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Taut Foliations of Torus Knot Complements

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Abstract. We show that for any torus knot $K(r, s)$, $|r| > s > 0$, there is a family of taut foliations of the complement of $K(r, s)$, which realizes all boundary slopes in $(-\infty, 1)$ when $r > 0$, or $(-1, \infty)$ when $r < 0$. This theorem is proved by a construction of branched surfaces and laminations which are used in the Roberts paper [5]. Applying this construction to a fibered knot K', we also show that there exists a family of taut foliations of the complement of the cable knot K of K' which realizes all boundary slopes in $(-\infty, 1)$ or $(-1, \infty)$. Further, we partially extend the theorem of Roberts to a link case.

1. Introduction

In this paper, we discuss taut foliations of the complement of a torus knot. A taut foliation of a 3-manifold is a codimension one foliation such that there is a circle which intersects every leaf transversely. There are a lot of studies on foliations of a 3-manifold, many of these indicate that the structure of foliations reflects well the topology of a manifold. Novikov [3] showed that if a 3-manifold other than $S^2 \times S^1$ possesses a foliation without Reeb components, its fundamental group is infinite, the second homotopy group π_2 is trivial and its leaves are all π_1-injective. Rosenberg [8] showed that if a 3-manifold possesses a foliation without Reeb components, then the manifold is irreducible. Combining theorems of Novikov and Rosenberg with that of Palmeira [4], one can see that if a 3-manifold possesses a foliation without Reeb components its universal cover is homeomorphic to $\mathbb{R}^3$. Therefore the existence of "Reebless" foliations plays an important role in studies of a 3-manifold. In fact, a Reeb component has no transverse circle which intersects all leaves, and hence a taut foliation has no Reeb component. Thus a taut foliation takes over the fruits of "Reebless" foliations with respect to the topological properties.

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Rachel Roberts showed the following theorem.

THEOREM 1.1 (Roberts [5]). *Let M be an orientable, fibered compact 3-manifold with single boundary component, whose fiber is a surface of negative Euler characteristic with one puncture. Then there is an interval $(-a, b)$ for some $a, b > 0$ such that for any $\rho \in (-a, b)$ there is a taut foliation which realizes a boundary slope ρ.*

The boundary of such manifold M is a torus, and the boundaries of leaves of these taut foliations are parallel simple curves on the torus. Since a torus is homeomorphic to the quotient space $\mathbb{R}^2/\mathbb{Z}^2$, a simple curve on a torus is regarded as a straight line on the quotient space. Then the boundary slope of a taut foliation means a slope of the simple curve which is a boundary of a leaf of the foliation. If one performs the Dehn filling to the manifold with that taut foliation by the slope ρ belonging to the interval $(-a, b) \cap \mathbb{Q}$, a taut foliation of a closed manifold is obtained. Hence one of the advantages of the theorem of Roberts is that one can estimate a range of slopes in which a taut foliation survives after doing the Dehn filling.

THEOREM 3.1 (Main Theorem). *For any (r, s)-type torus knot $K(r, s)$ in S^3, where r and s are relatively prime integers and $|r| > s > 0$, there is a family of taut foliations in the complement of $K(r, s)$ which realize all boundary slopes in $(-\infty, 1)$ when $r > 0$, or $(-1, \infty)$ when $r < 0$.*

Theorem 3.1 leads one to the conclusion that all the Dehn surgeries along any torus knot by the slope belonging to these interval yield closed 3-manifolds with a taut foliation.

In [6] R.Roberts showed a condition that a fibered hyperbolic manifold with a single boundary component has a family of taut foliations which realizes any boundary slope in $(-\infty, 1)$ or $(-1, \infty)$. Since a torus knot is not a hyperbolic knot, our main theorem indicates a condition for a non-hyperbolic case in comparison with the theorem of [6].

CORROLLARY 4.3. *Let K be a cable knot of a fibered knot in S^3. Then there is a family of taut foliations in the complement of K which realizes all boundary slopes in $(-\infty, 1)$ or $(-1, \infty)$ according to the embedded torus knot.*

Theorem 4.1. *Each iterated torus knot K_i is fibered, and moreover there is a family of taut foliations in the complement of K_i which realizes all boundary slopes in $(-\infty, 1)$ or $(-1, \infty)$ according to the last embedded torus knot.*

We partially extend the theorem of Roberts to a link case as follows.

Theorem 5.1. *Let M be an orientable, fibered compact 3-manifold with two boundary components, whose fiber is a surface with two punctures and its genus is more than two. If the monodromy of the fibration satisfies the condition (1) of Lemma 5.5, then there are intervals $(-a_i, b_i)$ for some $a_i, b_i > 0$ and $i = 1, 2$ such that there is a family of taut foliations which realizes all boundary slopes in each intervals, where i corresponds to each torus boundary component of M.*

2. Preliminaries

In this section, we review some definitions and explain backgrounds which are necessary to understand the main theorem of this paper. Throughout this paper, all manifolds and knots or links are oriented unless otherwise specified. For a manifold M and a submanifold B of M, $N(B)$ denotes the regular neighborhood of B in M.

A *branched surface* B is a compact space modelled locally on the object of Figure 1.

If B lies in a 3-manifold M, we denote a fibered regular neighbourhood of B in M by $N(B)$, locally modelled on Figure 1. When we regard that the

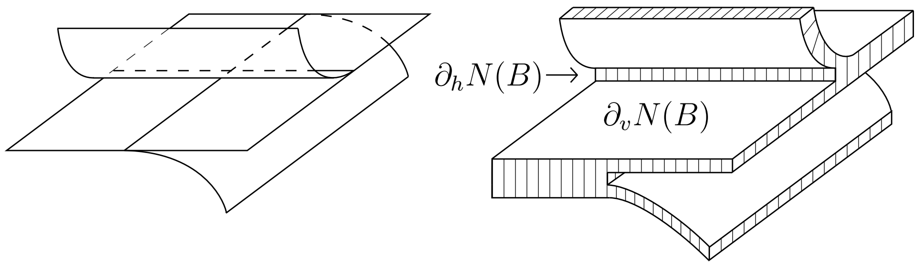


Figure 1.

branched surface B is embedded in $N(B)$, we consider that $N(B)$ is fibered by I-fibers normal to the branched surface B.

For such a fibered regular neighbourhood $N(B)$, we denote the part of $\partial N(B)$ which lies in the set of end points of the I-fibers of $N(B)$ by $\partial_h N(B)$, and the part of $\partial N(B)$ which contains sub arcs of the I-fibers by $\partial_v N(B)$ as in Figure 1. We call that $\partial_h N(B)$ is a *horizontal boundary*, and $\partial_v N(B)$ is a *vertical boundary.* If M has boundaries and the branched surface embedded in M intersects ∂M transversely, $\partial M \cap B$ is a *train track* τ, a space modelled locally on Figure 2. The train track on ∂M has also a fibered regular neighbourhood $N(\tau)$ locally modelled on Figure 2 with I-fiber, and then we denote similarly the part which intersects the endpoints of I-fibers by $\partial_h N(\tau)$ and the part which contains sub arcs of the I-fibers by $\partial_v N(\tau)$.

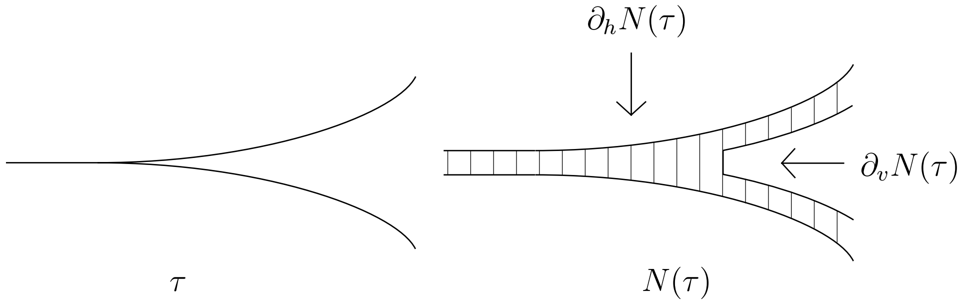


Figure 2.

If we denote the map which collapses all I-fibers by $\pi : N(B) \to B$, a *branch locus* is an arc on B which contains the image of the vertical boundary $\partial_v N(B)$ under the collapsing map π.

The *sectors* $\{S_i\}$ of B are the closures of the components of $B \setminus \{\text{branch locus}\}$. Now we put a weight $\{w_i \geqq 0\}$ on each sector $\{S_i\}$ of B, and we denote the correction of these weights by the vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. The *branch equation* is the equation among the sectors which intersect at the branch loci locally modelled in Figure 3. If we assign weights to sectors as in Figure 3, then the branch equations are $d = e + f$, $b = a + d$ and $c = a + e$. If the vector $\mathbf{w}$ satisfies the branch equations for all branches, we call the vector $\mathbf{w}$ an *invariant measure* of B. The branched

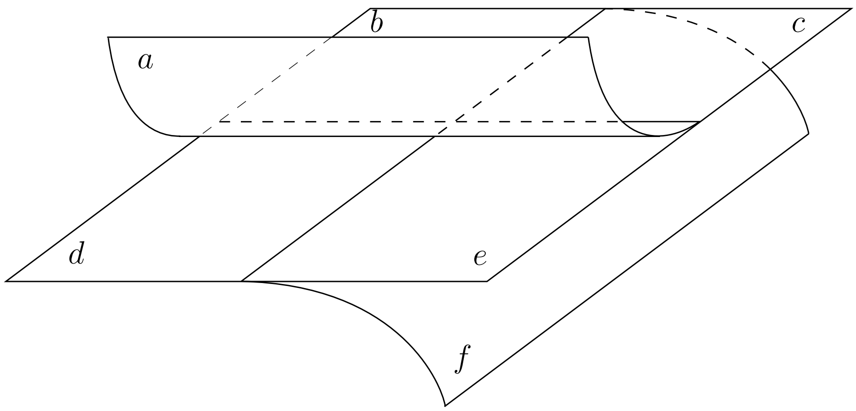


Figure 3.

surface B is called a *measured branched surface* if there is an invariant measure on B. The measures assigned on the sectors induce the measures on the train track τ on the boundary ∂M. Therefore, if B is measured then the train track τ has also an invariant measure. In this case we call that the train track is a *measured train track*.

For a 3-manifold M we say λ is *a lamination* of M if λ is a foliation on a closed subset of M. We see that the measured branched surface B with positive integer weight carries a compact surface, then if we extend these weights to real numbers there is a non-compact surface on $N(B)$. These non-compact surfaces are a source of a measured lamination on $N(B)$.

We define that a lamination λ is *carried* by a branched surface B if it can be isotoped into $N(B)$ everywhere transverse to the fiber of the I-bundle, λ is *fully carried* by B if it also intersects every fiber of the I-bundle.

Related to the main theorem of this paper, we introduce the definition of affinely measured branched surface.

Definition 2.1. Let M be a compact 3-manifold and B be a branched surface embedded in M. If there is a family of simple curves or simple properly embedded arcs $\{\gamma_i\}_{i=1,\dots,n}$ such that $B \setminus \bigcup_{i=1}^n \gamma_i$ has an invariant measure $\mathbf{w}$, then we call that B is *affinely measured with respect to* $\bigcup_{i=1}^n \gamma_i$.

Let M_h be a surface bundle with monodromy h whose fiber is a once punctured oriented surface F of genus g. In fact we see that

$$M_h = F \times [0,1]/(x,1) \sim (h(x),0).$$

We take a family of properly embedded arcs $\{\alpha_i\}_{i=1,\ldots,n}$ in a fiber F and n copies of the fiber,

$$F_0 = F \times \{0\}\,,\ F_1 = F \times \left\{\frac{1}{n}\right\},\ \ldots,\ F_{n-1} = F \times \left\{\frac{n-1}{n}\right\}.$$

For the family of arcs $\{\alpha_i\}_{i=1,\ldots,n}$, we define the family of disks

$$D_1 = \alpha_1 \times \left[0, \frac{1}{n}\right],\ D_2 = \alpha_2 \times \left[\frac{1}{n}, \frac{2}{n}\right],\ \ldots,\ D_n = \alpha_n \times \left[\frac{n-1}{n}, 1\right].$$

Then we construct a branched surface embedded in M_h by combining these copies of fibers and disks whose branch loci are the arcs $\{\alpha_i\}_{i=1,\ldots,n}$. We denote the branched surface B by

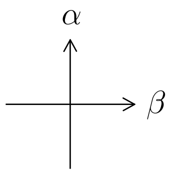
$$B = \langle F_0,\ F_1,\ \ldots,\ F_{n-1}\ ;\ D_1,\ D_2,\ \ldots,\ D_n\rangle.$$

For any knot or link K in S^3, there is a *Seifert surface* S of K such that the boundary of S is equivalent to K, more precisely S intersects $N(K)$ in annuli whose boundary consists of K and an essential, simple closed curves on $\partial N(K)$. We call the latter curves $S \cap \partial N(K)$ *longitude* of K. The longitude is characterized up to isotopy, then we define the *longitudinal slope* of K denoted by λ such that λ is represented by any longitude of K.

Now we prepare the convention for this paper. For given two oriented simple closed curves α and β properly embedded in a surface F, we denote the homological intersection number by $\langle\alpha,\beta\rangle$ with the sign convention for orientation such that if the positive vector of the first curve overlaps to the next one by rotating clockwise by angle $\frac{\pi}{2}$ then $\langle\alpha,\beta\rangle = 1$ (see Figure 4).

For a torus boundary T of a 3-manifold M, we take distinguished two simple closed curves μ and λ on T which satisfy $\langle\mu,\lambda\rangle = 1$. They are called meridian and longitude when M is an exterior of a knot. The pair μ and λ represents a basis of $H_1(T)$, then we also write this basis by μ and λ. Corresponding to this basis (μ,λ), for any given essential simple closed curve γ in T we define the corresponding fraction of the slope which represented by γ by the formula

$$\text{slope } \gamma = \frac{\langle\gamma,\lambda\rangle}{\langle\mu,\gamma\rangle}.$$



$$\langle \alpha, \beta \rangle = 1$$

Figure 4.

Note that by the above definition the slope of λ corresponds to $\frac{0}{1}$, and the slope of μ corresponds to $\frac{1}{0}$.

In this paper we mainly deal with a torus knot embedded in S^3. For a solid torus standardly embedded in S^3, a simple closed curve γ on the boundary T of this solid torus is represented by a form $r\mu + s\lambda \in H_1(T;\mathbb{Z})$. We call γ is a *torus knot or link* of type (r,s), and denote it $K(r,s)$.

Note that if r and s are relatively prime, then γ is one simple closed curve, thus $K(r,s)$ is a knot embedded in S^3. Otherwise $K(r,s)$ is a link embedded in S^3, whose number of components is equal to the greatest common divisor between r and s. $K(r,s)$ has properties that $K(r,s) \cong K(s,r)$ and $K(-r,-s) \cong K(r,s)$, therefore we suppose $|r| > s > 0$. Although $K(-r,s)$ is a mirror image of $K(r,s)$, we distinguish between these knots in this paper.

For a knot K embedded in S^3, K is called *fibered* if the exterior of K is a surface bundle over a circle. It is well known that a torus knot is fibered.

We shall prove this fact by constructing a fiber bundle directly in the exterior of the torus knot $K(r,s)$ in Section 3, but usually it is a well known fact by the theory of singularities of complex functions (see Milnor's book [2]).

3. Main Theorem

Theorem 3.1 (Main Theorem). *Let $K(r,s)$ be the torus knot of type (r,s), where (r,s) is a pair of relatively prime integers and $|r| > s > 0$. Then there is a family of taut foliations $\{\mathcal{F}_x\}$ of the exterior of $K(r,s)$ which realizes any boundary slope in the open interval $(-\infty, 1)$ when $r > 0$, or $(-1, \infty)$ when $r < 0$.*

This theorem is proved as follows. First we construct explicitly a fiber bundle structure of the exterior of the (r, s)-type torus knot $K(r, s)$. Next we choose an arc properly embedded in a fiber surface and then, by the explicit construction of the fibration, we can see the image of this arc under the action of the monodromy of this fibration. Finally, we shall prove that this properly embedded arc and its image is a "good pair" in the sense of the theorem of Roberts and we obtain the desired family of taut foliations $\{\mathcal{F}_x\}$ with parameter x which realizes all boundary slopes in the open interval $(-\infty, 1)$ when $r > 0$, or $(-1, \infty)$ when $r < 0$.

3.1. Constructing fibrations of the exterior of tours knots

In this subsection we construct a fibration as an extension of the example for the trefoil in Rolfsen's book (see [7] section 10.I). Let V be a solid torus standardly embedded in the 3-sphere S^3. We consider that the (r, s)-type torus knot $K(r, s)$ is a simple closed curve on the boundary ∂V of V. Cutting V by a meridian disk D and joining infinitely many copies of this piece, we get the universal cover $\tilde{V}$ of V and the covering $\tilde{K}$ of K on $\partial\tilde{V}$. $\tilde{V}$ becomes a cylinder of infinite length, so we put $\tilde{V}$ into $\mathbb{R}^3$ such that the x-axis is the core of this cylinder. Notice that the number of components of $\tilde{K}$ is s, and then let $k_1(x)$, $k_2(x)$, $\ldots$, $k_s(x)$ be components of $\tilde{K}$.

These components $k_1(x)$, $k_2(x)$, $\ldots$, $k_s(x)$ are the curves represented by following formulae;

$$k_i(x) = (x,\ \cos\frac{r}{s}(x + \frac{2(i-1)\pi}{r}),\ \sin\frac{r}{s}(x + \frac{2(i-1)\pi}{r}))\ \ (i = 1, \ldots, s).$$

Now we construct a surface in the cylinder $\tilde{V}$ as follows. Let ${G_B}^i$ be the twisted band embedded in the part of the cylinder $\tilde{V}$ where $x \in [0, \frac{2\pi}{|r|}]$ represented by following formulae;

$${G_B}^i = \left\{ r_i k_i(x) + (1 - r_i) k_{i-1}\left(\frac{2\pi}{|r|} - x\right) + \left(\frac{2\pi}{|r|}n, 0, 0\right) \;\middle|\; 0 \leqq x \leqq \frac{\pi}{|r|},\ 0 < r_i < 1,\ n = 0, \pm 1, \pm 2, \ldots \right\}$$
$$(i = 1, \ldots, s,\ \ k_0 = k_s) \text{ when } r > 0,$$

$$G_B{}^i = \left\{ r_i k_i(x) + (1-r_i)k_{i+1}\left(\frac{2\pi}{|r|} - x\right) + \left(\frac{2\pi}{|r|}n, 0, 0\right) \;\middle|\; 0 \leqq x \leqq \frac{\pi}{|r|},\ 0 < r_i < 1,\ n = 0, \pm 1, \pm 2, \dots \right\}$$

$$(i = 1, \dots, s,\quad k_{s+1} = k_1) \text{ when } r < 0.$$

For the parameter value $x = \frac{\pi}{|r|}$, there is a disk with s points removed from the boundary. It is the regular polygon with s edges which are parts of boundaries of these bands. Then let G_P be the set of regular polygonal disks embedded into the disks $\left\{(\frac{\pi}{|r|} + \frac{2\pi}{|r|}n,\ y,\ z) \;\middle|\; y^2 + z^2 \leqq 1,\ n = 0, \pm 1, \pm 2, \dots \right\}$ such that the boundary edges of one of these disks P_n are the arcs represented by the following formulae;

$$\partial P_n = \bigcup_{i=1}^{s} \left\{ r_i k_i\left(\frac{\pi}{|r|} + \frac{2\pi}{|r|}n\right) + (1-r_i)k_{i-1}\left(\frac{\pi}{|r|} + \frac{2\pi}{|r|}n\right) \;\middle|\; 0 < r_i < 1 \right\}$$

$$\text{when } r > 0,$$

$$\partial P_n = \bigcup_{i=1}^{s} \left\{ r_i k_i\left(\frac{\pi}{|r|} + \frac{2\pi}{|r|}n\right) + (1-r_i)k_{i+1}\left(\frac{\pi}{|r|} + \frac{2\pi}{|r|}n\right) \;\middle|\; 0 < r_i < 1 \right\}$$

$$\text{when } r < 0.$$

The regular polygonal disk P_n is bounded by the above arcs ∂P_n and embedded in the disk $\left\{(\frac{\pi}{|r|} + \frac{2\pi}{|r|}n\ , y\ , z) \mid y^2 + z^2 \leqq 1\right\}$, therefore $G_P = \bigcup_{n \in \mathbb{Z}} P_n$. Then the surface G which we want to construct in $\tilde{V}$ is defined as the union of the set $G_B = \{G_B^i\}_{i=1,\dots,s}$ and G_P.

Next we define the map $R_\theta : \tilde{V} \longrightarrow \tilde{V}$ given by

$$R_\theta(x, y, z) = \left(x + \frac{\theta}{r}\ ,\ y\cos\frac{\theta}{s} - z\sin\frac{\theta}{s}\ ,\ y\sin\frac{\theta}{s} + z\cos\frac{\theta}{s}\right).$$

LEMMA 3.2. *R_θ turns $\tilde{V}$ by the angle $\dfrac{\theta}{s}$ keeping components $k_1(x)$, $k_2(x)$, $\dots$, $k_s(x)$ of $\tilde{K}$ invariant.*

PROOF. Let $k_i(t) = \left(t,\ \cos\frac{r}{s}(t + \frac{2(i-1)\pi}{r}),\ \sin\frac{r}{s}(t + \frac{2(i-1)\pi}{r})\right)$ be a component of $\tilde{K}$. Then

$$\begin{aligned}
R_\theta(k_i(t)) &= R_\theta\left(t,\ \cos\frac{r}{s}(t + \frac{2(i-1)\pi}{r}),\ \sin\frac{r}{s}(t + \frac{2(i-1)\pi}{r})\right) \\
&= \left(t + \frac{\theta}{r},\ \cos\frac{r}{s}(t + \frac{2(i-1)\pi}{r})\cos\frac{\theta}{s}\right. \\
&\qquad\qquad - \sin\frac{r}{s}(t + \frac{2(i-1)\pi}{r})\sin\frac{\theta}{s}, \\
&\qquad \cos\frac{r}{s}(t + \frac{2(i-1)\pi}{r})\sin\frac{\theta}{s} \\
&\qquad\qquad \left. + \sin\frac{r}{s}(t + \frac{2(i-1)\pi}{r})\cos\frac{\theta}{s}\right) \\
&= \left(t + \frac{\theta}{r},\ \cos(\frac{r}{s}(t + \frac{2(i-1)\pi}{r}) + \frac{\theta}{s}),\right. \\
&\qquad \left. \sin(\frac{r}{s}(t + \frac{2(i-1)\pi}{r}) + \frac{\theta}{s})\right) \\
&= \left(t + \frac{\theta}{r},\ \cos\frac{r}{s}((t + \frac{\theta}{r}) + \frac{2(i-1)\pi}{r}),\right. \\
&\qquad \left. \sin\frac{r}{s}((t + \frac{\theta}{r}) + \frac{2(i-1)\pi}{r})\right) \\
&= k_i(t + \frac{\theta}{r}) \ \square
\end{aligned}$$

We define $G_\theta = R_\theta(G),\ 0 \leqq \theta \leqq 2\pi$.

LEMMA 3.3. *The family of surfaces* $\{G_\theta \mid 0 \leqq \theta \leqq 2\pi\}$ *fills up* $\tilde{V} \setminus \bigcup k_i$. *If* $\theta_i,\ \theta_j \in (0, 2\pi)$ *and* $\theta_i \neq \theta_j$, *then* $G_{\theta_i} \cap G_{\theta_j} = \emptyset$.

PROOF. Let $p = (t,\ u,\ v) \in \tilde{V} \subset \mathbb{R}^3$ be a point in $\tilde{V}$. At first we suppose $r > 0$ and it is sufficient to prove when $0 \leqq t \leqq \frac{2\pi}{r}$. Let D_t and $D_t{}'$ be the disks in $\tilde{V}$ given by

$$D_t = \{(t,\ y,\ z) \mid y^2 + z^2 \leqq 1\},\ \ D_t{}' = \{(t,\ y,\ z) \mid y^2 + z^2 < 1\}.$$

Now we define a flow ψ on $\tilde{V}$ by $\psi = \{R_\theta(w) \mid \theta \in \mathbb{R}\}_{w \in D_0}$. We denote ψ' the flow ψ restricted to $\tilde{V}' = \{(x,\ y,\ z) \mid x \in \mathbb{R},\ y^2 + z^2 < 1\}$, G' the surface G restricted to $\tilde{V}'$.

Claim. The intersection of a flow line l of $\psi'|_{0 \leqq \theta \leqq 2\pi}$ and a surface $G'|_{0 \leqq x \leqq \frac{2\pi}{r}}$ is one point.

Proof of Claim. Let $\mathrm{proj} : \tilde{V}|_{0 \leqq x \leqq \frac{2\pi}{r}} \longrightarrow {D_0}'$ be the perpendicular projection map given by $\mathrm{proj}(x,\ y,\ z) = (0,\ y,\ z)$. Then this map is shown to be one to one and onto when it is restricted to G' as follows. For a point $p \in G'$, if $p \in G_P$, the point p is written as $p = \left(\frac{\pi}{r},\ u,\ v\right)$ and then $\mathrm{proj}(p) = (u, v)$ which belongs to the regular polygonal disk P on D_0. If $p \notin G_P$, let ${G_B^i}'$ be the surface G_B^i restricted to $\tilde{V}'$ and we set $p \in {G_B^i}'$. We can write ${G_B^i}'$ and $k_i(x)$ as follows;

$$
\begin{aligned}
{G_B^i}' &= \left\{ r_i k_i(x) + (1 - r_i) k_{i-1}(\frac{2\pi}{r} - x) \;\middle|\; 0 < x < \frac{\pi}{r},\ 0 < r_i < 1 \right\} \\
k_i(x) &= \left(x,\ \cos \frac{r}{s}(x + \frac{2(i-1)\pi}{r}),\ \sin \frac{r}{s}(x + \frac{2(i-1)\pi}{r}) \right).
\end{aligned}
$$

Then there are real numbers r_p and t_p such that $0 < r_p < 1$ and $0 < t_p < \frac{\pi}{r}$, and we can write $\mathrm{proj}(p)$ as follows;

$$
\begin{aligned}
&\mathrm{proj}(p) \\
&= \mathrm{proj}\left(r_p k_i(t_p) + (1 - r_p) k_{i-1}(\frac{2\pi}{r} - t_p) \right) \\
&= \Bigg(r_p \cos \frac{r}{s}(t_p + \frac{2(i-1)\pi}{r}) + (1 - r_p) \cos \frac{r}{s}(\frac{2\pi}{r} - t_p + \frac{2(i-2)\pi}{r}), \\
&\qquad\qquad r_p \sin \frac{r}{s}(t_p + \frac{2(i-1)\pi}{r}) + (1 - r_p) \sin \frac{r}{s}(\frac{2\pi}{r} - t_p + \frac{2(i-2)\pi}{r}) \Bigg) \\
&= \Bigg(r_p \cos \frac{r}{s}(t_p + \frac{2(i-1)\pi}{r}) + (1 - r_p) \cos \frac{r}{s}(\frac{2(i-1)\pi}{r} - t_p), \\
&\qquad\qquad r_p \sin \frac{r}{s}(t_p + \frac{2(i-1)\pi}{r}) + (1 - r_p) \sin \frac{r}{s}(\frac{2(i-1)\pi}{r} - t_p) \Bigg).
\end{aligned}
$$

By putting $\gamma = r_p$ and $\xi = \frac{2(i-1)\pi}{r}$,

$$\begin{aligned}\mathrm{proj}(p) &= \Big(\gamma \cos \frac{r}{s}(t_p+\xi) + (1-\gamma)\cos\frac{r}{s}(\xi - t_p),\\ &\qquad\qquad \gamma \sin \frac{r}{s}(t_p+\xi) + (1-\gamma)\sin\frac{r}{s}(\xi - t_p)\Big)\\ &= \gamma\left(\cos\frac{r}{s}(t_p+\xi),\ \sin\frac{r}{s}(t_p+\xi)\right)\\ &\qquad + (1-\gamma)\left(\cos\frac{r}{s}(\xi-t_p),\ \sin\frac{r}{s}(\xi-t_p)\right).\end{aligned}$$

Let $\alpha(t)$ and $\beta(t)$ be the points on the boundary $\partial D_0 = \{(y,\ z) \in D_0 \mid y^2 + z^2 = 1\}$ such that

$$\begin{aligned}\alpha(t) &= \left(\cos\frac{r}{s}(t+\xi), \sin\frac{r}{s}(t+\xi)\right)\\ \beta(t) &= \left(\cos\frac{r}{s}(\xi-t), \sin\frac{r}{s}(\xi-t)\right).\end{aligned}$$

Then we can write the image of ${G_B^i}'$ under the map proj as follows;

$$\mathrm{proj}({G_B^i}') = \left\{\gamma\alpha(t) + (1-\gamma)\beta(t) \in D_0 \;\middle|\; 0 < t < \frac{\pi}{r},\ 0<\gamma<1\right\}.$$

Gathering this image for all $i = 1, 2, \ldots, s$, these fill the complement of P in ${D_0}'$. Thus we proved that the map proj is one to one and onto.

For a point $p = (t,\ u,\ v) \in \tilde{V}'|_{0 \leqq x \leqq \frac{2\pi}{r}}$, let l be the flow line of ψ' which contains the point p. By the constructions a flow line of ψ' intersects ${G_B^i}'$ and G_P transversely, and at its intersection point it always goes from a negative side to a positive side if the orientation of these surface is induced from the direction of x-axis. Since the perpendicular projection map is one to one and onto by the above argument, the number of intersections between a flow line and $G'|_{0 \leqq x \leqq \frac{2\pi}{r}}$ is odd. By comparing the gradient direction of a flow line and ${G_B^i}'$ it can be seen that they have only one intersection when $0 \leqq x \leqq \frac{2\pi}{r}$. Thus the flow line l intersects $G'|_{0 \leqq x \leqq \frac{2\pi}{r}}$ at one point $p' = (t',\ u',\ v')$. By the definition of the flow ψ', $R_{\frac{t-t'}{r}}(p') = p$. Therefore this point p exists on $R_{\frac{t-t'}{r}}(G') = {G_{\frac{t-t'}{t}}}'$, so there is an unique θ such that $p \in G_\theta$.

For a point $p = (t, u, v) = (t,\ \cos\tau,\ \sin\tau) \in (\tilde{V} \setminus \tilde{V}')|_{0 \leqq x \leqq \frac{2\pi}{r}} \setminus \bigcup k_i$, there exist some i and a path l on the boundary $\partial\tilde{V}$ such that

$$\begin{aligned} p \in l = \Bigg\{ & r_i k_i \left(\frac{s}{r}(\tau - \frac{2(i-1)\pi}{r}) \right) \\ & + (1-r_i) k_{i-1} \left(\frac{2\pi}{r} - \frac{s}{r}(\tau - \frac{2(i-1)\pi}{r}) \right) \;\Big|\; 0 < r_i < 1 \Bigg\} \\ = {} & R_{(\tau - \frac{2(i-1)\pi}{s})} \left(\left\{ r_i k_i(0) + (1-r_i) k_{i-1} \left(\frac{2\pi}{r} \right) \;\Big|\; 0 < r_i < 1 \right\} \right). \end{aligned}$$

Since the path l is contained in $R_{(\tau - \frac{2(i-1)\pi}{s})}(G_B^i)$, the point p is contained in this image. Therefore there exists an unique θ such that $p \in G_\theta$.

The case when $r < 0$ is proved by the similar argument, then this completes the proof of Lemma 3.3. □

Adding some points to G_θ and modifying G_θ in a neighbourhood of $\partial\tilde{V}$, we see that the boundary of G_θ consists of $k_1(x), \dots, k_s(x)$ and s lines $\{l_i\}_{i=1}^s$, where $l_i = \left\{ \left(x,\ \cos\left(\frac{2(i-1)\pi}{s} + \frac{\theta}{s}\right),\ \sin\left(\frac{2(i-1)\pi}{s} + \frac{\theta}{s}\right) \right) \;\Big|\; x \in \mathbb{R} \right\}$. By the above explicit construction of G_θ on $\tilde{V}$, every G_θ is invariant under the covering transformation of $\tilde{V}$. Then we can project G_θ to the surface ${F_\theta}' = q(G_\theta)$ on V by the covering map $q : \tilde{V} \to V$. The family of surfaces $\{{F_\theta}' = q(G_\theta) \mid 0 \leqq \theta < 2\pi\}$ fills up V since all G_θ are disjoint by Lemma 3.3, and each surfaces satisfy $\partial {F_\theta}' \supset K(r,s)$.

Next we define the s lines $\tilde{C}^1, \tilde{C}^2, \dots, \tilde{C}^i, \dots, \tilde{C}^s$ on $\partial\tilde{V}$ as follows;

$$\tilde{C}^i = \left\{ (x,\ \cos\frac{2(i-1)\pi}{s},\ \sin\frac{2(i-1)\pi}{s}) \;\Big|\; x \in \mathbb{R} \right\},\quad (i = 1, \dots, s).$$

We define the family of lines $\{\tilde{C}_\theta^i \mid 0 \leqq \theta < 2\pi\}$ on $\partial\tilde{V}$ by $\tilde{C}_\theta^i = R_\theta(\tilde{C}^i)$. Now we project this family to V, and get the family of curves $\{C_\theta^i = q(\tilde{C}_\theta^i) \mid 0 \leqq \theta < 2\pi\}$ on ∂V. The boundary of ${F_\theta}'$ on ∂V consists of the union of our torus knot $K(r,s)$ and this family of curves, that is,

$$\partial {F_\theta}' = K(r,s) \cup \left(\bigcup_{i=1}^s C_\theta^i \right).$$

By definition, V is a solid torus standardly embedded into S^3. So let W be the complement of V in S^3, then W also is a solid torus standardly embedded into S^3. By the above construction, a curve of this family

$\{\tilde{C_\theta^i} \mid 0 \leqq \theta < 2\pi\}$ is a longitude curve on ∂V, then it is a meridian curve on ∂W and we define the meridian disks D_θ^i such that $\partial D_\theta^i = C_\theta^i$.

Finally we define the surface

$$F_\theta = \left(F_\theta{}' \cup \left(\bigcup_{i=1}^{s} D_\theta^i \right) \right) \setminus K(r,s),$$

and the map $p : S^3 \setminus K(r,s) \longrightarrow S^1$ such that if $x \in F_\theta \subset S^3 \setminus K(r,s)$, $p(x) = e^{i\theta} \in S^1$.

LEMMA 3.4. *This map $p : S^3 \setminus K(r,s) \longrightarrow S^1$ defines a fibration on $S^3 \setminus K(r,s)$ whose fiber is F_θ.*

PROOF. Since the surfaces G_θ are disjoint in $\tilde{V}$ by Lemma 3.3 and q is a covering map, the surfaces F_θ are disjoint each other in $S^3 \setminus K(r,s)$. Let $I \subset S^1$ be an open interval on S^1. By the definition of the map p, for $x \in S^1$, $p^{-1}(x) = F_x$, and then $F_x \cap F_y = \emptyset$ when $x \neq y$. Thus $p^{-1}(I) = \coprod_{x \in I} F_x$ which is a disjoint union of fibers. For any $x \in S^1$, F_x is an open set in $S^3 \setminus K(r,s)$, so $p^{-1}(I)$ is open, thus the map p is continuous. As seen before there is the flow ψ on $\tilde{V}$. If we project it to V and denote this flow by $\hat{\psi}$, the flow $\hat{\psi}$ is transverse to $F_\theta{}'$ for any θ in V. The solid torus W is trivially foliated by disks D_θ^i, and there is a flow ϕ transverse to any disk D_θ^i which coincide with $\hat{\psi}$ on the boundary ∂W. By gathering these transverse flows $\hat{\psi}$ and ϕ, we obtain the flow φ on $S^3 \setminus K(r,s)$ transverse to F_θ for any θ. For any point $x \in S^1$ and any interval $x \in I \subset S^1$ we define a map $\eta : p^{-1}(I) \to F_x \times I$ such that $\eta(q) = (\varphi_\tau(q), t)$ where the point $\varphi_\tau(q)$ is the point on which the flow line of φ through q intersects $F_x = p^{-1}(x)$, and $t = p(q)$. Since φ is a transverse flow, the map η becomes a trivialization map of this fibration. $\square$

Thus we complete an explicit construction of a fibration for the torus knot $K(r,s)$.

3.2. Proof of Main theorem

We define the coordinate system on the torus boundary $\partial M = \partial N(K(r,s))$ by choosing two specific oriented simple closed curves λ and

μ as follows. Let λ be a curve such that $\lambda = \partial F_0$, and we call it a longitude. The orientation of λ is induced from the orientation of F_0. Let μ be a curve on ∂M such that it satisfies $\langle \mu, \lambda \rangle = 1$ and bounds an essential disk in $N(K(r,s))$. Let $h : F_0 \to F_0$ be the monodromy map of this fibration. This map is induced from the rotation map $R_{2\pi}$ on $\tilde{V}$.

From now on we shall construct a family of taut foliations by the same way taken in the section 3 of [5].

Now we consider the complement $M = \overline{S^3 \setminus N(K(r,s))}$ of the torus knot $K(r,s)$ as the quotient space of the product of the fiber F_0 and a unit interval $I = [0,1]$;

$$M = F_0 \times [0,1] \big/ (x,\ 1) \sim (h(x),\ 0).$$

Note that the boundary ∂M is homeomorphic to a torus since the boundary of F_0 is a circle and h maps this circle to itself. The positive side of F_0 is defined by a positive direction of the unit interval $[0,1]$.

Next we choose three properly embedded arcs α and $\beta_\pm$ on the fiber F_0. Let $\tilde{\alpha}$ and $\tilde{\beta}_\pm$ be the arcs on $\partial\tilde{V}$ such that

$$\begin{aligned}
\tilde{\alpha} &= \left\{ (t,\ 1,\ 0) \in \partial\tilde{V} \;\middle|\; 0 \leqq t \leqq \frac{2\pi}{|r|} \right\},\\
\tilde{\beta}_+ &= \left\{ (t,\ \cos\frac{2\pi}{s},\ \sin\frac{2\pi}{s}) \in \partial\tilde{V} \;\middle|\; \frac{2\pi}{|r|} \leqq t \leqq \frac{4\pi}{|r|} \right\},\\
\tilde{\beta}_- &= \left\{ (t,\ \cos\frac{2\pi}{s},\ \sin\frac{2\pi}{s}) \in \partial\tilde{V} \;\middle|\; -\frac{2\pi}{|r|} \leqq t \leqq 0 \right\}.
\end{aligned}$$

We define the three arcs α and $\beta_\pm$ on the fiber F_0 so that $\alpha = q(\tilde{\alpha})$, $\beta_+ = q(\tilde{\beta}_+)$ and $\beta_- = q(\tilde{\beta}_-)$, and give an orientation to these three arcs induced from the orientation of $\tilde{\alpha}$ and $\tilde{\beta}_\pm$ defined by the increasing direction of t. The fiber F_0 is an open surface, but attaching a copy of our torus knot $K(r,s)$ to it we regard it as a compact surface whose boundary is on $\partial N(K(r,s))$. If we regard the fiber as a compact surface, these three arcs α and $\beta_\pm$ are properly embedded arcs whose each end points $\partial\alpha$ and $\partial\beta_\pm$ sit on $\partial N(K(r,s))$. In the later argument, we always regard a fiber surface as a compact surface whose boundary is on $\partial N(K(r,s))$. By the construction of the fibration, the monodromy h maps α to $\beta_\pm$, i.e. $h(\alpha) = \beta_+$ when $r > 0$ and $h(\alpha) = \beta_-$ when $r < 0$.

Using the unit interval $[0,1]$, we define a disk D in M such that $D = \alpha \times [0,1]$. Note that the boundary ∂D of the disk consists of four arcs, α, β_+ or β_- on F_0 and $\partial\alpha \times [0,1]$ on ∂M.

Now we define the branched surface B_- such that $B_- = \langle F_0; D\rangle$. We shall prove that this branched surface B_- carries a family of laminations $\{\lambda_x\}$ which realize all boundary slopes in $(-\infty, 0]$, and then these laminations λ_x extend to taut foliations $\mathcal{F}_x$ by filling up complementary regions. In order to prove the Main theorem, we recall a definition of "*good pair*" stated in [5], and need some lemmas.

DEFINITION 3.5. Let F be a compact surface with a single circle boundary component and negative Euler characteristic, and δ and δ' be simple arcs properly embedded in F. The pair (δ, δ') is called *good* if δ and δ' are disjoint on F, and their endpoints alternate along ∂F as are shown in Figure 5.

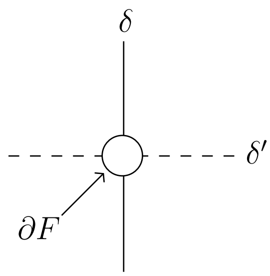


Figure 5.

Note that for a good pair (δ, δ'), each simple arc is non-separating on F.

LEMMA 3.6. *Let α and $\beta_\pm$ be the simple arcs on F_0 defined above, then (α, β_+) and (α, β_-) are a good pair.*

PROOF. The arc α is a part of the boundary of D_0^1 and both $\beta_\pm$ are a part of the boundary of D_0^2. Since $D_0^i \cap D_0^j = \emptyset$ if $i \neq j$, and each disk D_0^i is a meridian disk of W, then clearly α and $\beta_\pm$ have no self intersection and are disjoint each other.

Let $\partial^1\tilde{\alpha}$ be the point on $\partial\tilde{V}$ such that $\partial^1\tilde{\alpha} = (0,\ 1,\ 0) \in \tilde{V} \subset \mathbb{R}^3$. It is one of the end points of $\tilde{\alpha}$ and is on the component $k_1(x)$ of $\tilde{K}(r,s)$. $\partial^1\tilde{\alpha}$ can move along $k_1(x)$ in the direction induced by the rotation map R_θ, then $\partial^1\tilde{\alpha}$ meets one end point of $\tilde{\beta}_\pm$. We denote this point by $\partial^1\tilde{\beta}_\pm$. The other end point $\partial^2\tilde{\alpha}$ of $\partial\tilde{\alpha}$ also meet the other end point of $\tilde{\beta}_\pm$, denoting $\partial^2\tilde{\beta}_\pm$. Then on ∂V we can see that $\partial^1\beta_\pm = q(\partial^1\tilde{\beta}_\pm)$ is next to $\partial^1\alpha = q(\partial^1\tilde{\alpha})$ along the knot $K(r,s)$ and also $\partial^2\beta_\pm = q(\partial^2\tilde{\beta}_\pm)$ is next to $\partial^2\alpha = q(\partial^2\tilde{\alpha})$. Therefore the end points $\partial\alpha$ and $\partial\beta_\pm$ alternate along the knot $K(r,s)$. $\square$

By the construction of the fibration and the definition of the orientation of the arcs α and $\beta_\pm$, the situation of arcs with an orientation near the boundary ∂F_0 is pictured as in Figure 6.

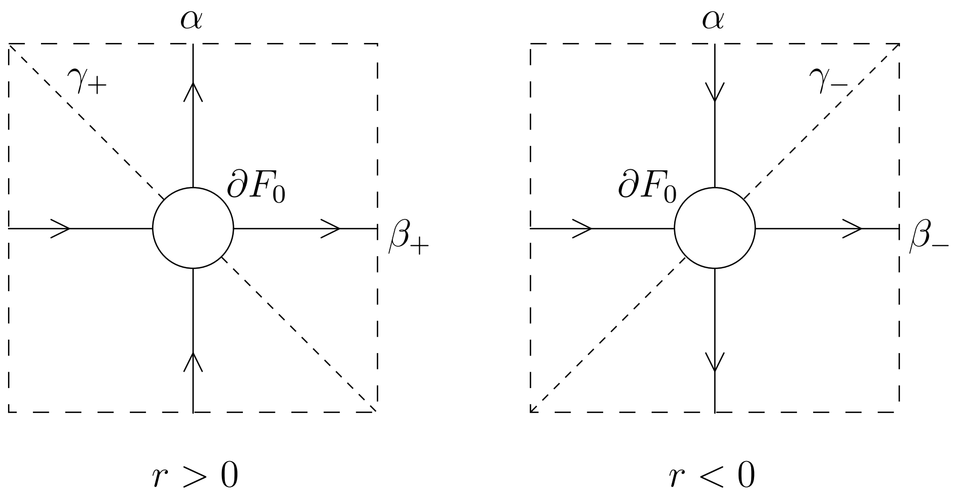


Figure 6.

In order to make branched surfaces which we need to prove the Main theorem we define a convention for the branching direction of the sector which made from an arc properly embedded in a fiber surface and an interval. For an arc α with a given orientation we can take a sector $D = \alpha \times I$ as defined in Section 2 where $I = [a, b]$ is an interval. We define a branching direction of D which corresponds to the orientation of $\alpha \times \partial I$ as in Figure 7. In the neighbourhood of $\alpha \times \{a\}$, the sector D comes down from the left

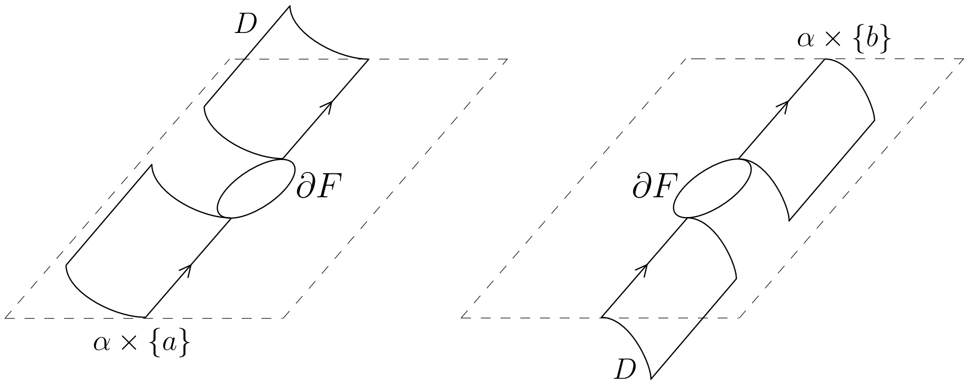


Figure 7.

side of the arc $\alpha \times \{a\}$, and in the neighbourhood of $\alpha \times \{b\}$, D comes up from the right side of the arc $\alpha \times \{b\}$.

Next we choose properly embedded arcs $\gamma_\pm$ on F_0 which satisfy a condition that

$$[\gamma_\pm] = -[\alpha] + [\beta_\pm] \in H_1(F_0, \partial F_0).$$

and its boundary points are on ∂F_0 as pictured also in Figure 6. Taking a disk $D = \alpha \times [0,1] \subset M = F_0 \times [0,1]/\sim$, we define two branched surface such that $B^+_- = \langle F_0; D\rangle$ when $r > 0$ and $B^-_- = \langle F_0; D\rangle$ when $r < 0$. Then by the construction we can see that these two branched surface $B^\pm_-$ are affinely measured with respect to $\gamma_\pm$ respectively in the same way as the proof of Lemma 4.3 of [5].

For these branched surface $B^\pm_-$ with measures, these boundaries $\tau^\pm_- = \partial B^\pm_-$ become a train track with the affine measure which induced from the modified measures on $B^\pm_-$ made by a scaling map on $\gamma_\pm \times I$ such that

$$f : \gamma_\pm \times [0,1] \to \gamma_\pm \times [0,1] : (p,t) \mapsto (p, (1+x)t)$$

where p is a point of $\gamma_\pm$, x is the weight parameter defined on $B^\pm_-$. We denote the lamination on the neighbourhood of the train track $\tau^\pm_-$ induced by these measures by $\tau^\pm_-(w)$. Then these measured train tracks $\tau^\pm_-$ are pictured in Figure 8.

Putting the meridian-longitude pair (μ, λ) as also in Figure 8, we can calculate the intersection number at each intersection point between (μ, λ)

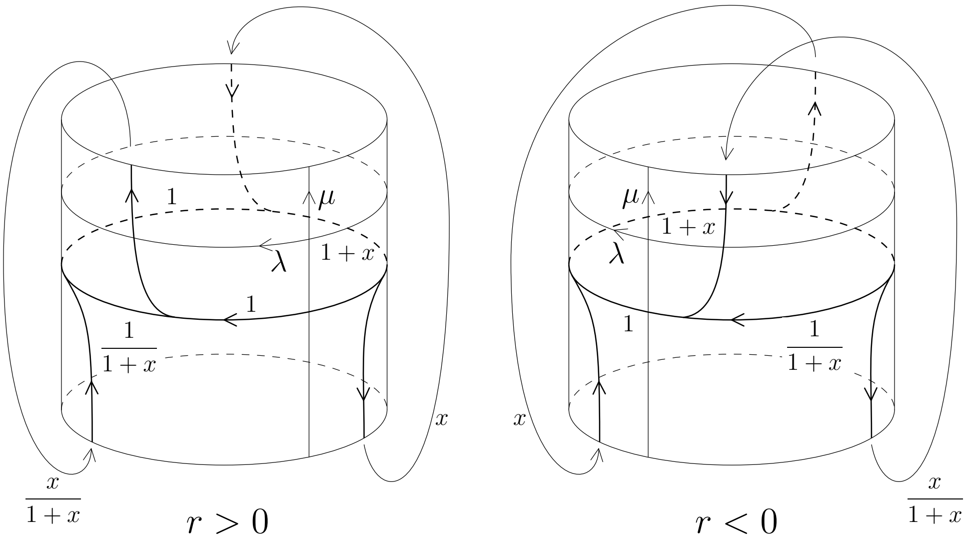


Figure 8.

and the measured lamination $\tau_-^{\pm}(w)$ as follows.

When $r > 0$,

$$\langle \mu, \tau_-^+(w) \rangle = 1, \ \ \langle \tau_-^+(w), \lambda \rangle = \frac{x}{1+x} - x, \ \ \text{then}$$
$$\text{slope } \tau_-^+(w) = \frac{\langle \tau_-^+(w), \lambda \rangle}{\langle \mu, \tau_-^+(w) \rangle} = \frac{x}{1+x} - x = \frac{-x^2}{1+x}.$$

When $r < 0$,

$$\langle \mu, \tau_-^-(w) \rangle = 1, \ \ \langle \tau_-^-(w), \lambda \rangle = -\frac{x}{1+x} + x, \ \ \text{then}$$
$$\text{slope } \tau_-^-(w) = \frac{\langle \tau_-^-(w), \lambda \rangle}{\langle \mu, \tau_-^-(w) \rangle} = -\frac{x}{1+x} + x = \frac{x^2}{1+x}.$$

Letting x range over $[0, \infty)$ for the above formulae, we conclude that the family of laminations $\{\lambda_x\} \subset N(B_-^{\pm})$ carried by $B_-^{\pm}$ with a measure parameter x realizes all boundary slopes in $(-\infty, 0]$ when $r > 0$, or in $[0, \infty)$ when $r < 0$. Similar to the proof of Theorem 4.1 of [5], the family of laminations $\{\lambda_x\}$ extends to the family of taut foliations which realize slopes in the same interval.

In order to complete the proof of Main theorem, we define two branched surfaces $B_+^\pm$ as follows. We take three fibers $F_0 = F_0 \times \{0\}$, $F_1 = F_0 \times \{\frac{1}{3}\}$ and $F_2 = F_0 \times \{\frac{2}{3}\}$. For the branched surface B_+^+, we take three pair of arcs on each F_i for $i = 0, 1, 2$ such that $(\beta_+, -\alpha)$ on F_0, $(-\alpha, -\beta_+)$ on F_1 and $(-\beta_+, \alpha)$ on F_2. Using these pairs of arcs we put three sectors such that $D_1^+ = -\alpha \times [0, \frac{1}{3}]$, $D_2^+ = -\beta_+ \times [\frac{1}{3}, \frac{2}{3}]$ and $D_3^+ = \alpha \times [\frac{2}{3}, 1]$. For the branched surface B_+^-, we also take three pairs of arcs $(\beta_-, -\alpha)$, $(-\alpha, -\beta_-)$ and $(-\beta_-, \alpha)$, and take three sectors $D_1^- = -\alpha \times [0, \frac{1}{3}]$, $D_2^- = -\beta_- \times [\frac{1}{3}, \frac{2}{3}]$ and $D_3^- = \alpha \times [\frac{2}{3}, 1]$. By these settings we define two branched surfaces such that

$$B_+^\pm = \langle F_0, F_1, F_2; D_1^\pm, D_2^\pm, D_3^\pm \rangle$$

whose branch direction on each sectors is defined by our convention. As same as the case of $B_-^\pm$ we can take properly embedded arcs $\gamma_0^\pm$ on F_0, $\gamma_1^\pm$ on F_1 and $\gamma_2^\pm$ on F_2, then we can see that the branched surfaces $B_+^\pm$ are affinely measured with respect to $\gamma_0^\pm \cup \gamma_1^\pm \cup \gamma_2^\pm$. The measured train tracks $\tau_+^\pm = \partial B_+^\pm$ which are the restriction of $B_+^\pm$ on the boundary ∂M are pictured in Figure 9.

Putting the meridian-longitude pair (μ, λ) as also in Figure 9, we can calculate the intersection number at each intersection point between (μ, λ) and the measured lamination $\tau_+^\pm(w)$ as follows.

When $r > 0$,

$$\begin{aligned}
\langle \mu, \tau_+^+(w) \rangle &= \frac{1}{1+x} + 1 + (1+x) = \frac{x^2+3x+3}{1+x} \\
\langle \tau_+^+(w), \lambda \rangle &= -\frac{x}{1+x} + x = \frac{x^2}{1+x} \quad \text{then,} \\
\text{slope } \tau_+^+(w) &= \frac{\langle \tau_+^+(w), \lambda \rangle}{\langle \mu, \tau_+^+(w) \rangle} = \frac{x^2}{x^2+3x+3}.
\end{aligned}$$

When $r < 0$,

$$\begin{aligned}
\langle \mu, \tau_+^-(w) \rangle &= \frac{1}{1+x} + 1 + (1+x) = \frac{x^2+3x+3}{1+x} \\
\langle \tau_+^-(w), \lambda \rangle &= \frac{x}{1+x} - x = \frac{-x^2}{1+x} \quad \text{then,} \\
\text{slope } \tau_+^-(w) &= \frac{\langle \tau_+^-(w), \lambda \rangle}{\langle \mu, \tau_+^-(w) \rangle} = \frac{-x^2}{x^2+3x+3}.
\end{aligned}$$

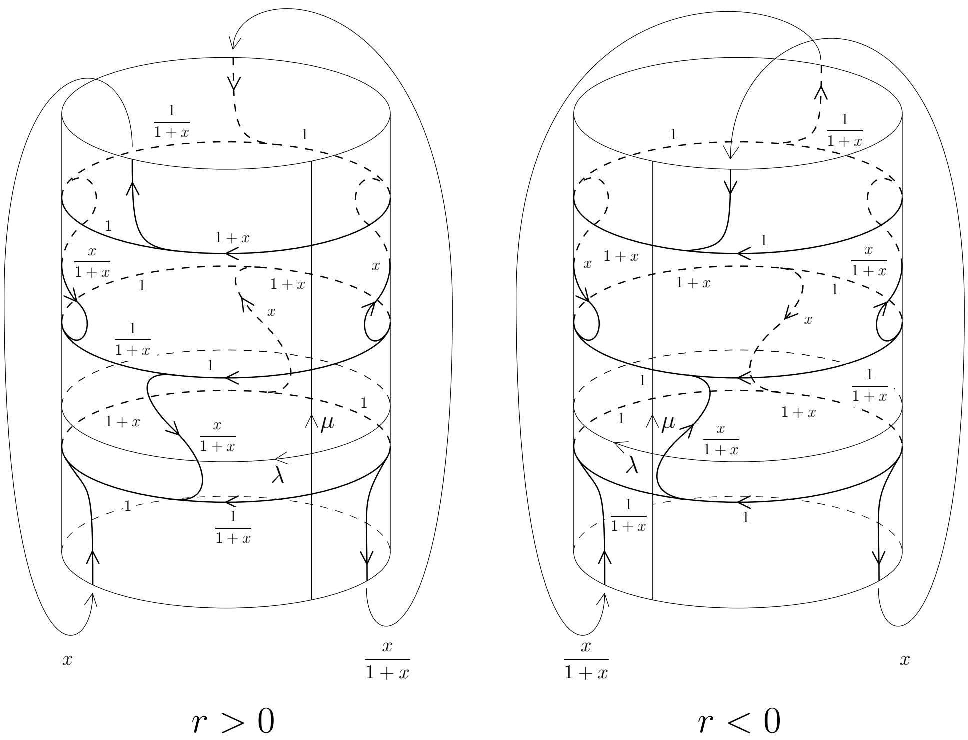


Figure 9.

Letting x range over $[0, \infty)$ for the above formulae, we conclude that the family of laminations $\{\lambda_x\} \subset N(B_+^{\pm})$ carried by $B_+^{\pm}$ with a measure parameter x realizes all boundary slopes in $[0, 1)$ when $r > 0$, or in $(-1, 0]$ when $r < 0$. This family of laminations $\{\lambda_x\}$ also extends to the family of taut foliations which realize slopes in the same interval. Combining the intervals formerly obtained with these intervals, the proof of Main theorem is completed.

4. Iterated Torus Knot Case

In this Section, we extend the result of section 3 to an iterated torus knot. To define an iterated torus knot, at first we define a sequence of solid tori $\{T_i\}$ and knots $\{K_i\}$ embedded in S^3 as follows. The first solid torus

T_0 is standardly embedded in S^3 and let K_0 be a simple closed curve on the boundary ∂T_0. We called K_0 a torus knot before, now we will call it a standard torus knot. A regular neighbourhood of K_0 is also a solid torus, and we denote this solid torus by T_1 which is embedded in S^3. Then we define a new knot K_1 which is a simple closed curve on the boundary ∂T_1. By iterating this construction, the knot K_{i-1} has a regular neighbourhood T_i homeomorphic to a solid torus and there is a new knot K_i which is a simple closed curve on the boundary ∂T_i. To avoid complicated arguments, we assume that each K_i is not homotopic to a meridian curve or a longitude curve on ∂T_i.

To construct a taut foliation made as a modification of fibration, we define these $\{K_i\}$ precisely and construct a fibration of its complement.

Let T_0 be a solid torus standardly embedded in S^3 and $K_0(r_0, s_0)$ a simple closed curve on ∂T_0 which has a homological representation $r_0 m_0 + s_0 l_0 \in H_1(\partial T_0)$, where m_0 is the standard meridian and l_0 is the standard longitude of ∂T_0. Let T_1 be a regular neighbourhood of K_0. The complement $M_0 = \overline{S^3 \setminus N(K_0)}$ has the fibration ξ_0 as seen before, then we define that the longitude l_1 of ∂T_1 is a simple closed curve which coincides with the boundary of a fiber of the fibration ξ_0, and we define the meridian m_1 such that m_1 intersects l_1 transversely at one point and m_1 bounds a disk in T_1. For this meridian-longitude pair we define a new knot $K_1(r_1, s_1)$ which is a simple closed curve on ∂T_1 and has a homological representation $r_1 m_1 + s_1 l_1 \in H_1(\partial T_1)$.

In Section 3.1, we construct a sub surface $F_\theta{}'$ in the solid torus V and prove that the family of surfaces $\{F_\theta{}' \mid 0 \leqq \theta < 2\pi\}$ fills up V. The boundaries of $F_\theta{}'$ consist of circles $\{C_\theta^i\}_{i=1,\dots,s}$ and the torus knot $K(r,s)$ on ∂V. By the construction, the circles $\{C_\theta^i\}_{i=1,\dots,s}$ are parallel on ∂V. Then we replace T_1 by this solid torus V so that the circles $\{C_\theta^i\}_{i=1,\dots,s}$ coincide with the curves parallel to the longitude l_1 and the torus knot $K(r,s)$ on ∂V coincides with $K_1(r_1,s_1)$, that is, $r = r_1$ and $s = s_1$. Since any boundary of fibers of ξ_0 is a curve on ∂T_1 which parallel to a longitude, any surface of the family $\{F_\theta{}' \mid 0 \leqq \theta < 2\pi\}$ is connected to a fiber of ξ_0 via the boundary circles $\{C_\theta^i \mid i = 1, \dots, s_1,\ 0 \leqq \theta < 2\pi\}$. Let F_θ^1 be one of the surfaces made by this construction. F_θ^1 consists of one sub surface $F_\theta{}'$ and s_1 copies of a fiber of ξ_0 which are connected to $F_\theta{}'$ on the circles $\{C_\theta^i\}_{i=1,\dots,s_1}$. Since the family $\{F_\theta{}'\}$ fills up the solid torus T_1

and M_0 is fibered, the family of surfaces $\{F_\theta^1 \mid 0 \leqq \theta < 2\pi\}$ fills up the complement $M_1 = \overline{S^3 \setminus N(K_1(r_1, s_1))}$. Similar to the proof of Lemma 3.3, we can see that surfaces of the family $\{F_\theta^1\}$ are disjoint. Then the map $p : M_1 \to S^1 : x \in F_\theta^1 \mapsto \theta$ defines the fibration ξ_1.

Therefore, $K_1(r_1, s_1)$ is a fibered knot embedded in S^3. Let T_2 be a regular neighbourhood of $K_1(r_1, s_1)$. We define the longitude l_2 on ∂T_2 so that its homology class coincides with the homology class of a curve which is the boundary of a fiber of the fibration ξ_1, and define the meridian m_2 so that it intersects l_2 at one point and bounds a disk in T_2. Then we define a new knot $K_2(r_2, s_2)$ which is a simple closed curve on ∂T_2 whose homology class is represented by $r_2 m_2 + s_2 l_2 \in H_1(\partial T_2)$. Replacing the solid torus T_2 by the same V, we can construct the fibration ξ_2 on $M_2 = \overline{S^3 \setminus N(K_2(r_2, s_2))}$.

Iterating this construction, we can get the sequence of knots $\{K_i(r_i, s_i)\}$, and then we call it an iterated torus knot sequence. Simply, we call $K_i(r_i, s_i)$ an iterated torus knot. By this construction, the complement M_i of every iterated torus knot is fibered with the fibration ξ_i.

For every iterated torus knot, we can extend the result of Theorem 3.1.

THEOREM 4.1. *Let $K_i(r_i, s_i)$ be the iterated torus knot defined as above and we assume $|r_i| > s_i > 0$. Then there is a family of taut foliations $\{\mathcal{F}_x\}$ of the exterior of $K_i(r_i, s_i)$ which realizes any boundary slope in the open interval $(-\infty, 1)$ when $r_i > 0$, or in $(-1, \infty)$ when $r_i < 0$.*

We shall prove this theorem by the same steps as in the proof of Theorem 3.1. First we define two arcs on the fiber of fibration ξ_i. Let $\tilde{\alpha}$ and $\tilde{\beta}_\pm$ be the arcs on $\partial \tilde{V}$ similarly in previous section such that

$$\tilde{\alpha} = \left\{(t,\ 1,\ 0) \in \partial \tilde{V} \;\middle|\; 0 \leqq t \leqq \frac{2\pi}{|r_i|}\right\},$$

$$\tilde{\beta}_+ = \left\{(t,\ \cos\frac{2\pi}{s_i},\ \sin\frac{2\pi}{s_i}) \in \partial \tilde{V} \;\middle|\; \frac{2\pi}{|r_i|} \leqq t \leqq \frac{4\pi}{|r_i|}\right\},$$

$$\tilde{\beta}_- = \left\{(t,\ \cos\frac{2\pi}{s_i},\ \sin\frac{2\pi}{s_i}) \in \partial \tilde{V} \;\middle|\; -\frac{2\pi}{|r_i|} \leqq t \leqq 0\right\}.$$

We define the three arcs α and $\beta_\pm$ on the boundary of ∂V so that $\alpha = q(\tilde{\alpha})$ and $\beta_\pm = q(\tilde{\beta}_\pm)$ where the map $q : \tilde{V} \to V$ is the covering map. Since α and $\beta_\pm$ are the arcs on C_0^1 and C_0^2 respectively, α and $\beta_\pm$ are properly embedded in the fiber F_0 of the fibration ξ_i, and these end points are on the boundary of a regular neighbourhood of the iterated torus knot $K_i(r_i, s_i)$.

Let $h : F_0 \to F_0$ be the monodromy map of the fibration ξ_i. The rotation map $R_\theta : \tilde{V} \to \tilde{V}$ defined in previous section induces the map $\widehat{R_\theta} : V \to V$ by composition with the covering map q, $\widehat{R_\theta} = q \circ R_\theta \circ q^{-1}$. We define a map $h' : V \to V$ by $h' = \widehat{R_{2\pi}}$. By the construction of the fibration ξ_i, the map h' maps the subsurface $F_0 \cap V$ to $F_0 \cap V$. If we define the map h' on complementary region of V such that for $k = 1, \ldots, s_i$, h' maps the fiber F_k^{i-1} of the fibration ξ_{i-1} connected with ${F_0}'$ via C_0^k to the fiber F_{k+1}^{i-1} of ξ_{i-1} connected with ${F_0}'$ via C_0^{k+1}, we can extend h' to the monodromy h. Because of this extension, we can see that $h(\alpha) = \beta_+$ when $r_i > 0$ or $h(\alpha) = \beta_-$ when $r_i < 0$.

Now we consider the complement $M_i = \overline{S^3 \setminus N(K_i(r_i, s_i))}$ of the torus knot $K_i(r_i, s_i)$ as the quotient space of the product of the fiber F_0 and a unit interval $I = [0, 1]$;

$$M_i = F_0 \times [0, 1]/(x, 1) \sim (h(x), 0).$$

We define the orientation of F_0 such that the positive side is the positive direction of the unit interval $[0, 1]$.

We choose the meridian-longitude pair (μ, λ) on $\partial M_i = \partial N(K_i(r_i, s_i))$ so that λ is the boundary of ∂F_0 and μ satisfies that $\langle \mu, \lambda \rangle = 1$ and μ bounds an essential disk in $N(K_i(r_i, s_i))$. In this definition, the orientation of λ is induced from the orientation of F_0.

LEMMA 4.2. *The pairs of properly embedded arcs $(\alpha, \beta_\pm)$ defined above are a good pair.*

PROOF. By the construction of ξ_i, for some parameter $k, k' \in [0, 2\pi)$, α is a part of the boundary of the fiber F_k^{i-1} of the fibration ξ_{i-1} and $\beta_\pm$ are a part of the boundary of the fiber $F_{k'}^{i-1}$. Since F_k^{i-1} and $F_{k'}^{i-1}$ are disjoint, α and $\beta_\pm$ have no self intersection and are disjoint.

Then similar to the proof of Lemma 3.6, tracing four end points $\partial\alpha$, $\partial\beta_\pm$ along $K_i(r_i, s_i)$ with the orientation induced by the orientation of $\tilde{V}$, we can see that the pairs are a good pair. □

Using these pairs of arcs (α, β_+) for $r_i > 0$ and (α, β_-) for $r_i < 0$, by the same steps of the proof of Theorem 3.1 we complete the proof of Theorem 4.1.

The method of the proof of Theorem 4.1 also gives the following Corollary.

COROLLARY 4.3. *Let K be a fibered knot embedded in S^3, and ∂T be a boundary of the regular neighbourhood T of K. Let $\hat{K}(r,s)$ be a simple closed curve on ∂T whose homology class is represented by $rm+sl \in H_1(\partial T)$ where (m,l) is the meridian-longitude pair of ∂T, r and s are relatively prime integers. We suppose $|r| > s > 0$. Then there is a family of taut foliations $\{\mathcal{F}_x\}$ of the exterior of $\hat{K}(r,s)$ which realizes any boundary slope in the open interval $(-\infty, 1)$ when $r > 0$, or in $(-1, \infty)$ when $r < 0$.*

PROOF. Since K is a fibered knot embedded in S^3, there is a fibration ξ in $M' = \overline{S^3 \setminus N(K)}$. We define a longitude curve l on ∂T as the boundary of a fiber of ξ and a meridian curve m such that m intersects l at one point and bounds a disk in $N(K)$. We replace the solid torus T by the solid torus V defined before such that the circles $\{C^i_\theta\}_{i=1,\dots,s}$ coincide with the parallel curves of the longitude l. By joining the internal surfaces in V and original fiber surfaces of ξ along the family of circles $\{C^i_\theta \mid i = 1, \dots, s,\ 0 \leqq \theta < 2\pi\}$, we obtain a fibration $\hat{\xi}$ on $M = \overline{S^3 \setminus \hat{K}(r,s)}$.

Next we take the pairs of arcs $(\alpha, \beta_\pm)$ properly embedded in a fiber of $\hat{\xi}$ as defined in this section. Then by the same proof of Theorem 4.1 we can conclude the statement of the Corollary. □

5. Extension to a Link Case

In this section, we partially extend the theorem of Rachel Roberts (**Theorem 4.1** in [5]) to a fibered link case.

We denote a surface whose genus is i and has j boundaries by $\Sigma_{i,j}$. Let M be an oriented, compact, fibered 3-manifold with a monodromy h and an orientable fiber $\Sigma_{i,j}$. Any boundary component of M is homeomorphic to a torus. We suppose that $j = 2$ and i is more than or equal to two, for simplicity we write $\Sigma_{i,2}$ by F, and the monodromy h maps each boundary to itself. We consider M as a quotient space; $M = F \times [0,1] \big/ (h(x), 0) \sim (x, 1)$ where $x \in F$. The orientation of F is defined by the increasing direction of this interval $[0,1]$. For this orientation of F we define a coordinate system (μ, λ) for each component of ∂M such that λ is a component of ∂F with the orientation induced from F and μ satisfies that $\langle \mu, \lambda \rangle = 1$.

Let α be a simple non-separating arc properly embedded in $F \times \{0\}$. Setting $D = \alpha \times [0,1]$ we consider that D is properly embedded in $F \times [0,1]$ such that ∂D consists of four arcs, $\partial\alpha \times [0,1]$ on $\partial F \times [0,1]$, α_+ on $F \times \{0\}$ and α_- on $F \times \{1\}$. In order to prepare for the later section where we will construct a branched surface by these fibers and disks, we take the same convention for the orientation of D defined in Section 3 as seen in Figure 7.

For this settings, we state the theorem extended to a link case.

THEOREM 5.1. *For $i = 1, 2$, let α^i be simple non-separating arcs properly embedded in F such that the end points of each α^i are on one component of ∂F and α^i are disjoint each other. Let D^i be disks in M such that $D^i = \alpha^i \times [0,1]$. If the arcs α^i and the monodromy h satisfy the condition* (1) *of Lemma 5.5, there is a branched surface which is made from a splitting of $B = \langle F; D_1, D_2\rangle$ such that it carries a family of laminations $\{\lambda_x\}$ realizing all boundary slopes in $(-a_i, b_i)$ for some $a_i, b_i > 0$, $i = 1, 2$ where i corresponds to the component of the torus boundaries of M. Moreover, these laminations λ_x extend to taut foliations $\mathcal{F}_x$ with same property of slopes.*

We shall prove Theorem 5.1 by the same steps in the proof of Theorem 4.1 of [5] with some modification.

Recall that a pair of arcs δ and δ' properly embedded in a surface F is good if they are disjoint and their end points alternate along the boundary of F. For the sequence $\sigma = (\alpha_0, \alpha_1, \dots, \alpha_n)$ of arcs properly embedded in F, if each pair (α_k, α_{k+1}) for $0 \leqq k < n$ is good we say that the sequence σ is a good sequence.

LEMMA 5.2. *Let $\sigma^i = (h(\alpha^i_n) = \alpha^i_0, \alpha^i_1, \dots, \alpha^i_n)$, $i = 1, 2$ be good sequences and we suppose that four arcs $(\alpha^1_k, \alpha^1_{k+1})$ and $(\alpha^2_k, \alpha^2_{k+1})$ are disjoint $(k = 0, \dots, n-1)$. For $1 \leqq k \leqq n$, we take disks $\{D^i_k\}$ in M such that*

$$D^i_k = \alpha^i_k \times \left[\frac{k-1}{n}, \frac{k}{n}\right].$$

We fix an orientation for each α^i_k and define the orientation on D^i_k by our convention. We define the branched surface

$$B = \langle F_0, F_1, \dots, F_{n-1}; D^1_1, D^2_1, D^1_2, D^2_2, \dots, D^1_n, D^2_n\rangle.$$

Then there is a family of simple arcs $\{\gamma_k^i\}$ *properly embedded in each* F_k *for* $0 \leqq k \leqq n-1, i = 1, 2$, *such that* B *is affinely measured with respect to* $\bigcup_{i=1,2} \bigcup_{k=0}^{n-1} \gamma_k^i$.

PROOF. We denote each boundary of the fiber F_k by $\partial^i F_k$ for $i = 1, 2$. For $0 \leqq k \leqq n-1$ and $i = 1, 2$, there is a regular neighbourhood T_k^i of $\alpha_k^i \cup \alpha_{k+1}^i \cup \partial^i F_k$ which is homeomorphic to a torus with two boundary components in each F_k. Since the pairs $(\alpha_k^i, \alpha_{k+1}^i)$ are mutually disjoint and non-separating, the region $F_k \setminus (T_k^1 \cup T_k^2)$ is connected for each k. Therefore, by assigning a weight 1 for each regions $F_k \setminus (T_k^1 \cup T_k^2)$ we can prove this as an extension of the proof of Lemma 4.3 of [5]. □

LEMMA 5.3. *Let* α^1 *and* α^2 *be two disjoint non-separating simple arcs properly embedded in* F *such that the boundary points* $\partial\alpha^1$ *are on* $\partial^1 F$ *and* $\partial\alpha^2$ *are on* $\partial^2 F$. *Then there are good sequences*

$$\sigma^i = (h(\alpha^i) = \alpha_0^i, \alpha_1^i, \dots, \alpha_n^i = \alpha^i), i = 1, 2,$$

such that for $0 \leqq k \leqq n-1$, *four arcs* $(\alpha_k^1, \alpha_{k+1}^1)$ *and* $(\alpha_k^2, \alpha_{k+1}^2)$ *are disjoint.*

PROOF. We suppose the arcs α^1 and α^2 are in the configuration shown in Figure 10.

Lickorish proved in [1] that the group of orientation preserving automorphisms $Aut_+(F)$ of the surface of genus g is generated by the set $\mathcal{D}$ of Dehn twists with respect to the curves $A_1, \dots, A_g, B_1, \dots, B_g, C_1, \dots, C_{g-1}$ in Figure 10. We denote the Dehn twist along these curves also by $A_1, \dots, A_g, B_1, \dots, B_g, C_1, \dots, C_{g-1}$, then $\mathcal{D} = \{A_1, \dots, A_g, B_1, \dots, B_g,$

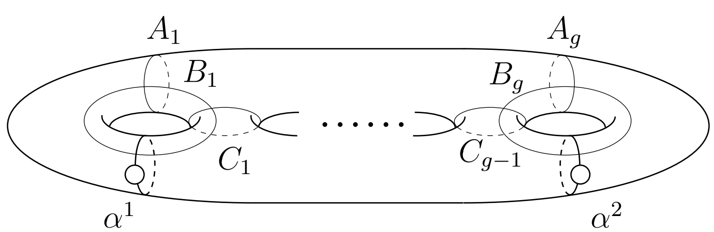


Figure 10.

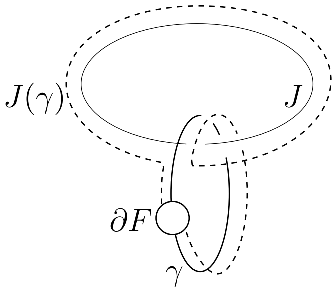


Figure 11.

$C_1, \ldots, C_{g-1}\}$. Let γ be a properly embedded arc in F whose endpoints are on one boundary component. If γ intersects only one of the curves of $\mathcal{D}$ which we denote by J, then we easily see that $(\gamma, J(\gamma))$ is a good pair by isotoping $J(\gamma)$ slightly in the neighbourhood of $\gamma \cup J$ (see Figure 11).

Let J be any element of $\mathcal{D}$. If $J = B_1$, $(\alpha^1, A_1 J(\alpha^1))$ is a good pair, and also $(A_1 J(\alpha^1), ({A_1}^{-1}) A_1 J(\alpha^1) = J(\alpha^1))$ is a good pair. Then $(\alpha^1, A_1 J(\alpha^1), J(\alpha^1))$ is a good sequence. By the same considerations, $(\alpha^2, B_g(\alpha^2), (J{B_g}^{-1}) B_g(\alpha^2) = J(\alpha^2))$ is a good sequence. If $J = B_g$, by a symmetrical argument, $(\alpha^1, B_1(\alpha^1), J(\alpha^1))$ and $(\alpha^2, A_g J(\alpha^2), J(\alpha^2))$ are good sequences. If $J \neq B_1$ and B_g, $(\alpha^1, B_1(\alpha^1), J(\alpha^1))$ and $(\alpha^2, B_g(\alpha^2), J(\alpha^2))$ are good sequences. In all cases, there are good sequences $\hat{\sigma^1} : \alpha^1 \to J(\alpha^1)$ and $\hat{\sigma^2} : \alpha^2 \to J(\alpha^2)$ with the same number of terms.

Now we decompose the monodromy h into compositions of elements of $\mathcal{D}$, $h = J_m \circ J_{m-1} \circ \cdots \circ J_1$ where $J_i \in \mathcal{D}$. By the above argument, there are good sequences

$$\hat{\sigma^i_k} : \alpha^i \to J_k(\alpha^i)$$

for each $i = 1, 2$ and $k = 1, \ldots, m$. Now we can apply the method of proof of Lemma 4.4 in [5] for each good sequences $\hat{\sigma^i_k} : \alpha^i \to J_k(\alpha^i)$, then the proof is completed. $\square$

In Lemma 5.4, we shall construct two branched surfaces such that the one of them carries the family of laminations which realizes all boundary slopes in negative part of the interval of the conclusion of Theorem 5.1, the other carries positive part. In order to define these specific branched surfaces, we recall some notation for orientations of simple arcs on the surface defined in [5].

Let α and β be simple arcs properly embedded in F and we suppose the pair (α, β) is good. If we give the orientations for α and β, there are two cases for the orientation of the pair (α, β) in the neighbourhood of ∂F as in Figure 12.

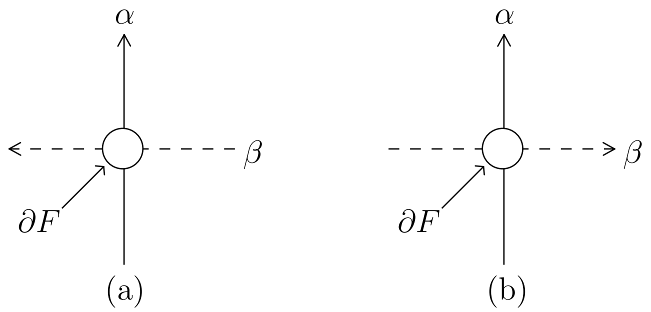


Figure 12.

According to the definition in [5] we call a good pair (α, β) is *a negatively oriented pair* if it is oriented as in Figure 12 (a), otherwise if it is oriented as in Figure 12 (b) we call it *a positively oriented pair.* For a good sequence $\sigma = (\alpha_0, \alpha_1, \ldots, \alpha_n)$, we call σ is *a negatively oriented good sequence* if each pair (α_{i-1}, α_i) is a negatively oriented pair for $i = 1, 2, \ldots, n$, and we call σ is *a positively oriented good sequence* if each pair (α_{i-1}, α_i) is a positively oriented pair.

For the pair of good sequences $\sigma = (\sigma^1, \sigma^2)$ defined in Lemma 5.2, we denote the branched surface defined in Lemma 5.2 by B_σ, and we consider that each sector of B_σ constructed from $\{D_k^i\}$ has the orientation induced from the arcs $\{\alpha_k^i\}$ with our convention defined before. We denote the two boundaries of M by $\partial^i M$ for $i = 1, 2$, which corresponds to $\partial^i F \times [0, 1]/(h(x), 0) \sim (x, 1)$ where $x \in \partial^i F$.

LEMMA 5.4. *For α^1 and α^2 defined in the proof of Lemma 5.3, if $\sigma^1 = (h(\alpha^1) = \alpha_0^1, \alpha_1^1, \ldots, \alpha_n^1 = \alpha^1)$ and $\sigma^2 = (h(\alpha^2) = \alpha_0^2, \alpha_1^2, \ldots, \alpha_n^2 = \alpha_2)$ are both negatively oriented good sequences, then the branched surface B_σ*

carries the family of laminations $\{\lambda_x\}$ which realizes all boundary slopes of $\partial^i M$ in $(-a_i, 0]$ for some $a_i > 0$. If σ^1 and σ^2 are both positively oriented good sequences, B_σ carries the family of laminations $\{\lambda_x\}$ which realize all boundary slopes of $\partial^i M$ in $[0, b_i)$ for some $b_i > 0$.

PROOF. Let τ_{σ^i} be the train track on $\partial^i M$ such that $\tau_{\sigma^i} = B_\sigma \cap \partial^i M$ for $i = 1, 2$. By Lemma 5.2 there is a family of properly embedded arcs $\{\gamma_k^i\}$ such that B_σ is affinely measured with respect to $\bigcup_{i=1,2} \bigcup_{k=0}^{n-1} \gamma_k^i$. Then each train track τ_{σ^i} for $i = 1, 2$ has the affine measure induced from B_σ and we see that there is an interval in which τ_{σ^i} realizes all slopes on each boundary component by the same way of proof of Lemma 4.5 in [5]. $\square$

LEMMA 5.5. *Let $\sigma^i = (h(\alpha_n^i) = \alpha_0^i, \alpha_1^i, \ldots, \alpha_n^i)$, $i = 1, 2$ be good sequences and we suppose that four arcs $(\alpha_k^1, \alpha_{k+1}^1)$ and $(\alpha_k^2, \alpha_{k+1}^2)$ are disjoint for $0 \leqq k \leqq n$. Then we can modify the sequence $\sigma = (\sigma^1, \sigma^2)$ into $\bar{\sigma} = (\bar{\sigma^1}, \bar{\sigma^2})$ with one of the following two properties,*

(1) *both of $\bar{\sigma^1}$ and $\bar{\sigma^2}$ are either positively oriented good sequences or negatively oriented good sequences,*

(2) *$\bar{\sigma} = (\bar{\sigma^1}, \bar{\sigma^2})$ has the property that $(\alpha_{n-1}^1, \alpha_n^1)$ is a positively oriented good pair and $(\alpha_{n-1}^2, \alpha_n^2)$ is a negatively oriented good pair, or $(\alpha_{n-1}^1, \alpha_n^1)$ is a negatively oriented good pair and $(\alpha_{n-1}^2, \alpha_n^2)$ is a positively oriented good pair. Other pairs $(\alpha_{k-1}^i, \alpha_k^i)$, $k = 1, \ldots, n-2$, $i = 1, 2$ are either all positive or all negative pair.*

PROOF. For the original good sequences

$$\begin{aligned} \sigma^1 &= (\alpha_0^1, \alpha_1^1, \ldots, \alpha_n^1) \\ \sigma^2 &= (\alpha_0^2, \alpha_1^2, \ldots, \alpha_n^2), \end{aligned}$$

there are the following eight cases:

$(NP)_k^P$: For $0 \leqq j < k$ and $i = 1, 2$, each pair $(\alpha_j^i, \alpha_{j+1}^i)$ is positively oriented; $(\alpha_k^1, \alpha_{k+1}^1)$ is negatively oriented and $(\alpha_k^2, \alpha_{k+1}^2)$ is positively oriented.

$(NP)_k^N$: For $0 \leqq j < k$ and $i = 1, 2$, each pair $(\alpha_j^i, \alpha_{j+1}^i)$ is negatively oriented; $(\alpha_k^1, \alpha_{k+1}^1)$ is negatively oriented and $(\alpha_k^2, \alpha_{k+1}^2)$ is positively oriented.

$(PN)^P_k$: For $0 \leqq j < k$ and $i = 1, 2$, each pair $(\alpha^i_j, \alpha^i_{j+1})$ is positively oriented; $(\alpha^1_k, \alpha^1_{k+1})$ is positively oriented and $(\alpha^2_k, \alpha^2_{k+1})$ is negatively oriented.

$(PN)^N_k$: For $0 \leqq j < k$ and $i = 1, 2$, each pair $(\alpha^i_j, \alpha^i_{j+1})$ is negatively oriented; $(\alpha^1_k, \alpha^1_{k+1})$ is positively oriented and $(\alpha^2_k, \alpha^2_{k+1})$ is negatively oriented.

$(NN)^P_k$: For $0 \leqq j < k$ and $i = 1, 2$, each pair $(\alpha^i_j, \alpha^i_{j+1})$ is positively oriented; $(\alpha^1_k, \alpha^1_{k+1})$ is negatively oriented and $(\alpha^2_k, \alpha^2_{k+1})$ is negatively oriented.

$(NN)^N_k$: For $0 \leqq j < k$ and $i = 1, 2$, each pair $(\alpha^i_j, \alpha^i_{j+1})$ is negatively oriented; $(\alpha^1_k, \alpha^1_{k+1})$ is negatively oriented and $(\alpha^2_k, \alpha^2_{k+1})$ is negatively oriented.

$(PP)^P_k$: For $0 \leqq j < k$ and $i = 1, 2$, each pair $(\alpha^i_j, \alpha^i_{j+1})$ is positively oriented; $(\alpha^1_k, \alpha^1_{k+1})$ is positively oriented and $(\alpha^2_k, \alpha^2_{k+1})$ is positively oriented.

$(PP)^N_k$: For $0 \leqq j < k$ and $i = 1, 2$, each pair $(\alpha^i_j, \alpha^i_{j+1})$ is negatively oriented; $(\alpha^1_k, \alpha^1_{k+1})$ is positively oriented and $(\alpha^2_k, \alpha^2_{k+1})$ is positively oriented.

For each of eight cases, we define operations as follows.

For the case $(NP)^P_k$, we replace the pair $(\alpha^1_k, \alpha^1_{k+1})$ by the sequence $(\alpha^1_k, -\alpha^1_{k+1}, -\alpha^1_k, \alpha^1_{k+1})$, and replace the pair $(\alpha^2_k, \alpha^2_{k+1})$ by the sequence $(\alpha^2_k, \alpha^2_{k+1}, -\alpha^2_k, -\alpha^2_{k+1})$ and rewrite $-\alpha^2_{k+1}$ to α^2_{k+1}, i.e. we reverse the orientation of α^2_{k+1}. Then we can see that $(\alpha^1_k, -\alpha^1_{k+1}, -\alpha^1_k, \alpha^1_{k+1})$ and $(\alpha^2_k, \alpha^2_{k+1}, -\alpha^2_k, \alpha^2_{k+1})$ are positively oriented good sequences. Then all pairs before α^i_{k+1} are positive good pairs. It means that we modify the cases $(NP)^P_k$ into the cases $(NP)^P_{k+1}$, $(PN)^P_{k+1}$, $(PP)^P_{k+1}$, or $(NN)^P_{k+1}$.

For other cases, the operations are as follows.

$(NP)^N_k$: $(\alpha^1_k, \alpha^1_{k+1}) \to (\alpha^1_k, \alpha^1_{k+1}, -\alpha^1_k, -\alpha^1_{k+1})$ and rewrite the last term, $(\alpha^2_k, \alpha^2_{k+1}) \to (\alpha^2_k, -\alpha^2_{k+1}, -\alpha^2_k, \alpha^2_{k+1})$.

$(PN)^P_k$: $(\alpha^1_k, \alpha^1_{k+1}) \to (\alpha^1_k, \alpha^1_{k+1}, -\alpha^1_k, -\alpha^1_{k+1})$ and rewrite the last term, $(\alpha^2_k, \alpha^2_{k+1}) \to (\alpha^2_k, -\alpha^2_{k+1}, -\alpha^2_k, \alpha^2_{k+1})$.

$(PN)_k^N$: $(\alpha_k^1, \alpha_{k+1}^1) \to (\alpha_k^1, -\alpha_{k+1}^1, -\alpha_k^1, \alpha_{k+1}^1)$,
$(\alpha_k^2, \alpha_{k+1}^2) \to (\alpha_k^2, \alpha_{k+1}^2, -\alpha_k^2, -\alpha_{k+1}^2)$ and rewrite the last term.

$(NN)_k^P$: $(\alpha_k^1, \alpha_{k+1}^1) \to (\alpha_k^1, -\alpha_{k+1}^1, -\alpha_k^1, \alpha_{k+1}^1)$,
$(\alpha_k^2, \alpha_{k+1}^2) \to (\alpha_k^2, -\alpha_{k+1}^2, -\alpha_k^2, \alpha_{k+1}^2)$.

$(NN)_k^N$: no operations.

$(PP)_k^P$: no operations.

$(PP)_k^N$: $(\alpha_k^1, \alpha_{k+1}^1) \to (\alpha_k^1, -\alpha_{k+1}^1, -\alpha_k^1, \alpha_{k+1}^1)$,
$(\alpha_k^2, \alpha_{k+1}^2) \to (\alpha_k^2, -\alpha_{k+1}^2, -\alpha_k^2, \alpha_{k+1}^2)$.

By doing these operations, in each case the resultant sequences satisfy the condition of one of the cases $(NP)_{k+1}^P$, $(NP)_{k+1}^N$, $(PN)_{k+1}^P$, $(PN)_{k+1}^N$, $(NN)_{k+1}^P$, $(NN)_{k+1}^N$, $(PP)_{k+1}^P$, $(PP)_{k+1}^N$. Therefore if we iterate these operations, finally we reach one of the following situations:

(1a) the resultant sequences $\bar{\sigma}^1$ and $\bar{\sigma}^2$ are both positively oriented.

(1b) the resultant sequences $\bar{\sigma}^1$ and $\bar{\sigma}^2$ are both negatively oriented.

(2a) For $0 \leqq k \leqq n-1$ and $i = 1, 2$ each pair $(\alpha_k^i, \alpha_{k+1}^i)$ is positively oriented but $(\alpha_{n-1}^1, \alpha_n^1)$ is negatively oriented and $(\alpha_{n-1}^2, \alpha_n^2)$ is positively oriented, or $(\alpha_{n-1}^1, \alpha_n^1)$ is positively oriented and $(\alpha_{n-1}^2, \alpha_n^2)$ is negatively oriented.

(2b) For $0 \leqq k \leqq n-1$ and $i = 1, 2$ each pair $(\alpha_k^i, \alpha_{k+1}^i)$ is negatively oriented but $(\alpha_{n-1}^1, \alpha_n^1)$ is negatively oriented and $(\alpha_{n-1}^2, \alpha_n^2)$ is positively oriented, or $(\alpha_{n-1}^1, \alpha_n^1)$ is positively oriented and $(\alpha_{n-1}^2, \alpha_n^2)$ is negatively oriented.

Hence the cases (1a) and (1b) is the case (1) of the conclusion of this lemma, and the cases (2a) and (2b) is the case (2). □

LEMMA 5.6. *Let λ_x be the lamination obtained in Lemma 5.4. Then the lamination λ_x extends to a taut foliation $\mathcal{F}_x$ with the same boundary slope property.*

PROOF. For any point x^1 on α_k^1 and any point x^2 on α_k^2, let δ be a simple arc on F_{k-1} whose end points are $\partial^1\delta = x^1$ and $\partial^2\delta = x^2$ and such that δ does not intersect α_k^1 and α_k^2. Since α_k^1 and α_k^2 are disjoint and both non-separating, there is such a simple arc δ. Let ${F_{k-1}}'$ be a sub surface which is a metrically completed surface of the open surface $F_{k-1} \setminus (\alpha_k^1 \cup \alpha_k^2)$. Then we can regard that δ is properly embedded in ${F_{k-1}}'$. The boundaries of the sub surface ${F_{k-1}}'$ has four components, two of them are copies of α_k^1 and the others are copies of α_k^2. We denote these boundaries by α_k^{i+} and α_k^{i-} for $i = 1, 2$, where the signs mean that the copy with $+$ sign is on the right side of the original arc with respect to the orientation of original arc, the sign $-$ means that it is on opposite side.

Since we construct the disks $\{D_k^i\}$ by using the sub interval $[\frac{k-1}{n}, \frac{k}{n}]$, we denote the image of $\partial^i \delta$ on F_k induced from this construction of disks by $\partial^i \bar{\delta}$ for $i = 1, 2$, and by the same construction we can consider the image of δ on F_k, we denote it by $\bar{\delta}$. The arcs α_k^1 and α_k^2 also separate the surface F_k into sub surface ${F_k}'$ with four boundary components α_k^{i+} and α_k^{i-} for $i = 1, 2$. In order to specify these arcs we denote them by $\bar{\alpha}_k^{i+}$ and $\bar{\alpha}_k^{i-}$. Because of our convention for the orientation of the disks $\{D_k^i\}$, we can see that α_k^{i-} corresponds to the vertical boundary $\partial_v N(B_\sigma)$ near the surface F_{k-1} and $\bar{\alpha}_k^{i+}$ corresponds to $\partial_v N(B_\sigma)$ near the surface F_k.

There are four cases related to the endpoints condition of δ. If $\partial^1 \delta \in \alpha_k^{1+}$ and $\partial^2 \delta \in \alpha_k^{2+}$, then $\partial^1 \bar{\delta} \in \bar{\alpha}_k^{1+}$ and $\partial^2 \bar{\delta} \in \bar{\alpha}_k^{2+}$. In this case, by the condition of the orientation of disks $\{D_k^i\}$, we can modify δ by sliding the end point $\partial^1 \delta$ to the point $\partial^1 \bar{\delta}$ along the disk D_k^1 and $\partial^2 \delta$ to the point $\partial^2 \bar{\delta}$ along the disk D_k^2. The resultant arc is smooth arc on B_σ with endpoints on $\bar{\alpha}_k^{1+}$ and $\bar{\alpha}_k^{2+}$. By the same argument, if $\partial^1 \delta \in \alpha_k^{1+}$ and $\partial^2 \delta \in \alpha_k^{2-}$, then there is a smooth arc on B_σ with endpoints on $\bar{\alpha}_k^{1+}$ and α_k^{2-}; if $\partial^1 \delta \in \alpha_k^{1-}$ and $\partial^2 \delta \in \alpha_k^{2+}$, then there is a smooth arc on B_σ with endpoints on α_k^{1-} and $\bar{\alpha}_k^{2+}$; and if $\partial^1 \delta \in \alpha_k^{1-}$ and $\partial^2 \delta \in \alpha_k^{2-}$, then there is a smooth arc on B_σ with endpoints on α_k^{1-} and α_k^{2-}. In all cases, each end points of the modified arc δ corresponds to the points on $\partial_v N(B)$.

Therefore we can foliate the complementary region $F_{k-1} \times [\frac{k-1}{n}, \frac{k}{n}] \setminus \mathring{N}(B_\sigma)$ by the product foliation ${F_{k-1}}' \times [0, 1]$ with the property that the vertical boundaries of ${F_{k-1}}' \times [0, 1]$ are connected to the vertical boundaries $\partial_v N(B_\sigma)$. Filling the complementary region of the lamination λ_x in $N(B_\sigma)$ with parallel leaves, we can extend λ_x to a foliation $\mathcal{F}_x$. In the boundary ∂M a meridian curve intersects all leaves of $\mathcal{F}_x$ transversely, thus $\mathcal{F}_x$ is a taut foliation. $\square$

In summary, we proved the existence of the good sequences $\sigma = (\sigma^1, \sigma^2)$ in Lemma 5.3 and we modify these sequences suitable for the assumption of Lemma 5.4. By Lemma 5.2 and Lemma 5.4, for these modified good sequences with good property there are two branched surfaces B_{σ_-} and B_{σ_+} which carry the families of laminations $\{\lambda_x\}$ which realize all boundary slope in $(-a_i, 0]$ and $[0, b_i)$ on $\partial^i M$ for some $a_i > 0$ and $b_i > 0$, $i = 1, 2$ respectively. Then the lamination λ_x is extended to the taut foliations $\mathcal{F}_x$ by Lemma 5.6, we complete the proof of Theorem 5.1.

Example 5.7. Now we calculate these intervals of slopes for the complement of $(6, 4)$-torus link. First we propose an explicit construction of the fibration on the complement of $(6, 4)$-torus knot as similar to the construction established in Section 3.1.

Let K be the $(6, 4)$-torus link which is a pair of simple closed curves on the solid torus V standardly embedded in S^3. We denote these components by K_1 and K_2. Taking the infinite cover $\tilde{V}$ of V with the covering map $q : \tilde{V} \to V$, we denote the cover of K on $\tilde{V}$ by $\tilde{K}$. Then $\tilde{V}$ is a cylinder of infinite length, and $\tilde{K}$ has four components. If we embed $\tilde{V}$ into $\mathbb{R}^3$ in the same way as in section 3.1, these components are the curves represented by the following formulae;

$$\begin{aligned} k_i^1(x) &= (x,\ \cos\frac{3}{2}(x + \frac{2(i-1)\pi}{3}),\ \sin\frac{3}{2}(x + \frac{2(i-1)\pi}{3}))\ \ (i = 1, 2), \\ k_i^2(x) &= (x,\ \cos\frac{3}{2}(x + \frac{2(i-1)\pi}{3} + \frac{\pi}{3}), \\ &\qquad\qquad \sin\frac{3}{2}(x + \frac{2(i-1)\pi}{3}) + \frac{\pi}{3})\ \ (i = 1, 2), \end{aligned}$$

where each $k_i^j(x)$ projects to K_j by the covering map q.

Now we construct a surface in $\tilde{V}$ as follows. We define twisted bands $G_B^{i,j}$ by the following formulae;

$$\begin{aligned} {G_B}^{1,1} = \Bigg\{ & rk_1^1(x) + (1-r)k_2^2\left(\frac{2\pi}{6} - x\right) + \left(\frac{2\pi}{3}n, 0, 0\right) \\ & \Bigg|\ 0 \leqq x \leqq \frac{\pi}{6},\ 0 < r < 1,\ n = 0, \pm 1, \pm 2, \dots \Bigg\}, \\ {G_B}^{1,2} = \Bigg\{ & rk_2^2(x + \frac{2\pi}{6}) + (1-r)k_2^1\left(\frac{4\pi}{6} - x\right) + \left(\frac{2\pi}{3}n, 0, 0\right) \\ & \Bigg|\ 0 \leqq x \leqq \frac{\pi}{6},\ 0 < r < 1,\ n = 0, \pm 1, \pm 2, \dots \Bigg\}, \end{aligned}$$

$$G_B{}^{2,1} = \left\{ rk_1^2(x) + (1-r)k_1^1\left(\frac{2\pi}{6} - x\right) + \left(\frac{2\pi}{3}n, 0, 0\right) \,\middle|\, 0 \leqq x \leqq \frac{\pi}{6},\ 0 < r < 1,\ n = 0, \pm 1, \pm 2, \dots \right\},$$

$$G_B{}^{2,2} = \left\{ rk_1^1(x + \frac{2\pi}{6}) + (1-r)k_2^2\left(\frac{4\pi}{6} - x\right) + \left(\frac{2\pi}{3}n, 0, 0\right) \,\middle|\, 0 \leqq x \leqq \frac{\pi}{6},\ 0 < r < 1,\ n = 0, \pm 1, \pm 2, \dots \right\},$$

$$G_B{}^{3,1} = \left\{ rk_2^1(x) + (1-r)k_1^2\left(\frac{2\pi}{6} - x\right) + \left(\frac{2\pi}{3}n, 0, 0\right) \,\middle|\, 0 \leqq x \leqq \frac{\pi}{6},\ 0 < r < 1,\ n = 0, \pm 1, \pm 2, \dots \right\},$$

$$G_B{}^{3,2} = \left\{ rk_1^2(x + \frac{2\pi}{6}) + (1-r)k_1^1\left(\frac{4\pi}{6} - x\right) + \left(\frac{2\pi}{3}n, 0, 0\right) \,\middle|\, 0 \leqq x \leqq \frac{\pi}{6},\ 0 < r < 1,\ n = 0, \pm 1, \pm 2, \dots \right\},$$

$$G_B{}^{4,1} = \left\{ rk_2^2(x) + (1-r)k_2^1\left(\frac{2\pi}{6} - x\right) + \left(\frac{2\pi}{3}n, 0, 0\right) \,\middle|\, 0 \leqq x \leqq \frac{\pi}{6},\ 0 < r < 1,\ n = 0, \pm 1, \pm 2, \dots \right\},$$

$$G_B{}^{4,2} = \left\{ rk_2^1(x + \frac{2\pi}{6}) + (1-r)k_1^2\left(\frac{4\pi}{6} - x\right) + \left(\frac{2\pi}{3}n, 0, 0\right) \,\middle|\, 0 \leqq x \leqq \frac{\pi}{6},\ 0 < r < 1,\ n = 0, \pm 1, \pm 2, \dots \right\}.$$

The boundaries of these bands bound the squares $\{P_k\}$ on each disk $D_k = \{(x, y, z) \,|\, x = \frac{2k+1}{6}\pi,\ y^2 + z^2 \leqq 1\}$, $k \in \mathbb{Z}$. Then we define a surface G in $\tilde{V}$ as the union of all $G_B^{i,j}$ and P_k. Next we define the map $R_\theta : \tilde{V} \to \tilde{V}$ given by

$$R_\theta(x, y, z) = \left(x + \frac{\theta}{6}\ ,\ y\cos\frac{\theta}{4} - z\sin\frac{\theta}{4}\ ,\ y\sin\frac{\theta}{4} + z\cos\frac{\theta}{4} \right).$$

As seen in Section 3.1, R_θ keeps the components $k_i^j(x)$ invariant and rotates $\tilde{V}$ by angle $\frac{\pi}{2}$, moreover if we set $G_\theta = R_\theta(G)$, $0 \leqq \theta \leqq 2\pi$, the family of surfaces $\{G_\theta \,|\, 0 \leqq \theta < 2\pi\}$ fills up $\tilde{V}$ and all G_θ are disjoint. G_θ has four line boundaries $\{\tilde{C_\theta^i}\}_{i=1,2,3,4}$ on $\partial\tilde{V}$. We set $F_\theta{}' = q(G_\theta)$ and then the family of surfaces $\{F_\theta{}' \,|\, 0 \leqq \theta < 2\pi\}$ fills up V. The images of $\tilde{C}_\theta$ are four longitudinal circles $\{C_\theta^i\}_{i=1,2,3,4}$ on ∂V. Since the complement of V is also a solid torus, we connect meridian disks of the complement to each C_θ along its boundaries, then we obtain a surface F_θ in $S^3 \setminus K$. The family of surfaces $\{F_\theta \,|\, 0 \leqq \theta < 2\pi\}$ fills up $S^3 \setminus K$, and as seen in Section 3.1, the map $p : S^3 \setminus K \to S^1$ defines a fibration.

Next we take two arcs. Let $\tilde{\alpha}^1$ and $\tilde{\alpha}^2$ be arcs on $\partial\tilde{V}$ such that

$$\tilde{\alpha}^1 = \left\{(t,\ 1,\ 0) \in \partial\tilde{V} \mid 0 \leqq t \leqq \frac{2\pi}{3}\right\}$$
$$\tilde{\alpha}^2 = \left\{(t,\ 1,\ 0) \in \partial\tilde{V} \mid \pi \leqq t \leqq \frac{5\pi}{3}\right\}.$$

We project each arc to V and denote their images by $\alpha^{1\prime}$ and $\alpha^{2\prime}$. These arcs are on the circle C_0, then we modify these arcs slightly in the neighbourhood of C_0 such that we fix each end points and shift each center of arcs forward to the direction of the center of the meridian disk whose boundary is C_0, so that each arc does not intersect the link K in its interior. We denote the resultant arcs by α^1 and α^2.

Let h be the monodromy of the fibration, and set $\beta^1 = h(\alpha^1)$ and $\beta^2 = h(\alpha^2)$. These arcs β^1 and β^2 are on the meridian disk whose boundary is $C_{2\pi}$, then four arcs are on the one fiber surface F_0. The four arcs are mutually disjoint, and each arc is non-separating on the fiber. The pair (α^1, β^1) is a good pair, and so is (α^2, β^2). Hence by tracing the method of Section 3.2, we can construct the branched surfaces, then we obtain a family of taut foliations $\{\mathcal{F}_x\}$ such that $\mathcal{F}_x$ realizes all boundary slope in $(-\infty, 1)$ on each boundary component.

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