J. Math. Sci. Univ. Tokyo
13 (2006), 545–593.

On a KM_2O-Langevin Equation with Continuous Time (1)

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Abstract. We treat a continuous time stationary Gaussian process $\mathbf{X}$ whose global time evolution is governed by $[\alpha,\beta,\gamma]$-Langevin equation and derive a KM_2O-Langevin equation which governs a local time evolution of the stochastic process $\mathbf{X}$. Moreover, we prove a fluctuation-dissipation theorem based upon the KM_2O-Langevin equation and derive a system of equations characterizing both the fluctuation and the drift coefficients in the KM_2O-Langevin equation from the covariance function of the stationary process $\mathbf{X}$.

1. Introduction

About twenty-five years ago, we treated in [12] and [16] a continuous time stationary Gaussian process with T-positivity and characterized T-positivity by deriving a stochastic differential equation that describes a global time evolution of the stochastic process, which is the first (resp. second) KMO-Langevin equation with infinite delay term and white noise (resp. Kubo noise) term. Furthermore, we showed the fluctuation-dissipation theorem, stating relations between the fluctuation and the drift terms in the KMO-Langevin equation, as a precision and a generalization of fluctuation-dissipation theorem in non-equilibrium statistical physics ([3], [10]). We note that the terminology of "KMO" comes from the initials of Kubo, Mori and Okabe.

Moreover, in [17], [18] and [24], we treated the Alder-Wainwright effect for Stokes-Boussinesq-Langevin equation which states a long time tail behavior ($\propto t^{-3/2}$) of the velocity autocorrelation function for hard spheres

2000 *Mathematics Subject Classification*. Primary 60G25; Secondary 60G12, 82C05.
Key words: $[\alpha,\beta,\gamma]$-Langevin equation, KMO-Langevin equation with continuous time, KM_2O-Langevin equation with continuous time, Fluctuation-dissipation theorem.
This work is partially supported by Grand-in-Aid for Science Research No.14340030 and Grand-in-Aid for Germination Research No.14654019, the Ministry of Education, Science, Sports and Culture, Japan and Superrobust Computation Project of the 21st Century COE Program "Information Science and Technology Strategic Core".

that had been studied in [1], [2], [4], [39] and [40]. Noting that the unique solution of Stokes-Boussinesq-Langevin equation does not have Markovian property, but has T-positivity, we showed how the Alder-Wainwright effect for Stokes-Boussinesq-Langevin equation can be generalized in the framework of the theory of KMO-Langevin equations with T-positivity. The case for discrete time stationary Gaussian process with T-positivity was investigated in [21]–[23].

On the other hand, we treated in [13] a continuous time stationary Gaussian process with finite multiple Markovian property and derived a stochastic differential equation with infinite delay and white noise terms which describes a global time evolution of the stochastic process. Furthermore, we showed the fluctuation-dissipation theorem based upon the stochastic differential equation.

As a unification of the stochastic differential equations stated above, we introduced in [12] an $[\alpha, \beta, \gamma]$-Langevin equation whose solution is a stationary Gaussian process with continuous time and studied the problem predicting the future and the past from a bounded part of the stochastic process. The case for stationary Gaussian processes with finite multiple Markovian property had been discussed in [11].

On the other hand, Miyoshi treated in [8] and [9] a multi-dimensional continuous time stationary Gaussian process associated with (k, l, m)-string and derived a stochastic differential equation with finite delay and white noise terms, to be called $(\alpha, \beta, \gamma, \delta)$-Langevin equation, which describes a local time evolution of the stochastic process, and showed a fluctuation-dissipation theorem, by using an inverse spectral theory.

As a discrete version of the stochastic differential equation derived in [8] and [9], we treated in [25]–[35] a general multi-dimensional square integrable stochastic process with discrete time and derived a stochastic difference equation, to be called $\mathrm{KM_2O}$-Langevin equation, describing a local time evolution of the stochastic process. Furthermore, we characterized the stationarity of the stochastic process as a fluctuation-dissipation theorem, which gives some relations between the fluctuation and the drift terms in the $\mathrm{KM_2O}$-Langevin equation. In [6] and [33], we solved some problems arising from Masani and Wiener's work ([5]) on the non-linear prediction problem for discrete time stochastic processes under Dobrushin-Minlos' regularity condition. We note that the terminology of "$\mathrm{KM_2O}$" comes from the initials of Kubo, Mori, Miyoshi and Okabe.

Furthermore, we applied the theory of $\mathrm{KM_2O}$-Langevin equations to non-linear time series analysis to develop an experimental mathematics with the spirit of " from data to model", and proposed five tests: one is Test(S) checking stationarity property of time series ([26]); the second is Test(ABN) detecting certain signs of abnormality of time series ([35]); the third is Test(D) searching determinacy property of time series ([29]); the fourth is Test(CS) looking for causality property between two kinds of time series ([28]); the fifth is Test(MOD) deriving a model describing a local time evolution of time series ([33], [34], [37]).

The first aim of this paper is to treat a general stationary solution $\mathbf{X}$ of $[\alpha, \beta, \gamma]$-Langevin equation with infinite delay and white noise terms in [12], and to derive the same type of stochastic differential equation with finite delay and white noise terms as the one in Miyoshi's work. We call such an equation $\mathrm{KM_2O}$-Langevin equation with continuous time. The second aim is to show the fluctuation-dissipation theorem based upon it. The third aim is to derive a system of equations characterizing both the fluctuation and the drift coefficients in the $\mathrm{KM_2O}$-Langevin equation with continuous time from the covariance function of the stationary process $\mathbf{X}$.

We shall state the contents of this paper. In Section 2, we shall recall the results for the stationary solution of $[\alpha, \beta, \gamma]$-Langevin equation in [12] and [14]. By a stationary solution of $[\alpha, \beta, \gamma]$-Langevin equation, we mean a one-dimensional stationary Gaussian process $\mathbf{X} = (X(t); t \in \mathbf{R})$ with continuous paths such that

(i) there exists a standard Brownian motion $\mathbf{B} = (B(t); t \in \mathbf{R})$ such that for any $t \in \mathbf{R}$

$$\sigma(X(s); s \le t) = \sigma(B(s_1) - B(s_2); s_1, s_2 \le t); \tag{1.1}$$

(ii) with probability one, $\mathbf{X}$ and $\mathbf{B}$ satisfy

$$\begin{aligned} X(t) - X(s) \; = \; & -\int_s^t (\beta X(u) + \int_{(-\infty,0)} X(u+\tau)\gamma(d\tau))du \\ & + \alpha(B(t) - B(s)) \end{aligned} \tag{1.2}$$

for any t, s $(s < t)$, where the triplet (α, β, γ) satisfies the following conditions:

$$\alpha > 0, \tag{1.3}$$

(1.4) $\beta \in \mathbf{R}$,

(1.5) $\gamma(d\tau)$ is a bounded signed Borel measure on $(-\infty, 0)$.

In Section 3, we shall recall the results in [14] for the forward (resp. backward) innovation processes $\nu_a^+ = (\nu_a^+(t); t \geq 0)$ (resp. $\nu_a^- = (\nu_a^-(t); t \geq 0)$) ($a \in \mathbf{R}$) and the local forward (resp. backward) prediction kernel $P = P(t,s)$ $(0 \leq s \leq t)$ (resp. $Q = Q(t,s)$ $(t, s \geq 0)$) associated with the stochastic process $\mathbf{X}$.

Under additional conditions for the bounded signed measure $\gamma(d\tau)$ in (1.5), in Section 4, we shall investigate the regularity properties of both the local prediction kernels P and Q.

In Section 5, we shall derive the following stochastic differential equation which governs a local time evolution of the stochastic process $\mathbf{X}$: for any $t \geq 0$

$$(1.6) \quad X(t) - X(0) = -\int_0^t \left(\beta(s)X(s) + \int_0^s \gamma(s,u)X(u)du + \delta(s)X(0)\right)ds + \alpha\nu_0^+(t).$$

We call equation (1.6) $\mathrm{KM_2O}$-Langevin equation with continuous time associated with the stochastic process $\mathbf{X}$. We note that equation (1.6) describes the local time evolution from the initil time $t = 0$ until time t of the stochastic process $\mathbf{X}$, while equation (1.2) describes the global time evolution from the initial time $t = -\infty$ until time t of the stochastic process $\mathbf{X}$.

In Section 6, we shall prove a fluctuation-dissipation theorem which gives a relation among the constant α, and three functions $\beta = \beta(t)$ $(t \geq 0), \gamma = \gamma(t,s)$ $(0 \leq s \leq t)$ and $\delta = \delta(t)$ $(t \geq 0)$ in the $\mathrm{KM_2O}$-Langevin equation (1.6). We will find that the fucnction $\delta = \delta(t)$ plays the same role in fluctuation-dissipation theorem as the partial autocorrelation function for stationary process with discrete time ([25], [30]–[32]). We denote such a quadruplet $(\alpha, \beta, \gamma, \delta)$ by $\mathcal{L}(\mathbf{X})$ and call it a $\mathrm{KM_2O}$-Langevin system associated with the stationary process $\mathbf{X}$.

In Section 7, we shall derive a system of equations for characterizing the $\mathrm{KM_2O}$-Langevin system $\mathcal{L}(\mathbf{X})$ from the covariance function R of the stationary process $\mathbf{X}$.

The author would like to express his gratitude to the referee for his valuable advice.

2. General Theory for $[\alpha, \beta, \gamma]$-Langevin Equation

Let $\mathbf{X}$ be any stationary solution of $[\alpha, \beta, \gamma]$-Langevin equation (1.2) with conditions (1.1) and (1.3)–(1.5). Let $R = R(t)$ be the covariance function of the stationary process $\mathbf{X}$:

$$R(t-s) \equiv E(X(t)X(s)) \qquad (t, s \in \mathbf{R}). \tag{2.1}$$

It follows from (1.1) that $\mathbf{X}$ is purely non-deterministic and so $\mathbf{X}$ has a spectral density function $\Delta = \Delta(\lambda)$ of Hardy class. We denote by $h = h(\zeta)$ the outer function of $\Delta = \Delta(\lambda)$:

$$h(\zeta) \equiv \exp\Big(\frac{1}{2\pi i} \int_{\mathbf{R}} \frac{1 + \lambda\zeta}{\lambda - \zeta} \frac{\log \Delta(\lambda)}{1 + \lambda^2} d\lambda\Big) \qquad (\zeta \in \mathbf{C}^+). \tag{2.2}$$

Since there exists a limit $h(\xi) \equiv \lim_{\eta \downarrow 0} h(\xi + i\eta)$ in $L^2(\mathbf{R})$, we can define a function $E = E(t)$ by the Fourier transform $\hat{h}$ of the L^2 function $h = h(\xi)$:

$$E(t) \equiv \hat{h}(t) = \int_{\mathbf{R}} e^{-it\xi} h(\xi) d\xi \qquad (t \in \mathbf{R}). \tag{2.3}$$

We note that

$$R(t) = \int_{\mathbf{R}} e^{-it\xi} \Delta(\xi) d\xi \qquad (t \in \mathbf{R}), \tag{2.4}$$

$$\Delta(\xi) = |h(\xi)|^2 \qquad (\xi \in \mathbf{R}), \tag{2.5}$$

$$E(t) = 0 \qquad (t < 0), \tag{2.6}$$

$$R(t) = \frac{1}{2\pi} \int_0^\infty E(|t| + s) E(s) ds \qquad (t \in \mathbf{R}). \tag{2.7}$$

It follows from Theorem 3.1 in [12] that the stochastic process $\mathbf{X}$ has the global forward canonical representation

$$X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t E(t-s) dB(s) \qquad (t \in \mathbf{R}) \tag{2.8}$$

and the outer function h can be represented by

$$h(\zeta) = \frac{\alpha}{\sqrt{2\pi}} \frac{1}{\beta - i\zeta + \hat{\gamma}(\zeta)} \qquad (\zeta \in \mathbf{C}^+ \cup \mathbf{R}), \tag{2.9}$$

where $\hat{\gamma}$ denotes the Fourier-Laplace transform of the bounded signed measure $\gamma(d\tau)$:

$$\hat{\gamma}(\zeta) \equiv \int_{(-\infty,0)} e^{-i\zeta\tau}\gamma(d\tau) \qquad (\zeta \in \mathbf{C}^+ \cup \mathbf{R}). \tag{2.10}$$

On the other hand, it follows from Proposition 2.2 in [14] that there exists a standard Brownian motion $\mathbf{B}_- = (B_-(t); t \in \mathbf{R})$ such that $\mathbf{X}$ has the global backward canonical representation

$$X(t) = \frac{1}{\sqrt{2\pi}}\int_t^\infty E(s-t)dB_-(s) \qquad (t \in \mathbf{R}), \tag{2.11}$$

$$\sigma(X(s); s \geq t) = \sigma(B_-(s_1) - B_-(s_2); s_1, s_2 \geq t) \qquad (t \in \mathbf{R}) \tag{2.12}$$

and $\mathbf{X}$ satisfies the following stochastic differential equation: for any t, s $(s < t)$

$$\begin{aligned} X(s) - X(t) \ &= \ -\int_s^t (\beta X(u) + \int_{(-\infty,0)} X(u-\tau)\gamma(d\tau))du \\ &\quad + \alpha(B_-(t) - B_-(s)). \end{aligned} \tag{2.13}$$

We note from the canonical representaion (2.8) (resp. (2.11)) with (1.1) (resp. (2.12)) that the function $E = E(t)$ is called the global canonical representation kernel. It follows from Propositions 3.1 and 3.2 in [12] that

(2.14) Re h is bounded and integrable in $\mathbf{R}$,

$$E(|t|) = 2\int_{\mathbf{R}} e^{-it\xi}(\mathrm{Re}\ h)(\xi)d\xi \qquad (t \in \mathbf{R}), \tag{2.15}$$

(2.16) E is bounded, continuous on $[0,\infty)$ and vanishes at infinity,

$$E(t) = \sqrt{2\pi}\alpha - \int_0^t (\beta E(s) + \int_{[-s,0)} E(s+\tau)\gamma(d\tau))ds \qquad (t \geq 0), \tag{2.17}$$

$$E(0) = \sqrt{2\pi}\alpha, \tag{2.18}$$

$$(D^+E)(t) = -\beta E(t) - \int_{[-t,0)} E(t+\tau)\gamma(d\tau) \qquad (t > 0), \tag{2.19}$$

$$(D^+E)(0+) = -\sqrt{2\pi}\alpha\beta, \tag{2.20}$$

where $(D^+E)(t)$ stands for the right derivative at t $(t > 0)$ of the function E.

Morerover, it follows from Proposition 3.4 in [12] that

$$(2.21)\quad R(t) = R(0) - \int_0^t (\beta R(s) + \int_{-\infty}^0 R(s+\tau)\gamma(d\tau))ds \qquad (t \geq 0),$$

$$(2.22)\quad R'(t) = -\beta R(t) - \int_{-\infty}^0 R(t+\tau)\gamma(d\tau) \qquad (t > 0),$$

$$(2.23)\quad R'(0+) = -\frac{\alpha^2}{2}.$$

We shall give two examples of stationary Gaussian processes.

Example 2.1. Let $\mathbf{X} = (X(t); t \in \mathbf{R})$ be a stationary Gaussian process with T-positivity whose covariance function R is given by the Laplace transform of a bounded Borel measure $\sigma(d\lambda)$ on $[0,\infty)$:

$$(2.24)\qquad R(t) = \int_{[0,\infty)} e^{-|t|\lambda}\sigma(d\lambda) \qquad (t \in \mathbf{R}).$$

We treat the case where the Borel measure $\sigma(d\lambda)$ satisfies the following conditions:

$$(2.25)\qquad \sigma(\{0\}) = 0 \quad \text{and} \quad \int_{[0,\infty)} (\lambda^{-1} + \lambda^2)\sigma(d\lambda) < \infty.$$

By Theorem 6.1 in [12], we see that there exist two positive constants α, β and a Borel measure $\mu(d\lambda)$ on $[0,\infty)$ such that

$$(2.26)\qquad h(\zeta) \;=\; \frac{\alpha}{\sqrt{2\pi}}\frac{1}{\beta - i\zeta - \hat{\gamma}(\zeta)} \qquad (\zeta \in \mathbf{C}^+ \cup \mathbf{R}),$$

$$(2.27)\qquad \gamma(d\tau) \;=\; (\int_{[0,\infty)} e^{\tau\lambda}\mu(d\lambda))d\tau,$$

where the Borel measure $\mu(d\lambda)$ satisfies the following conditions:

$$(2.28)\qquad \mu(\{0\}) = 0 \quad \text{and} \quad \int_{(0,\infty)} \lambda^{-1}\mu(d\lambda) < \beta.$$

Moreover, we note from Theorems 7.1 and 7.3 in [12] that there exists a bounded Borel measure $\nu(d\lambda)$ on $[0,\infty)$ such that

$$(2.29)\qquad E(t) = \chi_{[0,\infty)}(t)\int_0^\infty e^{-t\lambda}\nu(d\lambda),$$

$$(2.30)\qquad \nu(\{0\}) = 0 \quad \text{and} \quad \int_0^\infty (\lambda^{-1} + \lambda)\nu(d\lambda) < \infty,$$

$$(2.31)\qquad \sigma(d\lambda) = \frac{1}{2\pi}(\int_0^\infty \frac{1}{\lambda + \lambda'}\nu(d\lambda'))\nu(d\lambda).$$

For the future use in Section 4, we shall show two propositions.

PROPOSITION 2.1. *The following two conditions are equivalent each other*:

(i) $\int_0^\infty \lambda^3 \sigma(d\lambda) < \infty$;

(ii) $\int_0^\infty \lambda^2 \nu(d\lambda) < \infty$.

PROOF. It follows from (2.31) that

$$\begin{aligned}\int_0^\infty \lambda^3 \sigma(d\lambda) &= \frac{1}{4\pi}\int_0^\infty \int_0^\infty \frac{\lambda^3 + \lambda'^3}{\lambda + \lambda'} \nu(d\lambda')\nu(d\lambda) \\ &= \frac{1}{4\pi}\int_0^\infty \int_0^\infty (\lambda^2 - \lambda\lambda' + \lambda'^2)\nu(d\lambda')\nu(d\lambda).\end{aligned}$$

Therefore, noting from (2.30) that $\nu([0,\infty)) < \infty$ and $\int_0^\infty \lambda\nu(d\lambda) < \infty$, we see that (i) and (ii) are equivalent each other. $\square$

Next, we shall show

PROPOSITION 2.2. *The following three conditions are equivalent one another*:

(i) $\int_0^\infty \frac{1}{\lambda^2} \sigma(d\lambda) < \infty$;

(ii) $\int_0^\infty \frac{1}{\lambda^2} \nu(d\lambda) < \infty$;

(iii) $\int_0^\infty \frac{1}{\lambda^2} \mu(d\lambda) < \infty$.

PROOF. It follows from (7.6) in [12] that for any $\eta > 0$

$$h(i\eta) = \frac{1}{2\pi}\int_0^\infty \frac{1}{\eta + \lambda}\nu(d\lambda). \tag{2.32}$$

Differentiating the above equation with respect to η, we have

$$\frac{d}{d\eta}h(i\eta) = -\frac{1}{2\pi}\int_0^\infty \frac{1}{(\eta+\lambda)^2}\nu(d\lambda) \qquad (\eta > 0). \tag{2.33}$$

On the other hand, it follows from (2.26) that for any $\eta > 0$

$$h(i\eta) = \frac{\alpha}{\sqrt{2\pi}}\frac{1}{\beta+\eta-\int_0^\infty \frac{1}{\eta+\lambda}\mu(d\lambda)}. \tag{2.34}$$

Differentiating the above equation with respect to η, we have

$$\frac{d}{d\eta}h(i\eta) = -\frac{\alpha}{\sqrt{2\pi}}\frac{1+\int_0^\infty \frac{1}{(\eta+\lambda)^2}\mu(d\lambda)}{(\beta+\eta-\int_0^\infty \frac{1}{\eta+\lambda}\mu(d\lambda))^2} \qquad (\eta > 0). \tag{2.35}$$

First, we shall show that (i) and (ii) are equivalent each other. Let us assume that (i) holds. Since $\nu((0,\infty)) < \infty$, for the proof of (ii), we have only to prove that

$$\int_0^1 \frac{1}{\lambda^2}\nu(d\lambda) < \infty. \tag{2.36}$$

It follows from (2.31) that

$$\begin{aligned}
\int_0^\infty \frac{1}{\lambda^2}\sigma(d\lambda) &\geq \frac{1}{2\pi}\int_0^\infty\left(\int_0^1 \frac{1}{\lambda^2}\frac{1}{\lambda+\lambda'}\nu(d\lambda)\right)\nu(d\lambda') \\
&\geq \frac{1}{2\pi}\int_0^\infty\left(\int_0^1 \frac{1}{\lambda^2}\frac{1}{1+\lambda'}\nu(d\lambda)\right)\nu(d\lambda') \\
&= \frac{1}{2\pi}\left(\int_0^\infty \frac{1}{1+\lambda'}\nu(d\lambda')\right)\left(\int_0^1 \frac{1}{\lambda^2}\nu(d\lambda)\right).
\end{aligned}$$

Since $\nu((0,\infty)) > 0$, we see from (2.30) that $0 < \int_0^\infty \frac{1}{1+\lambda'}\nu(d\lambda') < \infty$. Therefore, the above inequality implies that (2.36) and so (ii) holds.

Let us assume that (ii) holds. Since $\int_0^1 \lambda^{-2}\nu(d\lambda) < \infty$, we see from (2.31) that

$$\begin{aligned}
\int_0^1 \frac{1}{\lambda^2}\sigma(d\lambda) &= \frac{1}{4\pi}\Big(\int_0^1\int_0^1 \frac{\lambda^2+\lambda'^2}{\lambda+\lambda'}\frac{1}{\lambda^2\lambda'^2}\nu(d\lambda')\nu(d\lambda) \\
&\quad + \int_0^1 \frac{1}{\lambda^2}\Big(\int_1^\infty \frac{1}{\lambda+\lambda'}\nu(d\lambda')\Big)\nu(d\lambda)\Big).
\end{aligned}$$

Since $\frac{\lambda^2+\lambda'^2}{\lambda+\lambda'} \le 1$ $(0 < \lambda, \lambda' \le 1)$ and $\frac{1}{\lambda+\lambda'} \le 1$ $(\lambda > 0, \lambda' \ge 1)$, we find that

$$\int_0^1 \frac{1}{\lambda^2}\sigma(d\lambda) \le \frac{1}{4\pi}\Big(\int_0^1 \frac{1}{\lambda^2}\nu(d\lambda)\Big)\Big(\int_0^1 \frac{1}{\lambda^2}\nu(d\lambda) + \nu([1,\infty))\Big) < \infty,$$

which with $\sigma((0,\infty)) < \infty$ implies that (i) holds.

Second, we shall show that (ii) and (iii) are equivalent each other. Let us assume that (ii) holds. We see from (2.33) that there exists a finite limit $\lim_{\eta\downarrow 0} \frac{d}{dt}h(i\eta)$ and it is given by

$$\lim_{\eta\downarrow 0} \frac{d}{dt}h(i\eta) = -\frac{1}{2\pi}\int_0^\infty \frac{1}{\lambda^2}\nu(d\lambda) \in \mathbf{R}. \tag{2.37}$$

Therefore, noting (2.25), we see from (2.35) that

$$\lim_{\eta\downarrow 0} \frac{d}{d\eta}h(i\eta) = -\frac{\alpha}{\sqrt{2\pi}}\frac{1 + \int_0^\infty \frac{1}{\lambda^2}\mu(d\lambda)}{(\beta - \int_0^\infty \frac{1}{\lambda}\mu(d\lambda))^2},$$

which with (2.28) and (2.37) implies that (iii) holds. Third, taking the procedure used above conversely, we can see that (iii) implies (ii). □

Remark 2.1. Let us consider the case where the measure $\sigma(d\lambda)$ in (2.24) is given by

$$\sigma(d\lambda) = \sum_{n=1}^{N} \sigma_n \delta_{p_n}, \tag{2.38}$$

where all σ_n $(1 \le n \le N)$ are positive and $0 < p_1 < p_2 < \cdots < p_N < \infty$. We see from Lemma 2.2 in [12] that the measure $\mu(d\lambda)$ in (2.27) has the following representation

$$\mu(d\lambda) = \sum_{n=1}^{N-1} \mu_n \delta_{q_n}(d\lambda), \tag{2.39}$$

where all μ_n $(1 \le n \le N-1)$ are positive and $0 < p_n < q_n < p_{n+1}$ $(1 \le n \le N-1)$. Therefore, all conditions in Propositions 2.1 and 2.2 hold.

Example 2.2. Let $\mathbf{X} = (X(t); t \in \mathbf{R})$ be a stationary Gaussian process with finite multiple Markovian property whose outer function h is given as follows:

$$h(\zeta) = \frac{\alpha}{\sqrt{2\pi}} \frac{Q(-\zeta)}{P(-\zeta)} \qquad (\zeta \in \mathbf{C}^+ \cup \mathbf{R}), \tag{2.40}$$

$$P(z) = \sum_{n=0}^{N} a_n z^n, \ a_n \in \mathbf{R} \ (0 \le n \le N-1), \ a_N = 1 \qquad (z \in \mathbf{C}), \tag{2.41}$$

$$Q(z) = \sum_{n=0}^{N-1} b_n z^n, \ b_n \in \mathbf{R} \ (0 \le n \le N-2), \ b_{N-1} = 1 \qquad (z \in \mathbf{C}), \tag{2.42}$$

$$V_P \subset \mathbf{C}^+, \ V_Q \subset \mathbf{C}^+, \ V_P \cap V_Q = \emptyset, \tag{2.43}$$

where V_P (resp. V_Q) denotes the set of zero points of the polynomial P (resp. Q).

By Lemma 2.1 and Proposition 2.1 in [13], we see that there exist a real constant β and a function $\gamma = \gamma(\tau) \in L^1((-\infty, 0)) \cap L^2((-\infty, 0))$ such that

$$h(\zeta) = \frac{\alpha}{\sqrt{2\pi}} \frac{1}{\beta - i\zeta - \hat{\gamma}(\zeta)} \qquad (\zeta \in \mathbf{C}^+ \cup \mathbf{R}), \tag{2.44}$$

where $\hat{\gamma}$ stands for the Fourier-Laplace transform of the L^1 function $\gamma = \gamma(\tau)$:

$$\hat{\gamma}(\zeta) = \int_{(-\infty,0)} e^{-i\zeta\tau} \gamma(\tau) d\tau \qquad (\zeta \in \mathbf{C}^+ \cup \mathbf{R}). \tag{2.45}$$

Defining a bounded signed Borel measure $\gamma(d\tau)$ by

$$\gamma(d\tau) \equiv \gamma(\tau) d\tau, \tag{2.46}$$

we find that the stochastic process $\mathbf{X}$ becomes a stationary solution of $[\alpha, \beta, \gamma]$-Langevin equation. We note from (2.13) and (2.15) in [13] that

$$\gamma = \gamma(\tau) \in C^1((-\infty, 0)) \cap C((-\infty, 0]), \tag{2.47}$$

$$|\tau|^n \gamma(\tau) \text{ is bounded in } (-\infty, 0) \text{ for any } n \in \mathbf{N}^*. \tag{2.48}$$

3. Innovation Process and Local Prediction Kernel

In this section, we shall treat the same stochastic process $\mathbf{X}$ as in Section 2 and recall the results of Sections 3, 4 and 5 in [14]. For each $a \in \mathbf{R}$, we define two reference families $(\mathbf{F}_a^+(t); t \geq 0)$ and $(\mathbf{F}_a^-(t); t \geq 0)$ by

$$
\begin{aligned}
&\mathbf{F}_a^+(t) \equiv \sigma(X(s); a \leq s \leq a+t) \quad \text{and} \\
&\mathbf{F}_a^-(t) \equiv \sigma(X(s); a-t \leq s \leq a).
\end{aligned} \tag{3.1}
$$

From Propositions 3.1 and 3.2 in [14], we can construct two standard Brownian motions $\nu_a^+ = (\nu_a^+(t); t \geq 0)$ and $\nu_a^- = (\nu_a^-(t); t \geq 0)$ such that for any $a \in \mathbf{R}$

$$\nu_a^{\pm}(0) = 0, \tag{3.2}$$

$$\nu_a^{\pm} \text{ is independent of } X(a), \tag{3.3}$$

$$\mathbf{F}_a^{\pm}(t) = \sigma(X(a), \nu_a^{\pm}(s); 0 \leq s \leq t) \qquad (t \geq 0), \tag{3.4}$$

$$
\begin{aligned}
X(a \pm t) - X(a) = -\int_0^t (\beta X(a \pm s) \\
+ \int_{-\infty}^0 E(X(a \pm s \pm \tau) | \mathbf{F}_a^{\pm}(s)) \gamma(d\tau)) ds \\
+ \alpha \nu_a^{\pm}(t) \qquad (t \geq 0).
\end{aligned} \tag{3.5}
$$

Defining a stochastic process $\mathbf{Y}_- = (Y_-(t); t \in \mathbf{R})$ and a function $P = P(t,s)$ $(0 \leq s \leq t < \infty)$ by

$$Y_-(t) \equiv \int_{(-\infty,0)} X(t+\tau) \gamma(d\tau), \tag{3.6}$$

$$
\begin{aligned}
P(t,s) \equiv\ & -\alpha^{-1} E(X(t)(Y_-(s) - E(Y_-(s) | \mathbf{F}_0^+(s)))) \\
& + \sqrt{2\pi}^{-1} E(t-s),
\end{aligned} \tag{3.7}
$$

we see from Lemmas 4.1, 4.2 and 4.4 in [14] that for any $a \in \mathbf{R}$

$$
\begin{aligned}
\nu_a^+(t) = -\alpha^{-1} \int_0^t (Y_-(a+s) - E(Y_-(a+s) | \mathbf{F}_a^+(s))) ds \\
+ B_+(a+t) - B_+(a),
\end{aligned} \tag{3.8}
$$

$$P(t,s) = \frac{\partial}{\partial s} E(X(a+t) \nu_a^+(s)), \tag{3.9}$$

$$P(t,s) = \frac{\partial}{\partial s} E(X(a-t) \nu_a^-(s)). \tag{3.10}$$

From Theorems 4.1 and 4.2 in [14], we have the local canonical representation for the process $\mathbf{X}$: for each $a \in \mathbf{R}$

$$X(a+t) = \Xi(t) X(a) + \int_0^t P(t,s) d\nu_a^+(s) \qquad (t \geq 0), \tag{3.11}$$

$$X(a-t) = \Xi(t)X(a) + \int_0^t P(t,s)d\nu_a^-(s) \qquad (t \geq 0), \tag{3.12}$$

where $\Xi = \Xi(t)$ $(t \in \mathbf{R})$ is the normalized covariance function of the stationary process $\mathbf{X}$ defined by

$$\Xi(t) \equiv R(t)R(0)^{-1}. \tag{3.13}$$

We call the function $P = P(t,s)$ the local canonical representation kernel associated with the stochastic process $\mathbf{X}$. It follows from Lemma 5.1 in [14] that

(3.14) the local canonical representation kernel P is bounded and continuous,

$$\sup_{0 \leq t < \infty} \int_0^t P(t,s)^2 ds < \infty. \tag{3.15}$$

Concerning the prediction formula for the problem predicting the future and the past from a bounded part of the stochastic process $\mathbf{X}$, we find from Theorem 4.3 in [14] that for each $a \in \mathbf{R}$

$$E(X(a+t)|\mathbf{F}_a^+(s)) = \Xi(t)X(a) + \int_0^s P(t,u)d\nu_a^+(u) \qquad (0 \leq s < t), \tag{3.16}$$

$$E(X(a-t)|\mathbf{F}_a^-(s)) = \Xi(t)X(a) + \int_0^s P(t,u)d\nu_a^-(u) \qquad (0 \leq s < t). \tag{3.17}$$

Taking account of the above formulae, therefore, we can call the function $P = P(t,s)$ the local forward prediction kernel associated with the stochastic process $\mathbf{X}$.

On the other hand, defining a stochastic process $\mathbf{Y}_+ = (Y_+(t); t \in \mathbf{R})$ and a function $Q = Q(t,s)$ $(0 \leq s, t < \infty)$ by

$$Y_+(t) \equiv \int_{(-\infty,0)} X(t-\tau)\gamma(d\tau), \tag{3.18}$$

$$Q(t,s) \equiv -\alpha^{-1}E(X(-t)(Y_-(s) - E(Y_-(s)|\mathbf{F}_0^+(s)))), \tag{3.19}$$

we see from Lemmas 4.3, 4.5 and 4.6 in [14] that for any $a \in \mathbf{R}$

$$(3.20) \qquad \nu_a^-(t) = -\alpha^{-1}\int_0^t (Y_+(a-s) - E(Y_+(a-s)|\mathbf{F}_a^-(s)))ds + B_-(a) - B_-(a-t),$$

$$(3.21) \qquad Q(t,s) = \frac{\partial}{\partial s}E(X(a+t)\nu_a^-(s)),$$

$$(3.22) \qquad Q(t,s) = \frac{\partial}{\partial s}E(X(a-t)\nu_a^+(s)).$$

From Theorem 4.4 in [14], we have another prediction formula for the problem predicting the future and the past from a bounded part of the stochastic process $\mathbf{X}$: for each $a \in \mathbf{R}$

$$(3.23) \qquad E(X(a+t)|\mathbf{F}_a^-(s)) = \Xi(t)X(a) + \int_0^s Q(t,u)d\nu_a^-(u) \qquad (t,s>0),$$

$$(3.24) \qquad E(X(a-t)|\mathbf{F}_a^+(s)) = \Xi(t)X(a) + \int_0^s Q(t,u)d\nu_a^+(u) \qquad (t,s>0).$$

We call the function $Q = Q(t,s)$ the local backward prediction kernel associated with the stochastic process $\mathbf{X}$. It follows from Lemma 5.1 in [14] that

(3.25) the local backward prediction kernel Q is bounded and continuous,

$$(3.26) \qquad \sup_{0\le t<\infty}\int_0^\infty Q(t,s)^2 ds < \infty.$$

4. Properties of Local Prediction Kernels P and Q

In this section, we shall treat any stationary solution $\mathbf{X}$ of $[\alpha,\beta,\gamma]$-Langevin equation (1.2) with conditions (1.1) and (1.3)–(1.5), and investigate the regularity properties of the local prediction kernels P and Q introduced in Section 3.

We define a function $\delta = \delta(t)$ $(t \ge 0)$ by

$$(4.1) \qquad \delta(t) \equiv \alpha^{-1}\int_{(-\infty,-t)} P(-\tau,t)\gamma(d\tau).$$

It follows from Lemma 6.3 in [14] that

$$\delta = \delta(t) \text{ is bounded on } [0,\infty), \tag{4.2}$$

$$\lim_{t\to\infty} \delta(t) = 0, \tag{4.3}$$

$$\int_0^\infty \delta(t)^2 dt < \infty. \tag{4.4}$$

From Lemmas 6.1 and 6.2 in [14], we have

$$Q(t,s) = -\int_s^{s+t} P(s+t,u)\delta(u)du \qquad (t,s\ge 0), \tag{4.5}$$

$$P(t,s) = -\int_s^\infty \delta(u)Q(t-s,u)du + \sqrt{2\pi}^{-1}E(t-s) \qquad (0\le s<t). \tag{4.6}$$

We shall give a complete proof of (4.6), because only a sketch of the proof of (4.6) was given in Lemma 6.1 in [14]. We fix any t,s $(0<s<t)$. It follows from (3.6) that

$$Y_-(s) - E(Y_-(s)|\mathbf{F}_0^+(s)) = \int_{(-\infty,-s)} (X(s+\tau) - E(X(s+\tau)|\mathbf{F}_0^+(s)))\gamma(d\tau).$$

Therefore, it follows from (3.7) that

$$\begin{aligned} P(t,s) \;=\; & -\frac{1}{\alpha}\int_{(-\infty,-s)} E(X(t)(X(s+\tau) - E(X(s+\tau)|\mathbf{F}_0^+(s))))\gamma(d\tau) \\ & + \frac{1}{\sqrt{2\pi}}E(t-s). \end{aligned}$$

Furthermore, for any τ $(\tau < -s)$, we see from (3.12) and (3.17) that $X(s+\tau) - E(X(s+\tau)|\mathbf{F}_0^+(s)) = \int_s^{-\tau} P(-\tau,u)d\nu_s^-(u)$. Since $\mathbf{F}_{s+\tau}^+(-\tau) = \mathbf{F}_s^-(-\tau)$, we see from (3.23) that

$$E(X(t)|\mathbf{F}_{s+\tau}^+(-\tau)) = \Xi(t-s)X(s) + \int_0^{-\tau} Q(t-s,u)d\nu_s^-(du).$$

Hence, noting that $\mathbf{F}_{s+\tau}^+(-\tau) \supset \mathbf{F}_0^+(s)$, we see that

$$\begin{aligned} & E(X(t)(X(s+\tau) - E(X(s+\tau)|\mathbf{F}_0^+(s)))) \\ = \;& E(E(X(t)|\mathbf{F}_{s+\tau}^+(-\tau))(X(s+\tau) - E(X(s+\tau)|\mathbf{F}_0^+(s)))) \\ = \;& \int_s^{-\tau} Q(t-s,u)P(-\tau,u)du. \end{aligned}$$

Thus, we see that $P(t,s) = -\alpha^{-1}\int_{-\infty}^{-s}(\int_s^{-\tau} P(-\tau,u)Q(t-s,u)du)\gamma(d\tau) + \sqrt{2\pi}^{-1}E(t-s)$. Noting (3.15) and (3.26), we can change the order of integration in the above equation to see from (4.1) that (4.6) holds.

By Theorem 5.5 and Lemma 6.5 in [14], we find that the partial derivative $P^{(1,0)}(t,s) \equiv \frac{\partial}{\partial t}P(t,s)$ $(0 < s < t)$ exists and it satisfies the following differential-integral equation:

$$\begin{aligned} (4.7) \quad P^{(1,0)}(t,s) = &-\beta P(t,s) - \int_{[s-t,0)} P(t+\tau,s)\gamma(d\tau) \\ &+ \int_s^\infty (\int_{(-\infty,s-t-v]} P(s-t-\tau,v)\gamma(d\tau))\delta(v)dv \\ &\qquad\qquad (0 \le s < t), \end{aligned}$$

$$(4.8) \quad P(s,s) = \alpha \quad (s \ge 0),$$

(4.9) $P^{(1,0)}(t,s)$ is bounded on the set $\{(t,s); 0 \le s < t < \infty\}$.

In the sequel, we shall treat the case where the bounded signed Borel measure $\gamma(d\tau)$ in (1.5) has a density function $\gamma = \gamma(\tau)$ on $(-\infty, 0)$ with the following conditions:

(4.10) $\gamma(d\tau) = \gamma(\tau)d\tau, \ \gamma(\tau) \in L^1((-\infty,0))$,

(4.11) $\gamma = \gamma(\tau)$ is continuous and has a finite limit $\lim_{\tau\uparrow 0}\gamma(\tau)$,

$$(4.12) \quad \int_{-\infty}^0 |\tau||\gamma(\tau)|d\tau < \infty.$$

Example 4.1. We shall show that if the measure $\sigma(d\lambda)$ in (2.24) satisfies the following conditions:

$$(4.13) \quad \int_0^\infty \lambda^3\sigma(d\lambda) < \infty,$$

$$(4.14) \quad \int_0^\infty \frac{1}{\lambda^2}\sigma(d\lambda) < \infty,$$

then the bounded signed Borel measure $\gamma(d\tau)$ in Example 2.1 satisfies conditions (4.10)–(4.12). First, we note from (2.27) that the bounded Borel measure $\gamma(d\tau)$ has a density funstion $\gamma = \gamma(\tau)$ $(\tau < 0)$ such that

$$\gamma(\tau) = \int_{[0,\infty)} e^{\tau\lambda}\mu(d\lambda) \qquad (\tau < 0)$$

and so the function $\gamma = \gamma(\tau)$ is continuous in $(0, \infty)$. Next, we shall show (4.11), that is, that the function $\gamma(\tau)$ has a finite limit $\lim_{\tau\uparrow 0} \gamma(\tau)$. Since it follows from (2.19) that $E'(t) = -\beta E(t) - \int_{-t}^{0} E(t+\tau)\gamma(\tau)d\tau$ for any $t > 0$, we can differentiate this with respect to t to see that for any $t > 0$

$$(4.15) \qquad E''(t) = -\beta E'(t) - \sqrt{2\pi}\alpha\gamma(-t) - \int_{-t}^{0} E'(t+\tau)\gamma(\tau)d\tau.$$

On the other hand, it follows from (2.29) and (4.13) that $E'(t), E''(t)$ $(t > 0)$ exist and they can be extended continuously on $[0, \infty)$. Therefore, we see from (4.15) that the function $\gamma = \gamma(\tau)$ $(\tau < 0)$ has a finite limit $\lim_{\tau\uparrow 0} \gamma(\tau)$.

Finally, we shall show (4.12). Since it follows that $\int_{-\infty}^{0} |\tau||\gamma(\tau)|d\tau = \int_{0}^{\infty} \lambda^{-2}\mu(d\lambda)$, we see from Proposition 2.2 that condition (4.12) holds.

Example 4.2. Let us consider Example 2.2. We find that conditions (4.10), (4.11) and (4.12) for the bounded signed Borel measure $\gamma(d\tau)$ in Example 2.2 come from (2.46)–(2.48), respectively.

Remark 4.1. In [14], we assumed the following condition (4.16) in place of condition (4.12):

$$(4.16) \qquad \int_{-\infty}^{0} \sqrt{|\tau|}|\gamma(\tau)|d\tau < \infty.$$

The point in [14] was that the integrability of the function $\delta = \delta(t)$ in (4.1) comes from condition (4.16).

We shall show that the statement noted in Remark 4.1 comes also from condition (4.12).

Lemma 4.1.(i) *$\delta = \delta(t)$ is continuous on $[0, \infty)$.*

(ii) *$\delta = \delta(t) \in L^1((0, \infty))$.*

Proof. (i) comes from (3.14), (4.1) and (4.10). It follows from (3.14) and (4.1) that

$$\begin{aligned} \int_{0}^{\infty} |\delta(t)|dt &\le \alpha^{-1} \int_{0}^{\infty} (\int_{-\infty}^{-t} |P(-\tau, t)||\gamma(\tau)|d\tau)dt \\ &= \alpha^{-1} \int_{-\infty}^{0} (\int_{0}^{-\tau} |P(-\tau, t)|dt)|\gamma(\tau)|d\tau \\ &\le \alpha^{-1} (\sup_{\tau<0, 0<t<-\tau} |P(-\tau, t)|) \int_{-\infty}^{0} |\tau||\gamma(\tau)|d\tau, \end{aligned}$$

which with (4.12) implies (ii) holds. □

Concerning the smoothness property of the global canonical representation kernel E in (2.3), we shall show

LEMMA 4.2. *The function* $E = E(t)$ $(t > 0)$ *is* C^2 *class. The first derivative* $E'(t)$ $(t > 0)$ *can be extended continuously and boundedly on* $[0, \infty)$ *and the second derivative* $E''(t)$ $(t > 0)$ *can be extended continuously on* $[0, \infty)$.

PROOF. Noting (2.16) and (4.10), we can differentiate equation (2.17) with respect to t to see that for any $t > 0$

$$E'(t) = -\beta E(t) - \int_{-t}^{0} E(t+\tau)\gamma(\tau)d\tau \tag{4.17}$$

and so the derivative $E'(t)$ $(t > 0)$ can be extended continuously and boundedly on $[0, \infty)$. Moreover, using condition (4.11), we can differentiate equation (4.17) with respect to t to see from (2.18) that $E''(t) = -\beta E'(t) - \sqrt{2\pi}\alpha\gamma(-t) - \int_{-t}^{0} E'(t+\tau)\gamma(\tau)d\tau$ and so the derivative $E''(t)$ $(t > 0)$ can be extended continuously on $[0, \infty)$. □

Concerning the smoothness property of the local prediction kernels P and Q, we shall show

LEMMA 4.3. (i) *The partial derivatives* $P^{(1,0)}(t,s)$, $P^{(0,1)}(t,s) \equiv \frac{\partial}{\partial s}P(t,s)$ $(0 < s < t)$ *exist and the function* $Q(t,s)$ $(t, s > 0)$ *is* C^1 *class.*

(ii) *The functions* $P^{(1,0)}(t,s), P^{(0,1)}(t,s)$ $(0 < s < t)$ *and the functions* $Q^{(1,0)}(t,s)$, $Q^{(0,1)}(t,s)$ $(t, s > 0)$ *can be extended continuously and boundedly on the set* $\{(t,s); 0 \le s \le t < \infty\}$ *and the set* $\{(t,s); 0 \le t, s < \infty\}$, *respectively.*

(iii) $P^{(1,0)}(t,s) + P^{(0,1)}(t,s) = -\delta(s)\int_{s}^{t} P(t,u)\delta(u)du \qquad (0 \le s < t).$

PROOF. We have already seen in (4.7) that the partial derivative $P^{(1,0)}(t,s)$ $(0 < s < t)$ exists. Furthermore, noting (3.14), (4.10) and Lemma 4.1(ii), we see from (4.7) that the function $P^{(1,0)}(t,s)$ $(0 < s < t)$ can be extended continuously and boundedly on the set $\{(t,s); 0 \le s \le t < \infty\}$.

Next, noting (3.14) and Lemma 4.1(i), we can differentiate equation (4.5) with respect to t or s to find from (4.8) that for any t, s $(t, s > 0)$, both the partial derivatives $Q^{(1,0)}(t,s)$ and $Q^{(0,1)}(t,s)$ exist and they satisfy

$$\begin{aligned} Q^{(1,0)}(t,s) &= -\alpha\delta(s+t) - \int_s^{s+t} P^{(1,0)}(s+t,u)\delta(u)du, \\ Q^{(0,1)}(t,s) &= -\alpha\delta(s+t) - \int_s^{s+t} P^{(1,0)}(s+t,u)\delta(u)du + P(s+t,s)\delta(s). \end{aligned}$$

Hence, it follows from (4.9) and Lemma 4.1(ii) that the function $Q(t,s)$ $(t, s > 0)$ is C^1 class and the partial derivatives $Q^{(1,0)}(t,s), Q^{(0,1)}(t,s)$ $(t, s > 0)$ can be extended continuously and boundedly on the set $\{(t,s); 0 \le t, s < \infty\}$.

Finally, from (4.2), Lemmas 4.1, 4.2 and the facts proved above, we can differentiate equation (4.6) with respect to s to see that the partial derivative $P^{(0,1)}(t,s)$ $(0 < s < t)$ exists and it satisfies

$$\begin{aligned} \text{(4.18)} \qquad P^{(0,1)}(t,s) &= \delta(s)Q(t-s,s) + \int_s^{\infty} \delta(u)Q^{(1,0)}(t-s,u)du \\ &\quad - \sqrt{2\pi}^{-1}E'(t-s). \end{aligned}$$

On the other hand, differentiating equation (4.6) with respect to t, we have

$$\text{(4.19)} \qquad P^{(1,0)}(t,s) = -\int_s^{\infty} \delta(u)Q^{(1,0)}(t-s)du + \sqrt{2\pi}^{-1}E'(t-s).$$

Hence, summing up (4.18) and (4.19), we see from (4.5) that (iii) holds. Using (iii) proved above, we see from (3.14) and Lemma 4.1 that the function $P^{(0,1)}(t,s)$ $(0 < s < t)$ can be extended continuously and boundedly on the set $\{(t,s); 0 \le s \le t < \infty\}$. Thus we have proved Lemma 4.3. □

Concerning the smoothness property of the function $\delta = \delta(t)$, we shall show

LEMMA 4.4. *The function* $\delta = \delta(t)$ $(t > 0)$ *is* C^1 *class and* $\delta' = \delta'(t)$ $(t > 0)$ *can be extended continuously on* $[0, \infty)$.

PROOF. Noting (3.14), (4.11) and Lemma 4.3(iii), we can differentiate equation (4.1) with respect to t to see from (4.8) that

$$\delta'(t) = -\gamma(-t) + \alpha^{-1}\int_{-\infty}^{-t} P^{(0,1)}(-\tau,t)\gamma(\tau)d\tau.$$

Therefore, it follows from (4.8), (4.10) and (4.11) and that $\delta' = \delta'(t)$ is continuous in $(0,\infty)$ and it can be extended continuously on $[0,\infty)$. $\square$

Finally, we shall show

LEMMA 4.5. (i) *The partial derivatives* $P^{(2,0)}(t,s) \equiv \frac{\partial}{\partial t}P^{(1,0)}(t,s)$ *and* $P^{(0,2)}(t,s) \equiv \frac{\partial}{\partial s}P^{(0,1)}(t,s)$ *exist. Furthermore, the partial derivatives* $\frac{\partial}{\partial s}P^{(1,0)}(t,s), \frac{\partial}{\partial t}P^{(0,1)}(t,s)$ $(0 < s < t)$ *exist and they are equal each other, which is denoted by* $P^{(1,1)}(t,s)$.

(ii) *The functions* $P^{(2,0)}(t,s), P^{(0,2)}(t,s)$ *and* $P^{(1,1)}(t,s)$ *can be also extended continuously on the set* $\{(t,s); 0 \le s \le t < \infty\}$.

PROOF. By (4.10), we can rewrite equation in (4.7) into the following equation: for any t, s $(0 < s < t)$

$$P^{(1,0)}(t,s) = -\beta P(t,s) - \varphi(t,s) + \int_{-\infty}^{-t} \psi(s, s-t-\tau)\gamma(\tau)d\tau, \tag{4.20}$$

where

$$\varphi(t,s) \equiv \int_{s-t}^{0} P(t+\tau, s)\gamma(\tau)d\tau, \tag{4.21}$$

$$\psi(s,t) \equiv \int_{s}^{t} P(t,\tau)\delta(\tau)d\tau. \tag{4.22}$$

First, we shall show the existence of the partial derivative $P^{(2,0)}(t,s)$ in (i). It follows from (4.8), (4.11) and Lemma 4.3(ii) that the partial derivative $\varphi^{(1,0)}(t,s)$ $(0 < s < t)$ exists and it is given by

$$\varphi^{(1,0)}(t,s) = \int_{s-t}^{0} P^{(1,0)}(t+\tau, s)\gamma(\tau)d\tau + \alpha\gamma(s-t). \tag{4.23}$$

On the other hand, it follows form (4.9), Lemmas 4.1(i) and 4.3(ii) that the partial derivative $\psi^{(0,1)}(s,u)$ $(0 < s < u)$ exists and it is given by

$$\psi^{(0,1)}(s,u) = \alpha\delta(u) + \int_{s}^{u} P^{(1,0)}(u,v)\delta(v)dv. \tag{4.24}$$

Therefore, using (4.10) and (4.11), we can differentiate the third term of the right-hand side of (4.20) with respect to t to see from (4.22) and (4.24) that

$$\begin{aligned}(4.25)\qquad &\frac{\partial}{\partial t}\int_{-\infty}^{-t}\psi(s,s-t-\tau)\gamma(\tau)d\tau \\ &= -\psi(s,s)\gamma(-t) - \int_{-\infty}^{-t}\psi^{(0,1)}(s,s-t-\tau)\gamma(\tau)d\tau \\ &= -\alpha\int_{-\infty}^{-t}\delta(s-t-\tau)\gamma(\tau)d\tau \\ &\quad - \int_{-\infty}^{-t}(\int_{s}^{s-t-\tau}P^{(1,0)}(s-t-\tau,v)\delta(v)dv)\gamma(\tau)d\tau.\end{aligned}$$

Hence, by Lemma 4.3(ii) and (4.23)–(4.25), we can differentiate equation (4.20) with respect to t to see that for any t, s $(0 < s < t)$

$$\begin{aligned}(4.26)\quad P^{(2,0)}(t,s) \;=\; &-\beta P^{(1,0)}(t,s) - \int_{s-t}^{0}P^{(1,0)}(t+\tau,s)\gamma(\tau)d\tau \\ &-\alpha\gamma(s-t) - \alpha\int_{-\infty}^{-t}\delta(s-t-\tau)\gamma(\tau)d\tau \\ &- \int_{-\infty}^{-t}(\int_{s}^{s-t-\tau}P^{(1,0)}(s-t-\tau,v)\delta(v)dv)\gamma(\tau)d\tau.\end{aligned}$$

In particular, we see from (4.2), (4.10), (4.11), Lemmas 4.1 and 4.3(ii) that the statement concerning $P^{(2,0)}(t,s)$ in (ii) holds.

Second, using (4.20) again, we shall show that the partial derivative $\frac{\partial}{\partial s}P^{(1,0)}(t,s)$ $(0 < s < t)$ exists and it is given by

$$\begin{aligned}(4.27)\qquad &\frac{\partial}{\partial s}P^{(1,0)}(t,s) \\ &= \beta P^{(0,1)}(t,s) + \int_{s-t}^{0}P^{(0,1)}(t+\tau,s)\gamma(\tau)d\tau + \alpha\gamma(s-t) \\ &\quad +\alpha\int_{-\infty}^{-t}\delta(s-t-\tau)\gamma(\tau)d\tau \\ &\quad + \int_{-\infty}^{-t}(\int_{s}^{s-t-\tau}P^{(1,0)}(s-t-\tau,v)\delta(v)dv)\gamma(\tau)d\tau \\ &\quad -\delta(s)(\int_{-\infty}^{-t}P(s-t-\tau,s)\gamma(\tau)d\tau - \beta\int_{s}^{t}P(t,u)\delta(u)du \\ &\qquad\qquad - \int_{s-t}^{0}(\int_{s}^{t+\tau}P(t+\tau,u)\delta(u)du)\gamma(\tau)d\tau).\end{aligned}$$

It follows from (4.8), (4.11) and Lemma 4.3(ii) that the partial derivative $\varphi^{(0,1)}(t,s)$ $(0<s<t)$ exists and it is given by

$$\varphi^{(0,1)}(t,s)=\int_{s-t}^{0}P^{(0,1)}(t+\tau,s)\gamma(\tau)d\tau-\alpha\gamma(s-t). \tag{4.28}$$

Moreover, it follows from (3.14) and Lemma 4.1(i) that the partial derivative $\psi^{(1,0)}(s,u)$ $(0<s<u)$ exists and it is given by

$$\psi^{(1,0)}(s,u)=-P(u,s)\delta(s). \tag{4.29}$$

Therefore, using (4.10) and (4.11), we can differentiate the third term of the right-hand side of (4.20) with respect to s to find from (4.22) and (4.24) that

$$\begin{aligned}
&\frac{\partial}{\partial s}\int_{-\infty}^{-t}\psi(s,s-t-\tau)\gamma(\tau)d\tau \\
&= \int_{-\infty}^{-t}\psi^{(1,0)}(s,s-t-\tau)\gamma(\tau)d\tau+\int_{-\infty}^{-t}\psi^{(0,1)}(s,s-t-\tau)\gamma(\tau)d\tau \\
&= -\delta(s)\int_{-\infty}^{-t}P(s-t-\tau,s)\gamma(\tau)d\tau+\alpha\int_{-\infty}^{-t}\delta(s-t-\tau)\gamma(\tau)d\tau \\
&\qquad +\int_{-\infty}^{-t}(\int_{s}^{s-t-\tau}P^{(1,0)}(s-t-\tau,v)\delta(v)dv)\gamma(\tau)d\tau.
\end{aligned} \tag{4.30}$$

Hence, by Lemma 4.3(ii) and (4.28)–(4.30), we can differentiate equation (4.20) with respect to s to find that for any t,s $(0<s<t)$

$$\begin{aligned}
\frac{\partial}{\partial s}P^{(1,0)}(t,s) &= -\beta P^{(0,1)}(t,s)-\int_{s-t}^{0}P^{(0,1)}(t+\tau,s)\gamma(\tau)d\tau+\alpha\gamma(s-t) \\
&\quad -\delta(s)\int_{-\infty}^{-t}P(s-t-\tau,s)\gamma(\tau)d\tau \\
&\quad +\alpha\int_{-\infty}^{-t}\delta(s-t-\tau)\gamma(\tau)d\tau \\
&\quad +\int_{-\infty}^{-t}(\int_{s}^{s-t-\tau}P^{(1,0)}(s-t-\tau,v)\delta(v)dv)\gamma(\tau)d\tau.
\end{aligned}$$

Moreover, by Lemma 4.3(iii), we can see that that (4.27) holds.

Third, we shall show that the partial derivative $\frac{\partial}{\partial t}P^{(0,1)}(t,s)$ $(0<s<t)$ exists and it is given by

$$
\begin{aligned}
(4.31)\quad \frac{\partial}{\partial t}P^{(0,1)}(t,s) \;=\;& \beta P^{(1,0)}(t,s) + \int_{s-t}^{0} P^{(1,0)}(t+\tau,s)\gamma(\tau)d\tau \\
& + \alpha\gamma(s-t) + \alpha\int_{-\infty}^{-t} \delta(s-t-\tau)\gamma(\tau)d\tau \\
& + \int_{-\infty}^{-t}\Big(\int_{s}^{s-t-\tau} P^{(1,0)}(s-t-\tau,v)\delta(v)dv\Big)\gamma(\tau)d\tau \\
& - \delta(s)\Big(\alpha\delta(t) + \int_{s}^{t} P^{(1,0)}(t,u)\delta(u)du\Big).
\end{aligned}
$$

Since we have proved that the function $P^{(2,0)}(t,s)$ $(0<s<t)$ can be extended continuously on the set $\{(t,s); 0\le s\le t<\infty\}$, we can differentiate the equation in Lemma 4.3(iii) with respect to t to see from (3.14), (4.8) and Lemma 4.1(i) that the partial derivative $\frac{\partial}{\partial t}P^{(0,1)}(t,s)$ $(0<s<t)$ exists and it is given by

$$
\begin{aligned}
& \frac{\partial}{\partial t}P^{(0,1)}(t,s) \\
=\;& -P^{(2,0)}(t,s) - \alpha\delta(s)\delta(t) - \delta(s)\int_{s}^{t} P^{(1,0)}(t,u)\delta(u)du \qquad (0<s<t).
\end{aligned}
$$

Substituting (4.26) into the above equation, we find that (4.31) holds.

Fourth, we shall show that both the partial derivatives $\frac{\partial}{\partial s}P^{(1,0)}(t,s)$ and $\frac{\partial}{\partial t}P^{(0,1)}(t,s)$ $(0<s<t)$ are equal each other. For that purpose, noting (4.27) and (4.31), we have only to prove for any t,s $(0<s<t)$

$$
\begin{aligned}
(4.32)\qquad & \alpha\delta(t) + \int_{s}^{t} P^{(1,0)}(t,u)\delta(u)du \\
&= \int_{-\infty}^{-t} P(s-t-\tau,s)\gamma(\tau)d\tau - \beta\int_{s}^{t} P(t,u)\delta(u)du \\
&\quad - \int_{s-t}^{0}\Big(\int_{s}^{t+\tau} P(t+\tau,u)\delta(u)du\Big)\gamma(\tau)d\tau.
\end{aligned}
$$

It follows from (4.20) that for any t,s $(0<s<t)$

$$
(4.33)\qquad \int_{s}^{t} P^{(1,0)}(t,u)\delta(u)du = -\beta\int_{s}^{t} P(t,u)\delta(u)du - \mathrm{I} - \mathrm{II},
$$

where

$$\mathrm{I} \equiv \int_s^t (\int_{u-t}^0 P(t+\tau,u)\gamma(\tau)d\tau)\delta(u)du,$$
$$\mathrm{II} \equiv \int_s^t (\int_{-\infty}^{-t} (\int_u^{u-t-\tau} P(u-t-\tau,v)\delta(v)dv)\gamma(\tau)d\tau)\delta(u)du.$$

Changing the order of integration, we can see that

$$(4.34)\quad \mathrm{I} = \int_{s-t}^0 (\int_s^{\tau+t} P(t+\tau,u)\delta(u)du)\gamma(\tau)d\tau,$$
$$\mathrm{II} = \int_{-\infty}^{-t} (\int_s^t (\int_u^{u-t-\tau} P(u-t-\tau,v)\delta(v)dv)\delta(u)du)\gamma(\tau)d\tau.$$

Using (4.5), (4.6) and noting (4.1), we have

$$\begin{aligned}(4.35)\quad \mathrm{II} &= -\int_{-\infty}^{-t} (\int_s^t Q(-t-\tau,u)\delta(u)du)\gamma(\tau)d\tau \\ &= -\int_{-\infty}^{-t} (\int_s^\infty Q(-t-\tau,u)\delta(u)du \\ &\qquad -\int_t^\infty Q(-t-\tau,u)\delta(u)du)\gamma(\tau)d\tau \\ &= -\int_{-\infty}^{-t} (-P(-t-\tau+s,s) + \sqrt{2\pi}^{-1}E(-t-\tau) \\ &\qquad -(-P(-\tau,t) + \sqrt{2\pi}^{-1}E(-t-\tau))\gamma(\tau)d\tau \\ &= -\alpha\delta(t) + \int_{-\infty}^{-t} P(-t-\tau+s,s)\gamma(\tau)d\tau.\end{aligned}$$

Hence, we see from (4.33)–(4.35) that $\frac{\partial}{\partial s}P^{(1,0)}(t,s) = \frac{\partial}{\partial t}P^{(0,1)}(t,s)$ $(0 < s < t)$. We denote it by $P^{(1,1)}(t,s)$ $(0 < s < t)$. Noting (4.27), we see from (4.2), (4.10), (4.11), Lemmas 4.1 and 4.3(iii) that the function $P^{(1,1)}(t,s)$ $(0 < s < t)$ can be extended continuously on the set $\{(t,s); 0 \le s \le t < \infty\}$.

Furthermore, by Lemma 4.4, we can differentiate the equation in Lemma 4.3(iii) with respect to s to find that the derivative $P^{(0,2)}(t,s)$ $(0 < s < t)$ exists and it is given by $P^{(0,2)}(t,s) = -P^{(1,1)}(t,s) - \delta'(s)\int_s^t P(t,u)\delta(u)du + \delta(s)^2 P(t,s)$. Hence, we see from (3.14), Lemmas 4.1 and 4.4 that the function $P^{(0,2)}(t,s)$ $(0 < s < t)$ can be extended continuously on the set $\{(t,s); 0 \le s \le t < \infty\}$. Thus, we have proved (ii). $\square$

5. KM_2O-Langevin Equation with Continuous Time

In this section, we shall treat any stationary solution $\mathbf{X}$ of $[\alpha, \beta, \gamma]$-Langevin equation (1.2) with conditions (1.1), (1.3), (1.4) and (4.10)–(4.12), and derive a KM_2O-Langevin equation with continuous time which governs a local time evolution of the stochastic process $\mathbf{X}$.

By virtue of (4.4), we can define a function $\beta = \beta(t)$ $(t \geq 0)$ by

$$\beta(t) \equiv \beta - \int_t^\infty \delta(s)^2 ds \qquad (t \geq 0). \tag{5.1}$$

We shall show

LEMMA 5.1. *There exists a unique continuous function $\gamma = \gamma(t, s)$ defined on the set $\{(t, s); 0 \leq s \leq t < \infty\}$ such that*

$$P^{(1,0)}(t, s) = -\beta(t)P(t, s) - \int_s^t \gamma(t, u)P(u, s)du \qquad (0 < s < t). \tag{5.2}$$

PROOF. First, we shall rewrite equation (5.2) into an integral equation of Volterra type. For that purpose, we shall assume that there exists a continuous function $\gamma = \gamma(t, s)$ defined on the set $\{(t, s); 0 \leq s \leq t < \infty\}$ satisfying equation (5.2). By Lemmas 4.3(ii) and 4.5(ii), we can differentiate equation (5.2) with respect to s to find that

$$\begin{aligned} & \gamma(t, s) - \int_s^t \gamma(t, u)(\alpha^{-1}P^{(0,1)}(u, s))du \\ = \ & \alpha^{-1}(P^{(1,1)}(t, s) + \beta(t)P^{(0,1)}(t, s)). \end{aligned} \tag{5.3}$$

We define a continuous kernel $K = K(u, s)$ $(0 \leq s \leq u < \infty)$ and for each $t > 0$, a continuous function $f_t = f_t(s)$ $(0 \leq s \leq t)$ by

$$K(u, s) \equiv \alpha^{-1}P^{(0,1)}(u, s) \qquad (0 \leq s \leq u < \infty), \tag{5.4}$$

$$f_t(s) \equiv \alpha^{-1}(P^{(1,1)}(t, s) + \beta(t)P^{(0,1)}(t, s)) \qquad (0 \leq s \leq t). \tag{5.5}$$

It folllows that (5.3) can be rewritten into

$$\gamma(t, s) - \int_s^t K(u, s)\gamma(t, u)du = f_t(s) \qquad (0 \leq s \leq t). \tag{5.6}$$

Therefore, applying a general theory of integral equations of Volterra type to equation (5.6) for each $t > 0$, we can obtain a resolvent kernel $C = C(u,s)$ defined on the set $\{(u,s); 0 \le s \le u < \infty\}$ to solve equation (5.6) as follows: for any $t > 0$

$$\gamma(t,s) = f_t(s) + \int_s^t C(u,s) f_t(u) du, \tag{5.7}$$

where the resolvent kernel $C = C(u,s)$ is defined by

$$C(u,s) \equiv \sum_{n=1}^{\infty} K^{(n)}(u,s), \tag{5.8}$$

$$K^{(1)}(u,s) \equiv K(u,s), \tag{5.9}$$

$$K^{(n)}(u,s) \equiv \int_s^u K^{(n-1)}(v,s) K(v,s) dv \qquad (n \ge 2). \tag{5.10}$$

Since the series in (5.8) converges uniformly on the set $\{(u,s); 0 \le s \le u \le t\}$ for any $t > 0$, we see that the resolvent kernel $C = C(u,s)$ is continuous on the set $\{(u,s); 0 \le s \le u < \infty\}$. It follows from (5.7) that a continuous function $\gamma = \gamma(t,s)$ satisfying equation (5.3) is uniquely determined.

Conversely, we shall show that the continuous function $\gamma = \gamma(t,s)$ defined by the right-hand side of (5.7) satisfies equation (5.2). Let us fix any $t > 0$. It follows from the relation between the kernel $K = K(u,s)$ and its resolvent kernel $C = C(u,s)$ that the function $\gamma = \gamma(t,s)$ satisfies equation (5.6) and so

$$\frac{\partial}{\partial s}(P^{(1,0)}(t,s) + \beta(t)P(t,s) + \int_s^t \gamma(t,u)P(u,s)du) = 0 \qquad (0 < s < t).$$

On the other hand, it follows from (4.1), (4.7) and (5.1) that $P^{(1,0)}(t,t) + \beta(t)P(t,t) = 0$. Hence, the continuous function $\gamma = \gamma(t,s)$ satisfies equation (5.2). $\square$

Moreover, we shall show

LEMMA 5.2. *The stochastic process* $\mathbf{X}$ *satisfies the following stochastic differential equation*:

$$\begin{aligned} X(t) - X(0) \ = \ & -\int_0^t (\beta(s)X(s) + \int_0^s \gamma(s,u)X(u)du + \delta(s)X(0))ds \\ & + \alpha\nu_0^+(t) \quad (t \ge 0). \end{aligned}$$

PROOF. It follows from (3.11) and (4.8) that for any $t \geq 0$

$$
\begin{aligned}
(5.11) \qquad X(t) &= \Xi(t)X(0) + \int_0^t (P(t,s) - P(s,s))d\nu_0^+(s) + \alpha\nu_0^+(t) \\
&= \Xi(t)X(0) + \int_0^t (\int_0^u P^{(1,0)}(u,s)d\nu_0^+(s))du + \alpha\nu_0^+(t).
\end{aligned}
$$

Using Lemma 5.1, we find that for any $u \geq 0$

$$
\begin{aligned}
\int_0^u P^{(1,0)}(u,s)d\nu_0^+(s) &= -\beta(u)\int_0^u P(u,s)d\nu_0^+(s) \\
&\quad - \int_0^u \gamma(u,v)(\int_0^v P(v,s)d\nu_0^+(s))dv.
\end{aligned}
$$

Therefore, using (3.11) again, we see that for any t, u $(0 \leq u \leq t)$

$$
\begin{aligned}
\int_0^u P^{(1,0)}(u,s)d\nu_0^+(s) &= -(\beta(u)X(u) + \int_0^u \gamma(u,v)X(v)dv) \\
&\quad +(\beta(u)\Xi(u) + \int_0^u \gamma(u,v)\Xi(v)dv)X(0).
\end{aligned}
$$

Substituting this into the second term of the right-hand side of (5.11), we find that for any $t \geq 0$

$$
\begin{aligned}
X(t) &= -\int_0^t (\beta(u)X(u) + \int_0^u \gamma(u,v)X(v)dv)du \\
&\quad +(\Xi(t) + \int_0^t (\beta(u)\Xi(u) + \int_0^u \gamma(u,v)\Xi(v)dv)du)X(0) + \alpha\nu_0^+(t).
\end{aligned}
$$

Hence, defining a function $\tilde{\delta} = \tilde{\delta}(t)$ $(t \geq 0)$ by

$$
(5.12) \qquad \tilde{\delta}(t) \equiv -(\Xi'(t) + \beta(t)\Xi(t) + \int_0^t \gamma(t,v)\Xi(v)dv),
$$

we see that for any $t \geq 0$

$$
\begin{aligned}
(5.13) \qquad X(t) - X(0) &= -\int_0^t (\beta(s)X(s) + \int_0^s \gamma(s,u)X(u)du)ds \\
&\quad - \int_0^t \tilde{\delta}(s)X(0)ds + \alpha\nu_0^+(t).
\end{aligned}
$$

Therefore, in order to complete the proof of Lemma 5.2, we have only to show

$$
(5.14) \qquad \delta(t) = \tilde{\delta}(t) \qquad (t \geq 0).
$$

For that purpose, we use the backward innovations ν_a^- ($a \in \mathbf{R}$) stated in Section 3. It follows from (3.5) that for any $t \geq 0$

$$X(t) - X(0) = -\int_0^t (\beta X(s) + \int_{-\infty}^0 E(X(s+\tau)|\mathbf{F}_0^+(s))\gamma(d\tau))ds + \alpha\nu_0^+(t). \tag{5.15}$$

Since for any $s \geq 0$

$$\int_{-\infty}^0 E(X(s+\tau)|\mathbf{F}_0^+(s))\gamma(d\tau) = \int_{-s}^0 X(s+\tau)\gamma(d\tau) + \int_{-\infty}^{-s} E(X(s+\tau)|\mathbf{F}_0^+(s))\gamma(d\tau),$$

we can apply (3.17) to the second term of the right-hand side of the above equation to get

$$\begin{aligned} &\int_{-\infty}^0 E(X(s+\tau)|\mathbf{F}_0^+(s))\gamma(d\tau) \\ &= \int_{-s}^0 X(s+\tau)\gamma(d\tau) + (\int_{-\infty}^{-s} \Xi(-\tau)\gamma(d\tau))X(s) \\ &\quad + \int_0^s (\int_{-\infty}^{-s} P(-\tau, u)\gamma(d\tau))d\nu_s^-(u). \end{aligned} \tag{5.16}$$

Taking account of the causality relation in (3.4), we shall construct three continuous functions $c(s), d(s), e(s,u)$ ($0 \leq u \leq s$) such that for any $s \geq 0$

$$\int_0^s (\int_{-\infty}^{-s} P(-\tau, u)\gamma(d\tau))d\nu_s^-(u) = c(s)X(0) + d(s)X(s) + \int_0^s e(s,u)X(u)du. \tag{5.17}$$

Since it follows from Theorem 3.1(ii) that

$$\begin{aligned} &\text{the right-hand side of (5.17)} \\ &= (c(s)\Xi(s) + d(s) + \int_0^s e(s,u)\Xi(s-u)du)X(s) \\ &\quad + \int_0^s (c(s)P(s,v) + \int_0^{s-v} P(s-u,v)e(s,u)du)d\nu_s^-(v), \end{aligned}$$

we can reduce the problem of the existence of such functions $c(s), d(s)$, $e(s,u)$ $(0 \leq u \leq s)$ satisfying (5.17) to that of the existence of three continuous functions $c(s), d(s), e(s,u)$ $(0 \leq u \leq s)$ such that for any s, v $(0 < v < s)$

$$c(s)\Xi(s) + d(s) + \int_0^s e(s,u)\Xi(s-u)du = 0, \tag{5.18}$$

$$\int_{-\infty}^{-s} P(-\tau, v)\gamma(d\tau) = c(s)P(s,v) + \int_0^{s-v} P(s-u,v)e(s,u)du. \tag{5.19}$$

First, let us assume the existence of three continuous functions $c(s), d(s)$, $e(s,u)$ $(0 \leq u \leq s)$ satisfying (5.18) and (5.19). We note from (5.18) that the existence of the function $d(s)$ comes from that of functions $c(s)$ and $e(s,u)$. Letting v tend to s in (5.19), we see from (4.1) and (4.8) that

$$c(s) = \delta(s) \qquad (s \geq 0). \tag{5.20}$$

Differentiating equation (5.19) with respect to v, we have

$$\begin{aligned}\int_{-\infty}^{-s} P^{(0,1)}(-\tau, v)\gamma(d\tau) \ &= \ c(s)P^{(0,1)}(s,v) - \alpha e(s, s-v) \\ &\quad + \int_v^s P^{(0,1)}(z,v)e(s,s-z)dz.\end{aligned}$$

Hence

$$\begin{aligned}& e(s, s-v) - \int_v^s \alpha^{-1}P^{(0,1)}(z,v)e(s,s-z)dz \\ &= \alpha^{-1}(c(s)P^{(0,1)}(s,v) - \int_{-\infty}^{-s} P^{(0,1)}(-\tau,v)\gamma(d\tau)).\end{aligned} \tag{5.21}$$

Therefore, using the same method which was used to solve equation (5.3), we find that a continuous function $e(s,z)$ $(0 \leq z \leq s)$ satisfying (5.21) exists uniquely.

Next, we show that the functions $c(s)$ and $e(s,u)$ determined by (5.20) and (5.21) satisfy equation (5.19). It follows from (5.20) and (5.21) that for any s, v $(0 < v < s)$

$$\frac{\partial}{\partial v}(\int_{-\infty}^{-s} P(-\tau, v)\gamma(d\tau) - c(s)P(s,v) - \int_0^{s-v} P(s-u,v)e(s,u)du) = 0,$$

$$\int_{-\infty}^{-s} P(-\tau, v)\gamma(d\tau) - c(s)P(s,s) = 0.$$

Hence, we find that two functions $c(s)$ and $e(s,u)$ satisfy equation (5.19). Therefore, it follows from (5.15)–(5.17) and (5.20) that

$$\begin{aligned} X(t)-X(0) &= -\int_0^t \{\beta X(s)+\int_{-s}^0 X(s+\tau)\gamma(\tau)d\tau \\ &\qquad +(\int_{-\infty}^{-s}\Xi(-\tau)\gamma(d\tau))X(s)+d(s)X(s)+\delta(s)X(0) \\ &\qquad +\int_0^s e(s,u)X(u)du\}ds+\alpha\nu_0^+(t) \\ &= -\int_0^t\{(\beta+\int_{-\infty}^{-s}\Xi(-\tau)\gamma(d\tau)+d(s))X(s) \\ &\qquad +\int_0^s(\gamma(u-s)+e(s,u))X(u)du+\delta(s)X(0)\}ds \\ &\quad +\alpha\nu_0^+(t). \end{aligned}$$

Consequently, comparing the above equation with equation (5.13), we find that

$$\beta(s) = \beta+\int_{-\infty}^{-s}\Xi(-\tau)\gamma(d\tau)+d(s) \qquad (s\geq 0), \tag{5.22}$$

$$\gamma(s,u) = \gamma(u-s)+e(s,u) \qquad (0\leq u\leq s). \tag{5.23}$$

In particular, it follows from (5.23) that Lemma 5.2 holds. □

Using (4.8) and Lemma 5.1, we have

LEMMA 5.3. $P^{(1,0)}(t,t)=-\alpha\beta(t) \qquad (t\geq 0).$

We define a set $\mathcal{L}$ by

$$\begin{aligned} \mathcal{L} \equiv \{(\alpha,\beta,\gamma,\delta) \mid\ & \alpha,\beta,\gamma \text{ and } \delta \text{ satisfy the following conditions;} \\ & \text{(i)} \quad \alpha>0, \\ & \text{(ii)} \quad \beta=\beta(t) \text{ is continuous on } [0,\infty) \text{ and } C^1\text{-class in } (0,\infty), \\ & \text{(iii)} \quad \gamma=\gamma(t,s) \text{ is continuous on } \{(t,s); 0\leq s\leq t<\infty\}, \\ & \text{(iv)} \quad \delta=\delta(t) \text{ is bounded and continuous on } [0,\infty), \\ & \qquad C^1\text{-class in } (0,\infty) \text{ and } \delta'=\delta'(t) \\ & \qquad \text{can be extended continuously on } [0,\infty)\}. \end{aligned} \tag{5.24}$$

We shall show one of main theorems in this paper.

THEOREM 5.1. *There exists a unique element $(\alpha, \beta, \gamma, \delta)$ of $\mathcal{L}$ such that the stochastic process* $\mathbf{X}$ *satisfies the following stochastic differential equation*: for any $t \geq 0$

$$(5.25)\qquad \begin{aligned} & X(t) - X(0) \\ = & -\int_0^t (\beta(s)X(s) + \int_0^s \gamma(s,u)X(u)du + \delta(s)X(0))ds + \alpha\nu_0^+(t). \end{aligned}$$

PROOF. From (4.2), (4.4), (5.1), Lemmas 4.1, 4.4, 5.1 and 5.2, we can prove the existence of an element $(\alpha, \beta, \gamma, \delta)$ of the set $\mathcal{L}$ satisfying (5.25). For the proof of the uniqueness, let us assume that there exists another element $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})$ of the set $\mathcal{L}$. First, we shall show the following statement obtained by replacing β (resp. γ) in (5.2) by $\tilde{\beta}$ (resp. $\tilde{\gamma}$):

$$(5.26)\qquad P^{(1,0)}(t,s) = -\tilde{\beta}(t)P(t,s) - \int_s^t \tilde{\gamma}(t,u)P(u,s)du \qquad (0 \leq s \leq t).$$

Let v be any fixed positive number less than t. By multiplying the both-hand sides of (5.25) by $\nu_0^+(v)$ and taking expectation, we see from (3.3) that

$$(5.27)\qquad \begin{aligned} & E(X(t)\nu_0^+(v)) \\ = & -\int_0^t (\tilde{\beta}(s)E(X(s)\nu_0^+(v)) + \int_0^s \tilde{\gamma}(s,u)E(X(u)\nu_0^+(v))du)ds \\ & +\alpha E(\nu_0^+(t)\nu_0^+(v)). \end{aligned}$$

Since $(\nu_0^+(t); t \geq 0)$ is the standard Brownian motion, we see from (3.11) that for any $w \geq 0$

$$(5.28)\qquad \begin{aligned} & E(X(w)\nu_0^+(v)) = \int_0^{w\wedge v} P(w,\tau)d\tau \quad \text{and} \\ & E(\nu_0^+(w)\nu_0^+(v)) = w \wedge v. \end{aligned}$$

Therefore, we find from (5.27) and (5.28) that

$$\begin{aligned} \int_0^v P(t,\tau)d\tau = & -\int_0^t (\tilde{\beta}(s)\int_0^{s\wedge v} P(s,\tau)d\tau \\ & + \int_0^s \tilde{\gamma}(s,u)(\int_0^{u\wedge v} P(u,\tau)d\tau)du)ds + \alpha v. \end{aligned}$$

Differentiating the above equation with respect to t, we have

$$\begin{aligned}\int_0^v P^{(1,0)}(t,\tau)d\tau &= -(\tilde{\beta}(t)\int_0^v P(t,\tau)d\tau + \int_0^t \tilde{\gamma}(t,u)(\int_0^{u\wedge v} P(u,\tau)d\tau)du) \\ &= -(\tilde{\beta}(t)\int_0^v P(t,\tau)d\tau + \int_0^v (\int_\tau^t \tilde{\gamma}(t,u)P(u,\tau)du)d\tau).\end{aligned}$$

Furthermore, differentiating the above equation with respect to v, we see that (5.26) holds.

Next, we shall show that

$$\alpha = \tilde{\alpha}. \tag{5.29}$$

It follows from (5.25) that for any $t \geq 0$

$$\begin{aligned}&\int_0^t ((\beta(s) - \tilde{\beta}(s))X(s) \\ &\qquad + \int_0^s (\gamma(s,u) - \tilde{\gamma}(s,u))X(u)du + (\delta(s) - \tilde{\delta}(s))X(0))ds \\ &= (\alpha - \tilde{\alpha})\nu_0^+(t),\end{aligned}$$

which implies that the stochastic process $((\alpha-\tilde{\alpha})\nu_0^+(t); t \geq 0)$ is a continuous martingale with bounded variation. Hence, we see that $\alpha = \tilde{\alpha}$.

Using (4.8) and (5.26), we have the same result as in Lemma 5.3 such that $P^{(1,0)}(t,t) = -\alpha\tilde{\beta}(t)$ $(t \geq 0)$. Therefore, we find that

$$\beta(t) = \tilde{\beta}(t) \qquad (t \geq 0). \tag{5.30}$$

Finally, noting (5.26), (5.27) and (5.29), we see from the proof of Lemma 5.1 that the function $\tilde{\gamma}(t,s)$ $(0 \leq s \leq t)$ satisfies the integral equation (5.6) with the same kernel $K = K(u,s)$ as in (5.4) and the same inhomogeneous term $f_t = f_t(s)$ $(0 \leq s \leq t)$ as in (5.5). Therefore, we find from (5.7) that

$$\tilde{\gamma}(t,s) = \tilde{\gamma}(t,s) \qquad (0 \leq s \leq t). \tag{5.31}$$

Thus, we see from (5.29)–(5.31) that Theorem 5.1 holds. □

As stated in Section 1, Miyoshi([8]) derived the stochastic differential equation (5.25) in Theorem 5.1 for the stationary Gaussian process associated with (l,m)-string and called it an $(\alpha,\beta,\gamma,\delta)$-Langevin equation. By taking account of the background stated in Section 1, we call the stochastic differential equation (5.25) in Theorem 5.1 a KM_2O-Langevin equation with continuous time associated with the stochastic process $\mathbf{X}$.

6. Fluctuation-Dissipation Theorem

Under the same situation as in Section 5, in this section, we shall obtain some relations which hold among the quadruplet $(\alpha, \beta, \gamma, \delta)$ in equation (5.25) associated with the stochastic process $\mathbf{X}$.

First, we shall derive a relation between the coefficient α in the fluctuation term of equation (5.25) and the coefficients $\beta = \beta(t), \delta = \delta(t)$ in the drift term of equation (5.25) which is called a fluctuation-dissipation theorem.

Immediately from (2.22) and (2.23), we have

LEMMA 6.1. $\beta = \dfrac{\alpha^2}{2R(0)} - \displaystyle\int_{-\infty}^{0} \Xi(\tau)\gamma(\tau)d\tau.$

Moreover, it follows from (5.12) and (5.14) that

LEMMA 6.2. $\delta(t) = -(\Xi'(t) + \beta(t)\Xi(t) + \displaystyle\int_{0}^{t} \gamma(t,u)\Xi(u)du) \quad (t \geq 0).$

From (2.23) and Lemma 6.2, we have

THEOREM 6.1. (Fluctuation-Dissipation Theorem)

$$\frac{\alpha^2}{2} = R(0)(\beta(0) + \delta(0)).$$

Second, we shall obtain a relation between the coefficients $\beta = \beta(t)$ and $\delta = \delta(t)$ in the drift term of equation (5.25) which is called a dissipation-dissipation theorem. By Lemma 4.1(i), we can differentiate equation (5.1) with respect to t to see that

THEOREM 6.2. (Dissipation-Dissipation Theorem) *The function $\beta = \beta(t)$ $(t > 0)$ is C^1 class and*

$$\frac{d}{dt}\beta(t) = \delta(t)^2 \qquad (t > 0).$$

Third, we shall show another dissipation-dissipation theorem which states that there exists a relation between the function $\gamma = \gamma(t,s)$ and the functin $\delta = \delta(t)$ in the drift term of equation (5.25). For that purpose, we shall show

LEMMA 6.3. $\delta(s)P(s,v) - \alpha\delta(v) = -\int_v^s P(\tau,v)\gamma(s,s-\tau)d\tau \quad (0 < v < s)$.

PROOF. By (4.1), the left-hand side of (5.19) $= \int_{-\infty}^{-v} P(-\tau,v)\gamma(d\tau) - \int_{-s}^{-v} P(-\tau,v)\gamma(d\tau) = \alpha\delta(v) - \int_v^s P(\tau,v)\gamma(-\tau)d\tau$. On the other hand, by (5.20), the right-hand side of (5.19) $= \delta(s)P(s,v) + \int_v^s P(\tau,v)e(s,s-\tau)d\tau$. Therefore, we see from (5.23) that Lemma 6.3 holds. □

LEMMA 6.4. $\gamma(t,0) = \beta(t)\delta(t) - \delta'(t) \qquad (t \geq 0)$.

PROOF. Differentiating the equation in Lemma 6.3 with respect to v, we see that for any s, v $(0 < v < s)$

$$\delta(s)P^{(0,1)}(s,v) - \alpha\delta'(v) = \alpha\gamma(s,s-v) - \int_v^s P^{(0,1)}(\tau,v)\gamma(s,s-\tau)d\tau.$$

Letting v tend to s in the above equation, we have

$$\delta(s)P^{(0,1)}(s,s) - \alpha\delta'(s) = \alpha\gamma(s,0),$$

which with Lemmas 4.3(iii) and 5.3 implies that Lemma 6.4 holds. □

We define a function $\Gamma = \Gamma(t,s)$ $(t, s \geq 0)$ by

$$\Gamma(t,s) \equiv \gamma(t+s,t) \qquad (t,s \geq 0). \tag{6.1}$$

THEOREM 6.3. (Dissipation-Dissipation Theorem) (i) *The function* $\Gamma = \Gamma(t,s)$ *is* C^1*-class with respect to* $t > 0$ *for any fixed* $s > 0$.
(ii) *For any fixed* $s > 0$, *the function* $\Gamma(*,s) = \Gamma(t,s)$ *of* t *satisfies the following equation*:

$$\text{(a)} \quad \frac{\partial\Gamma(t,s)}{\partial t} = \delta(t+s)\Gamma(s,t) \qquad (t > 0),$$
$$\text{(b)} \quad \Gamma(0,s) = \beta(s)\delta(s) - \delta'(s) \qquad (t = 0).$$

PROOF. It follows from (5.23) that

$$\Gamma(t,s) = \gamma(-s) + e(t+s,t). \tag{6.2}$$

For the proof of (i), we have only to show that the function $e(t+s,t)$ is C^1 class with respect to $t > 0$ for any fixed $s > 0$. Using the same Voltera kernel $K = K(z,v)$ $(0 \le v \le z < \infty)$ as in (5.4), we can rewrite equation (5.21) into

$$e(s,s-v) - \int_v^s K(z,v)e(s,s-z)dz = g_s(v) \qquad (0 \le v < s), \tag{6.3}$$

where the function $g_s = g_s(v)$ is defined by

$$g_s(v) \equiv \alpha^{-1}(\delta(s)P^{(0,1)}(s,v) - \int_{-\infty}^{-s} P^{(0,1)}(-\tau,v)\gamma(\tau)d\tau) \qquad (0 \le v < s). \tag{6.4}$$

Hence, we can use the same resolvent kernel $C = C(z,v)$ $(z \ge v \ge 0)$ as in (5.8) to get

$$e(s,s-v) = g_s(v) + \int_v^s C(z,v)g_s(z)dz.$$

For any t,s $(t,s>0)$, by replacing s and v in the above equation by $t+s$ and s, respectively, we have

$$e(t+s,t) = g_{t+s}(s) + \int_s^{t+s} C(z,s)g_{t+s}(z)dz. \tag{6.5}$$

We note that the resolvent kernel $C = C(z,s)$ is continuous with respect to z,s $(0 \le s \le z)$. Moreover, since it follows from (4.10), (5.20), Lemmas 4.5 and (6.4) that the function $g_{t+s}(s)$ $(t,s>0)$ is C^1 class with respect to t for any fixed $s > 0$, we can see from (6.5) that the function $e(t+s,t)$ is C^1 class with respect to t for any fixed $s > 0$. Hence, we have proved Theorem 6.3(i).

Next, we shall prove Theorem 6.3(ii). It follows from Lemma 6.4 and (6.1) that (b) in Theorem 6.1(ii) holds. By noting (6.1) again, we see from Lemma 6.3 that

$$\delta(s)P(s,v) - \alpha\delta(v) = -\int_v^s P(\tau,v)\Gamma(s-\tau,\tau)d\tau \qquad (0 < v < s).$$

Differentiating the above equation with respect to s, we find that for any s, v $(0 < v < s)$

$$\delta'(s)P(s,v)+\delta(s)P^{(1,0)}(s,v) = -P(s,v)\Gamma(0,s)-\int_v^s P(\tau,v)\Gamma^{(1,0)}(s-\tau,\tau)d\tau.$$

Applying Lemma 5.1 to the above equation, we have

$$\begin{aligned} &\delta'(s)P(s,v) - \delta(s)(\beta(s)P(s,v) + \int_v^s \gamma(s,u)P(u,v)du) \\ = &-P(s,v)\Gamma(0,s) - \int_v^s P(\tau,v)\Gamma^{(1,0)}(s-\tau,\tau)d\tau \qquad (0 < v < s). \end{aligned}$$

Therefore, we see from (b) in Theorem 6.3(ii) that

$$\int_v^s P(u,v)(\Gamma^{(1,0)}(s-u,u) - \delta(s)\gamma(s,u))du = 0 \qquad (0 < v < s).$$

Moreover, by differentiating the above equation with respect to v, we see from (4.8) that for any s, v $(0 < v < s)$

$$\begin{aligned} &\alpha(\Gamma^{(1,0)}(s-v,v) - \delta(s)\gamma(s,v)) \\ = &\int_v^s P^{(0,1)}(s,v)(\Gamma^{(1,0)}(s-u,u) - \delta(s)\gamma(s,u))du. \end{aligned}$$

Let us fix any $s > 0$. Defining a function $\phi_s(v)$ $(0 < v < s)$ by

$$\phi_s(v) \equiv \Gamma^{(1,0)}(s-v,v) - \delta(s)\gamma(s,v),$$

we see that

$$\phi_s(v) - \int_v^s (\alpha^{-1}P^{(0,1)}(s,v))\phi_s(u)du = 0.$$

Therefore, using the same method which was used to solve equation (5.6), we find that $\phi_s = 0$, which implies that

$$\Gamma^{(1,0)}(s-v,v) = \delta(s)\gamma(s,v) \qquad (0 < v < s).$$

For any t, s $(t, s > 0)$, we replace s (resp. v) in the above equation by $t+s$ (resp. s) to get from (6.1) that

$$\Gamma^{(1,0)}(t,s) = \delta(t+s)\Gamma(s,t),$$

which proves (a) in Theorem 6.3(ii). □

Concerning the uniqueness of equation in Theorem 6.3(ii), we shall show the following general lemma.

LEMMA 6.5. *Let us be given any real valued continuous function $\delta = \delta(t)$ defined on $[0,\infty)$. Let $F = F(t,s)$ be a real valued continuous function defined on $[0,\infty)\times[0,\infty)$ such that for any fixed $s > 0$*

(i) *the function $F(*,s) = F(t,s)$ is C^1 class with respect to t $(t > 0)$,*

(ii) *the function $F(*,\star)$ satisfies the following equation:*

$$\begin{aligned} &\text{(a)} \quad \frac{\partial F(t,s)}{\partial t} = \delta(t+s)F(s,t) \qquad (t>0), \\ &\text{(b)} \quad F(0,s) = 0 \qquad (t=0). \end{aligned}$$

Then, $F(t,s) = 0 \qquad (t,s \geq 0)$.

PROOF. Let us fix any positive number T and define two real constants C_1, C_2 by

$$C_1 \equiv \max_{0\leq t,s\leq T} |F(t,s)| \quad \text{and} \quad C_2 \equiv \max_{0\leq t,s\leq T} |\delta(t+s)|.$$

For the proof of Lemma 6.5, we have only to show the following inequality: for any $n \in \mathbf{N}^* \equiv \{0,1,2,\ldots\}$

$$\text{(6.6)} \quad |F(t,s)| \leq C_1 C_2^{4(n+1)} \frac{t^{2(n+1)}}{(2(n+1))!}\frac{s^{2(n+1)}}{(2(n+1))!} \qquad (0 \leq t,s \leq T).$$

We shall prove (6.6) by mathematical inducton with respect to n. Using (a) and (b), we have

$$\begin{aligned} F(t,s) &= \int_0^t \delta(u_1+s)F(s,u_1)du_1 \\ &= \int_0^t \delta(u_1+s)(\int_0^s \delta(u_2+u_1)F(u_1,u_2)du_2)du_1 \\ &= \int_0^t\int_0^s \delta(s+u_1)\delta(u_1+u_2)F(u_1,u_2)du_1du_2. \end{aligned}$$

Therefore, we have

$$(6.7)\quad F(t,s) = \int_0^t \int_0^s \delta(s+u_1)\delta(u_1+u_2) \cdot\left(\int_0^{u_1}\int_0^{u_2} \delta(u_2+u_3)\delta(u_3+u_4)F(u_3,u_4)du_3du_4\right)du_1du_2.$$

Thus, we can estimate it as follows:

$$\begin{aligned}|F(t,s)| &\le C_1C_2^4\int_0^t\int_0^s du_1du_2\left(\int_0^{u_1}\int_0^{u_2} du_3du_4\right)\\ &= C_1C_2^4\frac{t^2}{2!}\frac{s^2}{2!},\end{aligned}$$

which implies that inequality (6.6) holds for $n=0$.

Next, for any fixed $k\in\mathbf{N}^*$, we shall assume that inequality (6.6) holds for $n=k$. Using (6.7) again, we obtain the following estimate:

$$\begin{aligned}|F(t,s)| &\le C_1C_2^{4(k+1)}\int_0^t\int_0^s du_1du_2\left(\int_0^{u_1}\int_{u_2}\frac{u_3^{2(k+1)}}{(2(k+1))!}\frac{u_4^{2(k+1)}}{(2(k+1))!}du_3du_4\right)\\ &= C_1C_2^{4(k+1)}\int_0^t\int_0^s du_1du_2\frac{u_1^{2k+3}}{(2k+3)!}\frac{u_2^{2k+3}}{(2k+3)!}\\ &= C_1C_2^{4(k+1)}\frac{t^{2(k+2)}}{(2(k+2))!}\frac{s^{2(k+2)}}{(2(k+2))!},\end{aligned}$$

which implies that inequality (6.6) holds for $n=k+1$. Therefore, by mathematical induction with respect to n, we find that inequality (6.6) holds for any $n\in\mathbf{N}^*$. □

Finally, we shall obtain a concrete representation of the function $\Gamma=\Gamma(t,s)$ in (6.1) which is the unique solution of differential equation in Theorem 6.3 by virtue of Lemma 6.5.

THEOREM 6.4. *The function* $\Gamma=\Gamma(t,s)$ $(t,s\ge 0)$ *in* (6.1) *has the following representation*:

$$\begin{aligned}\Gamma(t,s) &= g(t,s)+\int_0^t\int_0^s \delta(s+u_1)\delta(u_1+u_2)g(u_1,u_2)du_1du_2\\ &\quad+\sum_{n=1}^{\infty}\int_0^t\int_0^s \delta(s+u_1)\delta(u_1+u_2)du_1du_2\end{aligned}$$

$$\cdot\left(\int_0^{u_1}\int_0^{u_2}\delta(u_2+u_3)\delta(u_3+u_4)du_3du_4\right.$$
$$(\cdots(\int_0^{u_{2n-1}}\int_0^{u_{2n}}\delta(u_{2n}+u_{2n+1})\delta(u_{2n+1}+u_{2n+2})$$
$$\left.\cdot g(u_{2n+1},u_{2n+2})du_{2n+1}du_{2n+2})\cdots)\right),$$

where the function $g = g(t,s)$ $(t, s > 0)$ *is given by*

$$g(t,s)=\beta(s)\delta(s)-\delta'(s)+\int_0^t\delta(s+u_1)(\beta(u_1)\delta(u_1)-\delta'(u_1))du_1.$$

Proof. It follows from Theorem 6.3(ii) that

$$\Gamma(t,s)=\Gamma(0,s)+\int_0^t\delta(s+u_1)\Gamma(s,u_1)du_1.$$

Using this equation again, we have

$$\begin{aligned}\Gamma(t,s) &= \Gamma(0,s)+\int_0^t\delta(s+u_1)\Gamma(0,u_1)du_1\\ &\quad+\int_0^t\int_0^s\delta(s+u_1)\delta(u_1+u_2)\Gamma(u_1,u_2)du_1du_2\\ &= g(t,s)+\int_0^t\int_0^s\delta(s+u_1)\delta(u_1+u_2)\Gamma(u_1,u_2)du_1du_2,\end{aligned}\tag{6.8}$$

where the function $g = g(t,s)$ is defined by

$$g(t,s)\equiv\Gamma(0,s)+\int_0^t\delta(s+u_1)\Gamma(0,u_1)du_1.$$

Therefore, for any $n\in\mathbf{N}$, using (6.8) n-times repeatedly, we have

$$\begin{aligned}\Gamma(t,s) &= g(t,s)+\int_0^t\int_0^s\delta(s+u_1)\delta(u_1+u_2)g(u_1,u_2)du_1du_2\\ &\quad+\sum_{k=1}^n\int_0^t\int_0^s\delta(s+u_1)\delta(u_1+u_2)du_1du_2\\ &\quad\cdot\left(\int_0^{u_1}\int_0^{u_2}\delta(u_2+u_3)\delta(u_3+u_4)du_3du_4\right.\\ &\quad(\cdots(\int_0^{u_{2k-1}}\int_0^{u_{2k}}\delta(u_{2k}+u_{2k+1})\delta(u_{2k+1}+u_{2k+2})\\ &\quad\left.\cdot g(u_{2k+1},u_{2k+2})du_{2k+1}du_{2k+2})\cdots)\right)\\ &\quad+\phi_n(t,s),\end{aligned}\tag{6.9}$$

where the function $\phi_n = \phi_n(t,s)$ $(t, s \geq 0)$ is given by

$$\begin{aligned}\phi_n(t,s) \equiv & \int_0^t \int_0^s \delta(s+u_1)\delta(u_1+u_2)du_1du_2 \\ & \cdot(\int_0^{u_1}\int_0^{u_2} \delta(u_2+u_3)\delta(u_3+u_4)du_3du_4 \\ & (\cdots(\int_0^{u_{2n-1}}\int_0^{u_{2n}} \delta(u_{2n}+u_{2n+1})\delta(u_{2n+1}+u_{2n+2}) \\ & \qquad \cdot\Gamma(u_{2n+1},u_{2n+2})du_{2n+1}du_{2n+2})\cdots)).\end{aligned}$$

Let us fix any positive number T and define two real constants C_3, C_4 by

$$C_3 \equiv \max_{0\leq t,s\leq T} |\Gamma(t,s)| \quad \text{and} \quad C_4 \equiv \max_{0\leq t,s\leq T} |\delta(t+s)|.$$

Similarly as in (6.6), we see that for any $n \in \mathbf{N}$

$$\max_{0\leq t,s\leq T} |\phi_n(t,s)| \leq C_3(\frac{(C_4T)^{2(n+1)}}{(2(n+1))!})^2.$$

Therefore, since it follows that $\lim_{n\to\infty} |\phi_n(t,s)| = 0$ for any t, s $(t, s \geq 0)$, we see from (6.9) that Theorem 6.4 holds. $\square$

We call Theorems 6.1, 6.2 and 6.3 together as the fluctuation-dissipation theorem based upon $\mathrm{KM_2O}$-Langevin equation (5.25) associated with the stochastic process $\mathbf{X}$, which wad proved in [8] for the continuous time stationary Gaussian process associated with (l,m)-string.

Moreover, we define the quadrupulet $(\alpha, \beta, \gamma, \delta)$ consisting of the coefficients $\alpha, \beta = \beta(t)(0 \leq t < \infty), \gamma = \gamma(t,s)(0 \leq s \leq t < \infty)$ and $\delta = \delta(t)(0 \leq t < \infty)$ in equation (5.25) associated with the stochastic process $\mathbf{X}$ by

$$\mathcal{L}(\mathbf{X}) \equiv (\alpha, \beta, \gamma, \delta) \tag{6.10}$$

and call it a $\mathrm{KM_2O}$-Langevin system associated with the stationary process $\mathbf{X}$.

7. $\mathrm{KM_2O}$-Langevin System and Covariance Function

Under the same situation as in Sections 5 and 6, in this section, we shall derive a system of equations for characterizing the $\mathrm{KM_2O}$-Langevin system $\mathcal{L}(\mathbf{X})$ from the covariance function R of the stationary process $\mathbf{X}$.

We define a subset $\mathcal{L}_0$ of the set $\mathcal{L}$ in (5.24) by

$$\begin{aligned}(7.1)\quad \mathcal{L}_0 \equiv \{(\alpha,\beta,\gamma,\delta)\in\mathcal{L} \mid \Gamma = \Gamma(t,s) \equiv \gamma(t+s,t)(t,s>0) \\ \text{is } C^1\text{-class with respect to } t>0 \text{ for any fixed } s>0\}.\end{aligned}$$

It then follows from Theorems 5.1 and 6.3 that

$$\mathcal{L}(\mathbf{X}) \in \mathcal{L}_0. \tag{7.2}$$

We shall show

Lemma 7.1. $R'(t-s) = -(\beta(t)R(t-s)+\int_0^t \gamma(t,u)R(s-u)du+\delta(t)R(s))$ $(0<s<t)$.

Proof. For any fixed $v>0$, multiplying the both-hand sides of KM_2O-Langevin equation (5.25) by $X(v)$ and taking expectation, we have

$$\begin{aligned}R(v-t)-R(v) &= -\int_0^t (\beta(s)R(v-s) \\ &\quad + \int_0^s \gamma(s,u)R(v-u)du + \delta(s)R(v))ds \\ &\quad +\alpha E(X(v)\nu_0^+(t)).\end{aligned} \tag{7.3}$$

It follows from (5.28) and (7.3) that for any t, v $(0<v<t)$

$$\begin{aligned}R(v-t)-R(v) &= -\int_0^t (\beta(s)R(v-s) \\ &\quad + \int_0^s \gamma(s,u)R(v-u)du + \delta(s)R(v))ds \\ &\quad +\alpha\int_0^v P(v,u)du.\end{aligned}$$

Differentiating the above equation with respect to t and then replacing v by s, we find that Lemma 7.1 holds. □

We shall show the second of main theorems in this paper.

Theorem 7.1. *The KM_2O-Langevin system $\mathcal{L}(\mathbf{X})$ is the unique solution in the set $\mathcal{L}_0$ satisfying the following system of equations*:

(i) $\alpha = \sqrt{-2R'(0+)}$;

(ii) $R'(t) = -(\beta(t)R(t) + \int_0^t \gamma(t,u)R(u)du + \delta(t)R(0)) \qquad (t > 0)$;

(iii) $\frac{d}{dt}\beta(t) = \delta(t)^2 \qquad (t > 0)$;

(iv) *for any fixed* $s > 0$

$$\begin{cases} \text{(a)} & \dfrac{\partial\Gamma(t,s)}{\partial t} = \delta(t+s)\Gamma(s,t) \qquad (t > 0), \\ \text{(b)} & \Gamma(0,s) = \beta(s)\delta(s) - \delta'(s) \qquad (t = 0). \end{cases}$$

PROOF. (i), (ii), (iii) and (iv) come from (2.23), Lemma 6.2, Theorems 6.2 and 6.3, respectively. Note that we can get (ii) also by letting s in Lemma 7.1 tend to 0.

Next, we shall prove the uniqueness. Let $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})$ be another element of the set $\mathcal{L}_0$ satisfying the equations (i)–(iv). Immediately from (i), we see

$$\alpha = \tilde{\alpha}. \tag{7.4}$$

Corresponding to Lemma 7.1, we shall show

$$\begin{aligned} R'(t-s) \;=\; & -(\tilde{\beta}(t)R(t-s) \\ & + \int_0^t \tilde{\gamma}(t,u)R(s-u)du + \tilde{\delta}(t)R(s)) \quad (0 < s < t). \end{aligned} \tag{7.5}$$

For that purpose, we define a function $f = f(t,s)$ $(t, s \geq 0)$ by

$$\begin{aligned} f(t,s) \;\equiv\; & R'(s) + \tilde{\beta}(t+s)R(s) \\ & + \int_0^{t+s} \tilde{\gamma}(t+s,u)R(t-u)du + \tilde{\delta}(t+s)R(t). \end{aligned} \tag{7.6}$$

In particular, we see from equation (ii) with β, γ and δ replaced by $\tilde{\beta}, \tilde{\gamma}$ and $\tilde{\delta}$ that for any $s \geq 0$

$$f(0,s) = 0. \tag{7.7}$$

Defining another function $h = h(t, s)$ by

$$h(t,s) \equiv \int_0^{t+s} \tilde{\gamma}(t+s,u)R(t-u)du, \tag{7.8}$$

we have

$$f(t,s) = R'(s) + \tilde{\beta}(t+s)R(s) + h(t,s) + \tilde{\delta}(t+s)R(t). \tag{7.9}$$

It follows that

$$\begin{aligned} h(t,s) &= \int_0^t \tilde{\gamma}(t+s,u)R(t-u)du + \int_t^{t+s} \tilde{\gamma}(t+s,u)R(u-t)du \\ &= \int_0^t \tilde{\Gamma}(t-u,s+u)R(u)du + \int_0^s \tilde{\Gamma}(t+u,s-u)R(u)du, \end{aligned}$$

where $\tilde{\Gamma}(t,s) \equiv \tilde{\gamma}(t+s,t)$. By using (a) in equation (iv) with Γ and δ replaced by $\tilde{\Gamma}$ and $\tilde{\delta}$, we can differentiate the above with respect to t to see that

$$\begin{aligned} h^{(1,0)}(t,s) &= \tilde{\Gamma}(0,s+t)R(t) + \int_0^t \tilde{\Gamma}^{(1,0)}(t-u,s+u)R(u)du \\ &\quad + \int_0^s \tilde{\Gamma}^{(1,0)}(t+u,s-u)R(u)du \\ &= \tilde{\Gamma}(0,s+t)R(t) + \tilde{\delta}(t+s)\int_0^{t+s} \tilde{\gamma}(t+s,u)R(u-s)du. \end{aligned}$$

Hence, it follows from (7.9) and (b) in equation (iv) with Γ and δ replaced by $\tilde{\Gamma}$ and $\tilde{\delta}$ that

$$\begin{aligned} f^{(1,0)}(t,s) &= \tilde{\beta}'(t+s)R(s) + \tilde{\delta}'(t+s)R(t) + \tilde{\delta}(t+s)R'(t) + h^{(1,0)}(t,s) \\ &= \tilde{\beta}'(t+s)R(s) + \tilde{\delta}(t+s)R'(t) \\ &\quad + \tilde{\beta}(s+t)\tilde{\delta}(s+t)R(t) + \tilde{\delta}(t+s)\int_0^{t+s} \tilde{\gamma}(t+s,u)R(u-s)du. \end{aligned}$$

Thus, by using equation (iii) with β and δ replaced by $\tilde{\beta}$ and $\tilde{\delta}$, we have

$$\begin{aligned} f^{(1,0)}(t,s) &= \tilde{\delta}(t+s)(\tilde{\delta}(t+s)R(s) + R'(t) + \tilde{\beta}(s+t)R(t) \\ &\quad + \int_0^{t+s} \tilde{\gamma}(t+s,u)R(u-s)du), \end{aligned}$$

which implies that

$$f^{(1,0)}(t,s) = \tilde{\delta}(t+s)f(s,t) \quad (t,s \geq 0). \tag{7.10}$$

By Lemma 6.5, we see from (7.7) and (7.10) that $f(t,s) = 0$ $(t,s \geq 0)$. Therefore, for any t,s $(0 \leq s < t)$, by replacing t and s in (7.6) by s and $t-s$, respectively, we find that (7.5) holds.

By using (7.5) proved above, we shall prove that $\beta = \tilde{\beta}, \gamma = \tilde{\gamma}$ and $\delta = \tilde{\delta}$. By Lemma 7.1 and (7.5), we see that for any t,s $(0 \leq s < t)$

$$\begin{aligned} & \beta(t)R(t-s) + \int_0^t \gamma(t,u)R(s-u)du + \delta(t)R(s) \\ = \ & \tilde{\beta}(t)R(t-s) + \int_0^t \tilde{\gamma}(t,u)R(s-u)du + \tilde{\delta}(t)R(s). \end{aligned}$$

This implies that for any t,s $(0 \leq s < t)$

$$\begin{aligned} & E((\beta(t) - \tilde{\beta}(t))X(t) \\ & \quad + \int_0^t (\gamma(t,u) - \tilde{\gamma}(t,u))X(u)du + (\delta(t) - \tilde{\delta}(t))X(0), X(s)) = 0. \end{aligned}$$

Therefore, we see that for any $t > 0$

$$(\beta(t) - \tilde{\beta}(t))X(t) + \int_0^t (\gamma(t,u) - \tilde{\gamma}(t,u))X(u)du + (\delta(t) - \tilde{\delta}(t))X(0) = 0.$$

Combining this with (7.7), we see that the quadruplet $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})$ satisfies the stochastic differential equation (5.25). Thus, by Theorem 5.1, we see that $\beta = \tilde{\beta}, \gamma = \tilde{\gamma}$ and $\delta = \tilde{\delta}$. $\square$

COROLLARY 7.1. *The constants $\beta(0)$ and $\delta(0)$ can be obtained as follows from the covariance function R :*

(v) $\quad \beta(0) = -\dfrac{1}{2}(\dfrac{R'(0+)}{R(0)} + \dfrac{R''(0+)}{R'(0+)});$

(vi) $\quad \delta(0) = -\dfrac{1}{2}(\dfrac{R'(0+)}{R(0)} - \dfrac{R''(0+)}{R'(0+)}).$

Proof. Letting t tend to 0 in Theorem 7.1(ii), we have

$$\beta(0) + \delta(0) = -\frac{R'(0)}{R(0)}. \tag{7.11}$$

We have proved (7.5) in the proof of the uniqueness part in Theorem 7.1. Letting s tend to t therein and noting (i) in Theorem 7.1, we have

$$\frac{\alpha^2}{2} = \beta(t)R(0) + \int_0^t \gamma(t,u)R(t-u)du + \delta(t)R(t) \quad (t > 0). \tag{7.12}$$

We rewrite (ii) in Theorem 7.1 into

$$R(0)\delta(t) = -R'(t) - \beta(t)R(t) - \int_0^t \Gamma(t-u,u)R(t-u)du \quad (t > 0).$$

Differentiating the above equation with respect to t, we see from (iv) in Theorem 7.1 that

$$\begin{aligned} R(0)\delta'(t) &= -(R''(t) + \beta(t)R'(t) + \int_0^t \Gamma(t-u,u)R'(t-u)du) \\ &\quad -\beta'(t)R(t) - \Gamma(0,t)R(0) - \int_0^t \Gamma^{(1,0)}(t-u,u)R(t-u)du \\ &= -(R''(t) + \beta(t)R'(t) + \int_0^t \gamma(t,u)R'(u)du) - \delta^2(t)R(t) \\ &\quad -(\beta(t)\delta(t) - \delta'(t))R(0) - \delta(t)\int_0^t \gamma(t,u)R(t-u)du. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\delta(t)(\delta(t)R(t) + \beta(t)R(0) + \int_0^t \gamma(t,u)R(t-u)du) \\ &= -(R''(t) + \beta(t)R'(t) + \int_0^t \gamma(t,u)R'(u)du) \end{aligned}$$

which with (7.12) implies that for any $t > 0$

$$\frac{\alpha^2}{2}\delta(t) = -(R''(t) + \beta(t)R'(t) + \int_0^t \gamma(t,u)R'(u)du). \tag{7.13}$$

In particular, letting t tend to 0 in (7.13) and noting (i) in Theorem 7.1, we have

$$\delta(0) - \beta(t) = -\frac{R''(0)}{R'(0)}. \tag{7.14}$$

Therefore, combining (7.11) with (7.14), we see that Corollary 7.1 holds. □

We shall investigate the solvability of the system of equations in Theorem 7.1 for certain given non-negative definite function R defined on the bounded interval $[-T,T]$ in the future.

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(Received September 10, 2003)
(Revised August 24, 2006)

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