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Remarks on Solvability of Pseudodifferential Operators in the Space of Hyperfunctions

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Abstract. Let X be an open subset, and let $p(x,\xi)$ be a pseudo-analytic symbol defined in $X \times \mathbb{R}^n$. Let U and V be open subsets of X satisfying $U \Subset V \Subset X$. In this paper we prove that $p(x,D)$: $\mathscr{A}'(\overline{V}) \to \mathscr{B}(U)$ is surjective under some conditions on propagation of analyticity for the transposed operator $({}^tp)(x,D)$ of $p(x,D)$. This result was proved for differential operators by Cordaro and Trépreau [2].

1. Introduction

In the framework of C^∞ and distributions it is well known that solvability of operators is related to propagation of regularities for their transposed operators (see Treves [9], Yoshikawa [13] and Hörmander [3] and [4]). Let X be an open subset of $\mathbb{R}^n$, and let P be a linear partial differential operator on X with analytic coefficients. Cordaro and Trépreau [2] proved that P: $\mathscr{B}(U) \to \mathscr{B}(U)$ is surjective if U is an open subset of X satisfying $U \Subset X$ and P and U satisfy the following condition:

(A) f is analytic in U if $f \in L^2(\mathbb{R}^n)$, f is analytic in a neighborhood of ∂U and ${}^tPf$ is analytic in U.

Here $\mathscr{B}(U)$ denotes the space of hyperfunctions in U, and ${}^tP$ denotes the transposed operator of P. Moreover, $A \Subset B$ means that the closure $\overline{A}$ of A is compact and included in the interior $\overset{\circ}{B}$ of B, and ∂U denotes the boundary of U. We should note that Cordaro and Trépreau studied the problems in a more general setting in [2], although they dealt with only differential operators. In this paper we shall extend the above result for pseudodifferential operators.

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First we shall explain briefly about analytic functionals, hyperfunctions and pseudodifferential operators acting on them. For the details we refer to [10] (see, also, [11]). Let $\varepsilon \in \mathbb{R}$, and denote $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, where $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$ and $|\xi| = (\sum_{j=1}^n |\xi_j|^2)^{1/2}$. We define

$$\hat{\mathscr{S}}_\varepsilon := \{v(\xi) \in C^\infty(\mathbb{R}^n);\ e^{\varepsilon\langle\xi\rangle} v(\xi) \in \mathscr{S}\},$$

where $\mathscr{S}$ ($\equiv \mathscr{S}(\mathbb{R}^n)$) denotes the Schwartz space. We introduce the topology to $\hat{\mathscr{S}}_\varepsilon$ in a natural way. Then the dual space $\hat{\mathscr{S}}_\varepsilon'$ of $\hat{\mathscr{S}}_\varepsilon$ can be identified with $\{v(\xi) \in \mathscr{D}';\ e^{-\varepsilon\langle\xi\rangle} v(\xi) \in \mathscr{S}'\}$. Let $\varepsilon \geq 0$. Then $\hat{\mathscr{S}}_\varepsilon$ is a dense subset of $\mathscr{S}$ and we can define $\mathscr{S}_\varepsilon := \mathscr{F}^{-1}[\hat{\mathscr{S}}_\varepsilon]$ ($= \mathscr{F}[\hat{\mathscr{S}}_\varepsilon]$) ($\subset \mathscr{S}$), where $\mathscr{F}$ and $\mathscr{F}^{-1}$ denote the Fourier transformation and the inverse Fourier transformation on $\mathscr{S}$ (or $\mathscr{S}'$), respectively. For example, $\mathscr{F}[u](\xi) = \int e^{-ix\cdot\xi} u(x)\, dx$ for $u \in \mathscr{S}$, where $x \cdot \xi = \sum_{j=1}^n x_j\xi_j$ for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$. We introduce the topology in $\mathscr{S}_\varepsilon$ so that $\mathscr{F} : \hat{\mathscr{S}}_\varepsilon \to \mathscr{S}_\varepsilon$ is homeomorphic. Denote by $\mathscr{S}_\varepsilon'$ the dual space of $\mathscr{S}_\varepsilon$. Since $\mathscr{S}_\varepsilon$ is dense in $\mathscr{S}$, we can regard $\mathscr{S}'$ as a subspace of $\mathscr{S}_\varepsilon'$. We can define the transposed operators ${}^t\mathscr{F}$ and ${}^t\mathscr{F}^{-1}$ of $\mathscr{F}$ and $\mathscr{F}^{-1}$, which map $\mathscr{S}_\varepsilon'$ and $\hat{\mathscr{S}}_\varepsilon'$ onto $\hat{\mathscr{S}}_\varepsilon'$ and $\mathscr{S}_\varepsilon'$, respectively. Since $\hat{\mathscr{S}}_{-\varepsilon} \subset \hat{\mathscr{S}}_\varepsilon'$ ($\subset \mathscr{D}'$), we can define $\mathscr{S}_{-\varepsilon} = {}^t\mathscr{F}^{-1}[\hat{\mathscr{S}}_{-\varepsilon}]$, and introduce the topology in $\mathscr{S}_{-\varepsilon}$ so that ${}^t\mathscr{F}^{-1} : \hat{\mathscr{S}}_{-\varepsilon} \to \mathscr{S}_{-\varepsilon}$ is homeomorphic. $\mathscr{S}_{-\varepsilon}'$ denotes the dual space of $\mathscr{S}_{-\varepsilon}$. We note that $\mathscr{F} = {}^t\mathscr{F}$ on $\mathscr{S}'$. So we also represent ${}^t\mathscr{F}$ by $\mathscr{F}$. Let $\mathscr{A}(\mathbb{C}^n)$ be the space of entire analytic functions on $\mathbb{C}^n$, and let K be a compact subset of $\mathbb{C}^n$. We denote by $\mathscr{A}'(K)$ the space of analytic functionals carried by K, *i.e.*, $u \in \mathscr{A}'(K)$ if and only if (i) $u : \mathscr{A}(\mathbb{C}^n) \ni \varphi \mapsto u(\varphi) \in \mathbb{C}$ is a linear functional, and (ii) for any neighborhood ω of K in $\mathbb{C}^n$ there is $C_\omega \geq 0$ such that $|u(\varphi)| \leq C_\omega \sup_{z\in\omega} |\varphi(z)|$ for $\varphi \in \mathscr{A}(\mathbb{C}^n)$. Define $\mathscr{A}'(\mathbb{R}^n) := \bigcup_{K \Subset \mathbb{R}^n} \mathscr{A}'(K)$, $\mathscr{S}_\infty := \bigcap_{\varepsilon\in\mathbb{R}} \mathscr{S}_\varepsilon$, $\mathscr{E}_0 := \bigcap_{\varepsilon>0} \mathscr{S}_{-\varepsilon}$ and $\mathscr{F}_0 := \bigcap_{\varepsilon>0} \mathscr{S}_\varepsilon'$. For $u \in \mathscr{A}'(\mathbb{R}^n)$ we can define the Fourier transform $\hat{u}(\xi)$ of u by

$$\hat{u}(\xi)\, (= \mathscr{F}[u](\xi)) = u_z(e^{-iz\cdot\xi}),$$

where $z\cdot\xi = \sum_{j=1}^n z_j\xi_j$ for $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$ and $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$. By definition we have $\hat{u}(\xi) \in \bigcap_{\varepsilon>0} \hat{\mathscr{S}}_{-\varepsilon}$ ($= \mathscr{F}[\mathscr{E}_0]$). Therefore, we can regard $\mathscr{A}'(\mathbb{R}^n)$ as a subspace of $\mathscr{E}_0$, *i.e.*, $\mathscr{A}'(\mathbb{R}^n) \subset \mathscr{E}_0 \subset \mathscr{F}_0$ (see Lemma 1.1.2 of [10]). Let Ω be an open subset of $\mathbb{C}^n$, and let $\mathscr{A}(\Omega)$ be the space of analytic functions in Ω. $\mathscr{A}(\Omega)$ is a Fréchet-Schwartz space ((FS) space) whose topology is defined by the family of the semi-norms $|\cdot|_L$ ($L \Subset \Omega$), where

$$|\varphi|_L := \sup_{z\in L} |\varphi(z)|.$$

Let K be a compact subset of $\mathbb{R}^n$. Then K has a fundamental system of complex neighborhoods consisting of Runge domain (see, *e.g.*, [5] and Lemma 1.1.1 of [10]). So $u(\varphi)$ can be defined for $u \in \mathscr{A}'(K)$ if Ω is an open neighborhood of K in $\mathbb{C}^n$ and $\varphi \in \mathscr{A}(\Omega)$. We denote

$$K_\varepsilon := \{z \in \mathbb{C}^n;\ |z - x| < \varepsilon \text{ for some } x \in K\}$$

for $\varepsilon \geq 0$, and define the (DFS) space $\mathscr{A}(K)$ by $\mathscr{A}(K) = \operatorname{inj\,lim}_{\varepsilon\to 0} \mathscr{A}(K_\varepsilon)$ (see, *e.g.*, [6]). Then $\mathscr{A}'(K)$ can be identified with the strong dual space of $\mathscr{A}(K)$ and $\mathscr{A}'(K)$ is a (FS) space. For $\delta > 0$ we have ($\mathscr{S}_\delta \subset$) $\mathscr{S}'_{-\delta} \subset \mathscr{A}(\mathbb{R}^n_\delta)$, where $\mathbb{R}^n_\delta = \{z \in \mathbb{C}^n;\ |\mathrm{Im}\ z| < \delta\}$ (see Lemma 1.1.3 of [10]). Moreover, we have $u(\varphi) = \langle u, \varphi\rangle_{\mathscr{A}'(K),\mathscr{A}(K)} = \langle u, \varphi\rangle$ for $u \in \mathscr{A}'(K)$ and $\varphi \in \mathscr{S}_\delta$, where $\delta > 0$ and $\langle\cdot,\cdot\rangle_{\mathscr{A}'(K),\mathscr{A}(K)}$ and $\langle\cdot,\cdot\rangle$ denote the duality of $\mathscr{A}'(K)$ and $\mathscr{A}(K)$ and that of $\mathscr{S}'_\delta$ and $\mathscr{S}_\delta$, respectively (see Lemma 1.1.2 of [10]). For a bounded open subset X of $\mathbb{R}^n$ we define the space $\mathscr{B}(X)$ of hyperfunctions in X by

$$\mathscr{B}(X) := \mathscr{A}'(\overline{X})/\mathscr{A}'(\partial X).$$

For $u \in \mathscr{F}_0$ we define

$$\begin{aligned}&\mathscr{H}(u)(x, x_{n+1}) := (\mathrm{sgn}\ x_{n+1}) \exp[-|x_{n+1}|\langle D\rangle]u(x)/2\\ &(= (\mathrm{sgn}\ x_{n+1})\mathscr{F}_\xi^{-1}[\exp[-|x_{n+1}|\langle\xi\rangle]\hat{u}(\xi)](x)/2 \in \mathscr{S}'(\mathbb{R}^n))\end{aligned}$$

when $x_{n+1} \in \mathbb{R} \setminus \{0\}$, and

$$\begin{aligned}\mathrm{supp}\ u := \bigcap\{F;\ &F \text{ is a closed subset of } \mathbb{R}^n \text{ and there is a real}\\ &\text{analytic function } U(x, x_{n+1}) \text{ in } \mathbb{R}^{n+1} \setminus F \times \{0\}\\ &\text{such that } U(x, x_{n+1}) = \mathscr{H}(u)(x, x_{n+1}) \text{ for } x_{n+1} \neq 0\}\end{aligned}$$

(see [10]). For a compact subset K of $\mathbb{R}^n$, $u \in \mathscr{A}'(K)$ if and only if u is an analytic functional and $\mathrm{supp}\ u \subset K$ (see Proposition 1.2.6 of [10]). From Theorem 1.3.3 of [10] it follows that for any $u \in \mathscr{F}_0$ and any compact subset K of $\mathbb{R}^n$ there is $v \in \mathscr{A}'(K)$ satisfying $\mathrm{supp}\ (u - v) \cap K \subset \partial K$. Therefore, we can define the restriction map from $\mathscr{F}_0$ to $\mathscr{A}'(K)/\mathscr{A}'(\partial K)$ ($= \mathscr{B}(\overset{\circ}{K})$). For an open subset X of $\mathbb{R}^n$ we define the space $\mathcal{B}(X)$ of hyperfunctions in X as a local space of $\mathcal{A}'(\mathbb{R}^n)$ (or $\mathcal{F}_0$) (see Definition 1.4.5 of [10]). Let X and U be open subsets of $\mathbb{R}^n$ satisfying $U \subset X$. Then the restriction map ρ_U^X : $\mathcal{B}(X) \ni u \mapsto u|_U \in \mathcal{B}(U)$ can be defined. By definition we can also define the restriction map from $\mathcal{F}_0$ to $\mathcal{B}(X)$, and we denote by $v|_X$ the restriction of $v \in \mathcal{F}_0$ to $\mathcal{B}(X)$ (or on X). For $x^0 \in \mathbb{R}^n$ we say that u is analytic at x^0 if $\mathcal{H}(u)(x, x_{n+1})$ can be continued analytically from $\mathbb{R}^n \times (0, \infty)$ to a neighborhood of $(x^0, 0)$ in $\mathbb{R}^{n+1}$.

Assume that $a(\xi, y, \eta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ satisfies the estimates

$$\begin{aligned}&|\partial_\xi^\alpha D_y^{\beta+\tilde{\beta}} \partial_\eta^\gamma a(\xi, y, \eta)| \\ &\le C_{|\alpha|+|\tilde{\beta}|+|\gamma|} (A/R)^{|\beta|} \langle\xi\rangle^{m_1+|\beta|} \langle\eta\rangle^{m_2} \exp[\delta_1\langle\xi\rangle + \delta_2\langle\eta\rangle]\end{aligned}$$

for any $\alpha, \beta, \tilde{\beta}, \gamma \in (\mathbb{Z}_+)^n$, $\xi, y, \eta \in \mathbb{R}^n$ with $\langle\xi\rangle \ge R|\beta|$, where $D_y = -i\partial_y$, C_k $(k \ge 0)$ are positive constants, $R \ge 1$, $A \ge 0$, $m_1, m_2, \delta_1, \delta_2 \in \mathbb{R}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We define pseudodifferential operators $a(D_x, y, D_y)$ and ${}^r a(D_x, y, D_y)$ by

$$a(D_x, y, D_y)u(x) = (2\pi)^{-n} \mathcal{F}_\xi^{-1} \Big[\int \Big(\int e^{-iy\cdot(\xi-\eta)} a(\xi, y, \eta) \hat{u}(\eta)\, d\eta \Big) dy \Big](x)$$

and ${}^r a(D_x, y, D_y)u = b(D_x, y, D_y)u$ for $u \in \mathcal{S}_\infty$, respectively, where $b(\xi, y, \eta) = a(\eta, y, \xi)$.

PROPOSITION 1.1 (Theorem 2.3.3 of [10] or Proposition 1.2 of [11]). *$a(D_x, y, D_y)$ can be extended to a continuous linear operator from $\mathcal{S}_{\varepsilon_2}$ to $\mathcal{S}_{\varepsilon_1}$ and from $\mathcal{S}'_{-\varepsilon_2}$ to $\mathcal{S}'_{-\varepsilon_1}$, respectively, if*

$$\begin{cases} \kappa > 1, \quad \varepsilon_2 - \delta_2 = \kappa(\varepsilon_1 + \delta_1)_+, \\ \varepsilon_1 + \delta_1 \le 1/R, \quad R \ge e\sqrt{n}\kappa A/(\kappa - 1), \end{cases} \tag{1.1}$$

where $c_+ = \max\{c, 0\}$. Similarly, ${}^r a(D_x, y, D_y)$ can be extended to a continuous linear operator from $\mathcal{S}_{-\varepsilon_1}$ to $\mathcal{S}_{-\varepsilon_2}$ and from $\mathcal{S}'_{\varepsilon_1}$ to $\mathcal{S}'_{\varepsilon_2}$, respectively, if (1.1) is valid.

DEFINITION 1.2. Let X be an open subset of $\mathbb{R}^n$, and let $R_0 \ge 0$.

(i) Let $R_0 \ge 1$, $m, \delta \in \mathbb{R}$ and $A, B \ge 0$, and let $a(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. We say that $a(x, \xi) \in S^{m,\delta}(R_0, A, B)$ if $a(x, \xi)$ satisfies

$$|a_{(\beta+\tilde{\beta})}^{(\alpha+\tilde{\alpha})}(x, \xi)| \le C_{|\tilde{\alpha}|+|\tilde{\beta}|} (A/R_0)^{|\alpha|} (B/R_0)^{|\beta|} \langle\xi\rangle^{m+|\beta|-|\tilde{\alpha}|} e^{\delta\langle\xi\rangle}$$

for any $\alpha, \tilde{\alpha}, \beta, \tilde{\beta} \in (\mathbb{Z}_+)^n$ and $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ with $\langle \xi \rangle \geq R_0(|\alpha| + |\beta|)$, where $a^{(\alpha)}_{(\beta)}(x, \xi) = \partial^\alpha_\xi D^\beta_x a(x, \xi)$ and the C_k are independent of α and β. We also write $S^m(R_0, A, B) = S^{m,0}(R_0, A, B)$ and $S^m(R_0, A) = S^m(R_0, A, A)$ and so on. We define $S^+(R_0, A, B) = \bigcap_{\delta>0} S^{0,\delta}(R_0, A, B)$.

(ii) Let $R_0 \geq 1$, $m_j, \delta_j \in \mathbb{R}$ ($j = 1, 2$), $A_j \geq 0$ ($j = 1, 2$) and $B \geq 0$, and let $a(\xi, y, \eta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$. We say that $a(\xi, y, \eta) \in S^{m_1, m_2, \delta_1, \delta_2}(R_0, A_1, B, A_2)$ if $a(\xi, y, \eta)$ satisfies

$$\begin{aligned} |\partial^{\alpha+\tilde{\alpha}}_\xi D^{\beta^1+\beta^2+\tilde{\beta}}_y \partial^{\gamma+\tilde{\gamma}}_\eta a(\xi, y, \eta)| &\leq C_{|\tilde{\alpha}|+|\tilde{\beta}|+|\tilde{\gamma}|} (A_1/R_0)^{|\alpha|} (B/R_0)^{|\beta^1|+|\beta^2|} \\ &\quad \times (A_2/R_0)^{|\gamma|} \langle \xi \rangle^{m_1+|\beta^1|-|\tilde{\alpha}|} \langle \eta \rangle^{m_2+|\beta^2|-|\tilde{\gamma}|} \exp[\delta_1 \langle \xi \rangle + \delta_2 \langle \eta \rangle] \end{aligned}$$

for any $\alpha, \tilde{\alpha}, \beta^1, \beta^2, \tilde{\beta}, \gamma, \tilde{\gamma} \in (\mathbb{Z}_+)^n$, $(\xi, y, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ with $\langle \xi \rangle \geq R_0(|\alpha| + |\beta^1|)$ and $\langle \eta \rangle \geq R_0(|\gamma| + |\beta^2|)$. We also write $S^{m_1, m_2, \delta_1, \delta_2}(R_0, A) = S^{m_1, m_2, \delta_1, \delta_2}(R_0, A, A, A)$. Similarly, we define $S^+(R_0, A_1, B, A_2) = \bigcap_{\delta>0} S^{0,0,\delta,\delta}(R_0, A_1, B, A_2)$.

(iii) Let $m, \delta \in \mathbb{R}$ and $A, B \geq 0$, and let $a(x, \xi) \in C^\infty(X \times \mathbb{R}^n)$. We say that $a(x, \xi) \in PS^{m,\delta}(X; R_0, A, B)$ if $a(x, \xi)$ satisfies

$$|a^{(\alpha+\tilde{\alpha})}_{(\beta)}(x, \xi)| \leq C_{|\tilde{\alpha}|} A^{|\alpha|} B^{|\beta|} |\alpha|! |\beta|! \langle \xi \rangle^{m-|\alpha|-|\tilde{\alpha}|} e^{\delta \langle \xi \rangle}$$

for any $\alpha, \tilde{\alpha}, \beta \in (\mathbb{Z}_+)^n$, $(x, \xi) \in X \times \mathbb{R}^n$ with $|\xi| \geq 1$ and $\langle \xi \rangle \geq R_0|\alpha|$. We also write $PS^+(X; R_0, A, B) = \bigcap_{\delta>0} PS^{0,\delta}(X; R_0, A, B)$ and $PS^+(X; R_0, A) = PS^+(X; R_0, A, A)$.

(iv) Let $m, \delta \in \mathbb{R}$ and $A, C_0 \geq 0$, and let $\{a_j(x, \xi)\}_{j \in \mathbb{Z}_+} \in \prod_{j \in \mathbb{Z}_+} C^\infty(X \times \mathbb{R}^n)$. We say that $a(x, \xi) \equiv \{a_j(x, \xi)\}_{j \in \mathbb{Z}_+} \in FPS^{m,\delta}(X; R_0, C_0, A)$ if $a(x, \xi)$ satisfies

$$|a^{(\alpha+\tilde{\alpha})}_{j(\beta)}(x, \xi)| \leq C_{|\tilde{\alpha}|} C_0^j A^{|\alpha|+|\beta|} j! |\alpha|! |\beta|! \langle \xi \rangle^{m-j-|\alpha|-|\tilde{\alpha}|} e^{\delta \langle \xi \rangle}$$

for any $j \in \mathbb{Z}_+$, $\alpha, \tilde{\alpha}, \beta \in (\mathbb{Z}_+)^n$, $(x, \xi) \in X \times \mathbb{R}^n$ with $|\xi| \geq 1$ and $\langle \xi \rangle \geq R_0(j + |\alpha|)$. We also write $a(x, \xi) = \sum_{j=0}^\infty a_j(x, \xi)$ formally. Moreover, we write $FPS^+(X; R_0, C_0, A) = \bigcap_{\delta>0} FPS^{0,\delta}(X; R_0, C_0, A)$.

(v) For $a(x, \xi) \equiv \sum_{j=0}^\infty a_j(x, \xi) \in FPS^+(X; R_0, C_0, A)$ we define the symbol $({}^t a)(x, \xi)$ by

$$({}^t a)(x, \xi) = \sum_{j=0}^\infty b_j(x, \xi), \quad b_j(x, \xi) = \sum_{k+|\alpha|=j} (-1)^{|\alpha|} a^{(\alpha)}_{k(\alpha)}(x, -\xi)/\alpha!.$$

Remark. (i) If $R_0 \leq R_1$, then $PS^{m,\delta}(X; R_0, A, B) \subset PS^{m,\delta}(X; R_1, A, B)$.

(ii) $a(x,\xi) \in PS^+(X; R_0, A)$ can be identified with the element $\{a_j(x,\xi)\}_{j\in\mathbb{Z}_+}$ in $FPS^+(X; R_0, C_0, A)$, where $a_0(x,\xi) = a(x,\xi)$ and $a_j(x,\xi) = 0$ ($j \geq 1$) and $C_0 > 0$.

(iii) It is easy to see that $({}^t a)(x,\xi) \in FPS^+(X; R_0, C_0', 2A)$ if $a(x,\xi) \in FPS^+(X; R_0, C_0, A)$, where $C_0' = \max\{C_0, 4nA^2\}$.

Let X be an open subset of $\mathbb{R}^n$, and assume that $a(x,\xi) \in PS^+(X; R_0, A)$, where $A \geq 0$ and $R_0 \geq 1$. Let U and V be open subsets of X satisfying $U \Subset V \Subset X$. It follows from Proposition 2.2.3 of [10] that there are symbols $\Phi^R(x,\xi) \in S^0(R, C_*, C(U,V))$ ($R \geq 4$) satisfying $0 \leq \Phi^R(x,\xi) \leq 1$, supp $\Phi^R \subset V \times \mathbb{R}^n$ and $\Phi^R(x,\xi) = 1$ in $U \times \mathbb{R}^n$. Put $a^R(x,\xi) = \Phi^R(x,\xi)a(x,\xi)$. Then we have $a^R(x,\xi) \in S^+(R, A+C_*, 2A+C(U,V))$ if $R \geq \max\{4, R_0\}$. Applying Proposition 1.1 with $a(\xi, y, \eta) = a^R(y,\xi)$ and noting that $a^R(x,D) = {}^r a(D_x, y, D_y)$, we can see that $a^R(x,D)u$ is well-defined and belongs to $\mathscr{F}_0$ if $u \in \mathscr{F}_0$ and $R \geq \max\{4, R_0, 2e\sqrt{n} \times (2A + C(U,V))\}$. Moreover, $a^R(x,D)u$ determines an element $(a^R(x,D)u)|_U \in \mathscr{B}(U)$. It follows from Theorem 2.6.1 (or Collorary 2.6.2) of [10] that $(a^R(x,D)u)|_U$ does not depend on the choice of $\Phi^R(x,\xi)$ if $u \in \mathscr{F}_0$, $\Phi^R(x,\xi) \in S^0(R,B)$ and $R \geq \max\{4, R_0, 8e\sqrt{n}(2A+B)\}$. Therefore, we can define the operator $a(x,D)$: $\mathscr{F}_0 \to \mathscr{B}(U)$ by $a(x,D)u = (a^R(x,D)u)|_U$ for $R \gg 1$, and the operator $a(x,D)$: $\mathscr{F}_0 \to \mathscr{B}(X)$. Let $u \in \mathscr{B}(U)$. Then there is $v \in \mathscr{A}'(\overline{U})$ such that $v|_U = u$ in $\mathscr{B}(U)$. By Theorem 2.6.5 of [10] we have $a^R(x,D)w \in \mathscr{A}(U)$ if $w \in \mathscr{F}_0$, $R \geq \max\{4, R_0, 16e\sqrt{n}(2A + C(U,V))\}$ and supp $w \cap U = \emptyset$, where $\mathscr{A}(U)$ denotes the space of (real) analytic functions in U. This implies that $(a^R(x,D)v)|_U$ ($\in \mathscr{B}(U)/\mathscr{A}(U)$) is uniquely determined, as an element of $\mathscr{B}(U)/\mathscr{A}(U)$, by u and does not depend on the choice of v. Therefore, we can also define the operator $a(x,D)$: $\mathscr{B}(U) \to \mathscr{B}(U)/\mathscr{A}(U)$ and the operator $a(x,D)$: $\mathscr{B}(X) \to \mathscr{B}(X)/\mathscr{A}(X)$ (see §2.7 of [10]). We note that the above definitions of the operator $a(x,D)$ coincides with usual ones if $a(x,D)$ is a differential operator with analytic coefficients in X (see Theorem 2.7.1 of [10]).

Next we assume that $a(x,\xi) \equiv \sum_{j=0}^{\infty} a_j(x,\xi) \in FPS^+(X;R_0,C_0,A)$. Choose $\{\phi_j^R(\xi)\}_{j\in\mathbb{Z}_+} \subset C^\infty(\mathbb{R}^n)$ so that $0 \le \phi_j^R(\xi) \le 1$,

$$\phi_j^R(\xi) = \begin{cases} 0 & \text{if } \langle\xi\rangle \le 2Rj, \\ 1 & \text{if } \langle\xi\rangle \ge 3Rj, \end{cases}$$
$$|\partial_\xi^{\alpha+\beta}\phi_j^R(\xi)| \le \widehat{C}_{|\beta|}(\widehat{C}/R)^{|\alpha|}\langle\xi\rangle^{-|\beta|} \quad \text{if } |\alpha| \le 2j,$$

where the $\widehat{C}_k$ and $\widehat{C}$ do not depend on j and R (see §2.2 of [10]). Then we have

$$\tilde{a}(x,\xi) := \sum_{j=0}^{\infty} \phi_j^{R/2}(\xi)a_j(x,\xi) \in PS^+(X;R,A+6\widehat{C},A)$$

if $R \ge 2R_0$ and $R \ge C_0$ (see Lemma 2.2.4 of [10]). So we can define $a(x,D)$: $\mathscr{F}_0 \to \mathscr{B}(X)/\mathscr{A}(X)$ and $\mathscr{B}(X) \to \mathscr{B}(X)/\mathscr{A}(X)$ by $a(x,D) = \tilde{a}(x,D)$. Indeed, applying the same argument as in §3.7 of [10] we can see that $a(x,D)u \in \mathscr{B}(X)/\mathscr{A}(X)$ does not depend on the choice of $\{\phi_j^R(\xi)\}$, where $u \in \mathscr{F}_0$ or $u \in \mathscr{B}(X)$.

Let $p(x,\xi) \in PS^+(X;R_0,A)$, where $A \ge 0$ and $R_0 \ge 1$. Moreover, let U, V and W be open subsets of X satisfying $U \Subset V \Subset W \Subset X$, and assume that

(A)′ f is analytic in U if $f \in L^2(\mathbb{R}^n)$, f is analytic in a neighborhood of $\overline{W} \setminus U$ and $(({}^tp)(x,D)f)|_V = 0$ in $\mathscr{B}(V)/\mathscr{A}(V)$,

instead of the condition (A). We note that (A)′ is satisfied if (A) is satisfied. Now we can state our main result.

THEOREM 1.3. *If* (A)′ *is satisfied, then the operator* $p(x,D) : \mathscr{A}'(\overline{V}) \to \mathscr{B}(U)$ *is surjective, i.e., for any* $f \in \mathscr{B}(U)$ *there is* $u \in \mathscr{A}'(\overline{V})$ *satisfying* $p(x,D)u = f$ *in* $\mathscr{B}(U)$.

In [12] we proved similar results in the space of microfunctions (see, also, [11]). In the framework of the Gevrey classes and the spaces of ultradistributions Albanese, Corli and Rodino [1] obtained similar results.

We shall give the proof of Theorem 1.3 in §2. In §3 we shall apply Theorem 1.3 to microhyperbolic operators.

2. Proof of Theorem 1.3

Assume that $p(x,\xi) \in PS^+(X;R_0,A)$ satisfies the condition (A)$'$. Choose $\varepsilon_0>0$, $\Phi^R(x,\xi)\in S^0(R,C_*,C(V',W))$ ($R\ge 4$) and $\Psi^R(\xi,y,\eta)\in S^{0,0,0,0}(R,C_*,C(V',W),C_*)$ ($R\ge 4$) so that $V'\equiv\{x\in\mathbb{R}^n;\ |x-y|<\varepsilon_0$ for some $y\in V\}\Subset W$ ($\Subset X$), $0\le\Phi^R\le 1$, $0\le\Psi^R\le 1$, supp $\Phi^R\subset W\times\mathbb{R}^n$, supp $\Psi^R\subset\mathbb{R}^n\times W\times\mathbb{R}^n$, $\Phi^R(x,\xi)=1$ in $V'\times\mathbb{R}^n$ and $\Psi^R(\xi,y,\eta)=1$ in $\mathbb{R}^n\times V'\times\mathbb{R}^n$. We put

$$\begin{aligned}
&p^R(x,\xi):=\Phi^R(x,\xi)p(x,\xi)\in S^+(R,A+C_*,2A+C(V',W)),\\
&\tilde p^R(\xi,y,\eta):=\Psi^R(\xi,y,\eta)p(y,\eta)\in S^+(R,C_*,2A+C(V',W),A+C_*)
\end{aligned}$$

for $R\ge\max\{4,R_0\}$. Then, for $\delta>0$ $p^R(x,D)$ and $\tilde p^R(D_x,y,D_y)$ map continuously $\mathscr{S}_\delta$ to $\mathscr{S}$ and, therefore, the transposed operators ${}^tp^R(x,D)$ and ${}^t\tilde p^R(D_x,y,D_y)$ map continuously $\mathscr{S}'$ to $\mathscr{S}'_\delta$. It is obvious that ${}^tp^R(x,D)=q(D_x,y,D_y)$ and ${}^t\tilde p^R(D_x,y,D_y)=\tilde q(D_x,y,D_y)$, where $q(\xi,y,\eta)=p^R(y,-\xi)$ and $\tilde q(\xi,y,\eta)=\tilde p^R(-\eta,y,-\xi)$ (see the proof of Lemma 2.1 below).

Lemma 2.1. *Let $a(\xi,y,\eta)$ be a symbol satisfying*

$$|\partial_\xi^{\alpha+\tilde\alpha}D_y^\beta\partial_\eta^\gamma a(\xi,y,\eta)|\le C_{|\tilde\alpha|+|\beta|+|\gamma|,\delta}(B/R)^{|\alpha|}\langle\eta\rangle^{m-|\gamma|}e^{\delta\langle\xi\rangle}$$

if $\langle\xi\rangle\ge R|\alpha|$ and $\delta>0$, and $a(\xi,y,\eta)=0$ if $y\in V'$, where $R>0$, $B\ge 0$ and $m\in\mathbb{R}$. Then $a(D_x,y,D_y)u$ ($\in\mathscr{F}_0$) is analytic in V for $u\in\mathscr{S}'$ if $R\ge 16enB/\varepsilon_0$.

Proof. Since for $\delta>0$

$$|\partial_\xi^\alpha D_y^\beta\partial_\eta^\gamma\{a(\xi,y,\eta)e^{-\delta\langle\xi\rangle}\}|\le C_{|\alpha|+|\beta|+|\gamma|,\delta}\langle\eta\rangle^{m-|\gamma|}e^{-\delta\langle\xi\rangle/2}$$

and $e^{-\delta\langle D\rangle}a(D_x,y,D_y)$: $\mathscr{S}'\to\mathscr{S}'$, $a(D_x,y,D_y)$ maps continuously $\mathscr{S}'$ to $\mathscr{F}_0$. Here we introduce the topology of $\mathscr{F}_0$ by $\mathscr{F}_0=\text{inj}\lim_{\varepsilon\downarrow 0}\mathscr{S}'_\varepsilon$. We shall prove the lemma, applying the same argument as in the proof of Lemma 2.3 of [11]. Let $u\in\mathscr{S}'$, $\mu=0,1$ and $0<\rho\le 1$. We put $\psi_j^R(\xi):=\phi_{j-1}^R(\xi)-\phi_j^R(\xi)$ ($j\in\mathbb{N}$), where the $\phi_j^R(\xi)$ are symbols as in §1. Then we have

$$\begin{aligned}
(2.1)\qquad &\langle D\rangle^\mu e^{-\rho\langle D\rangle}a(D_x,y,D_y)u\\
&=\sum_{j=1}^\infty\langle D\rangle^\mu e^{-\rho\langle D\rangle}\psi_j^{R'}(D)a(D_x,y,D_y)u\quad\text{in }\mathscr{S}',
\end{aligned}$$

where $R'>0$. A standard argument yields

$$\langle D\rangle^{\mu}e^{-\rho\langle D\rangle}\psi_j^{R'}(D)a(D_x,y,D_y)u(x)=\langle\hat u(\eta),f_{\mu,j}^{R'}(x,\eta;\rho)\rangle_\eta, \tag{2.2}$$

where $M,N\in\mathbb{Z}_+$, $2M>n$ and

$$\begin{aligned}f_{\mu,j}^{R'}(x,\eta;\rho)&=(2\pi)^{-2n}\int e^{i(x-y)\cdot\xi+iy\cdot\eta}\langle\xi-\eta\rangle^{-2N}\\&\quad\times\langle D_y\rangle^{2N}\{\langle x-y\rangle^{-2M}\langle D_\xi\rangle^{2M}(\langle\xi\rangle^{\mu}e^{-\rho\langle\xi\rangle}\psi_j^{R'}(\xi)a(\xi,y,\eta))\}\,d\xi dy.\end{aligned}$$

Indeed, for $\varphi\in\mathscr{S}(\mathbb{R}^n)$ we have

$$\begin{aligned}&\langle\langle D\rangle^{\mu}e^{-\rho\langle D\rangle}\psi_j^{R'}(D)a(D_x,y,D_y)u,\varphi\rangle\\&=\langle\hat u(\eta),\int f_{\mu,j}^{R'}(x,\eta;\rho)\varphi(x)\,dx\rangle_\eta,\end{aligned}$$

$$\sup_{|\alpha|+k\le\ell}|\langle\eta\rangle^k\partial_\eta^\alpha D_x^\beta f_{\mu,j}^{R'}(x,\eta;\rho)|\le C_{\ell,|\beta|,\rho,j,R'}\langle x\rangle^{\ell}. \tag{2.3}$$

This proves (2.2). Define L by

$${}^tL=|x-y|^{-2}\sum_{k=1}^n(\bar x_k-y_k)D_{\xi_k}$$

for $x\in\mathbb{C}^n$ with $\mathrm{Re}\ x\in V$ and $y\in\mathbb{R}^n\setminus V'$. A simple calculation gives

$$\begin{aligned}&|\partial_\eta^\alpha\langle D_y\rangle^{2N}L^{j+M}\{\langle\xi\rangle^{\mu}e^{-\rho\langle\xi\rangle}\psi_j^{R'}(\xi)a(\xi,y,\eta)\}|\\&\le C_{|\alpha|,N,M,\varepsilon_0,\delta,R'}|x-y|^{-M}\langle\eta\rangle^{m-|\alpha|}\langle\xi\rangle^{\mu-M}e^{\delta\langle\xi\rangle}\\&\quad\times\{8n(B/R+(\widehat C+6(1+\sqrt2))/R')/\varepsilon_0\}^j\chi_j^{R'}(\xi)\end{aligned} \tag{2.4}$$

if $\alpha\in(\mathbb{Z}_+)^n$, $M,N,j\in\mathbb{Z}_+$, $R'\ge R$, $x\in\mathbb{C}^n$, $\mathrm{Re}\ x\in V$ and $\delta>0$, where $\chi_j^{R'}(\xi)$ is the defining function of the set $\{\xi\in\mathbb{R}^n;\,2R'(j-1)\le\langle\xi\rangle\le 3R'j\}$. Here we have used Lemmas 2.1.1 and 2.1.7 of [10]. Therefore, we have

$$\sup_{k+|\alpha|\le\ell}|\langle\eta\rangle^k\partial_\eta^\alpha f_{\mu,j}^{R'}(x,\eta;\rho)|\le C_{\ell,\varepsilon_0,\rho_1,R'}j^{-2} \tag{2.5}$$

if $\ell\in\mathbb{Z}_+$, $x\in\mathbb{C}^n$, $\mathrm{Re}\ x\in V$, $|\mathrm{Im}\ x|\le\rho_1$ ($\le 1/2$) and

$$\begin{cases}R'\ge R,\quad R'\ge 16en(\widehat C+6(1+\sqrt2))/\varepsilon_0,\\ R\ge 16enB/\varepsilon_0,\quad \rho_1<1/(3R'),\end{cases} \tag{2.6}$$

taking $M > \ell + n$ and $N \geq \ell + m$ in (2.4). Since Re $(1 + (x-y)\cdot(x-y)) = 1 + |\text{Re}\ x - y|^2 - |\text{Im}\ x|^2$ for $x \in \mathbb{C}^n$ and $y \in \mathbb{R}^n$, $f^{R'}_{\mu,j}(x, \eta; \rho)$ is analytic in x if $|\text{Im}\ x| < 1$. We note that (2.3) is valid for $x \in \mathbb{C}^n$ with $|\text{Im}\ x| \leq 1/2$, where D_x means complex differentiation. So it follows from (2.2) and (2.5) that $\langle D\rangle^\mu e^{-\rho\langle D\rangle}\psi^{R'}_j(D)a(D_x, y, D_y)u(x)$ is analytic in x and

$$|\langle D\rangle^\mu e^{-\rho\langle D\rangle}\psi^{R'}_j(D)a(D_x, y, D_y)u(x)| \leq C_{\varepsilon_0,\rho_1,R'}(V, u)j^{-2} \tag{2.7}$$

if $u \in \mathscr{S}'$, $x \in \mathbb{C}^n$, Re $x \in V$, $|\text{Im}\ x| \leq \rho_1$ ($\leq 1/2$) and (2.6) is valid. Put

$$\mathscr{V}(x, x_{n+1}) = \mathscr{H}(a(D_x, y, D_y)u)(x, x_{n+1}),$$

and assume that

$$R \geq 16enB/\varepsilon_0,\ 0 < \rho_1 < \min\{1/2, 1/(3R), \varepsilon_0/(48en(\widehat{C} + 6(1+\sqrt{2})))\}.$$

Then it follows from (2.1) and (2.7) that $\langle D_x\rangle^\mu \mathscr{V}(x, \rho)$ ($\mu = 0, 1$) can be continued analytically to $\{x \in \mathbb{C}^n;\ \text{Re}\ x \in V \text{ and } |\text{Im}\ x| < \rho_1\}$. Applying Lemma 1.2.4 of [10] to the Cauchy problem

$$\begin{cases} (1 - \Delta_{x,x_{n+1}})v(x, x_{n+1}) = 0, \\ v(x, \rho) = \mathscr{V}(x, \rho),\ (\partial v/\partial x_{n+1})(x, \rho) = -\langle D_x\rangle \mathscr{V}(x, \rho), \end{cases}$$

we can show that $\mathscr{V}(x, x_{n+1})$ can be continued analytically from $\mathbb{R}^n \times (0, \infty)$ to $V \times (\rho - \rho_1, \infty)$. This implies that $a(D_x, y, D_y)u$ is analytic in V. □

Assume that $R \geq \max\{4, R_0, 16en(A + C_*)/\varepsilon_0\}$. From Lemma 2.1 we see that ${}^t p^R(x, D)u - {}^t\tilde{p}^R(D_x, y, D_y)u$ is analytic in V for $u \in \mathscr{S}'$. Let us apply Corollary 2.4.7 of [10] to ${}^t\tilde{p}^R(D_x, y, D_y)$. We note that $({}^t p)(x, \xi) \equiv \sum_{j=0}^\infty q_j(x, \xi) \in FPS^+(X; R_0, 4nA^2, 2A)$, where $q_j(x, \xi) = \sum_{|\alpha|=j}(-1)^{|\alpha|}p^{(\alpha)}_{(\alpha)}(x, -\xi)/\alpha!$. Let $R_0 \geq nA^2/2$, and put $q(x, \xi) := \sum_{j=0}^\infty \phi^{4R_0}_j(\xi)q_j(x, \xi)$. By definition $({}^t p)(x, D)$ coincides with $q(x, D)$ as the operator from $\mathscr{F}_0$ to $\mathscr{B}(X)/\mathscr{A}(X)$. Since ${}^t\tilde{p}^R(D_x, y, D_y) = a(D_x, y, D_y)$ if $a(\xi, y, \eta) = \tilde{p}^R(-\eta, y, -\xi)$, it follows from Corollary 2.4.7 of [10] that there are symbols $h(x, \xi)$ and $r(x, \xi)$ and $R(A, V', W) \geq \max\{4, R_0\}$ such that

$${}^t\tilde{p}^R(D_x, y, D_y) = h(x, D) + r(x, D) \quad \text{on } \mathscr{S}_\infty,$$

$h(x,\xi) \in S^+(4R, \widehat{C}_* + 10A_1)$ and

$$|r^{(\alpha)}_{(\beta)}(x,\xi)| \le C_{|\alpha|,R}(4R+1)^{|\beta|}|\beta|!e^{-\langle\xi\rangle/R}$$

if $R \ge R(A, V', W)$, where $A_1 = \max\{A + C_*, 2A + C(V', W)\}$. Moreover, we have

$$|\partial_\xi^\alpha D_x^\beta\{h(x,\xi) - q(x,\xi)\}| \le C_{|\alpha|,R}(R+1)^{|\beta|}|\beta|!\langle\xi\rangle^{-|\alpha|}e^{-\langle\xi\rangle/R}$$

if $x \in V'$ and $R \ge R(A, V', W)$. Now assume that $R \ge R(A, V', W)$. Proposition 1.1 implies that $r(x, D)u$ is analytic if $u \in \mathscr{F}_0$. It follows from Lemma 2.4 of [11] that $(h(x, D)u)|_X - q(x, D)u$ ($\in \mathscr{B}(X)$) is analytic in V for $u \in \mathscr{F}_0$, with a modification of $R(A, V', W)$ if necessary. This yields

(2.8)
$$({}^t p^R(x, D)u)|_V = ({}^t \tilde{p}^R(D_x, y, D_y)u)|_V = (({}^t p)(x, D)u)|_V \quad \text{in } \mathscr{B}(V)/\mathscr{A}(V)$$

for $u \in \mathscr{F}_0$.

Lemma 2.2. *Let $a(x,\xi)$ be a symbol in $S^+(R_0, A)$ satisfying* supp $a(x,\xi) \subset W \times \mathbb{R}^n$. *Then $a(x, D)u \in \mathscr{A}'(\overline{W})$ for $u \in \mathscr{F}_0$.*

Proof. We shall apply the same argument as in the proof of Theorem 3.3.6 of [10]. Put

$$a^R(x,\xi;y) := \sum_{k=1}^\infty \psi_k^R(\xi) \sum_{|\beta|\le k-1} (iy)^\beta \partial_x^\beta a(x,\xi)/\beta!$$

for $x, y \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ and $R \ge R_0$. Then we have

$$a^R(x,\xi;y) \in \bigcap_{\delta > \delta(y)/R_0} S^{0,\delta}(3R, 3\widehat{C} + 3AR/R_0, 3AR/R_0)$$

for any $y \in \mathbb{R}^n$, where $\delta(y) = \sqrt{n}A|y|$. Moreover, we have

$$\begin{aligned}|(\partial_{x_j} + i\partial_{y_j})\partial_\xi^{\alpha+\tilde\alpha} D_x^{\beta+\tilde\beta} a^R(x,\xi;y)| &\le C_{|\tilde\alpha|+|\tilde\beta|,\delta}(\widehat{C}/R + A/R_0)^{|\alpha|} \\ &\times (A/R_0)^{|\beta|}\langle\xi\rangle^{|\beta|-|\tilde\alpha|}\exp[(e\delta(y)/R_0 - 1/(3R) + \delta)\langle\xi\rangle]\end{aligned}$$

if $\langle\xi\rangle \ge 3R(|\alpha| + |\beta|)$. We choose open convex proper cones Γ_j ($1 \le j \le J$) in $\mathbb{R}^n \setminus \{0\}$ and $\{g_j^R(\xi)\} \subset C^\infty(\mathbb{R}^n)$ ($R \ge 2$, $1 \le j \le J$) so that $g_j^R(\xi)$

is positively homogeneous of degree 0 in $|\xi| \geq 1$, $\mathbb{R}^n \setminus \{0\} = \bigcup_{j=1}^J \Gamma_j$, supp $g_j^R \cap \{|\xi| \geq 1\} \subset \Gamma_j$, $\sum_{j=1}^J g_j^R(\xi) = 1$ for $\xi \in \mathbb{R}^n$ and $|\partial_\xi^{\alpha+\gamma} g_j^R(\xi)| \leq C_{|\gamma|}(C_*/R)^{|\alpha|}\langle\xi\rangle^{-|\gamma|}$ if $\langle\xi\rangle \geq R|\alpha|$. Let $u \in \mathscr{F}_0$, and put

$$U_j^R(x, x_{n+1}) := (\text{sgn } x_{n+1}) e^{-|x_{n+1}|\langle D\rangle} g_j^R(D) u(x)/2$$
$$(= g_j^R(D_x)\mathscr{H}(u)(x, x_{n+1})).$$

It is obvious that

$$U_j^R(x, x_{n+1}) = (2\pi)^{-n}\langle \hat{u}(\xi), e^{ix\cdot\xi - x_{n+1}\langle\xi\rangle} g_j^R(\xi)\rangle$$

for $x_{n+1} > 0$. We can choose $c > 0$ so that

$$\text{Im } z\cdot\xi \geq c|\text{Im } z|\,|\xi| \quad \text{for } 1 \leq j \leq J,\ z \in \mathbb{R}^n + i\Gamma_j^* \text{ and } \xi \in \text{supp } g_j^R \text{ with } |\xi| \geq 1,$$

where $\Gamma_j^* = \{y \in \mathbb{R}^n;\ y\cdot\xi \geq 0 \text{ for any } \xi \in \Gamma_j\}$. Now assume that $R_0 \geq 2e\sqrt{n}A/c$. Then Stokes' formula gives

$$\begin{aligned}
\langle a(x,D)u_\varepsilon(x), \varphi(x)\rangle &= 2\sum_{j=1}^J \langle a(x,D)U_j^{R_0}(x,\varepsilon), \varphi(x)\rangle \\
&= 2\sum_{j=1}^J \Big\{\int_W U_{j,1,\varepsilon}(x; y^j)\varphi(x + iy^j)\,dx \\
&\qquad + \int_0^1 \Big(\int_W U_{j,2,\varepsilon}(x; ry^j)\varphi(x + iry^j)\,dx\Big)\,dr\Big\}
\end{aligned}$$

for $\varphi \in \mathscr{S}_\infty$, $\varepsilon > 0$ and $y^k \in \Gamma_k^* \setminus \{0\}$ ($1 \leq k \leq J$), where $u_\varepsilon(x) = e^{-\varepsilon\langle D\rangle}u(x)$ and

$$\begin{aligned}
&U_{j,1,\varepsilon}(x;y) = (2\pi)^{-n}\langle \hat{u}(\xi), e^{i(x+iy)\cdot\xi - \varepsilon\langle\xi\rangle} g_j^{R_0}(\xi) a^R(x,\xi;y)\rangle_\xi/2, \\
&U_{j,2,\varepsilon}(x;y) \\
&\quad = (2\pi)^{-n}\langle \hat{u}(\xi), e^{i(x+iy)\cdot\xi - \varepsilon\langle\xi\rangle} g_j^{R_0}(\xi)\sum_{k=1}^n iy_k(\partial_{x_k} + i\partial_{y_k})a^R(x,\xi;y)\rangle_\xi/2
\end{aligned}$$

for $1 \leq j \leq J$ and $y \in \Gamma_j^* \setminus \{0\}$. It is easy to see that for each $y \in \Gamma_j^* \setminus \{0\}$

$$U_{j,1,\varepsilon}(x;y) \rightrightarrows U_{j,1,0}(x;y) \quad \text{on } \mathbb{R}^n \text{ as } \varepsilon \downarrow 0,$$

$$U_{j,2,\varepsilon}(x;ry) \rightrightarrows U_{j,2,0}(x;ry) \quad \text{in } (x,r) \in \mathbb{R}^n \times [0,1] \text{ as } \varepsilon \downarrow 0.$$

Therefore, we have

$$\langle a(x,D)u(x), \varphi(x)\rangle = 2\sum_{j=1}^{J}\Big\{\int_W U_{j,1,0}(x;y^j)\varphi(x+iy^j)\,dx + \int_0^1 \Big(\int_W U_{j,2,0}(x;ry^j)\varphi(x+iry^j)\,dx\Big)\,dr\Big\}$$

for $\varphi \in \mathscr{S}_\infty$ and $y^k \in \Gamma_k^* \setminus \{0\}$ ($1 \le k \le J$). This implies that $a(x,D)u(x) \in \mathscr{A}'(\overline{W})$. Indeed, $\mathscr{S}_\infty$ includes $\mathscr{P} := \{p(x)e^{-x^2};\ p(x) \text{ is a polynomial}\}$ and, therefore, $\mathscr{A}(\mathbb{C}^n)$ can be approximated locally uniformly by elements of $\mathscr{S}_\infty$. On the other hand, we have

$$|\langle a(x,D)u(x), \varphi(x)\rangle| \le C_\delta \sup_{x\in\overline{W},\, |y|\le\delta} |\varphi(x+iy)| \quad \text{for } \varphi \in \mathscr{S}_\infty$$

if $\delta > 0$, which gives $a(x,D)u(x) \in \mathscr{A}'(\overline{W})$. □

By Lemma 2.2 we can define an operator P: $\mathscr{A}'(\overline{V}) \to \mathscr{A}'(\overline{W})$ by $Pu = p^R(x,D)u$ for $u \in \mathscr{A}'(\overline{V})$ ($\subset \mathscr{F}_0$). Since the strong dual space of $\mathscr{A}'(K)$ is $\mathscr{A}(K)$, we can define the transposed operator ${}^tP$: $\mathscr{A}(\overline{W}) \to \mathscr{A}(\overline{V})$, *i.e.*,

$$\langle u, {}^tP\varphi\rangle_{\mathscr{A}'(\overline{V}),\mathscr{A}(\overline{V})} (= u({}^tP\varphi)) = \langle Pu, \varphi\rangle_{\mathscr{A}'(\overline{W}),\mathscr{A}(\overline{W})} (= (Pu)(\varphi))$$

for $u \in \mathscr{A}'(\overline{V})$ and $\varphi \in \mathscr{A}(\overline{W})$. On the other hand, we can define ${}^tp^R(x,D)\varphi(x)$ for $\varphi \in \mathscr{A}(\overline{W})$ by

$${}^tp^R(x,D)\varphi(x) = \mathscr{F}_\xi^{-1}\Big[\int e^{-iy\cdot\xi}p^R(y,-\xi)\varphi(y)\,dy\Big](x) (\in \mathscr{F}_0)$$

since supp $p^R \subset W \times \mathbb{R}^n$. Moreover, we can define ${}^tp^R(x,D)u \in \mathscr{F}_0$ for $u \in \mathscr{D}'(W)$. Assume that $R \ge 2e\sqrt{n}(2A + C(V',W))$. Then, from Proposition 1.1 we have ${}^tp^R(x,D)$: $\mathscr{S}_\infty \to \mathscr{S}_\delta\, (\subset \mathscr{A}(\overline{V}))$ if $\delta < 1/R$. By definition it is easy to see that

$${}^tP\varphi = {}^tp^R(x,D)\varphi \quad \text{in } \mathscr{A}(\overline{V}) \text{ for } \varphi \in \mathscr{S}_\infty.$$

Lemma 2.3. *Let $a(x,\xi)$ be a symbol satisfying* $\mathrm{supp}\ a \subset W \times \mathbb{R}^n$ *and*

$$|a_{(\beta)}(x,\xi)| \le C_\delta (A/R)^{|\beta|}\langle\xi\rangle^{|\beta|} e^{\delta\langle\xi\rangle}$$

if $\langle\xi\rangle \ge R|\beta|$ and $\delta > 0$. Let $\varepsilon > 0$, and assume that $u \in C^\infty(W)$ satisfies

$$|D^\alpha u(x)| \le C(u)\varepsilon^{-|\alpha|}|\alpha|! \quad \text{for } x \in W \text{ and } \alpha \in (\mathbb{Z}_+)^n,$$

where $C(u)$ is a positive constant. Then we have ${}^r a(x,D)u \in \mathscr{S}'_{-\delta}$ and

$$\sup_{z\in\mathbb{C}^n,\, |\mathrm{Im}\ z|\le\delta} |v(z)| \le C'_\delta C(u)$$

if $R \ge 2e\sqrt{n}A$ and $\delta < 1/(2e\sqrt{n}\max\{A, 1/\varepsilon\})$, where $v(z)$ denotes the analytic continuation of ${}^r a(x,D)u(x)$ to $\{z \in \mathbb{C}^n;\ |\mathrm{Im}\ z| \le \delta\}$ and C'_δ is a positive constant independent of u.

Proof. Put $K = |\xi|^{-2}\sum_{k=1}^n \xi_k D_{y_k}$. Then we have

$$\begin{aligned}&|K^j(a(y,\xi)u(y))| \le C_\delta C(u)|\xi|^{-j}\langle\xi\rangle^j\{\sqrt{n}(A/R + 1/(R_1\varepsilon))\}^j e^{\delta\langle\xi\rangle}\\ &\le C_\delta C(u) e^{1/R}\{\sqrt{n}(A/R + 1/(R_1\varepsilon))\}^j e^{\delta\langle\xi\rangle}\end{aligned}$$

if $R_1 \ge R$, $\langle\xi\rangle \ge R_1 j$ and $\delta > 0$. Therefore, we have

$$\begin{aligned}\Big|\int e^{-iy\cdot\xi}a(y,\xi)u(y)\,dy\Big| &\le \int |K^j(a(y,\xi)u(y))|\,dy\\ &\le C'_\delta C(u)\{e\sqrt{n}(A/R + 1/(R_1\varepsilon))\}^j \exp[(\delta - 1/R_1)\langle\xi\rangle]\end{aligned}$$

if $R_1 \ge R$, $R_1 j \le \langle\xi\rangle \le R_1(j+1)$ and $\delta > 0$. This yields

$$|\mathscr{F}[{}^r a(x,D)u(x)](\xi)| \le C''_\delta C(u) e^{-\delta\langle\xi\rangle} \tag{2.9}$$

if $R \ge 2e\sqrt{n}A$, $R_1 \ge R$, $R_1 \ge 2e\sqrt{n}/\varepsilon$ and $\delta < 1/R_1$. From (2.9) we can easily prove the lemma. □

We note that for $\varepsilon > 0$ and a compact subset K of $\mathbb{R}^n$

$$\widehat{K}_\varepsilon := \{z \in \mathbb{C}^n;\ |\mathrm{Re}\ z - x| + |\mathrm{Im}\ z| \le \varepsilon \text{ for some } x \in K\}$$

is polynomially convex and, therefore, $\widehat{K}^\circ_\varepsilon$ is a Runge domain, where $\widehat{K}^\circ_\varepsilon$ denotes the interior of $\widehat{K}_\varepsilon$ in $\mathbb{C}^n$ (see, *e.g.*, Lemma 1.1.1 of [10]). Let

$\varphi \in \mathscr{A}(\overline{W})$. Then there are $\varepsilon > 0$ and $\{\varphi_j\} \subset \mathscr{S}_\infty$ ($\subset \mathscr{A}(\mathbb{C}^n)$) such that $\varphi \in \mathscr{A}(\widehat{K}^\circ_{3\sqrt{n}\varepsilon})$ and

$$\sup_{z \in \widehat{K}_{2\sqrt{n}\varepsilon}} |\varphi(z) - \varphi_j(z)| \to 0 \quad \text{as } j \to \infty,$$

where $K = \overline{W}$. Since $\{z \in \mathbb{C}^n;\ |z_k - x_k| \le \varepsilon\ (\ 1 \le k \le n)$ for some $x \in K\} \subset \widehat{K}_{\sqrt{2n}\varepsilon}$, Cauchy's estimates give

$$\sup_{\alpha \in (\mathbb{Z}_+)^n} \sup_{x \in W} \varepsilon^{|\alpha|} |D^\alpha(\varphi(x) - \varphi_j(x))|/|\alpha|! \to 0 \quad \text{as } j \to \infty.$$

Therefore, it follows from Lemma 2.3 that

$$ {}^tP\varphi = {}^tp^R(x, D)\varphi \quad \text{for } \varphi \in \mathscr{A}(\overline{W}). \tag{2.10}$$

In order to prove Theorem 1.3 it suffices to apply the same argument as in [2] with slight modifications. For completeness we shall repeat their argument. Define $\widetilde{P}$: $\mathscr{A}'(\overline{V}) \to \mathscr{B}(U)$ by $\widetilde{P}u = p(x, D)u$ for $u \in \mathscr{A}'(\overline{V})$.

Lemma 2.4. *$\widetilde{P}$ is surjective if and only if $Q : \mathscr{A}'(\overline{V}) \times \mathscr{A}'(\overline{W} \setminus U) \ni (\varphi, \mu) \mapsto P\varphi + \mu \in \mathscr{A}'(\overline{W})$ is surjective.*

Remark. The above result was given in Schapira [8].

Proof. Assume that $\widetilde{P}$ is surjective. Let $g \in \mathscr{A}'(\overline{W})$, and put $f = g|_U \in \mathscr{B}(U)$. Then there is $\varphi \in \mathscr{A}'(\overline{V})$ such that $\widetilde{P}\varphi = f$. Therefore, we have $P\varphi - g \in \mathscr{A}'(\overline{W} \setminus U)$ since $(P\varphi)|_U = (p(x, D)\varphi)|_U$ ($= \widetilde{P}\varphi$). This proves that Q is surjective. Next assume that Q is surjective. Let $f \in \mathscr{B}(U)$. By definition there is $g \in \mathscr{A}'(\overline{U})$ satisfying $f = g|_U$. Then there are $\varphi \in \mathscr{A}'(\overline{V})$ and $\mu \in \mathscr{A}'(\overline{W} \setminus U)$ such that $g = P\varphi + \mu$. Therefore, we have $(P\varphi)|_U = g|_U = f$. □

Lemma 2.5. *Let Ω be a complex neighborhood of $\overline{W}$. Then $\widetilde{P}$ is surjective if and only if for any ε with $0 < \varepsilon < \operatorname{dis}(\overline{W}, \mathbb{C}^n \setminus \Omega)$ there are positive constants η and C such that*

$$|h|_{U_\eta} \le C(|{}^tPh|_{V_\varepsilon} + |h|_{(\overline{W} \setminus U)_\varepsilon}) \quad \textit{for any } h \in \mathscr{A}(\Omega), \tag{2.11}$$

where $\operatorname{dis}(A, B) := \inf\{|x - y|;\ x \in A \textit{ and } y \in B\}$.

Proof. Since the boundary of each connected component of U is included in ∂U, $\mathscr{A}(\overline{W}) \to \mathscr{A}(\overline{W} \setminus U)$ is injective and, therefore, $\mathscr{A}'(\overline{W} \setminus U)$ is dense in $\mathscr{A}'(\overline{W})$. So it suffices to prove that Q has closed range if and only if (2.11) holds, where Q is the operator defined in Lemma 2.4. On the other hand, it follows from Köthe [7, p18] that Q has closed range if and only if ${}^tQ$: $\mathscr{A}(\overline{W}) \ni h \mapsto ({}^tPh, h|_{\overline{W}\setminus U}) \in \mathscr{A}(\overline{V}) \times \mathscr{A}(\overline{W} \setminus U)$ has (sequentially) closed range. It is easy to see that ${}^tQ$ has closed range if (2.11) holds. Therefore, Q has closed range if (2.11) holds. Now assume that ${}^tQ$ (and Q) has closed range. Since ${}^tQ$ is injective, ${}^tQ$: $\mathscr{A}(\overline{W}) \to R({}^tQ)$ is an isomorphism, where $R({}^tQ)$ denotes the range of ${}^tQ$. This implies that $h_k \to 0$ in $\mathscr{A}(\overline{W})$ if ${}^tQh_k \to 0$ in $\mathscr{A}(\overline{V}) \times \mathscr{A}(\overline{W} \setminus U)$. Suppose that (2.11) does not hold. Then there are $\varepsilon > 0$ and a sequence $\{h_k\} \subset \mathscr{A}(\Omega)$ such that $|h_k|_{U_{1/k}} = 1$ and ${}^tQh_k \to 0$ in $\mathscr{A}^\infty(V_\varepsilon) \times \mathscr{A}^\infty((\overline{W} \setminus U)_\varepsilon)$, where $\mathscr{A}^\infty(\Omega) := \{\varphi \in \mathscr{A}(\Omega);\ |\varphi|_\Omega < \infty\}$ is a Banach space with the norm $|\varphi|_\Omega$. This leads us a contradiction. □

Now we can prove Theorem 1.3. It follows from the assumption (A)$'$ and (2.8) that f is analytic in U if $f \in L^2(\mathbb{R}^n)$, f is analytic in a neighborhood of $\overline{W} \setminus U$ and ${}^tp^R(x,D)f$ is analytic in V. Let Ω be a complex neighborhood of $\overline{W}$. Choose $\varepsilon > 0$ so that $\varepsilon < \mathrm{dis}(\overline{W}, \mathbb{C}^n \setminus \Omega)$, and put

$$E := \{(f,g,h) \in L^2(W) \times \mathscr{A}^\infty(V_\varepsilon) \times \mathscr{A}^\infty((W \setminus U)_\varepsilon);\\ g|_V = ({}^tp^R(x,D)f)|_V,\ h|_{W\cap(W\setminus U)_\varepsilon} = f|_{W\cap(W\setminus U)_\varepsilon}\}.$$

Then for any $(f,g,h) \in E$ there is $\hat{\varepsilon} > 0$ such that f can be continued analytically to $W_{\hat{\varepsilon}}$. Indeed, ${}^tp^R(x,D)f = {}^tp^R(x,D)\tilde{f}$ if $\tilde{f} \in L^2(\mathbb{R}^n)$, $\tilde{f} = h$ in $(W \setminus U)_\varepsilon \cap \mathbb{R}^n$ and $\tilde{f} = f$ in U. So f is analytic in U and $f \in \mathscr{A}(\overline{W})$. Let us prove that E is closed and, therefore, E is a Banach space. Assume that $\{(f_j,g_j,h_j)\} \subset E$ and $(f_j,g_j,h_j) \to (f,g,h)$ in $L^2(W) \times \mathscr{A}^\infty(V_\varepsilon) \times \mathscr{A}^\infty((W \setminus U)_\varepsilon)$. Let V_1 and V_2 be open subsets of V satisfying $U \Subset V_1 \Subset V_2 \Subset V$, and choose $\Phi_1^R(x,\xi) \in S^0(R, C_*, C(V_1,V_2))$ ($R \geq 4$) so that $0 \leq \Phi_1^R \leq 1$, $\mathrm{supp}\ \Phi_1^R \subset V_2 \times \mathbb{R}^n$ and $\Phi_1^R(x,\xi) = 1$ in $V_1 \times \mathbb{R}^n$. We put

$$p_1^R(x,\xi) := \Phi_1^R(x,\xi)p(x,\xi), \quad p_2^R(x,\xi) := p^R(x,\xi) - p_1^R(x,\xi).$$

Then we have $p_j^R(x,\xi) \in S^+(R, C_* + A, A_1)$ ($j = 1,2$), where A_1 is a positive constant depending on A, V_1, V_2, V' and W. From Lemma 2.3

we have ${}^t p_\ell^R(x,D) f_j \in \mathscr{A}(\mathbb{R}^n)$ ($\ell = 1,2$) if $R \ge 2e\sqrt{n}A_1$. Assume that $R \ge 2e\sqrt{n}A_1$. It is obvious that ${}^t p^R(x,D) f_j \to {}^t p^R(x,D) f$ in $\mathscr{F}_0$ and ${}^t p_\ell^R(x,D) f_j \to {}^t p_\ell^R(x,D) f$ in $\mathscr{F}_0$ ($\ell = 1,2$). Note that supp $p_2^R \subset (W \setminus V_1) \times \mathbb{R}^n$ and that $f_j|_{W\setminus V_1}$ can be continued analytically to $h_j \in \mathscr{A}^\infty((W \setminus U)_\varepsilon)$ which satisfies $C_j := \sup_{z \in (W\setminus U)_\varepsilon} |h_j(z) - h(z)| \to 0$ as $j \to \infty$. Cauchy's estimates give

$$\sup_{x \in W\setminus V_1} |D^\alpha (f_j(x) - h(x))| \le C_j (\sqrt{n}/\varepsilon)^{|\alpha|} |\alpha|!.$$

It follows from Lemma 2.3 and (2.9) that

$$\begin{gathered} {}^t p_2^R(x,D) f_j \to {}^t p_2^R(x,D)(h|_{W\setminus V_1}) \quad \text{in } \mathscr{S}'_{-\delta}, \\ \sup_{z \in \mathbb{C}^n,\, |\operatorname{Im}\ z| \le \delta} |v_j(z) - v(z)| \le C_\delta C_j \end{gathered}$$

if $\delta < 1/(2e\sqrt{n}\max\{A_1, \sqrt{n}/\varepsilon\})$, where $v_j(z)$ ($j \in \mathbb{N}$) and $v(z)$ denote the analytic continuations of ${}^t p_2^R(x,D) f_j$ ($j \in \mathbb{N}$) and ${}^t p_2^R(x,D)(h|_{W\setminus V_1})$, respectively. Moreover, we have $v(x) = {}^t p_2^R(x,D) f$ in $\mathscr{F}_0$. Since $g_j|_V = ({}^t p^R(x,D) f_j)|_V \rightrightarrows g|_V$ on V, we have

$$\text{(2.12)} \qquad ({}^t p_1^R(x,D) f_j)|_V \rightrightarrows g|_V - v|_V \quad \text{on } V.$$

We can write

$${}^t p_1^R(x,D) f_j(x) = \sum_{k=1}^\infty \psi_k^R(D)\ {}^t p_1^R(x,D) f_j \quad \text{in } \mathscr{F}_0.$$

For $x \in \mathbb{R}^n \setminus V$ we have

$$\begin{aligned} \psi_k^R(D)\, {}^t p_1^R(x,D) f_j &= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} \psi_k^R(\xi) p_1^R(y,-\xi) f_j(y)\, dy d\xi \\ &= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} L^k (\psi_k^R(\xi) p_1^R(y,-\xi)) f_j(y)\, dy d\xi, \end{aligned}$$

where $L = |x-y|^{-2} \sum_{\ell=1}^n (y_\ell - x_\ell) D_{\xi_\ell}$. Note that

$$|L^k(\psi_k^R(\xi) p_1^R(y,-\xi))| \le C_{\delta,R} (\sqrt{n}(\widehat{C} + C_* + A)/(\varepsilon_1 R))^k e^{\delta\langle\xi\rangle}$$

if $x \in \mathbb{R}^n \setminus V$ and $\delta > 0$, where $\varepsilon_1 = \operatorname{dis}(V_2, \mathbb{R}^n \setminus V)$ (> 0). Therefore, we have

$$|\psi_k^R(D)\, {}^t p_1^R(x,D) f_j| \le C_R k^{-2} \|f_j\|_{L^2(V_2)}$$

if $x \in \mathbb{R}^n \setminus V$ and $R \geq 2e\sqrt{n}(\widehat{C} + C_* + A)/\varepsilon_1$. Now assume that $R \geq 2e\sqrt{n}(\widehat{C} + C_* + A)/\varepsilon_1$. Then $\sum_{k=1}^\infty \psi_k^R(D)\, {}^t p_1^R(x, D) f_j$ converges uniformly to ${}^t p_1^R(x, D) f_j(x)$ on $\mathbb{R}^n \setminus V$ and

$$\sup_{x \in \mathbb{R}^n \setminus V} |{}^t p_1^R(x, D) f_j(x)| \leq C \|f_j\|_{L^2(V_2)} \quad (\, j = 1, 2, \cdots),$$

where $C > 0$. Therefore, we have

$${}^t p_1^R(x, D) f_j(x) \rightrightarrows {}^t p_1^R(x, D) f(x) \quad \text{on } \mathbb{R}^n \setminus V.$$

This, together with (2.12), gives

$${}^t p_1^R(x, D) f_j(x) \rightrightarrows w(x) \quad \text{on } \mathbb{R}^n,$$

where $w(x) = g(x) - v(x)$ for $x \in V$ and $w(x) = {}^t p_1^R(x, D) f(x)$ for $x \in \mathbb{R}^n \setminus V$. So we have ${}^t p_1^R(x, D) f(x) = w(x)$ in $\mathscr{F}_0$ and

$$({}^t p^R(x, D) f)|_V = g|_V - v|_V + v|_V = g|_V.$$

Since $f \in L^2(W)$ and $f|_{W \cap (W \setminus U)_\varepsilon} = h|_{W \cap (W \setminus U)_\varepsilon}$, this proves that E is closed. Put

$$E(k) := \{(f, g, h) \in E;\ f \text{ is the restriction of a function } \tilde{f} \in \mathscr{A}^\infty(W_{1/k}) \text{ with } |\tilde{f}|_{W_{1/k}} \leq k\}.$$

Then $E = \bigcup_{k=1}^\infty E(k)$ and $E(k)$ is a closed balanced convex subset of E since $\{\tilde{f}_j\}_{j=1,2,\cdots}$ is relatively compact in $\mathscr{A}(W_{1/k})$ if $\tilde{f}_j \in \mathscr{A}^\infty(W_{1/k})$ and $|\tilde{f}_j|_{W_{1/k}} \leq k$. By Baire's theorem there are $k \in \mathbb{N}$ and $c > 0$ such that f is the restriction of a function $\tilde{f} \in \mathscr{A}^\infty(W_{1/k})$ with $|\tilde{f}|_{W_{1/k}} \leq k$ if $(f, g, h) \in E$ and $\|f\|_{L^2(W)} + |g|_{V_\varepsilon} + |h|_{(W \setminus U)_\varepsilon} < c$. This, together with (2.10), yields

$$|h|_{U_{1/k}} \leq |h|_{W_{1/k}} \leq (k/c)(\|h|_W\|_{L^2(W)} + |{}^t Ph|_{V_\varepsilon} + |h|_{(W \setminus U)_\varepsilon}) \tag{2.13}$$

for $h \in \mathscr{A}(\Omega)$ ($\subset \mathscr{A}^\infty(W_\varepsilon)$). Let $\eta < 1/k$. Then (2.11) is valid. Indeed, suppose that (2.11) does not hold for some $\eta > 0$ with $\eta < 1/k$. Then there is a sequence $\{h_j\} \subset \mathscr{A}(\Omega)$ such that

$$|h_j|_{U_\eta} = 1, \quad |{}^t Ph_j|_{V_\varepsilon} + |h_j|_{(W \setminus U)_\varepsilon} \to 0.$$

Putting $\varepsilon' = \min\{\eta, \varepsilon\}$, we have

$$|h_j|_{W_{\varepsilon'}} \le |h_j|_{U_\eta} + |h_j|_{(W\setminus U)_\varepsilon} \le 2 \quad \text{if } j \gg 1.$$

Therefore, we have

$$\|h_j|_W\|_{L^2(W)} \le 2|W|^{1/2} \quad \text{if } j \gg 1,$$

where $|W|$ denotes the volume of W. This, together with (2.13), implies that $\{h_j|_{U_{1/k}}\}$ is bounded in $\mathscr{A}^\infty(U_{1/k})$ and that there are a subsequence $\{h_{j_\ell}\}$ of $\{h_j\}$ and $h \in \mathscr{A}^\infty(U_\eta)$ such that $h_{j_\ell}|_{U_\eta} \to h$ in $\mathscr{A}^\infty(U_\eta)$. Since $h_{j_\ell} \rightrightarrows 0$ on $\overline{W} \setminus U$ and $h(x) = 0$ in $U_\eta \cap (\overline{W} \setminus U)$, $h(z) = 0$ in U_η, which contradicts $|h|_{U_\eta} = 1$. It follows from Lemma 2.5 that $\widetilde{P} \equiv p(x, D)$: $\mathscr{A}'(\overline{V}) \to \mathscr{B}(U)$ is surjective.

3. Microhyperbolic Operators

First we shall give an immediate consequence of Theorem 1.3.

THEOREM 3.1. *Let X be an open subset of $\mathbb{R}^n$, and let $p(x,\xi) \in PS^+(X; R_0, A)$, where $A \ge 0$ and $R_0 \ge 1$. Let U be an open subset of X satisfying $U \Subset X$, and assume that f is analytic in U if $f \in L^2(\mathbb{R}^n)$ and $(({}^tp)(x,D)f)|_U = 0$ in $\mathscr{B}(U)/\mathscr{A}(U)$. Then $p(x,D) : \mathscr{A}'(\overline{V}) \to \mathscr{B}(U)$ is surjective for any open subset V of X with $U \Subset V \Subset X$. In particular, $p(x,D) : \mathscr{A}'(\overline{V}) \to \mathscr{B}(U)$ is surjective if $({}^tp)(x,D)$ is analytic hypoelliptic in U and V is an open subset of X satisfying $U \Subset V \Subset X$ (see, e.g.,* Definition 4.5.1 *of* [10]).

Let X be an open subset of $\mathbb{R}^n$, and let $p(x,\xi) \in PS^{m,0}(X; 0, A)$, where $m \in \mathbb{R}$ and $A \ge 0$. We assume that there are $p_0(x,\xi) \in PS^{m,0}(X; 0, A)$ and $p_1(x,\xi) \in PS^{m-1,0}(X; 0, A)$ such that $p_0(x,\xi)$ is positively homogeneous of degree m in ξ for $|\xi| \ge 1$ and $p(x,\xi) = p_0(x,\xi) + p_1(x,\xi)$. We define $q(x,\xi) \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ by $q(x,\xi) = |\xi|^m p_0(x, \xi/|\xi|)$. Note that $q(x,\xi) = p_0(x,\xi)$ if $|\xi| \ge 1$.

DEFINITION 3.2. Let $z^0 = (x^0, \xi^0) \in T^*X \setminus 0$ ($\simeq X \times (\mathbb{R}^n \setminus \{0\})$) and $\vartheta \in T_{z^0}(T^*X) \simeq \mathbb{R}^{2n}$.

(i) We say that $p(x,\xi)$ is microhyperbolic at z^0 with respect to ϑ if there are a neighborhood $\mathcal{U}$ of z^0 in $T^*X \setminus 0$ and $t_0 > 0$ such that

$$q(z - it\vartheta) \neq 0 \quad \text{for } z = (x,\xi) \in \mathcal{U} \text{ and } t \in (0, t_0].$$

(ii) Assume that $p(x,\xi)$ is microhyperbolic at z^0 with respect to ϑ. We define the localization polynomial $q_{z^0}(\zeta)$ of $q(z)$ at z^0 by

$$\begin{aligned} &q(z^0 + t\zeta) = t^\mu (q_{z^0}(\zeta) + o(1)) \quad \text{as } t \to 0, \\ &q_{z^0}(\zeta) \not\equiv 0 \quad \text{in } \zeta \in T_{z^0}(T^*X). \end{aligned}$$

We call the number μ the multiplicity of z^0 relative to q.

If $p(x,\xi)$ is microhyperbolic at $z^0 \in T^*X \setminus 0$ with respect to $\vartheta \in \mathbb{R}^{2n}$, then $q_{z^0}(\zeta)$ is hyperbolic, *i.e.*,

$$q_{z^0}(\zeta - i\vartheta) \neq 0 \quad \text{for any } \zeta \in \mathbb{R}^{2n},$$

and we can define $\Gamma(q_{z^0}, \vartheta)$ as the connected component of the set $\{\zeta \in T_{z^0}(T^*X);\ q_{z^0}(\zeta) \neq 0\}$ which contains ϑ (see, *e.g.*, §4.3 of [10]).

Let U be an open subset of X satisfying $U \Subset X$, and assume that there is a continuous vector field ϑ: $\overline{U} \times (\mathbb{R}^n \setminus \{0\}) \ni z \mapsto \vartheta(z) \in \mathbb{R}^{2n}$ such that $p(x,\xi)$ is microhyperbolic at each $z \in \overline{U} \times (\mathbb{R}^n \setminus \{0\})$ with respect to $\vartheta(z)$. A Lipschitz continuous curve $\{z(s)\}_{s \in (-a,0]}$ in $U \times (\mathbb{R}^n \setminus \{0\})$ is called a generalized semi-bicharacteristics of p_0 in the negative direction (with respect to ϑ) if

$$(d/ds)z(s) \in \Gamma(q_{z(s)}, \vartheta(z(s)))^\sigma \cap \{\delta z;\ |\delta z| = 1\} \quad \text{for} \quad a.e.\ s \in (-a, 0],$$

where $a > 0$, σ denotes the cannonical symplectic form on $T^*\mathbb{R}^n$ ($\simeq \mathbb{R}^n \times \mathbb{R}^n$), *i.e.*, $\sigma((\delta x, \delta\xi), (\delta y, \delta\eta)) = \delta y \cdot \delta\xi - \delta x \cdot \delta\eta$ for $(\delta x, \delta\xi), (\delta y, \delta\eta) \in \mathbb{R}^{2n} \equiv \mathbb{R}^n \times \mathbb{R}^n$, and

$$\Gamma^\sigma := \{\delta z \in T_z(T^*X);\ \sigma(\delta w, \delta z) \geq 0 \quad \text{for any } \delta w \in \Gamma\}$$

for $z \in T^*X$ and $\Gamma \subset T_z(T^*X)$. Moreover, we say that a generalized semi-bicharac- teristics $\{z(s)\}_{s \in (-a,0]}$ of p in the negative direction is maximally extended if there is no generalized semi-bicharacteristics $\{w(t)\}_{t \in (-b,0]}$ of p in the negative direction satisfying $z(0) = w(0)$ and $\{z(s)\}_{s \in (-a,0]} \subsetneq \{w(t)\}_{t \in (-b,0]}$. We assume the following condition:

(B) If $\{z(s)\}_{s\in(-a,0]}$ is a maximally extended generalized semi-bicharacteristics of p in the negative direction, where the parameter s of the curve is chosen so that $-s$ coincides with the arc length from $z(0)$ to $z(s)$, then $\lim_{s\to -a+0} z(s) \in (\partial U \times \mathbb{R}^n) \cup U \times (\mathbb{R}^n \setminus \{0\})$ when $a < \infty$, and $\lim_{s\to-\infty} z(s) \in (\partial U \times \mathbb{R}^n)$ when $a = \infty$.

Under the condition (B) it follows from Theorem 4.3.8 of [10] that there is a maximally extended generalized semi-bicharacteristics $\{z(s)\}_{s\in(-a,0]}$ of p in the negative direction with $z(0) = z^0$ satisfying $z(s) \in WF_A(f)$ for $s \in (-a, 0]$ and $\lim_{s\downarrow -a} z(s) \in \partial U \times \mathbb{R}^n$ if $f \in \mathscr{B}(U)$, $({}^tp)(x, D)f = 0$ in $\mathscr{B}(U)/\mathscr{A}(U)$ and $z^0 \in WF_A(f)$. Here the parameter s of the curve is chosen so that $-s$ coincides with the arc length from z^0 to $z(s)$. For $WF_A(f)$ we refer to §3.1 of [10]. So the condition (A)$'$ is satisfied for any open subsets V and W of X satisfying $U \Subset V \Subset W \Subset X$.

Theorem 3.3. *Under the condition* (B) $p(x, D) : \mathscr{A}'(\overline{V}) \to \mathscr{B}(U)$ *is surjective for any open subset* V *of* X *with* $U \Subset V \Subset X$.

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