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Singular Cauchy Problems for Perfect Incompressible Fluids

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Abstract. We study local Cauchy problems in a complex domain, for the Euler equation of incompressible fluids. We assume that the initial value of the velocity has singularities along a hyperplane, and prove that the singularities propagate toward characteristic directions of the flow.

1. Introduction

In this article we study the propagation of the singularities of the solution to the incompressible Euler equation in a complex domain. Let $x = (x_0, x') = (x_0, x_1, x_2, x_3) \in \mathbf{C}^4$. We define $X = \{x \in \mathbf{C}^4;\ x_0 = 0\}$ and $Y = \{(0, x') \in X;\ x_1 = 0\}$. We denote the velocity of a perfect incompressible flow by $u(x) = (u_1(x), u_2(x), u_3(x))$ and the pressure by $p(x)$. We consider the following Cauchy problem for them:

$$
(1) \qquad \begin{cases} \partial_{x_0} u + \sum_{1 \le k \le 3} u_k \partial_{x_k} u + \nabla_{x'} p = 0, \\ u(0, x') = u^0(x') \end{cases}
$$

in a neighborhood $\omega \subset \mathbf{C}^4$ of the origin. We also assume that the volume of the fluid is preserved by the flow:

$$
(2) \qquad \mathrm{div}_{x'} u = 0.
$$

We assume that the initial value $u^0 = (u_1^0, u_2^0, u_3^0)$ is holomorphic in a neighborhood of the origin outside of Y, and study the propagation of the singularities of the solution.

Let $\omega_X = \omega \cap X$. We assume that the initial value u^0 is holomorphic on the universal covering space $\mathcal{R}(\omega_X \setminus Y)$ of $\omega_X \setminus Y$. In addition, we assume that it belongs to the function space defined below.

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Let $\mathcal{O}$ denote the sheaf of holomorphic functions, and $\mathcal{O}_{\mathbf{C}^3,0}$ the set of germs of holomorphic functions at the origin of $\mathbf{C}^3$. If $n \in \mathbb{Z}_+ = \{0, 1, 2, \cdots\}$, we denote by $\mathcal{O}^n(\mathcal{R}(\omega_X \setminus Y))$ the set of $h(x') \in \mathcal{O}(\mathcal{R}(\omega_X \setminus Y))$ satisfying

$$|\partial_{x'}^{\alpha'} h(x')| \le \exists a, \quad 0 \le |\alpha'| \le n$$

uniformly on $\mathcal{R}(\omega_X \setminus Y)$.

REMARK. Let $n \ge 1$, $h(x') \in \mathcal{O}^n(\mathcal{R}(\omega_X \setminus Y))$, and $|\alpha'| \le n-1$. We have

$$\partial_{x'}^{\alpha'} h(x') = \int_{\varepsilon}^{x_1} \partial_{x_1} \partial_{x'}^{\alpha'} h(\tau, x_2, x_3) d\tau + \partial_{x'}^{\alpha'} h(\varepsilon, x_2, x_3)$$

for an appropriate $\varepsilon > 0$. Here we can let $x_1 \to 0$, and we can define $[\partial_{x'}^{\alpha'} h]_Y = \partial_{x'}^{\alpha'} h(0, x_2, x_3)$. We have

$$\begin{aligned} &\partial_{x'}^{\alpha'} h(0, x_2, x_3) \in \mathcal{O}(\omega), \\ &|\partial_{x'}^{\alpha'} h(0, x_2, x_3)| \le \exists a && \text{on } \omega, \\ &|\partial_{x'}^{\alpha'} h(x') - \partial_{x'}^{\alpha'} h(0, x_2, x_3)| \le \exists a |x_1| && \text{on } \mathcal{R}(\omega_X \setminus Y) \end{aligned}$$

for $|\alpha'| \le n-1$, shrinking ω if necessary.

We assume the following conditions:

$$u_j^0(x') \in \mathcal{O}^1(\mathcal{R}(\omega_X \setminus Y)), \qquad 1 \le j \le 3, \tag{3}$$

$$\mathrm{div}_{x'} u^0 = 0. \tag{4}$$

Under these assumptions, we want to solve (1), (2). To state the main result, we need to discuss the notion of characteristic hypersurface. Let $\varphi = \varphi(x_0, x_2, x_3) \in \mathcal{O}(\omega)$ satisfy $\varphi(0, x_2, x_3) = 0$. We define $Z = \{x \in \omega;\ x_1 = \varphi(x)\}$. Note that $X \cap Z = Y$. If $n \in \mathbb{Z}_+$, we denote by $\mathcal{O}^n(\mathcal{R}(\omega \setminus Z))$ the set of $h(x) \in \mathcal{O}(\mathcal{R}(\omega \setminus Z))$ satisfying

$$|\partial_x^{\alpha} h(x)| \le \exists a, \quad 0 \le |\alpha| \le n$$

uniformly on $\mathcal{R}(\omega \setminus Z)$. If $n \ge 1$, $h(x) \in \mathcal{O}^n(\mathcal{R}(\omega \setminus Z))$, and $|\alpha| \le n-1$, we can define $[\partial_x^{\alpha} h]_Z = (\partial_x^{\alpha} h)(x_0, \varphi, x_2, x_3)$ as before. We have

$$\begin{aligned} &[\partial_x^{\alpha} h]_Z \in \mathcal{O}(\omega), \\ &|[\partial_x^{\alpha} h]_Z| \le \exists a && \text{on } \omega, \end{aligned}$$

$$|\partial_x^{\alpha} h(x) - [\partial_x^{\alpha} h]_Z| \le \exists a |x_1 - \varphi(x_0, x_2, x_3)| \qquad \text{on } \mathcal{R}(\omega \setminus Z) \tag{5}$$

for $|\alpha| \leq n-1$, shrinking ω if necessary.

We shall prove that there exists a solution $u_1, u_2, u_3, p \in \mathcal{O}(\mathcal{R}(\omega \setminus Z))$ of (1), (2) for some φ, and that the singularity locus Z defined as above is a characteristic hypersurface corresponding to this solution. However, to give a precise statement of this fact, we need to give the definition of a characteristic hypersurface in the following way. Assume that $\psi(x)$ satisfies the eiconal equation:

$$\begin{cases} \partial_{x_0}\psi + \sum_{1\leq k\leq 3} u_k \partial_{x_k}\psi = 0, \\ \psi(0, x') = x_1. \end{cases} \tag{6}$$

We want to say that the singularity locus $Z = \{x_1 = \varphi\}$ is characteristic if it is also written in the following form:

$$Z = \{\psi(x) = 0\}. \tag{7}$$

But $u(x)$ is singular along Z, therefore at most we can only expect that the solution $\psi(x)$ of (6) is holomorphic outside of Z. Therefore the above expression (7) does not make sense. Fortunately, if $\psi \in \mathcal{O}^1(\mathcal{R}(\omega \setminus Z))$, we can define $[\psi]_Z$, and the expression (7) makes sense. Precisely speaking, if the solution ψ of (6) belongs to $\mathcal{O}^1(\mathcal{R}(\omega \setminus Z))$ and we have

$$Z = \{x \in \mathcal{R}(\omega \setminus Z) \cup Z;\ \psi(x) = 0\},$$

then we say that Z is a characteristic hypersurface corresponding to u.

Now we can give our main result:

THEOREM 1. *We assume* (3) *and* (4). *Let* $f(x) \in \mathcal{O}_{\mathbf{C}^4,0}$ *be arbitrarily given. Let* ω *be a small neighborhood of the origin. There exists unique* $\varphi(x_0, x_2, x_3) \in \mathcal{O}(\omega)$, $u_1(x), u_2(x), u_3(x) \in \mathcal{O}^1(\mathcal{R}(\omega \setminus Z))$, *and* $p(x) \in \mathcal{O}^2(\mathcal{R}(\omega \setminus Z))$ *where* $Z = \{x \in \omega;\ x_1 = \varphi(x_0, x_2, x_3)\}$ *is chararacteristic,* (u, p) *satisfies* (1), (2), *and*

$$p(0) = f(0),\ [\nabla_x p]_Z = [\nabla_x f]_Z. \tag{8}$$

REMARK. There are many articles studying the existence of the solution of (1), (2) in a real domain (See [1, 3] and the references cited there).

They usually discuss the problem globally for $x' \in \mathbf{R}^3$, under the assumption that u and p belong to some Sobolev spaces or Hölder spaces. In this case the last condition (8) is unnecessary, because they tacitly assume that p decreases at infinity, instead of (8). We are studying the problem locally, and need to assume (8) in addition. For example if $u^0 = 0$, then $u = 0$, $p = c$ is a local solution of (1), (2) for any constant c. In a global framework such as $p(x_0, \cdot) \in L^2(\mathbf{R}^3)$ we must set $c = 0$, and $(u, p) = 0$ is a unique solution of (1), (2) in such a framework. To the contrary, in our local framework each of $(u, p) = (0, c)$ is a solution of (1), (2), and to assure the uniqueness of the solution we need an additional condition (8).

The propagation of the singularities in a complex domain is a fundamental problem in the theory of linear partial differential equations. Y. Hamada [5], C. Wagschal [10], and many other people studied this problem. E. Leichtnam [6] studied this problem for semilinear equations, and the author [9] for quasilinear equations. However, we cannot apply these results to the Euler equation. J.-M. Delort [4] studied this problem for the Euler equation. His assumptions are different from ours. He assumes that u and p are defined globally for $x' \in \mathbf{R}^3$, they belong to some function spaces as above, and they are continued holomorphically to a complex neighborhood except for the singularity locus Z. We are studying a local theory, which requires different methods. In addition, the result of our local theory is not the same as [4] (i.e., [4] does not require (8) because of the above reason). We also refer to the important result of J.-Y. Chemin [2] for compressible fluids (in a real space). In this case the singularities propagate in a different way from incompressible fluids.

Plan of the paper. In section 2 we shall calculate $\varphi(x_0, x_2, x_3) \in \mathcal{O}(\omega)$ describing the singularity locus Z, together with the trace $[u]_Z$ of u along Z. This part is very easy because all these functions are holomorphic in a full neighborhood of the origin. In section 3, we shall calculate $v = u - [u]_Z$ and p, which are holomorphic on $\mathcal{R}(\omega \setminus Z)$. It is possible to do so because the difference $v = u - [u]_Z$ satisfies an inequality of the form (5). This fact was already pointed out by Delort [4], and sometimes such an idea is called 2-microlocalization. In section 4, we shall prove that the singularity locus Z is characteristic. This means that the singularities of the solution propagate together with the motion of the fluid.

2. Singularity Locus

In this section we calculate a holomorphic function $\varphi(x_0, x_2, x_3)$, and the trace $u_Z(x_0, x_2, x_3) = [u]_Z$ of the solution along $Z = \{x \in \omega;\ x_1 = \varphi(x_0, x_2, x_3)\}$. At this stage we do not calculate the solution u itself. Later we shall prove that the singularities of the solution propagate along Z, althogh this fact is not clear for the moment.

We require that $\varphi(x_0, x_2, x_3) \in \mathcal{O}(\omega)$ satisfies

$$[\partial_{x_0}(x_1 - \varphi) + \sum_{1 \le k \le 3} u_k \partial_{x_k}(x_1 - \varphi)]_Z = 0.$$

This requirement shall be justified in section 4. We can rewrite it in the following form:

$$\text{(9)} \qquad \begin{cases} \partial_{x_0}\varphi + \sum_{2 \le k \le 3} u_{Zk}\partial_{x_k}\varphi = u_{Z1}\ , \\ \varphi(0, x_2, x_3) = 0. \end{cases}$$

We next require that $u_{Zj}(x_0, x_2, x_3) = [u_j]_Z$ is holomorphic on ω and satisfies

$$\text{(10)} \qquad \begin{cases} \partial_{x_0}u_{Zj} + \sum_{2 \le k \le 3} u_{Zk}\partial_{x_k}u_{Zj} + [\partial_{x_j} f]_Z = 0\ , \\ u_{Zj}(0, x_2, x_3) = u_j^0(0, x_2, x_3) \end{cases}$$

for $1 \le j \le 3$. Here $u_{Zj}(0, x_2, x_3)$ denotes $[u_{Zj}(x_0, x_2, x_3)]_{x_0=0}$, $u_j^0(0, x_2, x_3)$ denotes $[u_j^0(x_1, x_2, x_3)]_{x_1=0}$, and $f(x)$ is a holomorphic function given in Theorem 1.

Remark. Equation (9) means that $x_1 - \varphi$ satisfies the eiconal equation on Z. The meaning of (10) is the following. For the sake of simplicity, assume that $u_1, u_2, u_3, p \in \mathcal{O}(\mathcal{R}(\omega \setminus Z))$ satisfies (1), (2) and are sufficiently regular along Z. For $k = 0, 2, 3$ we have

$$\partial_{x_k}u_{Zj} = \partial_{x_k}(u_j(x_0, \varphi(x_0, x_2, x_3), x_2, x_3)) = [\partial_{x_k}u_j]_Z + \partial_{x_k}\varphi \cdot [\partial_{x_1}u_j]_Z,$$

and thus $[\partial_{x_k}u_j]_Z = \partial_{x_k}u_{Zj} - \partial_{x_k}\varphi \cdot [\partial_{x_1}u_j]_Z$. It follows that

$$0 = [\partial_{x_0}u_j + \sum_{1 \le k \le 3} u_k\partial_{x_k}u_j + \partial_{x_j}p]_Z$$

$$= \partial_{x_0} u_{Zj} + \sum_{2\le k\le 3} u_{Zk}\partial_{x_k} u_{Zj}$$
$$+ (-\partial_{x_0}\varphi + u_{Z1} - \sum_{2\le k\le 3} u_{Zk}\cdot\partial_{x_k}\varphi)[\partial_{x_1} u_j]_Z + [\partial_{x_j} f]_Z$$

From (9) u_Z must satisfy (10). Therefore (10) means that u_Z satisfies the Euler equation on Z.

All the known functions appearing in (9) and (10) are holomorphic, and we obtain a unique solution (φ, u_Z) also holomorphic on a small neighborhood ω of the origin. This is due to the classical theorem of Cauchy-Kowalewski, which is applicable in our situation (See [8] for example).

From now on, we define $Z = \{x \in \omega;\ x_1 = \varphi(x_0, x_2, x_3)\}$ using this function φ. Note that we can rewrite (8) in the following form:

$$[\partial_{x_1}^k p]_Z = g_k(x_0, x_2, x_3),\ k = 0, 1,$$

where $g_k(x_0, x_2, x_3) = (\partial_{x_1}^k f)(x_0, \varphi(x_0, x_2, x_3), x_2, x_3)$.

3. Calculation of the Singularities

In this section we prove that the solution (u, p) is holomorphic on $\mathcal{R}(\omega \setminus Z)$. We denote $v = (v_1, v_2, v_3) = u - u_Z$, and calculate (v, p). As in [3, 4], we use the following classical result:

Lemma 1. *We assume* (1), (4). *Then* (2) *is equivalent to the following condition*:

(11) $$\Delta_{x'} p + \sum_{1\le j,k\le 3} \partial_{x_j} u_k \cdot \partial_{x_k} u_j = 0.$$

This is well known, and we omit the proof. Remarking $\partial_{x_1}(u_Z(x_0, x_2, x_3)) = 0$ and (10), we can rewrite (1) in the following form:

(12) $$\partial_{x_0} v_j + \sum_{1\le k\le 3} (v_k + u_{Zk})\partial_{x_k} v_j$$
$$+ \sum_{2\le k\le 3} v_k \partial_{x_k} u_{Zj} + \partial_{x_j} p - [\partial_{x_j} f]_Z = 0.$$

Therefore we need to calculate (v,p) satisfying (11) and (12). As is noted also by [4], we can solve these equations because v should be small (i.e., v should satisfy an inequality of the form (5)).

Let us introduce the following isomorphism:

$$\kappa : \omega \ni x \longmapsto y = (x_0, x_1 - \varphi(x_0,x_2,x_3), x_2, x_3) \in \kappa(\omega).$$

We can regard $\partial_{x_k} y_j(x)$, $\partial_{x_k} u_{Zj}(x)$ as functions of y, which we denote by $(\partial_{x_k} y_j)(y)$, $(\partial_{x_k} u_{Zj})(y)$ or simply by $\partial_{x_k} y_j$, $\partial_{x_k} u_{Zj}$. By this coordinate transformation we can rewrite (12) in the following form:

$$\begin{aligned} &\partial_{y_0} v_j + \{v_1 + u_{Z1} - \partial_{x_0}\varphi - \sum_{2\le k\le 3} (v_k + u_{Zk})\partial_{x_k}\varphi\}\partial_{y_1} v_j \\ + &\sum_{2\le k\le 3} (v_k + u_{Zk})\partial_{y_k} v_j + \sum_{2\le k\le 3} (\partial_{x_k} u_{Zj})(y)\cdot v_k \\ + &\sum_{1\le k\le 3} \{(\partial_{x_j} y_k)(y)\cdot \partial_{y_k} p - [(\partial_{x_j} y_k)(y)\cdot\partial_{y_k} f]_{y_1=0}\} = 0. \end{aligned}$$

Using (9) we can rewrite this in the following form:

$$\begin{aligned} &\partial_{y_0} v_j + \{v_1 - \sum_{2\le k\le 3} (\partial_{x_k}\varphi)(y)\cdot v_k\}\partial_{y_1} v_j \\ + &\sum_{2\le k\le 3} (v_k + u_{Zk})\partial_{y_k} v_j + \sum_{2\le k\le 3} (\partial_{x_k} u_{Zj})(y)\cdot v_k \\ + &\sum_{1\le k\le 3} \{(\partial_{x_j} y_k)(y)\cdot \partial_{y_k} p - [(\partial_{x_j} y_k)(y)\cdot\partial_{y_k} f]_{y_1=0}\} = 0. \end{aligned}$$

Note that here we can divide the coefficient of $\partial_{y_1} v_j$ by (v_1, v_2, v_3). We may regard $\partial_{x'}^{\alpha'}\varphi$, $\partial_{x'}^{\alpha'} u_Z$ as (known) functions of y, and we can rewrite (11) in the following form:

$$\begin{aligned} &\partial_{y_1}^2 p + \frac{1}{1 + (\partial_{x_2}\varphi)^2 + (\partial_{x_3}\varphi)^2} \\ \times\ &\Big\{\partial_{y_2}^2 p + \partial_{y_3}^2 p - (\partial_{x_2}^2\varphi + \partial_{x_3}^2\varphi)\partial_{y_1} p - 2\partial_{x_2}\varphi\cdot\partial_{y_1}\partial_{y_2} p - 2\partial_{x_3}\varphi\cdot\partial_{y_1}\partial_{y_3} p \\ &+ \sum_{1\le j,k\le 3} (\partial_{x_k} u_{Zj} + \sum_{1\le l\le 3} \partial_{x_k} y_l\cdot\partial_{y_l} v_j) \\ &\cdot (\partial_{x_j} u_{Zk} + \sum_{1\le l\le 3} \partial_{x_j} y_l \cdot \partial_{y_l} v_k)\Big\} = 0. \end{aligned}$$

Let us rewrite these equations once more. Let $1 \le j \le 3$. Regarding $(y,\ \partial_y^{\alpha'} v,\ \partial_y^{\beta'} p)$ with $|\alpha'| \le 1,\ |\beta'| = 1$ as independent variables, we define

$$
\begin{aligned}
F_j(y,\ \partial_y^{\alpha'} v,\ \partial_y^{\beta'} p) = & - \{v_1 - \sum_{2\le k\le 3} \partial_{y_k}\varphi(y)\cdot v_k\}\partial_{y_1} v_j \\
& - \sum_{2\le k\le 3}(v_k + u_{Zk}(y))\partial_{y_k} v_j - \sum_{2\le k\le 3}(\partial_{x_k} u_{Zj})(y)\cdot v_k \\
& - \sum_{1\le k\le 3}\{(\partial_{x_j} y_k)(y)\cdot\partial_{y_k} p - [(\partial_{x_j} y_k)(y)\cdot\partial_{y_k} f(y)]_{y_1=0}\}.
\end{aligned}
$$

Here we regard all the functions except for $\partial_y^{\alpha'} v, \partial_y^{\beta'} p$ as holomorphic functions of y, which are already known. Regarding $(y,\ \partial_y^{\alpha'} v,\ \partial_y^{\beta'} p)$ with $|\alpha'| \le 1,\ |\beta'| \le 2,\ \beta_1 \ne 2$ as independent variables, we define

$$
\begin{aligned}
& G(y,\ \partial_y^{\alpha'} v,\ \partial_y^{\beta'} p) = -\frac{1}{1 + (\partial_{x_2}\varphi)^2 + (\partial_{x_3}\varphi)^2} \\
& \times\Big\{\partial_{y_2}^2 p + \partial_{y_3}^2 p - (\partial_{x_2}^2\varphi + \partial_{x_3}^2\varphi)\partial_{y_1} p - 2\partial_{x_2}\varphi\cdot\partial_{y_1}\partial_{y_2} p - 2\partial_{x_3}\varphi\cdot\partial_{y_1}\partial_{y_3} p \\
& + \sum_{1\le j,k\le 3}(\partial_{x_k} u_{Zj} + \sum_{1\le l\le 3}\partial_{x_k} y_l\cdot\partial_{y_l} v_j)\cdot(\partial_{x_j} u_{Zk} + \sum_{1\le l\le 3}\partial_{x_j} y_l\cdot\partial_{y_l} v_k)\Big\}\ .
\end{aligned}
$$

Again we regard all the functions except for $\partial_y^{\alpha'} v, \partial_y^{\beta'} p$ as holomorphic functions of y, which are already known. Then we can rewrite (1), (8), (11) in the following form:

$$
(13)\qquad
\begin{cases}
\partial_{y_0} v_j = F_j(y, \partial_y^{\alpha'} v, \partial_y^{\beta'} p), & \\
v_j(0, y') = v_j^0(y'), & 1 \le j \le 3, \\
\partial_{y_1}^2 p = G(y,\ \partial_y^{\alpha'} v,\ \partial_y^{\beta'} p), & \\
\partial_{y_1}^k p(y_0, 0, y_2, y_3) = g_k(y_0, y_2, y_3), & 0 \le k \le 1.
\end{cases}
$$

Here we have defined $v^0 = (v_1^0, v_2^0, v_3^0) = u^0 - [u_Z]_{y_0=0}$. By definition, we can rewrite F_j and G in the following form:

$$
\begin{aligned}
F_j(y, \partial_y^{\alpha'} v, \partial_y^{\beta'} p) = & \sum_{1\le k,l,m\le 3} F_{jklm}(y) v_k\cdot\partial_{y_l} v_m + \sum_{\substack{1\le m\le 3\\ |\alpha'|\le 1\\ \alpha_1=0}} F_{j\alpha' m}(y)\partial_{y'}^{\alpha'} v_m \\
& + \sum_{1\le l\le 3}(F_{jl}(y)\partial_{y_l} p - [F_{jl}(y)\partial_{y_l} f(y)]_{y_1=0}),
\end{aligned}
$$

$$G(y, \partial_y^{\alpha'} v, \partial_y^{\beta'} p) = \sum_{\substack{|\beta'| \le 2 \\ \beta_1 \ne 2}} G_{\beta'}(y) \partial_{y'}^{\beta'} p + \sum_{1 \le j,k,l,m \le 3} G_{jklm}(y) \partial_{y_l} v_k \cdot \partial_{y_m} v_j$$
$$+ \sum_{1 \le j,m \le 3} G_{jm}(y) \partial_{y_m} v_j + G_0(y)$$

for some $F_{jklm}, F_{j\alpha' m}, F_{jl}, f, G_{\beta'}, G_{jklm}, G_{jm}, G_0 \in \mathcal{O}(\omega)$.

We solve (13) by iteration. We first define

$$\text{(14)} \qquad \begin{cases} v_j^{(0)}(y) = v_j^0(y'), \\ p^{(0)}(y) = g_0(y_0, y_2, y_3) + y_1 g_1(y_0, y_2, y_3). \end{cases}$$

If $i \ge 1$, we inductively define $(v^{(i)}, p^{(i)})$ as a solution of

$$\text{(15)} \qquad \begin{cases} \partial_{y_0} v_j^{(i)} = F_j(y, \partial_{y'}^{\alpha'} v^{(i-1)}, \partial_{y'}^{\beta'} p^{(i-1)}), \\ v_j^{(i)}(0, y') = v_j^0(y'), & 1 \le j \le 3, \\ \partial_{y_1}^2 p^{(i)} = G(y, \partial_y^{\alpha'} v^{(i-1)}, \partial_y^{\beta'} p^{(i-1)}), \\ \partial_{y_1}^k p^{(i)}(y_0, 0, y_2, y_3) = g_k(y_0, y_2, y_3), & 0 \le k \le 1. \end{cases}$$

Here v^0 denotes the initial value, and $(v^{(0)}, p^{(0)})$ denotes the 0-th approximation. Let proj : $\mathcal{R}(\omega \setminus Z) \longrightarrow \omega \setminus Z$ be a natural projection. If $\tilde{y} \in \mathcal{R}(\omega \setminus Z)$ satisfies $\mathrm{proj}(\tilde{y}) = y \in \omega \setminus Z$, then we may identify $\tilde{y}$ with $(y, \arg y_1)$. If $\theta = \arg y_1$, we may denote $\tilde{y}$ by y^θ or simply by y. If we have calculated $(v^{(k)}, p^{(k)})$ fot $0 \le k \le i-1$ on $\mathcal{R}(\omega \setminus Z)$, we can define the branch of $(v^{(i)}, p^{(i)})$ at $y^\theta \in \mathcal{R}(\omega \setminus Z)$ by

$$\text{(16)} \qquad v_j^{(i)}(y) = \int_0^{y_0} F_j(\tau, y', \partial_{y'}^{\alpha'} v^{(i-1)}(\tau, y'), \partial_{y'}^{\beta'} p^{(i-1)}(\tau, y')) d\tau + v_j^0(y'),$$

taking $\arg y_1 = \theta$, and

$$\begin{aligned} & p^{(i)}(y) \\ = & \int_0^{y_1} \int_0^{\sigma} G(y_0, \tau, y_2, y_3, \partial_{y'}^{\alpha'} v^{(i-1)}(y_0, \tau, y_2, y_3), \partial_{y'}^{\beta'} p^{(i-1)}(y_0, \tau, y_2, y_3)) d\tau d\sigma \\ & + g_0(y_0, y_2, y_3) + y_1 g_1(y_0, y_2, y_3), \end{aligned}$$

taking $\arg y_1 = \arg \tau = \arg \sigma = \theta$ (for a certain ω).

To prove the convergence of $(v^{(i)}, p^{(i)})$, we use the method of T. Nishida [7]. Let $M > 0$ be large, and let $0 < r << 1/M$. Let

$$\pi_i = (1 + 2^{-i})(1 + 2^{-i-1})(1 + 2^{-i-2}) \cdots$$

for $i = 0, 1, 2, \cdots$. It is easy to see

$$e^2 > \pi_0 > \pi_1 > \pi_2 > \cdots > 1, \quad \lim_{i\to\infty} \pi_i = 1.$$

We define

$$\begin{aligned} \rho_i(y) &= \pi_i r^3 - r|y_0| - |y_1| - r^2|y_2| - r^2|y_3|, \\ \rho(y) &= r^3 - r|y_0| - |y_1| - r^2|y_2| - r^2|y_3|, \\ \omega_i(r) &= \{y \in \mathbf{C}^4 :\ \rho_i(y) > 0,\ y_1 \neq 0\}, \\ \omega(r) &= \{y \in \mathbf{C}^4 :\ \rho(y) > 0,\ y_1 \neq 0\}. \end{aligned}$$

Then we have

$$\begin{aligned} &\rho_0(y) > \rho_1(y) > \rho_2(y) > \cdots > \rho(y), \\ &\omega_0(r) \supset \omega_1(r) \supset \omega_2(r) \supset \cdots \supset \omega(r). \end{aligned}$$

We shall use the following fact:

LEMMA 2. (a) *If* $i \geq 1$ *and* $y \in \omega_i(r)$, *then we have* $\rho_{i-1}(y) \geq 2^{-i+1}r^3$.
(b) *If* $y \in \omega_i(r)$ *and* $z' \in \mathbf{C}^3$ *satisfies*

$$(17) \qquad \begin{cases} |z_1| \leq \dfrac{\rho_i(y)}{8},\ \dfrac{|y_1|}{8}, \\ |z_j| \leq \dfrac{\rho_i(y)}{8r^2}, & j = 2, 3, \end{cases}$$

then we have $(y_0, y' + z') \in \omega_i(r)$, *and* $\rho_i(y_0, y' + z') \geq \rho_i(y)/2$.

The proof is easy, and we omit it. Let $\tilde{y} \in \mathcal{R}(\omega_i(r))$ and let $y = \mathrm{proj}(\tilde{y}) \in \omega_i(r)$. As before, we identify $\tilde{y}$ with $(y, \arg y_1)$. If $z' \in \mathbf{C}^3$ satisfies (17), then we can naturally define $\arg(y_1 + z_1)$ satisfying $|\arg y_1 - \arg(y_1 + z_1)| \leq \pi/6$. Therefore we may regard $(y_0, y' + z')$ as an element of $\mathcal{R}(\omega_i(r))$ in this sense. It is easy to see that we can define $(v^{(i)}, p^{(i)})$ inductively by (14) and (15) on $\mathcal{R}(\omega_0(r))(\supset \mathcal{R}(\omega_{i-1}(r)))$ for $i \geq 1$. To prove the convergence, we prepare the following fact:

PROPOSITION 1. (a) *Let* $i \geq 1$. *We have*

$$|\partial_{y'}^{\alpha'}(v_j^{(i)} - v_j^{(i-1)})| \leq 2^{-5i} M r^7 \frac{|y_1|}{\rho_{i-1}(y)^2}\Big(\frac{1}{|y_1|} + \frac{1}{\rho_{i-1}(y)}\Big)^{\alpha_1}\Big(\frac{r^2}{\rho_{i-1}(y)}\Big)^{\alpha_2+\alpha_3}, \tag{18}$$

$$|\partial_{y'}^{\beta'}(p^{(i)} - p^{(i-1)})| \leq 2^{-5i} M r^6 \frac{|y_1|^2}{\rho_{i-1}(y)^2}\Big(\frac{1}{|y_1|} + \frac{1}{\rho_{i-1}(y)}\Big)^{\beta_1}\Big(\frac{r^2}{\rho_{i-1}(y)}\Big)^{\beta_2+\beta_3} \tag{19}$$

for $|\alpha'| \leq 1,\ |\beta'| \leq 2$ *on* $\mathcal{R}(\omega_{i-1}(r))$.

(b) *Let* $i \geq 1$. *We have*

$$|\partial_{y'}^{\alpha'}(v_j^{(i)} - v_j^{(i-1)})| \leq 2^{-i} M |y_1|^{1-\alpha_1} \tag{20}$$

$$|\partial_{y'}^{\beta'}(p^{(i)} - p^{(i-1)})| \leq 2^{-i} M r^{-2} |y_1|^{2-\beta_1} \tag{21}$$

for $|\alpha'| \leq 1,\ |\beta'| \leq 2$ *on* $\mathcal{R}(\omega_i(r))$ $(\subset \mathcal{R}(\omega_{i-1}(r)))$.

We first remark that (b) is a consequence of (a). To see this, we only need to verify that the right hand side of (18) (resp. (19)) does not exceed that of (20) (resp. (21)) on $\mathcal{R}(\omega_i)$. Let $|\alpha'| \leq 1$. Using (a) of Lemma 2, we have

$$\begin{aligned} A &\stackrel{\text{def}}{=} 2^{-5i} M r^7 \frac{|y_1|}{\rho_{i-1}(y)^2}\Big(\frac{1}{|y_1|} + \frac{1}{\rho_{i-1}(y)}\Big)^{\alpha_1}\Big(\frac{r^2}{\rho_{i-1}(y)}\Big)^{\alpha_2+\alpha_3} \\ &\leq 2^{-5i} M r^7 \frac{|y_1|^{1-\alpha_1}}{(2^{-i+1}r^3)^2}\Big(1 + \frac{|y_1|}{2^{-i+1}r^3}\Big)^{\alpha_1}\Big(\frac{r^2}{2^{-i+1}r^3}\Big)^{\alpha_2+\alpha_3}. \end{aligned}$$

We have $|y_1| \leq e^2 r^3$ on $\mathcal{R}(\omega_i(r))$, and it follows that

$$A \leq 2^{-2i} M r^{1-\alpha_2-\alpha_3} e^{2\alpha_1} |y_1|^{1-\alpha_1} \leq 2^{-i} M |y_1|^{1-\alpha_1}.$$

Similarly we can compare (19) and (21).

We next prove (a) of Proposition 1 for $i=1$. Since M is large, we may assume

$$(22)\qquad \begin{cases} |F_{jklm}|, |F_{j\alpha' m}|, |F_{jl}|, |G_{\beta'}|, |G_{jklm}|, |G_{jm}| \le M^{1/10}, \\ |\partial_{y'}^{\alpha'} v_j^{(0)}| \le M^{1/10}|y_1|^{1-\alpha_1}, \\ |\partial_{y'}^{\beta'} p^{(0)}| \le M^{1/10} \end{cases}$$

for $|\alpha'| \le 1,\ |\beta'| \le 2$ on $\mathcal{R}(\omega_0(r))$. Therefore we have

$$|F_j(y, \partial_{y'}^{\alpha'} v^{(0)}(y), \partial_{y'}^{\beta} p^{(0)}(y))| \le \sqrt{M}|y_1|,$$
$$|G(y, \partial_{y'}^{\alpha'} v^{(0)}(y), \partial_{y'}^{\beta'} p^{(0)}(y))| \le \sqrt{M}.$$

From (16) we have

$$|v_j^{(1)} - v_j^{(0)}| \le \sqrt{M}|y_0 y_1| \le \sqrt{M}e^2 r^2 |y_1|.$$

By the Cauchy integration formula and (b) of Lemma 2, we have

$$\begin{aligned} &|\partial_{y'}^{\alpha'}(v_j^{(1)}(y) - v^{(0)}(y))| \\ \le\ & \alpha'!\ \inf_{z'}\Big\{\sqrt{M}e^2r^2|y_1+z_1|\Big(\frac{8}{|y_1|}+\frac{8}{\rho_{i-1}(y)}\Big)^{\alpha_1}\Big(\frac{8r^2}{\rho_{i-1}(y)}\Big)^{\alpha_2+\alpha_3}\Big\}. \end{aligned}$$

Here we take the infimum for $z' \in \mathbf{C}^3$ satisfying (17). If $i=1$, we obtain (18) from this, and similarly we obtain (19) (Therefore statement (b) of Proposition 1 is also true).

Let $i_0 \ge 2$. We next assume that (a) and (b) of Proposition 1 are true for $1 \le i \le i_0 - 1$. Let us prove (a) for $i = i_0$. We have (22) on $\mathcal{R}(\omega_{i-1}(r)) \subset \mathcal{R}(\omega_0(r))$. If $1 \le i' \le i-1$, we have

$$(23)\qquad |\partial_{y'}^{\alpha'} v_j^{(i')}| \le \sum_{1\le i''\le i'} |\partial_{y'}^{\alpha'}(v_j^{(i'')} - v_j^{(i''-1)})| + |\partial_{y'}^{\alpha'} v_j^{(0)}| \le 2M|y_1|^{1-\alpha_1},$$

$$(24)\qquad |\partial_{y'}^{\beta'} p^{(i')}| \le \sum_{1\le i''\le i'} |\partial_{y'}^{\beta'}(p^{(i'')} - p^{(i''-1)})| + |\partial_{y'}^{\beta'} p^{(0)}| \le 2M,$$

for $|\alpha'| \le 1,\ |\beta'| \le 2$ on $\mathcal{R}(\omega_{i-1}(r)) \subset \mathcal{R}(\omega_{i'}(r))$, by the assumption (b) of induction.

We assume $y \in \mathcal{R}(\omega_{i-1}(r))$. We have

$$\begin{aligned} &F_j(y,\ \partial_{y'}^{\alpha'} v^{(i-1)}(y),\ \partial_{y'}^{\beta'} p^{(i-1)}(y)) - F_j(y,\ \partial_{y'}^{\alpha'} v^{(i-2)}(y),\ \partial_{y'}^{\beta'} p^{(i-2)}(y)) \\ =& A + B + C + D, \end{aligned}$$

where

$$A = \sum_{1 \le k,l,m \le 3} F_{jklm}(y) \cdot v_k^{(i-1)}(y) \cdot \partial_{y_l}(v_m^{(i-1)}(y) - v_m^{(i-2)}(y)),$$

$$B = \sum_{1 \le k,l,m \le 3} F_{jklm}(y) \cdot (v_k^{(i-1)}(y) - v_k^{(i-2)}(y)) \cdot \partial_{y_l} v_m^{(i-2)}(y),$$

$$C = \sum_{\substack{1 \le m \le 3 \\ |\alpha'| \le 1 \\ \alpha_1 = 0}} F_{j\alpha' m}(y) \cdot \partial_{y'}^{\alpha'}(v_m^{(i-1)}(y) - v_m^{(i-2)}(y)),$$

$$D = \sum_{1 \le l \le 3} F_{jl}(y) \cdot \partial_{y_l}(p^{(i-1)}(y) - p^{(i-2)}(y)).$$

We estimate A, B, C, D in the following way. Each term in A, B, C, D contains one of $v^{(i-1)} - v^{(i-2)}$, $p^{(i-1)} - p^{(i-2)}$ or their derivatives once, and we apply inequality (18) or (19) to them. We apply (23) to the other parts, i.e., $v_k^{(i-1)}$ in A and $\partial_{y_l} v_m^{(i-2)}$ in B. Then we have

$$\begin{aligned} |A| &\le M \cdot 2M|y_1| \cdot 2^{-5i+5} M r^7 \frac{|y_1|}{\rho_{i-2}(y)^2} \Big(\frac{1}{|y_1|} + \frac{1}{\rho_{i-2}(y)} \Big) \\ &\le 2^{-5i+5} M^3 r^6 \frac{|y_1|}{\rho_{i-1}(y)^2} \Big(1 + \frac{|y_1|}{\rho_{i-1}(y)} \Big). \end{aligned}$$

Similarly we can prove

$$|B| \le 2^{-5i+5} M^3 r^6 \frac{|y_1|}{\rho_{i-1}(y)^2}\ ,$$

$$|C| \le 2^{-5i+5} M^3 r^6 \frac{|y_1|}{\rho_{i-1}(y)^2} \cdot \frac{r^3}{\rho_{i-1}(y)}\ ,$$

$$|D| \le 2^{-5i+5} M^3 r^6 \frac{|y_1|}{\rho_{i-1}(y)^2} \Big(1 + \frac{|y_1|}{\rho_{i-1}(y)} \Big)\ .$$

In the above estimate of C we need to note $\alpha_1 = 0$. We have $|y_1|, \rho_{i-1}(y) \le e^2 r^3$ on $\omega_{i-1}(r)$, and it follows that

$$\Big| F_j(y, \partial_{y'}^{\alpha'} v^{(i-1)}, \partial_{y'}^{\beta'} p^{(i-1)}) - F_j(y, \partial_{y'}^{\alpha'} v^{(i-2)}, \partial_{y'}^{\beta'} p^{(i-2)}) \Big|$$

$$
\begin{aligned}
&\le 4\cdot 2^{-5i+5}M^3r^6\frac{|y_1|}{\rho_{i-1}(y)^2}\Big(1+\frac{|y_1|}{\rho_{i-1}(y)}+\frac{r^3}{\rho_{i-1}(y)}\Big)\\
&\le 2^{-5i}M^4r^9\frac{|y_1|}{\rho_{i-1}(y)^3}.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
|v_j^{(i)}(y)-v_j^{(i-1)}(y)|\le&\quad 2^{-5i}M^4r^9\int_0^{y_0}\frac{|y_1|}{\rho_{i-1}(\tau,y')^3}|d\tau|\\
\le&\quad 2^{-5i}M^4r^8\frac{|y_1|}{\rho_{i-1}(y)^2}.
\end{aligned}
$$

Using (b) of Lemma 2 and Cauchy integration theorem, we have

$$
\begin{aligned}
&|\partial_{y'}^{\alpha'}(v_j^{(i)}(y)-v_j^{(i-1)}(y))|\\
\le&\quad \alpha'!\,2^{-5i}M^4r^8\frac{2|y_1|}{(\rho_{i-1}(y)/2)^2}\Big(\frac{8}{|y_1|}+\frac{8}{\rho_{i-1}(y)}\Big)^{\alpha_1}\Big(\frac{8r^2}{\rho_{i-1}(y)}\Big)^{\alpha_2+\alpha_3}\\
\le&\quad 2^{-5i}M^3r^7\frac{|y_1|}{\rho_{i-1}(y)^2}\Big(\frac{1}{|y_1|}+\frac{1}{\rho_{i-1}(y)}\Big)^{\alpha_1}\Big(\frac{r^2}{\rho_{i-1}(y)}\Big)^{\alpha_2+\alpha_3}
\end{aligned}
$$

for $|\alpha'|\le 1$, which gives (18). As for (19), we can similarly prove

$$
\begin{aligned}
&|G(y,\partial_{y'}^{\alpha'}v^{(i-1)},\partial_{y'}^{\beta'}p^{(i-1)})-G(y,\partial_{y'}^{\alpha'}v^{(i-2)},\partial_{y'}^{\beta'}p^{(i-2)})|\\
\le&2^{-5i}M^4r^7\Big(\frac{1}{\rho_{i-1}(y)^2}+\frac{|y_1|}{\rho_{i-1}(y)^3}+\frac{|y_1|^2}{\rho_{i-1}(y)^4}\Big).
\end{aligned}
$$

Let us denote $\rho_{i-1}(y)$ by $\rho_{i-1}(y_1)$, for the moment. From the above inequality we obtain

$$
|\partial_{y_1}^2(p^{(i)}(y)-p^{(i-1)}(y))|\le 2^{-5i}M^4r^7\frac{|y_1|^2}{\rho_{i-1}(y_1)^2}\Big(\frac{1}{|y_1|}+\frac{1}{\rho_{i-1}(y_1)}\Big)^2.
$$

It follows that

$$
\begin{aligned}
&|\partial_{y_1}(p^{(i)}(y)-p^{(i-1)}(y))|\\
\le&2^{-5i}M^4r^7\int_0^{y_1}\Big(\frac{1}{\rho_{i-1}(\tau)^2}+\frac{|\tau|}{\rho_{i-1}(\tau)^3}+\frac{|\tau|^2}{\rho_{i-1}(\tau)^4}\Big)|d\tau|\\
\le&2^{-5i}M^4r^7\Big(\frac{1}{\rho_{i-1}(y_1)^2}\int_0^{y_1}|d\tau|+|y_1|\int_0^{y_1}\frac{|d\tau|}{\rho_{i-1}(\tau)^3}+|y_1|^2\int_0^{y_1}\frac{|d\tau|}{\rho_{i-1}(\tau)^4}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\le 2\cdot 2^{-5i}M^4r^7\Big(\frac{|y_1|}{\rho_{i-1}(y_1)^2}+\frac{|y_1|^2}{\rho_{i-1}(y_1)^3}\Big)\\
&\le 2\cdot 2^{-5i}M^4r^7\frac{|y_1|^2}{\rho_{i-1}(y_1)^2}\Big(\frac{1}{|y_1|}+\frac{1}{\rho_{i-1}(y)}\Big).
\end{aligned}
$$

We also have

$$
\begin{aligned}
&|p^{(i)}(y)-p^{(i-1)}(y)|\\
\le& 2\cdot 2^{-5i}M^4r^7\int_0^{y_1}\Big(\frac{|\tau|}{\rho_{i-1}(\tau)^2}+\frac{|\tau|^2}{\rho_{i-1}(\tau)^3}\Big)|d\tau|\\
\le& 2\cdot 2^{-5i}M^4r^7\Big(\frac{|y_1|}{\rho_{i-1}(y_1)^2}\int_0^{y_1}|d\tau|+|y_1|^2\int_0^{y_1}\frac{|d\tau|}{\rho_{i-1}(\tau)^3}\Big)\\
\le& 4\cdot 2^{-5i}M^4r^7\frac{|y_1|^2}{\rho_{i-1}(y_1)^2}.
\end{aligned}
$$

Therefore we have

$$
|\partial_{y_1}^k(p^{(i)}(y)-p^{(i-1)}(y))|\le 4\cdot 2^{-5i}M^4r^7\frac{|y_1|^2}{\rho_{i-1}(y_1)^2}\Big(\frac{1}{|y_1|}+\frac{1}{\rho_{i-1}(y_1)}\Big)^k
$$

for $0\le k\le 2$. From (b) of Lemma 2, we obtain

$$
\begin{aligned}
&|\partial_{y'}^{\beta'}(p^{(i)}(y)-p^{(i-1)}(y))|\\
\le\;& \beta_2!\beta_3!\;4\cdot 2^{-5i}M^4r^7\frac{|y_1|^2}{(\rho_{i-1}(y)/2)^2}\Big(\frac{1}{|y_1|}+\frac{1}{\rho_{i-1}(y)}\Big)^{\beta_1}\Big(\frac{8r^2}{\rho_{i-1}(y)}\Big)^{\beta_2+\beta_3}\\
\le\;& 2^{-5i}M^3r^6\frac{|y_1|^2}{\rho_{i-1}(y)^2}\Big(\frac{1}{|y_1|}+\frac{1}{\rho_{i-1}(y)}\Big)^{\beta_1}\Big(\frac{r^2}{\rho_{i-1}(y)}\Big)^{\beta_2+\beta_3}
\end{aligned}
$$

for $|\beta'|\le 2$, which gives (19). Therefore (a) of Proposition 1 is true for $i=i_0$, and (b) is also true. The proof of Proposition 1 is completed.

Corollary. *Taking $r>0$ smaller, the sequence $(v^{(i)},p^{(i)})$ converges to (v,p) uniformly on $\mathcal{R}(\omega(r))$, and we obtain*

$$
v_j(y)\in\mathcal{O}^1(\mathcal{R}(\omega(r))),\ p(y)\in\mathcal{O}^2(\mathcal{R}(\omega(r)))
$$

satisfying (13).

We next prove the uniqueness of the solution of (13).

Proposition 2. *If*

$$v_j(y) \in \mathcal{O}^1(\mathcal{R}(\omega(r))),\ p(y) \in \mathcal{O}^2(\mathcal{R}(\omega(r)))$$

and

$$w_j(y) \in \mathcal{O}^1(\mathcal{R}(\omega(r))),\ q(y) \in \mathcal{O}^2(\mathcal{R}(\omega(r)))$$

satisfy (13), *then we have* $(v,p)=(w,q)$.

Proof. The proof is similar to that of Proposition 1. We have

$$\begin{cases} \partial_{y_0}(v_j - w_j) = F_j(y,\ \partial_{y'}^{\alpha'} v,\ \partial_{y'}^{\beta'} p) - F_j(y,\ \partial_{y'}^{\alpha'} w,\ \partial_{y'}^{\beta'} q), & \\ v_j(0,y') - w_j(0,y') = 0, & 1 \le j \le 3, \\ \partial_{y_1}^2 (p-q) = G(y,\ \partial_{y}^{\alpha'} v,\ \partial_{y}^{\beta'} p) - G(y, \partial_{y}^{\alpha'} w, \partial_{y}^{\beta'} q), & \\ \partial_{y_1}^k (p(y_0,0,y_2,y_3) - q(y_0,0,y_2,y_3) = 0, & 0 \le k \le 1. \end{cases}$$

We can similarly prove

$$|\partial_{y'}^{\alpha'}(v_j - w_j)| \le 2^{-i} M |y_1|^{1-\alpha_1},$$
$$|\partial_{y'}^{\beta'}(p-q)| \le 2^{-i} M r^{-2} |y_1|^{2-\beta_1}$$

for $|\alpha'| \le 1,\ |\beta'| \le 2$ on $\mathcal{R}(\omega_i(r))$, for an arbitrary $i \ge 1$. Therefore we have $(v,p)=(w,q)$. $\square$

4. Characteristic Hypersurface

To complete the proof of Theorem 1, it remains to prove that $Z = \{x \in \mathbf{C}^4;\ x_1 = \varphi(x_0,x_2,x_3)\}$ is characteristic (i.e., there exists a function ψ satisfying (6), vanishing precisely on Z). We denote $y = (x_0, x_1 - \varphi(x_0,x_2,x_3), x_2, x_3)$ as before. We first prepare two lemmas.

Lemma 3. *If* $h(x) \in \mathcal{O}^k(\mathcal{R}(\omega \backslash Z))$, *we have* $(x_1 - \varphi)h(x) \in \mathcal{O}^{k+1}(\mathcal{R}(\omega \backslash Z))$, *shrinking* ω *if necessary.*

Proof. Let $r>0$ be small. If $|y|<r,\ y_1 \ne 0$, then we have $|\partial_{y_1}^l h| \le M$ for $0 \le l \le k$ with some $M>0$. By the Cauchy integration theorem we may also assume $|\partial_{y_1}^{k+1} h| \le M|y_1|^{-1}$. It follows that

$$|\partial_{y_1}^l (y_1 h)| \le M', \qquad 0 \le l \le k+1$$

for some M'. Shrinking $r > 0$ we have

$$|\partial_y^\alpha (y_1 h)| \le M'', \qquad 0 \le |\alpha| \le k+1$$

for some M''. $\square$

LEMMA 4. *If $k \ge 1$ and $h(x) \in \mathcal{O}^k(\mathcal{R}(\omega \setminus Z))$ satisfies $[h]_Z = 0$, then we have $h(x) = (x_1 - \varphi)h'(x)$ for some $h' \in \mathcal{O}^{k-1}(\mathcal{R}(\omega \setminus Z))$, shrinking ω if necessary.*

PROOF. Let $r > 0$ be small enough. If $|y| < r$, $y_1 \neq 0$, then we define

$$h'(y) = y_1^{-1} h(y) = \int_0^1 (\partial_{y_1} h)(y_0, \theta y_1, y_2, y_3) d\theta.$$

If $|\alpha| \le k-1$, then we have

$$|\partial_y^\alpha h'(y)| = \left| \int_0^1 \theta^{\alpha_1} (\partial_y^\alpha \partial_{y_1} h)(y_0, \theta y_1, y_2, y_3) d\theta \right| \le M$$

for some M. Shrinking ω if necessary, we have

$$|\partial_x^\alpha ((x_1 - \varphi)^{-1} h(x))| \le M'$$

for some M' on $\mathcal{R}(\omega \setminus Z)$. $\square$

We need to show the following fact:

PROPOSITION 3. *There exists $h(x) \in \mathcal{O}^0(\mathcal{R}(\omega \setminus Z))$, such that $|h(x) - 1| \le 1/2$ and $\psi(x) = h(x)(x_1 - \varphi) \in \mathcal{O}^1(\mathcal{R}(\omega \setminus Z))$ satisfies* (6).

PROOF. From (9) we have

$$\begin{aligned}
& \partial_{x_0}(x_1 - \varphi(x_0, x_2, x_3)) + \sum_{1 \le j \le 3} u_j \partial_{x_j}(x_1 - \varphi(x_0, x_2, x_3)) \\
= \; & -\partial_{x_0}\varphi + u_1 - \sum_{2 \le j \le 3} u_j \partial_{x_j} \varphi \\
= \; & u_1 - u_{Z1} + \sum_{2 \le j \le 3} (u_{Zj} - u_j) \partial_{x_j} \varphi \\
= \; & v_1 - \sum_{2 \le j \le 3} v_j \partial_{x_j} \varphi.
\end{aligned}$$

We have $v_j \in \mathcal{O}^1(\mathcal{R}(\omega \setminus Z))$ and $[v_j]_Z = 0$ by Proposition 1 (and its Corollary). From Lemma 4, we have

$$\partial_{x_0}(x_1 - \varphi) + \sum_{1 \le j \le 3} u_j \partial_{x_j}(x_1 - \varphi) = (x_1 - \varphi)h'(x) \tag{25}$$

for some $h'(x) \in \mathcal{O}^0(\mathcal{R}(\omega \setminus Z))$. Setting $\psi(x) = h(x)(x_1 - \varphi)$, we may rewrite (6) in the following form:

$$\begin{aligned} & \partial_{x_0}\psi + \sum_{1 \le j \le 3} u_j \partial_{x_j}\psi \\ = \; & h\{\partial_{x_0}(x_1 - \varphi) + \sum_{1 \le j \le 3} u_j \partial_{x_j}(x_1 - \varphi\}\} \\ & + (x_1 - \varphi)\{\partial_{x_0}h + \sum_{1 \le j \le 3} u_j \partial_{x_j} h\} \\ = \; & (x_1 - \varphi)hh' + (x_1 - \varphi)\{\partial_{x_0}h + \sum_{1 \le j \le 3} u_j \partial_{x_j} h\} = 0. \end{aligned}$$

Therefore we need to solve

$$\begin{cases} \partial_{x_0}h(x) + \sum_{1 \le j \le 3} u_j(x)\partial_{x_j}h(x) = -h'(x)h(x), \\ h(0, x') = 1. \end{cases} \tag{26}$$

To complete the proof of Proposition 3, it suffices to prove that there exists a solution $h(x) \in \mathcal{O}^0(\mathcal{R}(\omega \setminus Z))$ of (26). From (25) we have

$$\begin{aligned} & \partial_{x_0} + \sum_{1 \le j \le 3} u_j \partial_{x_j} \\ = \; & \partial_{y_0} + \sum_{2 \le j \le 3} u_j \partial_{y_j} + (u_1 - \partial_{x_0}\varphi - \sum_{2 \le j \le 3} u_j \partial_{x_j}\varphi)\partial_{y_1} \\ = \; & \partial_{y_0} + \sum_{2 \le j \le 3} u_j \partial_{y_j} + (x_1 - \varphi)h'\partial_{y_1}. \end{aligned}$$

Therefore we can rewrite (26) in the following form:

$$\begin{cases} \partial_{y_0}h(y) + y_1 h'(y)\partial_{y_1}h(y) + \sum_{2 \le j \le 3} u_j(y)\partial_{y_j}h(y) = -h'(x)h(x), \\ h(0, y') = 1. \end{cases}$$

We set $h^{(0)}(y) = 1$, and solve

$$\partial_{y_0}h^{(i)}(y) = H^{(i-1)}(y), \quad h^{(i)}(0, y') = 1$$

inductively for $i \geq 1$, where

$$H^{(i-1)}(y) = -y_1 h'(y)\partial_{y_1} h^{(i-1)}(y) - \sum_{2\leq j\leq 3} u_j(y)\partial_{y_j} h^{(i-1)}(y) - h'(y)h^{(i-1)}(y).$$

As before, we assume that $M > 0$ is large, and $0 < r << 1/M$. We define $\Omega = \{y \in \mathbf{C}^4;\ y_1 \neq 0,\ |y_j| < r,\ 0 \leq j \leq 3\}$. We can inductively prove

$$|h^{(i)}(y) - h^{(i-1)}(y)| \leq M^{i+1}|y_0|^i\Big(\sum_{1\leq j\leq 3} \frac{1}{r - |y_j|}\Big)^i \tag{27}$$

for $i \geq 1$ on $\mathcal{R}(\Omega)$. If $i = 1$, we have $H^{(0)}(y) = -h'(y)$ and

$$h^{(1)}(y) - h^{(0)}(y) = -\int_0^{y_0} h'(\tau, y')d\tau,$$

and we obtain (27) for $i = 1$.

We next assume that $i_0 \geq 2$, and (27) is true for $i = i_0 - 1$. Let us consider the case $i = i_0$. We remark that if $y \in \Omega$ and $z' = (z_1, z_2, z_3) \in \mathbf{C}^3$ satisfies

$$\begin{cases} |z_1| \leq \dfrac{r - |y_1|}{i+1},\ \dfrac{|y_1|}{2}, \\ |z_j| \leq \dfrac{r - |y_j|}{i+1}, \qquad 2 \leq j \leq 3, \end{cases}$$

then $(y_0, y' + z') \in \Omega$. From the assumption of induction and the Cauchy integration theorem, we obtain

$$\begin{aligned} & |\partial_{y_1}(h^{(i-1)}(y) - h^{(i-2)}(y))| \\ \leq\ & M^i|y_0|^{i-1}\Big(\frac{2}{|y_1|} + \sum_{1\leq j\leq 3} \frac{i+1}{r - |y_j|}\Big) \\ & \times \Big(\sum_{1\leq j\leq 3} \frac{1}{r - |y_j| - (r - |y_j|)/(i+1)}\Big)^{i-1} \\ \leq\ & M^i|y_0|^{i-1}(i+1)\Big(\frac{i+1}{i}\Big)^{i-1}\Big(\frac{1}{|y_1|} + \sum_{1\leq j\leq 3} \frac{1}{r - |y_j|}\Big) \\ & \times \Big(\sum_{1\leq j\leq 3} \frac{1}{r - |y_j|}\Big)^{i-1} \\ \leq\ & (i+1)eM^i|y_0|^{i-1}|y_1|^{-1}\Big(\sum_{1\leq j\leq 3} \frac{1}{r - |y_j|}\Big)^i. \end{aligned}$$

Similarly we can prove

$$|\partial_{y_j}(h^{(i-1)}(y) - h^{(i-2)}(y))| \le (i+1)eM^i|y_0|^{i-1}\Big(\sum_{1\le j\le 3}\frac{1}{r-|y_j|}\Big)^i$$

for $j = 2, 3$. Therefore we have

$$|H^{(i-1)}(y) - H^{(i-2)}(y)| \le (i+1)M^{i+1}|y_0|^{i-1}\Big(\sum_{1\le j\le 3}\frac{1}{r-|y_j|}\Big)^i.$$

Integrating this term with respect to y_0, we obtain (27) for $i = i_0$.

The inequality (27) means the convergence of $h^{(i)}(y)$ on $\mathcal{R}(\Omega)$, shrinking Ω. We have $h(x) = \lim_{i\to\infty} h^{(i)}(x) \in \mathcal{O}^0(\mathcal{R}(\omega \setminus Z))$. □

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