1. The ϕ-exceptional (-2)-curves are contracted by the morphism $\mu\colon M \to \mathbb{F}_4$, since these curves do not intersect E_M.

Let $\sigma_\infty^{(4)} \subset \mathbb{F}_4$ be a section at infinity and let ℓ be a fiber of $\mathbb{F}_4 \to \mathbb{P}^1$. The contraction morphism $\mathbb{F}_4 \to \overline{\mathbb{F}}_4$ of the negative section $\sigma^{(4)}$ gives an isomorphism $\overline{\mathbb{F}}_4 \simeq \mathbb{P}(1,1,4)$. The image of ℓ in $\mathbb{P}(1,1,4)$ is a generating line and the image of $\sigma_\infty^{(4)}$ is a cross section of the cone $\mathbb{P}(1,1,4)$ over $\mathbb{P}^1$. The vertex v of the cone is a singular point of type K_1. For a homogeneous coordinate $(\mathtt{X},\mathtt{Y},\mathtt{Z})$ of $\mathbb{P}(1,1,4)$, v is the point $(0\!:\!0\!:\!1)$, $\operatorname{div}(\mathtt{Z})$ is a cross section, and $\operatorname{div}(\mathtt{X})$ and $\operatorname{div}(\mathtt{Y})$ are generating lines. Thus there is a birational morphism $q\colon \mathbb{F}_4 \to \mathbb{P}(1,1,4)$ such that $q(\sigma_4) = \{v\}$, $q(\sigma_\infty^{(4)}) = \operatorname{div}(\mathtt{Z})$, and $q(\ell) = \operatorname{div}(\mathtt{X})$.

Proposition 7.1. *Suppose that a log del Pezzo surface S of index two is of type $[n;1,0]_0$ for $0 \le n \le 4$. Then S is isomorphic to $\mathbb{P}(1,1,4)$ blown up along a zero-dimensional subscheme $\mathbb{D}$ satisfying*

(*) *$v \notin \mathbb{D}$, $\deg \mathbb{D} = 4-n$, and $\deg(\mathbb{D}\cap\overline{\ell}) \le 1$ for any generating line $\overline{\ell}$.*

Conversely, if $\mathbb{D} \subset \mathbb{P}(1,1,4)$ is a zero-dimensional subscheme satisfying (*) for $0 \le n \le 4$, then $\mathbb{D}$ is a Cartier divisor of a cross section, and $\mathbb{P}(1,1,4)$ blown up along $\mathbb{D}$ is a log del Pezzo surface of index two of type $[n;1,0]_0$.

Proof. Let (X,E,Δ) be a fundamental triplet defining S. Then $E = \sigma$ and $\deg\Delta = 4-n$. The total transform $\Theta_M = \phi^*(\sigma_\infty) \subset M$ of a section σ_∞ at infinity of X is a section of $M \to \mathbb{P}^1$. Since $K_X + \sigma + \sigma_\infty + 2\ell \sim 0$, we have $K_M + E_M + \Theta_M + 2\phi^*\ell \sim 0$. Since $E_M = \mu^*\sigma^{(4)}$, we infer that $\mu(\Theta_M) \subset \mathbb{F}_4$ is a section $\sigma_\infty^{(4)}$ at infinity and that μ is the elimination of the Cartier divisor $\mathbb{D}' \subset \sigma_\infty^{(4)}$, by Proposition 2.9. Here, $\mathbb{D}'$ is isomorphic to Δ under the isomorphism $\sigma_\infty^{(4)} \simeq E$ over $\mathbb{P}^1$. Let $\mathbb{D}$ be the image $q_*\mathbb{D}'$ for the birational morphism $q\colon \mathbb{F}_4 \to \mathbb{P}(1,1,4)$. Then $\mathbb{D}$ is a Cartier divisor of the cross section $\Theta = q(\sigma_\infty)$ satisfying (*). The induced morphism $S \to \mathbb{P}(1,1,4)$ is just the blowing-up along $\mathbb{D}$.

Conversely, if $\mathbb{D} \subset \mathbb{P}(1,1,4)$ is a zero-dimensional subscheme satisfying (*), then $\mathbb{D}$ is a Cartier divisor of a cross section Θ by Lemma 7.2 below. Let $\mathbb{D}'$ be the preimage $q^{-1}(\mathbb{D})$ for $q\colon \mathbb{F}_4 \to \mathbb{P}(1,1,4)$. The preimage $q^{-1}\Theta$ is a section at infinity. Let $\mu\colon M \to \mathbb{F}_4$ be the elimination of $\mathbb{D}'$. The proper transform $\Theta_M \subset M$ of Θ and the total transform $E_M \subset M$ of $\sigma^{(4)}$ are sections of $M \to \mathbb{P}^1$, where $K_M + \Theta_M + E_M + 2\mu^*\ell \sim 0$ and $\Theta_M^2 = 4 - (4-n) = n \geq 0$. We set $L_M = -2K_M - E_M$. Then $L_M \sim 2\Theta_M + E_M + 4\mu^*\ell$ and $K_M + L_M = \Theta_M + 2\mu^*\ell$ imply that (M, E_M) is a basic pair with $L_M E_M = 0$. The log del Pezzo surface S associated with (M, E_M) is just the blowing up of $\mathbb{P}(1,1,4)$ along $\mathbb{D}$. On the other hand, M is the elimination of (X, E, Δ) for $X = \mathbb{F}_n$, $E = \sigma$, and an effective divisor Δ of E with $\deg \Delta = 4 - n$. Hence, S is a log del Pezzo surface of index two of type $[n; 1, 0]_0$. $\square$

LEMMA 7.2. *Let Δ be a zero-dimensional subscheme of $\mathbb{F}_n$ such that $\Delta \cap \sigma = \emptyset$ for a minimal section σ and that* $\deg(\Delta \cap \ell) \leq 1$ *for any fiber ℓ of $\mathbb{F}_n \to \mathbb{P}^1$.*

(1) *If* $\deg \Delta \leq n+1$, *then Δ is a Cartier divisor of a section σ_∞ at infinity.*

(2) *If* $\deg \Delta = n+2$, *then Δ is a Cartier divisor of σ_∞ or of $\sigma_\infty \cup \ell$ for a section σ_∞ at infinity and for a fiber ℓ.*

In particular, $\nu_P(\Delta) = 1$ for any $P \in \operatorname{Supp} \Delta$ if $\deg \Delta \leq n+2$.

PROOF. (1) We may assume that $\deg \Delta = n+1$. From the exact sequence

$$0 \to \mathcal{I}_\Delta \mathcal{O}_X(\sigma + n\ell) \to \mathcal{O}_X(\sigma + n\ell) \to \mathcal{O}_\Delta \to 0$$

on $X = \mathbb{F}_n$ for the defining ideal $\mathcal{I}_\Delta$ of Δ, we infer that $\mathrm{H}^0(X, \mathcal{I}_\Delta \mathcal{O}_X(\sigma + n\ell)) \neq 0$ since $\dim \mathrm{H}^0(X, \sigma + n\ell) = n+2$. Thus $\mathcal{O}_X(-D) \subset \mathcal{I}_\Delta$ for an effective divisor $D \sim \sigma + n\ell$. If D is irreducible, then D is a section at infinity. We shall derive a contradiction by assuming that D is reducible. Then $n > 0$ and $D = \sigma + F$ for an effective divisor $F \sim n\ell$. Thus $\mathcal{O}_X(-F) \subset \mathcal{I}_\Delta$ since $\Delta \cap \sigma = \emptyset$. The non-empty intersection $\Delta \cap \ell$ for a fiber $\ell \subset F$ is supported on a point P. For a defining equation $\mathtt{t} \in \mathcal{O}_{X,P}$ of ℓ at P, let $\mathcal{O}_{\Delta,P} \to \mathcal{O}_{\Delta,P}$ be the multiplication map by $\mathtt{t}$. Then this is a nilpotent endomorphism with one-dimensional cokernel since $\deg(\Delta \cap \ell) = 1$. Hence, $\mathtt{t}^k \in \mathcal{I}_{\Delta,P}$ and $\mathtt{t}^{k-1} \notin \mathcal{I}_{\Delta,P}$ for $k = \operatorname{mult}_P \Delta = \dim_{\Bbbk} \mathcal{O}_{\Delta,P}$. Thus $\operatorname{mult}_P \Delta \leq \operatorname{mult}_\ell F$. Considering any fiber ℓ contained in F, we have $\deg \Delta \leq n$ which contradicts $\deg \Delta = n+1$.

(2) Let us fix a point $P \in \operatorname{Supp} \Delta$ and let ℓ be the fiber containing P. Suppose that $\operatorname{mult}_P(\Delta) = 1$. Then $\Delta = \Delta' \cup \{P\}$ for a subscheme Δ' with $\Delta' \cap \ell = \emptyset$. By (1), Δ' is a Cartier divisor of a section σ_∞ at infinity. Thus Δ is a Cartier divisor of $\sigma_\infty \cup \ell$ in this case.

Suppose that $k := \operatorname{mult}_P(\Delta) - 1 > 0$. Let $\mathtt{t} \in \mathcal{O}_{X,P}$ be a defining equation of ℓ at P. Then the multiplication map $\mathcal{O}_{\Delta,P} \to \mathcal{O}_{\Delta,P}$ by $\mathtt{t}$ is a nilpotent endomorphism with one-dimensional cokernel. Thus $\mathtt{t}^k \notin \mathcal{I}_{\Delta,P}$ and $\mathtt{t}^{k+1} \in \mathcal{I}_{\Delta,P}$. The image $\mathtt{t}\mathcal{O}_{\Delta,P}$ is isomorphic to $\mathcal{O}_{\Delta,P}/(\mathtt{t}^k)$. Thus the image of the homomorphism $\mathcal{O}_\Delta \to \mathcal{O}_\Delta$ obtained by tensoring $\mathcal{O}_\Delta$ with the inclusion $\mathcal{O}_X(-\ell) \to \mathcal{O}_X$ is isomorphic to $\mathcal{O}_{\Delta'}$ for a subscheme $\Delta' \subset \Delta$ with $\deg \Delta' = n+1$ and $\operatorname{mult}_P(\Delta') = k$. By (1), Δ' is a Cartier divisor of a section σ_∞ at infinity. Thus $\mathcal{I}_{\Delta',P}$ is generated by $(\mathtt{f}, \mathtt{t}^k)$ for a defining equation $\mathtt{f} \in \mathcal{O}_{X,P}$ of σ_∞ at P. Since $\mathtt{t}^k \notin \mathcal{I}_{\Delta,P}$, there is a constant $c \in \Bbbk$ with $\mathtt{f} + c\mathtt{t}^k \in \mathcal{I}_{\Delta,P}$. Thus $\mathcal{I}_{\Delta,P} = (\mathtt{f} + c\mathtt{t}^k, \mathtt{t}^{k+1})$. If $c = 0$, then $\mathcal{O}_X(-\sigma_\infty) \subset \mathcal{I}_\Delta$ and Δ is a Cartier divisor of σ_∞. If $c \neq 0$, then $\mathcal{I}_{\Delta,P} = (\mathtt{f} + c\mathtt{t}^k, \mathtt{f}\mathtt{t})$ and Δ is a Cartier divisor of $\sigma_\infty \cup \ell$. $\square$

Proposition 7.3. *Let S be a log del Pezzo surface of index two determined by a fundamental triplet (X, E, Δ) such that $X \simeq \mathbb{F}_n$ and E is a non-minimal section of $X \to \mathbb{P}^1$. Then the type of S is one of* $[0;1,1]_0$, $[1;1,1]_0$, *and* $[2;1,2]_0$.

(1) *If the type is $[0;1,1]_0$, then S is isomorphic to $\mathbb{P}(1,1,4)$ blown up along a zero-dimensional subscheme $\mathbb{D}$ satisfying the following conditions*:

(a) *$v \notin \mathbb{D}$, $\deg \mathbb{D} = 6$, and $\deg(\mathbb{D} \cap \overline{\ell}) \leq 1$ for any generating line $\overline{\ell}$*;

(b) *$\mathbb{D}$ is not a Cartier divisor of any cross section of $\mathbb{P}(1,1,4)$.*

Conversely, if a zero-dimensional subscheme $\mathbb{D}$ satisfies the conditions above, then $\mathbb{P}(1,1,4)$ blown up along $\mathbb{D}$ is a log del Pezzo surface of index two of type $[0;1,1]_0$.

(2) *If the type is $[1;1,1]_0$, then S is isomorphic to $\mathbb{P}(1,1,4)$ blown up along a zero-dimensional subscheme $\mathbb{D}$ such that $v \notin \mathbb{D}$, $\deg \mathbb{D} = 5$, and $\deg(\mathbb{D} \cap \overline{\ell}) \leq 1$ for any generating line $\overline{\ell}$. Conversely, if $\mathbb{D}$ is a zero-dimensional subscheme satisfying the same condition as above,*

then $\mathbb{D}$ is a Cartier divisor of a cross section, and $\mathbb{P}(1,1,4)$ blown up along $\mathbb{D}$ is a log del Pezzo surface of index two of type $[1;1,1]_0$.

(3) *Suppose that the type is* $[2;1,2]_0$. *Then there exist a cross section Θ of $\mathbb{P}(1,1,4)$, an effective Cartier divisor $\mathbb{D}$ of Θ of* $\deg \mathbb{D} = 6$, *and a birational morphism $\widehat{S} \to S$ for the blowing-up $\widehat{S} \to \mathbb{P}(1,1,4)$ along $\mathbb{D}$ such that $\widehat{S} \to S$ is the contraction morphism of the proper transform of Θ in $\widehat{S}$. Conversely, the surface S obtained from an effective Cartier divisor $\mathbb{D}$ of a cross section Θ as above is a log del Pezzo surface of index two of type* $[2;1,2]_0$.

PROOF. The case $[1;1,1]_0$ is proved by the same argument as in Proposition 7.1.

Case $[2;1,2]_0$. The negative section σ does not intersect E. The total transform Θ_M of σ in M is also a section satisfying $K_M + E_M + \Theta_M + 2\phi^*\ell \sim 0$. Since E_M is the total transform of the negative section $\sigma^{(4)}$, $\mu(\Theta_M)$ is a section $\sigma_\infty^{(4)}$ at infinity, and μ is the elimination of the divisor $\mathbb{D}' \subset \sigma_\infty^{(4)}$. Here $\mathbb{D}'$ is isomorphic to Δ under the isomorphism $\sigma_\infty^{(4)} \simeq E$ over $\mathbb{P}^1$. The image $\mathbb{D} = q_*\mathbb{D}' \subset \mathbb{P}(1,1,4)$ is a Cartier divisor of the cross section $\Theta = q(\sigma_\infty^{(4)})$ with $\deg \mathbb{D} = 6$. Let $\widehat{S} \to \mathbb{P}(1,1,4)$ be the blowing-up along $\mathbb{D}$. Then the induced birational morphism $M \to \widehat{S}$ contracts all the ϕ-exceptional (-2)-curves on M. Since $\alpha\colon M \to S$ contracts also the proper transform of σ in M, S is obtained by contracting the the proper transform $\widehat{\Theta}$ of σ in $\widehat{S}$. Conversely, if $\mathbb{D}$ is a Cartier divisor of a cross section Θ of $\deg \mathbb{D} = 6$, then for the elimination $\mu\colon M \to \mathbb{F}_4$ of $\mathbb{D}' = q^{-1}\mathbb{D}$, M is obtained as the elimination for a fundamental triplet (X, E, Δ) of type $[2;1,2]_0$, where E is the proper transform of Θ.

Case $[0;1,1]_0$. Since $\deg \Delta = 6$, we can take a minimal section σ such that $E \cap \sigma \subset \Delta$. Let $X' \to X$ be the blowing up at the point $E \cap \sigma$. Then the proper transform ℓ' of the fiber through the point $E \cap \sigma$ is a (-1)-curve. Let $X' \to X_1$ be the blowdown of ℓ'. Then the proper transform σ_1 of σ in X_1 is the negative section and the proper transform E_1 of E in X_1 is a section at infinity. Here, the image $Q \in X_1$ of ℓ' is not contained in $\sigma_1 \cup E_1$. The elimination $M \to X$ of Δ induces a morphism $M \to X_1$ which is regarded as the elimination of the zero-dimensional subscheme $\Delta'_1 \cup \{Q\}$ for a Cartier divisor Δ'_1 of E_1 with $\deg \Delta'_1 = 5$. The proper transform of E_1 in $\mathbb{F}_4$ by the rational map $\mu \circ \phi^{-1}\colon X \dashrightarrow M \to \mathbb{F}_4$ is the negative section

$\sigma^{(4)}$ and the proper transform of σ_1 in $\mathbb{F}_4$ is a section $\sigma_\infty^{(4)}$ at infinity. Let $\mathbb{D}'_1$ be the Cartier divisor of $\sigma_\infty^{(4)}$ isomorphic to Δ'_1 under the isomorphism $\sigma_\infty^{(4)} \simeq E_1$ over $\mathbb{P}^1$.

Suppose that Δ'_1 does not intersect the fiber ℓ_Q of $X_1 \to \mathbb{P}^1$ passing through Q. Then the rational map $X \dashrightarrow \mathbb{F}_4$ is an isomorphism at Q and let $Q' \in \mathbb{F}_4$ be the image of Q. The morphism $\mu\colon M \to \mathbb{F}_4$ is considered as the elimination of $\mathbb{D}'_1 \cup \{Q'\}$. The image $\mathbb{D}_1 = q(\mathbb{D}'_1) \subset \mathbb{P}(1,1,4)$ is a Cartier divisor of the cross section $\Theta = q(\sigma_\infty)$ and $q(Q') \notin \Theta$. Then the induced morphism $S \to \mathbb{P}(1,1,4)$ is the blowing-up along the zero-dimensional subscheme $\mathbb{D} = \mathbb{D}_1 \cup \{q(Q')\}$, which satisfies the condition (a).

Next, suppose that Δ'_1 intersects the fiber ℓ_Q. Then $X \dashrightarrow M \to \mathbb{F}_4$ is not isomorphic to Q. Let $\widehat{M} \to \mathbb{F}_4$ be the elimination of $\mathbb{D}'_1$. Then $M \to \widehat{M}$ is obtained as the blowing-up at a point $\widehat{Q}$ of the proper transform of ℓ_Q in $\widehat{M}$ lying over Q. Thus $\mu\colon M \to \mathbb{F}_4$ is the elimination of a Cartier divisor $\mathbb{D}'$ of $\sigma_\infty^{(4)} \cup \ell'_Q$ for the proper transform ℓ'_Q of ℓ_Q in $\mathbb{F}_4$, where $\mathbb{D}' \cap \sigma_\infty^{(4)}$ is isomorphic to Δ'_1 under the isomorphism $\sigma_\infty^{(4)} \simeq E_1$ over $\mathbb{P}^1$. The image $\mathbb{D} = q(\mathbb{D}') \subset \mathbb{P}(1,1,4)$ is a Cartier divisor of $\Theta \cup \overline{\ell}$ for the cross section $\Theta = q(\sigma_\infty)$ and the generating line $\overline{\ell} = q(\ell_{Q'})$. Then the induced morphism $S \to \mathbb{P}(1,1,4)$ is the blowing-up along $\mathbb{D}$, which satisfies the condition (a).

Let $\mathbb{D} \subset \mathbb{P}(1,1,4)$ be a zero-dimensional subscheme satisfying the condition (a). If it does not satisfy the other condition (b), $\mathbb{D}$ is a Cartier divisor of a cross section Θ, and the blowing-up $\widehat{S} \to \mathbb{P}(1,1,4)$ along $\mathbb{D}$ gives a birational morphism from $\widehat{S}$ into a log del Pezzo surface S of index two of type $[2;1,2]_0$ by (3). If $\mathbb{D}$ satisfies the condition (b), then, by Lemma 7.2 and by considering the inverse construction of $X_1 \dashrightarrow M \to \mathbb{F}_4$, we infer that $\mathbb{P}(1,1,4)$ blown up along $\mathbb{D}$ is a log del Pezzo surface of index two of type $[0;1,1]_0$. $\square$

PROPOSITION 7.4. *Let S be a log del Pezzo surface of index two of type $[1]_0$. Then there exist a zero-dimensional subscheme $\mathbb{D} \subset \mathbb{P}(1,1,4)$ of $\deg \mathbb{D} = 5$ and a cross section Θ containing $\mathbb{D}$ such that the proper transform $\widehat{\Theta}$ of Θ in the variety $\widehat{S}$ obtained as the blowing up of $\mathbb{P}(1,1,4)$ along $\mathbb{D}$ is a (-1)-curve and that S is obtained as the blowdown $\widehat{S} \to S$ of the (-1)-curve Θ.*

PROOF. Let $(X = \mathbb{P}^2, E, \Delta)$ be a fundamental triplet determining S. Let $\tau\colon X_1 \simeq \mathbb{F}_1 \to X$ be the blowing-up at a point $P \notin E$. Then

(X_1, E_1, Δ_1) is a fundamental triplet of type $[1;1,1]_0$ for the inverse images $E_1 = \tau^{-1}E$ and $\Delta_1 = \tau^{-1}\Delta$. By Proposition 7.3, the log del Pezzo surface $\widehat{S}$ determined by (X_1, E_1, Δ_1) is isomorphic to $\mathbb{P}(1,1,4)$ blown up along a Cartier divisor $\mathbb{D}$ of a cross section Θ with $\deg \mathbb{D} = 5$. Here, the proper transform $\widehat{\Theta} \subset \widehat{S}$ is a (-1)-curve since it is the proper transform of the negative section $\sigma_1 \subset X_1$. Thus the log del Pezzo surface S is obtained by contracting the (-1)-curve $\widehat{\Theta}$. $\square$

7.2. Remarks on weighted projective spaces

We insert here some notes on weighted projective spaces which are useful in the subsequent subsections. The results mentioned here are well known but we shall give proofs based on Demazure's construction [11] of normal graded rings.

LEMMA 7.5. *Let X be the weighted projective space $\mathbb{P}(a_0, a_1, \dots, a_d)$ with $a_0 = 1$ and let $\pi\colon \mathbb{P} = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(e)) \to X$ be the $\mathbb{P}^1$-bundle defined for a positive integer $e > 0$ divisible by $\operatorname{lcm}\{a_1, \dots, a_d\}$. Then there is a birational morphism $\mathbb{P} \to \mathbb{P}(a_0, \dots, a_d, e)$ such that the exceptional locus is the section $\Sigma \subset \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(e))$ of π corresponding to $\mathcal{O} \oplus \mathcal{O}(e) \to \mathcal{O}$ and that Σ is contracted to the point $(0:0:\cdots:0:1)$.*

PROOF. We fix a homogeneous coordinate $(\mathtt{X}_0, \dots, \mathtt{X}_d)$ of X of weight $(a_0, \dots, a_d)$. Let $\Sigma_\infty \subset \mathbb{P}$ be the section corresponding to a surjection $\mathcal{O} \oplus \mathcal{O}(e) \to \mathcal{O}(e)$. Then $\Sigma \cap \Sigma_\infty = \emptyset$ and $\Sigma_\infty \sim \Sigma + e\pi^*E_0$ for the Weil divisor $E_0 = \operatorname{div}(\mathtt{X}_0)$. Let us fix defining equations $\mathtt{g}$ and $\mathtt{f}$ of Σ and Σ_∞, respectively. We consider the $\mathbb{Q}$-divisor

$$H = \frac{1}{e}\Sigma + \pi^*E_0$$

on $\mathbb{P}$ and the graded ring $R = R(\mathbb{P}, H)$ (cf. Section 3.4). Here, $R_m = \mathrm{H}^0(\mathbb{P}, \llcorner mH \lrcorner)$ for $m \geq 0$. For a given positive integer m, we set $k = \llcorner m/e \lrcorner$. Then

$$\pi_*\mathcal{O}_{\mathbb{P}}(\llcorner mH \lrcorner) = \operatorname{Sym}^k(\mathcal{O}\mathtt{g} \oplus \mathcal{O}(-e)\mathtt{f}) \otimes \mathcal{O}(m) = \bigoplus\nolimits_{j=0}^{k} \mathcal{O}(m - je)\mathtt{f}^j\mathtt{g}^{k-j}.$$

Hence, we have

$$R_m = \bigoplus\nolimits_{j=0}^{k} \Bbbk[\mathtt{X}_0, \dots, \mathtt{X}_d]_{m-je}\mathtt{f}^j\mathtt{g}^{k-j}, \tag{7–1}$$

where $\Bbbk[\mathtt{X}_0, \dots, \mathtt{X}_d]_l$ denotes the homogeneous part of degree l of the graded polynomial ring $\Bbbk[\mathtt{X}_0, \dots, \mathtt{X}_d]$. Let $\mathtt{Y}_i \in R$ for $0 \le i \le d$ be the homogeneous element of degree a_i corresponding to $\mathtt{X}_i$ as the element of the right hand side of (7–1). Let $\mathtt{Y}_{d+1} \in R$ be the homogeneous element of degree e corresponding to $\mathtt{f}$ as the element of the right hand side of (7–1). Since

$$R_m = \bigoplus\nolimits_{j=0}^{k} \Bbbk[\mathtt{Y}_0, \dots, \mathtt{Y}_d]_{m-je}\mathtt{Y}_{d+1}^j,$$

we infer that $R = \Bbbk[\mathtt{Y}_0, \dots, \mathtt{Y}_d, \mathtt{Y}_{d+1}]$ and R is isomorphic to the graded polynomial ring of weight $(a_0, \dots, a_d, e)$. Since H is a semi-ample big $\mathbb{Q}$-divisor on $\mathbb{P}$, we have a natural birational morphism $\varphi \colon \mathbb{P} \to \operatorname{Proj} R \simeq \mathbb{P}(a_0, \dots, a_d, e)$ such that $\varphi^*\mathcal{O}(e) \simeq \mathcal{O}_{\mathbb{P}}(eH)$,

$$\varphi^*\mathtt{Y}_{d+1} = \mathtt{f}, \quad \text{and} \quad \varphi^*P_e(\mathtt{Y}_1, \dots, \mathtt{Y}_d) = P_e(\mathtt{X}_1, \dots, \mathtt{X}_d)\mathtt{g}$$

for any weighted homogeneous polynomial P_e of degree e. Here, Σ is the exceptional locus of φ and $\varphi(\Sigma) = \{(0:0:\cdots:0:1)\}$. □

LEMMA 7.6. *The Hirzebruch surface $X = \mathbb{F}_n$ is isomorphic to the divisor*

$$\{\mathtt{XW} = \mathtt{YZ}\} \subset \mathbb{P}(1, 1, n+1, n+1)$$

for a homogeneous coordinate $(\mathtt{X}, \mathtt{Y}, \mathtt{Z}, \mathtt{W})$ of weight $(1, 1, n+1, n+1)$, in which the restriction of $\mathcal{O}(n+1)$ is isomorphic to $\mathcal{O}_X(\sigma + (n+1)\ell)$.

PROOF. We consider the graded ring $R = R(X, H)$ for the ample $\mathbb{Q}$-divisor

$$H = \frac{1}{n+1}\sigma + \ell.$$

Then $X \simeq \operatorname{Proj} R$. Let $\mathtt{g}$ be a defining equation of a minimal section σ and let $\mathtt{f}$ be a defining equation of a section at infinity. For a non-negative integer m and $k = \llcorner m/(n+1) \lrcorner$, we have an equality

$$\pi_*\mathcal{O}_X(\llcorner mH \lrcorner) = \operatorname{Sym}^k(\mathcal{O}\mathtt{g} \oplus \mathcal{O}(-n)\mathtt{f}) \otimes \mathcal{O}(m) = \bigoplus\nolimits_{j=0}^{k} \mathcal{O}(m - nj)\mathtt{f}^j\mathtt{g}^{k-j}$$

for $\pi \colon X = \mathbb{F}_n \to \mathbb{P}^1$. In particular,

$$R_m = \bigoplus\nolimits_{j \ge 0} \Bbbk[\mathtt{s}, \mathtt{t}]_{m-nj}\mathtt{f}^j\mathtt{g}^{k-j} \tag{7–2}$$

for a homogeneous coordinate $(\mathtt{s},\mathtt{t})$ of $\mathbb{P}^1$. Let $\mathtt{X} \in R_1$ and $\mathtt{Y} \in R_1$ correspond to $\mathtt{s}\mathtt{g}^{\varepsilon}$ and $\mathtt{t}\mathtt{g}^{\varepsilon}$ as the elements of the right hand side of (7–2), respectively, where $\varepsilon = \llcorner 1/(n+1) \lrcorner$. Let $\mathtt{Z} \in R_{n+1}$ and $\mathtt{W} \in R_{n+1}$ correspond to $\mathtt{sf}$ and $\mathtt{tf}$ as the elements of the right hand side of (7–2). Then $\mathtt{XW} = \mathtt{YZ}$. Let i_1 and i_2 be non-negative integers with $m \geq (n+1)(i_1+i_2)$. Then the element $P(\mathtt{X},\mathtt{Y})\mathtt{Z}^{i_1}\mathtt{W}^{i_2} \in R_m$ for a homogeneous polynomial P of degree $m-(n+1)(i_1+i_2)$ corresponds to

$$P(\mathtt{s},\mathtt{t})\mathtt{s}^{i_1}\mathtt{t}^{i_2}\mathtt{f}^{i_1+i_2}\mathtt{g}^{k-(i_1+i_2)}$$

as the element of the right hand side of (7–2). Hence, R is generated by $\mathtt{X}$, $\mathtt{Y}$, $\mathtt{Z}$, $\mathtt{W}$ with the relation $\mathtt{XW} = \mathtt{YZ}$. Therefore, there is a closed immersion $\tau\colon X \simeq \operatorname{Proj} R \hookrightarrow \mathbb{P}(1,1,n+1,n+1)$ such that $\tau^*\mathcal{O}(n+1) \simeq \mathcal{O}_X(\sigma+(n+1)\ell)$ and $\tau(X) = \{\mathtt{XW} = \mathtt{YZ}\}$, since $\{\mathtt{XW} = \mathtt{YZ}\}$ is irreducible. □

Lemma 7.7. *For positive integers n_1, n_2, let $\mathbb{P}$ be the fiber product of $\mathbb{F}_{n_1}$ and $\mathbb{F}_{n_2}$ over $\mathbb{P}^1$. Let σ_1 and σ_2 be the negative sections of $\mathbb{F}_{n_1} \to \mathbb{P}^1$ and $\mathbb{F}_{n_2} \to \mathbb{P}^1$, respectively. Let H be the $\mathbb{Q}$-divisor on $\mathbb{P}$ defined by*

$$H = \frac{1}{n_1}p_1^*\sigma_1 + \frac{1}{n_2}p_2^*\sigma_2 + F$$

for the projections $p_1\colon \mathbb{P} \to \mathbb{F}_{n_1}$, $p_2\colon \mathbb{P} \to \mathbb{F}_{n_2}$, and for a fiber F of $\pi\colon \mathbb{P} \to \mathbb{P}^1$.

(1) *The graded ring $R = R(\mathbb{P}, H)$ is isomorphic to the graded polynomial ring of four variables with weight $(1,1,n_1,n_2)$.*

(2) *For the naturally defined birational map $\mathbb{P} \dashrightarrow \operatorname{Proj} R = \mathbb{P}(1,1,n_1,n_2)$, the composite $\mathbb{P} \dashrightarrow \mathbb{P}(1,1,n_i)$ with the projection $\mathbb{P}(1,1,n_1,n_2) \dashrightarrow \mathbb{P}(1,1,n_i)$ is just the composite $\mathbb{P} \to \mathbb{F}_{n_i} \to \overline{\mathbb{F}}_{n_i} \simeq \mathbb{P}(1,1,n_i)$ for $i = 1$, 2.*

Proof. (1): Let $(\mathtt{s},\mathtt{t})$ be the homogeneous coordinate of $\mathbb{P}^1$. Let $\mathtt{g}_i$ be a defining equation of $\sigma_i \subset \mathbb{F}_{n_i}$ for $i = 1$, 2. Let $\sigma_i^\infty \sim \sigma_i + n_i\ell$ be a section at infinity of $\mathbb{F}_{n_i} \to \mathbb{P}^1$ and let $\mathtt{f}_i$ be a defining equation of σ_i^∞ for $i = 1$, 2. For a fixed positive integer m, we set $k_i = \llcorner m/n_i \lrcorner$ for $i = 1$, 2.

Then

$$\begin{aligned}\pi_*\mathcal{O}_{\mathbb{P}}(\llcorner mH \lrcorner) &= \operatorname{Sym}^{k_1}(\mathcal{O}\mathtt{g}_1 \oplus \mathcal{O}(-n_1)\mathtt{f}_1) \\ &\quad \otimes \operatorname{Sym}^{k_2}(\mathcal{O}\mathtt{g}_2 \oplus \mathcal{O}(-n_2)\mathtt{f}_2) \otimes \mathcal{O}(m) \\ &= \bigoplus\nolimits_{0\le j_1\le k_1, 0\le j_2\le k_2} \mathcal{O}(m - j_1n_1 - j_2n_2)\mathtt{f}_1^{j_1}\mathtt{g}_1^{k_1-j_1}\mathtt{f}_2^{j_2}\mathtt{g}_2^{k_2-j_2}.\end{aligned}$$

In particular, we have

$$R_m = \bigoplus\nolimits_{0\le j_1\le k_1, 0\le j_2\le k_2} \Bbbk[\mathtt{s},\mathtt{t}]_{m-j_1n_1-j_2n_2}\mathtt{f}_1^{j_1}\mathtt{g}_1^{k_1-j_1}\mathtt{f}_2^{j_2}\mathtt{g}_2^{k_2-j_2}. \tag{7–3}$$

We set $\delta_i = \llcorner 1/n_i \lrcorner$ for $i = 1$, 2. Then $\delta_i = 0$ unless $n_i = 1$. Let $\mathtt{X}$ and $\mathtt{Y} \in R_1$ correspond to $\mathtt{s}\mathtt{g}_1^{\delta_1}\mathtt{g}_2^{\delta_2}$ and $\mathtt{t}\mathtt{g}_1^{\delta_1}\mathtt{g}_2^{\delta_2}$ as the elements of the right hand side of (7–3), respectively. We set $e_1 = \llcorner(n_2/n_1)\lrcorner$ and $e_2 = \llcorner(n_1/n_2)\lrcorner$. If $n_1 = n_2$, then $e_1 = e_2 = 1$; if $n_1 < n_2$, then $e_1 \ge 1$ and $e_2 = 0$. Let $\mathtt{Z}_1 \in R_{n_1}$ and $\mathtt{Z}_2 \in R_{n_2}$ correspond to $\mathtt{f}_1\mathtt{g}_2^{e_2}$ and $\mathtt{f}_2\mathtt{g}_1^{e_1}$ as the elements of the right hand side of (7–3), respectively. Then, for a pair of non-negative integers (j_1, j_2) with $j_1n_1 + j_2n_2 \le m$, the equality

$$\mathtt{f}_1^{j_1}\mathtt{g}_1^{k_1-j_1}\mathtt{f}_2^{j_2}\mathtt{g}_2^{k_2-j_2} = (\mathtt{f}_1\mathtt{g}_2^{e_2})^{j_1}(\mathtt{f}_2\mathtt{g}_1^{e_1})^{j_2}\mathtt{g}_1^{k_1-j_1-e_1j_2}\mathtt{g}_2^{k_2-j_2-e_2j_1}$$

holds, and $P(\mathtt{X},\mathtt{Y})\mathtt{Z}_1^{j_1}\mathtt{Z}_2^{j_2} \in R_m$ for a homogeneous polynomial P of degree $m - j_1n_1 - j_2n_2$ corresponds to

$$P(\mathtt{s},\mathtt{t})(\mathtt{f}_1\mathtt{g}_2^{e_2})^{j_1}(\mathtt{f}_2\mathtt{g}_1^{e})^{j_2}\mathtt{g}_1^{k_1-j_1-e_1j_2}\mathtt{g}_2^{k_2-j_2-e_2j_1}$$

as the element of the right hand side of (7–3). Therefore, $R = \Bbbk[\mathtt{X},\mathtt{Y},\mathtt{Z}_1,\mathtt{Z}_2]$ and R is isomorphic to the graded polynomial ring of weight $(1, 1, n_1, n_2)$.

(2): For $i = 1$, 2, we consider the semi-ample $\mathbb{Q}$-divisor

$$H_i = \frac{1}{n_i}\sigma_i + \ell$$

on $\mathbb{F}_{n_i}$ and the graded ring $R^{\sharp i} := R(\mathbb{F}_{n_i}, H_i)$. Then $\operatorname{Proj} R^{\sharp i} \simeq \mathbb{P}(1, 1, n_i)$ and the natural birational morphism $\mathbb{F}_{n_i} \to \operatorname{Proj} R^{\sharp i}$ is isomorphic to the contraction morphism $\mathbb{F}_{n_i} \to \overline{\mathbb{F}}_{n_i}$ of σ_i, by Lemma 7.5. Since $p_i^*H_i \le H$, $R^{\sharp i}$ is regarded as a graded subring of R. We infer that the inclusion $R^{\sharp i} \subset R$ induces the projection $\mathbb{P}(1, 1, n_1, n_2) \dashrightarrow \mathbb{P}(1, 1, n_i)$ from the calculation in (1). Thus we are done. $\square$

7.3. Embedding into weighted projective spaces, I

Let (X, E, Δ) be a fundamental triplet defining a log del Pezzo surface S of index two. For the blowing-up $V \to X$ along Δ and for the minimal desingularization $\lambda\colon M \to V$, the composite $\phi\colon M \to X$ is just the elimination $(M, E_M) \to (X, E, \Delta)$. By Lemma 2.18 and by the vanishing $\mathrm{H}^1(X, L-E) = 0$ (cf. Lemma 3.17), we infer that V is a Cartier divisor of $\mathbb{P} = \mathbb{P}(\mathcal{E})$ for the locally free sheaf $\mathcal{E} = \mathcal{O}_X(L-E) \oplus \mathcal{O}_X$, where $\lambda^*\mathcal{O}_{\mathcal{E}}(1)|_V \simeq \mathcal{O}_M(L_M)$. An irreducible curve $\gamma \subset M$ is λ-exceptional if and only if γ is ϕ-exceptional and $L_M\gamma = 0$. Thus the minimal desingularization $\alpha\colon M \to S$ of S induces a morphism $\varphi\colon V \to S$ with $\alpha = \varphi \circ \lambda$. In particular, $\mathcal{O}_{\mathcal{E}}(1)|_V \simeq \varphi^*\mathcal{O}_S(-2K_S)$.

Let $\mathtt{u} \in \mathcal{O}_{\mathcal{E}}(1)$ and $\mathtt{v} \in \mathcal{O}_{\mathcal{E}}(1) \otimes p^*\mathcal{O}(E-L)$ be the global sections over $\mathbb{P}$ defined by the natural homomorphisms

$$\begin{aligned}\mathtt{u}\colon \mathcal{O}_X \ni s \mapsto (0, s) \in \mathcal{O}_X(L-E) \oplus \mathcal{O}_X,\\ \mathtt{v}\colon \mathcal{O}_X(L-E) \ni s \mapsto (s, 0) \in \mathcal{O}_X(L-E) \oplus \mathcal{O}_X.\end{aligned}$$

Let $\eta \in \mathrm{H}^0(X, E)$ be a defining equation of E. There exists a section $\xi \in \mathrm{H}^0(X, L)$ such that $\operatorname{div}(\xi|_E) = \Delta$ and $V \simeq V(\xi, \eta) = \operatorname{div}(p^*(\xi)\mathtt{v} - p^*(\eta)\mathtt{u})$ by Proposition 2.19.

The linear system $|\mathcal{O}_{\mathcal{E}}(1)|$ is base point free since $\operatorname{Bs}|L-E| = \operatorname{Bs}|2(K_X+L)| = \emptyset$ by Lemma 3.17. Let $\Phi'\colon \mathbb{P} \to \mathbb{P}|\mathcal{O}_{\mathcal{E}}(1)|$ be the morphism associated with $|\mathcal{O}_{\mathcal{E}}(1)|$ and let $\Phi\colon \mathbb{P} \to W$ be induced morphism as the Stein factorization of Φ'. The Stein factorization of $V \subset \mathbb{P} \to W$ is expresses as the composite of $\varphi\colon V \to S$ and a finite morphism $S \to W$.

PROPOSITION 7.8. *Suppose that $K_X + L$ is big. Then W is a three-dimensional toric variety and $\Phi\colon \mathbb{P} \to W$ is a birational toric morphism. Moreover, the image $\Phi(V)$ is a divisor of W and $\Phi(V) \simeq S$.*

PROOF. The morphism $\Phi\colon \mathbb{P} \to W$ is birational since $\mathcal{O}_{\mathcal{E}}(1)^3 = (L-E)^2 > 0$. If $K_X + L$ is ample, then the Φ-exceptional locus is the divisor $\operatorname{div}(\mathtt{v})$, which is contracted to a point. Since $\mathbb{P}$ has a structure of toric variety and $\operatorname{div}(\mathtt{v})$ is a $\mathbb{T}$-invariant divisor for the open torus $\mathbb{T} \subset \mathbb{P}$, the variety W and the morphism Φ are toric. If $K_X + L$ is not ample but big, then $X \simeq \mathbb{F}_2$ and $\mathcal{E}$ is isomorphic to the pullback of the locally free sheaf $\mathcal{O}(4) \oplus \mathcal{O}$ of $\mathbb{P}(1,1,2)$ by the contraction morphism $X \to \overline{\mathbb{F}}_2 \simeq \mathbb{P}(1,1,2)$

of the negative section. Thus W is isomorphic to the weighted projective space $\mathbb{P}(1,1,2,4)$ by Lemma 7.5; hence W and Φ are also toric.

From the linear equivalences $V \sim \mathcal{O}_{\mathcal{E}}(1)+p^*E$, $\operatorname{div}(\mathrm{v}) \sim \mathcal{O}_{\mathcal{E}}(1)-p^*(L-E)$, $L-E \sim 2(K_X+L)$, and $K_{\mathbb{P}} \sim p^*(K_X+L-E)-2\mathcal{O}_{\mathcal{E}}(1)$, we infer that

$$-V-(1/2)\operatorname{div}(\mathrm{v})-K_{\mathbb{P}} \sim_{\mathbb{Q}} (1/2)\mathcal{O}_{\mathcal{E}}(1)$$

is relatively numerically trivial for $\Phi\colon \mathbb{P} \to W$. Hence, if $\operatorname{char} \Bbbk = 0$, then $\mathrm{R}^1\,\Phi_*\mathcal{O}_{\mathbb{P}}(-V) = 0$ by the relative Kawamata–Viehweg vanishing theorem. By Leray's spectral sequence, the vanishing $\mathrm{R}^1\,\Phi_*\mathcal{O}_{\mathbb{P}}(-V)=0$ is equivalent to the vanishing $\mathrm{H}^1(\mathbb{P}, m\Phi^*A-V)=0$ for $m \gg 0$ for a $\mathbb{T}$-invariant ample divisor A of W. Recall that the cohomology group of an invertible sheaf on a toric variety is described by combinatorial data. Hence the vanishing is independent of $\operatorname{char}\Bbbk$. Therefore, $\mathrm{R}^1\,\Phi_*\mathcal{O}_{\mathbb{P}}(-V)=0$ holds, and consequently, $\mathcal{O}_W \simeq \Phi_*\mathcal{O}_{\mathbb{P}} \to \Phi_*\mathcal{O}_V$ is surjective. It follows that $\Phi(V)$ is normal and $\varphi^*(-2K_S)$ comes from an ample divisor on $\Phi(V)$. Therefore $S \simeq \Phi(V)$ and $\varphi \simeq \Phi|_V$. $\square$

Lemma 7.9. *Suppose that K_X+L is not big, i.e., the type of (X,E,Δ) is one of $[1;2,2]_0$, $[3;2,4]_+$, $[3;2,4]_{++}(a,b)$, and $[4;2,4]_{00}$. If $X \simeq \mathbb{F}_1$ or $X \simeq \mathbb{F}_3$, then $W \simeq \mathbb{P}(1,1,2)$. If $X = \mathbb{F}_4$, then $W \simeq \mathbb{P}(1,1,4)$. In the both cases, the induced finite morphism $S \to W$ is a double-covering.*

Proof. Suppose that (X,E) is of type $[n;2,e]$. Then $L-E \sim 2(K_X+L) \sim 2(n+2-e)\ell$ for a fiber ℓ of $\pi\colon X \to \mathbb{P}^1$. Hence, $\mathbb{P} \simeq \mathbb{F}_{2d} \times_{\mathbb{P}^1} X$ for $d = n+2-e \geq 1$ and Φ is the composite of the first projection $\mathbb{P} \to \mathbb{F}_{2d}$ and the contraction morphism $\mathbb{F}_{2d} \to \overline{\mathbb{F}}_{2d} \simeq \mathbb{P}(1,1,2d)$ of the negative section. In particular, $W \simeq \mathbb{P}(1,1,2d)$. The isomorphisms $\Phi^*\mathcal{O}_W(2d) \simeq \mathcal{O}_{\mathcal{E}}(1)$ and $\lambda^*(\mathcal{O}_{\mathcal{E}}(1)|_V) \simeq \mathcal{O}_M(L_M)$ induce

$$L_M^2 = \deg(V/W)\mathcal{O}_W(2d)^2 = 2d\deg(V/W).$$

On the other hand, we have

$$L_M^2 = L^2 - \deg(\Delta) = L(L-E) = 4(n+2-e) = 4d.$$

Hence, $\deg(V/W) = \deg(S/W) = 2$. Note that $d=2$ for the type $[4;2,4]_{00}$, and $d=1$ for the rest. $\square$

In the rest of Section 7.3, we shall embed S into a weighted projective space and give an explicit defining equation of S in the case where $K_X + L$ is big and S is not of type $[n; 1, 0]_0$. The case of types $[n; 1, 0]_0$ is studied in Section 7.4 below by another method. In Section 7.5 below, we treat the case where $K_X + L$ is not big by using Lemma 7.9. The list of defining equations is given in TABLE 14 at the end of this paper.

Here, we use the following:

Notation 7.10.

(1) Let $(\mathtt{s}, \mathtt{t})$ denote a homogeneous coordinate of $\mathbb{P}^1$. For a morphism $p\colon Z \to \mathbb{P}^1$, the pullbacks $p^*\mathtt{s}$ and $p^*\mathtt{t}$ are global sections of $p^*\mathcal{O}(1)$. Here, we write $p^*\mathtt{s} = \mathtt{s}$ and $p^*\mathtt{t} = \mathtt{t}$ for simplicity.

(2) For the Hirzebruch surface $X = \mathbb{F}_n$ with a fixed projection $X \to \mathbb{P}^1$, let σ be a minimal section and let σ_∞ be a section at infinity. A defining equation of σ is denoted by the symbol $\mathtt{g}$ and a defining equation of σ_∞ is denoted by the symbol $\mathtt{f}$. Here, $\mathtt{f}$ and $\mathtt{g}$ are regarded as the natural injections

$$\mathtt{f}\colon \mathcal{O} \ni s \mapsto (s, 0) \in \mathcal{O} \oplus \mathcal{O}(n)$$
$$\mathtt{g}\colon \mathcal{O}(n) \ni s \mapsto (0, s) \in \mathcal{O} \oplus \mathcal{O}(n).$$

Similarly to $\mathtt{s}$ and $\mathtt{t}$ above, the pullbacks $p^*\mathtt{f}$ and $p^*\mathtt{g}$ by a morphism $p\colon Z \to X$ are expressed by the same symbols $\mathtt{f}$ and $\mathtt{g}$, respectively.

PROPOSITION 7.11. *Suppose that* $X = \mathbb{P}^2$. *Then* W *is isomorphic to the weighted projective space* $\mathbb{P}(1, 1, 1, 2w)$ *for* $w = (1/2)\deg(L - E) = 3 - \deg E \in \{1, 2\}$. *Let* $(\mathtt{X}, \mathtt{Y}, \mathtt{Z}, \mathtt{U})$ *be a homogeneous coordinate system of* $\mathbb{P}(1, 1, 1, 2w)$.

(1) *Suppose that the type is* $[1]_0$. *Then* S *is isomorphic to*

$$\{F_5(\mathtt{Y}, \mathtt{Z}) = \mathtt{XU}\} \subset \mathbb{P}(1, 1, 1, 4)$$

for a quintic homogeneous polynomial $F_5 \neq 0$.

(2) *Suppose that the type is* $[2]_0$. *Then* S *is isomorphic to*

$$\{F_4(\mathtt{X}, \mathtt{Y}) + F_3(\mathtt{X}, \mathtt{Y})\mathtt{Z} = (\mathtt{Z}^2 - \mathtt{XY})\mathtt{U}\} \subset \mathbb{P}(1, 1, 1, 2)$$

for a cubic homogeneous polynomial F_3 and a quartic homogeneous polynomial F_4 with $(F_3, F_4) \neq (0,0)$.

(3) *Suppose that the type is $[2]_+(0)$. Then S is isomorphic to*

$$\{F_3(\mathtt{X},\mathtt{Z})\mathtt{X} + G_3(\mathtt{Y},\mathtt{Z})\mathtt{Y} + \mathtt{Z}^4 = \mathtt{XYU}\} \subset \mathbb{P}(1,1,1,2)$$

for cubic homogeneous polynomials F_3 and G_3.

(4) *Suppose that the type is $[2]_+(b)$ for $1 \leq b \leq 4$. Then S is isomorphic to*

$$\{F_{4-b}(\mathtt{X},\mathtt{Z})\mathtt{X}^b + G_3(\mathtt{Y},\mathtt{Z})\mathtt{Y} = \mathtt{XYU}\} \subset \mathbb{P}(1,1,1,2)$$

for a homogeneous polynomial F_{4-b} of degree $4-b$ and a cubic homogeneous polynomial G_3 with $F_{4-b}(0,1) \neq 0$, $G_3(0,1) \neq 0$.

In the descriptions above, $(0\!:\!0\!:\!0\!:\!1) \in W$ is the unique non-Gorenstein point of S.

Proof. $W \simeq \mathbb{P}(1,1,1,2w)$ since $\mathcal{E} = \mathcal{O}(L-E) \oplus \mathcal{O} = \mathcal{O}(2w) \oplus \mathcal{O}$. Let $(\mathtt{x},\mathtt{y},\mathtt{z})$ be a homogeneous coordinate of $\mathbb{P}^2$. We denote the pullbacks of $\mathtt{x}$, $\mathtt{y}$, and $\mathtt{z}$ to $\mathbb{P}$ by the same symbols, respectively, for simplicity. Then Φ is regarded as a morphism determined by the properties: $\Phi^*\mathtt{U} = \mathtt{u}$ and $\Phi^* P_{2w}(\mathtt{X},\mathtt{Y},\mathtt{Z}) = P_{2w}(\mathtt{x},\mathtt{y},\mathtt{z})\mathtt{v}$ for any homogeneous polynomial P_{2w} of weight $2w$ and for the homogeneous coordinate $(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{U})$ of W. Since $\lambda^* \operatorname{div}(\mathtt{v}) = E_M$, S has the unique non-Gorenstein point $(0\!:\!0\!:\!0\!:\!1)$.

(1): We may assume that $\eta = \mathtt{x}$ and $\xi = F_5(\mathtt{y},\mathtt{z})$ for a quintic homogeneous polynomial $F_5 \neq 0$. Then $\xi\mathtt{v} - \eta\mathtt{u} = F_5(\mathtt{y},\mathtt{z})\mathtt{v} - \mathtt{xu}$ and S is isomorphic to the non-Cartier divisor $\{F_5(\mathtt{Y},\mathtt{Z}) = \mathtt{XU}\}$ of degree 5 of $\mathbb{P}(1,1,1,4)$.

(2): We may assume that $\eta = \mathtt{z}^2 - \mathtt{xy}$. Then $E \simeq \mathbb{P}^1$ has a coordinate $(\mathtt{s},\mathtt{t})$ such that $\mathtt{x}|_E = \mathtt{s}^2$, $\mathtt{y}|_E = \mathtt{t}^2$, and $\mathtt{z}|_E = \mathtt{st}$. Let $F_8(\mathtt{s},\mathtt{t}) \neq 0$ be an octic homogeneous polynomial such that $\Delta = \operatorname{div}(F_8(\mathtt{s},\mathtt{t})) \subset E$. We can write

$$F_8(\mathtt{s},\mathtt{t}) = F_4(\mathtt{s}^2,\mathtt{t}^2) + F_3(\mathtt{s}^2,\mathtt{t}^2)\mathtt{st}$$

for a cubic homogeneous polynomial F_3 and a quartic homogeneous polynomial F_4. Then $\operatorname{div}(\xi) \cap E = \Delta$ and $V = V(\xi,\eta)$ for the global section

$$\xi = F_4(\mathtt{x},\mathtt{y}) + F_3(\mathtt{x},\mathtt{y})\mathtt{z}$$

of $\mathcal{O}_X(L) \simeq \mathcal{O}(4)$. Since

$$\xi\mathtt{v} - \eta\mathtt{u} = (F_4(\mathtt{x},\mathtt{y}) + F_3(\mathtt{x},\mathtt{y})\mathtt{z})\mathtt{v} - (\mathtt{z}^2 - \mathtt{x}\mathtt{y})\mathtt{u},$$

S is isomorphic to the Cartier divisor

$$\{F_4(\mathtt{X},\mathtt{Y}) + F_3(\mathtt{X},\mathtt{Y})\mathtt{Z} = (\mathtt{Z}^2 - \mathtt{X}\mathtt{Y})\mathtt{U}\} \subset \mathbb{P}(1,1,1,2).$$

(3) and (4): We may assume that $\eta = \mathtt{xy}$. Then $\Delta = \operatorname{div}(\xi) \cap E$ for

$$\xi = F_3(\mathtt{x},\mathtt{z})\mathtt{x} + G_3(\mathtt{y},\mathtt{z})\mathtt{y} + c\mathtt{z}^4$$

for cubic homogeneous polynomials F_3 and G_3, and for a constant $c \in \mathbb{k}$. Here, $c \neq 0$ if and only if the type of (X, E, Δ) is $[2]_+(0)$. If $c \neq 0$, then we may assume $c = 1$ by replacing ξ by a non-zero multiple of ξ. If the type is $[2]_+(b)$ for $b > 0$, then $c = 0$ and we may assume that $\operatorname{mult}_P(\Delta \cap \operatorname{div}(\mathtt{y})) = b$ and $\operatorname{mult}_P(\Delta \cap \operatorname{div}(\mathtt{x})) = 1$. Thus $F_3(\mathtt{x},\mathtt{y}) = \mathtt{x}^{b-1}F_{4-b}(\mathtt{x},\mathtt{y})$ for a homogeneous polynomial F_{4-b} of degree $4-b$ with $F_{4-b}(0,1) \neq 0$, and $G_3(0,1) \neq 0$. Since

$$\xi\mathtt{v} - \eta\mathtt{u} = (F_3(\mathtt{x},\mathtt{z})\mathtt{x} + G_3(\mathtt{y},\mathtt{z})\mathtt{y} + c\mathtt{z}^4)\mathtt{v} - \mathtt{xyu},$$

S is isomorphic to the Cartier divisor of $\mathbb{P}(1,1,1,2)$ defined by

$$F_3(\mathtt{X},\mathtt{Z})\mathtt{X} + G_3(\mathtt{Y},\mathtt{Z})\mathtt{Y} + c\mathtt{Z}^4 = \mathtt{XYU}. \ \square$$

PROPOSITION 7.12. *Let* $(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{U})$ *be a homogeneous coordinate of the weighted projective space* $\mathbb{P}(1,1,2,4)$*.*

(1) *A log del Pezzo surface of index two of type* $[2;1,2]_0$ *is isomorphic to*

$$\{F_6(\mathtt{X},\mathtt{Y}) = \mathtt{ZU}\} \subset \mathbb{P}(1,1,2,4)$$

for a sextic homogeneous polynomial $F_6 \neq 0$*.*

(2) *A log del Pezzo surface of index two of type* $[2;1,2]_{++}$ *is isomorphic to*

$$\{\mathtt{Z}^3 + \mathtt{X}^2\mathtt{Z}F_1(\mathtt{Z},\mathtt{X}^2) + \mathtt{Y}^2\mathtt{Z}G_1(\mathtt{Z},\mathtt{Y}^2) = \mathtt{XYU}\} \subset \mathbb{P}(1,1,2,4)$$

for linear polynomials F_1 *and* G_1*.*

PROOF. For the fundamental triplet (X, E, Δ), we have $X \simeq \mathbb{F}_2$, $E \sim \sigma + 2\ell$, and $L \sim 3(\sigma + 2\ell)$. For a suitable homogeneous coordinate $(\mathtt{X}, \mathtt{Y}, \mathtt{Z})$ of $\mathbb{P}(1,1,2)$, the contraction morphism $q\colon X \to \mathbb{P}(1,1,2)$ of the negative section satisfies the following properties: $q^*\mathtt{Z} = \mathtt{f}$ and $q^* P_2(\mathtt{X}, \mathtt{Y}) = P_2(\mathtt{s}, \mathtt{t})\mathtt{g}$ for any quadric homogeneous polynomial P_2. Note that $q^*\mathcal{O}(2) \simeq \mathcal{O}_X(\sigma + 2\ell)$ and $\mathbb{P}_X(\mathcal{E}) \to X$ is isomorphic to the pullback of $\mathbb{P}(\mathcal{O}(4) \oplus \mathcal{O}) \to \mathbb{P}(1,1,2)$ by q. Hence $W \simeq \mathbb{P}(1,1,2,4)$ by Lemma 7.5. Thus the morphism $\Phi\colon \mathbb{P}_X(\mathcal{E}) \to W \simeq \mathbb{P}(1,1,2,4)$ satisfies the following properties:

- $\Phi^*\mathtt{U} = \mathtt{u}$;
- $\Phi^*(\mathtt{X}^i\mathtt{Y}^j\mathtt{Z}) = \mathtt{s}^i\mathtt{t}^j\mathtt{f}\mathtt{v}$ for $(i,j) = (1,0)$, $(0,1)$;
- $\Phi^* P_4(\mathtt{X}, \mathtt{Y}) = P_4(\mathtt{s}, \mathtt{t})\mathtt{g}\mathtt{v}$ for any quartic homogeneous polynomial P_4.

Case $[2;1,2]_0$. We may assume $\eta = \mathtt{f}$. There is a sextic homogeneous polynomial $F_6 \neq 0$ such that $\operatorname{div}(\xi) \cap E = \Delta$ for $\xi = F_6(\mathtt{s}, \mathtt{t})\mathtt{g}^3$. Since

$$\xi\mathtt{v} - \eta\mathtt{u} = F_6(\mathtt{s}, \mathtt{t})\mathtt{g}^3\mathtt{v} - \mathtt{f}\mathtt{u},$$

S is isomorphic to the divisor $\{F_6(\mathtt{X}, \mathtt{Y}) = \mathtt{Z}\mathtt{U}\} \subset \mathbb{P}(1,1,2,4)$.

Case $[2;1,2]_{++}$. We may assume $\eta = \mathtt{s}\mathtt{t}\mathtt{g}$. Moreover, we may assume that Δ contains the points $\{\mathtt{f} = \mathtt{s} = 0\}$ and $\{\mathtt{f} = \mathtt{t} = 0\}$. Then $\operatorname{div}(\xi) \cap E = \Delta$ for

$$\xi = \mathtt{f}^3 + \mathtt{s}^2\mathtt{f}\mathtt{g}F_1(\mathtt{f}, \mathtt{s}^2\mathtt{g}) + \mathtt{t}^2\mathtt{f}\mathtt{g}G_1(\mathtt{f}, \mathtt{t}^2\mathtt{g})$$

for certain linear polynomials F_1 and G_1. Since

$$\xi\mathtt{v} - \eta\mathtt{u} = \left(\mathtt{f}^3 + \mathtt{s}^2\mathtt{f}\mathtt{g}F_1(\mathtt{f}, \mathtt{s}^2\mathtt{g}) + \mathtt{t}^2\mathtt{f}\mathtt{g}G_1(\mathtt{f}, \mathtt{t}^2\mathtt{g})\right)\mathtt{v} - \mathtt{s}\mathtt{t}\mathtt{g}\mathtt{u},$$

S is isomorphic to

$$\{\mathtt{Z}^3 + \mathtt{X}^2\mathtt{Z}F_1(\mathtt{Z}, \mathtt{X}^2) + \mathtt{Y}^2\mathtt{Z}G_1(\mathtt{Z}, \mathtt{Y}^2) = \mathtt{X}\mathtt{Y}\mathtt{U}\} \subset \mathbb{P}(1,1,2,4). \square$$

PROPOSITION 7.13. *Let (X, E, Δ) be a fundamental triplet for $X \simeq \mathbb{F}_n$ and $E \sim \sigma + \ell$. Then W is isomorphic to the divisor*

$$\{\mathtt{X}\mathtt{W} = \mathtt{Y}\mathtt{Z}\} \subset \mathbb{P}(1,1,n+1,n+1,2(n+1))$$

for a homogeneous coordinate $(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{W},\mathtt{U})$ *of weight* $(1,1,n+1,n+1,2(n+1))$. *Moreover, the log del Pezzo surface* S *of index two associated with* (X,E,Δ) *is isomorphic to a subvariety of* W *defined by the following equations*:

Type $[0;1,1]_0$**:**

$$F_2(\mathtt{Z},\mathtt{W})\mathtt{W}+G_2(\mathtt{W},\mathtt{Y})\mathtt{Y}=(\mathtt{X}-\mathtt{W})\mathtt{U},$$

for quadric polynomials F_2 *and* G_2 *with* $(F_2,G_2)\neq(0,0)$.

Type $[0;1,1]_+(0)$**:**

$$\mathtt{W}^3+F_1(\mathtt{Z},\mathtt{W})\mathtt{ZW}=\mathtt{XU}-G_1(\mathtt{W},\mathtt{Y})\mathtt{YW},$$

for linear polynomials F_1 *and* G_1.

Type $[0;1,1]_+(1)$**:**

$$(\mathtt{W}+c\mathtt{Z})\mathtt{ZW}=\mathtt{XU}-(\mathtt{W}+c'\mathtt{Y})\mathtt{YW},$$

for constants c, $c'\in\Bbbk$.

Type $[0;1,1]_+(b)$ **for** $b>1$**:**

$$(\mathtt{W}+c\mathtt{Z})\mathtt{ZW}=\mathtt{XU}-\mathtt{W}^{3-b}\mathtt{Y}^b,$$

for a constant $c\in\Bbbk$.

Type $[1;1,1]_0$**:**

$$F_5(\mathtt{X},\mathtt{Y})\mathtt{X}=\mathtt{ZU},\quad F_5(\mathtt{X},\mathtt{Y})\mathtt{Y}=\mathtt{WU},$$

for a quintic homogeneous polynomial $F_5\neq0$.

Type $[1;1,1]_+(0,0)$**:**

$$\begin{aligned}(\mathtt{W}+c\mathtt{Z})\mathtt{ZW}&=(\mathtt{XU}-G_1(\mathtt{W},\mathtt{Y}^2)\mathtt{YW})\mathtt{X},\\(\mathtt{W}+c\mathtt{Z})\mathtt{W}^2&=(\mathtt{XU}-G_1(\mathtt{W},\mathtt{Y}^2)\mathtt{YW})\mathtt{Y},\end{aligned}$$

for a constant c *and a linear polynomial* G_1.

Type $[1;1,1]_+(1,1)$**:**

$$\mathtt{Z}^2\mathtt{W} = (\mathtt{XU} - (\mathtt{W} + c\mathtt{Y}^2)\mathtt{YW})\mathtt{X}, \quad \mathtt{ZW}^2 = (\mathtt{XU} - (\mathtt{W} + c\mathtt{Y}^2)\mathtt{YW})\mathtt{Y},$$

for a constant $c \in \Bbbk$.

Type $[1;1,1]_+(2,1)$**:**

$$\mathtt{Z}^3 = (\mathtt{XU} - (\mathtt{W} + c\mathtt{Y}^2)\mathtt{YW})\mathtt{X}, \quad \mathtt{Z}^2\mathtt{W} = (\mathtt{XU} - (\mathtt{W} + c\mathtt{Y}^2)\mathtt{YW})\mathtt{Y},$$

for a constant $c \in \Bbbk$.

Type $[1;1,1]_+(1,b)$ **for** $b > 1$**:**

$$\mathtt{Z}^2\mathtt{W} = (\mathtt{XU} - \mathtt{Y}^{2b-1}\mathtt{W}^{3-b})\mathtt{X}, \quad \mathtt{ZW}^2 = (\mathtt{XU} - \mathtt{Y}^{2b-1}\mathtt{W}^{3-b})\mathtt{Y}.$$

Type $[2;1,1]_+(0,0)$**:**

$$\mathtt{Z}^{2-i}\mathtt{W}^{i+1} = (\mathtt{XU} - G_1(\mathtt{W}, \mathtt{Y}^3)\mathtt{YW})\mathtt{X}^{2-i}\mathtt{Y}^i,$$

for $0 \le i \le 2$ *for a linear polynomial* G_1.

Type $[2;1,1]_+(1,1)$**:**

$$\mathtt{Z}^{3-i}\mathtt{W}^i = (\mathtt{XU} - (\mathtt{W} + c\mathtt{Y}^3)\mathtt{YW})\mathtt{X}^{2-i}\mathtt{Y}^i,$$

for $0 \le i \le 2$ *for a constant* $c \in \Bbbk$.

Type $[2;1,1]_+(1,b)$ **for** $b > 1$**:**

$$\mathtt{Z}^{3-i}\mathtt{W}^i = (\mathtt{XU} - \mathtt{Y}^{3b-2}\mathtt{W}^{3-b})\mathtt{X}^{2-i}\mathtt{Y}^i,$$

for $0 \le i \le 2$.

Type $[3;1,1]_+$**:**

$$\mathtt{Z}^{3-i}\mathtt{W}^i = (\mathtt{XU} - G_1(\mathtt{W}, \mathtt{Y}^4)\mathtt{YW})\mathtt{X}^{3-i}\mathtt{Y}^i,$$

for $0 \le i \le 3$ *for a linear polynomial* G_1.

Proof. Let $X \hookrightarrow \mathbb{P}(1,1,n+1,n+1)$ be the embedding of Lemma 7.6. Then $\mathcal{E}$ is isomorphic to the restriction of $\mathcal{O}(2(n+1))\oplus\mathcal{O}$ since $L-E \sim 2(\sigma+(n+1)\ell)$. Hence, W is isomorphic to $\{\mathtt{XW}=\mathtt{YZ}\}$ in $\mathbb{P}(1,1,n+1,n+1,2(n+1))$ by Lemma 7.5.

For a defining equation $\eta \in \mathrm{H}^0(X,\sigma+\ell)$ of E and for a section $\xi \in \mathrm{H}^0(X,3\sigma+(2n+3)\ell)$ with $\mathrm{div}(\xi)\cap E=\Delta$, S is isomorphic to the image of $V=V(\xi,\eta)$ under the morphism $\Phi\colon \mathbb{P}(\mathcal{E}) \to W \subset \mathbb{P}(1,1,n+1,n+1,2(n+1))$. Here, we have

$$\begin{aligned} \Phi^*\mathtt{U} &= \mathtt{u}, & \Phi^*Q_2(\mathtt{Z},\mathtt{W}) &= Q_2(\mathtt{s},\mathtt{t})\mathtt{f}^2\mathtt{v},\\ \Phi^*(Q_{n+1}(\mathtt{X},\mathtt{Y})\mathtt{Z}) &= Q_{n+1}(\mathtt{s},\mathtt{t})\mathtt{sfgv}, & \Phi^*(Q_{n+1}(\mathtt{X},\mathtt{Y})\mathtt{W}) &= Q_{n+1}(\mathtt{s},\mathtt{t})\mathtt{tfgv},\\ \Phi^*Q_{2(n+1)}(\mathtt{X},\mathtt{Y}) &= Q_{2(n+1)}(\mathtt{s},\mathtt{t})\mathtt{g}^2\mathtt{v}, \end{aligned}$$

for any homogeneous polynomial $Q_j(\mathtt{s},\mathtt{t})$ of degree $j\in\{2,n+1,2(n+1)\}$. The global section ξ is written as

$$\xi = P^{(0)}_{3-n}(\mathtt{s},\mathtt{t})\mathtt{f}^3 + P^{(1)}_{3}(\mathtt{s},\mathtt{t})\mathtt{f}^2\mathtt{g} + P^{(2)}_{n+3}(\mathtt{s},\mathtt{t})\mathtt{f}\mathtt{g}^2 + P^{(3)}_{2n+3}(\mathtt{s},\mathtt{t})\mathtt{g}^3 \tag{7–4}$$

for some homogeneous polynomials $P^{(i)}_j(\mathtt{s},\mathtt{t})$ of degree $j=3+n(i-1)$ for $0\le i\le 3$.

We first treat the case where E is non-singular, i.e., the type is $[0;1,1]_0$ or $[1;1,1]_0$.

Case $[0;1,1]_0$. We may assume $\eta=\mathtt{sg}-\mathtt{tf}$. We may assume that the point $E\cap\mathrm{div}(\mathtt{t})=\{\mathtt{g}=\mathtt{t}=0\}$ is contained in Δ. By (7–4), ξ is written as

$$\xi = \mathtt{t}\left(F_2(\mathtt{s},\mathtt{t})\mathtt{f}^3 + G_2(\mathtt{f},\mathtt{g})\mathtt{t}^2\mathtt{g}\right)$$

for certain quadric polynomials F_2 and G_2 with $(F_2,G_2)\neq(0,0)$. Thus

$$\xi\mathtt{v}-\eta\mathtt{u} = \mathtt{t}\left(F_2(\mathtt{s},\mathtt{t})\mathtt{f}^3 + G_2(\mathtt{f},\mathtt{g})\mathtt{t}^2\mathtt{g}\right)\mathtt{v} - (\mathtt{sg}-\mathtt{tf})\mathtt{u}.$$

We define a weighted homogeneous polynomial $\Xi=\Xi(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{W},\mathtt{U})$ of degree 3 by

$$\Xi := F_2(\mathtt{Z},\mathtt{W})\mathtt{W} + G_2(\mathtt{W},\mathtt{Y})\mathtt{Y} - (\mathtt{X}-\mathtt{W})\mathtt{U}.$$

Then we have

$$\begin{aligned} \Phi^*(\mathtt{X}\Xi) &= \mathtt{sgv}(\xi\mathtt{v}-\eta\mathtt{u}), & \Phi^*(\mathtt{Y}\Xi) &= \mathtt{tgv}(\xi\mathtt{v}-\eta\mathtt{u}),\\ \Phi^*(\mathtt{Z}\Xi) &= \mathtt{sfv}(\xi\mathtt{v}-\eta\mathtt{u}), & \Phi^*(\mathtt{W}\Xi) &= \mathtt{tfv}(\xi\mathtt{v}-\eta\mathtt{u}). \end{aligned}$$

Thus $\Phi(V(\xi,\eta))$ is the prime divisor of W defined by $\{\Xi = 0\}$.

Case $[1;1,1]_0$. We may assume $\eta = \mathtt{f}$ and $\xi = F_5(\mathtt{s},\mathtt{t})\mathtt{g}^3$ for a quintic homogeneous polynomial $F_5 \neq 0$ by (7–4). Then $\xi\mathtt{v} - \eta\mathtt{u} = F_5(\mathtt{s},\mathtt{t})\mathtt{g}^3\mathtt{v} - \mathtt{fu}$. We define weighted homogeneous polynomials $\Xi_i = \Xi_i(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{W},\mathtt{U})$ for $i = 1$, 2 of degree 6 by

$$\Xi_1 = F_5(\mathtt{X},\mathtt{Y})\mathtt{X} - \mathtt{ZU}, \quad \Xi_2 = F_5(\mathtt{X},\mathtt{Y})\mathtt{Y} - \mathtt{WU}.$$

Then we have

$$\text{(7–5)} \qquad \begin{aligned} \Phi^*(\mathtt{X}^2\Xi_1) &= \mathtt{s}^3\mathtt{gv}(\xi\mathtt{v} - \eta\mathtt{u}), & \Phi^*(\mathtt{Y}^2\Xi_2) &= \mathtt{t}^3\mathtt{gv}(\xi\mathtt{v} - \eta\mathtt{u}), \\ \Phi^*(\mathtt{Z}\Xi_1) &= \mathtt{s}^2\mathtt{fv}(\xi\mathtt{v} - \eta\mathtt{u}), & \Phi^*(\mathtt{W}\Xi_2) &= \mathtt{t}^2\mathtt{fv}(\xi\mathtt{v} - \eta\mathtt{u}). \end{aligned}$$

Thus the prime divisor $\Phi(V(\xi,\eta))$ of W is just the reduced part of the subscheme of $\mathbb{P}(1,1,2,2,4)$ defined by the ideal $J \subset \Bbbk[\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{W},\mathtt{U}]$ generated by $\mathtt{XW} - \mathtt{YZ}$, Ξ_1, and Ξ_2. We shall show that the subscheme is reduced and equals $\Phi(V(\xi,\eta))$. Let A be the affine ring of the open subset $\{\mathtt{U} \neq 0\}$ in $\mathbb{P}(1,1,2,2,4)$. Then A is regarded as a subring of the usual polynomial ring $R = \Bbbk[\mathtt{x},\mathtt{y},\mathtt{z},\mathtt{w}]$ of four variables by $\mathtt{X} \mapsto \mathtt{x}$, $\mathtt{Y} \mapsto \mathtt{y}$, $\mathtt{Z} \mapsto \mathtt{z}$, $\mathtt{W} \mapsto \mathtt{w}$, $\mathtt{U} \mapsto 1$. Let $I \subset R$ be the ideal generated by $\mathtt{xw} - \mathtt{yz}$, $F_5(\mathtt{x},\mathtt{y})\mathtt{x} - \mathtt{z}$, $F_5(\mathtt{x},\mathtt{y}) - \mathtt{w}$. Then $R/I \simeq \Bbbk[\mathtt{x},\mathtt{y}]$ and hence J is reduced on the open subset U. Combining with (7–5), we infer that $\Phi(V(\xi,\eta))$ is defined by the ideal J.

Next, we treat the case where E is singular. Then $E = \sigma + \ell$ for a minimal section σ and a fiber ℓ. We may assume that $\ell = \operatorname{div}(\mathtt{s})$, $\eta = \mathtt{sg}$, and

$$\xi = P_{3-n}(\mathtt{s},\mathtt{t})\mathtt{f}^3 + G_2(\mathtt{f},\mathtt{t}^n\mathtt{g})\mathtt{t}^3\mathtt{g}$$

for a homogeneous polynomial P_{3-n} of degree $3-n$ and for a quadric homogeneous polynomial G_2 by (7–4). Thus

$$\xi\mathtt{v} - \eta\mathtt{u} = \left(P_{3-n}(\mathtt{s},\mathtt{t})\mathtt{f}^3 + G_2(\mathtt{f},\mathtt{t}^n\mathtt{g})\mathtt{t}^3\mathtt{g}\right)\mathtt{v} - \mathtt{sgu}.$$

We define weighted homogeneous polynomials Ξ_i for $0 \leq i \leq n$ of degree $3(n+1)$ with respect to $(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{W},\mathtt{U})$ by

$$\Xi_i := P_{3-n}(\mathtt{Z},\mathtt{W})\mathtt{Z}^{n-i}\mathtt{W}^i + (G_2(\mathtt{W},\mathtt{Y}^{n+1})\mathtt{Y} - \mathtt{XU})\mathtt{X}^{n-i}\mathtt{Y}^i.$$

Then we have

$$\begin{aligned}\Phi^*(\mathtt{X}^{n+1}\Xi_i) &= \mathtt{s}^{2n+1-i}\mathtt{t}^i\mathtt{g}\mathtt{v}(\xi\mathtt{v}-\eta\mathtt{u}),\\ \Phi^*(\mathtt{Y}^{n+1}\Xi_i) &= \mathtt{s}^{n-i}\mathtt{t}^{n+1+i}\mathtt{g}\mathtt{v}(\xi\mathtt{v}-\eta\mathtt{u}),\\ \Phi^*(\mathtt{Z}\Xi_i) &= \mathtt{s}^{n+1-i}\mathtt{t}^i\mathtt{f}\mathtt{v}(\xi\mathtt{v}-\eta\mathtt{u}),\\ \Phi^*(\mathtt{W}\Xi_i) &= \mathtt{s}^{n-i}\mathtt{t}^{i+1}\mathtt{f}\mathtt{v}(\xi\mathtt{v}-\eta\mathtt{u}),\end{aligned} \tag{7–6}$$

for $0 \le i \le n$. $\square$

Claim. The subscheme $\Phi(V(\xi,\eta))$ of $\mathbb{P}(1,1,n+1,n+1,2(n+1))$ is defined by $\mathtt{XW}-\mathtt{YZ}=\Xi_0=\cdots=\Xi_n=0$.

Proof. Let A be the affine ring of $\{\mathtt{U}\neq 0\}$ in the weighted projective space $\mathbb{P}(1,1,n+1,n+1,2(n+1)) = \operatorname{Proj}\Bbbk[\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{W},\mathtt{U}]$. Then A is a subring of the usual polynomial ring $R=\Bbbk[\mathtt{x},\mathtt{y},\mathtt{z},\mathtt{w}]$ by $\mathtt{X}\mapsto\mathtt{x}$, $\mathtt{Y}\mapsto\mathtt{y}$, $\mathtt{Z}\mapsto\mathtt{z}$, $\mathtt{W}\mapsto\mathtt{w}$, $\mathtt{U}\mapsto 1$. Let $I\subset R$ be the ideal generated by $\mathtt{xw}-\mathtt{yz}$ and

$$\Xi_i(\mathtt{x},\mathtt{y},\mathtt{z},\mathtt{w}) = P_{3-n}(\mathtt{z},\mathtt{w})\mathtt{z}^{n-i}\mathtt{w}^i + G_2(\mathtt{w},\mathtt{y}^{n+1})\mathtt{x}^{n-i}\mathtt{y}^{i+1} - \mathtt{x}^{n+1-i}\mathtt{y}^i$$

for $0\le i\le n$. By (7–6), it is enough to check that R/I has no non-zero ideal supported at the origin. We set

$$\Psi_i = \Xi_i + G_2(\mathtt{w},\mathtt{y}^{n+1})\Xi_{i+1} + \cdots + G_2(\mathtt{w},\mathtt{y}^{n+1})^{n-i}\Xi_n$$

for $0\le i\le n$. We have an isomorphism $R/(\Psi_0)\simeq\bigoplus_{i=0}^n\Bbbk[\mathtt{y},\mathtt{z},\mathtt{w}]\mathtt{x}^i$ as a $\Bbbk[\mathtt{y},\mathtt{z},\mathtt{w}]$-module. Hence, $R/(\Xi_0,\Xi_1,\ldots,\Xi_n) = R/(\Psi_0,\Psi_1,\ldots,\Psi_n)$ is isomorphic to

$$\Bbbk[\mathtt{y},\mathtt{z},\mathtt{w}]\oplus\bigoplus\nolimits_{i=1}^{n}\left(\Bbbk[\mathtt{y},\mathtt{z},\mathtt{w}]/(\mathtt{y}^{n+1-i})\right)\mathtt{x}^i.$$

Therefore, we have an isomorphism

$$R/I\simeq\Bbbk[\mathtt{y},\mathtt{z},\mathtt{w}]\oplus\bigoplus\nolimits_{i=1}^{n}\left(\Bbbk[\mathtt{y},\mathtt{z},\mathtt{w}]/(\mathtt{y}^{n+1-i},\mathtt{w})\right)\mathtt{x}^i$$

as a $\Bbbk[\mathtt{y},\mathtt{z},\mathtt{w}]$-module. In particular, R/I is a torsion-free $\Bbbk[\mathtt{z}]$-module. Hence, R/I has no non-zero ideal supported at the origin. $\square$

Proof of Proposition 7.13 continued. In the next step, we shall normalize P_{3-n} and G_2. Let P be the node $\sigma\cap\ell=\{\mathtt{s}=\mathtt{g}=0\}$. If

$\Delta \cap \sigma \setminus \{P\} \neq \emptyset$, then we may assume that $\Delta \cap \sigma$ contains $\{\mathtt{t} = 0\} \cap \sigma$ by replacing $(\mathtt{s}, \mathtt{t})$ with $(\mathtt{s}, \mathtt{t} + c_1\mathtt{s})$ for a constant $c_1 \in \Bbbk$. If $\Delta \cap \ell \setminus \{P\} \neq \emptyset$, then we may assume that $\Delta \cap \ell$ contains $\{\mathtt{f} = 0\} \cap \ell$ by replacing $(\mathtt{f}, \mathtt{g})$ with $(\mathtt{f} + c_2\mathtt{t}^n\mathtt{g}, \mathtt{g})$ for a constant $c_2 \in \Bbbk$. We may also replace (P_{3-n}, G_2) with $(\lambda_1 P_{3-n}, \lambda_2 G_2)$ for any non-zero constants λ_1, $\lambda_2 \in \Bbbk$. The normalization is done as follows:

Case 1. $P \notin \Delta = \operatorname{div}(\xi) \cap E$: Then the type is one of $[0; 1, 1]_+(0)$, $[1; 1, 1]_+(0, 0)$, $[2; 1, 1]_+(0, 0)$, and $[3; 1, 1]_+$. Here, we have $P_{3-n}(0, 1) \neq 0$. If $n < 3$, then $P_{3-n}(1, 0) = 0$, by the assumption. Similarly, $G_2(0, 1) = 0$, by the assumption. Thus we can write

$$P_{3-n}(\mathtt{s}, \mathtt{t}) = \mathtt{t}^{3-n} + \mathtt{st}F_{1-n}(\mathtt{s}, \mathtt{t}) \quad \text{and} \quad G_2(x, y) = xG_1(x, y)$$

for a homogeneous polynomial F_{1-n} of degree $1-n$ and a linear polynomial G_1.

Case 2. $P \in \Delta$ and $\operatorname{mult}_P(\Delta \cap \sigma) > 1$: If $n = 0$, then we may change the first and second projections $\mathbb{F}_0 \to \mathbb{P}^1$ and may assume that $\operatorname{mult}_P(\Delta \cap \sigma) = 1$; thus the case $n = 0$ is treated in *Case* 3 below. Then we may assume $n > 0$, and hence the type $[1; 1, 1]_+(2, 1)$ remains only. Since $\operatorname{mult}_P(\Delta \cap \sigma) = 2$ and $\operatorname{mult}_P(\Delta \cap \ell) = 1$, we can write

$$P_{3-n}(\mathtt{s}, \mathtt{t}) = \mathtt{s}^2 \quad \text{and} \quad G_2(x, y) = x(x + cy)$$

for a constant $c \in \Bbbk$.

Case 3. $P \in \Delta$ and $\operatorname{mult}_P(\Delta \cap \sigma) = 1$: Then $0 \le n \le 2$ and $1 \le b \le 3$ for $b = \operatorname{mult}_P(\Delta \cap \ell)$. If $n \le 1$, then $P_{3-n}(1, 0) = 0$, and if $b < 3$, then $G_2(1, 0) = 0$, by assumption. Thus we can write

$$P_{3-n}(\mathtt{s}, \mathtt{t}) = \begin{cases} \mathtt{st}(\mathtt{t} + c\mathtt{s}), & \text{if } n = 0; \\ \mathtt{st}^{2-n}, & \text{if } n > 0, \end{cases} \quad \text{and}$$

$$G_2(x, y) = \begin{cases} x(x + c'y), & \text{if } b = 1; \\ x^{3-b}y^{b-1}, & \text{if } b > 1, \end{cases}$$

for constants c, $c' \in \Bbbk$.

Applying the normalization to each type, we have the list of defining equations of $\Phi(V(\xi, \eta))$. $\square$

REMARK 7.14. In Proposition 7.13, if $n = 1$, then S is defined by three equations in $\mathbb{P}(1, 1, 2, 2, 4)$ as a subvariety of codimension two. These equations are written as the 2×2-minors of a matrix of size 2×3. In particular, the description of S is the same style as in [28, Theorem 1] (cf. [7, Theorem 5.1]).

7.4. Embedding into weighted projective spaces, II

In Section 7.3, we do not consider the types $[n; 1, 0]_0$ for $0 \le n \le 4$ among the case where $K_X + L$ is big. The log del Pezzo surfaces of these types are described by:

THEOREM 7.15. *Let S be a log del Pezzo surface of the type $[n; 1, 0]_0$ for $0 \le n \le 4$.*

(1) *If $n = 4$, then $S \simeq \mathbb{P}(1, 1, 4)$.*

(2) *If $1 \le n \le 3$, then S is isomorphic to the subvariety of $\mathbb{P}(1, 1, n) \times \mathbb{P}(1, 1, 4)$ defined by the following equations:*

$$\mathtt{X}_0\mathtt{Y}_1 = \mathtt{X}_1\mathtt{Y}_0, \quad \mathtt{Z}_1\mathtt{X}_0^{n-i}\mathtt{Y}_0^i = \mathtt{Z}_0\mathtt{X}_1^{n-i}\mathtt{Y}_1^i F_{4-n}(\mathtt{X}_1, \mathtt{Y}_1) \quad \textit{for} \quad 0 \le i \le n,$$

where $(\mathtt{X}_0, \mathtt{Y}_0, \mathtt{Z}_0)$ and $(\mathtt{X}_1, \mathtt{Y}_1, \mathtt{Z}_1)$ are homogeneous coordinates of $\mathbb{P}(1, 1, n)$ and $\mathbb{P}(1, 1, 4)$, respectively, and F_j is a non-zero homogeneous polynomial of degree j.

(3) *If $n = 0$, then S is isomorphic to the subvariety of $\mathbb{P}^1 \times \mathbb{P}(1, 1, 4)$ defined by*

$$\mathtt{Z}_1\mathtt{X}_0 = \mathtt{Y}_0 F_4(\mathtt{X}_1, \mathtt{Y}_1)$$

for a quartic homogeneous polynomial $F_4 \neq 0$, where $(\mathtt{X}_0, \mathtt{Y}_0)$ is a coordinate of $\mathbb{P}^1$.

For the proof, we apply the result of Section 7.1. For a given S, the fundamental triplet (X, E, Δ) defining S is uniquely determined up to isomorphism. Here, $X \simeq \mathbb{F}_n$, $E = \sigma$, and $\deg \Delta = 4 - n$. For the elimination $\phi\colon (M, E_M) \to (X, E, \Delta)$, M is obtained also as the elimination $\mu\colon M \to \mathbb{F}_4$ of a zero-dimensional subscheme $\mathbb{D}'$ of a section $\sigma_\infty^{(4)}$ at infinity, by Section 7.1. Moreover, by Proposition 7.1, S is realized as the blowing up of $\mathbb{P}(1,1,4)$ along the zero-dimensional subscheme $\mathbb{D} = q(\mathbb{D}')$ for the contraction morphism $q\colon \mathbb{F}_4 \to \mathbb{P}(1,1,4)$ of the negative section $\sigma^{(4)}$ of $\mathbb{F}_4$. In order to prove Theorem 7.15, it suffices to consider the case: $n \neq 4$, since $\deg \mathbb{D} = 4-n$. There is an effective divisor $B \sim (4-n)\ell$ such that $\mathbb{D}' = \sigma^{(4)}_\infty \cap B$. Let $\mathtt{u}$ and $\mathtt{v}$ be the defining equation of $\sigma^{(4)}_\infty$ and $\sigma^{(4)}$, respectively. For the homogeneous coordinate $(\mathtt{s},\mathtt{t})$ of $\mathbb{P}^1$, let $F_d(\mathtt{s},\mathtt{t})$ be a homogeneous polynomial of degree $d = 4-n$ with $B = \operatorname{div}(F_d(\mathtt{s},\mathtt{t}))$ (cf. Notation 7.10). Then $\mathbb{D} = \operatorname{div}(\mathtt{u}) \cap \operatorname{div}(\mathtt{v}F_d(\mathtt{s},\mathtt{t}))$. The proper transform of $\sigma^{(4)}$ in $X \simeq \mathbb{F}_n$ by the birational map $\mu \circ \phi^{-1}\colon X \dashrightarrow M \dashrightarrow \mathbb{F}_4$ is just $E = \sigma$. Similarly, the proper transform of $\sigma^{(4)}_\infty$ in X is a section σ_∞ at infinity. We have fixed the defining equations $\mathtt{f}$ and $\mathtt{g}$ of σ_∞ and σ, respectively, of $X \simeq \mathbb{F}_n$ as in Notation 7.10. Then, the image of $(\phi,\mu)\colon M \to X \times_{\mathbb{P}^1} \mathbb{F}_4$ is a divisor V defined by

$$\mathtt{ug} = \mathtt{vf}F_d(\mathtt{s},\mathtt{t}). \tag{7–7}$$

We set $W = \mathbb{P}(1,1,n) \times \mathbb{P}(1,1,4)$ in case $n \neq 0$, and $W = \mathbb{P}^1 \times \mathbb{P}(1,1,4)$ in case $n = 0$. Let $h\colon X \times_{\mathbb{P}^1} \mathbb{F}_4 \to W$ be the natural morphism. We shall find explicit defining equations of the image $h(V)$, and show that $h(V) \simeq S$.

Suppose that $1 \leq n \leq 3$. Then the image of $h\colon X \times_{\mathbb{P}^1} \mathbb{F}_4 \to W$ is defined by $\mathtt{X}_0\mathtt{Y}_1 = \mathtt{X}_1\mathtt{Y}_0$. In fact, we can choose the homogeneous coordinates to satisfy $h^*P_n(\mathtt{X}_0,\mathtt{Y}_0) = P_n(\mathtt{s},\mathtt{t})\mathtt{g}$, $h^*\mathtt{Z}_0 = \mathtt{f}$, $h^*P_4(\mathtt{X}_1,\mathtt{Y}_1) = P_4(\mathtt{s},\mathtt{t})\mathtt{v}$, and $h^*\mathtt{Z}_1 = \mathtt{u}$, for homogeneous polynomials P_j of degree j. By the equation (7–7), $h(V)$ is contained in the subscheme $S' \subset W$ defined by

$$\mathtt{X}_0\mathtt{Y}_1 = \mathtt{X}_1\mathtt{Y}_0, \quad \mathtt{Z}_1\mathtt{X}_0^{n-i}\mathtt{Y}_i = \mathtt{Z}_0\mathtt{X}_1^{n-i}\mathtt{Y}_iF_d(\mathtt{X}_1,\mathtt{Y}_1) \quad \text{for} \quad 0 \leq i \leq n. \tag{7–8}$$

Lemma 7.16. *The subscheme S' is normal. In particular, $h(V) = S'$.*

Proof. We consider the following standard open covering $\{W_j\}$ of W:

$$\begin{aligned}
W_1 &= \{\mathtt{X}_0 \neq 0, \mathtt{X}_1 \neq 0\}, & W_2 &= \{\mathtt{X}_0 \neq 0, \mathtt{Y}_1 \neq 0\},\\
W_3 &= \{\mathtt{Y}_0 \neq 0, \mathtt{X}_1 \neq 0\}, & W_4 &= \{\mathtt{Y}_0 \neq 0, \mathtt{Y}_1 \neq 0\},\\
W_5 &= \{\mathtt{Z}_0 \neq 0, \mathtt{X}_1 \neq 0\}, & W_6 &= \{\mathtt{Z}_0 \neq 0, \mathtt{Y}_1 \neq 0\},\\
W_7 &= \{\mathtt{X}_0 \neq 0, \mathtt{Z}_1 \neq 0\}, & W_8 &= \{\mathtt{Y}_0 \neq 0, \mathtt{Z}_1 \neq 0\},\\
W_9 &= \{\mathtt{Z}_0 \neq 0, \mathtt{Z}_1 \neq 0\}.
\end{aligned}$$

On the open subset W_1, the regular functions $\mathtt{y}_0 = \mathtt{Y}_0/\mathtt{X}_0$, $\mathtt{z}_0 = \mathtt{Z}_0/\mathtt{X}_0^n$, $\mathtt{y}_1 = \mathtt{Y}_1/\mathtt{X}_1$, and $\mathtt{z}_1 = \mathtt{Z}_1/\mathtt{X}_1^4$ form a coordinate system, i.e.,

$$W_1 = \operatorname{Spec} \Bbbk[\mathtt{y}_0, \mathtt{z}_0, \mathtt{y}_1, \mathtt{z}_1] \simeq \mathbb{A}^4.$$

Here, $S' \cap W_1 \simeq \mathbb{A}^2$ is defined by

$$\mathtt{y}_1 = \mathtt{y}_0, \quad \mathtt{z}_1 = \mathtt{z}_0 F_d(1, \mathtt{y}_1).$$

Thus $S' \cap W_1 \simeq \mathbb{A}^2$. Applying a similar argument to the open set W_4, we have $W_4 \simeq \mathbb{A}^4$ and $S' \cap W_4 \simeq \mathbb{A}^2$.

On W_2, the regular functions $\mathtt{y}_0 = \mathtt{Y}_0/\mathtt{X}_0$, $\mathtt{z}_0 = \mathtt{Z}_0/\mathtt{X}_0^n$, $\mathtt{x}_1 = \mathtt{X}_1/\mathtt{Y}_1$, and $\mathtt{z}_1 = \mathtt{Z}_1/\mathtt{Y}_1^4$ form a coordinate system of $W_2 \simeq \mathbb{A}^4$. Here, $S' \cap W_2$ is defined by

$$1 = \mathtt{x}_1 \mathtt{y}_0, \quad \mathtt{z}_1 = \mathtt{z}_0 \mathtt{x}_1^n F_d(\mathtt{x}_1, 1).$$

Thus $S' \cap W_2 \simeq (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1$. Similarly, $W_3 \simeq \mathbb{A}^4$ and $S' \cap W_3 \simeq (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1$.

The open subset W_5 is isomorphic to

$$\operatorname{Spec} \left(\Bbbk[\mathtt{x}_0, \mathtt{y}_0]^{(n)} \otimes \Bbbk[\mathtt{y}_1, \mathtt{z}_1] \right),$$

where

- $\mathtt{y}_1 = \mathtt{Y}_1/\mathtt{X}_1$, $\mathtt{z}_1 = \mathtt{Z}_1/\mathtt{X}_1^4$,
- $\Bbbk[\mathtt{x}_0, \mathtt{y}_0]^{(n)}$ is the subring of the polynomial ring $\Bbbk[\mathtt{x}_0, \mathtt{y}_0]$ of two variables which is generated by the monomials of degree divisible by n,
- $P_n(\mathtt{x}_0, \mathtt{y}_0) = P_n(\mathtt{X}_0, \mathtt{Y}_0)/\mathtt{Z}_0$ for any homogeneous polynomial P_n of degree n.

Then $S' \cap W_5$ is defined by

$$\mathtt{y}_0 = \mathtt{x}_0 \mathtt{y}_1, \quad \mathtt{z}_1 \mathtt{x}_0^n = F_d(1, \mathtt{y}_1).$$

Therefore,

$$S' \cap W_5 \simeq \operatorname{Spec} \left(\Bbbk[\mathtt{x}_0^n, \mathtt{y}_1, \mathtt{z}_1] \,/\, (\mathtt{z}_1 \mathtt{x}_0^n - F_d(1, \mathtt{y}_1)) \right)$$

and hence $S' \cap W_5$ has at most rational double points of type A as singularities. The singularity of $S' \cap W_6$ is similar.

The open subset W_7 is isomorphic to

$$\operatorname{Spec}\left(\Bbbk[\mathtt{y}_0, \mathtt{z}_0] \otimes \Bbbk[\mathtt{x}_1, \mathtt{y}_1]^{(4)}\right),$$

where $\mathtt{y}_0 = \mathtt{Y}_0/\mathtt{X}_0$, $\mathtt{z}_0 = \mathtt{Z}_0/\mathtt{X}_0^n$, and $P_4(\mathtt{x}_1, \mathtt{y}_1) = P_4(\mathtt{X}_1, \mathtt{Y}_1)/\mathtt{Z}^4$ for any quartic homogeneous polynomial P_4. Thus $S' \cap W_7$ is defined by

$$\mathtt{y}_1 = \mathtt{x}_1\mathtt{y}_0, \quad 1 = \mathtt{z}_0\mathtt{x}_1^4 F_d(1, \mathtt{y}_0).$$

Therefore,

$$S' \cap W_7 \simeq \operatorname{Spec}\left(\Bbbk[\mathtt{y}_0, \mathtt{z}_0, \mathtt{x}_1^4] \;/\; (\mathtt{z}_0\mathtt{x}_1^4 F_d(1, \mathtt{y}_0) - 1)\right).$$

Thus $S' \cap W_7$ is non-singular. Similarly, $S' \cap W_8$ is non-singular.

The open subset W_9 is written as

$$\operatorname{Spec}\left(\Bbbk[\mathtt{x}_0, \mathtt{y}_0]^{(n)} \otimes \Bbbk[\mathtt{x}_1, \mathtt{y}_1]^{(4)}\right),$$

where $P_n(\mathtt{x}_0, \mathtt{y}_0) = P_n(\mathtt{X}_0, \mathtt{Y}_0)/\mathtt{Z}_0$ and $P_4(\mathtt{x}_1, \mathtt{y}_1) = P_4(\mathtt{X}_1, \mathtt{Y}_1)/\mathtt{Z}_1$ for homogeneous polynomials P_j of degree j. Then $S' \cap W_9$ is defined by "$\mathtt{x}_0\mathtt{y}_1 = \mathtt{x}_1\mathtt{y}_0$" and $\mathtt{x}_0^i\mathtt{y}_0^{n-i} = \mathtt{x}_1^i\mathtt{y}_1^{n-i}F_d(\mathtt{x}_1, \mathtt{y}_1)$ for $0 \le i \le n$. Therefore, $S' \cap W_9 \simeq \operatorname{Spec}\Bbbk[\mathtt{x}_1, \mathtt{y}_1]^{(4)}$, which is isomorphic to an open neighborhood of the vertex of the cone $\mathbb{P}(1,1,4)$. Therefore, S' is normal. $\square$

Proof of Theorem 7.15. Suppose that $1 \le n \le 3$. By construction of h, we have a birational morphism $S' \to S$ so that the composite $S' \to S \to \mathbb{P}(1,1,4)$ is induced from the second projection $W \to \mathbb{P}(1,1,4)$. By Lemma 7.16, $S' \to \mathbb{P}(1,1,4)$ is isomorphic outside $\mathbb{D} = \{F_d(\mathtt{X}_1, \mathtt{Y}_1) = \mathtt{Z}_1 = 0\}$, where $(\mathtt{X}_1, \mathtt{Y}_1, \mathtt{Z}_1)$ is regarded as a homogeneous coordinate of $\mathbb{P}(1,1,4)$. The description of $S' \cap W_5$ and $S' \cap W_6$ in Lemma 7.16 shows that $S' \to \mathbb{P}(1,1,4)$ is just the blowing up along $\mathbb{D}$. Hence, $S' \simeq S$. Therefore, S is isomorphic to the subvariety S' of $\mathbb{P}(1,1,n) \times \mathbb{P}(1,1,4)$ defined by (7–8). This finish the proof in the case $1 \le n \le 3$.

Finally suppose that $n = 0$. For the surjective morphism $h\colon X \times_{\mathbb{P}^1} \mathbb{F}_4 \to W = \mathbb{P}^1 \times \mathbb{P}(1,1,4)$, we can choose the homogeneous coordinates to satisfy $h^*\mathtt{X}_0 = \mathtt{g}$, $h^*\mathtt{Y}_0 = \mathtt{f}$, $h^*\mathtt{Z}_1 = \mathtt{u}$, and $h^*P_4(\mathtt{X}_1, \mathtt{Y}_1) = P_4(\mathtt{s}, \mathtt{t})\mathtt{v}$, for any quartic homogeneous polynomial P_4. By the equation (7–7), $h(V)$ is contained in the subscheme $S' \subset W$ defined by

$$\mathtt{Z}_1\mathtt{X}_0 = \mathtt{Y}_0 F_4(\mathtt{X}_1, \mathtt{Y}_1). \tag{7–9}$$

Let $W_1 \subset W$ be the open subset $\{\mathtt{X}_0 \neq 0\}$ and let $W_2 \subset W$ be $\{\mathtt{Y}_0 \neq 0\}$. Then

$$S' \cap W_1 \simeq \mathrm{Proj}\,\big((\Bbbk[\mathtt{y}_0])[\mathtt{X}_1, \mathtt{Y}_1, \mathtt{Z}_1] \,/\, (\mathtt{Z}_1 - \mathtt{X}_1^4 \mathtt{y}_0 F_4(1, \mathtt{y}_0))\big) \simeq \mathbb{A}^1 \times \mathbb{P}^1,$$

where $\mathtt{y}_0 = \mathtt{Y}_0/\mathtt{X}_0$. Moreover,

$$S' \cap W_2 \simeq \mathrm{Proj}\,\big((\Bbbk[\mathtt{x}_0])[\mathtt{X}_1, \mathtt{Y}_1, \mathtt{Z}_1] \,/\, (\mathtt{Z}_1\mathtt{x}_0 - F_4(\mathtt{X}_1, \mathtt{Y}_1))\big)$$

for $\mathtt{x}_0 = \mathtt{X}_0/\mathtt{Y}_0$. Thus S' is normal, $h(V) = S'$, and $S' \to \mathbb{P}(1,1,4)$ is the blowing-up along $\mathbb{D} = \{\mathtt{Z}_1 = F_4(\mathtt{X}_1, \mathtt{Y}_1) = 0\}$. In particular, $S' \simeq S$. Therefore S is defined by (7–9), and we are done. □

REMARK. If $\mathrm{Supp}\,\Delta$ consists of at most two points, then S is a toric variety. In fact, $S \to \mathbb{P}(1,1,4)$ is described as a toric blowup. In particular, S is toric if $n \leq 2$.

7.5. Embedding into weighted projective spaces, III

In the non-big case, $L - E \sim w\ell$ for $w = 2$ or 4 on $X = \mathbb{F}_n$ and hence

$$\mathbb{P} = \mathbb{P}_X(\mathcal{O}_X(L-E) \oplus \mathcal{O}_X) \simeq \mathbb{F}_w \times_{\mathbb{P}^1} \times X = \mathbb{F}_w \times_{\mathbb{P}^1} \mathbb{F}_n.$$

Let $p_1 \colon \mathbb{P} \to \mathbb{F}_w$ and $p_2 \colon \mathbb{P} \to X \simeq \mathbb{F}_n$ be the projections. The global sections $\mathtt{u}$ and $\mathtt{v}$ in Section 7.3 descend to global sections of $\mathcal{O}(\sigma^{(w)} + w\ell)$ and $\mathcal{O}(\sigma^{(w)})$ over $\mathbb{F}_w$, respectively, where $\sigma^{(w)}$ is the negative section and ℓ is a fiber on $\mathbb{F}_w$. The divisor $V = V(\xi, \eta) \subset \mathbb{P}$ is described by a quadric equation with respect to $(\mathtt{f}, \mathtt{g})$ over $\mathbb{F}_w$, since the mapping degree of $V \subset \mathbb{P} \to \mathbb{F}_w$ is two.

The morphism $\Phi \colon \mathbb{P} \to W$ is the composite of p_2 and the contraction morphism $q \colon \mathbb{F}_w \to \overline{\mathbb{F}}_w \simeq \mathbb{P}(1,1,w) \simeq W$ of the negative section $\sigma^{(w)}$. Let $(\mathtt{X}, \mathtt{Y}, \mathtt{U})$ be a homogeneous coordinate of $\mathbb{P}(1,1,w)$. We may assume that the morphism $q \colon \mathbb{F}_w \to \mathbb{P}(1,1,w)$ satisfies $q^*\mathtt{U} = \mathtt{u}$ and $q^* P_w(\mathtt{X}, \mathtt{Y}) = P_w(\mathtt{s}, \mathtt{t})\mathtt{v}$ for any homogeneous polynomial P_w of degree w.

Finding suitable sections ξ and η, we shall describe the surface S explicitly.

PROPOSITION 7.17. *A log del Pezzo surface of index two of type* $[4;2,4]_{00}$ *is isomorphic to the divisor*

$$\{\mathtt{UZ} = F_8(\mathtt{X},\mathtt{Y})\} \subset \mathbb{P}(1,1,4,4)$$

for a homogeneous coordinate $(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{U})$ *of weight* $(1,1,4,4)$ *and for an octic homogeneous polynomial* $F_8 \neq 0$.

PROOF. Since $E = \sigma + \sigma_\infty$ for a section σ_∞ at infinity, we may assume $\eta = \mathtt{fg}$. There is an octic homogeneous polynomial $F_8(\mathtt{s},\mathtt{t}) \neq 0$ such that $\pi_*(\Delta) = \operatorname{div}(F_8(\mathtt{s},\mathtt{t}))$. Thus

$$\xi = \mathtt{f}^2 + F_8(\mathtt{s},\mathtt{t})\mathtt{g}^2 \quad \in \mathrm{H}^0(X,L) = \mathrm{H}^0(X, 2\sigma + 8\ell)$$

satisfies $\operatorname{div}(\xi) \cap E = \Delta$. Since

$$\xi\mathtt{v} - \eta\mathtt{u} = \mathtt{vf}^2 - \mathtt{ufg} + F_8(\mathtt{s},\mathtt{t})\mathtt{vg}^2,$$

the first projection $p_1|_V : V \subset \mathbb{P} \to \mathbb{F}_4$ is a finite morphism. For the isomorphism

$$\mathbb{P} \simeq \mathbb{F}_w \times_{\mathbb{P}^1} X \simeq \mathbb{F}_4 \times_{\mathbb{P}^1} \mathbb{F}_4,$$

we have a birational map $\mathbb{P} \dashrightarrow \mathbb{P}(1,1,4,4)$ by Lemma 7.7. We set $\mathtt{U} := \mathtt{Z}_1$ and $\mathtt{Z} := \mathtt{Z}_2$ for the homogeneous coordinate $\mathtt{Z}_i$ defined in Lemma 7.7. Then the proper transform V' of V in $\mathbb{P}(1,1,4,4)$ is a Cartier divisor of degree 8 defined by

$$\Psi := \mathtt{Z}^2 - \mathtt{UZ} + F_8(\mathtt{X},\mathtt{Y}) = 0.$$

Note that V' is Cohen-Macaulay since so is $\mathbb{P}(1,1,4,4)$. The projection $(\mathtt{X}:\mathtt{Y}:\mathtt{U}:\mathtt{Z}) \mapsto (\mathtt{X}:\mathtt{Y}:\mathtt{U})$ induces a finite morphism $V' \to \mathbb{P}(1,1,4)$ which is birational to $\Phi|_V : V \subset \mathbb{P} \to W \simeq \mathbb{P}(1,1,4)$. Since

$$\frac{\partial}{\partial \mathtt{Z}}\Psi = 2\mathtt{Z} - \mathtt{U} \quad \text{and} \quad \frac{\partial}{\partial \mathtt{U}}\Psi = -\mathtt{Z},$$

and since $\operatorname{Sing}\mathbb{P}(1,1,4,4) \subset \{\mathtt{X} = \mathtt{Y} = 0\}$, we have

$$\operatorname{Sing} V' \subset \{\mathtt{Z} = \mathtt{U} = F_8(\mathtt{X},\mathtt{Y}) = 0\} \cup \{\mathtt{X} = \mathtt{Y} = \mathtt{Z}(\mathtt{Z}-\mathtt{U}) = 0\}.$$

Thus V' has only isolated singularities by $F_8 \neq 0$. Hence V' is normal, $V' \to W$ is the Stein factorization of $\Phi|_V \colon V \to W$, and thus $S \simeq V'$. Replacing $(\mathtt{U}, \mathtt{Z})$ with $(\mathtt{U}+\mathtt{Z}, \mathtt{Z})$, we have the expected equation. $\square$

PROPOSITION 7.18. *A log del Pezzo surface of index two of type* $[3; 2, 4]_+$ *is isomorphic to the divisor*

$$\{\mathtt{Z}^2 + (c\mathtt{Y}^3 + \mathtt{XU})\mathtt{Z} + F_6(\mathtt{X}, \mathtt{Y}) = \mathtt{Y}^4\mathtt{U}\} \subset \mathbb{P}(1, 1, 2, 3)$$

for a constant $c \in \Bbbk$ *and a sextic homogeneous polynomial* F_6*, where* $(\mathtt{X}, \mathtt{Y}, \mathtt{U}, \mathtt{Z})$ *is a homogeneous coordinate of weight* $(1, 1, 2, 3)$*.*

PROOF. In this type, $E = \sigma + D$ for a section $D \sim \sigma + 4\ell$. We may assume that the fiber ℓ passing through the intersection point $\sigma \cap D$ is defined by $\{\mathtt{s} = 0\}$. The divisor $\pi_*(\Delta) \subset \mathbb{P}^1$ of degree 8 does not contain $(0:1)$. Let $F_8(\mathtt{s}, \mathtt{t})$ be an octic homogeneous polynomial such that $\operatorname{div}(F_8(\mathtt{s}, \mathtt{t})) = \pi_*(\Delta)$. We may assume that

$$F_8(\mathtt{s}, \mathtt{t}) = \mathtt{t}^8 + c\mathtt{t}^7\mathtt{s} + F_6(\mathtt{s}, \mathtt{t})\mathtt{s}^2$$

for a constant $c \in \Bbbk$ and for a sextic homogeneous polynomial F_6. For the sections $\mathtt{f}$ and $\mathtt{g}$, we have $\pi_*\mathcal{O}_X(\sigma + 4\ell) = \mathcal{O}(4)\mathtt{g} \oplus \mathcal{O}(1)\mathtt{f}$ over $\mathbb{P}^1$. Hence, $D = \operatorname{div}(P_4(\mathtt{s}, \mathtt{t})\mathtt{g} - \mathtt{sf})$ for a quartic homogeneous polynomial P_4 with $P_4(0, 1) \neq 0$. We may replace $\mathtt{f}$ with $\mathtt{f} + P_3(\mathtt{s}, \mathtt{t})\mathtt{g}$ for any cubic polynomial P_3. Hence, we may assume that $P_4 = \mathtt{t}^4$. Therefore $D = \operatorname{div}(\mathtt{t}^4\mathtt{g} - \mathtt{sf})$ and $E = \operatorname{div}(\eta)$ for $\eta = (\mathtt{t}^4\mathtt{g} - \mathtt{sf})\mathtt{g}$. We consider a global section

$$\xi = F_6(\mathtt{s}, \mathtt{t})\mathtt{g}^2 + c\mathtt{t}^3\mathtt{gf} + \mathtt{f}^2 \quad \in \mathrm{H}^0(X, L) = \mathrm{H}^0(X, 2\sigma + 6\ell).$$

Then $\operatorname{div}(\xi) \cap \sigma = \emptyset$ and $\operatorname{div}(\xi) \cap D = \Delta$ by

$$s^2\xi \equiv \mathtt{g}^2(F_6(\mathtt{s}, \mathtt{t})\mathtt{s}^2 + c\mathtt{st}^7 + \mathtt{t}^8) \mod (\mathtt{t}^4\mathtt{g} - \mathtt{sf}).$$

Thus $V = V(\xi, \eta) \subset \mathbb{P}$. Since

$$\xi\mathtt{v} - \eta\mathtt{u} = \mathtt{vf}^2 + (c\mathtt{t}^3\mathtt{v} + \mathtt{su})\mathtt{fg} + (F_6(\mathtt{s}, \mathtt{t})\mathtt{v} - \mathtt{t}^4\mathtt{u})\mathtt{g}^2,$$

we infer that $p_1|_V \colon V \subset \mathbb{P} \to \mathbb{F}_2$ is a finite morphism. Applying Lemma 7.7, we have a birational map $\mathbb{P} \dashrightarrow \mathbb{P}(1, 1, 2, 3)$ such that the proper transform V' of V in $\mathbb{P}(1, 1, 2, 3)$ is a Cartier divisor of degree 6 given by

$$\Psi := \mathtt{Z}^2 + (c\mathtt{Y}^3 + \mathtt{XU})\mathtt{Z} + F_6(\mathtt{X}, \mathtt{Y}) - \mathtt{Y}^4\mathtt{U} = 0$$

for a homogeneous coordinate $(\mathtt{X}, \mathtt{Y}, \mathtt{U}, \mathtt{Z})$ of weight $(1, 1, 2, 3)$. Note that the projection $(\mathtt{X}:\mathtt{Y}:\mathtt{U}:\mathtt{Z}) \mapsto (\mathtt{X}:\mathtt{Y}:\mathtt{U})$ induces a finite morphism $V' \to W \simeq \mathbb{P}(1, 1, 2)$, which is birational to $\Phi|_V : V \to W$. Since

$$\frac{\partial}{\partial \mathtt{Z}}\Psi = 2\mathtt{Z} + (c\mathtt{Y}^3 + \mathtt{XU}) \quad \text{and} \quad \frac{\partial}{\partial \mathtt{U}}\Psi = \mathtt{XZ} - \mathtt{Y}^4,$$

the singular locus of V' is contained in

$$\{2\mathtt{Z} + (c\mathtt{Y}^3 + \mathtt{XU}) = \mathtt{XZ} - \mathtt{Y}^4 = \mathtt{Z}^2 + c\mathtt{Y}^3\mathtt{Z} + F_6(\mathtt{X}, \mathtt{Y}) = 0\} \cup \{\mathtt{X} = \mathtt{Y} = \mathtt{Z} = 0\}.$$

In particular, $\operatorname{Sing} V' \cap \{\mathtt{X} \neq 0\}$ is contained in the finite set

$$\{(1:\mathtt{y}:\mathtt{z}:\mathtt{u}) \mid \mathtt{z} = \mathtt{y}^4,\ \mathtt{u} = -c\mathtt{y}^3 - 2\mathtt{y}^4,\ \mathtt{y}^8 + c\mathtt{y}^7 + F_6(1, \mathtt{y}) = 0\}$$

and $\operatorname{Sing} V' \cap \{\mathtt{X} = 0\} \subset \{(0:0:0:1)\}$. Hence, V' has only isolated singular points and thus V' is normal. Thus $S \simeq V'$, since $V' \to W$ gives the Stein factorization of $V \to W$. $\square$

PROPOSITION 7.19. *Let S be a log del Pezzo surface of index two of type $[3; 2, 4]_{++}(a, b)$.*

(1) *If $(a, b) = (0, 0)$, then S is isomorphic to the divisor*

$$\{\mathtt{Z}^2 + (c\mathtt{Y}^3 + \mathtt{XU})\mathtt{Z} + \mathtt{Y}^6 + \mathtt{X}F_5(\mathtt{X}, \mathtt{Y}) = 0\} \subset \mathbb{P}(1, 1, 2, 3)$$

for a quintic homogeneous polynomial F_5 and for a constant c.

(2) *If $(a, b) = (2, 1)$, then S is isomorphic to the divisor*

$$\{\mathtt{Z}^2 + \mathtt{XUZ} + \mathtt{XY}^5 + \mathtt{X}^2F_4(\mathtt{X}, \mathtt{Y}) = 0\} \subset \mathbb{P}(1, 1, 2, 3)$$

for a quartic homogeneous polynomial F_4.

(3) *If $a = 1$, then $1 \leq b \leq 6$ and S is isomorphic to the divisor*

$$\{\mathtt{Z}^2 + (\mathtt{Y}^3 + \mathtt{XU})\mathtt{Z} + \mathtt{X}^b\mathtt{Y}^{6-b} + \mathtt{X}^{b+1}F_{5-b}(\mathtt{X}, \mathtt{Y}) = 0\} \subset \mathbb{P}(1, 1, 2, 3)$$

for a homogeneous polynomial F_{5-b} of degree $5 - b$.

PROOF. In this type, $E = \sigma + \sigma_\infty + \ell$ for a section σ_∞ at infinity and for a fiber ℓ. We may assume that $\sigma_\infty = \{\mathtt{f} = 0\}$, $\ell = \{\mathtt{s} = 0\}$, and $\eta = -\mathtt{fgs}$. A global section ξ of $\mathcal{O}_X(L) \simeq \mathcal{O}_X(2\sigma + 6\ell)$ with $\operatorname{div}(\xi) \cap E = \Delta$ can be written as

$$\xi = \mathtt{f}^2 + c\mathtt{t}^3\mathtt{fg} + F_6(\mathtt{s}, \mathtt{t})\mathtt{g}^2$$

for a constant $c \in \Bbbk$ and for a sextic homogeneous polynomial F_6. Since

$$\xi\mathtt{v} - \eta\mathtt{u} = \mathtt{vf}^2 + (c\mathtt{t}^3\mathtt{v} + \mathtt{su})\mathtt{fg} + F_6(\mathtt{s}, \mathtt{t})\mathtt{vg}^2,$$

we infer that $V \to \mathbb{F}_2$ is not finite along $\{\mathtt{v} = \mathtt{s} = 0\}$. We can normalize F_6 as follows:

Case $(a, b) = (0, 0)$. Then $F_6(0, 1) \neq 0$. Multiplying $\mathtt{t}$ by a non-zero constant, we may assume $F_6(\mathtt{s}, \mathtt{t}) = \mathtt{t}^6 + \mathtt{s}F_5(\mathtt{s}, \mathtt{t})$ for a quintic homogeneous polynomial F_5. Here, $c \neq 2$ if and only if $\operatorname{Supp}(\Delta \cap \ell)$ consists of two points.

Case $(a, b) = (2, 1)$. Then $c = 0$ and $F_6(\mathtt{s}, \mathtt{t}) = \mathtt{s}F_5(\mathtt{s}, \mathtt{t})$ for a quintic homogeneous polynomial F_5 with $F_5(0, 1) \neq 0$. Multiplying $\mathtt{t}$ by a non-zero constant, we may assume $F_6(\mathtt{s}, \mathtt{t}) = \mathtt{s}(\mathtt{t}^5 + \mathtt{s}F_4(\mathtt{s}, \mathtt{t}))$ for a quartic homogeneous polynomial F_4.

Case $(a, b) = (1, b)$. Then $1 \leq b \leq 6$, $c \neq 0$, and $F_6(\mathtt{s}, \mathtt{t}) = \mathtt{s}^b F_{6-b}(\mathtt{s}, \mathtt{t})$ for a homogeneous polynomial F_{6-b} with $F_{6-b}(0, 1) \neq 0$. Multiplying $\mathtt{s}$ and $\mathtt{t}$ by non-zero constants, we may assume $c = 1$ and $F_6(\mathtt{s}, \mathtt{t}) = \mathtt{s}^b(\mathtt{t}^{b-6} + \mathtt{s}F_{5-b}(\mathtt{s}, \mathtt{t}))$ for a homogeneous polynomial F_{5-b} of degree $5 - b$, where $F_{5-b} = 0$ in case $b > 5$.

Applying Lemma 7.7, we have a birational map $\mathbb{P} \dashrightarrow \mathbb{P}(1, 1, 2, 3)$ such that the proper transform V' of V in $\mathbb{P}(1, 1, 2, 3)$ is a Cartier divisor of degree 6 defined by

$$\Psi := \mathtt{Z}^2 + (c\mathtt{Y}^3 + \mathtt{XU})\mathtt{Z} + F_6(\mathtt{X}, \mathtt{Y}) = 0$$

for the homogeneous coordinate $(\mathtt{X}, \mathtt{Y}, \mathtt{U}, \mathtt{Z})$ of weight $(1, 1, 2, 3)$. Here, the projection $(\mathtt{X} : \mathtt{Y} : \mathtt{U} : \mathtt{Z}) \mapsto (\mathtt{X} : \mathtt{Y} : \mathtt{U})$ induces a finite morphism $V' \to W \simeq \mathbb{P}(1, 1, 2)$, which is birational to $\Phi|_V \colon V \to W$. By the calculation

$$\frac{\partial}{\partial \mathtt{Z}}\Psi = 2\mathtt{Z} + (c\mathtt{Y}^3 + \mathtt{XU}), \quad \frac{\partial}{\partial \mathtt{U}}\Psi = \mathtt{XZ}, \quad \frac{\partial}{\partial \mathtt{X}}\Psi = \mathtt{UZ} + \frac{\partial}{\partial \mathtt{X}}F_6(\mathtt{X}, \mathtt{Y}),$$

we infer that $\operatorname{Sing} V'$ is contained in

$$\begin{aligned}&\{F_6(\mathtt{X},\mathtt{Y}) = \mathtt{Z} = c\mathtt{Y}^3 + \mathtt{XU} = 0\}\\ &\qquad \cup \left\{\mathtt{X} = 2\mathtt{Z} + c\mathtt{Y}^3 = F_6(0,\mathtt{Y}) - \mathtt{Z}^2 = \mathtt{UZ} + \frac{\partial F_6}{\partial \mathtt{X}}(0,\mathtt{Y}) = 0\right\}.\end{aligned}$$

Here, $\operatorname{Sing} V' \cap \{\mathtt{X} = \mathtt{Z} = 0\}$ is contained in $\{(0:0:0:1)\}$. For, we have $F_6(0,\mathtt{Y}) = \mathtt{Y}^6$ in case $(a,b) \neq (0,0)$, $(\partial F_6/\partial \mathtt{X})(0,\mathtt{Y}) = \mathtt{Y}^5$ in case $(a,b) = (2,1)$, and $c = 1$ in case $(a,b) = (1,b)$. Furthermore, $\operatorname{Sing} V' \cap \{\mathtt{X} \neq 0\}$ is contained in the finite set

$$\{(1:\mathtt{y}:0:-c\mathtt{y}^3) \mid F_6(1,\mathtt{y}) = 0\}$$

and $\operatorname{Sing} V' \cap \{\mathtt{Z} \neq 0\}$ is contained in the finite set

$$\left\{(0:\mathtt{y}:1:\mathtt{u}) \mid 2 + c\mathtt{y}^3 = F_6(0,\mathtt{y}) - 1 = \mathtt{u} + \frac{\partial F_6}{\partial \mathtt{X}}(0,\mathtt{y}) = 0\right\}.$$

Hence, V' has only isolated singularities and thus V' is normal. Therefore $S \simeq V'$, since $V' \to W$ coincides with the Stein factorization of $V \to W$. Therefore, we have the expected defining equations. □

Proposition 7.20. *Let S be a log del Pezzo surface of index two of type $[1;2,2]_0$ and let (X,E,Δ) be a fundamental triplet defining S. Let $(\mathtt{X},\mathtt{Y},\mathtt{U},\mathtt{Z})$ be a homogeneous coordinate of $\mathbb{P}(1,1,2,3)$.*

(1) *Either if $\operatorname{char}\Bbbk \neq 2$ or if the double-covering $\pi|_E\colon E \subset X \to \mathbb{P}^1$ is inseparable, then S is isomorphic to the divisor of $\mathbb{P}(1,1,2,3)$ defined by*

$$\mathtt{Z}^2 = F_3(\mathtt{X},\mathtt{Y})\mathtt{Z} + F_4(\mathtt{X},\mathtt{Y})\mathtt{U} + \mathtt{XYU}^2$$

for a cubic polynomial F_3 and a quartic polynomial F_4 with $(F_3,F_4) \neq (0,0)$.

(2) *If $\operatorname{char}\Bbbk = 2$ and if $\pi|_E\colon E \subset X \to \mathbb{P}^1$ is separable, then S is isomorphic to the divisor of $\mathbb{P}(1,1,2,3)$ defined by*

$$\mathtt{Z}^2 = (F_3(\mathtt{X},\mathtt{Y}) + \mathtt{XU})\mathtt{Z} + F_4(\mathtt{X},\mathtt{Y})\mathtt{U} + \mathtt{Y}^2\mathtt{U}^2$$

for a cubic polynomial F_3 and a quartic polynomial F_4 with $(F_3,F_4) \neq (0,0)$.

PROOF. In this type, $E \sim 2\sigma + 2\ell$ is non-singular. By Lemma 4.11, we may assume

$$\eta = \begin{cases} \mathtt{f}^2 - \mathtt{stg}^2, & \text{in the case (1);} \\ \mathtt{f}^2 + \mathtt{sfg} + \mathtt{t}^2\mathtt{g}^2, & \text{in the case (2).} \end{cases}$$

Case (1). The fibers $\{\mathtt{s} = 0\}$ and $\{\mathtt{t} = 0\}$ intersect tangentially with E. Hence, $\mathtt{s}|_E = \mathtt{x}^2$ and $\mathtt{t}|_E = \mathtt{y}^2$ for a homogeneous coordinate $(\mathtt{x}, \mathtt{y})$ of $E \simeq \mathbb{P}^1$. Moreover we can identify $\mathtt{g}|_E$ with 1 and $\mathtt{f}|_E$ with $\mathtt{xy}$ by an isomorphism $\mathcal{O}_E(\sigma) \simeq \mathcal{O}_E$. Note that any homogeneous polynomial $P_{2m}(\mathtt{x}, \mathtt{y})$ of degree $2m$ is written as

$$P_{2m}(\mathtt{x}, \mathtt{y}) = P_m(\mathtt{x}^2, \mathtt{y}^2) + P_{m-1}(\mathtt{x}^2, \mathtt{y}^2)\mathtt{xy}$$

for some homogeneous polynomials P_j of degree j for $j = m$, $m - 1$. Thus we may assume

$$\xi = F_4(\mathtt{s}, \mathtt{t})\mathtt{g}^2 + F_3(\mathtt{s}, \mathtt{t})\mathtt{fg}$$

for a cubic polynomial F_3 and a quartic polynomial F_4, where $\Delta \subset E$ is defined by $F_4(\mathtt{x}^2, \mathtt{y}^2) + F_3(\mathtt{x}^2, \mathtt{y}^2)\mathtt{xy} = 0$. Since

$$\xi\mathtt{v} - \eta\mathtt{u} = (F_4(\mathtt{s}, \mathtt{t})\mathtt{v} + \mathtt{stu})\mathtt{g}^2 + F_3(\mathtt{s}, \mathtt{t})\mathtt{vfg} - \mathtt{uf}^2,$$

we infer that $V \to \mathbb{F}_2$ is not finite along $\{\mathtt{u} = F_3(\mathtt{s}, \mathtt{t}) = F_4(\mathtt{s}, \mathtt{t}) = 0\}$. Applying Lemma 7.7, we have a birational map $\mathbb{P} \dashrightarrow \mathbb{P}(1, 1, 1, 2)$ such that the proper transform V' of V in $\mathbb{P}(1, 1, 1, 2)$ is a Cartier divisor of degree 4 defined by

$$-\mathtt{U}_0\mathtt{Z}_0^2 + F_3(\mathtt{X}, \mathtt{Y})\mathtt{Z}_0 + F_4(\mathtt{X}, \mathtt{Y}) + \mathtt{XYU}_0 = 0$$

for the homogeneous coordinate $(\mathtt{X}, \mathtt{Y}, \mathtt{Z}_0, \mathtt{U}_0)$ of weight $(1, 1, 1, 2)$. Note that the projection $(\mathtt{X} : \mathtt{Y} : \mathtt{Z}_0 : \mathtt{U}_0) \mapsto (\mathtt{X} : \mathtt{Y} : \mathtt{U}_0)$ induces a rational map $V' \dashrightarrow W = \mathbb{P}(1, 1, 2)$ with non-empty locus of indeterminacy. We consider the birational map

$$\mathbb{P}(1, 1, 1, 2) \dashrightarrow \mathbb{P}(1, 1, 2, 3);$$
$$(\mathtt{X} : \mathtt{Y} : \mathtt{Z}_0 : \mathtt{U}_0) \mapsto (\mathtt{X} : \mathtt{Y} : \mathtt{U} : \mathtt{Z}) = (\mathtt{X} : \mathtt{Y} : \mathtt{U}_0 : \mathtt{Z}_0\mathtt{U}_0).$$

Then the proper transform V'' of V in $\mathbb{P}(1,1,2,3)$ is a Cartier divisor of degree 6 defined by

$$\Psi := -\mathtt{Z}^2 + F_3(\mathtt{X},\mathtt{Y})\mathtt{Z} + F_4(\mathtt{X},\mathtt{Y})\mathtt{U} + \mathtt{XYU}^2 = 0$$

and the projection $(\mathtt{X}:\mathtt{Y}:\mathtt{U}:\mathtt{Z}) \mapsto (\mathtt{X}:\mathtt{Y}:\mathtt{U})$ induces a finite morphism $V'' \to W$. By the calculation

$$\begin{aligned}\frac{\partial\Psi}{\partial\mathtt{Z}} &= -2\mathtt{Z} + F_3(\mathtt{X},\mathtt{Y}), \quad \frac{\partial\Psi}{\partial\mathtt{U}} = F_4(\mathtt{X},\mathtt{Y}) + 2\mathtt{XYU},\\ \frac{\partial\Psi}{\partial\mathtt{X}} &= \frac{\partial F_3}{\partial\mathtt{X}}(\mathtt{X},\mathtt{Y})\mathtt{Z} + \frac{\partial F_4}{\partial\mathtt{X}}(\mathtt{X},\mathtt{Y})\mathtt{U} + \mathtt{YU}^2,\\ \frac{\partial\Psi}{\partial\mathtt{Y}} &= \frac{\partial F_3}{\partial\mathtt{Y}}(\mathtt{X},\mathtt{Y})\mathtt{Z} + \frac{\partial F_4}{\partial\mathtt{Y}}(\mathtt{X},\mathtt{Y})\mathtt{U} + \mathtt{XU}^2,\end{aligned}$$

we infer that the singular locus of V'' is contained in the locus

$$\left\{F_3(\mathtt{X},\mathtt{Y}) - 2\mathtt{Z} = F_4(\mathtt{X},\mathtt{Y}) + 2\mathtt{XYU} = \mathtt{Z}^2 - \mathtt{XYU}^2 = \frac{\partial\Psi}{\partial\mathtt{X}} = \frac{\partial\Psi}{\partial\mathtt{Y}} = 0\right\}.$$

We shall show $\operatorname{Sing} V''$ is a finite set. Note that $\operatorname{Sing} V'' \cap \{\mathtt{X} = \mathtt{Y} = 0\} \subset \{(0:0:0:1)\}$. Thus it suffices to consider two subsets $\operatorname{Sing} V'' \cap \{\mathtt{X} \neq 0\}$ and $\operatorname{Sing} V'' \cap \{\mathtt{Y} \neq 0\}$. Suppose first that $\operatorname{char} \Bbbk \neq 2$. Then $\operatorname{Sing} V'' \cap \{\mathtt{X} \neq 0\}$ is contained in the finite set

$$\begin{aligned}\Big\{(1:\mathtt{y}:\mathtt{z}:\mathtt{u}) \;\Big|\; 2\mathtt{z} - F_3(1,\mathtt{y}) &= 2\mathtt{yu} + F_4(1,\mathtt{y})\\ &= \frac{\partial F_3}{\partial\mathtt{Y}}(1,\mathtt{y})\mathtt{z} + \frac{\partial F_4}{\partial\mathtt{Y}}(1,\mathtt{y})\mathtt{u} + \mathtt{u}^2 = 0\Big\}\end{aligned}$$

and $\operatorname{Sing} V'' \cap \{\mathtt{Y} \neq 0\}$ is contained in the finite set

$$\begin{aligned}\Big\{(\mathtt{x}:1:\mathtt{z}:\mathtt{u}) \;\Big|\; 2\mathtt{z} - F_3(\mathtt{x},1) &= 2\mathtt{xu} + F_4(\mathtt{x},1)\\ &= \frac{\partial F_3}{\partial\mathtt{X}}(\mathtt{x},1)\mathtt{z} + \frac{\partial F_4}{\partial\mathtt{X}}(\mathtt{x},1)\mathtt{u} + \mathtt{u}^2 = 0\Big\}.\end{aligned}$$

Next, suppose that $\operatorname{char} \Bbbk = 2$. Then there are finitely many $(\mathtt{x}:\mathtt{y}) \in \mathbb{P}^1$ satisfying $F_3(\mathtt{x},\mathtt{y}) = F_4(\mathtt{x},\mathtt{y}) = 0$. Hence, $\operatorname{Sing} V'' \cap \{\mathtt{X} \neq 0\}$ is contained in the finite set

$$\Big\{(1:\mathtt{y}:\mathtt{z}:\mathtt{u}) \;\Big|\; F_3(1,\mathtt{y}) = F_4(1;\mathtt{y}) = \mathtt{z}^2 - \mathtt{y}\mathtt{u}^2 = \frac{\partial F_3}{\partial \mathtt{Y}}(1,\mathtt{y})\mathtt{z} + \frac{\partial F_4}{\partial \mathtt{Y}}(1,\mathtt{y})\mathtt{u} + \mathtt{u}^2 = 0\Big\},$$

and $\operatorname{Sing} V'' \cap \{\mathtt{Y} \neq 0\}$ is contained in the finite set

$$\Big\{(\mathtt{x}:1:\mathtt{z}:\mathtt{u}) \;\Big|\; F_3(\mathtt{x},1) = F_4(\mathtt{x},1) = \mathtt{z}^2 - \mathtt{x}\mathtt{u}^2 = \frac{\partial F_3}{\partial \mathtt{X}}(\mathtt{x},1)\mathtt{z} + \frac{\partial F_4}{\partial \mathtt{X}}(\mathtt{x},1)\mathtt{u} + \mathtt{u}^2 = 0\Big\}.$$

Therefore, $\operatorname{Sing} V''$ is a finite set. Thus V'' is normal and $S \simeq V''$.

Case (2). We can choose a homogeneous coordinate $(\mathtt{x},\mathtt{y})$ of $E \simeq \mathbb{P}^1$ so that $\mathtt{s}|_E = \mathtt{x}^2$, $\mathtt{t}|_E = (\mathtt{x}+\mathtt{y})\mathtt{y}$ and that $\mathtt{g}|_E = 1$ and $\mathtt{f}|_E = \mathtt{y}^2$ under an isomorphism $\mathcal{O}_E(\sigma) \simeq \mathcal{O}_E$. Note that any homogeneous polynomial $P_{2m}(\mathtt{x},\mathtt{y})$ of degree $2m$ is written as

$$P_{2m}(\mathtt{x},\mathtt{y}) = P_m(\mathtt{x}^2,(\mathtt{x}+\mathtt{y})\mathtt{y}) + P_{m-1}(\mathtt{x}^2,(\mathtt{x}+\mathtt{y})\mathtt{y})\mathtt{y}^2$$

for homogeneous polynomials P_j of degree j for $j = m$, $m-1$. In fact, this is shown by using

$$\mathtt{x}\mathtt{y} = (\mathtt{x}+\mathtt{y})\mathtt{y} - \mathtt{y}^2 \quad \text{and} \quad \mathtt{y}^4 = -((\mathtt{x}+\mathtt{y})\mathtt{y})^2 + (2(\mathtt{x}+\mathtt{y})\mathtt{y} + \mathtt{x}^2)\mathtt{y}^2.$$

Thus we may assume that

$$\xi = F_4(\mathtt{s},\mathtt{t})\mathtt{g}^2 + F_3(\mathtt{s},\mathtt{t})\mathtt{f}\mathtt{g}$$

for a cubic polynomial F_3 and a quartic polynomial F_4, where Δ is defined by $F_4(\mathtt{x}^2,(\mathtt{x}+\mathtt{y})\mathtt{y}) + F_3(\mathtt{x}^2,(\mathtt{x}+\mathtt{y})\mathtt{y})\mathtt{y}^2 = 0$. Since

$$\xi\mathtt{v} - \eta\mathtt{u} = (F_4(\mathtt{s},\mathtt{t})\mathtt{v} - \mathtt{t}^2\mathtt{u})\mathtt{g}^2 + (F_3(\mathtt{s},\mathtt{t})\mathtt{v} - \mathtt{s}\mathtt{u})\mathtt{g}\mathtt{f} - \mathtt{u}\mathtt{f}^2,$$

we infer that $V \to \mathbb{F}_2$ is not finite over $\{\mathtt{u} = F_3(\mathtt{s},\mathtt{t}) = F_4(\mathtt{s},\mathtt{t}) = 0\}$. Applying Lemma 7.7, we have a birational map $\mathbb{P} \dashrightarrow \mathbb{P}(1,1,1,2)$ such that the proper transform V' of V in $\mathbb{P}(1,1,1,2)$ is a Cartier divisor of degree 4 defined by

$$-\mathtt{U}_0\mathtt{Z}_0^2 + (F_3(\mathtt{X},\mathtt{Y}) - \mathtt{X}\mathtt{U}_0)\mathtt{Z}_0 + F_4(\mathtt{X},\mathtt{Y}) - \mathtt{Y}^2\mathtt{U}_0 = 0$$

for the homogeneous coordinate $(\mathtt{X}, \mathtt{Y}, \mathtt{Z}_0, \mathtt{U}_0)$ of weight $(1,1,1,2)$. However, the projection $(\mathtt{X}:\mathtt{Y}:\mathtt{Z}_0:\mathtt{U}_0) \mapsto (\mathtt{X}:\mathtt{Y}:\mathtt{U}_0)$ induces a rational map $V' \dashrightarrow W = \mathbb{P}(1,1,2)$ with non-empty locus of indeterminacy. We consider the birational map

$$\begin{aligned}&\mathbb{P}(1,1,1,2) \dashrightarrow \mathbb{P}(1,1,2,3);\\ &(\mathtt{X}:\mathtt{Y}:\mathtt{Z}_0:\mathtt{U}_0) \mapsto (\mathtt{X}:\mathtt{Y}:\mathtt{U}:\mathtt{Z}) = (\mathtt{X}:\mathtt{Y}:\mathtt{U}_0:\mathtt{Z}_0\mathtt{U}_0).\end{aligned}$$

Then the proper transform V'' of V in $\mathbb{P}(1,1,2,3)$ is a Cartier divisor of degree 6 defined by

$$\Psi := -\mathtt{Z}^2 + (F_3(\mathtt{X},\mathtt{Y}) - \mathtt{XU})\mathtt{Z} + F_4(\mathtt{X},\mathtt{Y})\mathtt{U} - \mathtt{Y}^2\mathtt{U}^2 = 0$$

and the projection $(\mathtt{X}:\mathtt{Y}:\mathtt{U}:\mathtt{Z}) \mapsto (\mathtt{X}:\mathtt{Y}:\mathtt{U})$ induces a finite morphism $V'' \to W$. Since $\operatorname{char} \Bbbk = 2$, we have

$$\frac{\partial \Psi}{\partial \mathtt{Z}} = F_3(\mathtt{X},\mathtt{Y}) - \mathtt{XU}, \quad \frac{\partial \Psi}{\partial \mathtt{U}} = -\mathtt{XZ} + F_4(\mathtt{X},\mathtt{Y}),$$
$$\frac{\partial \Psi}{\partial \mathtt{X}} = \frac{\partial F_3}{\partial \mathtt{X}}(\mathtt{X},\mathtt{Y})\mathtt{Z} - \mathtt{UZ} + \frac{\partial F_4}{\partial \mathtt{X}}(\mathtt{X},\mathtt{Y})\mathtt{U}, \quad \frac{\partial \Psi}{\partial \mathtt{Y}} = \frac{\partial F_3}{\partial \mathtt{Y}}(\mathtt{X},\mathtt{Y})\mathtt{Z} + \frac{\partial F_4}{\partial \mathtt{Y}}(\mathtt{X},\mathtt{Y})\mathtt{U}.$$

Thus the singular locus of V'' is contained in the locus Σ defined by the following equations:

$$\text{(i)}\quad \Psi = 0; \qquad \text{(ii)}\quad \mathtt{XU} = F_3(\mathtt{X},\mathtt{Y}), \qquad \text{(iii)}\quad \mathtt{XZ} = F_4(\mathtt{X},\mathtt{Y}),$$
$$\text{(iv)}\quad \mathtt{UZ} = \frac{\partial F_3}{\partial \mathtt{X}}(\mathtt{X},\mathtt{Y})\mathtt{Z} + \frac{\partial F_4}{\partial \mathtt{X}}(\mathtt{X},\mathtt{Y})\mathtt{U}.$$

In order to show, $S \simeq V''$, we have only to check that Σ is a finite set. The four equations above induce the following (v) and (vi), where (v) follows from (i)–(iii), and (vi) follows from (ii), (iii), and (v) multiplied by $\mathtt{X}^2$:

$$\text{(v)}\quad \mathtt{Z}^2 - \mathtt{XUZ} + \mathtt{Y}^2\mathtt{U}^2 = 0;$$
$$\text{(vi)}\quad F_4(\mathtt{X},\mathtt{Y})^2 - \mathtt{X}F_3(\mathtt{X},\mathtt{Y})F_4(\mathtt{X},\mathtt{Y}) + \mathtt{Y}^2F_3(\mathtt{X},\mathtt{Y})^2 = 0.$$

Note that (vi) does not hold identically on $\mathbb{P}^1$. This is shown as follows: Assume the contrary. We may also assume that $F_3(\mathtt{X},\mathtt{Y})$ is not identically zero. Then the rational function $w = F_4(\mathtt{X},\mathtt{Y})\mathtt{X}^{-1}F_3(\mathtt{X},\mathtt{Y})^{-1}$ is related to the rational function $y = \mathtt{Y}/\mathtt{X}$ by the Artin-Schreier equation: $w^2 - w + y^2 =$

0. Here $\Bbbk(y)/\Bbbk(y^2)$ is inseparable but $\Bbbk(w)/\Bbbk(y^2)$ is separable. However $\Bbbk(w) \subset \Bbbk(y)$ by the assumption. Thus a contradiction is derived.

Therefore, $\Sigma \cap \{\mathtt{X} \neq 0\}$ is a finite set. If $(0:1:u:z) \in \Sigma$, then $u = z$ by (v) and

$$u\left(\frac{\partial F_3}{\partial \mathtt{X}}(0,1) + \frac{\partial F_4}{\partial \mathtt{X}}(0,1) - u\right) = 0$$

by (iv). Thus $\Sigma \cap \{\mathtt{X} = 0\}$ is also finite. Therefore we finished the proof. □

REMARK. The equations in Propositions 7.11–7.15 and 7.17–7.20 define log del Pezzo surfaces of index two. In fact, the subvariety defined by the equations is really constructed from a fundamental triplet (X, E, Δ) of the same type, where E and Δ are defined by the data of the equations.

Example 7.21. Let (X, E, Δ) be an extremal fundamental triplet of type $[1; 2, 2]_0$ with $\mathcal{D}(X, E, \Delta) = \mathsf{D}_8$. Then the associated log del Pezzo surface S is defined by

$$\begin{aligned} &\mathtt{Z}^2 = (\mathtt{X}^3 + \mathtt{YU})\mathtt{XU}, && \text{if } \operatorname{char}\Bbbk \neq 2 \text{ or } E \to \mathbb{P}^1 \text{ is inseparable,} \\ &\mathtt{Z}^2 = (\mathtt{XZ} + \mathtt{X}^4 + \mathtt{Y}^2\mathtt{U})\mathtt{U}, && \text{otherwise,} \end{aligned}$$

in $\mathbb{P}(1,1,2,3)$ for the homogeneous coordinate $(\mathtt{X}, \mathtt{Y}, \mathtt{U}, \mathtt{Z})$ of weight $(1,1,2,3)$. In fact, we can take $F_3 = 0$ and $F_4 = \mathtt{X}^4$ in Proposition 7.20.

The following example shows that the Smooth Divisor Theorem in [4] does not hold in general in characteristic two. This was pointed out by Ohashi in a special case.

Example 7.22. Suppose that $\operatorname{char}\Bbbk = 2$. Let S be a log del Pezzo surface defined by the equation of Proposition 7.20, (1), with $F_3 = 0$. Then $\pi|_E \colon E \to \mathbb{P}^1$ is inseparable. We can show that any member C of $|-2K_S|$ has a singular point, as follows: A general member C is defined by $\mathtt{U} - Q(\mathtt{X}, \mathtt{Y}) = 0$ for a quadric polynomial Q. Thus $C \subset \mathbb{P}(1,1,3)$ is defined by

$$\mathtt{Z}^2 = F_4(\mathtt{X}, \mathtt{Y})Q(\mathtt{X}, \mathtt{Y}) + \mathtt{XY}Q(\mathtt{X}, \mathtt{Y})^2$$

for the homogeneous coordinate $(\mathtt{X}, \mathtt{Y}, \mathtt{Z})$ of weight $(1,1,3)$. Let $(\mathtt{x}, \mathtt{z})$ be the coordinate system of the open subset $\{\mathtt{Y} \neq 0\} \simeq \mathbb{A}^2$ defined by $\mathtt{x} = \mathtt{X}/\mathtt{Y}$ and $\mathtt{z} = \mathtt{Z}/\mathtt{Y}^3$. Then $C \cap \{\mathtt{Y} \neq 0\}$ is defined by $\mathtt{z}^2 = \Phi(\mathtt{x})$ for the polynomial

$$\Phi(\mathtt{x}) = Q(\mathtt{x}, 1)F_4(\mathtt{x}, 1) + \mathtt{x}Q(\mathtt{x}, 1)^2.$$

Thus a point $(\mathtt{x}_0, \mathtt{z}_0) \in \mathbb{A}^2$ is contained in $\operatorname{Sing} C \cap \{\mathtt{Y} \neq 0\}$ if and only if $(\mathrm{d}\Phi/\mathrm{d}\mathtt{x})(\mathtt{x}_0) = 0$ and $\mathtt{z}_0^2 = \Phi(\mathtt{x}_0)$. Thus $\operatorname{Sing} C \neq \emptyset$.

Remark 7.23. We consider a fundamental triplet (X, E, Δ) is of type $[1; 2, 2]_0$ with $\Delta = 8P$ for a non-ramification point $P \in E$ for $\pi|_E$. Let $\phi\colon M \to X$ be the elimination of Δ. The dual graph $\mathit{\Gamma}[M] = \mathit{\Gamma}(X, E, \Delta)$ of negative curves on M is written in Table 12. We shall give further information on the set of negative curves by using the description of E and Δ in Proposition 7.20, in case $\operatorname{char} \Bbbk \geq 5$. Let $(\mathtt{x} : \mathtt{y})$ be the coordinate of $E \simeq \mathbb{P}^1$ used in *Case* (1) of the proof of Proposition 7.20. Then we may assume that $P \in E$ is defined by $\mathtt{x} + \mathtt{y} = 0$. Let us define homogeneous polynomials $P_n(\mathtt{s}, \mathtt{t})$ and $Q_n(\mathtt{s}, \mathtt{t}) \in \mathbb{Z}[1/2, \mathtt{s}, \mathtt{t}]$ of degree $n \geq 0$ by

$$(\mathtt{x} + \mathtt{y})^{2n} = P_n(\mathtt{x}^2, \mathtt{y}^2) + 2\mathtt{x}\mathtt{y}Q_{n-1}(\mathtt{x}^2, \mathtt{y}^2).$$

Here, $Q_{-1}(\mathtt{s}, \mathtt{t}) = 0$, $P_0(\mathtt{s}, \mathtt{t}) = Q_0(\mathtt{s}, \mathtt{t}) = 1$, and we have

$$\begin{aligned} 2P_n(\mathtt{x}^2, \mathtt{y}^2) &= (\mathtt{x} + \mathtt{y})^{2n} + (\mathtt{x} - \mathtt{y})^{2n} = \prod\nolimits_{k=0}^{2n-1} ((\mathtt{x} + \mathtt{y}) - \zeta^{2k+1}(\mathtt{x} - \mathtt{y})), \\ 4\mathtt{x}\mathtt{y}Q_{n-1}(\mathtt{x}^2, \mathtt{y}^2) &= (\mathtt{x} + \mathtt{y})^{2n} - (\mathtt{x} - \mathtt{y})^{2n} = \prod\nolimits_{k=0}^{2n-1} ((\mathtt{x} + \mathtt{y}) - \zeta^{2k}(\mathtt{x} - \mathtt{y})) \end{aligned}$$

for $\zeta = \exp(2\pi\sqrt{-1}/(4n))$ for $n \geq 1$. Therefore,

$$\begin{aligned} P_n(\mathtt{s}, \mathtt{t}) &= 2^{n-1} \prod\nolimits_{k=0}^{n-1} \left((\mathtt{s} + \mathtt{t}) - \cos\left(\tfrac{2k+1}{2n}\pi\right)(\mathtt{s} - \mathtt{t})\right), \\ Q_{n-1}(\mathtt{s}, \mathtt{t}) &= 2^{n-1} \prod\nolimits_{k=1}^{n-1} \left((\mathtt{s} + \mathtt{t}) - \cos\left(\tfrac{k}{n}\pi\right)(\mathtt{s} - \mathtt{t})\right). \end{aligned}$$

In particular, $P_n(\mathtt{s}, \mathtt{t})$ and $Q_n(\mathtt{s}, \mathtt{t})$ have only simple roots on $\mathbb{P}^1$ if $\gcd(\operatorname{char} \Bbbk, 2n) = 1$. We also have the equality

$$P_i(\mathtt{s}, \mathtt{t})Q_{j-1}(\mathtt{s}, \mathtt{t}) - P_j(\mathtt{s}, \mathtt{t})Q_{i-1}(\mathtt{s}, \mathtt{t}) = (\mathtt{s} - \mathtt{t})^{2i}Q_{j-i-1}(\mathtt{s}, \mathtt{t}) \tag{7–10}$$

for $0 < i < j$ by calculation. Let γ_j be the unique section of π with

$\gamma_j|_E = 2jP$ (cf. Proposition 6.2, (7g)) for $1 \le j \le 4$. Then

$$\gamma_j = \operatorname{div}(P_j(\mathtt{s},\mathtt{t})\mathtt{g} + 2Q_{j-1}(\mathtt{s},\mathtt{t})\mathtt{f}).$$

We set γ_0 to be σ. Then

$$\gamma_i \cap \gamma_j = \gamma_i \cap \operatorname{div}((\mathtt{s}-\mathtt{t})^{2i} Q_{j-i-1}(\mathtt{s},\mathtt{t}))$$

for $0 \le i < j \le 4$ by (7–10). Let $\gamma_{j,M} \subset M$ be the proper transform of γ_j in M for $0 \le j \le 4$, which is a (-1)-curve. If $\gamma_{i,M} \cap \gamma_{j,M} \ne \emptyset$ for $i < j$, then $i+1 < j$, and the following assertions hold:

- $\gamma_{i,M} \cap \gamma_{j,M}$ is a reduced point lying over the point $(1 : -1) \in \mathbb{P}^1$ for $j = i+2$,
- $\gamma_{i,M} \cap \gamma_{j,M}$ is reduced consisting of two points lying over $\{(3 : -1), (1 : -3)\} \subset \mathbb{P}^1$ for $j = i+3$,
- $\gamma_{0,M} \cap \gamma_{4,M}$ is reduced consisting of three points lying over

$$\left\{\left(1+\sqrt{2} : 1-\sqrt{2}\right), (1 : -1), \left(1-\sqrt{2} : 1+\sqrt{2}\right)\right\} \subset \mathbb{P}^1.$$

In particular, $\gamma_{0,M} \cap \gamma_{2,M} = \gamma_{2,M} \cap \gamma_{4,M}$ is a point P_M lying over the point $\{\mathtt{g} = \mathtt{s} + \mathtt{t} = 0\}$, and the union of negative curves on M is not normal crossing at the point P_M. From the dual graph $\Gamma[M]$ in TABLE 12, we can not obtain directly the property that $\gamma_{0,M}$, $\gamma_{2,M}$, and $\gamma_{4,M}$ meet at a point.

REMARK 7.24. For a log del Pezzo surface S of index two, we have proved in Theorem 3.32 that $-4K_S$ is very ample, and that $-2K_S$ is very ample if and only if $K_S^2 > 1$. We can check it by our explicit description of S as follows:

If S is one of surfaces treated in Section 7.3, i.e., $K_M + L_M$ is big and S is not of type $[n;1,0]_0$, then S is expressed as a subvariety of a weighted projective space. Here, $\mathcal{O}_S(-2K_S)$ is just the restriction of a very ample invertible sheaf of the weighted projective space, by construction.

The surfaces S of type $[n;1,0]_0$ are treated in Section 7.4, where $K_S^2 = 5+n > 1$. If $n = 4$, then $S \simeq \mathbb{P}(1,1,4)$ and $\mathcal{O}_S(-2K_S) = \mathcal{O}(4)$ is very ample. If $0 < n < 4$, then S is a subvariety of $\mathbb{P}(1,1,n) \times \mathbb{P}(1,1,4)$ where

$\mathcal{O}_S(-2K_S)$ is just the restriction of $\mathcal{O}(2n) \boxtimes \mathcal{O}(4)$ by Proposition 7.1; thus $-2K_S$ is very ample. If $n = 0$, then S is a subvariety of $\mathbb{P}^1 \times \mathbb{P}(1,1,4)$ and $\mathcal{O}_S(-2K_S)$ is just the restriction of the very ample invertible sheaf $\mathcal{O}(2) \boxtimes \mathcal{O}(4)$ also by Proposition 7.1.

If S is of type $[4;2,4]_{00}$, then $K_S^2 = 2$, and $\mathcal{O}_S(-2K_S)$ is the restriction of the very ample invertible sheaf $\mathcal{O}(4)$ of $\mathbb{P}(1,1,4,4)$ by Proposition 7.17.

Thus, the remaining types are $[3;2,4]_+$, $[3;2,4]_{++}(a,b)$, and $[1;2,2]_0$. These are just the cases of S with $K_S^2 = 1$. In this case, S is a prime divisor of $\mathbb{P}(1,1,2,3)$ not containing the point $(0:0:0:1)$ and $\mathcal{O}_S(-2K_S) \simeq \mathcal{O}(2)|_S$, by Propositions 7.18, 7.19, 7.20. We note that $-2K_S$ is not very ample, since $\mathcal{O}(2)|_S$ is the pullback of $\mathcal{O}(2)$ by the projection $S \to \mathbb{P}(1,1,2)$. It is enough to check that $\mathcal{O}(4)|_S \simeq \mathcal{O}_S(-4K_S)$ is very ample, since $\mathcal{O}(6)$ is very ample on $\mathbb{P}(1,1,2,3)$. Let $(\mathtt{X},\mathtt{Y},\mathtt{U},\mathtt{Z})$ be the homogeneous coordinate of $\mathbb{P}(1,1,2,3)$ as before. Then the vector space $\mathrm{H}^0(\mathbb{P}(1,1,2,3),\mathcal{O}(4))$ is generated by

$$\mathtt{X}^{4-i}\mathtt{Y}^i, \quad \mathtt{X}^{2-j}\mathtt{Y}^j\mathtt{U}, \quad \mathtt{XZ}, \quad \mathtt{YZ}$$

for $0 \le i \le 4$ and $0 \le j \le 2$. Now S is covered by three affine open subsets $\{\mathtt{X} \ne 0\}$, $\{\mathtt{Y} \ne 0\}$, $\{\mathtt{U} \ne 0\}$. The affine ring of $\{\mathtt{X} \ne 0\}$ is isomorphic to the polynomial ring of three variables generated by

$$\mathtt{Y}/\mathtt{X} = \mathtt{X}^3\mathtt{Y}/\mathtt{X}^4, \quad \mathtt{U}/\mathtt{X}^2 = \mathtt{X}^2\mathtt{U}/\mathtt{X}^4, \quad \mathtt{Z}/\mathtt{X}^3 = \mathtt{XZ}/\mathtt{X}^4.$$

Thus the linear system $|\mathcal{O}(4)|$ gives an embedding of the open subset $\{\mathtt{X} \ne 0\}$ into the projective space $|\mathcal{O}(4)|^\vee = \mathbb{P}(\mathrm{H}^0(\mathbb{P}(1,1,2,3),\mathcal{O}(4)))$. Similarly, it gives an embedding of $\{\mathtt{Y} \ne 0\}$. The affine ring of $\{\mathtt{U} \ne 0\}$ is isomorphic to the subring $\Bbbk[\mathtt{x},\mathtt{y},\mathtt{z}]^{(2)}$ generated by monomials of degree two of the polynomial ring $\Bbbk[\mathtt{x},\mathtt{y},\mathtt{z}]$ of three variables. This ring is generated by

$$\mathtt{x}^{2-j}\mathtt{y}^j = \mathtt{X}^{2-j}\mathtt{Y}^j/\mathtt{U}, \quad \mathtt{xz} = \mathtt{XZ}/\mathtt{U}^2, \quad \mathtt{yz} = \mathtt{YZ}/\mathtt{U}^2, \quad \mathtt{z}^2 = \mathtt{Z}^2/\mathtt{U}^3$$

for $0 \le j \le 2$. Since $\mathtt{Z}^2 \notin \mathrm{H}^0(\mathbb{P}(1,1,2,3),\mathcal{O}(4))$, the linear system $|\mathcal{O}(4)|$ does not give an embedding of $\{\mathtt{U} \ne 0\}$. However, the defining equations of S obtained in Propositions 7.18, 7.19, 7.20 express $\mathtt{z}^2 = \mathtt{Z}^2/\mathtt{U}^3$ by other generators of the affine ring. Hence, $\mathcal{O}(4)|_S$ is very ample.

Table 14. The list of defining equations of log del Pezzo surfaces of index two

Type	Equations (conditions)	coordinates ambient space
$[1]_0$	$\mathtt{XU} = F_5(\mathtt{Y},\mathtt{Z})$ $(F_5 \neq 0)$	$(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{U})$ $\mathbb{P}(1,1,1,4)$
$[2]_0$	$(\mathtt{Z}^2 - \mathtt{XY})\mathtt{U} = F_4(\mathtt{X},\mathtt{Y}) + F_3(\mathtt{X},\mathtt{Y})\mathtt{Z}$ $((F_3, F_4) \neq (0,0))$	$(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{U})$ $\mathbb{P}(1,1,1,2)$
$[2]_+(0)$	$\mathtt{XYU} = \mathtt{Z}^4 + F_3(\mathtt{X},\mathtt{Z})\mathtt{X} + G_3(\mathtt{Y},\mathtt{Z})\mathtt{Y}$	$(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{U})$ $\mathbb{P}(1,1,1,2)$
$[2]_+(b)$ $(1 \leq b \leq 4)$	$\mathtt{XYU} = F_{4-b}(\mathtt{X},\mathtt{Z})\mathtt{X}^b + G_3(\mathtt{Y},\mathtt{Z})\mathtt{Y}$ $(F_{4-b}(0,1) \neq 0, G_3(0,1) \neq 0)$	$(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{U})$ $\mathbb{P}(1,1,1,2)$
$[2;1,2]_0$	$\mathtt{ZU} = F_6(\mathtt{X},\mathtt{Y})$ $(F_6 \neq 0)$	$(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{U})$ $\mathbb{P}(1,1,2,4)$
$[2;1,2]_{++}$	$\mathtt{XYU} = \mathtt{Z}^3 + F_1(\mathtt{Z},\mathtt{X}^2)\mathtt{X}^2\mathtt{Z} + G_1(\mathtt{Z},\mathtt{Y}^2)\mathtt{Y}^2\mathtt{Z}$	$(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{U})$ $\mathbb{P}(1,1,2,4)$
$[0;1,1]_0$	$\mathtt{XW} = \mathtt{YZ}$ $(\mathtt{X}-\mathtt{W})\mathtt{U} = F_2(\mathtt{Z},\mathtt{W})\mathtt{W} + G_2(\mathtt{W},\mathtt{Y})\mathtt{Y}$ $((F_2, G_2) \neq (0,0))$	$(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{W},\mathtt{U})$ $\mathbb{P}(1,1,1,1,2)$
$[0;1,1]_+(0)$	$\mathtt{XW} = \mathtt{YZ}$ $\mathtt{XU} = \mathtt{W}^3 + F_1(\mathtt{Z},\mathtt{W})\mathtt{ZW} + G_1(\mathtt{W},\mathtt{Y})\mathtt{YW}$	$(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{W},\mathtt{U})$ $\mathbb{P}(1,1,1,1,2)$
$[0;1,1]_+(1)$	$\mathtt{XW} = \mathtt{YZ}$ $\mathtt{XU} = (\mathtt{W}+c\mathtt{Z})\mathtt{ZW} + (\mathtt{W}+c'\mathtt{Y})\mathtt{YW}$	$(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{W},\mathtt{U})$ $\mathbb{P}(1,1,1,1,2)$
$[0;1,1]_+(b)$ $(2 \leq b \leq 3)$	$\mathtt{XW} = \mathtt{YZ}$ $\mathtt{XU} = (\mathtt{W}+c\mathtt{Z})\mathtt{ZW} + \mathtt{W}^{3-b}\mathtt{Y}^b$	$(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{W},\mathtt{U})$ $\mathbb{P}(1,1,1,1,2)$
$[1;1,1]_0$	$\mathtt{XW} = \mathtt{YZ}$ $\mathtt{ZU} = F_5(\mathtt{X},\mathtt{Y})\mathtt{X}$ $\mathtt{WU} = F_5(\mathtt{X},\mathtt{Y})\mathtt{Y}$ $(F_5 \neq 0)$	$(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{W},\mathtt{U})$ $\mathbb{P}(1,1,2,2,4)$
$[1;1,1]_+(0,0)$	$\mathtt{XW} = \mathtt{YZ}$ $(\mathtt{W}+c\mathtt{Z})\mathtt{ZW} = (\mathtt{XU} - G_1(\mathtt{W},\mathtt{Y}^2)\mathtt{YW})\mathtt{X}$ $(\mathtt{W}+c\mathtt{Z})\mathtt{W}^2 = (\mathtt{XU} - G_1(\mathtt{W},\mathtt{Y}^2)\mathtt{YW})\mathtt{Y}$	$(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{W},\mathtt{U})$ $\mathbb{P}(1,1,2,2,4)$
$[1;1,1]_+(1,1)$	$\mathtt{XW} = \mathtt{YZ}$ $\mathtt{Z}^2\mathtt{W} = (\mathtt{XU} - (\mathtt{W}+c\mathtt{Y}^2)\mathtt{YW})\mathtt{X}$ $\mathtt{ZW}^2 = (\mathtt{XU} - (\mathtt{W}+c\mathtt{Y}^2)\mathtt{YW})\mathtt{Y}$	$(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{W},\mathtt{U})$ $\mathbb{P}(1,1,2,2,4)$
$[1;1,1]_+(2,1)$	$\mathtt{XW} = \mathtt{YZ}$ $\mathtt{Z}^3 = (\mathtt{XU} - (\mathtt{W}+c\mathtt{Y}^2)\mathtt{YW})\mathtt{X}$ $\mathtt{Z}^2\mathtt{W} = (\mathtt{XU} - (\mathtt{W}+c\mathtt{Y}^2)\mathtt{YW})\mathtt{Y}$	$(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{W},\mathtt{U})$ $\mathbb{P}(1,1,2,2,4)$
$[1;1,1]_+(1,b)$ $(2 \leq b \leq 3)$	$\mathtt{XW} = \mathtt{YZ}$ $\mathtt{Z}^2\mathtt{W} = (\mathtt{XU} - \mathtt{Y}^{2b-1}\mathtt{W}^{3-b})\mathtt{X}$ $\mathtt{ZW}^2 = (\mathtt{XU} - \mathtt{Y}^{2b-1}\mathtt{W}^{3-b})\mathtt{Y}$	$(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{W},\mathtt{U})$ $\mathbb{P}(1,1,2,2,4)$
$[2;1,1]_+(0,0)$	$\mathtt{XW} = \mathtt{YZ}$ $\mathtt{Z}^{2-i}\mathtt{W}^{i+1} = (\mathtt{XU} - G_1(\mathtt{W},\mathtt{Y}^3)\mathtt{YW})\mathtt{X}^{2-i}\mathtt{Y}^i$ (for $0 \leq i \leq 2$)	$(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{W},\mathtt{U})$ $\mathbb{P}(1,1,3,3,6)$

TABLE 14. (continued).

Type	Equations (conditions)	coordinates ambient space
$[2;1,1]_+(1,1)$	$\mathtt{XW} = \mathtt{YZ}$ $\mathtt{Z}^{3-i}\mathtt{W}^i = (\mathtt{XU}-(\mathtt{W}+c\mathtt{Y}^3)\mathtt{YW})\mathtt{X}^{2-i}\mathtt{Y}^i$ (for $0 \le i \le 2$)	$(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{W},\mathtt{U})$ $\mathbb{P}(1,1,3,3,6)$
$[2;1,1]_+(1,b)$ $(2 \le b \le 3)$	$\mathtt{XW} = \mathtt{YZ}$ $\mathtt{Z}^{3-i}\mathtt{W}^i = (\mathtt{XU}-\mathtt{Y}^{3b-2}\mathtt{W}^{3-b})\mathtt{X}^{2-i}\mathtt{Y}^i$ (for $0 \le i \le 2$)	$(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{W},\mathtt{U})$ $\mathbb{P}(1,1,3,3,6)$
$[3;1,1]_+$	$\mathtt{XW} = \mathtt{YZ}$ $\mathtt{Z}^{3-i}\mathtt{W}^i = (\mathtt{XU}-G_1(\mathtt{W},\mathtt{Y}^4)\mathtt{YW})\mathtt{X}^{3-i}\mathtt{Y}^i$ (for $0 \le i \le 3$)	$(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{W},\mathtt{U})$ $\mathbb{P}(1,1,4,4,8)$
$[0;1,0]_0$	$\mathtt{Z}_1\mathtt{X}_0 = \mathtt{Y}_0F_4(\mathtt{X}_1,\mathtt{Y}_1)$	$(\mathtt{X}_0,\mathtt{Y}_0)\times(\mathtt{X}_1,\mathtt{Y}_1,\mathtt{Z}_1)$ $\mathbb{P}^1 \times \mathbb{P}(1,1,4)$
$[n;1,0]_0$ $(1 \le n \le 3)$	$\mathtt{X}_0\mathtt{Y}_1 = \mathtt{X}_1\mathtt{Y}_0$ $\mathtt{Z}_1\mathtt{X}_0^{n-i}\mathtt{Y}_0^i = \mathtt{Z}_0\mathtt{X}_1^{n-i}\mathtt{Y}_1^iF_{4-n}(\mathtt{X}_1,\mathtt{Y}_1)$ (for $0 \le i \le n$)	$(\mathtt{X}_0,\mathtt{Y}_0,\mathtt{Z}_0)\times(\mathtt{X}_1,\mathtt{Y}_1,\mathtt{Z}_1)$ $\mathbb{P}(1,1,n) \times \mathbb{P}(1,1,4)$
$[4;1,0]_0$	no equation	$\mathbb{P}(1,1,4)$
$[4;2,4]_{00}$	$\mathtt{UZ} = F_8(\mathtt{X},\mathtt{Y})$ $(F_8 \ne 0)$	$(\mathtt{X},\mathtt{Y},\mathtt{Z},\mathtt{U})$ $\mathbb{P}(1,1,4,4)$
$[3;2,4]_+$	$\mathtt{Z}^2+(c\mathtt{Y}^3+\mathtt{XU})\mathtt{Z}+F_6(\mathtt{X},\mathtt{Y}) = \mathtt{Y}^4\mathtt{U}$	$(\mathtt{X},\mathtt{Y},\mathtt{U},\mathtt{Z})$ $\mathbb{P}(1,1,2,3)$
$[3;2,4]_{++}(0,0)$	$\mathtt{Z}^2+(c\mathtt{Y}^3+\mathtt{XU})\mathtt{Z}+\mathtt{Y}^6+\mathtt{X}F_5(\mathtt{X},\mathtt{Y}) = 0$	$(\mathtt{X},\mathtt{Y},\mathtt{U},\mathtt{Z})$ $\mathbb{P}(1,1,2,3)$
$[3;2,4]_{++}(2,1)$	$\mathtt{Z}^2+\mathtt{XUZ}+\mathtt{XY}^5+\mathtt{X}^2F_4(\mathtt{X},\mathtt{Y}) = 0$	$(\mathtt{X},\mathtt{Y},\mathtt{U},\mathtt{Z})$ $\mathbb{P}(1,1,2,3)$
$[3;2,4]_{++}(1,b)$ $(1 \le b \le 5)$	$\mathtt{Z}^2+(\mathtt{Y}^3+\mathtt{XU})\mathtt{Z}$ $+\mathtt{X}^b\mathtt{Y}^{6-b}+\mathtt{X}^{b+1}F_{5-b}(\mathtt{X},\mathtt{Y}) = 0$	$(\mathtt{X},\mathtt{Y},\mathtt{U},\mathtt{Z})$ $\mathbb{P}(1,1,2,3)$
$[3;2,4]_{++}(1,6)$	$\mathtt{Z}^2+(\mathtt{Y}^3+\mathtt{XU})\mathtt{Z}+\mathtt{X}^6 = 0$	$(\mathtt{X},\mathtt{Y},\mathtt{U},\mathtt{Z})$ $\mathbb{P}(1,1,2,3)$
$[1;2,2]_0$ char $\Bbbk \ne 2$ or $E \subset X \to \mathbb{P}^1$ is inseparable	$\mathtt{Z}^2 = F_3(\mathtt{X},\mathtt{Y})\mathtt{Z}+F_4(\mathtt{X},\mathtt{Y})\mathtt{U}+\mathtt{XYU}^2$ $((F_3,F_4) \ne (0,0))$	$(\mathtt{X},\mathtt{Y},\mathtt{U},\mathtt{Z})$ $\mathbb{P}(1,1,2,3)$
$[1;2,2]_0$ char $\Bbbk \ne 2$ and $E \subset X \to \mathbb{P}^1$ is separable	$\mathtt{Z}^2 = (F_3(\mathtt{X},\mathtt{Y})+\mathtt{XU})\mathtt{Z}$ $+F_4(\mathtt{X},\mathtt{Y})\mathtt{U}+\mathtt{Y}^2\mathtt{U}^2$ $((F_3,F_4) \ne (0,0))$	$(\mathtt{X},\mathtt{Y},\mathtt{U},\mathtt{Z})$ $\mathbb{P}(1,1,2,3)$

F_i, G_i are homogeneous polynomials of two variables of degree i.
c, c' are constants in $\Bbbk$.

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