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On the Galois Actions on Torsors of Paths I, Descent of Galois Representations

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Contents

Abstract. We are studying representations obtained from actions of Galois groups on torsors of paths on a projective line minus a finite number of points. Using these actions on torsors of paths, we construct geometrically representations of Galois groups which realize ℓ-adically the associated graded Lie algebra of the fundamental group of the tannakian category of mixed Tate motives over $\operatorname{Spec}\mathbb{Z}$, $\operatorname{Spec}\mathbb{Z}[i]$, $\operatorname{Spec}\mathbb{Z}[\frac{1}{q}]$, $\operatorname{Spec}\mathcal{O}_{\mathbb{Q}(\sqrt{-q})}$ for any prime number q ($q \neq 2$ in the last case) and over $\operatorname{Spec}\mathcal{O}_{\mathbb{Q}(\sqrt{-q})}[\frac{1}{q}]$ for any prime number q congruent to 3 modulo 4 and also for $q = 2$.

0. Introduction

0.1. In this paper we are studying actions of Galois groups on torsors of paths. Let V be an algebraic variety defined over a number field K. Let us fix two points or tangential points v and z of V defined over K. Let

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ℓ be a fixed prime. We denote by $\pi_1(V_{\bar{K}};v)$ the ℓ-completion of the étale fundamental group of $V_{\bar{K}}$ base at v and by $\pi(V_{\bar{K}};z,v)$ the $\pi_1(V_{\bar{K}};v)$-torsor of ℓ-adic paths from v to z. The Galois group G_K acts on $\pi_1(V_{\bar{K}};v)$ and on $\pi(V_{\bar{K}};z,v)$. Therefore we have two representations

$$\varphi_v : G_K \to \mathrm{Aut}\,(\pi_1(V_{\bar{K}};v))$$

and

$$\psi_{z,v} : G_K \to \mathrm{Aut}_{\mathrm{set}}(\pi(V_{\bar{K}};z,v))\,.$$

We mention some examples.

Let $V := \mathbb{P}^1_{\mathbb{Q}}\backslash\{0,1,\infty\}$. Then the action of $G_{\mathbb{Q}}$ on $\pi_1(V_{\bar{\mathbb{Q}}};\overrightarrow{01})$ and on $\pi_1(V_{\bar{\mathbb{Q}}};\overrightarrow{01})$-torsor $\pi(V_{\bar{\mathbb{Q}}};\overrightarrow{10},\overrightarrow{01})$ is "the same", *i.e.*,

$$\ker(G_{\mathbb{Q}} \to \mathrm{Aut}\,(\pi_1(V_{\bar{\mathbb{Q}}};\overrightarrow{01}))) \quad \text{and} \quad \ker(G_{\mathbb{Q}} \to \mathrm{Aut}_{\mathrm{set}}(\pi(V_{\bar{\mathbb{Q}}};\overrightarrow{10},\overrightarrow{01})))$$

are equal. One also shows that the actions of $G_{\mathbb{Q}}$ on $\pi_1\Big({\mathbb{P}^1}_{\bar{\mathbb{Q}}}\backslash\{0,1,-1,\infty\};\overrightarrow{01}\Big)$ and on $\pi(V_{\bar{\mathbb{Q}}};-1,\overrightarrow{01})$ have equal kernels after passing to associated graded Lie algebras (see [22]). On the other side the action of $G_{\mathbb{Q}}$ on $\pi_1(V_{\bar{\mathbb{Q}}};\overrightarrow{01})$ is unramified outside ℓ and the action of $G_{\mathbb{Q}}$ on $\pi(V_{\bar{\mathbb{Q}}};2,\overrightarrow{01})$ is ramified at ℓ and at 2.

0.2. Let $a_1,\dots,a_{n+1}$ be K-points of the projective line $\mathbb{P}^1_K$ and let

$$V := \mathbb{P}^1_K\backslash\{a_1,\dots,a_n,a_{n+1}\}\,.$$

For simplicity we assume that $a_{n+1}=\infty$. Let v be a K-point of V or a tangential point of V defined over K.

Let $f(T)\in K[T]$ be an irreducible polynomial and let $z_1,\dots,z_r$ be all roots of $f(T)$ in $\bar{K}$. Let $L := K(z_1,\dots,z_r)$ be an extension of K generated by all roots of $f(T)$. The Galois group G_L acts on the disjoint union of torsors $t := \coprod_{i=1}^{r}\pi(V_{\bar{K}};z_i,v)$, hence we get a representation

$$\psi_t : G_L \to \mathrm{Aut}_{\mathrm{set}}\left(\coprod_{i=1}^{r}\pi(V_{\bar{K}};z_i,v)\right)\,.$$

Our aim is to construct from the representation ψ_t a representation of G_K which is prounipotent and pro-ℓ on $G_{K(\mu_{\ell^\infty})}$.

Let $x_1, \dots, x_n$ be geometric generators of $\pi_1(V_{\bar{K}}; v)$ (see [18] section 2). In [18] we defined a continuous embedding

$$\pi(V_{\bar{K}}; z_i, v) \to \mathbb{Q}_\ell\{\{\mathbb{X}\}\},$$

where $\mathbb{X} := \{X_1, \dots, X_n\}$ and $\mathbb{Q}_\ell\{\{\mathbb{X}\}\}$ is a $\mathbb{Q}_\ell$-algebra of non-commutative formal power series on non-commuting variables $X_1, \dots, X_n$. The action of G_L on the disjoint union of torsors $\coprod_{i=1}^{r} \pi(V_{\bar{K}}; z_i, v)$ induces a representation

$$\psi_t : G_L \to \mathrm{GL}\left(\bigoplus_{i=1}^{r} \mathbb{Q}_\ell\{\{\mathbb{X}\}\}\right). \tag{0.2.1}$$

In [18] section 3 we defined a filtration $\{H_k(V_L; z_i, v)\}_{k\in\mathbb{N}}$ of G_L associated with the action of G_L on the $\pi_1(V_{\bar{K}}; v)$-torsor $\pi(V_{\bar{K}}; z_i, v)$.

Let us set

$$H_k := \bigcap_{i=1}^{r} H_k(V_L; z_i, v) \quad \text{for} \quad k \in \mathbb{N}.$$

Passing with the representation ψ_t to Lie algebras and then to associated graded Lie algebras we get a morphism of associated graded Lie algebras

$$\mathrm{gr}\,\mathrm{Lie}\,\psi_t : \bigoplus_{k=1}^{\infty} (H_k/H_{k+1}) \otimes \mathbb{Q} \to \left(\bigoplus_{i=1}^{r} \mathrm{Lie}\,(\mathbb{X})\right) \tilde{\times}\, \mathrm{Der}^*\mathrm{Lie}\,(\mathbb{X}),$$

where Lie $(\mathbb{X})$ is a free Lie algebra over $\mathbb{Q}_\ell$ on $\mathbb{X}$, $\mathrm{Der}^*\mathrm{Lie}\,(\mathbb{X})$ is a Lie subalgebra of the Lie algebra of derivations defined in [18] and $\left(\bigoplus_{i=1}^{r} \mathrm{Lie}\,(\mathbb{X})\right) \tilde{\times}$ $\mathrm{Der}^*\mathrm{Lie}\,(\mathbb{X})$ is a semi-direct product of Lie algebras.

Let $G := \mathrm{Gal}\,(L/K)$. Then the group G acts on the associated graded Lie algebra $\bigoplus_{k=1}^{\infty} (H_k/H_{k+1}) \otimes \mathbb{Q}$. We shall study the restriction of the morphism of associated graded Lie algebras $\mathrm{gr}\,\mathrm{Lie}\,\psi_t$ to the fixed part of the action of G, *i.e.*, the morphism of Lie algebras

$$(\mathrm{gr}\,\mathrm{Lie}\,\psi_t)^G : \bigoplus_{k=1}^{\infty} (H_k/H_{k+1}) \otimes \mathbb{Q}^G \to \left(\bigoplus_{i=1}^{r} \mathrm{Lie}\,(\mathbb{X})\right) \tilde{\times}\, \mathrm{Der}^*\mathrm{Lie}\,(\mathbb{X}).$$

We give a sample indicating that something interesting is going on. In most interesting applications we shall study Galois actions on torsors of paths on $V_{\mathbb{Q}}$, where $V := \mathbb{P}^1 \setminus \{0,1,\infty\}$. Then traditionally we denote by $\mathrm{Lie}(X,Y)$ a free Lie algebra on two generators X and Y.

Theorem A. *Let q be a prime number different from ℓ. Let $V := \mathbb{P}^1 \setminus \{0,1,\infty\}$ and let $\mathfrak{t} := \coprod_{0<\alpha<q} \pi(V_{\mathbb{Q}}; \xi_q^\alpha, \overrightarrow{01})$ be a disjoint union of torsors of paths, where ξ_q is a primitive q-th root of 1. Then the Galois group $G_{\mathbb{Q}(\mu_q)}$ acts on $\mathfrak{t}$ and in the image of the morphism of Lie algebras*

$$(\mathrm{grLie}\psi_{\mathfrak{t}})^{(\mathbb{Z}/q)^*} : \bigoplus_{k=1}^{\infty} (H_k/H_{k+1} \otimes \mathbb{Q})^{(\mathbb{Z}/q)^*} \to$$
$$\Big(\bigoplus_{0<\alpha<q} \mathrm{Lie}(X,Y)\Big) \tilde{\times} \mathrm{Der}^* \mathrm{Lie}(X,Y)$$

there are elements $D_1, D_3, D_5, \dots, D_{2n+1}, \dots$ homogenous of degree $1, 3, 5, \dots, 2n+1, \dots$ respectively and these elements generate freely a free Lie subalgebra of the image of $(\mathrm{grLie}\psi_{\mathfrak{t}})^{(\mathbb{Z}/q)^*}$.

Now let us observe the followings facts:

i) the representation of $G_{\mathbb{Q}(\mu_q)}$ on $\mathfrak{t}$ is unramified outside prime ideals of $\mathcal{O}_{\mathbb{Q}(\mu_q)}$ lying over prime ideals (q) or (ℓ) of $\mathbb{Z}$;

ii) the conjectural Lie algebra of the fundamental group of the tannakian category of mixed Tate motives over $\mathrm{Spec}\,\mathbb{Z}[\frac{1}{q}]$ is free, freely generated by generators $d_1, d_3, d_5, \dots, d_{2n+1}, \dots$ of degree $1, 3, 5, \dots, 2n+1, \dots$ respectively;

iii) the elements $D_1, D_3, D_5, \dots, D_{2n+1}, \dots$ are dual to $\frac{1}{q-1}\ell(q)$, $\frac{1-q^2}{(q-1)q^2}\ell_3(1)$, $\frac{1-q^4}{(q-1)q^4}\ell_5(1), \dots, \frac{1-q^{2n}}{(q-1)q^{2n}}\ell_{2n+1}(1), \dots$ respectively, i.e., to ℓ-adic polylogarithms evaluated at elements of $\mathbb{Z}[\frac{1}{q}]^*$ (the point iii) we shall see in the proof of the theorem).

Now we can pose the following question. Can we construct from the representation $\psi_{\mathfrak{t}}$ a new representation $\theta_{\mathfrak{t}}^{\mathbb{Q}(\mu_q)/\mathbb{Q}}$ of $G_{\mathbb{Q}}$ which is unramified outside prime ideals (q) and (l) of $\mathbb{Z}$ and such that the image of the

Lie algebra morphism $\mathrm{grLie}\theta_{\mathfrak{t}}^{\mathbb{Q}(\mu_q)/\mathbb{Q}}$ is free, freely generated by elements $D_1, D_3, D_5, \dots, D_{2n+1}, \dots$?

The representation $\psi_{\mathfrak{t}}$ is an ℓ-adic realization of the mixed Tate motive associated with the geometrical object $\mathfrak{t}$. We can hope that the representation $\theta_{\mathfrak{t}}^{\mathbb{Q}(\mu_q)/\mathbb{Q}}$ will be also motivic.

Let

$$\psi_t : G_L \to \mathrm{GL}\left(\bigoplus_{i=1}^{r} \mathbb{Q}_\ell\{\{\mathbb{X}\}\}\right)$$

be the representation (0.2.1). Below we describe the main idea how to construct the representation $\theta_t^{L/K}$ of G_K from the representation ψ_t of G_L.

0.3. Let K be a number field and let T be a finite set of prime ideals of $\mathcal{O}_K$. Let $\mathcal{O}_{K,T}$ be the ring of T-integers in K. Let us denote by $\mathcal{MM}_{\mathcal{O}_{K,T}}$ the conjectural tannakian category of mixed Tate motives over $\mathrm{Spec}\mathcal{O}_{K,T}$. The tannakian category $\mathcal{MM}_{\mathcal{O}_{K,T}}$ is equivalent to the category of representations of an affine proalgebraic group $\Pi(K,T)$ over $\mathbb{Q}$ in finite dimensional $\mathbb{Q}$-vector spaces. The group $\Pi(K,T)$ is an extension of the multiplicative group $\mathbb{G}_m$ over $\mathbb{Q}$ by an affine proalgebraic prounipotent group $U(K,T)$ over $\mathbb{Q}$.

Let L be a finite Galois extension of K and let $G := \mathrm{Gal}(L/K)$. Let S be a set of these prime ideals of $\mathcal{O}_L$ which lie over some element of T. The inclusion functor

$$\mathcal{MM}_{\mathcal{O}_{K,T}} \hookrightarrow \mathcal{MM}_{\mathcal{O}_{L,S}}$$

induces surjective morphisms of affine proalgebraic groups over $\mathbb{Q}$

$$\Pi(L,S) \to \Pi(K,T) \quad \text{and} \quad U(L,S) \to U(K,T).$$

The group $U(K,T)$ is free in the category of affine proalgebraic prounipotent groups over $\mathbb{Q}$, freely generated by $r_2(K)$ elements in each even positive degree, by $r_1(K) + r_2(K)$ elements in each odd and greater than 1 degree and by $\dim(\mathcal{O}_{K,T}^* \otimes \mathbb{Q})$ elements in degree 1.

Therefore the surjective morphism

$$U(L,S) \to U(K,T)$$

has a section

$$s_{L,S/K,T} : U(K,T) \to U(L,S).$$

Hence we have also a section

$$s_{L,S/K,T} : \Pi(K,T) \to \Pi(L,S).$$

We assume that the representation $\psi_t : G_L \to \mathrm{GL}(\oplus_{i=1}^r \mathbb{Q}_\ell\{\{\mathbb{X}\}\})$ is unramified outside S. The representation ψ_t being motivic factors through the universal map

$$a_{L,S,\ell} : G_L \to \Pi(L,S)(\mathbb{Q}_\ell)$$

with a Zariski dense image. Hence we have a commutative diagram

$$\begin{array}{ccc} G_L & \xrightarrow{\psi_t} & \mathrm{GL}(\oplus_{i=1}^r \mathbb{Q}_\ell\{\{\mathbb{X}\}\}) \\ =\big\uparrow & & \bar{\psi}_t\big\uparrow \\ G_L & \xrightarrow{a_{L,S,\ell}} & \Pi(L,S)(\mathbb{Q}_\ell) \end{array}$$

where $\bar{\psi}_t$ is uniquely determined by ψ_t. We define the representation $\theta_t^{L/K}$ to be the composition

$$G_K \xrightarrow{a_{K,T,\ell}} \Pi(K,T)(\mathbb{Q}_\ell) \xrightarrow{s_{L,S/K,T}} \Pi(L,S)(\mathbb{Q}_\ell) \xrightarrow{\bar{\psi}_t} \mathrm{GL}(\oplus_{i=1}^r \mathbb{Q}_\ell\{\{\mathbb{X}\}\}).$$

The category $\mathcal{MM}_{\mathcal{O}_{K,T}}$ being conjectural, we need to find substitutes for the groups $\Pi(K,T)$ and $U(K,T)$.

First however we introduce the following notation. For a number field K, we denote by $\mathcal{V}(K)$ the set of prime ideals of $\mathcal{O}_K$.

Hain and Matsumoto in [7] and [8] considered a category of continous weighted Tate representations of G_K unramified outside T and prime ideals of $\mathcal{O}_K$ lying over ℓ, in finite dimensional $\mathbb{Q}_\ell$-vector spaces. This category is tannakian over $\mathbb{Q}_\ell$. Its fundamental group, which we denote by $\mathcal{G}(K,T,\ell)$, is a proalgebraic group over $\mathbb{Q}_\ell$, an extension of the multiplicative group $\mathbb{G}_m$ by a proalgebraic prounipotent group $\mathcal{U}(K,T,\ell)$. They showed that the associated graded (with respect to weight filtration) Lie algebra of $\mathcal{U}(K,T,\ell)$ is free, freely generated by $r_2(K)$ elements in each even positive degree, by $r_1(K)+r_2(K)$ elements in each odd and greater than 1 degree and by $\dim(\mathcal{O}^*_{K,T'} \otimes \mathbb{Q})$ elements in degree 1, where $T' := T \cup \{\lambda \in \mathcal{V}(K) \mid \lambda \text{ divides } \ell\}$.

It follows from the result of Hain and Matsumoto that for any $M \in \mathcal{MM}_{\mathcal{O}_{K,T}}$, the representation $\varphi_{M_\ell} : G_K \to \mathrm{Aut}\, M_\ell$ factors through the homomorphism

$$G_K \to \mathcal{G}(K,T,\ell)$$

and the restriction of φ_{M_ℓ} to $G_{K(\mu_{\ell^\infty})}$ factors through the homomorphism

$$G_{K(\mu_{\ell^\infty})} \to \mathcal{U}(K,T,\ell).$$

The inclusion $G_L \subset G_K$ induces surjective morphisms of affine proalgebraic groups over $\mathbb{Q}_\ell$

$$\mathcal{G}(L,S,\ell) \to \mathcal{G}(K,T,\ell) \quad \text{and} \quad \mathcal{U}(L,S,\ell) \to \mathcal{U}(K,T,\ell).$$

We shall construct a section

$$s_{L,S,\ell/K,T,\ell} : \mathcal{U}(L,S,\ell) \to \mathcal{U}(K,T,\ell)$$

and using this section we shall define a representation $\theta_t^{L/K}$ to be the composition

$$G_{K(\mu_{\ell^\infty})} \longrightarrow \mathcal{U}(K,T,\ell) \overset{s_{L,S,\ell/K,T,\ell}}{\longrightarrow} \mathcal{U}(L,S,\ell) \overset{[\psi_t]}{\longrightarrow} \mathrm{GL}\left(\bigoplus_{i=1}^{r} \mathbb{Q}_\ell\{\{\mathbb{X}\}\}\right),$$

where the first arrow is the canonical morphism and the last arrow is induced by ψ_t.

Then we can prove the following result.

THEOREM B. *Let K be a number field and let $V = \mathbb{P}^1_K \setminus \{a_1, a_2, \dots, a_n, \infty\}$. Let a field L, a torsor t and a representation ψ_t be as in 0.2. Let $G = \mathrm{Gal}(L/K)$. Let us assume that the representation ψ_t is unramified outside a finite set S of finite places of L and that S is G-invariant. Let us assume that $L \cap K(\mu_{\ell^\infty}) = K$ and that ℓ does not divide the order of G. Then*

i) *the representation $\theta_t^{L/K}$ is unramified outside the set $T = \{\mathfrak{q} \in \mathcal{V}(K) \mid \exists\, \mathfrak{p} \in S,\ \mathfrak{q} = \mathfrak{p} \cap \mathcal{O}_K\}$ of finite places of K,*

ii) *the filtration $\{\oplus_{j=1}^r I(\mathbb{X})^n\}_{n \in \mathbb{N}}$ of $\bigoplus_{j=1}^r \mathbb{Q}_\ell\{\{\mathbb{X}\}\}$ by the powers of the augmentation ideal $I(\mathbb{X})$ of $\mathbb{Q}_\ell\{\{\mathbb{X}\}\}$ is a filtration by $G_{K(\mu_{\ell^\infty})}$-modules,*

iii) *any* $\sigma \in G_{K(\mu_{\ell^\infty})}$ *acts as the identity on* $\oplus_{j=1}^{r}(I(\mathbb{X})^n/I(\mathbb{X})^{n+1})$ *for any* $n \in \mathbb{N}$.

With the representation $\theta_t^{L/K}$ there is associated in a standard way a filtration $\{F_n(\theta_t^{L/K})\}_{n\in\mathbb{N}}$ of $G_{K(\mu_{\ell^\infty})}$. Passing to associated graded Lie algebras we get a morphism

$$\mathrm{gr\,Lie}\,\theta_t^{L/K} : \bigoplus_{k=1}^{\infty}(F_k(\theta_t^{L/K})/F_{k+1}(\theta_t^{L/K}))\otimes\mathbb{Q} \longrightarrow \left(\bigoplus_{j=1}^{r} L_{\mathrm{Lie}(\mathbb{X})}\right)\tilde{\times}\,\mathrm{Der}^*\,\mathrm{Lie}(\mathbb{X}).$$

Choosing suitably V and a disjoint union of torsors t we get examples where the image of $\mathrm{gr\,Lie}\,\theta_t^{L/K}$ is as big as possible, i.e., it is a free Lie algebra on the maximal possible number of generators depending on K and on the set of ramification places.

For example in the simplest case considered in Theorem A we have the following result.

THEOREM C. *Let q be a prime number different from ℓ. Let $V := \mathbb{P}^1 \setminus \{0,1,\infty\}$ and let $\mathfrak{t} := \coprod_{0<\alpha<q}\pi(V_{\bar{\mathbb{Q}}};\xi_q^\alpha,\overrightarrow{01})$ be a disjoint union of torsors of paths, where ξ_q is a primitive q-th root of 1. Then the image of the morphism of associated graded Lie algebras*

$$\mathrm{gr\,Lie}\,\theta_{\mathfrak{t}}^{\mathbb{Q}(\mu_q)/\mathbb{Q}} : \bigoplus_{k=1}^{\infty}(F_k(\theta_{\mathfrak{t}}^{\mathbb{Q}(\mu_q)/\mathbb{Q}})/F_{k+1}(\theta_{\mathfrak{t}}^{\mathbb{Q}(\mu_q)/\mathbb{Q}}))\otimes\mathbb{Q} \longrightarrow \left(\bigoplus_{0<\alpha<q} L_{\mathrm{Lie}(\mathbb{X})}\right)\tilde{\times}\,\mathrm{Der}^*\,\mathrm{Lie}(\mathbb{X}).$$

is a free Lie algebra, freely generated by elements $D_1, D_3, D_5, \ldots, D_{2n+1}, \ldots$ from Theorem A.

REMARK 0.3.1. The main problem is to show that the obtained representation $\theta_t^{L/K}$ is motivic. To show this one needs to show that section $s_{L,S,\ell/K,T,\ell} : \mathcal{U}(L,S,\ell) \to \mathcal{U}(K,T,\ell)$ is induced by a section $s_{L,S/K,T}$:

$U(L,S) \to U(K,T)$. Unfortunately we are not able to show this as we have no $U(L,S)$.

In order to show that $\theta_t^{K/L}$ is motivic one can also try to show that all coefficients of $\theta_t^{K/L}$ are motivic. Polylogarithmic coefficients are best understood, but in [5] we can only prove that in some special cases dilogarithm coefficients are motivic.

0.4. We indicate briefly why it is important to construct such motivic representations. Let K be a number field and let T be a finite set of prime ideals of $\mathcal{O}_K$. Let $\mathcal{O}_{K,T}$ be the ring of T-integers in K.

Let us denote by $\mathcal{MM}_{\mathcal{O}_{K,T}}$ the category of mixed Tate motives over $\operatorname{Spec}\mathcal{O}_{K,T}$. Then the conjectural associated graded Lie algebra of the unipotent part of the motivic fundamental group of the tannakian category $\mathcal{MM}_{\mathcal{O}_{K,T}}$ is free, freely generated by $r_2(K)$ elements in each even positive degree, by $r_1(K) + r_2(K)$ elements in each odd and greater than 1 degree and by $\dim(\mathcal{O}^*_{K,T} \otimes \mathbb{Q})$ elements in degree 1.

If $M \in \mathcal{MM}_{\mathcal{O}_{K,T}}$ then we denote by M_ℓ the ℓ-adic realization of M. It is a vector space over $\mathbb{Q}_\ell$ equipped with a continous representation $\varphi_{M_\ell}: G_K \to \operatorname{Aut}(M_\ell)$ unramified outside T and prime ideals of K lying over ℓ.

Let $\operatorname{Rep}_{\mathbb{Q}_\ell}(G_K)$ be the category of continous representations of G_K in finite dimensional vector spaces over $\mathbb{Q}_\ell$. The following conjecture seems to be universally accepted.

Conjecture 0.4.1. *The functor of ℓ-adic realization*

$$\mathrm{real}_\ell : \mathcal{MM}_{\mathcal{O}_{K,T}} \to \operatorname{Rep}_{\mathbb{Q}_\ell}(G_K)$$

is faithful.

Let $M \in \operatorname{Pro}\mathcal{MM}_{\mathcal{O}_{K,T}}$. We denote by $\mathcal{M}(M)$ a tannakian subcategory of $\mathcal{MM}_{\mathcal{O}_{K,T}}$ generated by M.

Definition 0.4.2. Let $M \in \operatorname{Pro}\mathcal{MM}_{\mathcal{O}_{K,T}}$. We say that M realizes the fundamental group of $\mathcal{MM}_{\mathcal{O}_{K,T}}$ if the inclusion functor $\mathcal{M}(M) \to \mathcal{MM}_{\mathcal{O}_{K,T}}$ induces an isomorphism of fundamental groups of the tannakian categories.

Definition 0.4.3. Let $M \in \operatorname{Pro}\mathcal{MM}_{\mathcal{O}_{K,T}}$. We say that M realizes ℓ-adically the fundamental group of $\mathcal{MM}_{\mathcal{O}_{K,T}}$ if the associated graded Lie algebra of the $\mathbb{Q}_\ell$-Zariski closure of the image of the Galois action homomorphism

$$\varphi_{M_\ell} : G_{K(\mu_{\ell^\infty})} \to \mathrm{Aut}\,(M_\ell)$$

is a free Lie algebra over $\mathbb{Q}_\ell$, freely generated by $r_2(K)$ elements in each even positive degree, by $r_1(K) + r_2(K)$ elements in each odd and greater than 1 degree and by $\dim(\mathcal{O}^*_{K,T} \otimes \mathbb{Q})$ elements in degree 1.

Notice that the results of Hain and Matsumoto mentioned above do not imply that there is a mixed Tate motive M (in Pro $\mathcal{MM}_{\mathcal{O}_{K,T}}$) over $\mathrm{Spec}\,\mathcal{O}_{K,T}$ which realizes the fundamental group of $\mathcal{MM}_{\mathcal{O}_{K,T}}$. They do not imply that there is a mixed Tate motive M in $\mathcal{MM}_{\mathcal{O}_{K,T}}$ such that $G_{K(\mu_{\ell^\infty})}$ acts non-trivially on M_ℓ.

To study the category $\mathcal{MM}_{\mathcal{O}_{K,T}}$ as well as arithmetic properties of G_K it is important to find objects M in $\mathrm{Pro}\mathcal{MM}_{\mathcal{O}_{K,T}}$ such that the representation φ_{M_ℓ} realizes ℓ-adically the fundamental group of $\mathcal{MM}_{\mathcal{O}_{K,T}}$ or even that it has a non-trivial image.

Here there is a sample why it is very important.

PROPOSITION 0.4.4. *Let us assume that Conjecture 0.4.1 holds. If $M \in \mathrm{Pro}\,\mathcal{MM}_{\mathcal{O}_{K,T}}$ realizes ℓ-adically the fundamental group of $\mathcal{MM}_{\mathcal{O}_{K,T}}$, then M realizes the fundamental group of the tannakian category $\mathcal{MM}_{\mathcal{O}_{K,T}}$.*

In a case of the category $\mathcal{MM}_{\mathbb{Z}}$ - mixed Tate motives over $\mathrm{Spec}\mathbb{Z}$ one hopes that the fundamental group of a projective line minus three points $0, 1, \infty$ and based at $\overrightarrow{01}$ realizes the fundamental group of the tannakian category $\mathcal{MM}_{\mathbb{Z}}$.

Looking only at ℓ-adic side one hopes that the associated graded Lie algebra of the $\mathbb{Q}_\ell$-Zariski closure of the image of

$$G_{\mathbb{Q}(\mu_{\ell^\infty})} \to \mathrm{Aut}\,(\pi_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0,1,\infty\}; \overrightarrow{01}))$$

is free, freely generated by one element in each odd degree greater than 1.

In a case of the category $\mathcal{MM}_{\mathbb{Z}[\frac{1}{2}]}$, the fundamental group $\pi_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0,1,-1,\infty\}; \overrightarrow{01})$ as well as the torsor of paths from $\overrightarrow{01}$ to -1 on $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0,1,\infty\}$ realize ℓ-adically the fundamental group of $\mathcal{MM}_{\mathbb{Z}[\frac{1}{2}]}$. This follows from a result of P.Deligne presented on the conference on Polylogarithms in Schloss Ringberg in a case of π_1 (see also [20] where ℓ-adic version is discussed). The torsor case is considered in [22].

The fundamental group $\pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus (\{0,\infty\} \cup \mu_4); \overrightarrow{01})$ as well as the torsor of paths $\pi(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\};\, i\,, \overrightarrow{01})$ (resp. $\pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus (\{0,\infty\} \cup \mu_3); \overrightarrow{01})$ as well as the torsor of paths $\pi(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\};\, \exp(\frac{2\pi i}{3})\,, \overrightarrow{01})$, resp. $\pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus (\{0,\infty\} \cup \mu_8); \overrightarrow{01})$) realize the fundamental group of the tannakian category $\mathcal{MM}_{\mathbb{Z}[i][\frac{1}{2}]}$ (resp. $\mathcal{MM}_{\mathbb{Z}[\exp(\frac{2\pi i}{3})][\frac{1}{3}]}$, resp. $\mathcal{MM}_{\mathbb{Z}[\exp(\frac{2\pi i}{8})][\frac{1}{2}]}$) (see [20] and [23]).

On the other side $\pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus (\{0,\infty\} \cup \mu_7); \overrightarrow{01})$ definitely does not realize the fundamental group of $\mathcal{MM}_{\mathbb{Z}[\exp(\frac{2\pi i}{7})]}$ or $\mathcal{MM}_{\mathbb{Z}[\exp(\frac{2\pi i}{7})][\frac{1}{7}]}$ (see [4]).

In this note we shall construct "geometrical objects" which realize the fundamental group of the tannakian categories $\mathcal{MM}_{\mathbb{Z}}$, $\mathcal{MM}_{\mathbb{Z}[\frac{1}{p}]}$ for any prime number p, $\mathcal{MM}_{\mathcal{O}_{\mathbb{Q}(\sqrt{-p})}}$ for any prime number p, $\mathcal{MM}_{\mathbb{Z}[i]}$ and $\mathcal{MM}_{\mathcal{O}_{\mathbb{Q}(\sqrt{-p})}[\frac{1}{p}]}$ for any prime number p congruent to 3 modulo 4 and also for $p = 2$. Observe that for all considered rings we have $r_1 + r_2 = 1$ and $\dim \mathcal{O}^*_{K,T} \otimes \mathbb{Q} \leq 1$.

We are working with torsors on $\mathbb{P}^1 \setminus \{0,1,\infty\}$ and in the free Lie algebra on two generators there is no sufficient place to deal with examples when $r_1 + r_2 > 1$ or $\dim \mathcal{O}^*_{K,T} \otimes \mathbb{Q} > 1$. In the future work we hope to study Galois actions on torsors of paths on $\mathbb{P}^1 \setminus \{0, \mu_n, \infty\}$ and on configuration spaces.

In our study we use the idea of Deligne to work modulo a prime number presented in his lecture in Schloss Ringberg (see [3]).

Now we indicate some motivic motivations of our construction of such geometrical objects.

Let $\ell_n(\overrightarrow{10}) : G_{\mathbb{Q}} \to \mathbb{Q}_\ell$ be the ℓ-adic polylogarithm evaluated at $\overrightarrow{10}$ and calculated along the canonical path from $\overrightarrow{01}$ to $\overrightarrow{10}$. Then $\ell_n(\overrightarrow{10})$ is a cocycle with values in $\mathbb{Q}_\ell(n)$ and we have a Galois representation

$$G_{\mathbb{Q}} \ni \sigma \to U(\sigma) \in \mathrm{Aut}\,(\mathbb{Q}^2_\ell),$$

where $U(\sigma)$ is the following matrice

$$\begin{pmatrix} 1 & 0 \\ \ell_n(\overrightarrow{10}) & \chi^n(\sigma) \end{pmatrix}$$

The corresponding Hodge-De Rham realization, which we denote by E_n, is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$. The rational lattice is generated by

$$(\,1\,, Li_n(1)\,) \text{ and } (\,0\,, (2\pi i)^n\,).$$

Let p be a prime number. We denote by F_n an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$, whose rational lattice is generated by

$$(\,1\,, \sum_{k=1}^{p-1} Li_n(\exp(\frac{2\pi ik}{p})\,) \text{ and } (\,0\,, (2\pi i)^n\,).$$

Observe that F_n is over $\operatorname{Spec}\mathbb{Q}$ though $\exp(\frac{2\pi ik}{p}) \notin \mathbb{Q}$ for $k = 1, \dots , p-1$. We have also an equality

$$(1 - p^{n-1})E_n = p^{n-1}F_n.$$

In fact it was observed by Zagier (see [26]) that some Galois invariant linear combinations of polylogarithms evaluated at elements of $\bar{K}$ behave like linear combinations of polylogarithms evaluated at elements of K. For example in [26] we found a relation

$$D_3(\frac{1+\sqrt{5}}{2}) + D_3(\frac{1-\sqrt{5}}{2}) = \frac{1}{5}\zeta(3),$$

where D_3 is a univalent version of Li_3. Let us denote by G_3 an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(3)$ in the category of mixed Hodge-De Rham structures over $\operatorname{Spec}\mathbb{Z}$, whose rational lattice is generated by

$$\big(1\,, Li_3(\frac{1+\sqrt{5}}{2}) + Li_3(\frac{1-\sqrt{5}}{2})\big) \text{ and } \big(0\,, (2\pi i)^3\big).$$

One would like to construct representations of $G_{\mathbb{Q}}$ corresponding to mixed Hodge-De Rham structures F_n and G_3. Equivalently one would like to consider $\sum_{k=1}^{p-1} \ell_n(\exp\frac{2\pi i}{p})$ and $\ell_3(\frac{1+\sqrt{5}}{2}) + \ell_3(\frac{1-\sqrt{5}}{2})$ as (ℓ-adic period) functions on $G_{\mathbb{Q}}$. Notice that a priori they are defined on $G_{\mathbb{Q}(\mu_p)}$ and on $G_{\mathbb{Q}(\sqrt{5})}$ respectively (see [19], where ℓ-adic polylogarithms $\ell_n(z)$ are defined).

In both cases considered here it is easy to find a corresponding function on $G_{\mathbb{Q}}$. One can take $\frac{(1-p^{n-1})}{p^{n-1}}\ell_n(\overrightarrow{10})$ in the first case and $\frac{1}{5}\ell_3(\overrightarrow{10})$ in the second case. It is however far from being obvious how to do this for other examples considered in [26].

The problem is studied in [5]. Unprecisely we can formulate the main results from [5] in the following way.

Let L be a finite Galois extension of K. Assume that a formal linear combination of elements of L, $\sum_{i=1}^{N} m_i[z_i]$ is G-invariant. Assume that $c_n := \sum_{i=1}^{N} m_i \ell_n(z_i)_{\gamma_i}$ is a cocycle on G_L. Then there is a cocycle s_n on G_K such that $c_n = s_{n|G_L}$ in $H^1(G_L; \mathbb{Q}_\ell(n))$.

However it is far from being obvious if s_n is motivic even if c_n is motivic. In fact we can show this only for $n = 2$ (see [5] Theorem 5.4).

More generally one can hope that some Galois invariant linear combinations of normalized iterated integrals (see [18] and [24]) evaluated at elements of $\bar{K}$ behave like linear combinations of normalized iterated integrals evaluated at elements of K.

Coefficients of a Lie algebra representation of $\mathrm{gr}\,\mathrm{Lie}\mathcal{U}(\mathcal{O}_{K,T})$ deduced from torsors of paths are given by symbols $\{z, v\}_e$, where $z, v \in K$ (see [18] and [24], where symbols $\{z, v\}_e$ are defined). The observation of Zagier and the hope expressed above suggest that some Galois invariant linear combinations with $\mathbb{Q}$-coefficients of symbols $\{a, b\}_e$ with $a, b \in \bar{K}$ behave like linear combinations of symbols $\{\alpha, \beta\}_e$ with $\alpha, \beta \in K$. Hence coefficients of a Lie algebra representation of $\mathrm{gr}\,\mathrm{Lie}\mathcal{U}(\mathcal{O}_{K,T})$ should be Galois invariant linear combinations with $\mathbb{Q}$-coefficients of symbols $\{a, b\}_e$ with $a, b \in \bar{K}$.

In this note we use this philosophy to construct representations of G_K starting from representations of G_L, where L is a finite Galois extension of K.

These informal motivic considerations we hope to make more precise in the second part of this paper.

During the conference in Acquafredda di Maratea I have given a talk entitle "Galois actions on torsors of paths". Before my lecture P.Deligne told me that he has studied the product of torsors of paths on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ (from $\overrightarrow{01}$) to ξ_6^1 and ξ_6^5 and that the Lie algebra (associated to the representation of $G_{\mathbb{Q}}$ or the mixed Hodge structure) is free on generators in degrees $3, 5, ..., 2n+1, \dots$. This paper is an attempt to generalize an example studied by Deligne.

Some results of this paper were presented on the conferences in Irvine 2002, in Sestri Levante 2004, in Banff 2005 (Regulators II) and on seminar talks in IHES, in MPI für Mathematik, Bonn and in Okayama University in 2004.

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1. Torsors of Paths

1.0. We recall here methods and results concerning actions of Galois groups on fundamental groups and on torsors of paths from [18].

Let K be a number field. Let $a_1, \dots , a_{n+1}$ be K-points of a projective line $\mathbb{P}^1$. Let

$$V := \mathbb{P}^1_K \setminus \{a_1, \dots , a_{n+1}\}\,.$$

We assume for simplicity that $a_{n+1} = \infty$. We denote by $\hat{V}(K)$ the set of K-points of V and of tangential points defined over K.

Let ℓ be a fixed prime and let $v \in \hat{V}(K)$. Let $\pi_1(V_{\bar{K}}; v)$ be the pro-ℓ completion of the étale fundamental group of $V_{\bar{K}}$ based at v. Let $x_1, \dots , x_n \in \pi_1(V_{\bar{K}}; v)$ be a sequence of geometric generators of the fundamental group associated with the family $\Gamma = \{\gamma_i\}_{i=1}^{n+1}$ of paths from v to tangential base points defined over K at a_i for $i = 1, \dots , n + 1$.

Let $\mathbb{X} := \{X_1, \dots , X_n\}$ and let

$$k : \pi_1(V_{\bar{K}}; v) \to \mathbb{Q}_\ell\{\{\mathbb{X}\}\}$$

be a continuous multiplicative embedding of the fundamental group into non-commutative power series on n non-commuting variables $X_1, \dots , X_n$ sending x_i into the power series e^{X_i}.

Let Lie $(\mathbb{X})$ be a free Lie algebra over $\mathbb{Q}_\ell$ on the set $\mathbb{X}$ and let $L(\mathbb{X}) := \varprojlim_i \mathrm{Lie}\,(\mathbb{X}) / \,\Gamma^i\,\mathrm{Lie}\,(\mathbb{X})$ be a completed free Lie algebra over $\mathbb{Q}_\ell$ on the set $\mathbb{X}$. We identify Lie $(\mathbb{X})$ (resp. $L(\mathbb{X})$) with Lie elements of finite length in $\mathbb{Q}_\ell\{\{\mathbb{X}\}\}$ (resp. Lie elements possibly of infinite length in $\mathbb{Q}_\ell\{\{\mathbb{X}\}\}$).

The action of G_K on the fundamental group $\pi_1(V_{\bar{K}}; v)$ induces an action of G_K on the $\mathbb{Q}_\ell$-algebra $\mathbb{Q}_\ell\{\{\mathbb{X}\}\}$, *i.e.*, we get a representation

$$\varphi_v := \varphi_{V_K,v} : G_K \to \mathrm{Aut}\,(\mathbb{Q}_\ell\{\{\mathbb{X}\}\})\,.$$

The representation φ_v factors through

$$\varphi_v : G_K \to \mathrm{Aut}^*\,(\mathbb{Q}_\ell\{\{\mathbb{X}\}\})\,,$$

where

$$\begin{aligned}&\mathrm{Aut}^*(\mathbb{Q}_\ell\{\{\mathbb{X}\}\}) =\\ &\{f \in \mathrm{Aut}(\mathbb{Q}_\ell\{\{\mathbb{X}\}\}) \mid \exists\, \alpha(f) \in \mathbb{Q}_\ell^* \,\forall\, X_i \in \mathbb{X}\; \exists\, l_i \in L(\mathbb{X}),\\ &\qquad\qquad f(X_i) = e^{-l_i} \cdot \alpha(f) X_i \cdot e^{l_i}\}\,.\end{aligned}$$

We also set

$$\begin{aligned}&\mathrm{Aut}_1^*(\mathbb{Q}_\ell\{\{\mathbb{X}\}\}) =\\ &\{f \in \mathrm{Aut}(\mathbb{Q}_\ell\{\{\mathbb{X}\}\}) \mid \forall\, X_i \in \mathbb{X}\; \exists\, l_i \in L(\mathbb{X}),\ f(X_i) = e^{-l_i} \cdot X_i \cdot e^{l_i}\}\,.\end{aligned}$$

With the action of G_K on $\pi_1(V_{\bar{K}}; v)$ there is associated a filtration $\{G_i(V_K, v)\}_{i\in\mathbb{N}}$ of G_K (see [18] section 3).

Passing to Lie algebras we get a morphism of Lie algebras

$$\mathrm{Lie}\,\varphi_v : \mathrm{Lie}\,(G_1(V_K, v)/G_\infty(V_K, v)) \to \mathrm{Der}\,(\mathbb{Q}_\ell\{\{\mathbb{X}\}\})\,,$$

where $G_\infty(V_K, v) := \bigcap_{i=1}^{\infty} G_i(V_K, v)$.

The morphism Lie φ_v factors through

$$\mathrm{Lie}\,\varphi_v : \mathrm{Lie}\,(G_1(V_K, v)/G_\infty(V_K, v)) \to \mathrm{Der}^*\, L(\mathbb{X})\,,$$

where

$$\mathrm{Der}^*\, L(\mathbb{X}) := \{D \in \mathrm{Der} L(\mathbb{X}) \mid \forall\, 1 \le i \le n\, \exists\, A_i \in L(\mathbb{X}),\ D(X_i) = [X_i, A_i]\}.$$

Let $z \in \hat{V}(K)$ and let p be a path from v to z. For any $\sigma \in G_K$ we set

$$\mathfrak{f}_p(\sigma) := p^{-1} \cdot \sigma(p) \qquad \text{and} \qquad \Lambda_p(\sigma) := k(\mathfrak{f}_p(\sigma))\,.$$

Let $\pi(V_{\bar{K}}; z, v)$ be a $\pi_1(V_{\bar{K}}; v)$-torsor of ℓ-adic paths from v to z. The Galois group G_K acts on $\pi(V_{\bar{K}}; z, v)$, hence we have a representation

$$\psi_{z,v} : G_K \to \mathrm{Aut}_{\mathrm{set}}(\pi(V_{\bar{K}}; z, v))\,.$$

We identify $\pi(V_{\bar{K}}; z, v)$ with $\pi_1(V_{\bar{K}}; v)$ sending a path q onto a loop $p^{-1} \cdot q$. After the identification of $\pi(V_{\bar{K}}; z, v)$ with $\pi_1(V_{\bar{K}}; v)$ and the embedding k, the Galois group G_K acts also on $\mathbb{Q}_\ell\{\{\mathbb{X}\}\}$. Therefore we get a representation

$$\psi_p : G_K \to \mathrm{GL}(\mathbb{Q}_\ell\{\{\mathbb{X}\}\})$$

given by
$$\psi_p(\sigma) = L_{\Lambda_p(\sigma)} \circ \varphi_v(\sigma)\,.$$
The representation ψ_p factors through
$$\psi_p : G_K \to L_{\exp L(\mathbb{X})} \tilde{\times} \mathrm{Aut}^*(\mathbb{Q}_\ell\{\{\mathbb{X}\}\})\,,$$
where $L_{\exp L(\mathbb{X})}$ is the subgroup of $\mathrm{GL}(\mathbb{Q}_\ell\{\{\mathbb{X}\}\})$ of left multiplications by elements of $\exp L(\mathbb{X})$ and $L_{\exp L(\mathbb{X})} \tilde{\times} \mathrm{Aut}^*(\mathbb{Q}_\ell\{\{\mathbb{X}\}\})$ is a subgroup of $\mathrm{GL}(\mathbb{Q}_\ell\{\{\mathbb{X}\}\})$ generated by $L_{\exp L(\mathbb{X})}$ and $\mathrm{Aut}^*(\mathbb{Q}_\ell\{\{\mathbb{X}\}\})$. This subgroup is a semi-direct product of these two subgroups.

It follows immediately from [18] Proposition 2.2.1 that
$$\varphi_v(\sigma)(e^{X_i}) = \Lambda_{\gamma_i}(\sigma)^{-1} \cdot e^{\chi(\sigma)X_i} \cdot \Lambda_{\gamma_i}(\sigma)$$
for any $\sigma \in G_K$. Hence we get

(1.0.0) $$\varphi_v(\sigma)(X_i) = \Lambda_{\gamma_i}(\sigma)^{-1} \cdot (\chi(\sigma)X_i) \cdot \Lambda_{\gamma_i}(\sigma)$$

and

(1.0.1) $$\psi_p(\sigma)(X_i) = \Lambda_p(\sigma) \cdot \varphi_v(\sigma)(X_i)$$

for any $\sigma \in G_K$.

For simplicity we shall also denote $\psi_p(\sigma)$ by σ_p and $\varphi_v(\sigma)$ by σ, hence $\sigma_p = L_{\Lambda_p(\sigma)} \circ \sigma$.

With the action of G_K on the $\pi_1(V_{\bar{K}}; v)$-torsor $\pi(V_{\bar{K}}; z, v)$ there is associated a filtration $\{H_i(V_K; z, v)\}_{i\in\mathbb{N}}$ of G_K (see [18] section 3).

The representation ψ_p induces a morphism of Lie algebras
$$\mathrm{Lie}\,\psi_p : \mathrm{Lie}\,(H_1(V_K; z, v)/H_\infty(V_K; z, v)) \to \mathrm{End}\,(\mathbb{Q}_\ell\{\{\mathbb{X}\}\})\,,$$
where $H_\infty(V_K; z, v) := \bigcap_{i=1}^{\infty} H_i(V_K; z, v)$. The morphism Lie ψ_p factors through
$$\mathrm{Lie}\,\psi_p : \mathrm{Lie}\,(H_1(V_K; z, v)/H_\infty(V_K; z, v)) \to L_{L(\mathbb{X})} \tilde{\times} \mathrm{Der}^* L(\mathbb{X})\,,$$
where $L_{L(\mathbb{X})}$ is the Lie algebra of left multiplications in $\mathbb{Q}_\ell\{\{\mathbb{X}\}\}$ by elements of $L(\mathbb{X})$. The Lie algebra $\mathrm{Der}^* L(\mathbb{X})$ acts on $L_{L(\mathbb{X})}$ via its action on $L(\mathbb{X})$ and $L_{L(\mathbb{X})} \tilde{\times} \mathrm{Der}^* L(\mathbb{X})$ is a semi-direct product.

One shows that for any $\sigma \in H_1(V_K; z, v) = G_{K(\mu_{\ell^\infty})}$ we have

$$\log \sigma_p = L_{(\log \sigma_p)(1)} + \log \sigma$$

and

$$(\log \sigma)(X_i) = [X_i, (\log \sigma_{\gamma_i})(1)]$$

for $i = 1, 2, \dots, n$ (see [18]).

Passing to associated graded Lie algebras with the morphisms Lie φ_v and Lie ψ_p we get morphisms of associated graded Lie algebras

$$\Phi_v := \mathrm{gr}\,\mathrm{Lie}\,\varphi_v : \mathrm{gr}\,\mathrm{Lie}\,(G_1(V_K, v)/G_\infty(V_K, v)) \otimes \mathbb{Q} \approx \bigoplus_{i=1}^{\infty} G_i(V_K, v)/G_{i+1}(V_K, v) \otimes \mathbb{Q} \to \mathrm{Der}^*\mathrm{Lie}\,(\mathbb{X})$$

and

$$\mathbf{\Psi}_{z,v} := \mathrm{gr}\,\mathrm{Lie}\,\psi_p : \mathrm{gr}\,\mathrm{Lie}\,(H_1(V_K; z, v)/H_\infty(V_K; z, v)) \otimes \mathbb{Q} \approx \bigoplus_{i=1}^{\infty} H_i(V_K; z, v)/H_{i+1}(V_K; z, v) \otimes \mathbb{Q} \to L_{\mathrm{Lie}(\mathbb{X})} \tilde{\times} \mathrm{Der}^*\mathrm{Lie}(\mathbb{X})\,.$$

The derivation $D \in \mathrm{Der}^*\mathrm{Lie}(\mathbb{X})$ such that $D(X_i) = [X_i, A_i]$ for $i = 1, \dots, n$ we shall denote by $D_{(A_i)_{i=1}^n}$.

The vector space $\bigoplus_{i=1}^{n} \mathrm{Lie}\,(\mathbb{X})/\langle X_i\rangle$ we equip with a Lie bracket $\{\ ,\ \}$ given by

$$\{(a_i)_{i=1}^n, (b_i)_{i=1}^n\} := ([a_i, b_i] + D_{(a_j)_{j=1}^n}(b_i) - D_{(b_j)_{j=1}^n}(a_i))_{i=1}^n\,.$$

(See [11] , where this bracket is introduced in the case of $\pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,\, 1,\, \infty\}, \overrightarrow{01})$.)

The obtained Lie algebra we denote by $\left(\bigoplus_{i=1}^{n} \mathrm{Lie}\,(\mathbb{X})/\langle X_i\rangle; \{\ ,\ \}\right)$. The Lie algebras $\mathrm{Der}^*\mathrm{Lie}(\mathbb{X})$ and $\left(\bigoplus_{i=1}^{n} \mathrm{Lie}\,(\mathbb{X})/\langle X_i\rangle; \{\ ,\ \}\right)$ are isomorphic and an isomorphism is given by sending $D_{(A_i)_{i=1}^n}$ to $(A_i)_{i=1}^n$.

Hence the representation $\mathbf{\Psi}_{z,v}$ can be regarded as a morphism of Lie algebras

$$\mathbf{\Psi}_{z,v} : \bigoplus_{k=1}^{\infty} H_k(V_K; z, v)/H_{k+1}(V_K; z, v) \otimes \mathbb{Q} \to$$
$$\mathrm{Lie}\,(\mathbb{X})\,\tilde{\times}\left(\bigoplus_{i=1}^{n} \mathrm{Lie}\,(\mathbb{X})/\langle X_i\rangle; \{\ ,\ \}\right).$$

If $\sigma \in H_k(V_K; z, v)$ then the coordinates of $\mathbf{\Psi}_{z,v}(\sigma)$ are

$$\left(\log \Lambda_p(\sigma) \operatorname{mod} \Gamma^{k+1}L(\mathbb{X}), (\log \Lambda_{\gamma_i}(\sigma) \operatorname{mod} \langle X_i\rangle + \Gamma^{k+1}L(\mathbb{X}))_{i=1}^{n}\right).$$

We finish this section by showing that the representations $\varphi_v : G_K \to \mathrm{Aut}(\mathbb{Q}_\ell\{\{\mathbb{X}\}\})$ and $\psi_p : G_K \to \mathrm{GL}(\mathbb{Q}_\ell\{\{\mathbb{X}\}\})$ are ℓ-adic mixed Tate modules (see [8] section 6, where ℓ-adic mixed Tate modules are defined).

Let

$$I(\mathbb{X}) := ker(\varepsilon : \mathbb{Q}_\ell\{\{\mathbb{X}\}\} \to \mathbb{Q}_\ell)$$

be the augmentation ideal. Observe that the powers of the augmentation ideal $\{I(\mathbb{X})^n\}_{n\in\mathbb{N}\cup\{0\}}$ (we set $\{I(\mathbb{X})^0 := \mathbb{Q}_\ell\{\{\mathbb{X}\}\}$) define a filtration of $\mathbb{Q}_\ell\{\{\mathbb{X}\}\}$. It follows immediately from the formulas (1.0.0) and (1.0.1) that

i) $\varphi_v(\sigma)(I(\mathbb{X})^n) \subset I(\mathbb{X})^n$ and $\psi_p(\sigma)(I(\mathbb{X})^n) \subset I(\mathbb{X})^n$,

ii) φ_v and ψ_p act on $I(\mathbb{X})^n/I(\mathbb{X})^{n+1}$ by the n-th power χ^n of the cyclotomic character $\chi : G_K \to \mathbb{Z}_\ell^*$.

Definition 1.0.2. (see also [24]) We define a weight filtration of $\mathbb{Q}_\ell\{\{\mathbb{X}\}\}$ setting

$$W_{-2k+1}(\mathbb{Q}_\ell\{\{\mathbb{X}\}\}) = W_{-2k}(\mathbb{Q}_\ell\{\{\mathbb{X}\}\}) := I(\mathbb{X})^k$$

for $k \in \mathbb{N} \cup \{0\}$.

Proposition 1.0.3. *We have*

i) $\bigcap_{i=0}^{\infty} W_{-i}(\mathbb{Q}_\ell\{\{\mathbb{X}\}\}) = 0$ *and* $\bigcup_{i=0}^{\infty} W_{-i}(\mathbb{Q}_\ell\{\{\mathbb{X}\}\}) = \mathbb{Q}_\ell\{\{\mathbb{X}\}\}$,

ii) *$\varphi_v(\sigma)(W_{-i}(\mathbb{Q}_\ell\{\{\mathbb{X}\}\})) \subset W_{-i}(\mathbb{Q}_\ell\{\{\mathbb{X}\}\})$ and $\psi_p(\sigma)(W_{-i}(\mathbb{Q}_\ell\{\{\mathbb{X}\}\})) \subset W_{-i}(\mathbb{Q}_\ell\{\{\mathbb{X}\}\})$ for any $\sigma \in G_K$,*

iii) *φ_v and ψ_p act on $W_{-2i}(\mathbb{Q}_\ell\{\{\mathbb{X}\}\})/W_{-2i-2}(\mathbb{Q}_\ell\{\{\mathbb{X}\}\})$ by the i-th power χ^i of the cyclotomic character χ.*

Proof. The proposition follows immediately from the properties of the filtration $\{I(\mathbb{X})^n\}_{n\in\mathbb{N}\cup\{0\}}$ of $\mathbb{Q}_\ell\{\{\mathbb{X}\}\}$. □

Let us equip $I(\mathbb{X})$ with the induced weight filtration by setting

$$W_{-2i+1}I(\mathbb{X}) = W_{-2i}I(\mathbb{X}) := I(\mathbb{X})^i$$

for $i \in \mathbb{N}$. Obviously the representations

$$\varphi_v : G_K \to \mathrm{GL}(I(\mathbb{X})) \text{ and } \psi_p : G_K \to \mathrm{GL}(I(\mathbb{X}))$$

(obtained from restriction of φ_v and ψ_p to $I(\mathbb{X})$) respect the filtration of $I(\mathbb{X})$. We define two filtrations of G_K associated with φ_v and ψ_p by setting

$$\mathcal{G}_{i-1}(f) := \{\sigma \in G_K \mid f(\sigma) \equiv Id_{I(\mathbb{X})} \mod I(\mathbb{X})^i\}$$

for $i \in \mathbb{N}$ and where $f = \varphi_v$ or $f = \psi_p$.

We set also

$$\mathcal{G}_\infty(f) := \bigcap_{i=0}^{\infty} \mathcal{G}_{i-1}(f)$$

.

One sees easily that

$$\mathcal{G}_i(\varphi_v) = G_i(V_K, v) \text{ and } \mathcal{G}_i(\psi_p) = H_i(V_K; z, v).$$

1.1. Let $z_1, \dots, z_r$ be K-points of V or tangential points defined over K. We shall study the action of G_K on the disjoint union $\coprod_{j=1}^{r} \pi(V_{\bar{K}}; z_j, v)$ and on the product $\prod_{j=1}^{r} \pi(V_{\bar{K}}; z_j, v)$ of $\pi_1(V_{\bar{K}}; v)$-torsors.

We start with some linear algebra. Let $f_1, \dots, f_r \in L(\mathbb{X})$. Then $L_{\exp f_1} \oplus \dots \oplus L_{\exp f_r}$ is an automorphism of the vector space $\bigoplus_{j=1}^{r} \mathbb{Q}_\ell\{\{\mathbb{X}\}\}$. The set of linear automorphisms of $\bigoplus_{j=1}^{r} \mathbb{Q}_\ell\{\{\mathbb{X}\}\}$ of the form $L_{\exp f_1} \oplus \ldots \oplus L_{\exp f_r}$, where $f_1, \ldots, f_r \in L(\mathbb{X})$ is a group which we denote by $\bigoplus_{j=1}^{r} L_{\exp L(\mathbb{X})}$.

Similarly $L_{\exp f_1} \otimes \ldots \otimes L_{\exp f_r}$ is an automorphism of the vector space $\bigotimes_{j=1}^{r} \mathbb{Q}_\ell\{\{\mathbb{X}\}\}$. The set of linear automorphisms of $\bigotimes_{j=1}^{r} \mathbb{Q}_\ell\{\{\mathbb{X}\}\}$ of the form $L_{\exp f_1} \otimes \ldots \otimes L_{\exp f_r}$, where $f_1, \ldots, f_r \in L(\mathbb{X})$ is a group which we denote by $\bigotimes_{j=1}^{r} L_{\exp L(\mathbb{X})}$.

The group $\mathrm{Aut}^* \mathbb{Q}_\ell\{\{\mathbb{X}\}\}$ acts on $\bigoplus_{j=1}^{r} \mathbb{Q}_\ell\{\{\mathbb{X}\}\}$ and on $\bigotimes_{j=1}^{r} \mathbb{Q}_\ell\{\{\mathbb{X}\}\}$ diagonally. The subgroup of $\mathrm{GL}\left(\bigoplus_{j=1}^{r} \mathbb{Q}_\ell\{\{\mathbb{X}\}\}\right)$ (resp. $\mathrm{GL}\left(\bigotimes_{j=1}^{r} \mathbb{Q}_\ell\{\{\mathbb{X}\}\}\right)$) generated by $\bigoplus_{j=1}^{r} L_{\exp L(\mathbb{X})}$ (resp. $\bigotimes_{j=1}^{r} L_{\exp L(\mathbb{X})}$) and $\mathrm{Aut}^* \mathbb{Q}_\ell\{\{\mathbb{X}\}\}$ is a semi-direct product of these two subgroups and we denote it by

$$\left(\bigoplus_{j=1}^{r} L_{\exp L(\mathbb{X})}\right) \tilde{\times} \mathrm{Aut}^* \mathbb{Q}_\ell\{\{\mathbb{X}\}\}$$

$$\text{(resp.}\quad \left(\bigotimes_{j=1}^{r} L_{\exp L(\mathbb{X})}\right) \tilde{\times} \mathrm{Aut}^* \mathbb{Q}_\ell\{\{\mathbb{X}\}\}).$$

The Lie algebra of

$$\left(\bigoplus_{j=1}^{r} L_{\exp L(\mathbb{X})}\right) \tilde{\times} \mathrm{Aut}_1^* \mathbb{Q}_\ell\{\{\mathbb{X}\}\}$$

$$\text{(resp.}\quad \left(\bigotimes_{j=1}^{r} L_{\exp L(\mathbb{X})}\right) \tilde{\times} \mathrm{Aut}_1^* \mathbb{Q}_\ell\{\{\mathbb{X}\}\})$$

is

$$\left(\bigoplus_{j=1}^{r} L_{L(\mathbb{X})}\right) \tilde{\times} \mathrm{Der}^* L(\mathbb{X})$$

(resp. $\left(\bigoplus_{j=1}^{r}(\mathrm{Id}\otimes\ldots\otimes L_{L(\mathbb{X})}\otimes\ldots\otimes\mathrm{Id})\right)\tilde{\times}\,\mathrm{Der}^*L(\mathbb{X})$). The above facts are proved in [18] section 5 for $r=1$. Proofs for $r>1$ are similar and we omit them.

We return to study Galois representations on the disjoint union $t := \coprod_{j=1}^{r}\pi(V_{\bar{K}};z_j,v)$ and on the product $T := \prod_{j=1}^{r}\pi(V_{\bar{K}};z_j,v)$ of $\pi_1(V_{\bar{K}};v)$-torsors.

We fix paths p_j from v to z_j for $j=1,\ldots,r$. We recall that $k : \pi_1(V_{\bar{K}};v)\to\mathbb{Q}_\ell\{\{\mathbb{X}\}\}$ is a fixed continuous multiplicative embedding. Each $\pi_1(V_{\bar{K}};v)$-torsor $\pi(V_{\bar{K}};z_j,v)$ we embed into $\mathbb{Q}_\ell\{\{\mathbb{X}\}\}$ sending the path q onto $k(p_j^{-1}\cdot q)$.

It follows from the formalism presented in section 1.0 that the action of G_K on $\coprod_{j=1}^{r}\pi(V_{\bar{K}};z_j,v)$ (resp. $\prod_{j=1}^{r}\pi(V_{\bar{K}};z_j,v)$) yields a Galois representation

$$\psi_t : G_K \to \left(\bigoplus_{j=1}^{r} L_{\exp L(\mathbb{X})}\right)\tilde{\times}\,\mathrm{Aut}^*\mathbb{Q}_\ell\{\{\mathbb{X}\}\}$$

(resp. $\psi_T : G_K \to \left(\bigotimes_{j=1}^{r} L_{\exp L(\mathbb{X})}\right)\tilde{\times}\,\mathrm{Aut}^*\mathbb{Q}_\ell\{\{\mathbb{X}\}\}$).

Let $\mathcal{S} := \{z_1,\ldots,z_r\}$. We define a filtration

$$\{H_i(V_K;\mathcal{S},v)\}_{i\in\mathbb{N}}$$

of the group G_K setting

$$H_i(V_K;\mathcal{S},v) := \bigcap_{j=1}^{r} H_i(V_K;z_j,v)$$

for $i\in\mathbb{N}$. Observe that the subgroup

$$H_\infty(V_K;\mathcal{S},v) := \bigcap_{i=1}^{\infty} H_i(V_K;\mathcal{S},v)$$

is the kernel of representations ψ_t and ψ_T. Passing with the representations ψ_t and ψ_T to associated graded Lie algebras we get morphisms of associated graded Lie algebras

$$\operatorname{gr}\operatorname{Lie}\psi_t : \bigoplus_{i=1}^{\infty} H_i(V_K;\mathcal{S},v)/H_{i+1}(V_K;\mathcal{S},v)\otimes\mathbb{Q}\to$$
$$\left(\bigoplus_{j=1}^{r}\operatorname{Lie}(\mathbb{X})\right)\tilde{\times}\operatorname{Der}^*\operatorname{Lie}(\mathbb{X})$$

and

$$\operatorname{gr}\operatorname{Lie}\psi_T : \bigoplus_{i=1}^{\infty} H_i(V_K;\mathcal{S},v)/H_{i+1}(V_K;\mathcal{S},v)\otimes\mathbb{Q}\to$$
$$\left(\bigoplus_{j=1}^{r}\operatorname{Id}\otimes\ldots\otimes\operatorname{Id}\otimes L_{\operatorname{Lie}(\mathbb{X})}\otimes\operatorname{Id}\otimes\ldots\otimes\operatorname{Id}\right)\tilde{\times}\operatorname{Der}^*\operatorname{Lie}(\mathbb{X}).$$

Lemma 1.1.0. *Let $\sigma\in H_k(V_K;\mathcal{S},v)$. The coordinates of* $\operatorname{gr}\operatorname{Lie}\psi_t(\sigma)$ *are*

$$\Big((\log\Lambda_{p_j}(\sigma)\operatorname{mod}\Gamma^{k+1}L(\mathbb{X}))_{j=1}^{r};(\log\Lambda_{\gamma_i}(\sigma)\operatorname{mod}\langle X_i\rangle+\Gamma^{k+1}L(\mathbb{X}))_{i=1}^{n}\Big).$$

We introduce the following notation. If A and B are elements of a Lie algebra then we set

$$[A,B^0]:=A \quad\text{and}\quad [A,B^n]:=[[A,B^{n-1}],B] \quad\text{for}\quad n>0.$$

2. Ramification

In this section we shall study ramification properties of Galois representations on fundamental groups and on torsors of paths. (See also [23] where more detailed results are presented.) We recall that $V=\mathbb{P}^1_K\backslash\{a_1,\ldots,a_n,a_{n+1}\}$ and v is a K-point or a tangential point defined over K. For simplicity we assume that $a_{n+1}=\infty$.

We denote by $\mathcal{V}(K)$ the set of finite places of the field K. Let $\mathfrak{p}\in\mathcal{V}(K)$. We denote by $\mathbf{v}_{\mathfrak{p}}:K^*\to\mathbb{Z}$ the valuation associated with $\mathfrak{p}$.

Definition 2.0. Let $\mathfrak{p}$ be a finite place of K. We say that a pair (V,v) has good reduction at $\mathfrak{p}$ if

i) $\mathbf{v}_{\mathfrak{p}}(a_i) \geq 0$ for all $i \in \{1, \dots, n\}$;

ii) if v is a K-point of V then $\mathbf{v}_{\mathfrak{p}}(v) \geq 0$, if $v = \overrightarrow{a_{i_0} y}$ is a tangential point defined over K then $\mathbf{v}_{\mathfrak{p}}(y) \geq 0$;

iii) $\mathbf{v}_{\mathfrak{p}}(a_i - a_j) = 0$ for all pair $i \neq j$;

iv) if v is a K-point of V then $\mathbf{v}_{\mathfrak{p}}(a_i - v) = 0$ for all $i \in \{1, \dots, n\}$, if $v = \overrightarrow{a_{i_0} y}$ then $\mathbf{v}_{\mathfrak{p}}(a_{i_0} - y) = 0$.

If any of these conditions is not satisfied then we say that a pair (V, v) has potentially bad reduction at $\mathfrak{p}$.

The next result generalizes Theorem from [9].

Theorem 2.1. *Let S' be a set of finite places of K where a pair (V, v) has potentially bad reduction. Let $S = S' \cup \{\lambda \in \mathcal{V}(K) \mid \lambda$ divides $\ell\}$. The representation*

$$\varphi_v : G_K \to \mathrm{Aut}\,(\pi_1(V_{\bar{K}}; v))$$

is unramified outside S.

Proof. We shall imitate the proof of Theorem in [9]. Let us take $\mathfrak{p} \notin S$. Let $K_{\mathfrak{p}}$ be the $\mathfrak{p}$-completion of K, let $\mathcal{O}_{K_{\mathfrak{p}}}$ be the ring of integers of $K_{\mathfrak{p}}$ and let $k_{\mathfrak{p}}$ be the residue field. Let $\bar{K}_{\mathfrak{p}}$ and $\bar{k}_{\mathfrak{p}}$ be algebraic closures of $K_{\mathfrak{p}}$ and $k_{\mathfrak{p}}$ respectively. Let $K_{\mathfrak{p}}^{ur}$ be the maximal unramified extension of $K_{\mathfrak{p}}$ contained in $\bar{K}_{\mathfrak{p}}$ and let $\mathcal{O}_{K_{\mathfrak{p}}}^{ur}$ be the ring of integers of $K_{\mathfrak{p}}^{ur}$.

We define a smooth projective scheme $\mathcal{X}$ over $\mathcal{O}_{K_{\mathfrak{p}}}$ by a system of equations

$$T_i^{\ell^m} - T_j^{\ell^m} = a_j - a_i$$

for all pairs (i, j) with $i \neq j$. The morphism $p_m : \mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_{K_{\mathfrak{p}}}}$ given by $z \to T_i^{\ell^m} + a_i$ is an étale, abelian $(\mathbb{Z}/\ell^m)^n$-covering outside the set $\{a_1, \dots, a_n, a_{n+1}\}$ and with ramification indices ℓ^m at each point a_i.

The K-point v is defined over $\mathcal{O}_{K_{\mathfrak{p}}}$ and its reduction modulo $\mathfrak{p}$ we denote by $\bar{v}$. Let us consider morphisms of pointed schemes

$$(X_{\bar{k}_{\mathfrak{p}}} := \mathcal{X} \otimes_{\mathcal{O}_{K_{\mathfrak{p}}}} \bar{k}_{\mathfrak{p}}, \bar{v}) \to (\mathcal{X} \otimes_{\mathcal{O}_{K_{\mathfrak{p}}}} \mathcal{O}_{K_{\mathfrak{p}}}^{ur}, v) \leftarrow (X_{\bar{K}_{\mathfrak{p}}} := \mathcal{X} \otimes_{\mathcal{O}_{K_{\mathfrak{p}}}} \bar{K}_{\mathfrak{p}}, v)\,.$$

By the Grothendieck comparison theorem the categories of finite ℓ-coverings of $X_{\bar{k}_{\mathfrak{p}}}$ and $X_{\bar{K}_{\mathfrak{p}}}$ are equivalent. Moreover the fiber functors defined by $\bar{v}$ and

v are isomorphic. Finite étale ℓ-coverings of $X_{\bar{k}_\mathfrak{p}}$ and their morphisms are obtained from finite étale ℓ-coverings of $\mathcal{X} \otimes_{\mathcal{O}_{K_\mathfrak{p}}} \mathcal{O}^{ur}_{K_\mathfrak{p}}$ and their morphisms. After tensoring with $\bar{K}_\mathfrak{p}$ one gets finite étale ℓ-coverings of $X_{\bar{K}_\mathfrak{p}}$ and morphisms of these coverings over $X_{\bar{K}_\mathfrak{p}}$. Any finite étale ℓ-covering of $V_{\bar{K}}$ is dominated by the composition of a finite étale ℓ-covering of $X_{\bar{K}_\mathfrak{p}}$ for some m and the projection $p_m : X_{\bar{K}_\mathfrak{p}} \to V_{\bar{K}_\mathfrak{p}}$. Hence the action of $\mathrm{Gal}(\bar{K}_\mathfrak{p}/K_\mathfrak{p})$ on $\pi_1(V_{\bar{K}}; v)$ factors through the action of $\mathrm{Gal}(K^{ur}_\mathfrak{p}/K_\mathfrak{p}) = \mathrm{Gal}(\bar{k}_\mathfrak{p}/k_\mathfrak{p})$. □

Let z be also a K-point of V or a tangential point defined over K.

Definition 2.2. Let $\mathfrak{p}$ be a finite place of K. We say that a triple (V, z, v) has good reduction at $\mathfrak{p}$ if both pairs (V, z) and (V, v) have good reduction at $\mathfrak{p}$.

If this condition is not satisfied then we say that a triple (V, z, v) has potentially bad reduction at $\mathfrak{p}$.

Proposition 2.3. *Let T' be a set of finite places of K where a triple (V, z, v) has potentially bad reduction. Let $T = T' \cup \{\lambda \in \mathcal{V}(K) \mid \lambda \text{ divides } \ell\}$. The representation*

$$\psi_{z,v} : G_K \to \mathrm{Aut}_{\mathrm{set}}(\pi(V_{\bar{K}}; z, v))$$

is unramified outside T.

Proof. We repeat the proof of Proposition 2.1. □

If R is a set of finite places of K then we define

$$\mathcal{O}_{K,R} := \{x \in K \mid \forall\, \mathfrak{p} \notin R,\ \mathbf{v}_\mathfrak{p}(x) \geq 0\}.$$

Corollary 2.4. *The representation*

$$\varphi_v : G_K \to \mathrm{Aut}(\pi_1(V_{\bar{K}}; v)) \quad (\textit{resp. } \psi_{z,v} : G_K \to \mathrm{Aut}_{\mathrm{set}}(\pi(V_{\bar{K}}; z, v)))$$

factors through the epimorphism

$$G_K \to \pi_1(\mathrm{Spec}\, \mathcal{O}_{K,S}; \mathrm{Spec}\, \bar{K}) \quad (\textit{resp. } G_K \to \pi_1(\mathrm{Spec}\, \mathcal{O}_{K,T}; \mathrm{Spec}\, \bar{K}))$$

induced by the inclusion $\mathcal{O}_{K,S} \hookrightarrow K$ *(resp.* $\mathcal{O}_{K,T} \hookrightarrow K$*).*

Proof. The étale fundamental group of $\operatorname{Spec} \mathcal{O}_{K,R}$ is the Galois group $\operatorname{Gal}(F/K)$, where F is a maximal Galois extension of K unramified outside R. $\square$

In [20] we have studied representations of Galois groups on $\pi_1(V_{\bar{\mathbb{Q}}}; \overrightarrow{01})$, where $V := \mathbb{P}^1_{\mathbb{Q}(\mu_n)} \backslash (\{0, \infty\} \cup \mu_n)$. We have constructed a family of derivations in the image of the homomorphism

$$\operatorname{gr}\operatorname{Lie}\varphi_{\overrightarrow{01}} : \operatorname{gr}\operatorname{Lie}(G_1(V_{\mathbb{Q}(\mu_n)}, \overrightarrow{01})/G_\infty(V_{\mathbb{Q}(\mu_n)}, \overrightarrow{01})) \to \operatorname{Der}^*\operatorname{Lie}(\mathbb{X}).$$

We raised a question if these derivations generate the image of $\operatorname{gr}\operatorname{Lie}\varphi_{\overrightarrow{01}}$. The next result is the first step to give an affirmative answer to this question (see also [23]).

Corollary 2.5. *Let* $V = \mathbb{P}^1_{\mathbb{Q}(\mu_n)} \backslash (\{0, \infty\} \cup \mu_n)$*. Let* S *be a set of finite places of* $\mathbb{Q}(\mu_n)$ *which divide* n *or* ℓ*. Then the representation*

$$\varphi_{\overrightarrow{01}} : G_{\mathbb{Q}(\mu_n)} \to \operatorname{Aut}(\pi_1(V_{\overline{\mathbb{Q}(\mu_n)}}; \overrightarrow{01}))$$

is unramified outside S*. The representation* $\varphi_{\overrightarrow{01}}$ *factors through the epimorphism*

$$G_{\mathbb{Q}(\mu_n)} \to \pi_1(\operatorname{Spec}\mathcal{O}_{\mathbb{Q}(\mu_n),S}; \operatorname{Spec}\overline{\mathbb{Q}(\mu_n)}).$$

3. ℓ-adic Polylogarithms

3.0. ℓ-adic polylogarithms are very important in this paper hence we recall here their definition (see [19], [13] and [21]). In fact we shall give three slightly different definitions.

Let $V := \mathbb{P}^1_K \backslash \{0, 1, \infty\}$. We denote by x and y standard generators of $\pi_1(V_{\bar{K}}; \overrightarrow{01})$. Let $k : \pi_1(V_{\bar{K}}; \overrightarrow{01}) \to \mathbb{Q}_\ell\{\{X, Y\}\}$ be the standard continuous multiplicative embedding given by $k(x) = e^X$ and $k(y) = e^Y$. Let z be a K-point of V_K or a tangential point defined over K. Let p be a path from $\overrightarrow{01}$ to z.

We recall that $\operatorname{Lie}(X, Y)$ is a free Lie algebra over $\mathbb{Q}_\ell$ on X and Y and $L(X, Y) := \varprojlim_n \operatorname{Lie}(X, Y)/\Gamma^n \operatorname{Lie}(X, Y)$ is a completed free Lie algebra over $\mathbb{Q}_\ell$ on X and Y.

Let I_r be a closed Lie ideal of $L(X,Y)$ generated by Lie brackets which contain r or more Y's.

DEFINITION 3.0.1. (see [19]) We define functions $\ell(z)_p$ and $\ell_n(z)_p$ from G_K to $\mathbb{Q}_\ell$ by the congruence

$$\log \Lambda_p(\sigma) \equiv \ell(z)_p\,(\sigma)\,X + \sum_{n=1}^{\infty} \ell_n(z)_p\,(\sigma)\,[Y, X^{n-1}] \bmod I_2\,.$$

The function $\ell(z)_p$ we call ℓ-adic logarithm and $\ell_n(z)_p$ we call ℓ-adic polylogarithm of degree n.

This definition is very useful to study Galois representations on fundamental groups of $\mathbb{P}^1_{\overline{\mathbb{Q}(\mu_n)}} \setminus (\{0,\infty\} \cup \mu_n)$ (see [20]).

The next version of ℓ-adic polylogarithms is more close to the definition given in [1].

DEFINITION 3.0.2. We define functions $\ell i_n(z)_p$ from G_K to $\mathbb{Q}_\ell$ by the congruence

$$\log(k(\mathfrak{f}_p(\sigma)\cdot x^{-\ell(z)_p(\sigma)})) \equiv \sum_{n=1}^{\infty} \ell i_n(z)_p\,(\sigma)\,[Y, X^{n-1}] \bmod I_2\,.$$

Now we shall give the third definition. Observe that the path p defines a compatible family $\{z^{1/\ell^n}\}_{n\in\mathbb{N}}$ of ℓ^n-th roots of z by the analytic continuation of the constant family $\{1\}_{n\in\mathbb{N}}$ of ℓ^n-th roots of 1. Using the development of $(1-\xi^i_{\ell^n}\, z^{1/\ell^n})^{1/\ell^n}$ into a power series of $\xi^i_{\ell^n}\, z^{1/\ell^n}$ for small z and analytic continuation in general case we get also a compatible family $\{(1-\xi^i_{\ell^n}\, z^{1/\ell^n})^{1/\ell^n}\}_{n\in\mathbb{N}}$ of ℓ^n-th roots of $1-\xi^i_{\ell^n}\, z^{1/\ell^n}$.

DEFINITION 3.0.3. We define functions $\boldsymbol{\ell}_k(z)_p$ ($k \in \mathbb{N}$) from G_K to $\mathbb{Z}_\ell$ by the identity

$$\xi_{\ell^n}^{\boldsymbol{\ell}_k(z)_p(\sigma)} = \frac{\sigma\left(\prod_{i=0}^{\ell^n-1}(1-\xi^i_{\ell^n}\, z^{1/\ell^n})^{\frac{i^{k-1}}{\ell^n}}\right)}{\prod_{i=0}^{\ell^n-1}(1-\xi_{\ell^n}^{\chi(\sigma)i+\kappa(z)(\sigma)}\cdot z^{1/\ell^n})^{\frac{i^{k-1}}{\ell^n}}}\,,$$

where $\kappa(z) : G_K \to \mathbb{Z}_\ell$ is the Kummer character associated with the family $\{z^{1/\ell^n}\}_{n\in\mathbb{N}}$.

PROPOSITION 3.0.4. *We have*

i) $\ell_n(z)_p(\sigma) = \ell i_n(z)_p(\sigma)$ *for* $\sigma \in H_2(V_K; z, \overrightarrow{01})$;

ii) $\ell i_n(z)_p(\sigma) = \frac{(-1)^{n-1}}{(n-1)!}\, \boldsymbol{\ell}_n(z)_p(\sigma)$ *for* $\sigma \in G_K$.

PROOF. If $\sigma \in H_2(V_K; z, \overrightarrow{01})$ then $\ell(z)_p(\sigma) = 0$. This implies the first part of the proposition. The second part is proved in [13]. □

3.1. In the next section we shall use coinvariant group functor. We recall here its definition and its elementary properties.

We start with the following general situation. Let a group G act on a group $\mathcal{N}$ by automorphisms. Let $\mathcal{I}(\mathcal{N}, G)$ be a normal subgroup of $\mathcal{N}$ generated by elements $g(n) \cdot n^{-1}$, where $n \in \mathcal{N}$ and $g \in G$. Then the coinvariant group $\mathcal{N}_G$ is defined by setting

$$\mathcal{N}_G := \mathcal{N}/\mathcal{I}(\mathcal{N}, G).$$

The coinvariant group functor is right exact.

Let us denote by

$$\pi_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}_G$$

the quotient map.

PROPOSITION 3.1.1. *Let*

$$1 \to \mathcal{C} \xrightarrow{i} \mathcal{B} \xrightarrow{pr} \mathcal{A} \to 1$$

be an exact sequence of groups. We assume that a group G acts on $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ by automorphisms and that i and pr are G-maps. Then

i) *the sequence of coinvariants*

$$\mathcal{C}_G \xrightarrow{i_G} \mathcal{B}_G \xrightarrow{pr_G} \mathcal{A}_G \to 1$$

is exact;

ii) *$ker(pr_G) = pr^{-1}(ker\,\pi_{\mathcal{A}})/ker\,\pi_{\mathcal{B}}$;*

iii) *the natural map $\mathcal{C}_G \to ker(pr_G)$ is surjective and its kernel is the group $(\mathcal{C} \cap ker\,\pi_{\mathcal{B}})/ker\,\pi_{\mathcal{C}}$.*

Proof. The point i) is well known. The point ii) is clear. We shall show the point iii). Let $a \in pr^{-1}(ker\,\pi_{\mathcal{A}})$. We recall that $ker\,\pi_{\mathcal{A}}$ is a normal subgroup of $\mathcal{A}$ generated by elements of the form $g(\alpha)\cdot\alpha^{-1}$ for $g \in G$ and $\alpha \in \mathcal{A}$. Hence the assumption that the map $pr : \mathcal{B} \to \mathcal{A}$ is surjective implies that we can find $b \in ker\,\pi_{\mathcal{B}}$ such that $pr(b) = pr(a)$. Then $pr(a\cdot b^{-1}) = 1$. Hence $a \cdot b^{-1} \in \mathcal{C}$ and $a\cdot b^{-1} \equiv a \bmod ker\,\pi_{\mathcal{B}}$. Hence we have shown that the natural map from $\mathcal{C}_G$ to $ker(pr_G)$ is surjective. Obviously its kernel is the group $(\mathcal{C} \cap ker\,\pi_{\mathcal{B}})/ker\,\pi_{\mathcal{C}}$. □

We give an example when the sequence of coinvariants is also exact.

Corollary 3.1.2. *Let*

$$1 \to \mathcal{C} \xrightarrow{i} \mathcal{B} \xrightarrow{pr} \mathcal{A} \to 1$$

be an exact sequence of groups. We assume that a group G acts on $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ by automorphisms and that i and pr are G-maps. We assume also that the group G acts trivially on $\mathcal{A}$ and that the set $H^1(G, \mathcal{C})$ has one element. Then the sequence of coinvariants

$$1 \to \mathcal{C}_G \xrightarrow{i_G} \mathcal{B}_G \xrightarrow{pr_G} \mathcal{A} \to 1$$

is also exact.

Proof. We have $ker\,\pi_{\mathcal{B}} \subset \mathcal{C}$ because G acts trivially on $\mathcal{A}$. The subgroup $ker\,\pi_{\mathcal{B}}$ is a normal subgroup of $\mathcal{B}$ generated by elements $g(b)\cdot b^{-1}$ for $g \in G$ and $b \in \mathcal{B}$. Hence $ker\,\pi_{\mathcal{B}}$ is a subgroup of $\mathcal{B}$ generated by elements $b_1 \cdot g(b) \cdot b^{-1} \cdot b_1^{-1}$ for $g \in G$ and $b, b_1 \in \mathcal{B}$. Observe that $b_1 \cdot g(b) \cdot b^{-1} \cdot b_1^{-1} = g(b') \cdot g(b) \cdot b^{-1} \cdot g(b')^{-1} = g(b' \cdot b) \cdot (b' \cdot b)^{-1} \cdot b' \cdot g(b')^{-1}$ for some $b' \in \mathcal{B}$. Therefore $ker\,\pi_{\mathcal{B}}$ is a subgroup of $\mathcal{B}$ generated by elements $g(b) \cdot b^{-1}$ for $g \in G$ and $b \in \mathcal{B}$.

Let $b \in \mathcal{B}$. Observe that a function $f : G \to \mathcal{C}$ given by $f(g) := g(b)\cdot b^{-1}$ is a cocycle. Hence there is $c \in \mathcal{C}$ such that $f(g) := g(c)\cdot c^{-1}$ for any $g \in G$. This implies that $ker\,\pi_{\mathcal{B}} = ker\,\pi_{\mathcal{C}}$. Now Proposition 3.1.1 implies that the sequence of coinvariants is exact. □

4. Descent

4.0. Let us fix a number field K and let $a_1, \dots, a_{n+1}$ be K-points of $\mathbb{P}^1_K$. We set

$$V_K := \mathbb{P}^1_K \backslash \{a_1, \dots, a_{n+1}\}\,.$$

We assume for simplicity that $a_{n+1} = \infty$. Let $f(T) \in K[T]$ be a polynomial and let $z_1, \dots, z_r$ be all roots of $f(T)$ in the algebraic closure $\bar{K}$ of K. We define L to be a field obtained from K by adjoining all roots of the polynomial $f(T)$, *i.e.*,

$$L := K(z_1, \dots, z_r)\,.$$

Then L is a Galois extension of K and we set

$$G := \mathrm{Gal}\,(L/K)\,.$$

The Galois group G permutes roots of the polynomial $f(T)$. If $g \in G$ then $g(z_i) = z_{g(i)}$ for some $g(i) \in \{1, \dots, r\}$.

Let v be a fixed K-point of V_K or a tangential point of V_K defined over K. We consider a disjoint union (resp. a product) of torsors of paths

$$t := \coprod_{j=1}^{r} \pi(V_{\bar{K}}; z_j, v) \quad (\text{resp. } T := \prod_{j=1}^{r} \pi(V_{\bar{K}}; z_j, v))\,.$$

We shall study actions of Galois groups G_L and G_K on the disjoint union and the product of torsors of paths.

Let $\mathcal{S} := \{z_1, \dots, z_r\}$. With the action of G_L on the disjoint union of torsors t or on the product of torsors T there is associated a filtration $\{H_k\}_{k \in \mathbb{N}}$ of G_L defined by

$$H_k := H_k(V_L; \mathcal{S}, v) := \bigcap_{j=1}^{r} H_k(V_L; z_j, v)$$

(see section 1). We recall that each subgroup H_k of G_L is a normal subgroup of G_L, $H_1 = G_{L(\mu_{l^\infty})}$, the group $\Gamma := \mathrm{Gal}\,(L(\mu_{l^\infty})/L) \subset \mathbb{Z}_l^*$ acts on H_k/H_{k+1} by conjugation and we have an isomorphism of Γ-modules $H_k/H_{k+1} \approx \mathbb{Z}_\ell(k)^{m_k}$ (see [18] section 3).

Now we shall try to explore the action of $G = \mathrm{Gal}\,(L/K)$ on the tower of groups $\{H_k\}_{k\in\mathbb{N}}$. We shall assume that $L \cap K(\mu_{l^\infty}) = K$. Then $\mathrm{Gal}\,(L(\mu_{l^\infty})/K(\mu_{l^\infty})) = G$. Observe that we have the following tower of groups

$$\begin{aligned} H_\infty \subset \ldots \subset H_{j+1} \subset H_j \subset \ldots \subset H_3 \subset H_2 \subset H_1 \\ = G_{L(\mu_{l^\infty})} \subset G_{K(\mu_{l^\infty})} \twoheadrightarrow G\,. \end{aligned}$$

We shall lift an element $\tau \in G$ to $\tilde{\tau} \in G_{K(\mu_{l^\infty})}$. We shall study the action of $\tilde{\tau}$ by conjugation on the tower $\{H_k\}_{k\in\mathbb{N}}$.

LEMMA 4.0.1. *Let us assume that $L \cap K(\mu_{l^\infty}) = K$. Let $\tau \in G$ and let $\tilde{\tau}$ be a lifting of τ to $G_{K(\mu_{l^\infty})}$.*

i) *If $\sigma \in H_j$ then $\tilde{\tau}\cdot\sigma\cdot\tilde{\tau}^{-1} \in H_j$.*

ii) *The class of $\tilde{\tau}\cdot\sigma\cdot\tilde{\tau}^{-1}$ modulo H_{j+1} does not depend on a choice of lifting $\tilde{\tau}$ of τ.*

iii) *The group G acts on H_j/H_{j+1} by conjugation, i.e., the formula $\tau(\sigma\cdot H_{j+1}) := \tilde{\tau}\cdot\sigma\cdot\tilde{\tau}^{-1}\cdot H_{j+1}$ defines an action of G on H_j/H_{j+1}.*

PROOF. The element σ belongs to H_j if and only if $\sigma \in G_j(V_L, v)$ and for any path γ from x to any z_i we have $\mathfrak{f}_\gamma(\sigma) \equiv 1 \bmod \Gamma^j\pi_1(V_{\bar{K}}; v)$.

The lower central series filtration is preserved by any automorphism, hence $\tilde{\tau}\cdot\sigma\cdot\tilde{\tau}^{-1} \in G_j(V_L, v)$ if $\sigma \in G_j(V_L, v)$. Let γ be a path from v to z_i. Then $\mathfrak{f}_\gamma(\tilde{\tau}\cdot\sigma\cdot\tilde{\tau}^{-1}) = \tilde{\tau}(\mathfrak{f}_{\tilde{\tau}^{-1}(\gamma)}(\sigma))$ Observe that $\tilde{\tau}^{-1}(\gamma)$ is a path from v to $z_{\tau^{-1}(i)}$, hence $\mathfrak{f}_{\tilde{\tau}^{-1}(\gamma)}(\sigma) \equiv 1 \bmod \Gamma^j\pi_1(V_{\bar{K}}; v)$. This implies that $\mathfrak{f}_\gamma(\tilde{\tau}\cdot\sigma\cdot\tilde{\tau}^{-1}) = \tilde{\tau}(\mathfrak{f}_{\tilde{\tau}^{-1}(\gamma)}(\sigma)) \equiv 1 \bmod \Gamma^j\pi_1\ (V_{\bar{K}}; v)$. Hence $\tilde{\tau}\cdot\sigma\cdot\tilde{\tau}^{-1} \in H_j$.

Let $\tau' = \tilde{\tau}\cdot\eta$, where $\eta \in H_1$, be another lifting of τ. Then $\tau'\cdot\sigma\cdot\tau'^{-1} = \tilde{\tau}\cdot\eta\cdot\sigma\cdot\eta^{-1}\cdot\tilde{\tau}^{-1} = \tilde{\tau}\cdot\sigma\cdot\sigma_{j+1}\cdot\tilde{\tau}^{-1} \equiv \tilde{\tau}\cdot\sigma\cdot\tilde{\tau}^{-1} \bmod H_{j+1}$, because $\eta\cdot\sigma\cdot\eta^{-1} = \sigma \bmod H_{j+1}$ and $\tilde{\tau}\cdot\sigma_{j+1}\cdot\tilde{\tau}^{-1} \in H_{j+1}$ if $\sigma_{j+1} \in H_{j+1}$.

Therefore the points i) and ii) are proved. The point iii) follows immediately from i) and ii). $\square$

We recall that $\bigoplus_{k=1}^{\infty} H_k/H_{k+1} \otimes \mathbb{Q}$ is a Lie algebra over $\mathbb{Q}_\ell$, whose Lie bracket is induced by the commutator, *i.e.*, if $\bar{a} \in H_k/H_{k+1}$ and $\bar{b} \in H_j/H_{j+1}$ then $[\bar{a}, \bar{b}] := \overline{a\cdot b\cdot a^{-1}\cdot b^{-1}} \in H_{k+j}/H_{k+j+1}$. It follows from Lemma 4.0.1 that the group G acts on each $\mathbb{Q}_\ell$-vector space $H_k/H_{k+1} \otimes \mathbb{Q}$.

Hence we can decompose the G-module $H_k/H_{k+1} \otimes \mathbb{Q}$ into a direct sum of irreducible representations of G. We get

$$H_k/H_{k+1} \otimes \mathbb{Q} \approx (H_k/H_{k+1} \otimes \mathbb{Q})^G \oplus \bigoplus_{\chi_i} (H_k/H_{k+1} \otimes \mathbb{Q})_{\chi_i} \,,$$

where $(H_k/H_{k+1} \otimes \mathbb{Q})^G$ is a fixed part and the sum is over all non-trivial irreducible representations of G.

Lemma 4.0.2. *The fixed part of the action of G, $\bigoplus_{k=1}^{\infty} (H_k/H_{k+1} \otimes \mathbb{Q})^G$ is a Lie subalgebra of a Lie algebra $\bigoplus_{k=1}^{\infty} (H_k/H_{k+1} \otimes \mathbb{Q})$.*

Proof. Let $\bar{a} \in H_k/H_{k+1}$ and $\bar{b} \in H_j/H_{j+1}$ be fixed by the action of G. Then $g(a) = a{\cdot}c$ and $g(b) = b{\cdot}d$, where $c \in H_{k+1}$ and $d \in H_{j+1}$. It follows immediately from the Witt-Hall identities that $\overline{g(a) \cdot g(b) \cdot g(a)^{-1} \cdot g(b)^{-1}} = \overline{a \cdot b \cdot a^{-1} \cdot b^{-1}}$. $\square$

4.1. We shall study subfields of an algebraic closure $\bar{L}$ of the field L associated with the filtration $\{H_k\}_{k\in\mathbb{N}}$ of G_L. We define subfields of $\bar{L}$ fixed by H_k setting

$$L_k := \bar{L}^{H_k} \qquad \text{for} \quad k \in \mathbb{N}$$

and

$$L_\infty := \bigcup_{k=1}^{\infty} L_k \,.$$

Observe that $L_\infty = \bar{L}^{H_\infty}$. We have got a tower of fields

$$\begin{array}{ccl} L \subset L_1 & = & L(\mu_{\ell^\infty}) \subset L_2 \subset \ldots \subset L_n \subset L_{n+1} \subset \ldots \subset L_\infty \subset \bar{L} \\ \cup & & \cup \\ K \subset K_1 & = & K(\mu_{\ell^\infty}) \end{array}$$

such that

$$\mathrm{Gal}\,(\bar{L}/L_k) = H_k \quad \text{and} \quad \mathrm{Gal}\,(L_k/L_j) = H_j/H_k$$

for $k, j \in \mathbb{N} \cup \{\infty\}$ and $k \geq j$.

For any $k \in \mathbb{N} \cup \{\infty\}$ we have an exact sequence of Galois groups

$$\begin{array}{ccccccccc} & 1 & \longrightarrow & \mathrm{Gal}\,(L_k/L_1) & \longrightarrow & \mathrm{Gal}\,(L_k/K_1) & \xrightarrow{\mathrm{pr}_k} & \mathrm{Gal}\,(L_1/K_1) & \longrightarrow & 1\,. \\ & & & \| & & & & \| & & \\ (*'_k) & & & G_{L(\mu_{\ell^\infty})}/H_k & & & & G & & \end{array}$$

We assume that ℓ does not divide the order of G. The group $\mathrm{Gal}\,(L_k/L_1)$ is a pro-ℓ group. Therefore the exact sequence $(*'_\infty)$ splits, *i.e.*, there is a homomorphism $\theta'_\infty : G \to \mathrm{Gal}\,(L_\infty/K_1)$ such that $\mathrm{pr}_\infty \circ \theta'_\infty = \mathrm{id}_G$. The composition of θ'_∞ with the inclusion $\mathrm{Gal}\,(L_\infty/K_1) \to \mathrm{Gal}\,(L_\infty/K)$ defines a splitting $\theta_\infty : G \to \mathrm{Gal}\,(L_\infty/K)$ of the exact sequence $(*_\infty)$, where

$$\begin{array}{ccccccccc} & 1 & \longrightarrow & \mathrm{Gal}\,(L_k/L) & \longrightarrow & \mathrm{Gal}\,(L_k/K) & \xrightarrow{\mathrm{pr}_k} & \mathrm{Gal}\,(L/K) & \longrightarrow & 1\,. \\ & & & \| & & & & \| & & \\ (*_k) & & & G_L/H_k & & & & G & & \end{array}$$

for $k \in \mathbb{N} \cup \{\infty\}$. Then we define a splitting θ_k of the sequence $(*_k)$ for $k \in \mathbb{N}$ to be a composition of θ_∞ with the projection $\mathrm{Gal}\,(L_\infty/K) \to \mathrm{Gal}\,(L_k/K)$. Therefore the group G acts on $\mathrm{Gal}\,(L_k/L)$ by $g(\sigma) := \theta_k(g) \cdot \sigma \cdot \theta_k(g)^{-1}$ for $k \in \mathbb{N} \cup \{\infty\}$. These actions are compatible in the sense that the projection $\mathrm{Gal}\,(L_k/L) \to \mathrm{Gal}\,(L_j/L)$ is a G-map for $k > j$. The group $\mathrm{Gal}\,(L_k/K)$ is a semi-direct product of $\mathrm{Gal}\,(L_k/L)$ by G, *i.e.*, $\mathrm{Gal}\,(L_k/K) = \mathrm{Gal}\,(L_k/L) \tilde{\times} G$ for $k \in \mathbb{N} \cup \{\infty\}$.

We recall that $\mathrm{Gal}\,(L_k/L)_G$ is the maximal quotient of $\mathrm{Gal}\,(L_k/L)$ on which G acts by the identity. Let $\pi_k : \mathrm{Gal}\,(L_k/L) \to \mathrm{Gal}\,(L_k/L)_G$ be the natural projection. Then we also have a projection

$$\pi'_k : \mathrm{Gal}\,(L_k/K) = \mathrm{Gal}\,(L_k/L) \tilde{\times} G \to \mathrm{Gal}\,(L_k/L)_G \times G\,.$$

Observe that we have $\ker \pi_k = \ker \pi'_k$. We define subfields L'_k of $\bar{L}$ by setting

$$L'_k := L_k^{\ker \pi_k}\,.$$

Then the field L'_k is a Galois extension of L and also of K because $\ker \pi_k$ (resp. $\ker \pi'_k$) is a normal subgroup of $\mathrm{Gal}\,(L_k/L)$ (resp. $\mathrm{Gal}(L_k/K)$). Therefore we have isomorphisms

(4.1.0) $\mathrm{Gal}\,(L'_k/L) = \mathrm{Gal}\,(L_k/L)_G$ and $\mathrm{Gal}\,(L'_k/K) = \mathrm{Gal}\,(L_k/L)_G \times G\,.$

Lemma 4.1.1. *Let us assume that ℓ does not divide the order of G. We have*

$$\mathrm{Gal}\,(L'_k/L_1) = \mathrm{Gal}\,(L_k/L_1)_G.$$

Proof. The sequence of coinvariants of Galois groups of a tower of fields

$$L \subset L_1 \subset L_k$$

is exact by Corollary 3.1.2. This exact sequence maps into exact sequence of Galois groups of a tower of fields

$$L \subset L_1 \subset L'_k$$

(see Diagram 1).

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Gal}(L_k/L_1)_G & \longrightarrow & \mathrm{Gal}(L_k/L)_G & \longrightarrow & \mathrm{Gal}(L_1/L)_G & \longrightarrow & 1 \\
 & & \alpha\big\downarrow & & =\big\downarrow & & =\big\downarrow & & \\
1 & \longrightarrow & \mathrm{Gal}(L'_k/L_1) & \longrightarrow & \mathrm{Gal}(L'_k/L) & \longrightarrow & \mathrm{Gal}(L_1/L) & \longrightarrow & 1\,.
\end{array}$$

Diagram 1

Hence it follows that the natural map $\mathrm{Gal}(L_k/L_1)_G \to \mathrm{Gal}(L'_k/L)$ is an isomorphism. □

Lemma 4.1.2. *Let $j < k$. The field L'_k is a Galois extension of L'_j and*

$$\mathrm{Gal}\,(L'_k/L'_j) = pr_{k,j}^{-1}(ker\,\pi_j)/ker\,\pi_k,$$

where $pr_{k,j} : \mathrm{Gal}\,(L_k/L) \to \mathrm{Gal}\,(L_j/L)$ is the natural projection.

Proof. Observe that $L'_j \subset L'_k$ because any element of $\mathrm{Gal}\,(L_k/L)$ which fixes L'_k, fixes also L'_j. The fields L'_k and L'_j are Galois extensions of L. Hence L'_k is a Galois extension of L'_j. We have an exact sequence of Galois groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Gal}\,(L'_k/L'_j) & \longrightarrow & \mathrm{Gal}\,(L'_k/L) & \longrightarrow & \mathrm{Gal}\,(L'_j/L) & \longrightarrow & 1 \\
 & & & & \| & & \| & & \\
 & & & & \mathrm{Gal}\,(L_k/L)_G & \xrightarrow{(\mathrm{pr}_{k,j})_G} & \mathrm{Gal}\,(L_j/L)_G\,. & &
\end{array}$$

It follows from Proposition 3.1.1 that

$$ker\,\big((\mathrm{pr}_{k,j})_G\big) = \mathrm{pr}_{k,j}^{-1}(\,ker\,\pi_j)/ker\,\pi_k.$$

Hence we get

$$\mathrm{Gal}\,(L'_k/L'_j) \approx \mathrm{pr}_{k,j}^{-1}(\,ker\,\pi_j)/ker\,\pi_k. \ \square$$

Let $k \in \mathbb{N} \cup \{\infty\}$. Let K_k be a subfield of L'_k fixed by G, *i.e.*,

$$K_k := L'^G_k.$$

Then it follows from (4.1.0) that

$$\mathrm{Gal}\,(K_k/K) = \mathrm{Gal}\,(L'_k/L) = \mathrm{Gal}\,(L_k/L)_G \ \text{ and } \ \mathrm{Gal}\,(L'_k/K_k) = G\,. \tag{4.1.3}$$

DEFINITION 4.1.4. We define a filtration $\{F_k(V_{L/K};\mathcal{S},v)\}_{k\in\mathbb{N}}$ of G_K by setting

$$F_k(V_{L/K};\mathcal{S},v) := \mathrm{Gal}\,(\bar{K}/K_k)\,.$$

To simplify the notation we denote $F_k(V_{L/K};\mathcal{S},v)$ by F_k. We set

$$F_\infty := F_\infty(V_{L/K};\mathcal{S},v) := \bigcap_{k=1}^{\infty} F_k\,.$$

Observe that $F_\infty = \mathrm{Gal}\,(\bar{K}/K_\infty)$. Observe that each F_k is a normal subgroup of G_K and that for any $j < k$, F_j is a normal subgroup of F_k.

We indicate ramification properties of the fields L_∞ and K_∞.

PROPOSITION 4.1.5. *Let S'_j be the set of finite places of L, where the triple (V_L, z_j, v) has potentially bad reduction. Let $S = \bigcup_{j=1}^{r} S'_j \cup \{\lambda \in \mathcal{V}(L) \mid \lambda$ divides $\ell\}$. Then the representation*

$$\psi_t : \mathrm{Gal}(\bar{L}/L) \to \mathrm{Aut}_{\mathrm{set}}\left(\coprod_{j=1}^{r} \pi(V_{\bar{L}};z_j,v)\right)$$

is unramified outside S.

PROOF. The proposition follows immediately from Proposition 2.3. $\square$

COROLLARY 4.1.6. *The field extension L_∞ of L is unramified outside S.*

COROLLARY 4.1.7. *The representation*

$$\psi_t : \mathrm{Gal}(\bar{L}/L) \to \mathrm{Aut}_{\mathrm{set}}\left(\coprod_{j=1}^{r} \pi(V_{\bar{L}}; z_j, v)\right)$$

factors through the epimorphism

$$\mathrm{Gal}(\bar{L}/L) \to \pi_1(\mathrm{Spec}\,\mathcal{O}_{L,S}; \mathrm{Spec}\,\bar{L})$$

induced by the inclusion $\mathcal{O}_{L,S} \subset L$.

We recall that $\mathcal{V}(K)$ is the set of finite places of the field K. Let us set

$$T := \{\mathfrak{q} \in \mathcal{V}(K) \mid \exists\, \mathfrak{p} \in \mathcal{S}\,,\ \mathfrak{q} \ = \ \mathfrak{p} \cap \mathcal{O}_K\}.$$

PROPOSITION 4.1.8. *The field extension K_∞ of K is unramified outside T.*

PROOF. The extension L'_∞ of L is unramified outside S because $L'_\infty \subset L_\infty$. In the diagram of fields

$$\begin{array}{ccc} L'_\infty & \longleftarrow & L \\ \uparrow & & \uparrow \\ K_\infty & \longleftarrow & K \end{array}$$

we have $L'_\infty = K_\infty L$ (the composition of fields K_∞ and L), $K = K_\infty \cap L$ and $\mathrm{Gal}(L/K) = \mathrm{Gal}(L'_\infty/K_\infty)$. Hence if a finite place $\mathfrak{q}$ of K not belonging to T ramifies in L, then it ramifies when passing from K_∞ to L'_∞ with the same ramification index. Therefore $\mathfrak{q}$ does not ramify when passing from K to K_∞. $\square$

Corollary 4.1.9. *There is an epimorphism*

$$\pi_1(\operatorname{Spec}\mathcal{O}_{K,T};\operatorname{Spec}\bar{K})\to \operatorname{Gal}(K_\infty/K)\,.$$

Let $K(\mu_{\ell^\infty})_T^{(\ell)}$ be the maximal pro-ℓ extension of $K(\mu_{\ell^\infty})$ unramified outside places of $K(\mu_{\ell^\infty})$ lying over T.

Corollary 4.1.10. *The field K_∞ is contained in $K(\mu_{\ell^\infty})_T^{(\ell)}$, i.e., we have an epimorphism of Galois groups*

$$\operatorname{Gal}(K(\mu_{\ell^\infty})_T^{(\ell)}/K)\to \operatorname{Gal}(K_\infty/K).$$

4.2. We shall review briefly some results from [8] and [7] and from [25] about weighted Tate completion of Galois groups.

Definition 4.2.0. Let F be a number field and let R be a finite set of finite places of F containing all finite places lying over ℓ. We denote by $M(F)_R$ a maximal pro-ℓ extension of $F(\mu_{\ell^\infty})$ unramified outside R, i.e., unramified outside finite places of $F(\mu_{\ell^\infty})$ lying over any place of R.

In [8] and [7] Hain and Matsumoto defined weighted Tate completion of Galois groups. We shall study weighted Tate completion of the group $\operatorname{Gal}(M(F)_R/F)$.

Let $\mathcal{G}(F,R,\ell)$ be the weighted Tate completion of the group $\operatorname{Gal}(M(F)_R/F)$. The group $\mathcal{G}(F,R,\ell)$ is an affine proalgebraic group over $\mathbb{Q}_\ell$, an extension of the multiplicative group over $\mathbb{Q}_\ell$ by an affine proalgebraic prounipotent group $\mathcal{U}(F,R,\ell)$. There is a map

$$i_{F,R,\ell}:\operatorname{Gal}(M(F)_R/F)\to \mathcal{G}(F,R,\ell)$$

of the group $\operatorname{Gal}(M(F)_R/F)$ into $\mathcal{G}(F,R,\ell)$ with a Zariski dense image making the following diagram commutative

$$\begin{array}{ccc}
\mathrm{Gal}(M(F)_R/F(\mu_{\ell^\infty})) & \stackrel{i_{F,R,\ell}}{\longrightarrow} & \mathcal{U}(F,R,\ell) \\
\Big\downarrow & & \Big\downarrow \\
\mathrm{Gal}(M(F)_R/F) & \stackrel{i_{F,R,\ell}}{\longrightarrow} & \mathcal{G}(F,R,\ell) \\
\Big\downarrow & & \Big\downarrow \\
\Gamma_F := \mathrm{Gal}(F(\mu_{\ell^\infty})/F) & \longrightarrow & \mathbb{G}_m(\mathbb{Q}_\ell).
\end{array}$$

The group $\mathcal{G}(F,R,\ell)$ is equipped with the weight filtration $\{W_{-i}\mathcal{G}(F,R,\ell)\}_{i\in\mathbb{N}\cup\{0\}}$ such that $W_{-2i+1}\mathcal{G}(F,R,\ell) = W_{-2i}\mathcal{G}(F,R,\ell)$ and $W_{-2}\mathcal{G}(F,R,\ell) = \mathcal{U}(F,R,\ell)$. The group $\mathbb{G}_m(\mathbb{Q}_\ell)$ acts by conjugation on $W_{-2i}\mathcal{G}(F,R,\ell)/W_{-2i-2}\mathcal{G}(F,R,\ell)$ and $g \in \mathbb{Q}_\ell^*$ acts as a multiplication by g^i.

Let $\varphi : G_F \to \mathrm{Aut}M$ be an ℓ-adic mixed Tate module unramified outside R. Then it follows from the definition of the field $M(F)_R$ that φ factors through $\mathrm{Gal}(M(F)_R/F)$, i.e., we have a commutative diagram

$$\begin{array}{ccc}
G_F & \stackrel{\varphi}{\longrightarrow} & \mathrm{Aut}M \\
\Big\downarrow & & id\Big\downarrow \\
\mathrm{Gal}(M(F)_R/F) & \stackrel{\varphi}{\longrightarrow} & \mathrm{Aut}M,
\end{array}$$

where the map from $\mathrm{Gal}(M(F)_R/F)$ to $\mathrm{Aut}M$ induced by φ we denote also by φ. It follows from the universal property of weighted Tate completion that $\varphi : \mathrm{Gal}(M(F)_R/F) \to \mathrm{Aut}M$ factors through $\mathcal{G}(F,R,\ell)$, i.e., we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Gal}(M(F)_R/F) & \stackrel{\varphi}{\longrightarrow} & \mathrm{Aut}M \\
i_{F,R,\ell}\Big\downarrow & & id\Big\downarrow \\
\mathcal{G}(F,R,\ell) & \stackrel{[\varphi]}{\longrightarrow} & \mathrm{Aut}M,
\end{array}$$

where $[\varphi]$ is the unique morphism making the diagram commutative.

If t is a coefficient of the representation φ then we denote by $[t]$ the corresponding coefficient of the representation $[\varphi]$.

Now we shall review the results from [25]. There we are studying relations between weighted Tate completions of Galois groups of two different fields.

4.2.1. Let L be a finite Galois extension of K and let $G := \mathrm{Gal}(L/K)$. Let T be a finite set of finite places of K containing all places over ℓ. Let us define a set S of finite places of L by

$$S := \{\, \mathfrak{p} \in \mathcal{V}(L) \mid \exists\, \mathfrak{q} \in T,\ \mathfrak{p} \cap \mathcal{O}_K = \mathfrak{q} \,\}.$$

Observe that the set S is G-invariant.

Let us assume also that

i) ℓ does not divide the order of G;

ii) $K(\mu_{\ell^\infty}) \cap L = K$.

Lemma 4.2.2. *(see [25] section 2). Let us assume 4.2.1. Then the group G acts on $\mathrm{Gal}(M(L)_S/L)$ by conjugations.*

Proof. The assumptions i) and ii) imply that the exact sequence of Galois groups

$$1 \to \mathrm{Gal}(M(L)_S/L(\mu_{\ell^\infty})) \to \mathrm{Gal}(M(L)_S/K(\mu_{\ell^\infty})) \to G \to 1$$

has a section $s' : G \to \mathrm{Gal}(M(L)_S/K(\mu_{\ell^\infty}))$. Composing s' with the inclusion $\mathrm{Gal}(M(L)_S/K(\mu_{\ell^\infty})) \to \mathrm{Gal}(M(L)_S/K)$ we get a section $s : G \to \mathrm{Gal}(M(L)_S/K)$ of the exact sequence of Galois groups

$$1 \to \mathrm{Gal}(M(L)_S/L) \to \mathrm{Gal}(M(L)_S/K) \to G \to 1.$$

Hence we get an action of G by conjugations ($\sigma \to s(g) \cdot \sigma \cdot s(g)^{-1}$) on $\mathrm{Gal}(M(L)_S/L)$. $\square$

The construction of weighted Tate completion is functorial hence the action of G on $\mathrm{Gal}(M(L)_S/L)$ implies that G acts on $\mathcal{G}(L, S, \ell)$ preserving weight filtration. This implies that G acts by Lie algebra automorphisms on the associated graded Lie algebra

$$grLie\mathcal{U}(L,S,\ell) := \bigoplus_{i=1}^{\infty} W_{-2i}\mathcal{U}(L,S,\ell)/W_{-2i-2}\mathcal{U}(L,S,\ell).$$

Let us set

$$grLie\mathcal{U}(L,S,\ell)_i := W_{-2i}\mathcal{U}(L,S,\ell)/W_{-2i-2}\mathcal{U}(L,S,\ell).$$

Definition 4.2.3. We denote by D_i the image of the Lie bracket

$$[\,,\,] : \bigwedge^2 \Big(\bigoplus_{k=1}^{i-1} grLie\mathcal{U}(L,S,\ell)_k\Big) \to grLie\mathcal{U}(L,S,\ell)_i$$

and we call D_i the space of decomposable elements in degree i of $grLie\mathcal{U}(L,S,\ell)$.

Observe that the subspace D_i of $grLie\mathcal{U}(L,S,\ell)_i$ is G-invariant.

Definition 4.2.4. We denote by I_i the G-invariant complement of D_i in $grLie\mathcal{U}(L,S,\ell)_i$ and we call I_i the space of indecomposable elements in degree i of $grLie\mathcal{U}(L,S,\ell)$. We denote by W_i the fixed part of I_i and by V_i the G-invariant complement of W_i in I_i, i.e., we have

$$W_i := I_i^G \quad \text{and} \quad I_i = W_i \oplus V_i$$

as G-modules.

If it will be necessary to indicate dependence on the field L then we shall write $D_i(L)$, $I_i(L)$, $W_i(L)$ and $V_i(L)$.

Now we shall define two Lie algebras associated with the action of G on $grLie\mathcal{U}(L,S,\ell)$.

Definition 4.2.5. We denote by $grLie\mathcal{U}(L,S,\ell)^{[G]}$ a Lie subalgebra of $grLie\mathcal{U}(L,S,\ell)$ generated by $\oplus_{i=1}^{\infty} W_i$.

Proposition 4.2.6. *The Lie subalgebra $grLie\mathcal{U}(L,S,\ell)^{[G]}$ of $grLie\mathcal{U}(L,S,\ell)$ is free, freely generated by a base of $\oplus_{i=1}^{\infty} W_i$.*

Proof. The associated graded Lie algebra $grLie\mathcal{U}(L,S,\ell)$ is free, freely generated by a base of $\oplus_{i=1}^{\infty} I_i$. Hence it follows from the Sirsov-Witt theorem (see [10] p. 331) that $grLie\mathcal{U}(L,S,\ell)^{[G]}$ is free, freely generated by a base of $\oplus_{i=1}^{\infty} W_i$. □

Definition 4.2.7. Let $I(\oplus_{i=1}^{\infty} V_i)$ be a Lie ideal of the Lie algebra $grLie\mathcal{U}(L,S,\ell)$ generated by $\oplus_{i=1}^{\infty} V_i$. Let us set

$$grLie\mathcal{U}(L,S,\ell)_{[G]} := grLie\mathcal{U}(L,S,\ell)/I(\oplus_{i=1}^{\infty} V_i).$$

We denote by

$$\alpha : grLie\mathcal{U}(L,S,\ell)^{[G]} \to grLie\mathcal{U}(L,S,\ell)$$

the inclusion of the Lie algebra $grLie\mathcal{U}(L,S,\ell)^{[G]}$ into $grLie\mathcal{U}(L,S,\ell)$. Let

$$pr : grLie\mathcal{U}(L,S,\ell) \to grLie\mathcal{U}(L,S,\ell)_{[G]}$$

be the projection and let

$$\beta := pr \circ \alpha : grLie\mathcal{U}(L,S,\ell)^{[G]} \to grLie\mathcal{U}(L,S,\ell)_{[G]}$$

be the composition of the inclusion α with the projection pr.

Observe that G acts trivially on $grLie\mathcal{U}(K,T,\ell)$. Hence the natural morphism of associated graded Lie algebras

$$p_G : grLie\mathcal{U}(L,S,\ell) \to grLie\mathcal{U}(K,T,\ell)$$

induced by the inclusion $G_L \subset G_K$ induces a morphism of Lie algebras

$$\bar{p}_G : grLie\mathcal{U}(L,S,\ell)_{[G]} \to grLie\mathcal{U}(K,T,\ell)$$

which we denote also by $p_{K,T}^{L,S}$.

Proposition 4.2.8. *We have*

i) *the Lie algebra* $grLie\mathcal{U}(L,S,\ell)_{[G]}$ *is free, freely generated by a base of* $\oplus_{i=1}^{\infty} I_i/V_i$*;*

ii) *the morphism of Lie algebras*

$$\beta : grLie\mathcal{U}(L,S,\ell)^{[G]} \to grLie\mathcal{U}(L,S,\ell)_{[G]}$$

is an isomorphism;

iii) *the morphism of Lie algebras*

$$\bar{p}_G : grLie\mathcal{U}(L,S,\ell)_{[G]} \to grLie\mathcal{U}(K,T,\ell)$$

is an isomorphism.

Proof. Let us choose a base of $\oplus_{i=1}^{\infty} W_i$ and a base of $\oplus_{i=1}^{\infty} V_i$. Let us write a Hall base of a vector space $grLie\mathcal{U}(L,S,\ell)$ in terms of these two chosen bases. Then it is clear that we have a direct sum decomposition

$$grLie\mathcal{U}(L,S,\ell) = grLie\mathcal{U}(L,S,\ell)^{[G]} \oplus I(\oplus_{i=1}^{\infty} V_i).$$

This implies immediately points i) and ii).

The inclusion of Galois groups $G_L \subset G_K$ induces a map of cohomology groups

$$H^1(G_K;\mathbb{Q}_\ell(i)) \to H^1(G_L;\mathbb{Q}_\ell(i))$$

for $i > 1$. It follows from the Lyndon spectral sequence of the exact sequence of Galois groups $1 \to G_L \to G_K \to G \to 1$ that

$$H^1(G_K;\mathbb{Q}_\ell(i)) \simeq H^1(G_L;\mathbb{Q}_\ell(i))^G$$

for $i > 1$. Hence we get that

$$I(K)_i \simeq (I(L)_i)_G = I(L)_i / V(L)_i$$

for $i > 1$.

For $i = 1$ we use the Lyndon spectral sequence of the exact sequence of Galois groups

$$1 \to \pi_1(\mathrm{Spec}\,\mathcal{O}_{L,S};\ \mathrm{Spec}\bar{L}) \to \pi_1(\mathrm{Spec}\,\mathcal{O}_{K,T};\ \mathrm{Spec}\bar{L}) \to G \to 1.$$

We can also observe that the isomorphism $I(K)_1 \simeq (I(L)_1)_G$ follows from the equality

$$\mathcal{O}^*_{K,T} \otimes \mathbb{Q} \simeq (\mathcal{O}^*_{L,S} \otimes \mathbb{Q})^G.$$

This finishes the proof of the proposition. □

Definition 4.2.9. Let $L = \bigoplus_{i=1}^{\infty} L_i$ be a graded Lie algebra. We define a completion $c(L)$ of L setting

$$c(L) := \varprojlim{}_m (L/ \oplus_{i=m}^{\infty} L_i).$$

The completed Lie algebra $c(L)$ equipped with a group law given by the Baker-Campbell-Hausdorff formula we denote by $g(L)$.

PROPOSITION 4.2.10. *We have*

i) $c(grLie\mathcal{U}(L,S,\ell)) = Lie\mathcal{U}(L,S,\ell)$*;*

ii) *the isomorphism* β *induces an isomorphism of Lie algebras*

$$c(\beta) : c(grLie\mathcal{U}(L,S,\ell)^{[G]}) \to c(grLie\mathcal{U}(L,S,\ell)_{[G]})\,;$$

iii) *the isomorphism* $\bar{p}_G$ *induces an isomorphism of Lie algebras*

$$c(\bar{p}_G) : c(grLie\mathcal{U}(L,S,\ell)_{[G]}) \to c(grLie\mathcal{U}(K,T,\ell)) = Lie\mathcal{U}(K,T,\ell).$$

PROOF. The point i) follows from the fact that the Lie algebra $grLie\mathcal{U}(L,S,\ell)$ is free. The points ii) and iii) are obvious. □

DEFINITION 4.2.11. Let us set

$$\mathcal{U}(L,S,\ell)_{[G]} := g(grLie\mathcal{U}(L,S,\ell)_{[G]})$$

and

$$\mathcal{U}(L,S,\ell)^{[G]} := g(grLie\mathcal{U}(L,S,\ell)^{[G]}).$$

PROPOSITION 4.2.12. *Let us assume 4.2.1. Then we have:*

i) *there is a natural isomorphism of affine proalgebraic prounipotent groups*

$$a_{L,S} : g(grLie\mathcal{U}(L,S,\ell)) \simeq \mathcal{U}(L,S,\ell);$$

ii) *there is a natural isomorphism of affine proalgebraic prounipotent groups*

$$\mathcal{U}(L,S,\ell)_{[G]} \simeq (g(grLie\mathcal{U}(L,S,\ell)))_G;$$

iii) *the isomorphism* $\bar{p}_G : grLie\mathcal{U}(L,S,\ell)_{[G]} \to grLie\mathcal{U}(K,T,\ell)$ *induces an isomorphism of affine proalgebraic prounipotent groups*

$$g(\bar{p}_G) : \mathcal{U}(L,S,\ell)_{[G]} \simeq g(grLie\mathcal{U}(K,T,\ell))\,;$$

iv) *the isomorphism* $\beta : grLie\mathcal{U}(L,S,\ell)^{[G]} \to grLie\mathcal{U}(L,S,\ell)_{[G]}$ *induces an isomorphism of affine proalgebraic prounipotent groups*

$$g(\beta) : \mathcal{U}(L,S,\ell)^{[G]} \to \mathcal{U}(L,S,\ell)_{[G]}.$$

Proof. We shall identify the Lie algebras $c(grLie\mathcal{U}(L,S,\ell))$ and $Lie\mathcal{U}(L,S,\ell)$. Hence we have the identity isomorphism between the group $g(grLie\mathcal{U}(L,S,\ell))$ and $Lie\mathcal{U}(L,S,\ell)$ equipped with a group law given by the Baker-Campbell-Hausdorff formula. On the other side there is a canonical isomorphism between the Lie algebra $Lie\mathcal{U}(L,S,\ell)$ equipped with a group law given by Baker-Campbell-Hausdorff formula and the affine proalgebraic prounipotent group $\mathcal{U}(L,S,\ell)$. The isomorphism $a_{L,S}$ is the composition of these two isomorphisms.

Closed ideals of the Lie algebra $c(grLie\mathcal{U}(L,S,\ell))$ are exactly closed normal subgroups of the Lie group $g(grLie\mathcal{U}(L,S,\ell))$. Observe that the closed Lie ideal of $c(grLie\mathcal{U}(L,S,\ell))$ generated by elements $gv-v$ is exactly the closed normal subgroup of $g(grLie\mathcal{U}(L,S,\ell))$ generated by elements $(gv)\cdot v^{-1}$. This implies the point ii) of the proposition.

The points iii) and iv) are obvious. □

Definition 4.2.13. We set

$$s^{L,S}_{K,T} := g(\alpha) \circ g(\beta)^{-1} \circ g(\bar{p}_G)^{-1}.$$

Lemma 4.2.14. *The morphism*

$$s^{L,S}_{K,T} : g(grLie\mathcal{U}(K,T,\ell)) \to g(grLie\mathcal{U}(L,S,\ell))$$

is a section of the natural projection

$$g(p_G) : g(grLie\mathcal{U}(L,S,\ell)) \to g(grLie\mathcal{U}(K,T,\ell)).$$

Proof. Observe that $g(p_G)\circ s^{L,S}_{K,T} = g(p_G)\circ g(\alpha)\circ g(\beta)^{-1}\circ g(\bar{p}_G)^{-1} = g(\bar{p}_G)\circ g(pr)\circ g(\alpha)\circ g(\beta)^{-1}\circ g(\bar{p}_G)^{-1} = g(\bar{p}_G)\circ g(\beta)\circ g(\beta)^{-1}\circ g(\bar{p}_G)^{-1} = \mathrm{Id}_{g(grLie\mathcal{U}(K,T,\ell))}$. □

Definition 4.2.15. Let us set

$$S^{L,S}_{K,T} := a_{L,S} \circ s^{L,S}_{K,T} \circ a^{-1}_{K,T}.$$

We need to understand the morphism induced by $S^{L,S}_{K,T}$ on associated graded Lie algebras.

Lemma 4.2.16. *We have*

i) $grLie(a_{L,S}) = \mathrm{Id}_{grLie\mathcal{U}(L,S,\ell)}$;

ii) $grLie(g(f)) = f$.

Definition 4.2.17. Let $\rho : G_L \to \mathrm{Aut}M$ be an ℓ-adic mixed Tate module unramified outside S. We define a representation $\theta(\rho)$ of $G_{K(\mu_{\ell^\infty})}$ by setting

$$\theta(\rho) := [\rho] \circ S^{L,S}_{K,T} \circ i_{K,T,\ell}$$

(see Diagram 2).

$$\begin{array}{ccccc}
\rho : G_{L(\mu_{\ell^\infty})} & \xrightarrow{i_{L,S,\ell}} & \mathcal{U}(L,S,\ell) & \xrightarrow{[\rho]} & \mathrm{Aut}M \\
 & & a_{L,S}\Big\uparrow & & \\
 & & g(grLie\mathcal{U}(L,S,\ell)) & & \\
 & & s^{L,S}_{K,T}\Big\uparrow & & \\
 & & g(grLie\mathcal{U}(K,T,\ell)) & & \\
 & & a^{-1}_{K,T}\Big\uparrow & & \\
G_{K(\mu_{\ell^\infty})} & \xrightarrow{i_{K,T,\ell}} & \mathcal{U}(K,T,\ell) & &
\end{array}$$

Diagram 2.

We would like to express coefficients of the representation $\theta(\rho)$ by the coefficients of ρ.

Proposition 4.2.18. *Let $\rho : G_L \to \mathrm{Aut}M$ be an ℓ-adic mixed Tate module unramified outside S which is an extension of $\mathbb{Q}_\ell(0)$ by $\mathbb{Q}_\ell(n)$, i.e.,*

$$\rho(\sigma) = \begin{pmatrix} 1 & t(\sigma) \\ 0 & \chi^n(\sigma) \end{pmatrix}$$

for some $t : G_L \to \mathbb{Q}_\ell$. Then

$$[\theta(\rho)](\tau) = \begin{pmatrix} 1 & \frac{1}{|G|}\sum_{g\in G}([t]\circ g)(\tilde{\tau}) \\ 0 & 1 \end{pmatrix}$$

where $\tilde{\tau} \in \mathcal{U}(L, S, \ell)$ is a lifting of $\tau \in \mathcal{U}(K, T, \ell)$.

Proof. We recall that $[t]$ is a homomorphism from $\mathcal{U}(L, S, \ell)$ to $\mathbb{Q}_\ell$ iduced by $t : G_L \to \mathbb{Q}_\ell$. Observe that $[t]$ and $\frac{1}{|G|}\sum_{g\in G}([t]\circ g)$ restricted to $\mathcal{U}(L, S, \ell)^{[G]}$ coincide. The homomorphism $\frac{1}{|G|}\sum_{g\in G}([t]\circ g) : \mathcal{U}(L, S, \ell) \to \mathbb{Q}_\ell$ is G-invariant and, when viewed as a homomorphism from $grLie\mathcal{U}(L, S, \ell)$, it vanishes on decomposable elements. Hence it induces a homomorphism from $\mathcal{U}(L, S, \ell)_{[G]}$ to $\mathbb{Q}_\ell$, i.e., a homomorphism from $\mathcal{U}(K, T, \ell)$ to $\mathbb{Q}_\ell$. □

In [25] we have shown that $\mathrm{Gal}(M(L)_S/L)_G \simeq \mathrm{Gal}(M(K)_T/K)$. Hence we get the following corollary.

Corollary 4.2.19. *Let $\rho : G_L \to \mathrm{Aut}M$ be an ℓ-adic mixed Tate module unramified outside S, which is an extension of $\mathbb{Q}_\ell(0)$ by $\mathbb{Q}_\ell(n)$, i.e.,*

$$\rho(\sigma) = \begin{pmatrix} 1 & t(\sigma) \\ 0 & \chi^n(\sigma) \end{pmatrix}$$

for some $t : G_L \to \mathbb{Q}_\ell$. Then

$$\theta(\rho)(\tau) = \begin{pmatrix} 1 & \frac{1}{|G|}\sum_{g\in G}(t\circ g)(\tilde{\tau}) \\ 0 & 1 \end{pmatrix}$$

where $\tau \in G_{K(\mu_{\ell^\infty})}$ *and* $\tilde{\tau} \in \mathrm{Gal}(M(L)_S/L(\mu_{\ell^\infty}))$ *is a lifting to* $\mathrm{Gal}(M(L)_S/L(\mu_{\ell^\infty}))$ *of the image of* τ *in* $\mathrm{Gal}(M(K)_T/K(\mu_{\ell^\infty}))$.

Proof. The corollary follows immediately from [25] Corollary 2.5. □

Proposition 4.2.20. *Let* $\rho : G_L \to \mathrm{Aut}M$ *be an* ℓ*-adic mixed Tate module unramified outside* S*. Let* $\alpha \in (\mathrm{End}(M))^*$ *be such that* $\alpha \circ grLie[\rho]$ *vanishes on decomposable elements of* $grLie\mathcal{U}(L,S,\ell)$*. Then* $\alpha \circ grLie[\theta(\rho)](\tau) = \frac{1}{|G|}\sum_{g\in G}(\alpha \circ grLie[\rho] \circ g)(\tilde{\tau})$*, where* $\tilde{\tau} \in grLie\mathcal{U}(L,S,\ell)$ *is a lifting of* $\tau \in grLie\mathcal{U}(K,T,\ell)$ *to* $grLie\mathcal{U}(L,S,\ell)$.

Proof. The linear forms $\alpha \circ grLie[\rho]$ and $\frac{1}{|G|}\sum_{g\in G}(\alpha \circ grLie[\rho] \circ g)$ coincide when restricted to $grLie\mathcal{U}(L,S,\ell)^{[G]}$. The homomorphism $\frac{1}{|G|}\sum_{g\in G}(\alpha \circ grLie[\rho] \circ g) : grLie\mathcal{U}(L,S,\ell) \to \mathbb{Q}_\ell$ is G-invariant and it vanishes on decomposable elements. Hence it induces a homomorphism from $grLie\mathcal{U}(L,S,\ell)_{[G]}$, i.e., from $grLie\mathcal{U}(K,T,\ell)$ to $\mathbb{Q}_\ell$. This homomorphism is of course $\alpha \circ grLie[\theta(\rho)]$. □

4.3. Let us choose a sequence of geometric generators $x_1,\dots,x_n,x_{n+1}$ of $\pi_1(V_{\bar{K}};v)$ associated with a family $\Gamma = \{\gamma_i\}_{i=1}^{n+1}$ of paths from v to (a tangent vector defined over K at) each a_i. Let $\mathbb{V} = \{X_1,\dots,X_n\}$. As usual we define a continuous multiplicative embedding

$$k : \pi_1(V_{\bar{K}};v) \to \mathbb{Q}_\ell\{\{\mathbb{V}\}\}$$

setting $k(x_i) = e^{X_i}$ for $i = 1,\dots,n$.

We fix paths p_j from v to z_j for $j = 1,\dots,r$. The action of G_L on the disjoint union of torsors $t = \coprod_{j=1}^{r} \pi(V_{\bar{K}};z_j,v)$ induces a representation

$$\psi_t : G_L \to \left(\bigoplus_{j=1}^{r} L_{\exp L(\mathbb{V})}\right) \tilde{\times}\, \mathrm{Aut}^*\,\mathbb{Q}_\ell\{\{\mathbb{V}\}\}\,.$$

We assume that the representation ψ_t is unramified outside the finite set S of finite places of L. We assume that S contains all finite places lying over (ℓ) and that S is G-invariant.

We assume also that

i) ℓ does not divide the order of G;

ii) $K(\mu_{\ell^\infty}) \cap L = K$.

Let us set

$$T := \{\mathfrak{q} \in \mathcal{V}(K) \mid \exists\, \mathfrak{p} \in S,\, \mathfrak{q} = \mathfrak{p} \cap \mathcal{O}_K\}.$$

We recall from section 4.2 that the representation

$$\theta(\psi_t) : G_{K(\mu_{\ell^\infty})} \to \oplus_{j=1}^{r} \mathrm{GL}(\mathbb{Q}_\ell\{\{\mathbb{V}\}\})$$

is defined by the formula

$$\theta(\psi_t) := [\psi_t] \circ S_{K,T}^{L,S} \circ i_{K,T,\ell}$$

(see Definition 4.2.17). Further we shall also denote the representation $\theta(\psi_t)$ by $\theta_t^{L/K}$.

The representation $\theta_t^{L/K}$ respects the filtration $\{\oplus_{j=1}^{r} I(\mathbb{V})^m\}_{m\in\mathbb{N}\cup\{0\}}$ of $\oplus_{j=1}^{r}\mathbb{Q}\{\{\mathbb{V}\}\}$. We define a filtration $\{F_m(\theta_t^{L/K})\}_{m\in\mathbb{N}\cup\{0\}}$ of $G_{K(\mu_{\ell^\infty})}$ by setting

$$F_m(\theta_t^{L/K}) := \{\sigma \in G_{K(\mu_{\ell^\infty})} \mid \theta_t^{L/K}(\sigma) \text{ induces the identity of } \oplus_{j=1}^{r} (I(\mathbb{V})/I(\mathbb{V})^{m+1})\}.$$

We state below elementary properties of the representation $\theta_t^{L/K}$.

Proposition 4.3.0. *The representation $\theta_t^{L/K}$ has the following properties:*

i) *it is compatible with the filtration $\{F_m(\theta_t^{L/K})\}_{m\in\mathbb{N}\cup\{0\}}$ of $G_{K(\mu_{\ell^\infty})}$ and the filtration induced by the powers of the augmentation ideal $I(\mathbb{V})$ on $\oplus_{j=1}^{r}\mathrm{GL}(\mathbb{Q}_\ell\{\{\mathbb{V}\}\})$;*

ii) *it is unramified outside the set T of finite places of K;*

iii) *any $\sigma \in G_{K(\mu_{\ell^\infty})}$ acts on $\oplus_{j=1}^{r}(I(\mathbb{V})^m/I(\mathbb{V})^{m+1})$ by the identity.*

The main object of our study is the representation induced by $\theta_t^{L/K}$ on the associated graded Lie algebras, i.e., the representation

$$\mathrm{grLie}\theta_t^{L/K} : \oplus_{i=1}^{\infty} F_i(\theta_t^{L/K})/F_{i+1}(\theta_t^{L/K}) \otimes \mathbb{Q} \to \big(\oplus_{j=1}^{r} L_{\mathrm{Lie}(\mathbb{V})}\big) \tilde{\times} \mathrm{Der}^*\mathrm{Lie}(\mathbb{V}).$$

Let us set

$$\mathbf{\Theta}_t^{\mathrm{Gal}(L/K)} := \mathrm{grLie}\theta_t^{L/K}$$

To calculate the image of the Lie algebra representation $\mathbf{\Theta}_t^{\mathrm{Gal}(L/K)}$ first we need to calculate coordinates of the Lie algebra homomorphism $\mathbf{\Psi}_t$. We recall that

$$\mathbf{\Psi}_t := \mathrm{grLie}\psi_t : \oplus_{k=1}^{\infty} (H_k/H_{k+1}) \otimes \mathbb{Q} \to \big(\oplus_{j=1}^{r} L_{\mathrm{Lie}(\mathbb{V})}\big) \tilde{\times} \mathrm{Der}^*\mathrm{Lie}(\mathbb{V}).$$

and $\mathbf{\Psi}_t^G$ is the restriction of $\mathbf{\Psi}_t$ to $\oplus_{k=1}^{\infty}(H_k/H_{k+1}) \otimes \mathbb{Q}^G$.

We recall that the representation ψ_t induces

$$[\psi_t] : \mathcal{U}(L,S,\ell) \to \big(\oplus_{j=1}^{r} L_{\exp L(\mathbb{V})}\big) \tilde{\times} \mathrm{Aut}^*(\mathbb{Q}_\ell\{\{\mathbb{V}\}\}).$$

We set

$$[\mathbf{\Psi}_t] := \ grLie[\psi_t]$$

and we denote by $[\mathbf{\Psi}_t]^{[G]}$ the morphism $[\mathbf{\Psi}_t]$ restricted to $grLie\mathcal{U}(L,S,\ell)^{[G]}$.

Let $\sigma \in H_k/H_{k+1}$. We recall that the coordinates of $\mathbf{\Psi}_t(\sigma)$ are

$$\Big(\big(\log \Lambda_{p_j}(\sigma) \bmod \Gamma^{k+1} L(\mathbb{V})\big)_{j=1,\dots,r};\ \big(\log \Lambda_{\gamma_i}(\sigma) \bmod \langle X_i\rangle + \Gamma^{k+1} L(\mathbb{V})\big)_{i=1,\dots,n}\Big) .$$

Definition 4.3.1. We denote by $\mathcal{J}$ a closed Lie ideal of $L(\mathbb{V})$ generated by Lie bracket which contain at least three different X_i's and by Lie brackets containing two different X_i's at least twice.

In the next proposition we shall calculate coefficients of $\log \Lambda_{p_j}(\sigma)$ modulo the Lie ideal $\mathcal{J}$.

Proposition 4.3.2. *Let $\sigma \in H_k$. If $k > 2$ then*

$$\log \Lambda_{p_j}(\sigma) \equiv \sum_{\alpha\neq\beta} \left(\ell_k\left(\frac{z_j - a_\alpha}{a_\beta - a_\alpha}\right)(\sigma) - \ell_k\left(\frac{v - a_\alpha}{a_\beta - a_\alpha}\right)(\sigma)\right) [\dots[X_\beta, X_\alpha], X_\alpha^{k-2}] \bmod \mathcal{J} + \Gamma^{k+1}L(\mathbb{V}) .$$

If $k=2$ then

$$\log \Lambda_{p_j}(\sigma) \equiv \sum_{\alpha<\beta} \left(\ell_k \left(\frac{z_j - a_\alpha}{a_\beta - a_\alpha} \right)(\sigma) - \ell_k \left(\frac{v - a_\alpha}{a_\beta - a_\alpha} \right)(\sigma) \right) [X_\beta, X_\alpha] \bmod \Gamma^3 L(\mathbb{V})\,.$$

If $k=1$ then

$$\log \Lambda_{p_j}(\sigma) \equiv \sum_{\alpha=1}^{n} \ell \left(\frac{z_j - a_\alpha}{v - a_\alpha} \right)(\sigma)\, X_\alpha \bmod \Gamma^2 L(\mathbb{V})\,.$$

Proof. Let $f : V \to \mathbb{P}^1_K \setminus \{0,1,\infty\}$ be given by $f(z) = \frac{z-a_\alpha}{a_\beta - a_\alpha}$. It follows from [19] Proposition 11.0.15 that

$$\log \Lambda_{f(p_j)}(\sigma) \equiv \left(\ell_k \left(\frac{z_j - a_\alpha}{a_\beta - a_\alpha} \right)(\sigma) - \ell_k \left(\frac{v - a_\alpha}{a_\beta - a_\alpha} \right)(\sigma) \right) [Y, X^{k-1}] \bmod \mathcal{J} + \Gamma^{k+1} L(X,Y)\,.$$

Observe that $f(a_\alpha) = 0$, $f(a_\beta) = 1$ and $f(\infty) = \infty$. This implies that $f_*(X_\alpha) = X$, $f_*(X_\beta) = Y$ and $f_*(X_i) = 0$ for $i \neq \alpha, \beta$. Now the proposition follows from the equality $f_* \log \Lambda_{p_j}(\sigma) = \log \Lambda_{f(p_j)}(\sigma)$. $\square$

Let $\mathcal{B}$ be a Hall base of a free Lie algebra Lie($\mathbb{V}$) and let $\mathcal{B}_k$ be elements of degree k in $\mathcal{B}$. Let p be a path from v to a K-point z of V. Let $\sigma \in H_k(V_K; z, v)$. We recall that in [18] section 5 we defined functions $\mathcal{L}^e(z,v)$ from $H_k(V_K; z, v)$ to $\mathbb{Q}_\ell$ by the following congruence

$$\log \Lambda_p(\sigma) \equiv \sum_{e \in \mathcal{B}_k} \mathcal{L}^e(z,v)\,(\sigma) \cdot e \quad \bmod \Gamma^{k+1} L(\mathbb{V})\,.$$

Proposition 4.3.3. *Let $\sigma \in H_k$ and let $\tilde{\tau} \in G_{K(\mu_{\ell^\infty})}$ be a lifting of $\tau \in G = \mathrm{Gal}(L(\mu_{\ell^\infty})/K(\mu_{\ell^\infty}))$. Then for any $e \in \mathcal{B}_k$ we have*

$$\mathcal{L}^e(z_j, v)\,(\tilde{\tau} \cdot \sigma \cdot \tilde{\tau}^{-1}) = \mathcal{L}^e(z_{\tau^{-1}(j)}, v)\,(\sigma)\,.$$

PROOF. After calculations we get

$$\mathfrak{f}_{p_j}(\tilde{\tau}\cdot\sigma\cdot\tilde{\tau}^{-1}) = \tilde{\tau}(\mathfrak{f}_{\tilde{\tau}^{-1}(p_j)}(\sigma)),$$

where $\tilde{\tau}^{-1}(p_j)$ is a path from v to $z_{\tilde{\tau}^{-1}(j)}$. Hence

$$\log\Lambda_{p_j}(\tilde{\tau}\cdot\sigma\cdot\tilde{\tau}^{-1}) = \tilde{\tau}(\log\Lambda_{\tilde{\tau}^{-1}(p_j)}(\sigma)).$$

The assumption $\sigma\in H_k$ implies that

$$\tilde{\tau}(\log\Lambda_{\tilde{\tau}^{-1}(p_j)}(\sigma)) \equiv \log\Lambda_{\tilde{\tau}^{-1}(p_j)}(\sigma) \mod \Gamma^{k+1}L(\mathbb{V}).$$

Therefore we get the congruence

$$\log\Lambda_{p_j}(\tilde{\tau}\cdot\sigma\cdot\tilde{\tau}^{-1}) \equiv \log\Lambda_{\tilde{\tau}^{-1}(p_j)}(\sigma) \mod \Gamma^{k+1}L(\mathbb{V})$$

which implies the lemma. □

COROLLARY 4.3.4. *Let $\sigma\in H_k$ and let $\tilde{\tau}\in G_{K(\mu_{\ell^\infty})}$ be a lifting of $\tau\in G$. Then*

$$\ell_k\left(\frac{z_j-a_\alpha}{a_\beta-a_\alpha}\right)(\tilde{\tau}\cdot\sigma\cdot\tilde{\tau}^{-1}) = \ell_k\left(\frac{z_{\tau^{-1}(j)}-a_\alpha}{a_\beta-a_\alpha}\right)(\sigma)$$

for $k>1$ and

$$\ell\left(\frac{z_j-a_\alpha}{v-a_\alpha}\right)(\tilde{\tau}\cdot\sigma\cdot\tilde{\tau}^{-1}) = \ell\left(\frac{z_{\tau^{-1}(j)}-a_\alpha}{v-a_\alpha}\right)(\sigma)$$

for $k=1$.

4.4. It is well known that

$$\dim_{\mathbb{Q}_\ell} H^1(\operatorname{Spec}\mathcal{O}_{K,T};\mathbb{Q}_\ell(i)) = r_1+r_2 \quad \text{for } i>1 \text{ and odd;}$$

$$\dim_{\mathbb{Q}_\ell} H^1(\operatorname{Spec}\mathcal{O}_{K,T};\mathbb{Q}_\ell(i)) = r_2 \quad \text{for } i>0 \text{ and even;}$$

and

$$\dim_{\mathbb{Q}_\ell} H^1(\operatorname{Spec}\mathcal{O}_{K,T};\mathbb{Q}_\ell(1)) = \dim_{\mathbb{Q}}(\mathcal{O}^*_{K,T}\otimes\mathbb{Q}).$$

Generators of $H^1(\operatorname{Spec}\mathcal{O}_{K,T};\mathbb{Q}_\ell(1))$ are given by Kummer classes of generators of the group $\mathcal{O}^*_{K,T}$. Generators of $H^1(\operatorname{Spec}\mathcal{O}_{K,T};\mathbb{Q}_\ell(i))$ for $i>1$ are given by Soulé classes (see [15]). For $K=\mathbb{Q}(\mu_n)$ Soulé classes can be expressed by ℓ-adic polylogarithms (see [21]). In general case one can hope that Soulé classes can be expressed by linear combinations of ℓ-adic polylogarithms (surjectivity in Zagier conjecture for polylogarithms) or by linear combination of ℓ-adic iterated integrals or at least by Galois invariant linear combination of ℓ-adic polylogarithms or ℓ-adic iterated integrals.

In most applications $L = \mathbb{Q}(\mu_n)$ and S is the set of finite places of $\mathbb{Q}(\mu_n)$ which divide the product $n \cdot \ell$. Then we have

$$\dim_{\mathbb{Q}_\ell} H^1(\operatorname{Spec} \mathcal{O}_{\mathbb{Q}(\mu_n),S}; \mathbb{Q}_\ell(i)) = \frac{1}{2}\varphi(n)$$

($\varphi(n)$ is the order of $\mathbb{Z}/n^*$) for $i > 1$ and

$$\dim_{\mathbb{Q}_\ell} H^1(\operatorname{Spec} \mathcal{O}_{\mathbb{Q}(\mu_n),S}; \mathbb{Q}_\ell(1)) = \dim_{\mathbb{Q}} (\mathcal{O}^*_{\mathbb{Q}(\mu_n),S} \otimes \mathbb{Q}).$$

Conjecture 4.4.1. *Let i be greater than 1. The ℓ-adic polylogarithms $\ell_i(\xi_n^k)$ for $0 < k < \frac{n}{2}$ and $(k, n) = 1$ are linearly independent over $\mathbb{Q}_\ell$ and generate $H^1(G_{\mathbb{Q}(\mu_n)}; \mathbb{Q}_\ell(i)) = H^1(\operatorname{Spec} \mathcal{O}_{\mathbb{Q}(\mu_n),S}; \mathbb{Q}_\ell(i))$.*

5. Torsors of Paths on $\mathbb{P}^1_{\bar{\mathbb{Q}}} \backslash \{0, 1, \infty\}$

5.0. In this section we shall study torsors of paths on

$$V := \mathbb{P}^1_{\mathbb{Q}} \backslash \{0, 1, \infty\}\,.$$

First we fix a notation. We denote by $\mathbb{Q}_\ell\{\{X, Y\}\}$ a $\mathbb{Q}_\ell$-algebra of non-commutative formal power series on non-commuting variables X and Y. The Lie algebra Lie (X, Y) is a free Lie algebra over $\mathbb{Q}_\ell$ on X and Y and $L(X, Y)$ is a completion of Lie (X, Y) with respect to lower central series. If R is a commutative ring we denote by Lie $(X, Y; R)$ a free Lie algebra over R on X and Y.

We assume that $\bar{\mathbb{Q}} \subset \mathbb{C}$. Let $\xi_q = e^{\frac{2\pi i}{q}}$ be a primitive q-th root of 1. We shall study Galois actions on a disjoint union and on a product of $\pi_1(V_{\bar{\mathbb{Q}}}; \overrightarrow{01})$-torsors

$$t := \coprod_{\substack{(\alpha,q)=1 \\ 0<\alpha<q}} \pi(V_{\bar{\mathbb{Q}}}; \xi_q^\alpha, \overrightarrow{01}) \quad \text{and} \quad T := \prod_{\substack{(\alpha,q)=1 \\ 0<\alpha<q}} \pi(V_{\bar{\mathbb{Q}}}; \xi_q^\alpha, \overrightarrow{01}))\,.$$

Let us set

$$H_i := \bigcap_{\substack{(\alpha,q)=1\\ 0<\alpha<q}} H_i(V_{\mathbb{Q}(\mu_q)}; \xi_q^\alpha, \overrightarrow{01})\,.$$

The action of $G_{\mathbb{Q}(\mu_q)}$ on t and T yields Lie algebra representations of associated graded Lie algebras

$$\mathbf{\Psi}_t : \bigoplus_{k=1}^{\infty}(H_k/H_{k+1})\otimes\mathbb{Q} \to (\bigoplus_{\substack{(\alpha,q)=1\\ 0<\alpha<q}} \mathrm{Lie}(X,Y))\,\tilde{\times}\,\mathrm{Der}^*\,\mathrm{Lie}(X,Y)$$

and

$$\mathbf{\Psi}_T : \bigoplus_{k=1}^{\infty}(H_k/H_{k+1})\otimes\mathbb{Q} \to$$

$$(\bigoplus_{\substack{(\alpha,q)=1\\ 0<\alpha<q}} \mathrm{Id}\otimes\ldots\otimes\mathrm{Id}\otimes L_{\mathrm{Lie}(X,Y)}\otimes\ldots\otimes\mathrm{Id})\,\tilde{\times}\,\mathrm{Der}^*\,\mathrm{Lie}(X,Y)\,,$$

where

$$\mathrm{Der}^*\,\mathrm{Lie}(X,Y) := \{D\in\mathrm{Der}\,\mathrm{Lie}(X,Y) \mid \exists\, A(X,Y)\in\mathrm{Lie}(X,Y),\ D(X)=0$$
$$\text{and } D(Y)=[Y,A(X,Y)]\}\,.$$

A derivation $D\in\mathrm{Der}^*\,\mathrm{Lie}(X,Y)$ such that $D(Y)=[Y,A]$ we denote by D_A. The vector space $\mathrm{Lie}(X,Y)/\langle Y\rangle$ we equip with a Lie bracket $\{\ \}$ given by

$$\{A,B\} = [A,B] + D_A(B) - D_B(A).$$

The obtained Lie algebra we denote by $(\mathrm{Lie}(X,Y)/\langle Y\rangle,\ \{\,\})$. The map

$$\mathrm{Der}^*\,\mathrm{Lie}(X,Y) \longrightarrow (\mathrm{Lie}(X,Y)/\langle Y\rangle, \{\,\})$$

sending D_A to A is an isomorphism of Lie algebras.

Hence we get a Lie algebra representation

$$\mathbf{\Psi}_t : \bigoplus_{k=1}^{\infty}(H_k/H_{k+1})\otimes\mathbb{Q} \to (\bigoplus_{\substack{(\alpha,q)=1\\ 0<\alpha<q}} \mathrm{Lie}(X,Y))\,\tilde{\times}\,(\mathrm{Lie}(X,Y)/\langle Y\rangle, \{\,\})\,.$$

We denote by I_r the vector subspace of $\mathrm{Lie}(X,Y)$ generated by Lie brackets in X and Y which contain at least r Y's.

Let us set

$$\mathcal{I}_r := \Big(\bigoplus_{\substack{(\alpha,q)=1\\ 0<\alpha<q}} I_r\Big) \oplus I_r\,.$$

$\mathcal{I}_r$ is a vector subspace of $\big(\bigoplus_{\substack{(\alpha,q)=1\\ 0<\alpha<q}} \mathrm{Lie}(X,Y)\big)\tilde{\times}(\mathrm{Lie}(X,Y)/\langle Y\rangle,\{\,\})$ and one can easily verify that $\mathcal{I}_r$ is a Lie ideal of the Lie algebra $\big(\bigoplus_{\substack{(\alpha,q)=1\\ 0<\alpha<q}} \mathrm{Lie}(X,Y)\big)\tilde{\times}$ $(\mathrm{Lie}(X,Y)/\langle Y\rangle,\{\,\})$.

To simplify the notation we denote the Lie algebra $\big(\bigoplus_{\substack{(\alpha,q)=1\\ 0<\alpha<q}} \mathrm{Lie}(X,Y)\big)\tilde{\times}$ $(\mathrm{Lie}(X,Y)/\langle Y\rangle,\{\,\})$ by $\mathcal{L}$.

Let S be a set of finite places of $\mathbb{Q}(\mu_q)$ which divide q or ℓ.

Lemma 5.0.1. *We have:*

i) *Let $\sigma \in H_k$. The coordinates of $\mathbf{\Psi}_t(\sigma)$ calculated modulo $I_2 + \Gamma^{k+1}L(X,Y)$ on each component are*

$$\Big(\big(\ell_k(\xi_q^\alpha)(\sigma)\,[Y,X^{k-1}]\big)_{\substack{(\alpha,q)=1\\ 0<\alpha<q}};\ell_k(1)(\sigma)\,[Y,X^{k-1}]\Big)$$

for $k>1$ and

$$\Big(\big(\ell_1(\xi_q^\alpha)(\sigma)Y\big)_{\substack{(\alpha,q)=1\\ 0<\alpha<q}};0\Big)$$

for $k=1$.

ii) *Let $\sigma \in grLie\mathcal{U}(L,S,\ell)_k$. The coordinates of $grLie[\psi_t](\sigma)$ calculated modulo $I_2 + \Gamma^{k+1}L(X,Y)$ on each component are*

$$\Big(\big([\ell_k(\xi_q^\alpha)](\sigma)\,[Y,X^{k-1}]\big)_{\substack{(\alpha,q)=1\\ 0<\alpha<q}};[\ell_k(1)](\sigma)\,[Y,X^{k-1}]\Big)$$

for $k>1$ and

$$\Big(\big([\ell_1(\xi_q^\alpha)](\sigma)Y\big)_{\substack{(\alpha,q)=1\\ 0<\alpha<q}};0\Big)$$

for $k=1$.

Proof. The first part of the lemma follows from Lemma 1.1.0 and from Definition 3.0.1 of ℓ-adic polylogarithms.

The universal property of the weighted Tate completion and the fact that all morphisms are strict imply that we have a surjective morphism of Lie algebras

$$grLie\mathcal{U}(L,S,\ell)\to\bigoplus_{k=1}^{\infty}(H_k/H_{k+1})\otimes\mathbb{Q}.$$

Hence it follows immediately the second part of the lemma. □

Further we shall apply formalism developed in section 4 in order to get representations of Galois groups of $\mathbb{Q}$ and other fields K intermediary between $\mathbb{Q}$ and $\mathbb{Q}(\mu_q)$ with prescribed ramifications. In some cases we shall be able to show that the image of the morphism $\mathbf{\Theta}_t^{\mathrm{Gal}(\mathbb{Q}(\mu_q)/K)}$ is a free Lie algebra on a maximal possible, depending on K and ramifications, number of generators.

We finish this section with some technical definitions and results. First we recall that

$$\mathrm{Gal}(\mathbb{Q}(\mu_q)/\mathbb{Q})=\mathbb{Z}/q^*.$$

The group $\mathbb{Z}/q^*=\mathrm{Gal}(\mathbb{Q}(\mu_q)/\mathbb{Q})$ acts by conjugation on the abelianization of the Galois group $G_{\mathbb{Q}(\mu_q)(\mu_{\ell^\infty})}$. Hence it acts also on $\mathrm{Hom}_{\mathbb{Z}_\ell}\big((G_{\mathbb{Q}(\mu_q)(\mu_{\ell^\infty})})^{ab};\mathbb{Q}_\ell(m)\big)$.

Lemma 5.0.2. *Let $\tau\in\mathbb{Z}/q^*$ and let $\tilde\tau$ be a lifting of τ to $G_{\mathbb{Q}(\mu_{\ell^\infty})}$. Then for any $\sigma\in G_{\mathbb{Q}(\mu_q)(\mu_{\ell^\infty})}$ we have*

$$\ell_m(\xi_q^{\alpha})(\tilde\tau\cdot\sigma\cdot\tilde\tau^{-1})=\ell_m(\xi_q^{\tau^{-1}\cdot\alpha})(\sigma).$$

Proof. Let γ be a path from $\overrightarrow{01}$ to ξ_q^{α} and let $\sigma':=\tilde\tau\cdot\sigma\cdot\tilde\tau^{-1}$. Then $\sigma'\cdot\gamma\cdot\sigma'^{-1}$ transforms

$$f(z)=\prod_{i=0}^{\ell^n-1}(1-\xi_{\ell^n}^{i}\,z^{1/\ell^n})^{\frac{i^{m-1}}{\ell^n}}$$

into $\xi_{\ell^n}^{\boldsymbol{\ell}_m(\xi_q^{\alpha})(\sigma')}\cdot f(z)$.

On the other side applying terms of the product $\tilde\tau\cdot\sigma\cdot\tilde\tau^{-1}\cdot\gamma\cdot\tilde\tau\cdot\sigma^{-1}\cdot\tilde\tau^{-1}$ one after another to $f(z)$, we get that the product transforms $f(z)$ into

$\xi_{\ell^n}^{\boldsymbol{\ell}_m(\xi_q^{\tau^{-1}\cdot\alpha})(\sigma)} \cdot f(z)$. Hence we get that

$$\boldsymbol{\ell}_m(\xi_q^\alpha)(\tilde{\tau}\cdot\sigma\cdot\tilde{\tau}^{-1}) = \boldsymbol{\ell}_m(\xi_q^{\tau^{-1}\cdot\alpha})(\sigma).$$

Observe that $\ell(\xi_q^\alpha)(\sigma) = 0$ for $\sigma \in G_{\mathbb{Q}(\mu_q)(\mu_{\ell^\infty})}$. Hence the lemma follows from Proposition 3.0.4. □

For each m we define a $\mathbb{Z}/q^*$-module $V_m(q)$ in the following way. Let $m > 1$. In the group ring $\mathbb{Q}_\ell[\mathbb{Z}/q]$ we consider a $\mathbb{Q}_\ell$-vector subspace $R_m(q)$ generated by elements

$$[\alpha] + (-1)^m[-\alpha] \quad \text{for} \quad [\alpha] \in \mathbb{Z}/q$$

and

$$r^{m-1}\left(\sum_{i=0}^{r-1}[ip]\right) - [0] \quad , \quad p^{m-1}\left(\sum_{j=0}^{p-1}[k+jr]\right) - [pk] \quad \text{if} \quad q = p\cdot r.$$

(The class of α in $\mathbb{Z}/q$ we denote by $[\alpha]$.)

If $m = 1$ we denote by $R_1(q)$ a $\mathbb{Q}_\ell$-vector subspace of $\mathbb{Q}_\ell[\mathbb{Z}/q]$ generated by elements

$$[0] \quad \text{and} \quad [\alpha] - [-\alpha] \quad \text{for} \quad [\alpha] \in \mathbb{Z}/q$$

and

$$\left(\sum_{i=0}^{r-1}[j+ip]\right) - [rj] \quad \text{with} \quad j \not\equiv 0 \bmod p \quad \text{if} \quad q = p\cdot r.$$

We set

$$V_m(q) := \mathbb{Q}_\ell[\mathbb{Z}/q]/R_m(q)\,.$$

The group $\mathbb{Z}/q^*$ acts on $\mathbb{Q}_\ell[\mathbb{Z}/q]$ permuting elements of $\mathbb{Z}/q$. Observe that the vector subspace $R_m(q)$ of $\mathbb{Q}_\ell[\mathbb{Z}/q]$ is preserved by the action of $\mathbb{Z}/q^*$, hence the quotient space $V_m(q)$ is also a $\mathbb{Z}/q^*$-module.

LEMMA 5.0.3. *There is a $\mathbb{Z}/q^*$-equivariant map*

$$s_m^q : V_m(q) \to H^1(G_{\mathbb{Q}(\mu_q)}; \mathbb{Q}_\ell(m))$$

for $m > 1$ and

$$s_1^q : V_1(q) \to H^1(\mathrm{Spec}\mathcal{O}_{\mathbb{Q}(\mu_q),S}; \mathbb{Q}_\ell(1))$$

for $m = 1$ such that $s^q_m([\alpha]) = \ell_m(\xi_q^\alpha)$.

PROOF. It is clear that the formula $\bar{s}^q_m([\alpha]) := \ell_m(\xi_q^\alpha)$ defines a $\mathbb{Q}_\ell$-linear map $\bar{s}^q_m$ from $V_m(q)$ to $\mathrm{Hom}_{\mathbb{Z}_\ell}\big((G_{\mathbb{Q}(\mu_q)(\mu_{\ell^\infty})})^{ab}; \mathbb{Q}_\ell(m)\big)$ for $m > 1$ and from $V_1(q)$ to $\mathrm{Hom}_{\mathbb{Z}_\ell}\big(\pi_1(\mathrm{Spec}\mathcal{O}_{\mathbb{Q}(\mu_q)(\mu_{\ell^\infty}),S}; \mathrm{Spec}\bar{\mathbb{Q}})^{ab}; \mathbb{Q}_\ell(1)\big)$ for $m = 1$ because generators of $R_m(q)$ correspond to the distribution and inversion relations of ℓ-adic polylogarithms.

Observe that we have $\mathbb{Z}/q^*$-isomorphisms

$$t_m : \mathrm{Hom}_{\mathbb{Z}_\ell}\big((G_{\mathbb{Q}(\mu_q)(\mu_{\ell^\infty})})^{ab}; \mathbb{Q}_\ell(m)\big) \longrightarrow H^1(G_{\mathbb{Q}(\mu_q)}; \mathbb{Q}_\ell(m))$$

for $m > 1$ and

$$\begin{aligned} t_1 : \mathrm{Hom}_{\mathbb{Z}_\ell}\big(&\pi_1(\mathrm{Spec}\mathcal{O}_{\mathbb{Q}(\mu_q)(\mu_{\ell^\infty}),S}; \mathrm{Spec}\bar{\mathbb{Q}})^{ab}; \mathbb{Q}_\ell(1)\big) \\ &\to H^1(\mathrm{Spec}\,\mathcal{O}_{\mathbb{Q}(\mu_q),S}; \mathbb{Q}(1)) \end{aligned}$$

for $m = 1$.

We set $s^q_m := t_m \circ \bar{s}^q_m$. It follows from Lemma 5.0.2 that the map $\bar{s}^q_m$ is $\mathbb{Z}/q^*$-equivariant hence s^q_m is also $\mathbb{Z}/q^*$-equivariant. □

LEMMA 5.0.4. *Let us assume that Conjecture 4.4.1 holds. Then we have*

i) *the $\mathbb{Z}/q^*$-equivariant map*

$$s^q_m : V_m(q) \to H^1(G_{\mathbb{Q}(\mu_q)}; \mathbb{Q}_\ell(m))$$

is an isomorphism for $m > 1$;

ii) *the $\mathbb{Z}/q^*$-equivariant map*

$$s^q_1 : V_1(q) \to H^1(\mathrm{Spec}\mathcal{O}_{\mathbb{Q}(\mu_q),S}; \mathbb{Q}_\ell(1))$$

is injective.

PROOF. The result of Soulé implies that $\dim H^1(G_{\mathbb{Q}(\mu_q)}; \mathbb{Q}_\ell(m)) = \frac{1}{2}\varphi(q)$ for $m > 1$ (see [15]). It follows from [12] that $\dim V_m(q) = \frac{1}{2}\varphi(q)$ for $m > 1$. We have assumed that Conjecture 4.4.1 holds. This implies that the map s^q_m is an isomorphism for $m > 1$.

Now we assume that $m = 1$. Let us denote by $C_{\mathbb{Q}(\mu_q)}$ a subgroup of $\mathbb{Q}(\mu_q)^*$ generated by elements of the form $1-\xi_q^{\alpha}$ and by μ_{2q}. It follows from the Bass theorem (see [17] Theorem 8.9) that the only relations between these elements are

$$-\xi_q^{-\alpha}(1-\xi_q^{\alpha}) = 1-\xi_q^{-\alpha}$$

and

$$\prod_{i=0}^{r-1}(1-\xi_q^{j+ip}) = 1-\xi_q^{rj} \ \text{ if } \ q = rp.$$

These relations correspond of course to generators of $R_1(q)$.

On the other side the homomorphism

$$\mathcal{O}^*_{\mathbb{Q}(\mu_q),S}/\mu_{2q} \longrightarrow H^1(\mathrm{Spec}\mathcal{O}_{\mathbb{Q}(\mu_q),S};\mathbb{Q}_\ell(1))$$

which to a unite u associates a corresponding Kummer character is injective. This implies that the map s_1^q is injective. □

Lemma 5.0.5. *Let us assume that Conjecture 4.4.1 holds. Let G be a subgroup of Gal$(\mathbb{Q}(\mu_q)/\mathbb{Q}) = \mathbb{Z}/q^*$. Then the map s_m^q induces*

i) *an isomorphism*

$$(s_m^q)^G : V_m(q)^G \to H^1(G_{\mathbb{Q}(\mu_q)};\mathbb{Q}_\ell(m))^G$$

for $m > 1$;

ii) *an injective map*

$$(s_1^q)^G : V_1(q)^G \to H^1(\mathrm{Spec}\mathcal{O}_{\mathbb{Q}(\mu_q),S};\mathbb{Q}_\ell(1))^G$$

for $m = 1$.

Proof. The lemma follows immediately from Lemma 5.0.4. □

5.1. Let q be a prime number. In this section we shall construct geometrically ℓ-adic realization of the associated graded Lie algebra of the fundamental group of the tannakien category of the mixed Tate motives over Spec $\mathbb{Z}\left[\frac{1}{q}\right]$. We recall that conjecturally this Lie algebra is free, freely generated by elements in degree $1, 3, 5, \dots, 2n+1, \dots$. Generator in degree 1 corresponds to $\log q$ and generator in degree $2n+1$ corresponds to $\zeta(2n+1)$ for $n>0$.

We assume that q is a prime number different from ℓ. We assume also that ℓ does not divide $q-1$. Then

$$\mathbb{Q}(\mu_q)\cap\mathbb{Q}(\mu_{\ell^\infty})=\mathbb{Q}$$

and ℓ does not divide the order of the group

$$\mathbb{Z}/q^* = \mathrm{Gal}(\mathbb{Q}(\mu_q)/\mathbb{Q})\,.$$

We consider actions of Galois groups on

$$t := \coprod_{0<\alpha<q} \pi(V_{\mathbb{Q}};\xi_q^\alpha,\overrightarrow{01}) \quad \text{and} \quad T := \prod_{0<\alpha<q} \pi(V_{\mathbb{Q}};\xi_q^\alpha,\overrightarrow{01})\,.$$

We shall apply formalism from section 4 for

$$K=\mathbb{Q} \ \text{ and } \ L=\mathbb{Q}(\mu_q).$$

We recall that the Galois group $\mathbb{Z}/q^* = \mathrm{Gal}(\mathbb{Q}(\mu_q)/\mathbb{Q})$ acts on $(H_k/H_{k+1})\otimes\mathbb{Q}$ and that $((H_k/H_{k+1})\otimes\mathbb{Q})^{\mathbb{Z}/q^*}$ is a fixed point subspace.

Observe that the triple $(V,\xi_q^\alpha,\overrightarrow{01})$ has good reduction at every finite place of $\mathbb{Q}(\mu_q)$ not dividing q. Let

$$S := \{\mathfrak{p}\in\mathcal{V}(\mathbb{Q}(\mu_q)) \mid \mathfrak{p} \ \text{ divides } \ q \ \text{ or } \ \ell\}.$$

Then it follows from Proposition 2.3 that the representation ψ_t is unramified outside S.

Lemma 5.1.1. *We have:*

i) *Let* $\sigma\in((H_k/H_{k+1})\otimes\mathbb{Q})^{\mathbb{Z}/q^*}$. *The coordinates of* $\mathbf{\Psi}_t^{\mathbb{Z}/q^*}(\sigma)$ *calculated modulo* $I_2+\Gamma^{k+1}L(X,Y)$ *on each component are*

$$\left(\left(\frac{1-q^{k-1}}{(q-1)q^{k-1}}\,\ell_k(1)(\sigma)\,[Y,X^{k-1}]\right)_{0<\alpha<q};\ell_k(1)(\sigma)\,[Y,X^{k-1}]\right)$$

for $k>1$ *and odd;*

$$(\,(\,0\,)_{0<\alpha<q}\,;\,0\,)$$

for $k>1$ and even;

$$\left(\left(\frac{1}{q-1}\,\ell(q)(\sigma)\,Y\right)_{0<\alpha<q};0\right)$$

for $k=1$.

ii) *Let $\sigma\in(grLie\mathcal{U}(L,S,\ell)^{[\mathbb{Z}/q^*]})_k$. The coordinates of $[\mathbf{\Psi}_t]^{[\mathbb{Z}/q^*]}(\sigma)$ calculated modulo $I_2+\Gamma^{k+1}L(X,Y)$ on each component are given by the same formulas as in the point i) if we replace $\ell_k(1)$ and $\ell(q)$ by $[\ell_k(1)]$ and $[\ell(q)]$ respectively.*

Proof. Let $\sigma\in((H_k/H_{k+1})\otimes\mathbb{Q})^{\mathbb{Z}/q^*}$ and let $0<\alpha<q$. It follows from Lemma 5.0.1 that the α coordinate of $\mathbf{\Psi}_t^{\mathbb{Z}/q^*}(\sigma)$ is $\ell_k(\xi_q^\alpha)(\sigma)[Y,X^{k-1}]$ modulo $I_2+\Gamma^{k+1}L(X,Y)$. It follows from Corollary 4.3.4 or Lemma 5.0.2 that $\ell_k(\xi_q^\alpha)(\tilde{\tau}\cdot\sigma\cdot\tilde{\tau}^{-1})=\ell_k(\xi_q^{\tau^{-1}\cdot\alpha})(\sigma)$, where $\tilde{\tau}$ is a lifting of $\tau\in\mathbb{Z}/q^*$ to $G_{\mathbb{Q}(\mu_{\ell^\infty})}$. Hence we get that $\ell_k(\xi_q^\alpha)(\sigma)=\ell_k(\xi_q)(\sigma)$. The distribution relation for ℓ-adic polylogarithms (see [19] Corollaries 11.2.3 or [21] Theorems 2.1) and the vanishing of $\ell_{2n}(1)$ (see [19] Corollary 11.2.11) imply that for any $\sigma\in((H_k/H_{k+1})\otimes\mathbb{Q})^{\mathbb{Z}/q^*}$ and any $0<\alpha<q$ we have

$$\ell_k(\xi_q^\alpha)(\sigma)=\frac{1-q^{k-1}}{(q-1)q^{k-1}}\,\ell_k(1)(\sigma)\ \text{ for }\ k>1\ \text{ and }\ \text{odd},$$

$$\ell_k(\xi_q^\alpha)(\sigma)=0\ \text{ for }\ k>1\ \text{ and }\ \text{even},$$

$$\ell(1-\xi_q^\alpha)(\sigma)=\frac{1}{q-1}\,\ell(q)(\sigma)\ \text{ for }\ k=1.$$

This implies the first part of the lemma.

The functions $[\ell_k(\xi_q^\alpha)]$ (for $k>1$ and $0\leqq\alpha<q$) and $[\ell_1(\xi_q^\alpha)]$ (for $0<\alpha<q$) from $grLie\mathcal{U}(L,S,\ell)$ to $\mathbb{Q}_\ell$ vanish on the subspace of decomposable elements D_k of $grLie\mathcal{U}(L,S,\ell)_k$. The restriction of $[\ell_k(\xi_q^\alpha)]$ (for $0<\alpha<q$) to $W_k:=I_k^{\mathbb{Z}/q^*}$ is equal $\frac{1}{q-1}\sum_{i\in\mathbb{Z}/q^*}[\ell_k(\xi_q^{i\alpha})]=\frac{1}{q-1}\sum_{i=1}^{q-1}[\ell_k(\xi_q^i)]$. It follows from the inversion relation for ℓ-adic polylogarithms that the last expresion vanishes for k even.

For $k>1$ and odd it follows from the distribution relations that $\frac{1}{q-1}\sum_{i=1}^{q-1}[\ell_k(\xi_q^i)]=\frac{1-q^{k-1}}{(1-q)q^{k-1}}[\ell_k(1)]$.

Finally it follows from the equality $\prod_{i=1}^{q-1}(1-\xi_q^i)=q$ that $\frac{1}{q-1}\sum_{i=1}^{q-1}[\ell_1(\xi_q^i)]=\frac{1}{q-1}[\ell_1(q)]$.

The universal property of the weighted Tate completion implies that the morphism $grLie[\psi_t]$ factors through a surjective morphism of Lie algebras

$$grLie\mathcal{U}(L,S,\ell)\to\oplus_{k=1}^{\infty}(H_k/H_{k+1})\otimes\mathbb{Q}.$$

Hence the morphism $[\mathbf{\Psi}_t]^{[\mathbb{Z}/q^*]}$ factors through

$$grLie\mathcal{U}(L,S,\ell)^{[\mathbb{Z}/q^*]}\to\oplus_{k=1}^{\infty}\big((H_k/H_{k+1})\otimes\mathbb{Q}\big)^{\mathbb{Z}/q^*}.$$

This implies the second part of the lemma. □

PROPOSITION 5.1.2. *In the image of the morphism of Lie algebras*

$$[\mathbf{\Psi}_t]^{[\mathbb{Z}/q^*]}:grLie\mathcal{U}(L,S,\ell)^{[\mathbb{Z}/q^*]}\\ \to\left(\bigoplus_{0<\alpha<q}\mathrm{Lie}(X,Y)\right)\tilde{\times}\,(\mathrm{Lie}(X,Y)/\langle Y\rangle,\{\,\})$$

there are elements $D_1,D_3,D_5,\dots,D_{2n+1},\dots$ *homogeneous of degree* $1,3,5,\dots,2n+1,\dots$ *respectively whose coordinates calculated modulo* I_2 *on each component are*

$$\big(((1-q^{2n})\,[Y,X^{2n}])_{0<\alpha<q};(q-1)\,q^{2n}[Y,X^{2n}]\big)$$

for $k=2n+1$ *and greater than* 1 *and*

$$\big((Y)_{0<\alpha<q};0\big)$$

for $k=1$.

PROOF. The ℓ-adic polylogarithm $\ell_{2n+1}(1)=\ell_{2n+1}(\overrightarrow{10})$ is a rational non-zero multiple of the Soulé class – the generator of $H^1(G_{\mathbb{Q}};\mathbb{Q}_\ell(2n+1))$. Hence $\ell_{2n+1}(1)$ restricted to $H^1(G_L;\mathbb{Q}_\ell(2n+1))$ is also non-zero. Therefore $[\ell_{2n+1}(1)]$ is a non-zero homomorphism from $grLie\mathcal{U}(\mathbb{Q},\{\ell,q\},\ell)_{2n+1}$ to $\mathbb{Q}_\ell$ and hence also from $grLie\mathcal{U}(L,S,\ell)_{2n+1}$ to $\mathbb{Q}_\ell$. It follows from Proposition 4.2.8 ii) and iii) that $[\ell_{2n+1}(1)]$ restricted to $(grLie\mathcal{U}(L,S,\ell)^{[\mathbb{Z}/q^*]})_{2n+1}$ is non-zero. Therefore there is $\sigma_{2n+1} \in (grLie\mathcal{U}(L,S,\ell)^{[\mathbb{Z}/q^*]})_{2n+1}$ such that $[\ell_{2n+1}(1)](\sigma_{2n+1}) = 1$. We set

$$D_{2n+1} := (q-1)\, q^{2n}\, [\mathbf{\Psi}_t]^{[\mathbb{Z}/q^*]}\, (\sigma_{2n+1})$$

for $2n+1 > 1$.

Now we consider the case $m = 1$. The cohomology class of the cocycle $\ell(q)$ is one of generators of $H^1(\mathrm{Spec}\mathbb{Z}[\frac{1}{q}, \frac{1}{\ell}]; \mathbb{Q}_\ell(1))$. Therefore the homomorphism $[\ell(q)] : grLie\mathcal{U}(\mathbb{Q}, \{\ell, q\}, \ell)_1 \to \mathbb{Q}_\ell$ is non-zero. It follows from Proposition 4.2.8 ii) and iii) that the homomorphism $[\ell(q)] : grLie\mathcal{U}(L,S,\ell)_1 \to \mathbb{Q}_\ell$ restricted to $(grLie\mathcal{U}(L,S,\ell)^{[\mathbb{Z}/q^*]})_1$ is non-zero. Hence we can find $\sigma_1 \in (grLie\mathcal{U}(L,S,\ell)^{[\mathbb{Z}/q^*]})_1$ such that $[\ell(q)](\sigma_1) = 1$. Then we set

$$D_1 := (q-1)\, [\mathbf{\Psi}_t]^{[\mathbb{Z}/q^*]}\, (\sigma_1)\,.$$

It is clear from the construction and from Lemma 5.1.1 that the elements $D_1, D_3, D_5, \dots, D_{2n+1}, \dots$ have the required coordinates. □

PROPOSITION 5.1.3. *The Lie algebra* $\mathrm{Im}\,([\mathbf{\Psi}_t]^{[\mathbb{Z}/q^*]})$ *is free, freely generated by elements* $D_1, D_3, D_5, \dots, D_{2n+1}, \dots$.

PROOF. First we shall show that elements $D_1, D_3, D_5, \dots$ generate freely a free Lie subalgebra of the image of the morphism $[\mathbf{\Psi}_t]^{[\mathbb{Z}/q^*]}$. To show this it is sufficient to show that the basic Lie elements in $D_1, D_3, D_5, \dots$ are linearly independent over $\mathbb{Q}_\ell$.

We recall that a Lie bracket in the Lie algebra $\big(\bigoplus_{0<\alpha<q} \mathrm{Lie}(X,Y)\big)\tilde{\times}$ $(\mathrm{Lie}(X,Y)/\langle Y\rangle,\ \{\,\})$, which further we denote by $\mathcal{L}$, is given by the formula

$$\begin{aligned} & [((f_\alpha)_{0<\alpha<q};\varphi), ((g_\alpha)_{0<\alpha<q};\psi)] \\ = \ & (([f_\alpha, g_\alpha] + D_\varphi(g_\alpha) - D_\psi(f_\alpha))_{0<\alpha<q}; [\varphi,\psi] + D_\varphi(\psi) - D_\psi(\varphi))\,. \end{aligned}$$

Let us set

$$z_{2n+1} := (((1-q^{2n})\,[Y, X^{2n}])_{0<\alpha<q}; (q-1)\, q^{2n}[Y, X^{2n}])$$

for $n > 0$ and let

$$z_1 := ((Y)_{0<\alpha<q}; 0)\,.$$

We recall that $\mathcal{I}_r := (\bigoplus_{0<\alpha<q} I_r) \oplus I_r$ is a Lie ideal of the Lie algebra $\mathcal{L}$. For any Lie bracket of length r in elements $D_1, D_3, D_5, \ldots$ in the Lie algebra $\mathcal{L}$ we have

$$[\ldots[D_{i_1}, D_{i_2}], \ldots D_{i_r}] \equiv [\ldots[z_{i_1}, z_{i_2}], \ldots z_{i_r}] \bmod \mathcal{I}_r\,. \tag{5.1.4}$$

We recall that $\mathrm{Lie}\,(X, Y; \mathbb{Z})$ denote a free Lie algebra over $\mathbb{Z}$ freely generated by X and Y. The elements z_i have integer coefficients hence they belong to the semi-direct product $(\bigoplus_{0<\alpha<q} \mathrm{Lie}\,(X, Y; \mathbb{Z}))\tilde{\times}$ $(\mathrm{Lie}\,(X, Y; \mathbb{Z})/\langle Y\rangle, \{\,\})$ of Lie algebras over $\mathbb{Z}$, which we denote by $\mathcal{L}(\mathbb{Z})$. Observe that $q\mathcal{L}(\mathbb{Z})$ is a Lie ideal of the Lie algebra $\mathcal{L}(\mathbb{Z})$. The quotient Lie algebra $\mathcal{L}(\mathbb{Z})/q\mathcal{L}(\mathbb{Z})$ is $(\bigoplus_{0<\alpha<q} \mathrm{Lie}\,(X, Y; \mathbb{Z}/q))\tilde{\times}\,(\mathrm{Lie}\,(X, Y; \mathbb{Z}/q)/\langle Y\rangle, \{\,\})$. The classes modulo q, $\bar{z}_1 = ((Y)_{0<\alpha<q}; 0)$ and $\bar{z}_{2n+1} = (([Y, X^{2n}])_{0<\alpha<q}; 0)$ for $n > 0$, generate freely a free Lie subalgebra of the Lie algebra $(\bigoplus_{0<\alpha<q} \mathrm{Lie}\,(X, Y; \mathbb{Z}/q))\tilde{\times}(\mathrm{Lie}\,(X, Y; \mathbb{Z}/q)/\langle Y\rangle, \{\,\})$. This follows from Shirshow-Witt Theorem applied to a free Lie algebra $\mathrm{Lie}\,(X, Y; \mathbb{Z}/q)$ over a field $\mathbb{Z}/q$ (see [10] page 331).

Hence basic Lie elements of length r in $\bar{z}_1, \bar{z}_3, \bar{z}_5, \ldots$ are linearly independent over $\mathbb{Z}/q$. This implies that basic Lie elements of length r in $z_1, z_3, z_5, \ldots$ are linearly independent over $\mathbb{Z}$, hence also over $\mathbb{Q}$ and $\mathbb{Q}_\ell$. It follows from the congruence (5.1.4) that basic Lie elements of length r in $D_1, D_3, D_5, \ldots$ are linearly independent.

Assume that a non-zero linear combination of basic Lie elements of length r in $D_1, D_3, D_5, \ldots$ is equal to a linear combination of basic Lie elements of length greater than r. Working modulo the Lie ideal $\mathcal{I}_r$ one sees immediately that this is impossible. Therefore basic Lie elements in $D_1, D_3, D_5, \ldots$ are linearly independent, hence the elements $D_1, D_3, D_5, \ldots$ generate freely a free Lie subalgebra of $\mathrm{Im}\,([\mathbf{\Psi}_t]^{[\mathbb{Z}/q^*]})$.

It follows from results of section 4 (Proposition 4.2.6) that the Lie algebra $grLie\mathcal{U}(\mathbb{Q}(\mu_q), S, \ell)^{[\mathbb{Z}/q^*]}$ is free, freely generated by elements $\sigma_3, \sigma_5, \ldots,$ $\sigma_{2n+1}, \ldots$ constructed in the proof of Proposition 5.1.2 and by two elements in degree 1. One of these elements is σ_1 and the other is mapped to zero by $[\mathbf{\Psi}_t]^{[\mathbb{Z}/q^*]}$. This implies that the elements $D_1, D_3, D_5, \ldots, D_{2n+1}, \ldots$ generate $\mathrm{Im}\,([\mathbf{\Psi}_t]^{[\mathbb{Z}/q^*]})$. $\Box$

THEOREM 5.1.5. *Let q be a prime number different from ℓ. Let $V :=$*

$\mathbb{P}^1 \setminus \{0,1,\infty\}$ *and let*

$$t := \coprod_{0<\alpha<q} \pi(V_{\mathbb{Q}}; \xi_q^\alpha, \overrightarrow{01}).$$

The representation

$$\theta_t^{\mathbb{Q}(\mu_q)/\mathbb{Q}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\mu_{\ell^\infty})) \longrightarrow \bigoplus_{0<\alpha<q} \mathrm{GL}(\mathbb{Q}_\ell\{\{X,Y\}\})$$

has the following properties:

i) *it is unramified outside finite places of* $\mathbb{Q}(\mu_{\ell^\infty})$ *lying over* ℓ *or* q*;*

ii) *it respects the filtration* $\{\bigoplus_{0<\alpha<q} I(\mathbb{Q}_\ell\{\{X,Y\}\})^i\}_{i\in\mathbb{N}}$ *of* $\bigoplus_{0<\alpha<q} \mathbb{Q}_\ell\{\{X,Y\}\}$*;*

iii) *any* $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q}(\mu_{\ell^\infty}))$ *acts on* $\bigoplus_{0<\alpha<q} \big(I(\mathbb{Q}_\ell\{\{X,Y\}\})^i/ I(\mathbb{Q}_\ell\{\{X,Y\}\})^{i+1}\big)$ *as the identity;*

iv) *the image of the associated graded Lie algebra representation*

$$\begin{aligned}\mathbf{\Theta}_t^{\mathrm{Gal}(\mathbb{Q}(\mu_q)/\mathbb{Q})} : \bigoplus_{i=1}^{\infty} \big(F_i(\theta_t^{\mathbb{Q}(\mu_q)/\mathbb{Q}})/F_{i+1}(\theta_t^{\mathbb{Q}(\mu_q)/\mathbb{Q}})\big) \otimes \mathbb{Q} \\ \longrightarrow \bigoplus_{0<\alpha<q} \mathrm{End}(\mathbb{Q}_\ell\{\{X,Y\}\})\end{aligned}$$

is a free Lie algebra, freely generated by D_1, D_3, $D_5, \dots, D_{2n+1}, \dots$.

Proof. The representation ψ_t is unramified outside S, where $S = \{\mathfrak{p} \in \mathcal{V}(\mathbb{Q}(\mu_q)) \mid \mathfrak{p}\,\text{divides}\ q \text{ or } \mathfrak{p} \text{ divides } \ell\,\}$. Let us set $T := \{q,\,\ell\}$. Hence the representation $\theta_t^{\mathbb{Q}(\mu_q)/\mathbb{Q}}$, which is equal by definition $[\psi_t] \circ S_{K,T}^{L,S} \circ i_{K,T,\ell}$, is unramified outside finite places of $\mathbb{Q}(\mu_{\ell^\infty})$ lying over ℓ or q.

The points ii) and iii) follow immediately from Proposition 4.3.0.

By Proposition 5.1.3 the image of $[\mathbf{\Psi}_t]^{[\mathbb{Z}/q^*]}$ is a free Lie algebra, freely generated by D_1, D_3, $D_5, \dots, D_{2n+1}, \dots$. This implies that the image of the composition $[\mathbf{\Psi}_t]^{[\mathbb{Z}/q^*]} \circ \beta^{-1} \circ (\bar{p}_{\mathbb{Z}/q^*})^{-1}$, and hence also the image of

$grLie[\theta_t^{\mathbb{Q}(\mu_q)/\mathbb{Q}}]$, is a free Lie algebra, freely generated by D_1, D_3, $D_5, \dots$, $D_{2n+1}, \dots$. The morphism

$$grLie[\theta_t^{\mathbb{Q}(\mu_q)/\mathbb{Q}}] : grLie\mathcal{U}(K,T,\ell) \longrightarrow \bigoplus_{0<\alpha<q} \mathrm{GL}(\mathbb{Q}_\ell\{\{X,Y\}\})$$

factors through a surjective morphism

$$grLie\mathcal{U}(K,T,\ell) \longrightarrow \bigoplus_{i=1}^{\infty}(F_i(\theta_t^{\mathbb{Q}(\mu_q)/\mathbb{Q}})/F_{i+1}(\theta_t^{\mathbb{Q}(\mu_q)/\mathbb{Q}})) \otimes \mathbb{Q}.$$

Hence the image of $\boldsymbol{\Theta}_t^{\mathbb{Z}/q^*} = grLie\theta_t^{\mathbb{Q}(\mu_q)/\mathbb{Q}}$ is a free Lie algebra, freely generated by D_1, D_3, $D_5, \dots, D_{2n+1}, \dots$. $\square$

5.2. We assume as in section 5.1 that q is a prime number different from ℓ and that ℓ does not divide $q-1$.

We assume also that $q \equiv 3 \mod 4$. Then the field $\mathbb{Q}(i\sqrt{q})$ is a subfield of $\mathbb{Q}(\mu_q)$. The field $\mathbb{Q}(i\sqrt{q})$ is fixed by the kernel of the homomorphism

$$h_q : \mathbb{Z}/q^* \to \{1,-1\}$$

given by $h_q(x) = x^{\frac{q-1}{2}}$. Let us set

$$G(q) := \ker h_q \, .$$

As in section 5.1 we are studying actions of Galois groups on

$$t := \coprod_{0<\alpha<q} \pi(V_{\bar{\mathbb{Q}}}; \xi_q^\alpha, \overrightarrow{01}) \quad \text{and} \quad T := \prod_{0<\alpha<q} \pi(V_{\bar{\mathbb{Q}}}; \xi_q^\alpha, \overrightarrow{01}))\, .$$

We shall apply formalism from section 4 for

$$K = \mathbb{Q}(i\sqrt{q}), \;\; L = \mathbb{Q}(\mu_q)$$
$$\text{and the Galois group } G(q) = \mathrm{Gal}(\mathbb{Q}(\mu_q)/\mathbb{Q}(i\sqrt{q})).$$

We recall that $S = \{\, \mathfrak{p} \in \mathcal{V}(\mathbb{Q}(\mu_q)) \mid \mathfrak{p} \text{ divides } q \text{ or } \ell \,\}$ and therefore in the case considered now

$$T := \{\, \mathfrak{p} \in \mathcal{V}(\mathbb{Q}(i\sqrt{q})) \mid \mathfrak{p} \text{ divides } q \text{ or } \ell \,\}.$$

We shall construct geometrically ℓ-adic realization of the associated graded Lie algebra of the fundamental group of the tannakien category of the mixed Tate motives over $\operatorname{Spec} \mathcal{O}_{\mathbb{Q}(i\sqrt{q})[\frac{1}{q}]}$.

LEMMA 5.2.1. *We have:*

i) *Let $\sigma \in ((H_k/H_{k+1}) \otimes \mathbb{Q})^{G(q)}$. The coordinates of $\mathbf{\Psi}_t^{G(q)}(\sigma)$ calculated modulo $I_2 + \Gamma^{k+1} L(X,Y)$ on each component are*

$$\left(\left(\frac{1-q^{k-1}}{(q-1)q^{k-1}}\ \ell_k(1)(\sigma)[Y, X^{k-1}]\right)_{0<\alpha<q}; \ell_k(1)(\sigma)[Y, X^{k-1}]\right)$$

for $k > 1$ and odd;

$$\left(\left(\left(\frac{2}{q-1}\sum_{j\in G(q)} \ell_k(\xi_q^j)(\sigma)\right)[Y, X^{k-1}]\right)_{\alpha\in G(q)},\right.$$

$$\left.\left(\left(-\frac{2}{q-1}\sum_{j\in G(q)} \ell_k(\xi_q^j)(\sigma)\right)[Y, X^{k-1}]\right)_{\alpha\notin G(q)}; 0\right)$$

for $k > 1$ and even;

$$\left(\left(\left(\frac{1}{q-1}\sum_{0<j<q} \ell(1-\xi_q^j)(\sigma)\right) Y\right)_{0<\alpha<q}; 0\right)$$

for $k = 1$.

ii) *Let $\sigma \in (grLie\mathcal{U}(L, S, \ell))_k^{[G(q)]}$. The coordinates of $[\mathbf{\Psi}_t]^{[G(q)]}(\sigma)$ calculated modulo $I_2 + \Gamma^{k+1}L(X,Y)$ on each component are given by the same formula as in point i) if we replace $\ell_k(1)$, $\ell_k(\xi_q^j)$ and $\ell(1-\xi_q^j)$ by $[\ell_k(1)]$, $[\ell_k(\xi_q^j)]$ and $[\ell(1-\xi_q^j)]$ respectively.*

PROOF. The lemma is proved in the same way as Lemma 5.1.1. □

PROPOSITION 5.2.2. *Let us assume that Conjecture 4.3.1 holds. Let q be a prime number satisfying $q \equiv 3 \mod 4$. In the image of the morphism* *of Lie algebras*

$$[\mathbf{\Psi}_t]^{[G(q)]} : grLie\mathcal{U}(L,S,\ell)^{[G(q)]} \\ \to \left(\bigoplus_{0<\alpha<q} \mathrm{Lie}(X,Y)\right) \tilde{\times} (\mathrm{Lie}(X,Y)/\langle Y\rangle, \{\,\})$$

there are elements $D_1, D_2, D_3, \dots, D_m, \dots$ *homogeneous of degree* $1, 2, 3, \dots, m, \dots$. *respectively whose coordinates calculated modulo* $\mathcal{I}_2$ *are*

$$\big(((1-q^{m-1})[Y,X^{m-1}])_{0<\alpha<q}; (q-1)\, q^{m-1}\, [Y,X^{m-1}]\big)$$

for $m>1$ *and odd;*

$$\big((([Y,X^{m-1}])_{\alpha\in G(q)}, (-[Y,X^{m-1}])_{\alpha\notin G(q)}); 0\big)$$

for $m>1$ *and even;*

$$((Y)_{0<\alpha<q}; 0)$$

for $m=1$.

Proof. Observe that for all $m\in\mathbb{N}$ we have $\dim_{\mathbb{Q}_\ell} V_m(q)^{G(q)} = 1$. The space $V_m(q)^{G(q)}$ is generated by $[0]$ if $m>1$ and odd, by $\sum\limits_{\alpha\in G(q)} [\alpha]$ if $m>1$ and even and by $\sum\limits_{\alpha\in\mathbb{Z}/q^*} [\alpha]$ if $m=1$.

Let us assume that k is even. It follows from Lemma 5.0.5 that $\sum_{j\in G(q)} \ell_k(\xi_q^j)$ is a non-zero element of $H^1(G_{\mathbb{Q}(\mu_q)}; \mathbb{Q}_\ell(k))^{G(q)}$. Hence there is a unique, non-zero element t_k of $H^1(G_{\mathbb{Q}(i\sqrt{q})}; \mathbb{Q}_\ell(k))$ such that t_k restricted to $H^1(G_{\mathbb{Q}(\mu_q)}; \mathbb{Q}_\ell(k))$ is equal $\sum_{j\in G(q)} \ell_k(\xi_q^j)$. The homomorphism $[t_k]$ from $grLie\mathcal{U}(\mathbb{Q}(i\sqrt{q}), T, \ell)_k$ to $\mathbb{Q}_\ell$ is non-zero because t_k is non-zero. Therefore it follows from Proposition 4.2.8 ii) and iii) that $[t_k]$ restricted to $(grLie\mathcal{U}(L,S,\ell)^{[G(q)]})_k$ is non-zero. Hence the restriction of $\sum_{j\in G(q)}[\ell_k(\xi_q^j)]$ to $(grLie\mathcal{U}(L,S,\ell)^{[G(q)]})_k$ is non-zero. Hence there is $\sigma_k \in grLie\mathcal{U}(L,S,\ell)_k^{[G(q)]}$ such that $\sum_{j\in G(q)}[\ell_k(\xi_q^j)](\sigma_k) = 1$. We set

$$D_k := \frac{q-1}{2}[\mathbf{\Psi}_t]^{[G(q)]}(\sigma_k)$$

for $k>0$ and even.

Let us assume that $k = 1$. It follows from Lemma 5.0.5 that the element $\sum_{0<j<q} \ell(1-\xi_q^j)$ is a non-zero element of $H^1(\operatorname{Spec}\mathcal{O}_{\mathbb{Q}(\mu_q),S};\mathbb{Q}_\ell(1))^{G(q)}$. Observe that $\sum_{0<j<q} \ell(1-\xi_q^j)$ and $2\ell(i\sqrt{q})$ define the same cohomology class in $H^1(G_{\mathbb{Q}(\mu_q)};\mathbb{Q}_\ell(1))$. The element $2\ell(i\sqrt{q})$ is non-zero in $H^1(G_{\mathbb{Q}(i\sqrt{q})};\mathbb{Q}_\ell(1))$. In the same way as before, it follows from Proposition 4.2.8 that $\sum_{0<j<q}[\ell(1-\xi_q^j)]$ restricted to $(grLie\mathcal{U}(L,S,\ell)^{[G(q)]})_1$ is non-zero. Hence there is $\sigma_1 \in (grLie\mathcal{U}(L,S,\ell)^{[G(q)]})_1$ such that $\sum_{0<j<q}[\ell(1-\xi_q^j)](\sigma_1) = 1$. We set

$$D_1 := (q-1)[\mathbf{\Psi}_t]^{[G(q)]}(\sigma_1).$$

If $k > 1$ and odd we can find $\sigma_k \in grLie\mathcal{U}(L,S,\ell)_k^{[G(q)]}$ such that $[\ell_k(1)](\sigma_k) = 1$. Then we set

$$D_k := \frac{(q-1)q^k}{1-q^{k-1}}[\mathbf{\Psi}_t]^{[G(q)]}(\sigma_k).$$

One verifies easily that the constructed elements have required properties. □

Proposition 5.2.3. *The Lie algebra* $\operatorname{Im}([\mathbf{\Psi}_t]^{[G(q)]})$ *is free, freely generated by elements* $D_1, D_2, D_3, \ldots, D_m, \ldots$.

Proof. The proof of the proposition repeats arguments of the proof of Proposition 5.1.3 and we omit it. □

Theorem 5.2.4. *Let us assume that Conjecture 4.3.1 holds. Let q be a prime number different from ℓ and congruent to* 3 *modulo* 4. *We assume also that ℓ does not divide $q-1$. Let $V := \mathbb{P}^1 \setminus \{0,1,\infty\}$ and let*

$$t := \coprod_{0<\alpha<q} \pi(V_{\bar{\mathbb{Q}}};\xi_q^\alpha,\overrightarrow{01}).$$

The representation

$$\theta_t^{\mathbb{Q}(\mu_q)/\mathbb{Q}(i\sqrt{q})} : \operatorname{Gal}(\overline{\mathbb{Q}(i\sqrt{q})}/\mathbb{Q}(i\sqrt{q})(\mu_{\ell^\infty})) \to \bigoplus_{0<\alpha<q} \operatorname{GL}(\mathbb{Q}_\ell\{\{X,Y\}\})$$

has the following properties:

i) *it is unramified outside finite places of* $\mathbb{Q}(i\sqrt{q})(\mu_{\ell^\infty})$ *dividing ℓ or q,*

ii) *the representation* $\theta_t^{\mathbb{Q}(\mu_q)/\mathbb{Q}(i\sqrt{q})}$ *respects the filtration* $\{\bigoplus_{0<\alpha<q} I(\mathbb{Q}_\ell\{\{X,Y\}\})^i\}_{i\in\mathbb{N}}$ *of* $\bigoplus_{0<\alpha<q}\mathbb{Q}_\ell\{\{X,Y\}\}$;

iii) *any* $\sigma\ \in Gal(\bar{\mathbb{Q}}/\mathbb{Q}(i\sqrt{q})(\mu_{\ell^\infty}))$ *acts on* $\bigoplus_{0<\alpha<q}\big(I(\mathbb{Q}_\ell\{\{X,Y\}\})^i/I(\mathbb{Q}_\ell\{\{X,Y\}\})^{i+1}\big)$ *as the identity;*

iv) *the image of the associated graded Lie algebra representation*

$$\mathbf{\Theta}_t^{Gal(\mathbb{Q}(\mu_q)/\mathbb{Q})} : \bigoplus_{i=1}^{\infty}\big(F_i(\theta_t^{\mathbb{Q}(\mu_q)/\mathbb{Q}(i\sqrt{q})})/F_{i+1}(\theta_t^{\mathbb{Q}(\mu_q)/\mathbb{Q}(i\sqrt{q})})\big) \longrightarrow \bigoplus_{0<\alpha<q}\mathrm{End}\,(\mathbb{Q}_\ell\{\{X,Y\}\})$$

is a free Lie algebra, freely generated by elements $D_1, D_2, D_3, \dots, D_m, \dots$.

Proof. The proof of the theorem is the same as the proof of Theorem 5.1.5. □

The ℓ-adic representation from Theorem 5.2.4 represents geometrically the ℓ-adic realization (for $\ell\neq q$) of the associated graded Lie algebra of the fundamental group of the tannakien category of the mixed Tate motives over $\mathrm{Spec}\,\mathcal{O}_{\mathbb{Q}(i\sqrt{q})}[\frac{1}{q}]$.

Now we shall construct geometrically an ℓ-adic realization of the fundamental group of the tannakien category of mixed Tate motives over $\mathrm{Spec}\,\mathcal{O}_{\mathbb{Q}(i\sqrt{q})}$.

Let us set

$$\tau := \coprod_{0<j<q}\pi(V_{\bar{\mathbb{Q}}}; -\xi_q^j, \overrightarrow{01}) \quad\text{and}\quad T := \prod_{0<j<q}\pi(V_{\bar{\mathbb{Q}}}; -\xi_q^j, \overrightarrow{01}).$$

Observe that the triple $(V_{\mathbb{Q}(i\sqrt{q})}; -\xi_q^j, \overrightarrow{01})$ has good reduction at every finite place of $\mathbb{Q}(\mu_q)$. Therefore we set

$$S := \{\mathfrak{p}\in\mathcal{V}(\mathbb{Q}(\mu_q)) \mid \mathfrak{p}\ \text{ divides }\ \ell\}\ \text{ and}$$
$$T := \{\mathfrak{p}\in\mathcal{V}(\mathbb{Q}(i\sqrt{q})) \mid \mathfrak{p}\ \text{ divides }\ \ell\}.$$

We shall apply formalism from section 4 to

$$K = \mathbb{Q}(i\sqrt{q}),\ \ L = \mathbb{Q}(\mu_q)\ \text{ and }\ G(q) = \mathrm{Gal}\,(\mathbb{Q}(\mu_q)/\mathbb{Q}(i\sqrt{q})).$$

Lemma 5.2.7. *We have*

i) *Let* $\sigma \in ((H_k/H_{k+1})\otimes\mathbb{Q})^{G(q)}$. *The coordinates of* $\mathbf{\Psi}_\tau^{G(q)}(\sigma)$ *calculated modulo* $I_2+\Gamma^{k+1}L(X,Y)$ *on each component are*

$$\Big(\big(\frac{1-q^{k-1}-2^{k-1}+2^{k-1}\cdot q^{k-1}}{(q-1)\cdot 2^{k-1}\cdot q^{k-1}}\ell_k(1)(\sigma)[Y,X^{k-1}]\big)_{0<\alpha<q};$$
$$\ell_k(1)(\sigma)[Y,X^{k-1}]\Big)$$

for $k>1$ *and odd;*

$$\Big(\big(\frac{2}{q-1}\big(\sum_{j\in G(q)}\ell_k(-\xi_q^j)(\sigma)\big)[Y,X^{k-1}]\big)_{\alpha\in G(q)},$$
$$\big(-\frac{2}{q-1}\big(\sum_{j\in G(q)}\ell_k(-\xi_q^j)(\sigma)\big)[Y,X^{k-1}]\big)_{\alpha\notin G(q)};\ 0\Big)$$

for $k>1$ *and even;*

$$(\,(\,0\,)_{0<\alpha<q};\,0\,)$$

for $k=1$.

ii) *Let* $\sigma \in (grLie\mathcal{U}(L,S,\ell))_k^{[G(q)]}$. *The coordinates of* $[\mathbf{\Psi}_t]^{[G(q)]}(\sigma)$ *calculated modulo* $I_2+\Gamma^{k+1}L(X,Y)$ *on each component are given by the same formula as in point i) if we replace* $\ell_k(1)$ *and* $\ell_k(-\xi_q^j)$ *by* $[\ell_k(1)]$ *and* $[\ell_k(-\xi_q^j)]$ *respectively.*

Proof. First we show the point i) of the lemma. Observe that the set $\{-\xi_q^j\}_{0<j<q}$ is the set of primitive $2q$-th roots of 1. Hence we can apply formalism from section 5.0. In particular we can apply Lemma 5.0.1 to calculate coordinates of $\mathbf{\Psi}_\tau(\sigma)$.

First we suppose that k is odd. We have the following relations between ℓ-adic polylogarithms

$$2^{k-1}(\ell_k(-\xi_q^j)+\ell_k(\xi_q^j))=\ell_k(\xi_q^{2j});$$

$$q^{k-1}(\sum_{j=1}^{q-1}\ell_k(\xi_q^j))=(1-q^{k-1})\ell_k(1);$$

$$\ell_k(\xi_q^j) = \ell_k(\xi_q^{-j}) \quad \text{and} \quad \ell_k(-\xi_q^j) = \ell_k(-\xi_q^{-j}).$$

Observe that $(\mathbb{Z}/2q)^* = \mathbb{Z}/q^*$, hence $G(q)$ acts on $V_k(2q)$.

The symbol $\frac{1}{q-1}\sum_{j=1}^{q-1}[q+2j] \in V_k(2q)$ is invariant under the action of $G(q)$. Observe that $s_k^{2q}\left(\frac{1}{q-1}\sum_{j=1}^{q-1}[q+2j]\right) = \frac{1}{q-1}\sum_{j=1}^{q-1}\ell_k(-\xi_q^j)$. Using the relations between ℓ-adic polylogarithms we get that

$$\frac{1}{q-1}\sum_{j=1}^{q-1}\ell_k(-\xi_q^j) = \frac{1-q^{k-1}-2^{k-1}+2^{k-1}\cdot q^{k-1}}{(q-1)\cdot 2^{k-1}\cdot q^{k-1}}\ell_k(1).$$

Then the formula for k odd follows from Corollary 4.3.4 or Lemma 5.0.2.

Now we suppose that k is even. We recall that for k even

$$\ell_k(-\xi_q^j) = -\ell_k(-\xi_q^{-j}) \quad \text{and} \quad \ell_k(1) = 0.$$

We have also that

$$\mathbb{Z}/q^* = G(q) \cup (-1)G(q).$$

The symbols $\frac{2}{q-1}\sum_{j\in G(q)} j([1])$ and $\frac{2}{q-1}\sum_{j\in G(q)} j([-1])$ are non-trivial in $V_k(2q)$ and invariant under the action of $G(q)$. Moreover we have

$$\frac{2}{q-1}\sum_{j\in G(q)} j([1]) + \frac{2}{q-1}\sum_{j\in G(q)} j([-1]) = 0$$

in $V_k(2q)$. This implies the formula for k even.

Now we assume that $k=1$. We have

$$(1+\xi_q^j) = \xi_q^j\cdot(1+\xi_q^{-j}) \ \text{ and } \ \prod_{j=1}^{q-1}(1+\xi_q^j) = 1.$$

Therefore the formula for $k=1$ follows from Corollary 4.3.4. One can also use the fact that $V_1(2q)^{G(q)} = 0$.

The point ii) of the lemma is proved in the same way as the point ii) of Lemma 5.1.1. □

Let

$$\alpha(k) := \mathbf{v}_q(1-2^{k-1})$$

be the exponent of the highest power of q which divides $1-2^{k-1}$. It is clear that $\alpha(k) < k-1$ and we have

$$1-2^{k-1} = a(k)\cdot q^{\alpha(k)},$$

where $a(k)$ is not divisible by q.

PROPOSITION 5.2.8. *Let us assume that Conjecture 4.3.1 holds. Let q be a prime number congruent to* 3 *modulo* 4. *In the image of the morphism of Lie algebras*

$$[\mathbf{\Psi}_\tau]^{[G(q)]} : grLie\mathcal{U}(L,S,\ell)^{[G(q)]} \\ \to \left(\bigoplus_{0<\alpha<q} \mathrm{Lie}(X,Y)\right) \tilde{\times} (\mathrm{Lie}(X,Y)/\langle Y\rangle, \{\,\})$$

there are elements $D_2, D_3, \ldots, D_k, \ldots$ homogeneous of degree $2,3,\ldots,k,\ldots$ respectively whose coordinates calculated modulo $\mathcal{I}_2$ are

$$\Big((a(k)\cdot(1-q^{k-1})[Y,X^{k-1}])_{0<\alpha<q};(q-1)\cdot 2^{k-1}\cdot q^{k-1-\alpha(k)}\,[Y,X^{k-1}]\Big)$$

for $k>1$ and odd;

$$\big((([Y,X^{k-1}])_{\alpha\in G(q)},(-[Y,X^{k-1}])_{\alpha\notin G(q)});0\big)$$

for $k>1$ and even.

PROOF. First we consider the case when k is odd. The ℓ-adic polylogarithm $\ell_k(1)$ is a generator of $H^1(G_{\mathbb{Q}};\mathbb{Q}_\ell(k))$. Hence $\ell_k(1)$ restricted to $H^1(G_L;\mathbb{Q}_\ell(k))$ is non-zero. Therefore $[\ell_k(1)]$ is a non-zero, $G(q)$-equivariant homomorphism from $grLie\mathcal{U}(L,S,\ell)_k$ to $\mathbb{Q}_\ell$ vanishing on decomposable elements. This implies that the restriction of $[\ell_k(1)]$ to $grLie\mathcal{U}(L,S,\ell)_k^{[G(q)]}$ is non-zero. Hence there is $\sigma_k \in grLie\mathcal{U}(L,S,\ell)_k^{[G(q)]}$ such that $[\ell_k(1)](\sigma_k)\neq 0$. We set

$$D_k := [\mathbf{\Psi}_\tau]^{[G(q)]}\left(\frac{(q-1)\cdot 2^{k-1}\cdot q^{k-1-\alpha(k)}}{\ell_k(1)(\sigma_k)}\cdot\sigma_k\right).$$

Let us assume that k is even. The symbol $\sum_{j\in G(q)} j([1])$ is non-trivial in $V_k(2q)$ and belongs to $V_k(2q)^{G(q)}$. Hence the symbol $\sum_{j\in G(q)} j([1])$ is a generator of $V_k(2q)^{G(q)}$. Hence it follows from Lemma 5.0.5 that $\sum_{j\in G(q)} \ell_k(-\xi_q^j)$

is a generator of $H^1(G_{\mathbb{Q}(\mu_q)};\mathbb{Q}_\ell(k))^{G(q)}$. Therefore the restriction of $\sum_{j\in G(q)}[\ell_k(-\xi_q^j)]$ to $grLie\mathcal{U}(L,S,\ell)_k^{[G(q)]}$ is non-zero. Hence there is $\sigma_k \in grLie\mathcal{U}(L,S,\ell)_k^{[G(q)]}$ such that $\sum_{j\in G(q)}[\ell_k(-\xi_q^j)](\sigma_k)\neq 0$. We set

$$D_k := [\mathbf{\Psi}_\tau]^{[G(q)]}\left(\frac{q-1}{2}\cdot\frac{1}{\sum_{j\in G(q)}\ell_k(-\xi_q^j)(\sigma_k)}\cdot\sigma_k\right).$$

It is clear that the constructed elements $D_2, D_3, D_4, \dots, D_n, \dots$ have the required properties. □

PROPOSITION 5.2.9. *The Lie algebra Im* $([\mathbf{\Psi}_\tau]^{[G(q)]})$ *is free, freely generated by the elements* $D_2,\ D_3, \dots, D_m, \dots$.

PROOF. One only need to notice that $\alpha(k) < k-1$. We left the details, which are the same as in the proof of Proposition 5.1.3, to readers. □

THEOREM 5.2.10. *Let us assume that Conjecture 4.3.1 holds. Let q be a prime number different from ℓ and congruent to* 3 *modulo* 4. *We assume that ℓ does not divide $q-1$. Let $V := \mathbb{P}^1\setminus\{0,1,\infty\}$ and let*

$$\tau := \coprod_{0<j<q}\pi(V_{\bar{\mathbb{Q}}}; -\xi_q^j, \overrightarrow{01}).$$

The representation

$$\theta_\tau^{\mathbb{Q}(\mu_q)/\mathbb{Q}(i\sqrt{q})} : \mathrm{Gal}\,(\overline{\mathbb{Q}(i\sqrt{q})}/\mathbb{Q}(i\sqrt{q})(\mu_{\ell^\infty})) \to \bigoplus_{0<\alpha<q}\mathrm{GL}\,(\mathbb{Q}_\ell\{\{X,Y\}\})$$

has the following properties:

i) *it is unramified outside finite places of* $\mathbb{Q}(i\sqrt{q})(\mu_{\ell^\infty})$ *lying over* ℓ*;*

ii) *the representation* $\theta_\tau^{\mathbb{Q}(\mu_q)/\mathbb{Q}(i\sqrt{q})}$ *respects the filtration* $\{\bigoplus_{0<\alpha<q} I(\mathbb{Q}_\ell\{\{X,Y\}\})^i\}_{i\in\mathbb{N}}$ *of* $\bigoplus_{0<\alpha<q}\mathbb{Q}_\ell\{\{X,Y\}\}$*;*

iii) *any* $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q}(i\sqrt{q})(\mu_{\ell^\infty}))$ *acts on* $\bigoplus_{0<\alpha<q}\big(I(\mathbb{Q}_\ell\{\{X,Y\}\})^i/ I(\mathbb{Q}_\ell\{\{X,Y\}\})^{i+1}\big)$ *as the identity;*

iv) *the image of the associated graded Lie algebra representation*

$$\mathbf{\Theta}_\tau^{Gal(\mathbb{Q}(\mu_q)/\mathbb{Q}(i\sqrt{q}))} : \bigoplus_{i=1}^{\infty} \big(F_i(\theta_\tau^{\mathbb{Q}(\mu_q)/\mathbb{Q}(i\sqrt{q})})/F_{i+1}(\theta_\tau^{\mathbb{Q}(\mu_q)/\mathbb{Q}(i\sqrt{q})})\big) \longrightarrow \bigoplus_{0<\alpha<q} \mathrm{End}\,(\mathbb{Q}_\ell\{\{X,Y\}\})$$

is a free Lie algebra, freely generated by elements $D_2, D_3, \ldots, D_m, \ldots$.

Proof. We omit the detailed proof as it is the same as the proofs of Theorems 5.1.5 and 5.2.4. We only notice that the Galois representation

$$G_{\mathbb{Q}(i\sqrt{q})(\mu_{\ell^\infty})} \to \mathrm{Aut}_{\,\mathrm{set}}(\tau)$$

is unramified outside finite places of $\mathbb{Q}(i\sqrt{q})(\mu_{\ell^\infty})$ lying over ℓ. This implies that there is no generator in degree 1. $\square$

5.3. We assume that q is a prime number different from ℓ and that ℓ does not divide $q-1$. We assume also that $q \equiv 1 \bmod 4$.

We shall construct geometrically ℓ-adic realization of the associated graded Lie algebra of the fundamental group of the tannakien category of the mixed Tate motives over $\mathrm{Spec}\,\mathcal{O}_{\mathbb{Q}(i\sqrt{q})}$.

Observe that the Galois group

$$\mathrm{Gal}\,(\mathbb{Q}(\mu_{4q})/\mathbb{Q}) = \mathrm{Gal}\,(\mathbb{Q}(i)/\mathbb{Q}) \times \mathrm{Gal}\,(\mathbb{Q}(\mu_q)/\mathbb{Q}) = \mathbb{Z}/4^* \times \mathbb{Z}/q^*.$$

Let $s \in \mathrm{Gal}\,(\mathbb{Q}(\mu_q)/\mathbb{Q}) = \mathbb{Z}/q^*$ be a generator and let $c \in \mathrm{Gal}\,(\mathbb{Q}(i)/\mathbb{Q}) = \mathbb{Z}/4^*$ be the complex conjugation.

We denote by $H(q)$ the subgroup of $\mathrm{Gal}\,(\mathbb{Q}(\mu_{4q})/\mathbb{Q})$ generated by $(c,s) \in \mathbb{Z}/4^* \times \mathbb{Z}/q^*$. Then the subfield of $\mathbb{Q}(\mu_{4q})$ fixed by $H(q)$,

$$\mathbb{Q}(\mu_{4q})^{H(q)} = \mathbb{Q}(i\sqrt{q}).$$

We shall study the action of $G_{\mathbb{Q}(\mu_{4q})}$ on the disjoint union of torsors of paths

$$\mathfrak{t} := \coprod_{\substack{0<j<4q \\ (j,4q)=1}} \pi(V_{\overline{\mathbb{Q}}}; \xi_{4q}^j, \overrightarrow{01}).$$

We set

$$L = \mathbb{Q}(\mu_{4q}) \ \text{ and } \ K = \mathbb{Q}(i\sqrt{q}).$$

Observe that a triple $(V_{\mathbb{Q}(\mu_{4q})}; \xi_{4q}^{j}, \overrightarrow{01})$ (for $0 < j < 4q$ and $(j, 4q) = 1$) has good reduction everywhere, hence we set

$$S = \{\, \mathfrak{p} \in \mathcal{V}(\mathbb{Q}(\mu_{4q})) \ \mid \ \mathfrak{p} \ \text{ divides } \ \ell \,\}.$$

LEMMA 5.3.1. *We have*

i) *Let $\sigma \in (H_n/H_{n+1}) \otimes \mathbb{Q}^{H(q)}$. The coordinates of $\mathbf{\Psi}_{\mathfrak{t}}^{H(q)}(\sigma)$ calculated modulo $I_2 + \Gamma^{n+1}L(X,Y)$ on each component are*

$$\Big(\big((\frac{1}{q-1}\sum_{k=1}^{q-1}\ell_n((-1)^k i\xi_q^{s^k})(\sigma))[Y, X^{n-1}]\big)_{t\in H(q)},$$
$$\big((\frac{-1}{q-1}\sum_{k=1}^{q-1}\ell_n((-1)^k i\xi_q^{s^k})(\sigma))[Y, X^{n-1}]\big)_{t\notin H(q)}\,; 0\Big)$$

for $n > 1$ and even;

$$\Big(\big(\frac{1-q^{n-1}}{(1-q)q^{n-1}}\cdot\frac{1-2^{n-1}}{2^{2n-1}}\ell_n(1)(\sigma)[Y, X^{n-1}]\big)_{t\in(\mathbb{Z}/4q)^*}\,;$$
$$\ell_n(1)(\sigma)[Y, X^{n-1}]\Big)$$

for $n > 1$ and odd;

$$\big((\,0\,)_{t\in(\mathbb{Z}/4q)^*}\,;\,0\big)$$

for $n = 1$.

ii) *Let $\sigma \in (grLie\mathcal{U}(L, S, \ell))_n^{[H(q)]}$. The coordinates of $[\mathbf{\Psi}_t]^{[H(q)]}(\sigma)$ calculated modulo $I_2 + \Gamma^{n+1}L(X,Y)$ on each component are given by the same formula as in point i) if we replace $\ell_n(1)$ and $\ell_n((-1)^k i\xi_q^{s^k})$ by $[\ell_n(1)]$ and $[\ell_n((-1)^k i\xi_q^{s^k})]$ respectively.*

PROOF. Let $n = 1$. We have the identity

$$\frac{x^{4q}-1}{x-1} = \frac{x^q-1}{x-1}\cdot(x^q+1)\cdot(x^2+1)\cdot\prod_{k=0}^{q-2}(x-(-1)^k i\xi_q^{s^k})\cdot\prod_{k=0}^{q-2}(x+(-1)^k i\xi_q^{s^k}).$$

Hence we get

$$\prod_{k=0}^{q-2}(1-(-1)^k i\xi_q^{s^k})\cdot\prod_{k=0}^{q-2}(1+(-1)^k i\xi_q^{s^k})=1.$$

Therefore $\prod_{t\in H(q)}(1-t(i\xi_q))=\prod_{k=0}^{q-2}(1-(-1)^k i\xi_q^{s^k})$ and $\prod_{t\in H(q)}(1-t(-i\xi_q))=\prod_{k=0}^{q-2}(1+(-1)^k i\xi_q^{s^k})$ are units of the ring $\mathcal{O}_{\mathbb{Q}(i\sqrt{q})}$. The only units of the ring $\mathcal{O}_{\mathbb{Q}(i\sqrt{q})}$ are roots of unity. This implies the lemma for $n=1$.

Let us assume that $n>1$. Observe that $-i\xi_q^{-1}$ does not belong to the $H(q)$-orbit of $i\xi_q$. Hence the set of primitive $4q$-th roots of 1 is a union of $H(q)$-orbits of $i\xi_q$ and $-i\xi_q^{-1}$. Observe also that all primitive q-th roots of 1 belong to one orbit of $H(q)$. Now the first part of the lemma follows from Corollary 4.3.4 and from distribution and inversion relations for ℓ-adic polylogarithms and we leave it for readers.

The second part of the lemma follows in the same way as in Lemma 5.1.1. □

Proposition 5.3.2. *Let q be a prime number different from ℓ and congruent to 1 modulo 4. We assume that ℓ does not divide $q-1$. We assume also that Conjecture 4.3.1 holds. In the image of the morphism of Lie algebras*

$$[\mathbf{\Psi_t}]^{[H(q)]}: grLie\mathcal{U}(L,S,\ell)^{[H(q)]}$$
$$\to\left(\bigoplus_{\substack{0<\alpha<4q\\(\alpha,4q)=1}}\mathrm{Lie}(X,Y)\right)\tilde{\times}(\mathrm{Lie}(X,Y)/\langle Y\rangle,\{\ \})$$

there are elements $D_2,D_3,\dots,D_m,\dots$ homogeneous of degree $2,3,\dots,m,\dots$ respectively whose coordinates calculated modulo $\mathcal{I}_2$ are

$$\big(((([Y,X^{n-1}])_{t\in H(q)},(-[Y,X^{n-1}])_{t\in(\mathbb{Z}/4q)^*\setminus H(q)}\big)\ ;\ 0\ \big)$$

for $n>1$ and even;

$$\big((a(n)(1-q^{n-1})[Y,X^{n-1}])_{t\in(\mathbb{Z}/4q)^*}\ ;\ (1-q)\cdot q^{n-1-\alpha(n)}\cdot 2^{2n-1}[Y,X^{n-1}]\big)$$

for $n>1$ and odd;

$$(\,(0)_{t\in(\mathbb{Z}/4q)^*}\,;0\,)$$

for $n = 1$.

PROOF. The proposition follows from Lemma 5.3.1. The details which are similar to the proof of Proposition 5.1.2 we omit. □

PROPOSITION 5.3.3. *The Lie algebra Im* $([\mathbf{\Psi}_{\mathfrak{t}}]^{[H(q)]})$ *is free, freely generated by the elements* $D_2, D_3, \dots, D_m, \dots$.

PROOF. As in the proof of Proposition 5.2.9 one need to notice that $n-1-\alpha(n) > 0$. The details of the proof we left to readers. □

THEOREM 5.3.4. *Let us assume that Conjecture 4.3.1 holds. Let* q *be a prime number different from* ℓ *and congruent to* 1 *modulo* 4. *We assume that* ℓ *does not divide* $q-1$. *Let* $V := \mathbb{P}^1 \setminus \{0, 1, \infty\}$ *and let*

$$\mathfrak{t} := \coprod_{\substack{0<j<4q \\ (j,4q)=1}} \pi(V_{\bar{\mathbb{Q}}}; \xi_{4q}^j, \overrightarrow{01}).$$

The representation

$$\theta_{\mathfrak{t}}^{\mathbb{Q}(\mu_{4q})/\mathbb{Q}(i\sqrt{q})} : \mathrm{Gal}\,(\overline{\mathbb{Q}(i\sqrt{q})}/\mathbb{Q}(i\sqrt{q})(\mu_{\ell^\infty})) \to \bigoplus_{\substack{0<\alpha<4q \\ (\alpha,4q)=1}} \mathrm{GL}\,(\mathbb{Q}_\ell\{\{X, Y\}\})$$

has the following properties:

i) *it is unramified outside finite places of* $\mathbb{Q}(i\sqrt{q})(\mu_{\ell^\infty})$ *lying over* ℓ;

ii) *it respects the filtration* $\{\bigoplus_{\substack{0<\alpha<4q \\ (\alpha,4q)=1}} I(\mathbb{Q}_\ell\{\{X, Y\}\})^i\}_{i\in\mathbb{N}}$ *of* $\bigoplus_{\substack{0<\alpha<q \\ (\alpha,4q)=1}} \mathbb{Q}_\ell\{\{X, Y\}\}$;

iii) *any* $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q}(i\sqrt{q})(\mu_{\ell^\infty}))$ *acts on* $\bigoplus_{\substack{0<\alpha<q \\ (\alpha,4q)=1}} \big(I(\mathbb{Q}_\ell\{\{X, Y\}\})^i / I(\mathbb{Q}_\ell\{\{X, Y\}\})^{i+1}\big)$ *as the identity;*

iv) *the image of the associated graded Lie algebra representation*

$$\mathbf{\Theta}_{\mathfrak{t}}^{\mathrm{Gal}(\mathbb{Q}(\mu_{4q})/\mathbb{Q}(i\sqrt{q}))} : \bigoplus_{i=1}^{\infty} \big(F_i(\theta_{\mathfrak{t}}^{\mathbb{Q}(\mu_{4q})/\mathbb{Q}(i\sqrt{q})})/F_{i+1}(\theta_{\mathfrak{t}}^{\mathbb{Q}(\mu_{4q})/\mathbb{Q}(i\sqrt{q})})\big) \otimes \mathbb{Q} \longrightarrow \bigoplus_{\substack{0<\alpha<q \\ (\alpha,4q)=1}} \mathrm{End}(\mathbb{Q}_\ell\{\{X, Y\}\})$$

is a free Lie algebra, freely generated by D_2, D_3, $D_4, \dots, D_n, \dots$.

5.4. We continue to study Galois actions on torsors of paths. In this section we shall construct geometrically an ℓ-adic realization of the associated graded Lie algebra of the motivic fundamental group of the category of mixed Tate motives over $\operatorname{Spec}\mathbb{Z}$.

Let

$$V := \mathbb{P}^1 \setminus \{0, 1, \infty\}.$$

Let q be a prime number greater than 2 and different from ℓ. Let $\xi_q = \exp\left(\frac{2\pi i}{q}\right)$ be a primitive q-th root of 1. We shall study actions of Galois groups on a disjoint union and on a product of $\pi_1(V_{\mathbb{Q}}; \overrightarrow{10})$-torsors

$$t_q := \coprod_{0<\alpha<q} \pi_1(V_{\mathbb{Q}}; -\xi_q^\alpha, \overrightarrow{10}) \text{ and } T_q := \prod_{0<\alpha<q} \pi_1(V_{\mathbb{Q}}; -\xi_q^\alpha, \overrightarrow{10}).$$

We have $\mathbb{Q}(\mu_{2q}) = \mathbb{Q}(\mu_q)$. Observe that the Galois group $G_{\mathbb{Q}(\mu_q)}$ acts on t_q and on T_q. Observe that a triple $(V_{\mathbb{Q}(\mu_q)}; -\xi_q^\alpha, \overrightarrow{01})$ has good reduction everywhere, hence we set

$$S := \{\mathfrak{p} \in \mathcal{V}(\mathbb{Q}(\mu_q)) \mid \mathfrak{p} \text{ divides } \ell\}.$$

We shall apply formalism from section 4 to

$$L := \mathbb{Q}(\mu_{2q}), \quad K := \mathbb{Q} \text{ and } \mathbb{Z}/q^* = \operatorname{Gal}(\mathbb{Q}(\mu_{2q})/\mathbb{Q}).$$

The action of $G_{\mathbb{Q}(\mu_q)}$ on t_q yields a Lie algebra representation

$$\mathbf{\Psi}_{t_q} : \bigoplus_{k=1}^{\infty} ((H_k/H_{k+1}) \otimes \mathbb{Q}) \to \left(\bigoplus_{0<\alpha<q} \operatorname{Lie}(X, Y) \right) \tilde{\times} (\operatorname{Lie}(X, Y)/\langle Y \rangle, \{\,\}),$$

where

$$H_k := \bigcap_{0<\alpha<q} H_k(V_{\mathbb{Q}(\mu_q)}; -\xi_q^\alpha, \overrightarrow{10})$$

for $k \in \mathbb{N}$.

Lemma 5.4.0. *We have:*

i) *Let* $\sigma \in H_k$. *The coordinates of* $\mathbf{\Psi}_{t_q}$ *calculated modulo* $I_2 + \Gamma^{k+1}L(X,Y)$ *on each component are*

$$\big((\ell_k(-\xi_q^\alpha)(\sigma)[Y,X^{k-1}])_{0<\alpha<q}\,;\,\ell_k(1)(\sigma)[Y,X^{k-1}]\big)$$

for $k>1$ *and*

$$\big((\ell_1(-\xi_q^\alpha)(\sigma)Y\,)_{0<\alpha<q}\,;\,0\,\big)$$

for $k=1$;

ii) *Let* $\sigma \in (grLie\mathcal{U}(L,S,\ell)^{[\mathbb{Z}/q^*]})_k$. *The coordinates of* $[\mathbf{\Psi}_{t_q}]^{[\mathbb{Z}/q^*]}(\sigma)$ *calculated modulo* $I_2+\Gamma^{k+1}L(X,Y)$ *on each component are given by the same formulas as in the point i) if we replace* $\ell_k(1)$, $\ell_k(-\xi_q^\alpha)$ *and* $\ell_1(-\xi_q^\alpha)$ *by* $[\ell_k(1)]$, $[\ell_k(-\xi_q^\alpha)]$ *and* $[\ell_1(-\xi_q^\alpha)]$ *respectively.*

Proof. The lemma follows from Lemma 1.1.0 and Definition 3.0.1 of ℓ-adic polylogarithms. □

The group $\mathbb{Z}/q^* = \mathrm{Gal}\,(\mathbb{Q}(\mu_q)/\mathbb{Q})$ acts on the associated graded Lie algebra $grLie\mathcal{U}(L,S,\ell)$. We shall study the Lie algebra homomorphism

$$[\mathbf{\Psi}_{t_q}]^{[\mathbb{Z}/q^*]} : (grLie\mathcal{U}(L,S,\ell))^{[\mathbb{Z}/q^*]} \\ \to \left(\bigoplus_{0<\alpha<q} \mathrm{Lie}(X,Y)\right)\tilde{\times}(\mathrm{Lie}(X,Y)/\langle Y\rangle,\{\,\}).$$

Lemma 5.4.1. *Let q be a prime number greater than 2 and different from ℓ. We assume that ℓ does not divide $q-1$. We have*

i) *Let* $\sigma \in ((H_k/H_{k+1})\otimes\mathbb{Q})^{\mathbb{Z}/q^*}$. *The coordinates of* $\mathbf{\Psi}_{t_q}^{\mathbb{Z}/q^*}(\sigma)$ *calculated modulo* $I_2+\Gamma^{k+1}L(X,Y)$ *on each component are*

$$\Big(\big(\frac{(1-2^{k-1})\cdot(1-q^{k-1})}{2^{k-1}\cdot q^{k-1}\cdot(q-1)}\ell_k(1)(\sigma)[Y,X^{k-1}]\,\big)_{0<\alpha<q}\,; \\ \ell_k(1)(\sigma)[Y,X^{k-1}]\Big)$$

for $k>1$ *and odd;*

$$(\,(\,0\,)\,)_{0<\alpha<q}\,;\,0\,)$$

for $k > 1$ and even;

$$((0))_{0<\alpha<q}\,;\,0)$$

for $k = 1$.

ii) *Let $\sigma \in (grLie\mathcal{U}(L,S,\ell))_k^{[\mathbb{Z}/q^*]}$. The coordinates of $[\mathbf{\Psi}_t]^{[\mathbb{Z}/q^*]}(\sigma)$ calculated modulo $I_2 + \Gamma^{k+1}L(X,Y)$ on each component are given by the same formula as in point i) if we replace $\ell_k(1)$ by $[\ell_k(1)]$.*

Proof. The first part of the lemma follows from Lemma 5.4.0, Corollary 4.3.4 and the distribution and inversion relations for ℓ-adic polylogarithms. For $k = 1$ we use the relation $\prod_{k=1}^{q-1}(1+\xi_q^k) = 1$.

The second part of the lemma is proved in the same way as part ii) of Lemma 5.1.1. □

We recall that

$$1 - 2^{k-1} = a(k) \cdot q^{\alpha(k)},$$

where $a(k)$ is an integer not divisible by q.

Proposition 5.4.2. *In the image of the morphism $[\mathbf{\Psi}_{t_q}]^{[\mathbb{Z}/q^*]}$ there are elements $D_3, D_5, D_7, \ldots, D_{2n+1}, \ldots$ homogenous of degree $3, 5, 7, \ldots, 2n+1, \ldots$ respectively whose coordinates calculated modulo $\mathcal{I}_2$ are*

$$\Big(\big(a(2n+1)\cdot(1-q^{2n})[Y,X^{2n}]\big)_{0<\alpha<q}\,;\, 2^{2n}\cdot q^{2n-\alpha(2n+1)}\cdot(q-1)[Y,X^{2n}] \Big)$$

for $n = 1, 2, 3, \ldots$.

Proof. The ℓ-adic polylogarithm $\ell_{2n+1}(\overrightarrow{10})$ is a rational non-zero multiple of the Soule class - the generator of $H^1(G_{\mathbb{Q}};\mathbb{Q}_\ell(2n+1))$. Therefore $\ell_{2n+1}(\overrightarrow{10})$ restricted to $H^1(G_L;\mathbb{Q}_\ell(2n+1))$ is also non-zero. It follows from Proposition 4.2.8 that $[\ell_{2n+1}(1)]$ restricted to $(grLie\mathcal{U}(L,S,\ell)^{[\mathbb{Z}/q^*]})_{2n+1}$ is non-zero. Therefore there is $\sigma_{2n+1} \in (grLie\mathcal{U}(L,S,\ell)^{[\mathbb{Z}/q^*]})_{2n+1}$ such that $[\ell_{2n+1}(\overrightarrow{10})](\sigma_{2n+1}) = 1$. We set

$$D_{2n+1} := (q-1)\cdot 2^{2n}\cdot q^{2n-\alpha(2n+1)}\cdot [\mathbf{\Psi}_{t_q}]^{[\mathbb{Z}/q^*]}(\sigma_{2n+1}). \;\square$$

PROPOSITION 5.4.3. *The Lie algebra Im* $([\mathbf{\Psi}_{t_q}]^{[\mathbb{Z}/q^*]})$ *is free freely generated by elements* $D_3, D_5, \dots, D_{2n+1}, \dots$.

PROOF. The proof is identical to the proof of Proposition 5.1.3 and we omit it. □

THEOREM 5.4.4. *Let q be a prime number different from 2 and from ℓ. We assume also that ℓ does not divide $q-1$. Let $V = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and let*

$$t_q := \coprod_{0<\alpha<q} \pi_1(V_{\overline{\mathbb{Q}}}; -\xi_q^\alpha, \overrightarrow{10}).$$

Then we have:

i) *the representation*

$$\theta_{t_q}^{\mathbb{Q}(\mu_{2q})/\mathbb{Q}} : \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_{\ell^\infty})) \longrightarrow \bigoplus_{0<j<q} \mathrm{GL}\,(\mathbb{Q}_\ell\{\{X, Y\}\})$$

is unramified outside finite places of $\mathbb{Q}(\mu_{\ell^\infty})$ lying over ℓ;

ii) *the representation $\theta_{t_q}^{\mathbb{Q}(\mu_{2q})/\mathbb{Q}}$ respects the filtration $\{\bigoplus_{\substack{0<\alpha<4q \\ (\alpha,4q)=1}} I(\mathbb{Q}_\ell\{\{X, Y\}\})^i\}_{i\in\mathbb{N}}$ of $\bigoplus_{\substack{0<\alpha<q \\ (\alpha,4q)=1}} \mathbb{Q}_\ell\{\{X, Y\}\}$;*

iii) *any $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q}(\mu_{\ell^\infty}))$ acts on $\bigoplus_{\substack{0<\alpha<q \\ (\alpha,4q)=1}} \big(I(\mathbb{Q}_\ell\{\{X, Y\}\})^i / I(\mathbb{Q}_\ell\{\{X, Y\}\})^{i+1}\big)$ as the identity;*

iv) *the image of the associated graded Lie algebra representation $gr\,\mathrm{Lie}\,\theta_{t_q}^{\mathbb{Q}(\mu_{2q})/\mathbb{Q}}$ is a free Lie algebra, freely generated by $D_3, D_5, \dots, D_{2n+1}, \dots$.*

PROOF. Observe that $-\xi_q \equiv \xi_q \bmod 2$ and $-\xi_q \equiv -1 \bmod q$. Hence the triple $(\mathbb{P}^1 \setminus \{0, 1, \infty\}; \xi_q^j, \overrightarrow{10})$ has good reduction everywhere. Proposition 4.1.7 implies that the representation

$$\psi_{t_q} : \mathrm{Gal}\,(\overline{\mathbb{Q}(\mu_{2q})}/\mathbb{Q}(\mu_{2q})) \longrightarrow \mathrm{Aut}_{\mathrm{set}}(t_q)$$

is unramified outside finite places of $\mathbb{Q}(\mu_{2q})$ dividing ℓ. The rest of the proof repeats arguments of the proof of Theorem 5.1.5. □

The representation of $G_{\mathbb{Q}}$ on $\pi_1(V_{\bar{\mathbb{Q}}};\overrightarrow{10})$ was studied by Ihara, Deligne, Grothendieck and others. This representation is unramified outside ℓ (see [9]). We do not know if the image of the associated graded Lie algebra representation is free, freely generated by single generators in degrees 3, 5, 7, ... , $2n+1$, ... The representation of $G_{\mathbb{Q}}$ on $\pi_1(V_{\bar{\mathbb{Q}}};\overrightarrow{10})$ is isomorphic to the representation of $G_{\mathbb{Q}}$ on $\pi_1(V_{\bar{\mathbb{Q}}};\overrightarrow{10})$-torsor of paths $\pi(V_{\bar{\mathbb{Q}}};\overrightarrow{10},\overrightarrow{01})$ (see [22]).

We hope that Theorem 5.4.4 will help to understand the representation of $G_{\mathbb{Q}}$ on $\pi\,(V_{\bar{\mathbb{Q}}};\overrightarrow{10},\overrightarrow{01})$ and hence also on $\pi_1(V_{\bar{\mathbb{Q}}};\overrightarrow{10})$. Notice that for any prime q greater than 2 the image of the morphism $\operatorname{gr}\operatorname{Lie}\theta_{t_q}^{\mathbb{Q}(\mu_{2q})/\mathbb{Q}}$ from Theorem 5.4.4 is free, freely generated by elements D_3, $D_5, \ldots, D_{2n+1}, \ldots$ of degrees 3, 5, ... , $2n+1, \ldots$ respectively.

To realize ℓ-adically the associated graded Lie algebra of the motivic fundamental group of the tannakian category of mixed Tate motives over $\operatorname{Spec}\mathbb{Z}[i]$ (resp. $\operatorname{Spec}\mathbb{Z}[i\sqrt{2}]$, resp. $\operatorname{Spec}\mathbb{Z}[i\sqrt{2}][\frac{1}{2}]$) one needs to study Galois action on torsors of paths $\coprod_{k=1,5,7,11}\pi(V_{\bar{\mathbb{Q}}};\xi_{12}^i,\overrightarrow{01})$ (resp. $\coprod_{\substack{0<j<8\\ j\text{ odd}}}\pi(V_{\bar{\mathbb{Q}}};\xi_8^j,\overrightarrow{01})$, resp. $\coprod_{\substack{0<j<24\\ (j,24)=1}}\pi(V_{\bar{\mathbb{Q}}};\xi_{24}^j,\overrightarrow{01})$).

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