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Large Deviation for Periodic Markov Process on Square Lattice

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Abstract. We discuss large deviations for the pinned motion of a periodic Markov chain on the d-dimensional square lattice $\mathbb{Z}^d$. Making use of the harmonic transform based on a positive principal eigenfunction of the difference operator related to the Markov chain, a nice large deviation principle is established.

1. Introduction

Let (X_n, P_x) be a periodic, reversible Markov chain on the $d-$dimensional square lattice $\mathbb{Z}^d$. In general, the n-step transition probabilities of the Markov chain decay exponentially in large time. For such a process we are interested in the asymptotic behavior of the expectation of the type:

$$E^{P_{(0,x)}^{(n,y)}}\left[\exp\left(-\sum_{k=0}^{n-1} m(X_k)\right)\right],$$

where $P_{(0,x)}^{(n,y)}$ is the probability law of the motion of $X_\cdot$ pinned as $X_0 = x, X_n = y$ and m is a periodic function with the same periodicity as the above Markov chain. The first step in this direction is to establish a large deviation principle for the pinned process. Such a result is obtained in this paper.

Large deviation for the occupation time distribution of a Markov chain has been established in a series of papers by Donsker and Varadhan [2] with the strong ergodicity assumptions. Several subsequent results have been obtained since then weakening the ergodicity conditions. However their results cannot be applied to our case, since our process is not expected any longer to have a nice condtion like ergodicity. In fact, the n-step transition probabilities of the processes generally decay exponentially in time. Consequently we are not able to prove the usual large derivations for the process.

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Nevertheless it is possible to prove a nice large deviation as far as the pinned processes are concerned if appropriate rate functions are adopted, which is our main assertion in this paper. It is to be noted that one of the authors has discussed this type of large deviation for a class of pinned covering diffusions(See [4]).

The organization of this paper is as follows. In Section 2 some notations are introduced and main results are stated. The principal eigenvalue problem for the difference operator corresponding to the original Markov chain is discussed in Section 3. With the help of of the harmonic transform by a positive principal eigenfunction obtained in Section 3, upper and lower estimates for the transition probabilities of the Markov chain are given. Large deviation result is proved in Section 4. Section 5 is devoted to a discussion of the positivity of the bottom of the spectrum of the above difference operator.

2. Notations and Statements of Main Results

Suppose we are given an irreducible Markov chain X_n on $\mathbb{Z}^d$ which has a one-step transition probability $\{p(x,y)\}$. We assume the following conditions:

(A.1) There exists a positive integer m_0 such that

$$p(x+m_0e_i, y+m_0e_i) = p(x,y), \quad x,y \in \mathbb{Z}^d, \quad i = 1, \cdots, n$$

where $e_i = (0, \cdots, \overset{i}{\check{1}}, \cdots, 0)$.

(A.2) It holds that with a positive constant r_0,

$$\{y \in \mathbb{Z}^d; p(x,y) \neq 0\} \subseteq B^1(x, r_0), \quad x \in \mathbb{Z}^d$$

where $B^1(x,r_0) = \{y; \|x-y\| < r_0\}, \quad \|x\| = \sum_{i=1}^d |x_i|$.

(A.3) There exists a positive function $a = a(x)$ on $\mathbb{Z}^d$ satisfying

$$a(x)p(x,y) = a(y)p(y,x), \quad x,y \in \mathbb{Z}^d.$$

(A.4) $$\inf\{p(x,y); \|x-y\| \leq 1,\ x,y \in \mathbb{Z}^d\} > 0.$$

Define a difference operator L for a function $u(x)$ on $\mathbb{Z}^d$ by

$$Lu(x) = \sum_{y\in\mathbb{Z}^d} p(x,y)(u(y)-u(x))$$

and set $\mathbb{T} = \mathbb{Z}/m_0\mathbb{Z}$, $\mathbb{T}^d = \overbrace{\mathbb{T}\times\cdots\times\mathbb{T}}^{d}$. Denote by c the covering map from $\mathbb{Z}$ onto $\mathbb{T}$, i.e.,

$$c(x) = x \pmod{m_0}$$

and set

$$c_0(x) = (c(x_1),\cdots,c(x_d)), \quad x=(x_1,\cdots,x_d)\in\mathbb{Z}^d$$

Let $\mathcal{M}$ be the set of probability measures on $\mathbb{T}^d$ endowed with the topology induced by the metric

$$d(\mu,\lambda) = \sqrt{\sum_{x\in\mathbb{T}^d} |\mu(x)-\lambda(x)|^2}, \quad \mu,\lambda\in\mathcal{M}.$$

Evidently $(\mathcal{M},d)$ is a compact metric space. Let Ω_x be the space of all sequences $X_0, X_1, X_2, \cdots$ with $X_0 = x$ and $X_i \in \mathbb{Z}^d$. We have a probability measure on Ω_x induced by $p(\cdot,\cdot)$ which we will denote by P_x. For each $\omega\in\Omega_x$, each positive integer n and a subset A of $\mathbb{T}^d$, define

$$L_n\big(\omega, A\big) = \frac{1}{n}\sum_{k=0}^{n-1}\chi_A(c_0(X_k(\omega)))$$

and for a measurable subset $B\subseteq\mathcal{M}$,

$$Q_{(0,x)}^{(n,y)}(B) = P_{(0,x)}^{(n,y)}\big(\omega; L_n(\omega,\cdot)\in B\big),$$

where $P_{(0,x)}^{(n,y)}$ denotes the probability law of the Markov chain $X.$ pinned as $X_0 = x$ and $X_n = y$.

In this paper a large deviation principle for $Q_{(0,x)}^{(n,y)}$ will be investigated. In order to introduce an appropriate rate function for the present case, we first discuss an eigenvalue problem for the difference operator L.

THEOREM 1. *There exist a nonnegative number* $\lambda_0 < 1$ *and a positive, periodic function* φ_0 *of period* m_0 *such that*

$$L\Big(\frac{\varphi_0}{\sqrt{a}}\Big)(x) + \lambda_0\Big(\frac{\varphi_0}{\sqrt{a}}\Big)(x) = 0, \quad x \in \mathbb{Z}^d,$$

where φ_0 *is unique up to the multiplication by positive constants.*

Making use of the function $u_0(x) = \varphi_0(x)/\sqrt{a(x)}$, a new transition probability function $p^0(x, y)$ is defined by

$$p^0(x, y) = \frac{p(x, y)u_0(y)}{(1 - \lambda_0)u_0(x)}, \quad x, y \in \mathbb{Z}^d. \tag{1}$$

Let (X_n^0, P_x^0) be the Markov chain on $\mathbb{Z}^d$ induced by $\{p^0(x, y)\}$. Denote by $p_n^0(x, y)$ the n-step transition probability of X_n^0. We introduce a distance on $\mathbb{Z}^d$ by

$$d_1(x, y) = \inf\{n; p_n^0(x, y) > 0\}.$$

Through upper and lower estimates of the n-step transition probability $p_n^0(x, y)$ of the Markov chain X_n^0, we can get upper and lower bounds for the n-step transition probability $p_n(x, y)$ for X_n.

THEOREM 2. *There exist some positive constants* $C_i, i = 1, 2, 3, 4$ *such that*

$$\begin{aligned}\frac{C_1(1-\lambda_0)^n u_0(x)\varphi_0(y)^2}{u_0(y)n^{d/2}}e^{-\frac{C_2 d_1(x,y)^2}{n}} &\le p_n(x, y)\\ &\le \frac{C_3(1-\lambda_0)^n u_0(x)\varphi_0(y)^2}{u_0(y)n^{d/2}}e^{-\frac{C_4 d_1(x,y)^2}{n}}\end{aligned}$$

for all $x, y \in \mathbb{Z}^d$ *with* $d_1(x, y) \le n$.

It should be remarked here that the process X_n^0 is also periodic of period m_0. Set

$$\pi_0 u(x) = \sum_{y\in\mathbb{Z}^d} p^0(x, y)u(y). \tag{2}$$

Denote by $\mathcal{U}_0$ the set of periodic functions of period m_0 on $\mathbb{Z}^d$. The rate function I_0 on $\mathcal{M}$ is defined by

$$I_0(\mu) = -\inf_{\substack{u\in\mathcal{U}_0\\ u>0}} \sum_{x\in\mathbb{T}^d} \log\Big(\frac{\pi_0 u}{u}\Big)(x)\mu(x). \tag{3}$$

Then we are able to show the following:

Theorem 3. (i) *For any closed $F \subseteq \mathcal{M}$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log Q_{(0,x)}^{(n,y)}(F) \le -\inf_{\mu\in F} I_0(\mu).$$

(ii) *For any open $G \subseteq \mathcal{M}$,*

$$\liminf_{n\to\infty} \frac{1}{n} \log Q_{(0,x)}^{(n,y)}(G) \ge -\inf_{\mu\in G} I_0(\mu).$$

A corollary of the above theorem is stated as follows.

Corollary 3.1. *If Φ is a real-valued weakly continuous functional on $\mathcal{M}$, then*

$$\lim_{n\to\infty} \frac{1}{n} \log E^{Q_{(0,x)}^{(n,y)}}[e^{-n\Phi(\mu)}] = -\inf_{\mu\in\mathcal{M}}[\Phi(\mu) + I_0(\mu)]$$

for any $x, y \in \mathbb{Z}^d$.

3. Principal Eigenvalue Problem and Gaussian Estimates

In this section we shall give the proof of Theorem 1 and Theorem 2. Consider the difference equation

$$L\Big(\frac{\varphi}{\sqrt{a}}\Big)(x) + \lambda\Big(\frac{\varphi}{\sqrt{a}}\Big)(x) = 0, \quad x \in \mathbb{Z}^d. \tag{4}$$

It follows from the assumption (A.3) that

$$\sum_{y\in\mathbb{Z}^d} p(x,y)\Big(\frac{\varphi}{\sqrt{a}}\Big)(y) = \frac{1}{\sqrt{a(x)}} \sum_{y\in\mathbb{Z}^d} \sqrt{p(x,y)p(y,x)}\varphi(y).$$

Hence

$$
\begin{aligned}
&L\Big(\frac{\varphi}{\sqrt{a}}\Big)(x)+\lambda\Big(\frac{\varphi}{\sqrt{a}}\Big)(x)\\
&=\frac{1}{\sqrt{a(x)}}\Big\{\sum_{y\in\mathbb{Z}^d}\sqrt{p(x,y)p(y,x)}\varphi(y)+(\lambda-1)\varphi(x)\Big\}.
\end{aligned}
$$

Thus (4) is equivalent to

$$
\sum_{y\in\mathbb{Z}^d}\sqrt{p(x,y)p(y,x)}\varphi(y)+(\lambda-1)\varphi(x)=0. \tag{5}
$$

Set $q(x,y)=\sqrt{p(x,y)p(y,x)}$, then $q(x,y)$ is symmetric in x,y and is a periodic function of period m_0, i.e., the condition (A.1) is fulfilled with $\{q(x,y)\}$.

Define for any $x,y\in\mathbb{T}^d$,

$$
\overline{q}(x,y)=\sum_{k_1,\cdots,k_d\in\mathbb{Z}}q(x,y+k_1m_0e_1+\cdots+k_dm_0e_d).
$$

Since φ is a periodic function of period m_0, (5) is reduced to

$$
\sum_{y\in\mathbb{T}^d}\overline{q}(x,y)\varphi(y)+(\lambda-1)\varphi(x)=0,\quad x\in\mathbb{T}^d. \tag{6}
$$

Set $\overline{q}(x)=\sum_{y\in\mathbb{T}^d}\overline{q}(x,y)$. It should be remarked that the Markov chain on $\mathbb{T}^d$ induced by $\{\overline{q}(x,y)/\overline{q}(x)\}$ is irreducible.

Let λ_1 be the biggest nonnegative eigenvalue of the matrix $Q=\big(\overline{q}(x,y)\big)$. Making use of the irreducibility mentioned above and the Perron-Frobenius Theorem, we can easily see that λ_1 is strictly positive and that the corresponding eigenspace is one-dimensional. It is also possible to choose its eigenvector whose components are all positive. Thus setting $\lambda_0=1-\lambda_1$, all the assertions in Theorem 1 except the nonnegativity of λ_0 have been verified.

Next we shall discuss upper and lower bounds for the n-step transition probability $p_n(x,y)$ of the Markov chain X_n. In order to perform this, we make use of the n-step transition probability $p_n^0(x,y)$ of the transformed process X_n^0. A simple computation shows that, by (1),

$$p_n^0(x,y) = \sum_{x_1,\cdots,x_{n-1}\in\mathbb{Z}^d} p^0(x,x_1)p^0(x_1,x_2)\cdots p^0(x_{n-1},y) = \frac{p_n(x,y)u_0(y)}{(1-\lambda_0)^n u_0(x)}. \tag{7}$$

Note that the process X_n^0 is reversible with respect to the measure $\{\varphi_0(x)^2\}$, i.e.,

$$\varphi_0(x)^2 p_n^0(x,y) = \varphi_0(y)^2 p_n^0(y,x).$$

Set for $x \in \mathbb{Z}^d$ and $r > 0$,

$$B(x,r) = \{y \in \mathbb{Z}^d; d_1(x,y) \le r\}, \quad V(x,r) = \sum_{y\in B(x,r)} \varphi_0(y)^2.$$

We can easily verify that for any $x \in \mathbb{Z}^d$ and $r > 1$,

$$C_5 r^d \le V(x,r) \le C_6 r^d$$

with some positive constants C_i, $i = 5,6$. With these notations, we can apply Theorem 1.7, Delmotte[1] for the Markov chain X_n^0 under our assumptions. Thus we get the following Gaussian upper and lower estimates for $p_n^0(x,y)$.

PROPOSITION 1. *There exist positive constants C_i, $i = 7,8,9,10$ such that for all n,*

$$\frac{C_7\varphi_0(y)^2}{V(x,\sqrt{n})} e^{-\frac{C_8 d_1(x,y)^2}{n}} \le p_n^0(x,y) \le \frac{C_9\varphi_0(y)^2}{V(x,\sqrt{n})} e^{-\frac{C_{10} d_1(x,y)^2}{n}}$$

for all $x, y \in \mathbb{Z}^d$ satisfying $d_1(x,y) \le n$.

Proposition 1 combined with the relationship between $p_n(x,y)$ and $p_n^0(x,y)$ gives Theorem 2. Note that the nonnegativity of λ_0 in Theorem 1 is immediate from Theorem 2.

4. Proof of Theorem 3

We shall first prove the upper bound in the theorem. Set, for a strictly positive $u \in \mathcal{U}_0$,

$$V(x) = \pi_0 u(x) \quad \text{and} \quad W(x) = \log\Big(\frac{V(x)}{u(x)}\Big).$$

where π_0 is given by (2). From the definition of the pinned process, noting (7), we have

$$\begin{aligned}
&E^{P_{(0,x)}^{(n,y)}}\big[V(X_{n-1})\exp\big(-\sum_{k=0}^{n-1} W(X_k)\big)\big] \\
&= \frac{1}{p_n(x,y)} E^{P_x}\big[V(X_{n-1})\exp\big(-\sum_{k=0}^{n-1} W(X_k)\big)p(X_{n-1},y)\big] \\
&= \frac{1}{p_n(x,y)} \sum_{x_1,\cdots,x_{n-1}\in\mathbb{Z}^d} V(x_{n-1})\exp\big(-\sum_{k=0}^{n-1} W(x_k)\big)p(x_{n-1},y) \\
&\quad \times p(x,x_1)p(x_1,x_2)\cdots p(x_{n-2},x_{n-1}) \\
&= \frac{(1-\lambda_0)^{n-1}u_0(x)}{p_n(x,y)} E^{P_x^0}\big[V(X_{n-1}^0)\exp\big(-\sum_{k=0}^{n-1} W(X_k^0)\big)\frac{p(X_{n-1}^0,y)}{u_0(X_{n-1}^0)}\big].
\end{aligned}$$

Hence, since $z \to p(z,y)/u_0(z)$ is bounded above by a positive constant depending only on y, we have, by virtue of Theorem 2,

$$\begin{aligned}
&E^{P_{(0,x)}^{(n,y)}}\big[V(X_{n-1})\exp\big(-\sum_{k=0}^{n-1} W(X_k)\big)\big] \\
&\le C_{11}\frac{(1-\lambda_0)^{n-1}u_0(x)}{p_n(x,y)} E^{P_x^0}\big[V(X_{n-1}^0)\exp\big(-\sum_{k=0}^{n-1} W(X_k^0)\big)\big] \\
&\le C_{12} n^{d/2} u_0(y) e^{C_{13}d_1(x,y)^2/n} E^{P_x^0}\big[V(X_{n-1}^0)\exp\big(-\sum_{k=0}^{n-1} W(X_k^0)\big)\big] \\
&= C_{12} n^{d/2} u_0(y) e^{C_{13}d_1(x,y)^2/n} u(x)
\end{aligned}$$

for some $C_{11} > 0$, $C_{12} > 0$ and $C_{13} > 0$. Here, the last equality follows as in the arguments in Donsker and Varadhan[2], page 8. Thus we have obtained

$$E^{P^{(n,y)}_{(0,x)}}\big[\exp\big(-\sum_{k=0}^{n-1} W(X_k)\big)\big] \le C_{12}\frac{n^{d/2}u_0(y)u(x)e^{C_{13}\frac{d_1(x,y)^2}{n}}}{\inf_{x\in\mathbb{Z}^d} V(x)}. \tag{8}$$

On the other hand, by the definition of $Q^{(n,y)}_{(0,x)}$, we have

$$\begin{aligned} E^{P^{(n,y)}_{(0,x)}}\big[\exp\big(-\sum_{k=0}^{n-1} W(X_k)\big)\big] &= E^{P^{(n,y)}_{(0,x)}}\big[\exp\big(-n\sum_{k=0}^{n-1}\frac{W(c_0(X_k))}{n}\big)\big] \\ &= E^{P^{(n,y)}_{(0,x)}}\big[\exp\big(-n\sum_{z\in\mathbb{T}^d} W(z)L_n(\omega,\{z\})\big)\big] \\ &= E^{Q^{(n,y)}_{(0,x)}}\big[\exp\big(-n\sum_{z\in\mathbb{Z}^d} W(z)\mu(z)\big)\big], \end{aligned}$$

and by noting that for any measurable subset C of $\mathcal{M}$,

$$\begin{aligned} &E^{Q^{(n,y)}_{(0,x)}}\big[\exp\big(-n\sum_{z\in\mathbb{Z}^d} W(z)\mu(z)\big)\big] \\ &\ge Q^{(n,y)}_{(0,x)}(C)\exp\big(-n\sup_{\mu\in C}\sum_{z\in\mathbb{Z}^d} W(z)\mu(z)\big). \end{aligned} \tag{9}$$

Combining (8) and (9), we obtain

$$\begin{aligned} Q^{(n,y)}_{(0,x)}(C) \le C_{12}&\frac{n^{d/2}u_0(y)u(x)e^{C_{13}d_1(x,y)^2/n}}{\inf_{x\in\mathbb{Z}^d} V(x)} \\ &\times\exp\big(n\sup_{\mu\in C}\sum_{z\in\mathbb{T}^d}\log\big(\frac{\pi_0 u}{u}\big)(z)\mu(z)\big), \end{aligned}$$

and hence

$$\limsup_{n\to\infty}\frac{1}{n}\log Q^{(n,y)}_{(0,x)}(C) \le \inf_{\substack{u\in\mathcal{U}_0\\ u>0}}\sup_{\mu\in C}\sum_{z\in\mathbb{T}^d}\log\big(\frac{\pi_0 u}{u}\big)(z)\mu(z). \tag{10}$$

Following the same arguments as in Donsker and Varadhan [2], it follows from (10) that for any closed subset F of $\mathcal{M}$,

$$\limsup_{n\to\infty}\frac{1}{n}\log Q^{(n,y)}_{(0,x)}(F) \le -\inf_{\mu\in F} I_0(\mu),$$

which completes the proof of the upper bound of Theorem 3.

In order to prove the lower bound, some preliminaries are required. We shall start with

LEMMA 1. *Let μ be a probability measure on $\mathbb{T}^d$ of which support coincides with $\mathbb{T}^d$. Then there exists a positive function V_0 in $\mathcal{U}_0$ such that*

$$I_0(\mu) = -\sum_{z\in\mathbb{T}^d} \log\Big(\frac{\pi_0 V_0}{V_0}\Big)(z)\mu(z).$$

PROOF. We first introduce a one-step transition probability $\tilde{p}^0$ on $\mathbb{T}^d$ by

$$\tilde{p}^0(c_0(x), c_0(y)) = \sum_{k_1,\cdots,k_d\in\mathbb{Z}} p^0(x, y + k_1 m_0 e_1 + \cdots + k_d m_0 e_d) \tag{11}$$

for $x, y \in \mathbb{Z}^d$ and define an operator $\tilde{\pi}_0$ for a function u on $\mathbb{T}^d$ as

$$\tilde{\pi}_0 u(x) = \sum_{y\in\mathbb{T}^d} \tilde{p}^0(x, y)u(y), \quad x \in \mathbb{T}^d.$$

Then it is evident that

$$I_0(\mu) = -\inf_{u\in K} \sum_{x\in\mathbb{T}^d} \log\Big(\frac{\tilde{\pi}_0 u}{u}\Big)(x)\mu(x)$$

where

$$K = \Big\{u = (u(x))_{x\in\mathbb{T}^d};\ u(x) > 0 \text{ for all } x \in \mathbb{T}^d \text{ and } \sum_{x\in\mathbb{T}^d} u(x) = 1\Big\}.$$

Set, for all $u \in K$,

$$f(u) = \sum_{x\in\mathbb{T}^d} \log\Big(\frac{\tilde{\pi}_0 u}{u}\Big)(x)\mu(x).$$

Obviously f is continuous in u.

For the proof of our lemma, it suffices to show that f attains its minimal value on K. Since K is regarded as a subset of a hyperplane in an Euclidean space, denote by $\bar{K}$ the usual closure of K i.e.

$$\bar{K} = \Big\{u = \big(u(x)\big)_{x\in\mathbb{T}^d}; u(x) \geq 0,\ \sum_{x\in\mathbb{T}^d} u(x) = 1\Big\}.$$

Noting the irreducibility of the Markov chain associated with the one-step transition probability $\tilde{p}^0$, it is easy to verify that for any $u \in \bar{K} \setminus K$,

$$\lim_{K \ni v \to u} f(v) = \infty.$$

Combining this fact with the continuity of f on K, it can be proved that f attains its minimum on K. This completes the proof of Lemma 1. □

Now, with the help of V_0 in Lemma 1, let a new transition probability $\tilde{p}^1(x, y)$ on $\mathbb{T}^d$ be defined by

$$\tilde{p}^1(x, y) = \frac{\tilde{p}^0(x, y)V_0(y)}{\tilde{\pi}_0 V_0(x)}, \quad x, y \in \mathbb{T}^d, \tag{12}$$

where $\tilde{p}^0$ is given by (11). Then we have

LEMMA 2. $\{\tilde{p}^1(x, y)\}$ *constitutes the one-step transition probability of a reversible Markov chain on* $\mathbb{T}^d$ *having* μ *as its unique invariant probability measure.*

PROOF. Let, for all $u \in \mathcal{U}_o$,

$$\tilde{\pi}_1 u(x) = \sum_{y \in \mathbb{T}^d} \tilde{p}^1(x, y)u(y)$$

and set I_1 on $\mathcal{M}$ by

$$I_1(\nu) = -\inf_{\substack{u \in \mathcal{U}_0 \\ u > 0}} \sum_{x \in \mathbb{T}^d} \log\big(\frac{\tilde{\pi}_1 u}{u}\big)(x)\nu(x).$$

Following the same argument as in Donsker and Varadhan [2], it is shown that $I_1(\nu_0) = 0$ if and only if ν_0 is the invariant probability measure for $\{\tilde{p}^1(x, y)\}$.

Assume μ to be the one given in Lemma 1. For a positive $u \in \mathcal{U}_0$,

$$\begin{aligned} &\sum_{x \in \mathbb{T}^d} \log\big(\frac{\tilde{\pi}_1 u}{u}\big)(x)\mu(x) \\ &= \sum_{x \in \mathbb{T}^d} \log\big(\frac{\tilde{\pi}_0 (V_0 u)}{V_0 u}\big)(x)\mu(x) - \sum_{x \in \mathbb{T}^d} \log\big(\frac{\tilde{\pi}_0 V_0}{V_0}\big)(x)\mu(x). \end{aligned}$$

Taking the definition of V_0 into account, we see,

$$\begin{aligned} I_1(\mu) &= -\inf_{\substack{u\in\mathcal{U}_0\\u>0}}\sum_{x\in\mathbb{T}^d}\log\big(\frac{\tilde{\pi}_1 u}{u}\big)(x)\mu(x) \\ &= -\inf_{\substack{u\in\mathcal{U}_0\\u>0}}\sum_{x\in\mathbb{T}^d}\log\big(\frac{\tilde{\pi}_0(V_0 u)}{V_0 u}\big)(x)\mu(x) + \sum_{x\in\mathbb{T}^d}\log\big(\frac{\tilde{\pi}_0 V_0}{V_0}\big)(x)\mu(x) \\ &= 0. \end{aligned}$$

In order to prove the reversibility of μ, we introduce another measure μ' on $\mathbb{T}^d$ by

$$\mu'(x) = \varphi_0(x)^2 V_0(x)\tilde{\pi}_0 V_0(x).$$

It is easily seen that μ' is a reversible measure for $\{\tilde{p}^1(x,y)\}$ i.e. $\mu'(x)\tilde{p}^1(x,y) = \mu'(y)\tilde{p}^1(y,x)$, $x, y \in \mathbb{T}^d$. Note that a reversible measure is also an invariant measure. Combining these with the fact that the Markov chain has the unique invariant probability measure, we see that there exists a positive constant C such that $\mu'(x) = C\mu(x), x \in \mathbb{T}^d$. Thus μ is reversible. This completes the proof of Lemma 2. □

We are now ready to prove the lower bound. Let G be an open subset of $\mathcal{M}$ and μ an arbitrary element in G whose support coincides with $\mathbb{T}^d$. First note that $I_0(\mu) < \infty$. We want to show for μ,

$$\liminf_{n\to\infty}\frac{1}{n}\log P_{(0,x)}^{(n,y)}\big(\omega; L_n(\omega,\cdot)\in G\big) \geq -I_0(\mu). \tag{13}$$

Denote by $S(\mu,\epsilon)$ the open sphere of radius $\epsilon > 0$ in $\mathcal{M}$ centered at μ. Suppose ϵ is sufficiently small such that $S(\mu,\epsilon) \subseteq G$. For the proof of (13), it suffices to verify

$$\liminf_{n\to\infty}\frac{1}{n}\log P_{(0,x)}^{(n,y)}\big(\omega; L_n(\omega,\cdot)\in S(\mu,\epsilon)\big) \geq -I_0(\mu). \tag{14}$$

From the definition of the pinned process, we have

$$\begin{aligned} &P_{(0,x)}^{(n,y)}(L_n\in S(\mu,\epsilon)) \\ &= \frac{1}{p_n(x,y)}E^{P_x}\big[p(X_{n-1},y); L_n\in S(\mu,\epsilon)\big], \\ &= \frac{(1-\lambda_0)^{n-1}u_0(x)}{p_n(x,y)}E^{P_x^0}\big[\frac{p(X_{n-1}^0,y)}{u_0(X_{n-1}^0)}; L_n\in S(\mu,\epsilon)\big]. \end{aligned} \tag{15}$$

Making use of the function V_0 in Lemma 1, a new transition probability on $\mathbb{Z}^d$ is defined by

$$p^1(x,y) = \frac{p^0(x,y)V_0(y)}{\pi_0 V_0(x)}, \quad x, y \in \mathbb{Z}^d.$$

Let (X_n^1, P_x^1) be the Markov chain on $\mathbb{Z}^d$ associated with the transition probability $\{p^1(x,y)\}$. Note that p^1 is periodic and $c_0(X_n^1)$ is the Markov chain on $\mathbb{T}^d$ whose one-step transition probability coincides with $\tilde{p}^1(x,y)$ given by (12). Since

$$\begin{aligned}
&E^{P_x^0}\big[\frac{p(X_{n-1}^0, y)}{u_0(X_{n-1}^0)}; L_n \in S(\mu,\epsilon)\big] \\
&= \sum_{x_1,\cdots,x_{n-1}\in\mathbb{Z}^d} \frac{p(x_{n-1},y)}{u_0(x_{n-1})}\chi_{\{L_n\in S(\mu,\epsilon)\}} p^0(x,x_1)p^0(x_1,x_2)\cdots p^0(x_{n-2},x_{n-1}) \\
&= \sum_{x_1,\cdots,x_{n-1}\in\mathbb{Z}^d} \frac{p(x_{n-1},y)}{u_0(x_{n-1})}\chi_{\{L_n\in S(\mu,\epsilon)\}} \big(\frac{\pi_0 V_0(x) p^1(x,x_1)}{V_0(x_1)}\big) \\
&\qquad \times \big(\frac{\pi_0 V_0(x_1) p^1(x_1,x_2)}{V_0(x_2)}\big)\cdots\big(\frac{\pi_0 V_0(x_{n-2}) p^1(x_{n-2},x_{n-1})}{V_0(x_{n-1})}\big) \\
&= E^{P_x^1}\big[\frac{p(X_{n-1}^1, y)}{u_0(X_{n-1}^1)}\big(\prod_{k=0}^{n-1}\frac{\pi_0 V_0(X_k^1)}{V_0(X_k^1)}\big)\frac{V_0(x)}{\pi_0 V_0(X_{n-1}^1)}; L_n \in S(\mu,\epsilon)\big],
\end{aligned}$$

by (15), we see that

$$\begin{aligned}
&P_{(0,x)}^{(n,y)}(L_n \in S(\mu,\epsilon)) \\
&\geq \frac{(1-\lambda_0)^{n-1}u_0(x)}{p_n(x,y)} \frac{V_0(x)}{\sup_{x\in\mathbb{Z}^d}\pi_0 V_0(x)} \\
&\qquad \times E^{P_x^1}\big[\frac{p(X_{n-1}^1, y)}{u_0(X_{n-1}^1)}\exp\big(\sum_{k=0}^{n-1}\log\frac{\pi_0 V_0(X_k^1)}{V_0(X_k^1)}\big); L_n \in S(\mu,\epsilon)\big].
\end{aligned} \tag{16}$$

Now we introduce, for $\epsilon' > 0$,

$$\begin{aligned}
S_1(n,\epsilon') = \big\{\omega;\ \big|&\sum_{x\in\mathbb{T}^d}\log\big(\frac{\pi_0 V_0}{V_0}\big)(x)L_n(\omega,\{x\}) \\
&- \sum_{x\in\mathbb{T}^d}\log\big(\frac{\pi_0 V_0}{V_0}\big)(x)\mu(x)\big| < \epsilon'\big\},
\end{aligned}$$

$$S_2(n,\epsilon') = \{\omega;\ L_n(\omega,\cdot) \in S(\mu,\epsilon) \cap S_1(\mu,\epsilon')\},$$

and

$$\Omega_1 = \Big\{\omega; \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \log\Big(\frac{\pi_0 V_0}{V_0}\Big)(X_k^1) = \sum_{x\in\mathbb{T}^d} \log\Big(\frac{\pi_0 V_0}{V_0}\Big)(x)\mu(x)\Big\},$$
$$\Omega_2 = \{\omega;\ L_n(\omega,\cdot) \text{ is convergent to } \mu\}.$$

Since $\pi_0 V_0/V_0$ is periodic, the ergodic theorem applied to $c_0(X_n^1)$ implies

$$P_x^1(\Omega_1) = P_x^1(\Omega_2) = 1.$$

Therefore, noting Lemma 1,

$$\begin{aligned} &E^{P_x^1}\Big[\frac{p(X_{n-1}^1,y)}{u_0(X_{n-1}^1)} \exp\Big(\sum_{k=0}^{n-1} \log \frac{\pi_0 V_0(X_k^1)}{V_0(X_k^1)}\Big);\ L_n \in S(\mu,\epsilon)\Big] \\ &\geq \exp\Big\{n\Big(\sum_{x\in\mathbb{T}^d} \log\Big(\frac{\pi_0 V_0}{V_0}\Big)(x)\mu(x) - \epsilon'\Big)\Big\} E^{P_x^1}\Big[\frac{p(X_{n-1}^1,y)}{u_0(X_{n-1}^1)}; S_2(n,\epsilon')\Big] \qquad (17) \\ &= \exp(-nI_0(\mu) - n\epsilon') E^{P_x^1}\Big[\frac{p(X_{n-1}^1,y)}{u_0(X_{n-1}^1)}; S_2(n,\epsilon')\Big]. \end{aligned}$$

From the assumption (A.2), the set $\{x \in \mathbb{Z}^d; d_1(x,y) \leq 1\}$ is finite. Denote this set by $\{y_1,\cdots,y_m\}$. Set $\alpha = \inf_{1\leq j\leq m} \frac{p(y_j,y)}{u_0(y_j)}$. Then,

$$\begin{aligned} &E^{P_x^1}\Big[\frac{p(X_{n-1}^1,y)}{u_0(X_{n-1}^1)}; S_2(n,\epsilon')\Big] \\ &= E^{P_x^1}\Big[\frac{p(X_{n-1}^1,y)}{u_0(X_{n-1}^1)}; S_2(n,\epsilon') \cap \{d_1(X_{n-1}^1,y) \leq 1\}\Big] \\ &= \sum_{j=1}^m E^{P_x^1}\Big[\frac{p(y_j,y)}{u_0(y_j)}; S_2(n,\epsilon') \cap \{X_{n-1}^1 = y_j\}\Big] \qquad (18) \\ &\geq \alpha \sum_{j=1}^m P_x^1(S_2(n,\epsilon') \cap \{X_{n-1}^1 = y_j\}) \\ &= \alpha \sum_{j=1}^m P_x^1(X_{n-1}^1 = y_j) - \alpha \sum_{j=1}^m P_x^1(\{X_{n-1}^1 = y_j\} \cap \{\Omega \setminus S_2\}) \\ &= (I) - (II), \quad \text{say.} \end{aligned}$$

Regarding $\mu(x)$ as a periodic function of period d_0 on $\mathbb{Z}^d$, it can be easily checked with the help of Lemma 2 that the transition probability $p_n^1(x,y)$ of the chain X_n^1 is also reversible with respect to μ. Thus we can again apply Theorem 1.7 in Delmotte[1] to the transition probability $p_n^1(x,y)$ with the distance $d_1(x,y)$ and obtain a Gaussian lower estimate for $p_n^1(x,y)$. Thus we get

$$P_x^1(X_{n-1}^1 = y_j) = p_{n-1}^1(x, y_j) \geq \frac{C_{14} e^{-\frac{C_{15} d_1(x,y_j)^2}{n-1}}}{\sqrt{(n-1)^d}}$$

for some $C_{14} > 0$ and $C_{15} > 0$, and thus we have

$$(I) \quad \geq C_{16} \sum_{j=1}^{m} \frac{e^{-\frac{C_{15} d_1(x,y_j)^2}{n-1}}}{\sqrt{(n-1)^d}}. \tag{19}$$

for some $C_{16} > 0$. As for the second term (II), since the process $c_0(X_n^1)$ is an irreducible Markov chain on $\mathbb{T}^d$, the process is ergodic on $\mathbb{T}^d$. Consequently the large deviation prinsiple in Donsker and Varadhan[2] is proved to hold for the process $c_0(X_n^1)$, which implies

$$(II) \leq C_{17} P_x^1(\Omega \setminus S_2) \leq C_{18} e^{-C_{19} n} \tag{20}$$

with some positive constants C_{17}, C_{18} and C_{19}. Combining (16), (17), (18), (19) and (20), along with Theorem 2, it follows that

$$\liminf_{n\to\infty} \frac{1}{n} \log P_{(0,x)}^{(n,y)}(\omega;\, L_n(\omega,\cdot) \in S(\mu,\epsilon)) \geq -I_0(\mu) - \epsilon'.$$

Since ϵ' is arbitrary, the above implies

$$\liminf_{n\to\infty} \frac{1}{n} \log P_{(0,x)}^{(n,y)}(\omega;\, L_n(\omega,\cdot) \in S(\mu,\epsilon)) \geq -I_0(\mu)$$

and thus

$$\liminf_{n\to\infty} \frac{1}{n} \log P_{(0,x)}^{(n,y)}(\omega;\, L_n(\omega,\cdot) \in G) \geq -\inf_{G\cap\mathcal{M}_1} I_0(\mu),$$

where $\mathcal{M}_1$ denotes the set of probability measures μ on $\mathbb{T}^d$ which have the full support. By the same argument as in Donsker and Varadhan[2], we have

$$\inf_{\mu\in G\cap\mathcal{M}_1} I_0(\mu) = \inf_{\mu\in G} I_0(\mu).$$

This completes the proof of Theorem 3.

5. Positivity of the Bottom of the Spectrum

In this section we shall discuss the positivity of the bottom of the spectrum of the difference operator L. To perform this, define for $\Phi(x) =^t (x_1, \cdots, x_d) \in \mathbb{Z}^d$,

$$Y_n = \Phi(X_n) - \sum_{k=0}^{n-1} L\Phi(X_k)$$

and $\mathcal{F}_n = \sigma(X_1.X_2, \cdots, X_n)$, $n = 1, 2, \cdots$ for the above Markov chain X_n.

It is easy to see the following.

LEMMA 3. *Under the assumptions (A.1), (A.2), $(Y_n, \mathcal{F}_n)_{n \geq 1}$ is a d-dimensional vector-valued martingale which satisfies the following conditions:*

(1) $E|Y_n|^2 < \infty, \quad n = 1, 2, \cdots,$

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{E|Y_n - Y_{n-1}|^2}{n^2} < \infty.$

Applying a strong law of large numbers for martingales(see e.g. Shiryaev[5], Corollary 2, page 471-472) to $\{Y_n\}$, we get

$$\lim_{n\to\infty} \frac{Y_n}{n} = 0.$$

Note that

$$L\Phi(x) = \sum_{y \in \mathbb{Z}^d} p(x, y)(\Phi(y) - \Phi(x))$$

is a periodic function of period m_0. Combining these facts with the ergodicity of the Markov chain $c_0(X_n)$ on $\mathbb{T}^d$, we have;

PROPOSITION 2. *It holds that*

$$\lim_{n\to\infty} \frac{X_n}{n} = \sum_{x \in \mathbb{T}^d} L\Phi(x)\mu_0(x) \quad a.e.\ P.$$

where μ_0 *is the unique invariant probability measure of the process* $c_0(X_n)$*.*

The following theorem gives a criterion for the positivity of the bottom of the spectrum.

THEOREM 4. *The following two conditions are equivalent;*

(1) $\lambda_0 > 0$.

(2) $\sum_{x \in \mathbb{T}^d} L\Phi(x)\mu_0(x) \neq 0$.

PROOF. For the proof of Theorem 4, a large deviation result due to Ellis[3] is required. Note that, for all $t \in \mathbb{Z}^d$,

$$\begin{aligned} &E_x\big[\exp\langle t, X_n\rangle\big] \\ &\quad = \sum_{y \in \mathbb{Z}^d} p_n(x,y)e^{\langle t,y\rangle} \\ &\quad = \sum_{y_1,\cdots,y_n \in \mathbb{Z}^d} e^{\langle t,x\rangle}\frac{p(x,y_1)e^{\langle t,y_1\rangle}}{e^{\langle t,x\rangle}}\frac{p(y_1,y_2)e^{\langle t,y_2\rangle}}{e^{\langle t,y_1\rangle}}\cdots\frac{p(y_{n-1},y_n)e^{\langle t,y_n\rangle}}{e^{\langle t,y_{n-1}\rangle}} \\ &\quad = \sum_{y_1,\cdots,y_n \in \mathbb{Z}^d} e^{\langle t,x\rangle}q_t(x,y_1)q_t(y_1,y_2)\cdots q_t(y_{n-1},y_n), \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product on $\mathbb{R}^d$ and $q_t(x,y) = p(x,y)e^{\langle t,y\rangle}/e^{\langle t,x\rangle}$. Since $q_t(x,y)$ is periodic of period m_0 in x, y, setting

$$\tilde{q}_t(c_0(x),c_0(y)) = \sum_{k_1,\cdots,k_d \in \mathbb{Z}} q_t(x, y + k_1 m_0 e_1 + \cdots + k_d m_0 e_d), \quad x,y \in \mathbb{Z}^d,$$

we have

$$E_x\big[e^{\langle t,X_n\rangle}\big] = e^{\langle t,x\rangle}\sum_{\tilde{y}_1,\cdots,\tilde{y}_n \in \mathbb{T}^d}\tilde{q}_t(c_0(x),\tilde{y}_1)\cdots\tilde{q}_t(\tilde{y}_{n-1},\tilde{y}_n).$$

Regarding $\tilde{q}_t$ as the matrix having elements $\tilde{q}_t(\tilde{x},\tilde{y})$, we have

$$E_x\big[e^{\langle t,X_n\rangle}\big] = e^{\langle t,x\rangle}\sum_{\tilde{y} \in \mathbb{T}^d}(\tilde{q}_t)^n(c_0(x),\tilde{y}). \tag{21}$$

We now apply Perron-Frobenius Theorem for matrices with non-negative components $\big(\tilde{q}(\tilde{x},\tilde{y})\big)_{\tilde{x},\tilde{y}\in\mathbb{T}^d}$ to obtain:

$$\lim_{n\to\infty}\frac{1}{n}\log E_x\big[e^{\langle t,X_n\rangle}\big] \stackrel{\text{def}}{=} c(t) \tag{22}$$

exists and c(t) is equal to the logarithm of the positive maximal eigenvalue of $\tilde{q}_t$. From the fact that the stochastic matrix $(p(x,y)) = (q_0(x,y))$ is irreducible, it is easily checked that the positive maximal eigenvalue of $\tilde{q}_0$ is simple. Therefore the implicit function theorem combined with Perron-Frobenius Theorem applied to (26) implies that the function $c(t)$ is differentiable in a neighbourhood of the origin $t = 0$. Accordingly Theorem IV in Ellis[3] along with Proposition 2 gives that for any $\epsilon > 0$, there exists a positive number M_ϵ such that

$$P_x\big(\big|\frac{X_n}{n} - \sum_{x\in\mathbb{T}^d} L\Phi(x)\mu_0(x)\big| \geq \epsilon\big) \leq \exp(-M_\epsilon n). \tag{23}$$

Now notice that we have the following inequality for any positive constant C_{20},

$$\begin{aligned} &P_x\big(|X_n| \leq C_{20}\big) \\ &\leq P_x\big(\big|\frac{X_n}{n} - \sum_{x\in\mathbb{T}^d} L\Phi(x)\mu_0(x)\big| \geq \big|\sum_{x\in\mathbb{T}^d} L\Phi(x)\mu_0(x)\big| - \frac{C_{20}}{n}\big). \end{aligned} \tag{24}$$

Thus (27) shows that the left-hand side in (28)is exponentially decaying as n tends to ∞ if $\sum_{x\in\mathbb{T}^d} L\Phi(x)\mu_0(x) \neq 0$. Thus (2) in Theorem 4 implies (1).

Conversely, suppose the condition (1) in Theorem 4 holds. Then it can be easily checked by means of Theorem 2 that there exist two positive constants C_{21} and C_{22} for a sufficiently small, positive constant ϵ_0 such as

$$P_0\big(\big|X_n\big| \leq \epsilon_0 n\big) \leq C_{21}e^{-C_{22}n}.$$

This together with (27) asserts that

$$\sum_{x\in\mathbb{T}^d} L\Phi(x)\mu_0(x) \neq 0.$$

Thus the conditions (1) and (2) are mutually equivalent. □

6. Example

In this section we shall give an example which illustrates our theorems. Suppose $\{p(x,y)\}$ to be a one-step transition probability on $\mathbb{Z}^d$ assigned as

$$p(x,y) = \begin{cases} p_i, & y = x + e_i \\ q_i, & y = x - e_i \\ r, & y = x \\ 0, & |y - x| > 1 \end{cases}$$

where $p_i, q_i, r > 0$ and $\sum_{i=1}^d (p_i + q_i) + r = 1$. It is evident that the condition (A.1) holds with any positive integer m_0. Setting $a(x) = \Pi_{i=1}^d \big(p_i/q_i\big)^{x_i}$, $x = (x_1, \cdots, x_d)$, it is easy to check that

$$a(x)p(x,y) = a(y)p(y,x), \quad x, y \in \mathbb{Z}^d.$$

Thus the Markov chain associated with $\{p(x,y)\}$ above fulfills all the conditions (A.1)-(A.4). Simple computations show:

$$\lambda_0 = 1 - r - \sum_{i=1}^d 2\sqrt{p_i q_i} = \sum_{i=1}^d (\sqrt{p_i} - \sqrt{q_i})^2,$$
$$u_0(x) = \frac{1}{\sqrt{a(x)}},$$

and $p^0(x,y) = \frac{p(x,y)u_0(y)}{(1-\lambda_0)u_0(x)}$ is as follows:

$$p^0(x,y) = \begin{cases} \frac{\sqrt{p_i q_i}}{r + 2\sum_{i=1}^d \sqrt{p_i q_i}}, & y = x + e_i \\ \frac{\sqrt{p_i q_i}}{r + 2\sum_{i=1}^d \sqrt{p_i q_i}}, & y = x - e_i \\ \frac{r}{r + 2\sum_{i=1}^d \sqrt{p_i q_i}}, & y = x \\ 0, & |y - x| > 1. \end{cases}$$

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