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The Littlewood-Paley-Stein Inequality for Diffusion Processes on General Metric Spaces

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Abstract. In this paper, we establish the Littlewood-Paley-Stein inequality on general metric spaces under a weaker condition than the lower boundedness of Bakry-Emery's Γ_2. We also discuss Riesz transforms. As examples, we deal with diffusion processes on a path space associated with stochastic partial differential equations (SPDEs in short) and a class of superprocesses with immigration.

1. Framework and Results

After the Meyer's celebrated work [16], many authors studied the Littlewood-Paley-Stein inequality by a probabilistic approach. Especially, Shigekawa-Yoshida [20] studied it for symmetric diffusion processes on a general state space. In [20], they assumed the existence of a suitable core $\mathcal{A}$ which is not only a ring but also stable under the operation of the semigroup and the infinitesimal generator to employ Bakry-Emery's Γ_2-method in the proof, and established the Littlewood-Paley-Stein inequality under that Γ_2 is bounded from below. However, it is very difficult to check the existence of such a good core $\mathcal{A}$ when we consider problems of infinite dimensional diffusion processes.

In this paper, we show that the Littlewood-Paley-Stein inequality holds on general metric spaces under the *gradient estimate condition* **(G)** below even if we do not assume the existence of such a core $\mathcal{A}$. Our condition seems somewhat weaker than the lower boundedness of Γ_2. We mention that Coulhon-Duong [5] and Li [14] also discussed the Littlewood-Paley-Stein inequality under similar conditions on finite dimensional Riemannian manifolds. In contrast to these papers, we work in a more general framework to handle certain infinite dimensional diffusion processes in Section 4.

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We introduce the framework. Let X be a complete separable metric space. Suppose we are given a Borel probability measure μ on X and a μ-symmetric local quasi-regular Dirichlet form $\mathcal{E}$ in $L^2(\mu)$ with the domain $\mathcal{D}(\mathcal{E})$. See Ma-Röckner [15] for the terminologies of quasi-regular Dirichlet forms. Then by Theorem 1.1 of Chapter V in [15], there exists a μ-symmetric diffusion process $\mathbb{M} := (X_t, \{P_x\}_{x\in X})$ associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. We denote the infinitesimal generator and the transition semigroup by L and $\{P_t\}_{t\geq 0}$, respectively. Since $\{P_t\}_{t\geq 0}$ is μ-symmetric, it can be extended to the semigroup on $L^p(\mu)$, $p \geq 1$. We denote the semigroup and its generator by $\{P_t\}_{t\geq 0}$ and L again. If we need to specify the acting space, we denote the generator L in $L^p(\mu)$ by L_p and the domain by $\mathrm{Dom}(L_p)$, respectively. We assume that $\mathbf{1} \in \mathrm{Dom}(L_p)$ and $L_p\mathbf{1} = 0$ for all $p \geq 1$, where $\mathbf{1}$ denotes the function that is identically equal to 1. In particular, the diffusion process $\mathbb{M}$ is conservative.

Here we introduce the following conditions:

(A): There exists a subspace $\mathcal{A}$ of $\mathrm{Dom}(L_2)$ consisting of bounded continuous functions which is dense in $\mathcal{D}(\mathcal{E})$ and $f^2 \in \mathrm{Dom}(L_1)$ holds for any $f \in \mathcal{A}$.

Under this condition, the form $\mathcal{E}$ admits a carré du champ, namely, there exists a unique positive symmetric and continuous bilinear form Γ from $\mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E})$ into $L^1(\mu)$ such that

$$\mathcal{E}(fh, g) + \mathcal{E}(gh, f) - \mathcal{E}(h, fg) = 2\int_X h\Gamma(f, g)\, d\mu$$

holds for any $f, g, h \in \mathcal{D}(\mathcal{E}) \cap L^\infty(\mu)$. In particular, for $f, g \in \mathrm{Dom}(L_2)$, $fg \in \mathrm{Dom}(L_1)$ and

$$\Gamma(f, g) = \frac{1}{2}\big\{L_1(fg) - (L_2 f)g - f(L_2 g)\big\}$$

hold. For further informations, see Theorem 4.2.2 of Chapter I in Bouleau-Hirsch [4]. In the sequel, we use the notation $\Gamma(f) := \Gamma(f, f)$ for simplicity.

The following *gradient estimate condition* is crucial in this paper.

(G): There exist constants $K > 0$ and $R \in \mathbb{R}$ such that the following inequality holds for any $f \in \mathcal{A}$ and $t \geq 0$:

$$\Gamma(P_t f) \leq K e^{2Rt} P_t\big\{\Gamma(f)\big\}. \tag{1.1}$$

Throughout this paper, we always assume **(A)** and **(G)**.

Remark 1.1. If $\mathcal{A}$ is stable under the operations of $\{P_t\}$ and L,

$$\Gamma_2(f) \geq -R\Gamma(f), \quad f \in \mathcal{A} \tag{1.2}$$

implies (1.1) with $K = 1$, where $\Gamma_2(f) := \frac{1}{2}\big(L_1\Gamma(f) - 2\Gamma(L_2 f, f)\big)$. Hence our condition **(G)** is weaker than (1.2). When X is a finite dimensional complete Riemannian manifold, (1.2) is equivalent to that the Ricci curvature is bounded by $-R$ from below. See Proposition 2.3 in Bakry [2] for details.

Let us introduce the Littlewood-Paley G-functions. To do this, we recall the subordination of a semigroup. For $t \geq 0$, we define a probability measure λ_t on $[0, +\infty)$ by

$$\lambda_t(ds) := \frac{t}{2\sqrt{\pi}} e^{-t^2/4s} s^{-3/2} ds.$$

In terms of the Laplace transform, this measure is characterized as

$$\int_0^\infty e^{-\gamma s} \lambda_t(ds) = e^{-\sqrt{\gamma} t}, \quad \gamma > 0.$$

For $\alpha \geq 0$, we define the subordination $\{Q_t^{(\alpha)}\}_{t\geq 0}$ of $\{P_t\}_{t\geq 0}$ by

$$Q_t^{(\alpha)} f := \int_0^\infty e^{-\alpha s} P_s f \ \lambda_t(ds), \quad f \in L^p(\mu).$$

Then we can easily see that

$$\begin{aligned} \|Q_t^{(\alpha)} f\|_{L^p(\mu)} &\leq \int_0^\infty e^{-\alpha s} \|P_s f\|_{L^p(\mu)} \lambda_t(ds) \\ &\leq \Big(\int_0^\infty e^{-\alpha s} \lambda_t(ds)\Big) \|f\|_{L^p(\mu)} = e^{-\sqrt{\alpha} t} \|f\|_{L^p(\mu)}, \end{aligned} \tag{1.3}$$

and hence $\{Q_t^{(\alpha)}\}_{t\geq 0}$ is a strongly continuous contraction semigroup on $L^p(\mu)$. The infinitesimal generator of $\{Q_t^{(\alpha)}\}_{t\geq 0}$ is denoted by $-\sqrt{\alpha - L_p}$. We may omit the subscript p for simplicity.

For $f \in L^2 \cap L^p(\mu)$ and $\alpha > 0$, we define Littlewood-Paley's G-functions by

$$g_f^{\rightarrow}(x,t) := \left|\frac{\partial}{\partial t}(Q_t^{(\alpha)} f)(x)\right|, \qquad G_f^{\rightarrow}(x) := \left(\int_0^\infty t g_f^{\rightarrow}(x,t)^2 \, dt\right)^{1/2},$$

$$g^{\uparrow}_f(x,t) := \big(\Gamma(Q^{(\alpha)}_t f)\big)^{1/2}(x), \qquad G^{\uparrow}_f(x) := \left(\int_0^\infty t g^{\uparrow}_f(x,t)^2\,dt\right)^{1/2},$$
$$g_f(x,t) := \sqrt{(g^{\rightarrow}_f(x,t))^2 + (g^{\uparrow}_f(x,t))^2}, \quad G_f(x) := \left(\int_0^\infty t g_f(x,t)^2\,dt\right)^{1/2}.$$

Now we present the Littlewood-Paley-Stein inequality. In what follows, the notation $\|u\|_{L^p(\mu)} \lesssim \|v\|_{L^p(\mu)}$ stands for $\|u\|_{L^p(\mu)} \le C\|v\|_{L^p(\mu)}$, where C is a positive constant depending only on K in condition **(G)** and p.

THEOREM 1.2. *For any $1 < p < \infty$ and $\alpha > R \vee 0$, the following inequalities hold for $f \in L^2 \cap L^p(\mu)$:*

$$\|G_f\|_{L^p(\mu)} \lesssim \|f\|_{L^p(\mu)}, \tag{1.4}$$
$$\|f\|_{L^p(\mu)} \lesssim \|G^{\rightarrow}_f\|_{L^p(\mu)}. \tag{1.5}$$

Before closing this section, we give an application of Theorem 1.2. It plays an important role in the regularity theory of parabolic PDEs on general metric spaces.

THEOREM 1.3. *Let $1 < p < \infty, q \ge 1$ and $\alpha > R \vee 0$. We define*

$$R^{(q)}_\alpha(L)f := \Gamma\big((\sqrt{\alpha - L_p})^{-q} f\big)^{1/2}, \qquad f \in L^p(\mu).$$

Then we have the following statements:
(1) *For any $p \ge 2$ and $q > 1$, $R^{(q)}_\alpha(L)$ is bounded on $L^p(\mu)$. The operator norm $\|R^{(q)}_\alpha(L)\|_{p,p}$ depends only on K, p, q and $\alpha_R := (\alpha - R) \wedge \alpha$. This implies the inclusion*

$$\mathrm{Dom}\big((\sqrt{1 - L_p})^q\big) \subset W^{1,p}(\mu) := \big\{ f \in L^p(\mu) \cap \mathcal{D}(\mathcal{E}) \mid \Gamma(f)^{1/2} \in L^p(\mu) \big\}.$$

(2) *For any $p \ge 2$ and $1 < q < 2$, there exists a positive constant $C_{p,q}$ such that*

$$\big\|\Gamma(P_t f)^{1/2}\big\|_{L^p(\mu)} \le C_{p,q} \|R^{(q)}_\alpha(L)\|_{p,p} \big(\alpha^{q/2} + t^{-q/2}\big) \|f\|_{L^p(\mu)}, \qquad t > 0,\ f \in L^p(\mu). \tag{1.6}$$

Remark 1.4. We do not know whether our gradient estimate condition **(G)** is sufficient or not to establish the item (1) of Theorem 1.3 for $q = 1$, i.e., so-called the boundedness of the Riesz transform $R_\alpha(L) := R_\alpha^{(1)}(L)$ on $L^p(\mu)$. Recently, Shigekawa [18] discussed the boundedness of $R_\alpha(L)$ under the intertwining condition for the diffusion semigroup in a general framework. We remark that the intertwining condition implies **(G)**. Hence one way to establish the boundedness of $R_\alpha(L)$ is to show the intertwining condition for each concrete problem.

2. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by a probabilistic method. The original idea is due to Meyer [16]. The reader is referred to Bakry [1], Shigekawa-Yoshida [20] and Yoshida [24]. In these papers, they expanded $L(Q_t^{(\alpha)} f)^p, f \in \mathcal{A}$, by employing the usual functional analytic argument in the proof of the Littlewood-Paley-Stein inequality. In that calculations, they needed to assume the existence of a good core $\mathcal{A}$ described in Section 1. However, we cannot follow their proof directly since we do not impose such good properties on $\mathcal{A}$. To overcome this difficulty, we replace the functional analytic argument by probabilistic one based on Itô's formula. We give details and prove Theorem 1.2 for $1 < p < 2$ in the second subsection. In the third subsection, we introduce the notion of H-functions to prove Theorem 1.2 for $p > 2$. Our gradient estimate condition **(G)** plays a crucial role when we compare G-functions with H-functions. For the case $p = 2$, (1.4) is proved as equality by using spectral resolution of L. See Proposition 3.1 in [20] for the proof. We note that (1.5) is derived from (1.4) by using the standard duality argument. See Theorem 4.4 in [20] for the detail.

2.1. Preparations

In this subsection, we make some preparations. We have already used the notation $\{P_x\}_{x\in X}$ to denote the diffusion measure of $\mathbb{M}$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. In this subsection, we use the notation $P_x^{\uparrow}$ in place of P_x. Let $(B_t, P_a^{\rightarrow})$ be one-dimensional Brownian motion starting at $a \in \mathbb{R}$ with the generator $\frac{\partial^2}{\partial a^2}$. We set $Y_t := (X_t, B_t)$, $t \geq 0$, and $\mathbb{P}_{(x,a)} := P_x^{\uparrow} \otimes P_a^{\rightarrow}$. Then $\tilde{\mathbb{M}} := (Y_t, \{\mathbb{P}_{(x,a)}\})$ is a $\mu \otimes m$-symmetric diffusion process on $X \times \mathbb{R}$ with the (formal) generator $L + \frac{\partial^2}{\partial a^2}$, where m is one-dimensional

Lebesgue measure. We put $P^{\uparrow}_{\mu} := \int_X P^{\uparrow}_x \mu(dx)$, $\mathbb{P}_{\mu\otimes\delta_a} := \int_X \mathbb{P}_{(x,a)}\mu(dx)$ and denote the integration with respect to $P^{\uparrow}_x, P^{\rightarrow}_a, \mathbb{P}_{(x,a)}$ and $\mathbb{P}_{\mu\otimes\delta_a}$ by $\mathbb{E}^{\uparrow}_x, \mathbb{E}^{\rightarrow}_a, \mathbb{E}_{(x,a)}$ and $\mathbb{E}_{\mu\otimes\delta_a}$, respectively.

We denote the semigroup on $L^p(X\times\mathbb{R};\mu\otimes m)$ associated with the diffusion process $\{Y_t\}_{t\geq 0}$ by $\{\hat{P}_t\}_{t\geq 0}$ and its generator by $\hat{L}_p$. We also denote the Dirichlet form on $L^2(X\times\mathbb{R};\mu\otimes m)$ associated with $\hat{L}_2$ by $(\hat{\mathcal{E}}, \mathcal{D}(\hat{\mathcal{E}}))$. That is,

$$\mathcal{D}(\hat{\mathcal{E}}) = \big\{u \in L^2(X\times\mathbb{R};\mu\otimes m) \;\big|\; \lim_{t\searrow 0}\frac{1}{t}\big(u - \hat{P}_t u, u\big)_{L^2(X\times\mathbb{R};\mu\otimes m)} < +\infty\big\},$$

$$\hat{\mathcal{E}}(u,v) = \lim_{t\searrow 0}\frac{1}{t}\big(u - \hat{P}_t u, v\big)_{L^2(X\times\mathbb{R};\mu\otimes m)} \quad \text{for } u,v\in\mathcal{D}(\hat{\mathcal{E}}).$$

We denote by $\hat{\mathcal{C}} := \mathcal{A}\otimes C_0^\infty(\mathbb{R})$ the totality of all linear combinations of $f\otimes\varphi, f\in\mathcal{A}, \varphi\in C_0^\infty(\mathbb{R})$, where $(f\otimes\varphi)(x,a) := f(x)\varphi(a)$. Meanwhile, the spaces $L^2(\mu)\otimes L^2(m)$ and $\mathcal{D}(\mathcal{E})\otimes H^{1,2}(\mathbb{R})$ are usual tensor products of Hilbert spaces, where $H^{1,2}(\mathbb{R})$ is the Sobolev space which consists of all functions $\varphi\in L^2(m)$ such that the weak derivative φ' exists and belongs to $L^2(m)$. Then we have

Lemma 2.1. *$\hat{\mathcal{C}}$ is dense in $\mathcal{D}(\hat{\mathcal{E}})$. Moreover for $u,v\in\mathcal{D}(\mathcal{E})\otimes H^{1,2}(\mathbb{R})$, we have*

$$\begin{aligned}\hat{\mathcal{E}}(u,v) &= \int_{\mathbb{R}}\mathcal{E}\big(u(\cdot,a),v(\cdot,a)\big)m(da) \\ &\quad + \int_X \mu(dx)\int_{\mathbb{R}}\frac{\partial u}{\partial a}(x,a)\frac{\partial v}{\partial a}(x,a)m(da).\end{aligned} \tag{2.1}$$

Proof. We denote by $\{T_t\}_{t\geq 0}$ the transition semigroup associated with $(B_t, \{P^{\rightarrow}_a\}_{a\in\mathbb{R}})$. We can regard it as the semigroup on $L^2(m)$. First, we note that the following identity holds:

$$\hat{P}_t(f\otimes\varphi) = (P_t f)\otimes(T_t\varphi), \quad f\in L^2(\mu), \varphi\in L^2(m). \tag{2.2}$$

By (2.2), we can see $\hat{\mathcal{C}}\subset\mathcal{D}(\mathcal{E})\otimes H^{1,2}(\mathbb{R})\subset\mathcal{D}(\hat{\mathcal{E}})$ and the identity (2.1). We also have

$$\begin{aligned}\hat{\mathcal{E}}_1(f\otimes\varphi, f\otimes\varphi) &\leq \mathcal{E}_1(f,f)\|\varphi\|^2_{L^2(m)} \\ &\quad + \|f\|^2_{L^2(\mu)}\big(\|\varphi'\|^2_{L^2(m)} + \|\varphi\|^2_{L^2(m)}\big)\end{aligned} \tag{2.3}$$

holds for $f \in \mathcal{D}(\mathcal{E}), \varphi \in H^{1,2}(\mathbb{R})$. By (2.3), we see that $\hat{\mathcal{C}}$ is dense in $\mathcal{D}(\mathcal{E}) \otimes H^{1,2}(\mathbb{R})$ with respect to $\hat{\mathcal{E}}_1$-topology, because $\mathcal{A}$ and $C_0^\infty(\mathbb{R})$ are dense in $\mathcal{D}(\mathcal{E})$ and $H^{1,2}(\mathbb{R})$, respectively.

Hence it is sufficient to show $\mathcal{D}(\mathcal{E}) \otimes H^{1,2}(\mathbb{R})$ is dense in $\mathcal{D}(\hat{\mathcal{E}})$. Since $L^2(\mu) \otimes L^2(m)$ is dense in $L^2(X \times \mathbb{R}; \mu \otimes m)$, $\bigcup_{t>0} \hat{P}_t\big(L^2(\mu) \otimes L^2(m)\big)$ is dense in $\mathcal{D}(\hat{\mathcal{E}})$. On the other hand, (2.2) also leads us to

$$\begin{aligned}\bigcup_{t>0} \hat{P}_t\big(L^2(\mu) \otimes L^2(m)\big) &= \bigcup_{t>0} \big(P_t(L^2(\mu))\big) \otimes \big(P_t(L^2(m))\big) \\ &\subset \mathcal{D}(\mathcal{E}) \otimes H^{1,2}(\mathbb{R}) \subset \mathcal{D}(\hat{\mathcal{E}}).\end{aligned}$$

Therefore the proof is completed. □

Here we note that, due to Fitzsimmons [6], the Dirichlet form $(\hat{\mathcal{E}}, \mathcal{D}(\hat{\mathcal{E}}))$ is quasi-regular. Thus we can apply the general theory of quasi-regular Dirichlet forms in [15].

Now we fix a function $f \in \mathcal{A}$. We set $u(x,a) := Q_a^{(\alpha)} f(x), a \geq 0$. Then it holds that

$$\left(\frac{\partial^2}{\partial a^2} + L - \alpha\right) u(\cdot, a) = 0 \quad \text{ in } L^2(\mu).$$

Furthermore for $a \in \mathbb{R}$, we consider $v(x,a) := u(x,|a|) = Q_{|a|}^{(\alpha)} f(x)$. Then by (1.3), we have

$$\|v\|_{L^2(X\times\mathbb{R};\mu\otimes m)} \leq \Big(\int_{\mathbb{R}} e^{-2\sqrt{\alpha}|a|} \|f\|^2_{L^2(\mu)} da\Big)^{1/2} = \alpha^{-1/4} \|f\|_{L^2(\mu)}. \tag{2.4}$$

The main purpose of this subsection is to discuss the semi-martingale decomposition of $v(X_{t\wedge\tau}, B_{t\wedge\tau}), t \geq 0$, where $\tau := \inf\{t > 0 \mid B_t = 0\}$. As the first step, we give the following fundamental lemma:

Lemma 2.2. *$v \in \mathcal{D}(\hat{\mathcal{E}})$ holds.*

Proof. At the beginning, we note $L^2(X \times \mathbb{R}; \mu \otimes m) \cong L^2(\mathbb{R}, L^2(X;\mu); m)$. According to Fubini's theorem, we have

$$\hat{P}_t v(x,a) = \mathbb{E}_{(x,a)}\big[u(X_t, |B_t|)\big] = \mathbb{E}_x^{\uparrow}\Big[\mathbb{E}_a^{\rightarrow}\big[u(\cdot, |B_t|)\big](X_t)\Big]. \tag{2.5}$$

We recall Tanaka's formula

$$|B_t| = |B_0| + \int_0^t \operatorname{sgn}(B_s)dB_s + L_t(0), \quad t \geq 0, \quad \mathbb{P}_a^{\rightarrow}\text{-a.s.},$$

where $\{L_t(0)\}_{t\geq 0}$ is the local time of one-dimensional Brownian motion $\{B_t\}_{t\geq 0}$ at the origin. Then by using Itô's formula, we have

$$\begin{aligned}
(2.6) \qquad u(\cdot, |B_t|) &= u(\cdot, |B_0|) + \int_0^t \frac{\partial u}{\partial a}(\cdot, |B_s|)\operatorname{sgn}(B_s)dB_s \\
&\quad + \int_0^t \frac{\partial u}{\partial a}(\cdot, |B_s|)dL_s(0) + \int_0^t \frac{\partial^2 u}{\partial a^2}(\cdot, |B_s|)ds \\
&= u(\cdot, |B_0|) - \int_0^t \sqrt{\alpha - L}u(\cdot, |B_s|)\operatorname{sgn}(B_s)dB_s \\
&\quad - \int_0^t \sqrt{\alpha - L}u(\cdot, |B_s|)dL_s(0) \\
&\quad + \int_0^t (\alpha - L)u(\cdot, |B_s|)ds.
\end{aligned}$$

Hence (2.6) leads us to

$$\begin{aligned}
(2.7) \qquad \mathbb{E}_a^{\rightarrow}[u(\cdot, |B_t|)] &= u(\cdot, |a|) - \mathbb{E}_a^{\rightarrow}\Big[\int_0^t \sqrt{\alpha - L}u(\cdot, |B_s|)dL_s(0)\Big] \\
&\quad + \mathbb{E}_a^{\rightarrow}\Big[\int_0^t (\alpha - L)u(\cdot, |B_s|)ds\Big].
\end{aligned}$$

On the other hand, since $f \in \mathcal{A}$, it holds that $u(\cdot, |a|) = Q_{|a|}^{(\alpha)} f(\cdot) \in \operatorname{Dom}(L_2)$. Hence

$$M_t^{[u(\cdot,|a|)]} := (Q_{|a|}^{(\alpha)} f)(X_t) - (Q_{|a|}^{(\alpha)} f)(X_0) - \int_0^t L(Q_{|a|}^{(\alpha)} f)(X_s)ds, \quad t \geq 0,$$

is an $L^2(P_\mu^{\uparrow})$-martingale. Then we have

$$\begin{aligned}
(2.8) \qquad \mathbb{E}_x^{\uparrow}\big[u(X_t, |a|)\big] &= (Q_{|a|}^{(\alpha)} f)(x) \\
&\quad + \int_0^t P_s(LQ_{|a|}^{(\alpha)} f)(x)ds, \qquad \mu\text{-a.e. } x \in X.
\end{aligned}$$

By summarizing (2.5), (2.7) and (2.8), we can proceed as

$$
\begin{aligned}
(2.9)\quad & \frac{1}{t}\big(v-\hat{P}_t v, v\big)_{L^2(X\times\mathbb{R};\mu\otimes m)} \\
= \quad & -\frac{1}{t}\int_{\mathbb{R}} da \int_X \Big\{\int_0^t P_s(LQ^{(\alpha)}_{|a|}f)(x)ds\Big\}\cdot Q^{(\alpha)}_{|a|}f(x)\mu(dx) \\
& +\frac{1}{t}\int_{\mathbb{R}} da \int_X \mathbb{E}^{\uparrow}_x\Big[\mathbb{E}^{\rightarrow}_a\big[\int_0^t \sqrt{\alpha-L}u(\cdot,|B_s|)dL_s(0)\big](X_t)\Big] \\
& \qquad\qquad \times Q^{(\alpha)}_{|a|}f(x)\mu(dx) \\
& -\frac{1}{t}\int_{\mathbb{R}} da \int_X \mathbb{E}^{\uparrow}_x\Big[\mathbb{E}^{\rightarrow}_a\big[\int_0^t (\alpha-L)u(\cdot,|B_s|)ds\big](X_t)\Big] \\
& \qquad\qquad \times Q^{(\alpha)}_{|a|}f(x)\mu(dx) \\
= \quad & -\frac{1}{t}\int_{\mathbb{R}} da \int_0^t \big(P_s LQ^{(\alpha)}_{|a|}f, Q^{(\alpha)}_{|a|}f\big)_{L^2(\mu)}ds \\
& +\frac{1}{t}\int_{\mathbb{R}} da \int_X \mathbb{E}^{\rightarrow}_a\big[\int_0^t \sqrt{\alpha-L}u(x,|B_s|)dL_s(0)\big] \\
& \qquad\qquad \times P_t(Q^{(\alpha)}_{|a|}f)(x)\mu(dx) \\
& -\frac{1}{t}\int_{\mathbb{R}} da \int_X \mathbb{E}^{\rightarrow}_a\big[\int_0^t (\alpha-L)u(x,|B_s|)ds\big] \\
& \qquad\qquad \times P_t(Q^{(\alpha)}_{|a|}f)(x)\mu(dx) \\
=: \quad & -I_1(t)+I_2(t)-I_3(t),
\end{aligned}
$$

where we used symmetry of $\{P_t\}_{t\geq 0}$ on $L^2(\mu)$. For the terms $I_1(t)$ and $I_2(t)$, we see the following estimates by using contractivity of $\{P_t\}_{t\geq 0}$ on $L^2(\mu)$ and (1.3):

$$
\begin{aligned}
(2.10)\qquad |I_1(t)| &\leq \frac{1}{t}\int_{\mathbb{R}} da \int_0^t \|LQ^{(\alpha)}_{|a|}f\|_{L^2(\mu)}\cdot\|Q^{(\alpha)}_{|a|}f\|_{L^2(\mu)}ds \\
&\leq \int_{\mathbb{R}} e^{-2\sqrt{\alpha}|a|}\|Lf\|_{L^2(\mu)}\cdot\|f\|_{L^2(\mu)}da \\
&= \frac{1}{\sqrt{\alpha}}\|Lf\|_{L^2(\mu)}\cdot\|f\|_{L^2(\mu)},
\end{aligned}
$$

$$
|I_2(t)| = \Big|\frac{1}{t}\int_{\mathbb{R}} da \int_X \Big(\sqrt{\alpha-L}u(x,0)\mathbb{E}^{\rightarrow}_a\big[L_t(0)\big]\Big)\cdot P_t(Q^{(\alpha)}_{|a|}f)(x)\mu(dx)\Big|
$$

$$
\begin{aligned}
&= \frac{1}{t}\Big|\int_{\mathbb{R}} \big(\sqrt{\alpha - L} f, P_t Q_{|a|}^{(\alpha)} f\big)_{L^2(\mu)} \mathbb{E}_a^{\rightarrow}\big[L_t(0)\big] da\Big| \\
&\leq \frac{2}{t}\|\sqrt{\alpha - L} f\|_{L^2(\mu)} \cdot \|f\|_{L^2(\mu)} \int_0^{\infty} e^{-\sqrt{\alpha} a} \mathbb{E}_a^{\rightarrow}\big[L_t(0)\big] da.
\end{aligned}
$$

Here we recall

$$
P_a^{\rightarrow}(L(t,r) \in dy) = \frac{1}{\sqrt{\pi t}} \exp\Big\{-\frac{(y+|r-a|)^2}{4t}\Big\} dy, \qquad y > 0.
$$

See page 155 of Borodin-Salminen [3]. Then we can continue as

$$
\begin{aligned}
(2.11) \qquad |I_2(t)| \;&\leq\; \frac{2}{t}\|\sqrt{\alpha - L} f\|_{L^2(\mu)} \cdot \|f\|_{L^2(\mu)} \\
&\quad \times \int_0^{\infty} e^{-\sqrt{\alpha} a}\Big\{\int_0^{\infty} y \frac{1}{\sqrt{\pi t}} \exp\Big(-\frac{(a+y)^2}{4t}\Big) dy\Big\} da \\
&\leq\; 8\|\sqrt{\alpha - L} f\|_{L^2(\mu)} \cdot \|f\|_{L^2(\mu)} \\
&\quad \times \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2}} da \int_0^{\infty} y e^{-\frac{y^2}{2}} dy \\
&=\; 4\|\sqrt{\alpha - L} f\|_{L^2(\mu)} \cdot \|f\|_{L^2(\mu)}.
\end{aligned}
$$

For the term $I_3(t)$, we have

$$
\begin{aligned}
(2.12) \qquad |I_3(t)| \;&\leq\; \frac{1}{t}\int_{\mathbb{R}} \Big\|\mathbb{E}_a^{\rightarrow}\Big[\int_0^t (\alpha - L) u(\cdot, |B_s|) ds\Big]\Big\|_{L^2(\mu)} \\
&\qquad \times \|Q_{|a|}^{(\alpha)} f\|_{L^2(\mu)} da \\
&\leq\; \frac{1}{t}\int_{\mathbb{R}} \mathbb{E}_a^{\rightarrow}\Big[\int_0^t \|(\alpha - L) Q_{|B_s|}^{(\alpha)} f(\cdot)\|_{L^2(\mu)} ds\Big] \\
&\qquad \times \Big(e^{-\sqrt{\alpha}|a|}\|f\|_{L^2(\mu)}\Big) da \\
&\leq\; \frac{1}{t}\int_{\mathbb{R}} \mathbb{E}_a^{\rightarrow}\Big[\int_0^t (\alpha\|f\|_{L^2(\mu)} + \|Lf\|_{L^2(\mu)}) ds\Big] \\
&\qquad \times \Big(e^{-\sqrt{\alpha}|a|}\|f\|_{L^2(\mu)}\Big) da \\
&=\; 2\sqrt{\alpha}\|f\|_{L^2(\mu)}^2 + \frac{2}{\sqrt{\alpha}}\|Lf\|_{L^2(\mu)} \cdot \|f\|_{L^2(\mu)}.
\end{aligned}
$$

Finally, we substitute estimates (2.10), (2.11) and (2.12) into (2.9). Then we can easily see

$$
\lim_{t \searrow 0} \frac{1}{t}\big(v - \hat{P}_t v, v\big)_{L^2(X \times \mathbb{R}; \mu \otimes m)} = \sup_{t > 0} \frac{1}{t}\big(v - \hat{P}_t v, v\big)_{L^2(X \times \mathbb{R}; \mu \otimes m)} < +\infty.
$$

This and (2.4) complete the proof. □

By Lemma 2.2, we can apply Fukushima's decomposition theorem. That is, there exist a martingale additive functional of finite energy $M^{[v]}$ and a continuous additive functional of zero energy $N^{[v]}$ such that

$$\tilde{v}(X_t, B_t) - \tilde{v}(X_0, B_0) = M_t^{[v]} + N_t^{[v]}, \qquad t \geq 0, \quad \mathbb{P}_{(x,a)}\text{-a.s. for q.e.-}(x,a), \tag{2.13}$$

where $\tilde{v}$ is an $\hat{\mathcal{E}}$-quasi-continuous modification of $v \in \mathcal{D}(\hat{\mathcal{E}})$. See Theorem 5.2.2 of Fukushima-Oshima-Takeda [7]. We note that, since $\hat{\mathcal{E}}$ has the strong local property, $M^{[v]}$ is continuous. Due to Theorem 5.2.3 of [7], we know that

$$\langle M^{[v]} \rangle_t = \int_0^t \Big\{ \Gamma(v,v)(X_s, B_s) + \Big(\frac{\partial v}{\partial a}(X_s, B_s) \Big)^2 \Big\} \, ds. \tag{2.14}$$

See also Theorem 5.1.3 and Example 5.1.1 of [7] for details.

From now, we discuss the explicit expression of $N^{[v]}$. Let us define a signed measure ν on $X \times \mathbb{R}$ by

$$\nu(dxda) := 2\sqrt{\alpha - L}v(x,a)\mu(dx)\delta_0(da),$$

where δ_0 is Dirac measure on $\mathbb{R}$ with mass at the origin. The total variation of ν is given by

$$|\nu|(dxda) := 2\big|\sqrt{\alpha - L}v(x,a)\big|\mu(dx)\delta_0(da).$$

Then we have

LEMMA 2.3. *There exists a constant $C > 0$ such that*

$$\iint_{X \times \mathbb{R}} \big|(g \otimes \varphi)(x,a)\big| \cdot |\nu|(dxda) \leq C\sqrt{\hat{\mathcal{E}}_1(g \otimes \varphi, g \otimes \varphi)}, \qquad g \in \mathcal{A}, \varphi \in C_0^\infty(\mathbb{R}).$$

That is, ν is of finite 1-order energy integral. (For definition of measures of finite 1-order energy integral, see Sections 2.2 and 5.4 of [7].)

PROOF. We take a positive constant a_0 such that $\mathrm{supp}(\varphi) \subset [-a_0, a_0]$. We first consider the case of $\varphi(0) \le 0$. Let $\varepsilon > 0$. Then for μ-a.e. $x \in X$, we have

$$\begin{aligned}
&\int_{\mathbb{R}} |\varphi(a)| \sqrt{\big(\sqrt{\alpha - L}v(x,a)\big)^2 + \varepsilon}\, \delta_0(da) \\
&= -\varphi(0) \sqrt{\big(\sqrt{\alpha - L}v(x,0)\big)^2 + \varepsilon} \\
&= \varphi(a_0) \sqrt{\big(\sqrt{\alpha - L}v(x,a_0)\big)^2 + \varepsilon} - \varphi(0) \sqrt{\big(\sqrt{\alpha - L}v(x,0)\big)^2 + \varepsilon} \\
&= \int_0^{a_0} \frac{\partial}{\partial a} \Big\{ \varphi(a) \sqrt{\big(\sqrt{\alpha - L}v(x,a)\big)^2 + \varepsilon} \Big\} da \\
&= \int_0^{a_0} \varphi'(a) \sqrt{\big(\sqrt{\alpha - L}v(x,a)\big)^2 + \varepsilon} da \\
&\quad - \int_0^{a_0} \varphi(a)\, \frac{\sqrt{\alpha - L}v(x,a) \cdot (\alpha - L)v(x,a)}{\sqrt{\big(\sqrt{\alpha - L}v(x,a)\big)^2 + \varepsilon}} da \\
&\le \int_{\mathbb{R}} |\varphi'(a)| \sqrt{\big(\sqrt{\alpha - L}v(x,a)\big)^2 + \varepsilon} da + \int_{\mathbb{R}} |\varphi(a)| \cdot |(\alpha - L)v(x,a)| da.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\iint_{X \times \mathbb{R}} \big|(g \otimes \varphi)(x,a)\big| \cdot |\nu|(dxda) \\
&= 2 \lim_{\varepsilon \searrow 0} \int_X |g(x)| \Big(\int_{\mathbb{R}} |\varphi(a)| \sqrt{\big(\sqrt{\alpha - L}v(x,a)\big)^2 + \varepsilon}\, \delta_0(da) \Big) \mu(dx) \\
&\le 2 \limsup_{\varepsilon \searrow 0} \int_X |g(x)| \Big(\int_{\mathbb{R}} |\varphi'(a)| \sqrt{\big(\sqrt{\alpha - L}v(x,a)\big)^2 + \varepsilon} da \Big) \mu(dx) \\
&\quad + 2 \int_X |g(x)| \Big(\int_{\mathbb{R}} |\varphi(a)| \cdot |(\alpha - L)v(x,a)| da \Big) \mu(dx) \\
&\le 2 \Big(\big\| \sqrt{\alpha - L}v \big\|_{L^2(X \times \mathbb{R}; \mu \otimes m)} \|\varphi'\|_{L^2(m)} \\
&\qquad + \big\| (\alpha - L)v \big\|_{L^2(X \times \mathbb{R}; \mu \otimes m)} \|\varphi\|_{L^2(m)} \Big) \|g\|_{L^2(\mu)} \\
&\le 2\sqrt{2}\alpha^{-1/4} \big(\big\| \sqrt{\alpha - L}f \big\|_{L^2(\mu)} + \|(\alpha - L)f\|_{L^2(\mu)} \big) \sqrt{\hat{\mathcal{E}}_1(g \otimes \varphi, g \otimes \varphi)} \\
&=: C \sqrt{\hat{\mathcal{E}}_1(g \otimes \varphi, g \otimes \varphi)},
\end{aligned}$$

where we used (2.4) and

$$\hat{\mathcal{E}}(g \otimes \varphi, g \otimes \varphi) = \mathcal{E}(g,g) \|\varphi\|_{L^2(m)}^2 + \|g\|_{L^2(\mu)}^2 \|\varphi'\|_{L^2(m)}^2$$

for the last line. This is the desired result.

In the case of $\varphi(0) \geq 0$, we easily see

$$\begin{aligned}(2.15) \qquad & \int_{\mathbb{R}} |\varphi(a)| \sqrt{\left(\sqrt{\alpha - L}v(x,a)\right)^2 + \varepsilon}\, \delta_0(da) \\ & = \int_{-a_0}^{0} \frac{\partial}{\partial a} \left\{ \varphi(a) \sqrt{\left(\sqrt{\alpha - L}v(x,a)\right)^2 + \varepsilon} \right\} da.\end{aligned}$$

By using (2.15), we can follow the same argument as the case where $\varphi(0) \leq 0$. Therefore the proof is completed. □

Due to Lemma 2.3, ν is of finite 1-order energy integral. Then for each $\beta > 0$, there exists a unique $U_\beta \nu \in \mathcal{D}(\hat{\mathcal{E}})$ such that the following relation holds:

$$\begin{aligned}(2.16) \qquad & \hat{\mathcal{E}}_\beta(U_\beta \nu, g \otimes \varphi) \\ & = \iint_{X \times \mathbb{R}} (g \otimes \varphi)(x,a) \nu(dxda), \qquad g \in \mathcal{A}, \varphi \in C_0^\infty(\mathbb{R}).\end{aligned}$$

Lemma 2.4. (1) $U_\alpha \nu = v$.
(2) *$U_\beta \nu = v - (\beta - \alpha)\hat{R}_\beta v$ holds, where $\{\hat{R}_\beta\}_{\beta > 0}$ is the resolvent of $\{\hat{P}_t\}_{t \geq 0}$.*

Proof. (1) We need to show (2.16). By using the integration by parts formula, for μ-a.e. $x \in X$, we have

$$\begin{aligned}(2.17) \quad & \int_{\mathbb{R}} \frac{\partial v}{\partial a}(x,a)\, \varphi'(a) da \\ & = -\int_0^\infty \sqrt{\alpha - L}u(x,a)\varphi'(a) da + \int_0^\infty \sqrt{\alpha - L}u(x,a)\varphi'(-a)\, da \\ & = -\int_0^\infty \sqrt{\alpha - L}u(x,a) \frac{d}{da}\big(\varphi(a) + \varphi(-a)\big) da \\ & = 2\sqrt{\alpha - L}u(x,0)\varphi(0) \\ & \quad + \int_0^\infty \frac{\partial}{\partial a}\sqrt{\alpha - L}u(x,a)\big(\varphi(a) + \varphi(-a)\big) da \\ & = 2\sqrt{\alpha - L}u(x,0)\varphi(0) - \int_0^\infty (\alpha - L)u(x,a)\big(\varphi(a) + \varphi(-a)\big) da \\ & = 2\sqrt{\alpha - L}v(x,0)\varphi(0) - \int_{\mathbb{R}} (\alpha - L)v(x,a)\varphi(a) da.\end{aligned}$$

Then (2.17) leads us to our desired equality as follows:

$$\begin{aligned}\hat{\mathcal{E}}_\alpha(v, g\otimes\varphi) &= \int_{\mathbb{R}} da\,\varphi(a)\int_X \sqrt{\alpha-L}v(x,a)\sqrt{\alpha-L}g(x)\,\mu(dx)\\ &\quad+\int_X \mu(dx)g(x)\Big(2\sqrt{\alpha-L}v(x,0)\varphi(0)\\ &\qquad\qquad -\int_{\mathbb{R}}(\alpha-L)v(x,a)\varphi(a)da\Big)\\ &= 2\int_X \sqrt{\alpha-L}v(x,0)g(x)\varphi(0)\mu(dx)\\ &= \iint_{X\times\mathbb{R}}(g\otimes\varphi)(x,a)\nu(dxda).\end{aligned}$$

(2) We recall $\hat{\mathcal{E}}_\beta(\hat{R}_\beta v, g\otimes\varphi) = \big(v, g\otimes\varphi\big)_{L^2(X\times\mathbb{R};\mu\otimes m)}$. Then we have

$$\begin{aligned}&\hat{\mathcal{E}}_\beta\big(v-(\beta-\alpha)\hat{R}_\beta v, g\otimes\varphi\big)\\ &\quad= \hat{\mathcal{E}}_\beta(v, g\otimes\varphi) - (\beta-\alpha)\cdot\big(v, g\otimes\varphi\big)_{L^2(X\times\mathbb{R};\mu\otimes m)}\\ &\quad= \hat{\mathcal{E}}_\alpha(v, g\otimes\varphi)\\ &\quad= \iint_{X\times\mathbb{R}}(g\otimes\varphi)(x,a)\nu(dxda),\end{aligned}$$

where we used (1) for the last line. Hence the proof of (2) is also completed. □

Due to Lemma 5.4.1 of [7] and the lemma above, we have

$$N_t^{[v]} = \alpha\int_0^t \tilde{v}(X_s, B_s)\,ds - A_t, \quad t\ge 0,$$

where $\tilde{v}$ is an $\hat{\mathcal{E}}$-quasi-continuous modification of v and A is the continuous additive functional corresponding to ν. Since ν does not charge out of $X\times\{0\}$, due to Theorem 5.1.5 of [7], $A_{t\wedge\tau}=0$ holds. Thus we get

$$N_{t\wedge\tau}^{[v]} = \alpha\int_0^{t\wedge\tau}\tilde{v}(X_s, B_s)\,ds. \tag{2.18}$$

By summarizing (2.13), (2.14), and (2.18), we have the following proposition which plays a crucial role later.

PROPOSITION 2.5. *We have the semi-martingale decomposition*

$$\tilde{v}(X_{t\wedge\tau}, B_{t\wedge\tau}) - \tilde{v}(X_0, B_0) = M^{[v]}_{t\wedge\tau} + \alpha\int_0^{t\wedge\tau} \tilde{v}(X_s, B_s)\, ds, \quad t \geq 0, \tag{2.19}$$

under $\mathbb{P}_{(x,a)}$ for q.e.-(x, a). Moreover it holds

$$\langle M^{[v]}\rangle_{t\wedge\tau} = \int_0^{t\wedge\tau} \Big\{\Gamma(v, v)(X_s, B_s) + \Big(\frac{\partial v}{\partial a}(X_s, B_s)\Big)^2\Big\}\, ds. \tag{2.20}$$

Since $v(x, a) = u(x, a)$ holds for $a \geq 0$, this proposition also gives the semi-martingale decomposition of $u(X_{t\wedge\tau}, B_{t\wedge\tau})$.

Before closing this subsection, we need the following lemma to allow $\mu \otimes \delta_a$ as an initial distribution.

LEMMA 2.6. *$\mu \otimes \delta_a$ does not charge any set of zero capacity for m-almost all $a \in \mathbb{R}$.*

PROOF. Let $N \subset X \times \mathbb{R}$ be a set of zero capacity with respect to $\hat{\mathcal{E}}_1$. Then by the item (4) in Theorem 4.1 of Okura [17], N_a is a set of zero capacity with respect to $\mathcal{E}_1$ for m-a.e. $a \in \mathbb{R}$, where the set $N_a \subset X$ is defined by $N_a := \{x \in X | (x, a) \in N\}, a \in \mathbb{R}$. Thus we have

$$(\mu \otimes \delta_a)(N) = \mu(N_a) \leq \mathrm{Cap}_{\mathcal{E}_1}(N_a) = 0.$$

This completes the proof. □

2.2. Proof of Theorem 1.2 ($1 < p < 2$)

In this subsection, we return to the proof of Theorem 1.2 in the case of $1 < p < 2$. Here we recall the following identities for our later use. See [16] for the proof.

LEMMA 2.7. *Let $\eta : X \times [0, +\infty) \to [0, +\infty)$ be a measurable function. Then*

$$\mathbb{E}_{\mu\otimes\delta_a}\Big[\int_0^\tau \eta(X_t, B_t) dt\Big] = \int_X \mu(dx) \int_0^\infty (a \wedge t)\eta(x, t)\, dt \tag{2.21}$$

and

$$\mathbb{E}_{\mu\otimes\delta_a}\Big[\int_0^\tau \eta(X_t,B_t)dt\Big|X_\tau=x\Big]=\int_0^\infty (a\wedge t)Q_t^{(0)}\eta(\cdot,t)(x)\,dt. \tag{2.22}$$

Since $\{X_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ are mutually independent under $\mathbb{P}_{\mu\otimes\delta_a}$ and μ is the invariant measure of $\{X_t\}_{t\geq 0}$, we can see the following identity for any bounded Borel measurable function h on X:

$$\mathbb{E}_{\mu\otimes\delta_a}\big[h(X_\tau)\big]=\int_X h(x)\mu(dx). \tag{2.23}$$

Hereafter, we abbreviate $M_{t\wedge\tau}^{[v]}$ as M_t for simplicity. By Proposition 2.5 and Lemma 2.6, there exists a non-negative sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} a_n=\infty$, (2.19) and (2.20) hold under $\mathbb{P}_{\mu\otimes\delta_{a_n}}$ for any $n\in\mathbb{N}$.

We set $V_t:=\tilde{v}(X_{t\wedge\tau},B_{t\wedge\tau})$. We apply Itô's formula to V_t^2. Proposition 2.5 implies

$$\begin{aligned} d(V_t^2) &= 2V_t dM_t+2\alpha V_t^2dt+d\langle M\rangle_t \\ &= 2V_t dM_t+2\big(g_f(X_t,B_t)^2+\alpha V_t^2\big)dt. \end{aligned} \tag{2.24}$$

Let $\varepsilon>0$. By applying Itô's formula to $(V_t^2+\varepsilon)^{p/2}$ again, we also have

$$\begin{aligned} d\big(V_t^2+\varepsilon\big)^{p/2} &= p\big(V_t^2+\varepsilon\big)^{p/2-1}V_t dM_t \\ &\quad + p\big(V_t^2+\varepsilon\big)^{p/2-1}\big(g_f(X_t,B_t)^2+\alpha V_t^2\big)dt \\ &\quad + \frac{p(p-2)}{2}\big(V_t^2+\varepsilon\big)^{p/2-2}V_t^2 d\langle M\rangle_t \\ &\geq p\big(V_t^2+\varepsilon\big)^{p/2-1}V_t dM_t \\ &\quad + p(p-1)\big(V_t^2+\varepsilon\big)^{p/2-1}g_f(X_t,B_t)^2dt, \end{aligned}$$

where we used $p<2$ for the last line.

Hence by taking the expectation of the inequality above and using

$u(x,a) = v(x,a)$ for $a \geq 0$, we have

$$\begin{aligned}
(2.25) \qquad & \mathbb{E}_{\mu\otimes\delta_{a_n}}\Big[p(p-1)\int_0^{\tau} \big(V_t^2+\varepsilon\big)^{p/2-1} g_f(X_t,B_t)^2\,dt\Big] \\
&\leq \mathbb{E}_{\mu\otimes\delta_{a_n}}\Big[(V_\tau^2+\varepsilon)^{p/2} - (V_0^2+\varepsilon)^{p/2}\Big] \\
&\leq \mathbb{E}_{\mu\otimes\delta_{a_n}}\Big[\big(V_\tau^2+\varepsilon\big)^{p/2}\Big] \\
&= \mathbb{E}_{\mu\otimes\delta_{a_n}}\Big[\big(u(X_\tau,B_\tau)^2+\varepsilon\big)^{p/2}\Big] \\
&= \mathbb{E}_{\mu\otimes\delta_{a_n}}\Big[\big(f(X_\tau)^2+\varepsilon\big)^{p/2}\Big] = \int_X (|f(x)|^2+\varepsilon)^{p/2}\,\mu(dx),
\end{aligned}$$

where we used (2.23) for the last line. Here, by recalling (2.21), the left hand side of (2.25) is equal to

$$p(p-1)\int_X \mu(dx)\int_0^\infty (t\wedge a_n)(u(x,t)^2+\varepsilon)^{p/2-1} g_f(x,t)^2\,dt.$$

Therefore, by letting $\varepsilon \to 0$ and $n \to \infty$, we have

$$(2.26) \qquad p(p-1)\int_X \mu(dx)\int_0^\infty t|u(x,t)|^{p-2} g_f(x,t)^2\,dt \leq \int_X |f(x)|^p\,\mu(dx).$$

Now we recall the maximal ergodic inequality

$$\Big\|\sup_{t\geq 0}|P_t f|\Big\|_{L^p(\mu)} \leq \frac{p}{p-1}\|f\|_{L^p(\mu)}, \qquad p>1.$$

See Theorem 3.3 in Shigekawa [19] for details. It leads us to

$$\begin{aligned}
\|G_f\|_{L^p(\mu)}^p &= \int_X \mu(dx)\Big\{\int_0^\infty t|u(x,t)|^{2-p}|u(x,t)|^{p-2} g_f(x,t)^2\,dt\Big\}^{p/2} \\
&\leq \int_X \mu(dx)\Big\{\int_0^\infty t\big(\sup_{t\geq 0}|P_t f(x)|\big)^{2-p}|u(x,t)|^{p-2} g_f(x,t)^2\,dt\Big\}^{p/2} \\
&\leq \Big\{\int_X \big(\sup_{t\geq 0}|P_t f(x)|\big)^p\,\mu(dx)\Big\}^{\frac{2-p}{2}} \\
&\qquad \times \Big\{\int_X\int_0^\infty t|u(x,t)|^{p-2} g_f(x,t)^2\,dt\,\mu(dx)\Big\}^{p/2} \\
&\lesssim \Big\{\int_X |f(x)|^p\,\mu(dx)\Big\}^{\frac{2-p}{2}} \Big\{\int_X |f(x)|^p\,\mu(dx)\Big\}^{p/2} = \|f\|_{L^p(\mu)}^p,
\end{aligned}$$

where we used (2.26) for the last line. This completes the proof.

2.3. Proof of Theorem 1.2 ($p > 2$)

In the case of $p > 2$, we need additional functions, namely H-functions defined by

$$\begin{aligned} H_f^{\rightarrow}(x) &:= \left\{ \int_0^\infty t Q_t^{(0)}(g_f^{\rightarrow}(\cdot, t)^2)(x)\, dt \right\}^{1/2}, \\ H_f^{\uparrow}(x) &:= \left\{ \int_0^\infty t Q_t^{(0)}(g_f^{\uparrow}(\cdot, t)^2)(x)\, dt \right\}^{1/2}, \\ H_f(x) &:= \left\{ \int_0^\infty t Q_t^{(0)}(g_f(\cdot, t)^2)(x)\, dt \right\}^{1/2}. \end{aligned}$$

We begin by the following proposition:

Proposition 2.8. *For $p > 2$, the following inequality holds for any $f \in \mathcal{A}$:*

$$\|H_f\|_{L^p(\mu)} \lesssim \|f\|_{L^p(\mu)}.$$

Proof. By a slight modification, we can prove in the same way as the proof of Proposition 4.2 in Shigekawa-Yoshida [20]. However we give the proof for the reader's convenience.

Let us recall that, due to (2.24), we have

$$V_{t\wedge\tau}^2 - V_0^2 = 2\int_0^{t\wedge\tau} V_s\, dM_s + 2\int_0^{t\wedge\tau} \big(\alpha V_s^2 + g_f(X_s, B_s)^2\big) ds. \tag{2.27}$$

Since $A_t := 2\int_0^{t\wedge\tau} \big(\alpha V_s^2 + g_f(X_s, B_s)^2\big) ds,\ t \geq 0$, is a continuous increasing process, (2.27) implies that $Z_t := V_{t\wedge\tau}^2 - V_0^2,\ t \geq 0$ is a submartingale.

Now we need an inequality for submartingales. Let $\{Z_t\}_{t\geq 0}$ be a continuous submartingale with the Doob-Meyer decomposition $Z_t = M_t + A_t$, where $\{M_t\}_{t\geq 0}$ is a continuous martingale and $\{A_t\}_{t\geq 0}$ is a continuous increasing process with $A_0 = 0$. Due to Lenglart-Lépingle-Pratelli [13], it holds that

$$\mathbb{E}[A_\infty^p] \leq (2p)^p \mathbb{E}\Big[\sup_{t\geq 0} |Z_t|^p\Big], \quad p > 1. \tag{2.28}$$

Then by using (2.28) and Doob's inequality, we have

$$
\begin{aligned}
(2.29)\qquad & \mathbb{E}_{\mu\otimes\delta_{a_n}}\Big[\big\{2\int_0^\tau \big(\alpha V_s^2+g_f(X_s,B_s)^2\big)ds\big\}^{p/2}\Big] \\
&\lesssim \mathbb{E}_{\mu\otimes\delta_{a_n}}\Big[\sup_{t\ge 0}|V_{t\wedge\tau}^2-V_0^2|^{p/2}\Big] \\
&\lesssim \mathbb{E}_{\mu\otimes\delta_{a_n}}\Big[|V_\tau^2-V_0^2|^{p/2}\Big] \\
&= \mathbb{E}_{\mu\otimes\delta_{a_n}}\Big[|u(X_\tau,B_\tau)^2-u(X_0,B_0)^2|^{p/2}\Big] \\
&= \mathbb{E}_{\mu\otimes\delta_{a_n}}\Big[|(Q_0^{(\alpha)}f(X_\tau))^2-(Q_{a_n}^{(\alpha)}f(X_0))^2|^{p/2}\Big] \\
&\lesssim \mathbb{E}_{\mu\otimes\delta_{a_n}}\Big[|(Q_0^{(\alpha)}f(X_\tau)|^p\Big]+\mathbb{E}_{\mu\otimes\delta_{a_n}}\Big[|Q_{a_n}^{(\alpha)}f(X_0)|^p\Big] \\
&= \|f\|_{L^p(\mu)}^p+\|Q_{a_n}^{(\alpha)}f\|_{L^p(\mu)}^p \lesssim \|f\|_{L^p(\mu)}^p.
\end{aligned}
$$

On the other hand, by using (2.22), (2.29) and Jensen's inequality, we have

$$
\begin{aligned}
\|H_f\|_{L^p(\mu)}^p &= \Big\|\Big\{\int_0^\infty tQ_t^{(0)}\big(g_f(\cdot,t)^2\big)dt\Big\}^{p/2}\Big\|_{L^1(\mu)} \\
&= \lim_{n\to\infty}\Big\|\Big\{\int_0^\infty (a_n\wedge t)Q_t^{(0)}\big(g_f(\cdot,t)^2\big)dt\Big\}^{p/2}\Big\|_{L^1(\mu)} \\
&= \lim_{n\to\infty}\mathbb{E}_{\mu\otimes\delta_{a_n}}\Big[\Big\{\int_0^\infty (a_n\wedge t)Q_t^{(0)}\big(g_f(\cdot,t)^2\big)(X_\tau)dt\Big\}^{p/2}\Big] \\
&= \lim_{n\to\infty}\mathbb{E}_{\mu\otimes\delta_{a_n}}\Big[\mathbb{E}_{\mu\otimes\delta_{a_n}}\Big[\int_0^\tau g_f(X_s,B_s)^2ds\big|X_\tau\Big]^{p/2}\Big] \\
&\le \liminf_{n\to\infty}\mathbb{E}_{\mu\otimes\delta_{a_n}}\Big[\mathbb{E}_{\mu\otimes\delta_{a_n}}\Big[\big(\int_0^\tau g_f(X_s,B_s)^2ds\big)^{p/2}\big|X_\tau\Big]\Big] \\
&= \liminf_{n\to\infty}\mathbb{E}_{\mu\otimes\delta_{a_n}}\Big[\big(\int_0^\tau g_f(X_s,B_s)^2ds\big)^{p/2}\Big] \\
&\le \liminf_{n\to\infty}\mathbb{E}_{\mu\otimes\delta_{a_n}}\Big[\big\{\int_0^\tau \big(\alpha V_s^2+g_f(X_s,B_s)^2\big)ds\big\}^{p/2}\Big] \lesssim \|f\|_{L^p(\mu)}^p.
\end{aligned}
$$

This completes the proof. □

Next we study the relationship between G-functions and H-functions. In the proof of this proposition, condition **(G)** plays a key role.

PROPOSITION 2.9. (1) *For any $f \in \mathcal{A}$ and $\alpha > R \vee 0$, the following inequality holds:*

$$G_f^{\uparrow} \leq 2\sqrt{K} H_f^{\uparrow}.$$

(2) *For any $f \in \mathcal{A}$, the following inequality holds:*

$$G_f^{\rightarrow} \leq 2 H_f^{\rightarrow}.$$

PROOF. We give a proof of the item (1) only. The item (2) can be proved in the same way. By condition **(G)** and Schwarz's inequality, we have the following estimate for any $\alpha > R \vee 0$ and $f \in \mathcal{A}$:

$$\begin{aligned} \Gamma(Q_t^{(\alpha)} f) &\leq \Big(\int_0^\infty e^{-\alpha s} \Gamma(P_s f)^{1/2} \lambda_t(ds) \Big)^2 \\ &\leq \Big(\int_0^\infty e^{-(\alpha - R)s} \lambda_t(ds) \Big) \\ &\quad \times \Big(\int_0^\infty e^{-(\alpha+R)s} \Gamma\big(P_s f\big) \lambda_t(ds) \Big) \\ &\leq K e^{-\sqrt{\alpha - R} t} \Big(\int_0^\infty e^{-(\alpha-R)s} P_s\big(\Gamma(f)\big) \lambda_t(ds) \Big) \\ &\leq K Q_t^{(\alpha - R)}\big(\Gamma(f)\big). \end{aligned} \tag{2.30}$$

Then (2.30) yields

$$\begin{aligned} g_f^{\uparrow}(x, 2t)^2 &= \Gamma(Q_{2t}^{(\alpha)} f)(x) \\ &= \Gamma\Big(Q_t^{(\alpha)} (Q_t^{(\alpha)} f) \Big)(x) \\ &\leq K Q_t^{(\alpha - R)} \Big(\Gamma(Q_t^{(\alpha)} f) \Big)(x) \leq K Q_t^{(0)} \big(g_f^{\uparrow}(\cdot, t)^2 \big)(x), \end{aligned} \tag{2.31}$$

Therefore we have

$$\begin{aligned} (G_f^{\uparrow}(x))^2 &= 4 \int_0^\infty t g_f^{\uparrow}(x, 2t)^2 \, dt \\ &\leq 4K \int_0^\infty t Q_t^{(0)} \big(g_f^{\uparrow}(\cdot, t)^2 \big)(x) \, dt = 4K (H_f^{\uparrow}(x))^2, \end{aligned}$$

where we changed the variable t to $2t$ in the first line and used (2.31) for the second line. This completes the proof. $\square$

It is clear that Propositions 2.8 and 2.9 conclude the desired inequality (1.4). Therefore the proof of Theorem 1.2 is completed.

3. Proof of Theorem 1.3

Before giving the proof of Theorem 1.3, we make a preparation following Yoshida [24]. Let ν be a finite signed measure on $[0,\infty)$. We denote by $\hat{\nu}$ and $\|\nu\| := \int_0^\infty |\nu|(ds)$ the Laplace transform and the total variation of ν, respectively. For $\alpha > 0$, we define a bounded operator $\hat{\nu}(\alpha - L)$ on $L^p(\mu)$, $1 \le p < \infty$, by

$$\hat{\nu}(\alpha - L)f := \int_{[0,\infty)} e^{-\alpha s} P_s f \, \nu(ds).$$

Thus we easily have

$$\|\hat{\nu}(\alpha - L)f\|_{L^p(\mu)} \le \|\nu\| \cdot \|f\|_{L^p(\mu)}, \;\; f \in L^p(\mu). \tag{3.1}$$

Here we give a remark in the case of $p = 2$. In this case, this operator is represented by

$$\hat{\nu}(\alpha - L) := \int_{[0,\infty)} \hat{\nu}(\alpha + \lambda) dE_\lambda,$$

where $\{E_\lambda\}_{\lambda \ge 0}$ is the spectral decomposition of $-L$ in $L^2(\mu)$.

By Lemma 2.3 in [1], there exist finite signed measures ν_1 and ν_2 such that the Laplace transform are given by $\hat{\nu}_1(\lambda) = \frac{\sqrt{1+\lambda}}{1+\sqrt{\lambda}}$ and $\hat{\nu}_2(\lambda) = \frac{1+\sqrt{\lambda}}{\sqrt{1+\lambda}}$, respectively. For $\varepsilon > 0$, we denote by $\nu_i^{(\varepsilon)}$, $i = 1, 2$, the image measure of ν_i under the mapping $\lambda \mapsto \lambda/\varepsilon$. Then we have

$$\hat{\nu}_1^{(\varepsilon)}(\lambda) = \frac{\sqrt{\varepsilon + \lambda}}{\sqrt{\varepsilon} + \sqrt{\lambda}}, \quad \|\nu_1^{(\varepsilon)}\| \le \|\nu_1\|, \tag{3.2}$$

$$\hat{\nu}_2^{(\varepsilon)}(\lambda) = \frac{\sqrt{\varepsilon} + \sqrt{\lambda}}{\sqrt{\varepsilon + \lambda}}, \quad \|\nu_2^{(\varepsilon)}\| \le \|\nu_2\|. \tag{3.3}$$

(3.1), (3.2) and (3.3) imply the operators $\frac{\sqrt{\varepsilon+(\alpha-L)}}{\sqrt{\varepsilon}+\sqrt{\alpha-L}}$ and $\frac{\sqrt{\varepsilon}+\sqrt{\alpha-L}}{\sqrt{\varepsilon+(\alpha-L)}}$ on $L^p(\mu)$

have the operator norms less than $\|\nu_1\|$ and $\|\nu_2\|$, respectively. We also have

$$\begin{aligned}
&\Big(\frac{\sqrt{\varepsilon+(\alpha-L)}}{\sqrt{\varepsilon}+\sqrt{\alpha-L}}\Big)\Big(\frac{\sqrt{\varepsilon}+\sqrt{\alpha-L}}{\sqrt{\varepsilon+(\alpha-L)}}\Big)\\
&\qquad=\Big(\frac{\sqrt{\varepsilon}+\sqrt{\alpha-L}}{\sqrt{\varepsilon+(\alpha-L)}}\Big)\Big(\frac{\sqrt{\varepsilon+(\alpha-L)}}{\sqrt{\varepsilon}+\sqrt{\alpha-L}}\Big)=I.
\end{aligned}$$

Then we obtain the following relation for $q>1$:

$$\begin{aligned}
\big(\sqrt{\varepsilon+(\alpha-L)}\big)^{-q} &= (\sqrt{\varepsilon}+\sqrt{\alpha-L})^{-q}\Big(\frac{\sqrt{\varepsilon+(\alpha-L)}}{\sqrt{\varepsilon}+\sqrt{\alpha-L}}\Big)^{-q}\\
&= (\sqrt{\varepsilon}+\sqrt{\alpha-L})^{-q}\Big(\frac{\sqrt{\varepsilon}+\sqrt{\alpha-L}}{\sqrt{\varepsilon+(\alpha-L)}}\Big)^{q}.
\end{aligned}\tag{3.4}$$

Now we are in a position to give the proof of Theorem 1.3.

PROOF OF THEOREM 1.3. First, we set $\beta\in\mathbb{R}$ and $\varepsilon>0$ such that $\alpha=\beta+\varepsilon$ and $\beta>R$. Note $0<\varepsilon<\alpha_R$. Let $f\in L^2\cap L^p(\mu)$ and we consider

$$g:=\Big(\frac{\sqrt{\varepsilon}+\sqrt{\beta-L}}{\sqrt{\varepsilon+(\beta-L)}}\Big)^{q}f.$$

By (3.4), we have

$$\begin{aligned}
\Gamma\big((\sqrt{\alpha-L})^{-q}f\big) &= \Gamma\big((\sqrt{\varepsilon}+\sqrt{\beta-L})^{-q}g\big)\\
&\le \left(\frac{1}{\Gamma(q)}\int_0^\infty t^{q-1}e^{-\sqrt{\varepsilon}t}\Gamma(Q_t^{(\beta)}g)^{1/2}\,dt\right)^2.
\end{aligned}$$

Here we use Theorem 1.2. By recalling $q>1$, we have the following estimate:

$$\begin{aligned}
&\big\|\Gamma\big((\sqrt{\alpha-L})^{-q}f\big)^{1/2}\big\|_{L^p(\mu)}\\
&\quad\le \frac{1}{\Gamma(q)}\Big\|\int_0^\infty t^{q-1}e^{-\sqrt{\varepsilon}t}\Gamma(Q_t^{(\beta)}g)^{1/2}dt\Big\|_{L^p(\mu)}\\
&\quad\le \frac{1}{\Gamma(q)}\Big\|\Big(\int_0^\infty t^{2q-3}e^{-2\sqrt{\varepsilon}t}dt\Big)^{1/2}\Big(\int_0^\infty t\Gamma(Q_t^{(\beta)}g)dt\Big)^{1/2}\Big\|_{L^p(\mu)}\\
&\quad= \frac{1}{\Gamma(q)}\cdot\Big(\frac{\Gamma(2q-2)}{(4\varepsilon)^{q-1}}\Big)^{1/2}\|G_g^{\uparrow}\|_{L^p(\mu)}\\
&\quad\lesssim (4\varepsilon)^{-(q-1)/2}\frac{\Gamma(2q-2)^{1/2}}{\Gamma(q)}\cdot\|g\|_{L^p(\mu)}.
\end{aligned}\tag{3.5}$$

However the left hand side of (3.5) does not depend on ε. Hence we can let $\varepsilon \nearrow \alpha_R$ on the right hand side, and it leads us to

$$\big\|\Gamma\big((\sqrt{\alpha - L})^{-q} f\big)^{1/2}\big\|_{L^p(\mu)} \le C_{K,p,q}\, \alpha_R^{-(q-1)/2} \cdot \|g\|_{L^p(\mu)}. \tag{3.6}$$

On the other hand, we have

$$\|g\|_{L^p(\mu)} \le \|\nu_1\|^q \cdot \|f\|_{L^p(\mu)}. \tag{3.7}$$

Then by combining (3.6) with (3.7), we complete the proof of the item (1).

For the proof of the item (2), we use the same argument as used in Kawabi [11]. Since $\{P_t\}_{t\ge 0}$ is an analytic semigroup on $L^p(\mu)$ (see Chapter III of Stein [23] for details), there exists a positive constant C_p such that

$$\big\|LP_t f\big\|_{L^p(\mu)} \le C_p t^{-1} \big\|f\big\|_{L^p(\mu)}, \quad f \in L^p(\mu), \tag{3.8}$$

and hence $P_t^{(\alpha)} := e^{-\alpha t} P_t$ also satisfies

$$\big\|(\alpha - L)P_t^{(\alpha)} f\big\|_{L^p(\mu)} \le e^{-\alpha t}\big(C_p t^{-1} + \alpha\big)\big\|f\big\|_{L^p(\mu)}, \quad f \in L^p(\mu). \tag{3.9}$$

Then by noting $1 < q < 2$ and (3.9), the left hand side of (1.6) is dominated as

$$\begin{aligned}
&\big\|\Gamma(P_t f)^{1/2}\big\|_{L^p(\mu)} \\
&\quad = e^{\alpha t}\big\|\Gamma\big(P_t^{(\alpha)} f\big)^{1/2}\big\|_{L^p(\mu)} \\
&\quad \le e^{\alpha t}\|R_\alpha^{(q)}(L)\|_{p,p}\big\|(\sqrt{\alpha - L})^q P_t^{(\alpha)} f\big\|_{L^p(\mu)} \\
&\quad = e^{\alpha t}\|R_\alpha^{(q)}(L)\|_{p,p}\big\|(\sqrt{\alpha - L})^{q-2}(\alpha - L) P_t^{(\alpha)} f\big\|_{L^p(\mu)} \\
&\quad \le \frac{e^{\alpha t}\|R_\alpha^{(q)}(L)\|_{p,p}}{\Gamma(1 - q/2)} \int_0^\infty s^{-q/2}\big\|(\alpha - L)P_{s+t}^{(\alpha)} f\big\|_{L^p(\mu)} ds \\
&\quad \le \frac{e^{\alpha t}\|R_\alpha^{(q)}(L)\|_{p,p}}{\Gamma(1 - q/2)} \\
&\qquad \times \int_0^\infty s^{-q/2}\Big\{e^{-\alpha(s+t)}\big(\frac{C_p}{s+t} + \alpha\big)\big\|f\big\|_{L^p(\mu)}\Big\} ds,
\end{aligned} \tag{3.10}$$

where we used the item (1) for the second line.

Moreover, we have

$$
\begin{aligned}
(3.11)\qquad & \frac{e^{\alpha t}}{\Gamma(1-q/2)}\int_0^\infty s^{-q/2}e^{-\alpha(s+t)}\Big(\frac{C_p}{s+t}+\alpha\Big)ds \\
&\le \frac{C_p}{\Gamma(1-q/2)}\int_0^\infty s^{-q/2}(s+t)^{-1}ds \\
&\qquad + \frac{\alpha}{\Gamma(1-q/2)}\int_0^\infty s^{-q/2}e^{-\alpha s}ds \\
&= \frac{C_p}{\Gamma(1-q/2)}t^{-q/2}\int_0^\infty \tau^{-q/2}(1+\tau)^{-1}d\tau + \alpha^{q/2} \\
&\le C_{p,q}\big(t^{-q/2}+\alpha^{q/2}\big),
\end{aligned}
$$

where we changed the variable s to $t\tau$ in the third line.

Hence by combining (3.10) with (3.11), we obtain our desired estimate (1.6). This completes the proof. □

4. Examples

4.1. Diffusion processes on a path space with Gibbs measures

In this subsection, we present an example in an infinite dimensional setting. This is studied in Kawabi [10], [12]. We consider diffusion processes on an infinite volume path space $C(\mathbb{R}, \mathbb{R}^d)$ with Gibbs measures associated with the (formal) Hamiltonian

$$\mathcal{H}(w) := \frac{1}{2}\int_{\mathbb{R}} |w'(x)|^2_{\mathbb{R}^d}dx + \int_{\mathbb{R}} U(w(x))dx,$$

where $U : \mathbb{R}^d \to \mathbb{R}$ is an interaction potential. Our diffusion processes are defined through the time dependent Ginzburg-Landau type SPDE

$$(4.1)\quad dX_t(x) = \big\{\Delta_x X_t(x) - \nabla U(X_t(x))\big\}dt + \sqrt{2}dW_t(x), \quad x \in \mathbb{R},\ t > 0,$$

where $\Delta_x = d^2/dx^2$, $\nabla = (\partial/\partial z_i)_{i=1}^d$ and $(W_t)_{t\ge 0}$ is a white noise process. This dynamics is called the $P(\phi)_1$-time evolution.

In what follows, we describe the framework. We introduce some spaces of functions to control the growth of $X_t(x)$ as $|x| \to \infty$. For fixed $\lambda > 0$, we consider a Hilbert space $E := L^2(\mathbb{R}, \mathbb{R}^d; e^{-2\lambda\chi(x)}dx)$, $\lambda > 0$, where $\chi \in$

$C^\infty(\mathbb{R},\mathbb{R})$ is a positive symmetric convex function satisfying $\chi(x)=|x|$ for $|x|\geq 1$. We also consider

$$\mathcal{C}:=\Big\{X(\cdot)\in C(\mathbb{R},\mathbb{R}^d)\ \big|\sup_{x\in\mathbb{R}}|X(x)|_{\mathbb{R}^d}e^{-\lambda\chi(x)}<\infty\ \text{ for every } \lambda>0\Big\}.$$

We regard these spaces as state spaces of our dynamics.

Let μ be a (U-)Gibbs measure. This means that the regular conditional probability satisfies the following DLR-equation for every $r\in\mathbb{N}$ and μ-a.e. $\xi\in\mathcal{C}$:

$$\mu(dw|\mathcal{B}_r^*)(\xi)=Z_{r,\xi}^{-1}\exp\big(-\int_{-r}^{r}U(w(x))dx\big)\mathcal{W}_{r,\xi}(dw),$$

where $\mathcal{B}_r^*$ is the σ-field generated by $\mathcal{C}|_{[-r,r]^c}$, $\mathcal{W}_{r,\xi}$ is the path space measure of the Brownian bridge on $[-r,r]$ with a boundary condition $\mathcal{W}_{r,\xi}\big(w(r)=\xi(r),w(-r)=\xi(-r)\big)=1$ and $Z_{r,\xi}$ is the normalization constant.

We impose the following conditions on the potential function U:

(U1) $U\in C^1(\mathbb{R}^d,\mathbb{R})$ and there exists a constant $K_1\in\mathbb{R}$ such that

$$\big(\nabla U(z_1)-\nabla U(z_2),z_1-z_2\big)_{\mathbb{R}^d}\geq -K_1|z_1-z_2|_{\mathbb{R}^d}^2,\quad z_1,z_2\in\mathbb{R}^d.$$

(U2) There exist $K_2>0$ and $p>0$ such that

$$|\nabla U(z)|_{\mathbb{R}^d}\leq K_2(1+|z|_{\mathbb{R}^d}^p),\quad z\in\mathbb{R}^d.$$

(U3) $\lim_{|z|_{\mathbb{R}^d}\to\infty}U(z)=\infty$.

As examples of U satisfying above conditions, we are interested in a square potential and a double-well potential. Those are, $U(z)=a|z|_{\mathbb{R}^d}^2$ and $U(z)=a(|z|_{\mathbb{R}^d}^4-|z|_{\mathbb{R}^d}^2)$, $a>0$, respectively. We remark that conditions **(U1)** and **(U2)** imply that SPDE (4.1) has a unique (mild) solution living in $C([0,\infty),\mathcal{C})$ for initial datum $w\in\mathcal{C}$. See Theorems 5.1 and 5.2 in Iwata [9] for the proof. We note that condition **(U3)** is sufficient for the existence of a Gibbs measure. Moreover it is known that Gibbs measures are reversible under the solution $X:=\{X_t(x)\}_{t\geq 0}$ of SPDE (4.1). See Proposition 2.7 and Lemma 2.9 in Iwata [8] for details. We denote by $\{P_t\}_{t\geq 0}$ the transition semigroup associated with the diffusion process X.

Now we introduce the relationship between our dynamics and a certain Dirichlet form. We define $H := L^2(\mathbb{R}, \mathbb{R}^d; dx)$ and

$$\begin{aligned}\mathcal{FC}_b^\infty := \Big\{ f(w) = \tilde{f}(\langle w, \phi_1\rangle, \cdots, \langle w, \phi_n\rangle) \;\Big|\; & n \in \mathbb{N},\ \{\phi_k\}_{k=1}^n \subset C_0^\infty(\mathbb{R}, \mathbb{R}^d), \\ & \tilde{f} = \tilde{f}(\alpha_1, \ \cdots, \alpha_n) \in C_b^\infty(\mathbb{R}^n), \\ & \langle w, \phi_k\rangle := \int_{\mathbb{R}} (w(x), \phi_k(x))_{\mathbb{R}^d} dx \ \Big\}.\end{aligned}$$

For $f \in \mathcal{FC}_b^\infty$, we define the Fréchet derivative $Df : E \longrightarrow H$ by

$$Df(w) := \sum_{k=1}^n \frac{\partial \tilde{f}}{\partial \alpha_k}(\langle w, \phi_1\rangle, \cdots, \langle w, \phi_n\rangle)\phi_k. \tag{4.2}$$

We consider a symmetric bilinear form $\mathcal{E}$ which is given by

$$\mathcal{E}(f) = \int_E |Df(w)|_H^2 \mu(dw), \quad f \in \mathcal{FC}_b^\infty.$$

We set $\mathcal{E}_1(f) := \mathcal{E}(f) + \|f\|_{L^2(\mu)}^2$ and denote by $\mathcal{D}(\mathcal{E})$ the completion of $\mathcal{FC}_b^\infty$ with respect to $\mathcal{E}_1^{1/2}$-norm. For $f \in \mathcal{D}(\mathcal{E})$, we denote by Df the closed extension of (4.2).

By virtue of the $C_0^\infty(\mathbb{R}, \mathbb{R}^d)$-quasi-invariance and the strictly positive property of the Gibbs measure μ, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form on $L^2(\mu)$. Hence by putting $\mathcal{A} = \mathcal{FC}_b^\infty$, we see that condition **(A)** holds. Moreover our diffusion process X is associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. See Proposition 2.3 in [10] for the detail. We note that $\Gamma(f) = |Df|_H^2$ in this case.

Then the following gradient estimate of the transition semigroup $\{P_t\}_{t\geq 0}$ holds for any $f \in \mathcal{D}(\mathcal{E})$:

$$|D(P_t f)(w)|_H \leq e^{K_1 t} P_t\big(|Df|_H\big)(w) \quad \text{for } \mu\text{-a.e. } w \in E.$$

See Proposition 2.4 in [10] and Proposition 2.1 in [12] for details. Therefore Theorems 1.2 and 1.3 hold for $\alpha > K_1 \vee 0$. These results play important roles when we study analytic properties for SPDEs containing rotation. See Theorem 4.4 in Kawabi [11] for details.

4.2. Superprocesses with immigration

In this subsection, we give a simple example which comes from superprocesses (or Dawson-Watanabe processes) with immigration. Recently, Stannat [21], [22] studied these measure-valued processes from analytic view points. Following [21] and [22], we consider the one of the most elementary superprocesses. In what follows, we introduce the framework precisely. We assume that the type space S is a finite set $\{1, \cdots, d\}$ and the mutation $A = 0$. Let $E := \mathcal{M}_+(S)$ be the set of finite positive Borel measures on S. Note that we can identify $E \cong \mathbb{R}^d_+ := \{x \in \mathbb{R}^d : x_i \geq 0, 1 \leq i \leq d\}$ with the usual topology. For immigration $\nu \in E$, we use the notation $\nu_i := \nu(\{i\}),\ 1 \leq i \leq d$. The branching mechanism is given by

$$\Psi(i, \lambda) := -a_i \lambda^2 - b_i \lambda, \quad \lambda \geq 0,$$

where $a_i, b_i > 0$ for every $i \in S$.

We consider a $(0, \Psi)$-superprocess $\mathbb{M}$ on E with immigration $\nu \in E$. It is a diffusion process on E whose generator is given by

$$Lf(x) = \sum_{i=1}^d a_i x_i \frac{\partial^2 f}{\partial x_i^2}(x) + \sum_{i=1}^d (\nu_i - b_i x_i) \frac{\partial f}{\partial x_i}(x),$$
$$f \in C_0^2(E),\ x = (x_i)_{i=1}^d \in E.$$

We may think of the diffusion process $\mathbb{M}$ as a continuous time limit of rescaled Galton-Watson processes modelling the random evolution of a given population where each individual $i \in S$, independently of the others, produces a random number of children distributed according to a given offspring distribution and an additional immigration rate ν. The immigration ν induces an additional state-independent drift.

We define a Gamma measure m^Ψ_ν on E by

$$m^\Psi_\nu(dx) := \prod_{i=1}^d \Big(\frac{b_i}{a_i}\Big)^{\nu_i/a_i} \Gamma(\nu_i/a_i)^{-1} x_i^{\nu_i/a_i - 1} e^{-b_i x_i/a_i} dx_i,$$

and consider a symmetric bilinear form

$$\mathcal{E}^\Psi_\nu(f) = \int_E \sum_{i=1}^d a_i x_i \big(\frac{\partial f}{\partial x_i}(x)\big)^2 m^\Psi_\nu(dx), \quad f \in C_0^2(E).$$

Then by Theorem 3.1 in [22], the closure of $(\mathcal{E}_\nu^\Psi, C_0^2(E))$ in $L^2(m_\nu^\Psi)$ is a Dirichlet form and it corresponds to the m_ν^Ψ-symmetric diffusion process $\mathbb{M}$. We denote by $(P_t^{\nu,\Psi})_{t\geq 0}$ its transition semigroup. We note that condition **(A)** holds by putting $\mathcal{A} = C_0^2(E)$ and

$$\Gamma(f)(x) = \sum_{i=1}^d a_i x_i \big(\frac{\partial f}{\partial x_i}(x)\big)^2, \quad x = (x_i)_{i=1}^d \in E.$$

Here we assume

$$\min_{1\leq i\leq d} \frac{\nu_i}{a_i} \geq \frac{1}{2}, \tag{4.3}$$

and set $a_0 := \min_{1\leq i\leq d} a_i$, $a_{d+1} := \max_{1\leq i\leq d} a_i$ and $b_0 := \min_{1\leq i\leq d} b_i$. Then by Theorem 2.9 in [21], we can see condition **(G)**

$$\Gamma(P_t^{\nu,\Psi} f) \leq \big(\frac{a_{d+1}}{a_0}\big) \cdot e^{-b_0 t} P_t^{\nu,\Psi}\big\{\Gamma(f)\big\}, \quad f \in C_b^1(E),$$

holds under (4.3). Therefore Theorems 1.2 and 1.3 hold for all $\alpha > 0$.

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