J. Math. Sci. Univ. Tokyo
13 (2006), 439–523.

Open Book Structures on $(n-1)$-Connected $(2n+1)$-Manifolds

By Oziride Manzoli Neto, Sadao Massago* and Osamu Saeki†

Abstract. We completely classify simple open book structures on $(n-1)$-connected closed $(2n+1)$-dimensional manifolds for $n \geq 4$, $n \neq 7$, and on $(n-1)$-connected rational homology $(2n+1)$-spheres for $n = 3, 7$, using their algebraic topological invariants. This generalizes some known results about the classification of fibered knots in spheres and the existence of open book structures on manifolds. We also give applications and examples so as to show the effectiveness of our classification.

1. Introduction

In the topological study of isolated singularities of complex hypersurfaces in $\mathbb{C}^{n+1}$, a special kind of codimension two submanifolds in S^{2n+1}, called *simple fibered knots*, play an important role [Mil68]. These submanifolds are highly connected and their complements fiber over the circle.

A natural generalization of simple fibered knots, called an *open book* (or an *open book structure*), consists of a highly connected codimension two submanifold K in a highly connected closed manifold M, and a fibration of $M - K$ over the circle S^1, satisfying certain conditions (see Definition 2.1 for details). Note that the case discussed by Milnor [Mil68] is a particular case where $M = S^{2n+1}$. As a generalization of Milnor's fibration theorem, Hamm [Ham71] and Lê [Lê92] have shown that open book structures on manifolds not necessarily diffeomorphic to S^{2n+1} appear naturally around isolated singularities of complex analytic functions on certain complex varieties (see

*The second author has been partially supported by CNPq, Brazil.
†The third author has been partially supported by Grant-in-Aid for Scientific Research (No. 13640076), Ministry of Education, Science and Culture, Japan.
2000 *Mathematics Subject Classification*. Primary 57Q45; Secondary 57R40.
Key words: Fibered knot, isotopy of embeddings, Milnor fibration, open book, rational Seifert form, spinnable structure.

also [Kin97]). This gives us a good motivation for the study of open book structures on general manifolds.

Historically, the terminology "open book" was introduced by Winkelnkemper [Win73], who proved that an arbitrary simply connected closed $(2n+1)$-dimensional manifold with $n > 3$ admits an open book structure. Independently, Tamura [Tam73] proved a similar result, although he used the terminology "spinnable structure". Then, Lawson [Law78] proved that the simply connectedness is not necessary for $n \geq 3$, and Quinn [Qui79] generalized the result of Lawson, for $n \geq 2$, and studied open book structures on manifolds with boundary. On the other hand, for simply connected 5-dimensional manifolds A'Campo [A'C72] obtained an existence theorem, and for 3-dimensional manifolds Alexander [Ale23] obtained an existence theorem. These theorems have been used to obtain certain interesting properties of these manifolds, and the authors have not worked over the classification problem in general cases.

The special case of open book structures on the sphere S^{2n+1}, $n \geq 2$, called "simple fibered knots", have been studied by several authors ([Ker65, Lev70, Dur74, Kat74, Sae99]) and classification theorems have been obtained for $n \geq 3$. In the classification, Seifert linking forms associated with a fiber of the fibration over the circle has played an essential role. Note that linking numbers can be naturally defined in S^{2n+1}, but not in general manifolds.

The purpose of this paper is to classify completely the open book structures on highly connected odd dimensional manifolds, using certain invariants, much more sophisticated than the Seifert linking form for the sphere case.

Let M be an $(n-1)$-connected closed $(2n+1)$-dimensional manifold. If it admits an open book structure, then the closure F of a fiber, called a *page*, of the fibration over the circle is a codimension one submanifold, and we can consider the following invariants associated with F: the homomorphism $i_{F*} : H_n(F;\mathbb{Z}) \to H_n(M;\mathbb{Z})$ induced by the inclusion $i_F : F \to M$, the intersection form Q_F on $H_n(F;\mathbb{Z})$, and the tangential invariant $\alpha_F : H_n(F;\mathbb{Z}) \to \pi_{n-1}(SO(n))$, where the last invariant measures the twist of the normal bundle of an embedded sphere representing a given homology class and has been introduced and studied by Wall [Wal62, Wal63]. Furthermore, we can define a Seifert linking form Γ_F over the *rational* numbers on the smallest direct summand of $H_n(F;\mathbb{Z})$ containing $\operatorname{Ker} i_{F*}$. These are the invariants that we consider in our work. It will turn out that they should satisfy certain relations among themselves and also some relations to the ambient manifold M. We call the set of the above invariants a *system of open book invariants* associated with an open book structure.

Given a manifold M as above, we can define a system of open book invariants purely algebraically, without the use of open book structures. Namely, we consider a finitely generated free $\mathbb{Z}$-module G, a homomorphism $i_G : G \to H_n(M;\mathbb{Z})$, a bilinear form Q_G on G, a certain map $\alpha_G : G \to \pi_{n-1}(SO(n))$, and a bilinear form Γ_G over the rational numbers defined on the smallest direct summand of G containing $\operatorname{Ker} i_G$. These should satisfy certain properties. We can also define a natural equivalence relation for such systems of open book invariants. Let $\mathcal{A}(M)$ denote the set of all equivalence classes of systems of open book invariants defined purely algebraically as above, for a given manifold M.

For open book structures on a manifold M, we can define the natural equivalence relation as follows. Two open book structures on M are *structurally isotopic* (or *isotopic through open books*), if there exists an ambient isotopy of M sending the fibration structure of one open book to that of the other. Such an equivalence has already been considered by Durfee [Dur74] in the case of M being the sphere. Let $\mathcal{OB}(M)$ denote the set of all structural isotopy classes of open book structures on M.

The main results of this paper are as follows. In the following, we denote by (K,φ) an open book structure on a manifold M, where K is the codimension two submanifold, called a *binding*, and $\varphi : M - K \to S^1$ is the fibration.

THEOREM 5.15. *Let M be an $(n-1)$-connected closed oriented $(2n+1)$-dimensional manifold with $n \geq 4, n \neq 7$, or an $(n-1)$-connected oriented rational homology $(2n+1)$-sphere with $n = 3, 7$. Then the map*

$$\mathcal{S} : \mathcal{OB}(M) \to \mathcal{A}(M)$$

defined by sending each structural isotopy class of a simple and oriented open book structure (K,φ) on M to the equivalence class $\mathcal{S}(K,\varphi)$ of its system of open book invariants establishes a one-to-one correspondence between the set $\mathcal{OB}(M)$ of all structural isotopy classes of simple and oriented open book *structures on M and the set $\mathcal{A}(M)$ of all equivalence classes of systems of open book invariants with respect to M.*

THEOREM 6.4. *Suppose that K is an $(n-2)$-connected closed oriented $(2n-1)$-dimensional manifold embedded in an $(n-1)$-connected closed oriented $(2n+1)$-dimensional manifold M with $n \geq 4, n \neq 7$, or in an $(n-1)$-connected oriented rational homology $(2n+1)$-sphere with $n = 3, 7$. Then all simple and oriented open book structures on M with binding K are structurally isotopic.*

THEOREM 6.6. *Let K be an $(n-2)$-connected closed $(2n-1)$-dimensional manifold embedded in an $(n-1)$-connected closed $(2n+1)$-dimensional manifold M with $n \geq 3$. Then we have the following.*

(1) *The submanifold K is the binding of some open book structure (which is not necessarily simple) on M with simply connected page F, if and only if the normal bundle of K in M is trivial (or equivalently, the tubular neighborhood $N(K)$ of K is trivial), $\pi_1(E) \cong \mathbb{Z}$, and $\pi_i(E)$ are finitely generated for all i, where $E = \overline{M - N(K)}$.*

(2) *The above open book is simple, if and only if $\pi_i(E) = 0$ for $i = 2, 3, \ldots, n-1$.*

Theorem 6.4 gives a uniqueness of an open book structure associated with a fixed binding. Due to this theorem, we can consider the system of invariants of an open book structure as a complete invariant of the binding as a codimension two embedding. Theorem 6.6 gives necessary and sufficient conditions for a codimension two embedding to be a binding of a simple open book structure.

The present paper is organized as follows.

In §2, we introduce the concept of an open book and review some results of fiber bundles over spheres which will be used in this paper.

In §3, we define and analyze the invariants associated with an open book structure which will be used in the subsequent sections.

In §4, we give an isotopy criterion for open book structures, using the invariants introduced in §3. This shows that the map $\mathcal{S}$ of Theorem 5.15 is injective.

In §5, we construct an open book structure on M, corresponding to a given system of open book invariants (Theorem 5.15), obtaining the realization of invariants introduced in §3. This shows that the map $\mathcal{S}$ of Theorem 5.15 is surjective, and hence it is bijective.

In §6, we analyze open book structures associated with a given binding, obtaining the uniqueness of the associated open book structure (Theorem 6.4), which has been known for open book structures on spheres S^{2n+1} for $n \geq 3$ (see [Dur74]). Another important result is a characterization of codimension two embeddings which are bindings of some open book structures (Theorem 6.6).

In §7, we study decompositions of open books with respect to connected sum, as an application of our classification theorem of open book structures. As an example of open books which are not decomposable, we introduce the notion of a minimal open book structure, and prove its existence. We also give some examples which have interesting properties with respect to the decomposition. These show that our classification is effective in a sense that the elements of $\mathcal{A}(M)$ can be computable.

Finally, in §8, we introduce the notion of a variation map associated with a diffeomorphism of a manifold with boundary which is the identity on the boundary. When applied to the monodromy diffeomorphism of an open book, this defines an invariant of an open book. It turns out that giving the variation map is equivalent to giving the rational Seifert form for an open book, which has been known for the spherical case [Kau74]. Furthermore, we use variation maps together with our classification theorem of open book structures to give an isotopy criterion for certain diffeomorphisms of manifolds with boundary.

Note that the third author [Sae99, Sae02] has developed a theory of open book structures on simply connected 5-dimensional manifolds; however, the results obtained therein are not complete as in this paper because of the difficulty in dealing with manifolds of dimensions three and four.

Throughout the paper all (co)homology groups are with integer coefficients, and manifolds and maps are differentiable of class C^∞, unless otherwise mentioned. The symbol "$\cong$" denotes a diffeomorphism between manifolds, or an appropriate isomorphism between algebraic objects, and "id" the identity map.

Observe that an open book and its system of invariants are denoted by (M, K, φ) and $\mathcal{S}(M, K, \varphi)$ respectively (see Definition 3.23) until §5, where $\varphi : M - K \to S^1$ is the associated fibration. However, from §6, sometimes we denote them simply by (M, K) and $\mathcal{S}(M, K)$ respectively, due to Theorem 6.4. When the ambient manifold M is obvious, sometimes we denote an open book by (K, φ) and its system of invariants by $\mathcal{S}(K, \varphi)$.

A large part of this paper has been written in the second author's PhD thesis [Mas00]. The first and the second authors would like to thank the people at Hiroshima University during their stay there. The third author would like to thank the people at ICMC, University of São Paulo, for their hospitality during his visit there. Finally, the authors would like to thank the referee for carefully reading the manuscript and for various helpful comments.

2. Preliminaries

In this section, we shall recall some basic definitions and properties of open book structures on closed odd dimensional manifolds. We also recall some facts about fibrations over spheres which will be necessary in the subsequent sections.

2.1. Open book structures

DEFINITION 2.1. Let K be a smoothly embedded closed $(2n-1)$-dimensional manifold in a closed $(2n+1)$-dimensional manifold M. Suppose that there exist a trivialization $\tau : K \times D^2 \to N(K)$ of the tubular neighborhood $N(K)$ of K in M and a smooth fibration $\varphi : M - K \to S^1$ such that the following diagram is commutative:

$$\begin{array}{ccc} K \times (D^2 - \{0\}) & \xrightarrow{\tau} & N(K) - K \\ & {}_p\searrow \quad \swarrow_{\varphi} & \\ & S^1, & \end{array}$$

where p denotes the obvious projection. Then the triple (M, K, φ) is called an *open book* and the pair (K, φ) is called an *open book structure* on M. Furthermore, K is called the *binding* and the closure in M of each fiber $F_t = \overline{\varphi^{-1}(t)}, t \in S^1$, is called a *page*. We call $F = F_0$, $0 \in S^1 = \mathbb{R}/\mathbb{Z}$, the *typical page* of the open book. Note that each page F_t is a compact

$2n$-dimensional manifold with boundary $\partial F_t = K$: in other words, it can be regarded as a *Seifert manifold* for the embedded manifold K.

DEFINITION 2.2. An open book (M, K, φ) is said to be *simple*, if K is $(n-2)$-connected, and both M and F are $(n-1)$-connected, where F denotes a page.

We will often use the following lemma, which can be proved by using standard arguments in algebraic topology together with Smale's result [Sma62]. See [Mas00] for details.

LEMMA 2.3. *Let F be an $(n-1)$-connected compact $2n$-dimensional manifold with boundary $\partial F = K \neq \emptyset$. Then for $n > 2$, the following three are equivalent to each other.*

(1) *The manifold F is $(n-1)$-connected and K is $(n-2)$-connected.*

(2) *The manifold F is homotopy equivalent to a bouquet of n-spheres.*

(3) *The manifold F decomposes as $D^{2n} \cup h_1 \cup \cdots \cup h_r$, where $r = \operatorname{rank} H_n(F)$ and h_i are n-handles attached to the zero handle D^{2n} simultaneously along an $(n-1)$-dimensional link in ∂D^{2n}.*

DEFINITION 2.4. We say that an open book (M, K, φ) is *oriented*, if M is oriented, and the pages have orientations compatible with the fibration $\varphi : M - K \to S^1$, where we fix an orientation of S^1 once and for all.

DEFINITION 2.5. Let F be the typical page of an open book (M, K, φ). We identify S^1 with $\mathbb{R}/\mathbb{Z}$ and for the fibration $\varphi : M - K \to S^1$, set $F_t = \overline{\varphi^{-1}(t)}, t \in S^1$. The vector field on M obtained as a pull-back of the canonical vector field on S^1 determines a one-parameter family of diffeomorphisms $\nu_t : M \to M$, $t \in \mathbb{R}$, such that $\nu_0 = \mathrm{id} : M \to M$, $\nu_t|_{F_0} : F_0 \to F_t$ and $\nu_t|_K = \mathrm{id} : K \to K$. The diffeomorphism $h = \nu_1 : F \to F$ is called the *characteristic map* of the fibration, which is determined uniquely by the fibration φ up to isotopy. We also call h the (geometric) *monodromy* of the open book.

An open book can always be obtained by the following construction.

DEFINITION 2.6 ([Win73, Qui79]). Let F be a connected compact oriented $2n$-dimensional manifold with boundary K and $h : F \to F$ an orientation preserving diffeomorphism such that $h|_K = \mathrm{id}$. Then the *mapping torus* of h is defined as $E = F \times I/\{(x,1) \sim (h(x),0)\}$, where $I = [0,1]$, and its boundary is naturally identified with $K \times S^1$. Gluing $K \times D^2$, using the natural identification $K \times S^1 = \partial(K \times D^2)$, to the mapping torus of $h|_K = \mathrm{id}$, we obtain the *relative mapping torus*

$$M = F \times I/\{(x,1) \sim (h(x),0)\} \cup_{K\times S^1} K \times D^2 = E \cup_{K\times S^1} K \times D^2.$$

By extending the projection $p : F \times I/\{(x,1) \sim (h(x),0)\} \to S^1 \cong \mathbb{R}/\mathbb{Z}$ defined by $p(x,t) = t$ for $x \in F$ and $t \in S^1 = \mathbb{R}/\mathbb{Z}$, we can construct the fibration $\varphi : M - (K \times \{0\}) \to S^1$, which is possible, since $K \times (D^2 - \{0\})$ is a collar neighborhood of $\partial E = K \times S^1$. Then the triple $(M, K \times \{0\}, \varphi)$ is an open book. Such a construction is called an *open book construction*. Note that its typical page can be identified with $F = F \times \{0\}$ and that its geometric monodromy coincides with h. It is easy to see that an arbitrary open book can be constructed in this way and that the isotopy class relative to boundary of the monodromy diffeomorphism h completely determines the open book.

It is easy to show that if F and K are $(n-1)$- and $(n-2)$-connected respectively, then M is $(n-1)$-connected, and hence the open book is simple.

REMARK 2.7. Open books have been first defined and studied independently by Winkelnkemper [Win73] and Tamura [Tam73], although Tamura called them *spinnable structures* (see also [Kat74]). In the special case where the ambient manifold M is the $(2n+1)$-sphere [Dur74] or the binding is the $(2n-1)$-sphere [Tam93], an open book structure is called a *fibered knot*.

It is known that every closed $(2n+1)$-dimensional manifold with $n \geq 1$ admits an open book structure [Ale23, Win73, Tam73, A'C72, Law78, Qui79].

For a later use, we present the following notion of trivial open books.

DEFINITION 2.8. A simple open book structure (M, K, φ) on a $(2n+1)$-dimensional manifold M with $n \geq 3$ is said to be *trivial* if $H_n(F) = 0$ for a page F.

It is a well-known fact that trivial open book structures on S^{2n+1}, $n \neq 2$, exist and are unique up to isotopy, by the classification theorem of simple fibered knots [Ker65, Lev70, Dur74, Kat74], where the isotopy means the isotopy through open books (see Definition 3.21). This trivial open book presents the trivial embedding of S^{2n-1} in S^{2n+1} as its binding. Note that M may not necessarily be the standard $(2n+1)$-sphere for a trivial open book (M, K, φ). It is a homotopy $(2n+1)$-sphere in general. For details, see §7.2.

2.2. Bundles over spheres

In this subsection, we recall some facts about the relationship between disk bundles over spheres and the homotopy groups of $SO(n)$, which will be used in the subsequent sections. For general terminologies, refer to [Ste51].

It is a well-known fact that $SO(n+1)$ fibers over S^n with fiber and structure group $SO(n)$. We have the following homotopy exact sequence associated with this fiber bundle:

$$\pi_n(S^n) \xrightarrow{\partial} \pi_{n-1}(SO(n)) \xrightarrow{i_*} \pi_{n-1}(SO(n+1)) \xrightarrow{p_*} \pi_{n-1}(S^n), \tag{2.1}$$

where ∂ is the boundary homomorphism, $i : SO(n) \to SO(n+1)$ is the inclusion map defined by $i(A) = A \oplus (1)$, and p is the projection defined by $p(B) = B \cdot e_{n+1}$ with e_{n+1} being the north pole of S^n. The boundary homomorphism ∂ sends the generator of $\pi_n(S^n) \cong \mathbb{Z}$ to the characteristic map of the fibration [Ste51]. We will often use the following lemmas, which are well-known (see [Wal65], [Ker60], or [Mas00]).

LEMMA 2.9. *For the boundary homomorphism $\partial : \pi_n(S^n) \to \pi_{n-1}(SO(n))$ as above with $n \geq 2, n \neq 3, 7$, we have $\operatorname{Im}\partial \cong \mathbb{Z}$ for n even and $\operatorname{Im}\partial \cong \mathbb{Z}_2$ for n odd. For $n = 3, 7$, we have $\partial = 0$.*

LEMMA 2.10. *Let E be the total space of an oriented D^n-bundle $\mathcal{E}$ over S^n associated with an oriented n-plane bundle over S^n $(n \geq 2)$. Note that its structure group is $SO(n)$. If $\xi \in H_n(E)$ denotes the class represented by the zero section $S^n \times \{0\}$, then the self-intersection number $\xi \cdot \xi$ in E coincides with $p_*(\chi) \in \pi_{n-1}(S^{n-1}) \cong \mathbb{Z}$, where $p_* : \pi_{n-1}(SO(n)) \to \pi_{n-1}(S^{n-1})$ is the homomorphism induced by the projection $p : SO(n) \to S^{n-1}$ of the fibration $SO(n-1) \to SO(n) \to S^{n-1}$, and χ is the characteristic map of the bundle $\mathcal{E}$.*

REMARK 2.11. Consider the homomorphism $p_* : \pi_{n-1}(SO(n)) \to \pi_{n-1}(S^{n-1})$ induced by the projection $p : SO(n) \to S^{n-1}$ and the boundary homomorphism $\partial : \pi_n(S^n) \to \pi_{n-1}(SO(n))$ of the fibration $SO(n) \to SO(n+1) \to S^n$ as above, with $n \geq 2$. Then $p_* \circ \partial : \pi_n(S^n) \to \pi_{n-1}(S^{n-1})$ is the multiplication by two for n even and $p_* \circ \partial = 0$ for n odd (see [Ste51, Theorem 23.4]).

LEMMA 2.12. *Consider $S^n \times D^{n+1}$ as the unit disk bundle associated with the trivial $(n+1)$-plane bundle over S^n and suppose that $\mathcal{E}$ is the unit D^n-bundle over S^n embedded as a subbundle of $S^n \times D^{n+1}, n \geq 2$. Thus the total space E of $\mathcal{E}$ is determined by the section v of the trivial bundle $S^n \times \partial D^{n+1} \to S^n$, where v is orthogonal to E in each fiber $\{*\} \times D^{n+1}$. If $\partial : \pi_n(S^n) \to \pi_{n-1}(SO(n))$ denotes the boundary homomorphism of* (2.1), *and $\chi \in \pi_{n-1}(SO(n))$ denotes the characteristic map of the bundle $\mathcal{E}$, then $\partial v = \chi$, where v is considered as an element of $\pi_n(S^n) = \pi_n(\partial D^{n+1})$.*

3. System of Invariants of an Open Book Structure

In this section, we define several invariants for a given open book structure on a manifold, among which is a generalization of the Seifert linking form in the case of open book structures on (or fibered knots in) spheres. In our general case, we need more materials than just the linking form, so that we define a system of such invariants.

In the rest of the paper, we assume that all open books are simple and oriented, unless otherwise specified.

3.1. Tangential invariant

DEFINITION 3.1 ([Wal62]). Suppose that F is an $(n-1)$-connected compact $2n$-dimensional manifold. Each element of $H_n(F)$ can be represented by an n-sphere embedded in F, uniquely determined up to isotopy, for $n \geq 4$ [Hae61, Wal62]. Define the map $\alpha_F : H_n(F) \to \pi_{n-1}(SO(n))$, called the *tangential invariant* of F, so that for each $\xi \in H_n(F) \cong \pi_n(F)$, $\alpha_F(\xi)$ is the characteristic map of the normal disk bundle of the embedded n-sphere which represents the element ξ. When $n = 3$, we have $\pi_{n-1}(SO(n)) = 0$, and we define α_F as the zero map. Thus, the tangential invariant of F is defined for all $n \geq 3$.

Remark 3.2. The above tangential invariant satisfies the addition rule given by

$$\alpha_F(\xi + \zeta) = \alpha_F(\xi) + \alpha_F(\zeta) + Q_F(\xi, \zeta)\partial t_n,$$

where $\partial : \pi_n(S^n) \to \pi_{n-1}(SO(n))$ is the boundary homomorphism of Lemma 2.9, t_n is the generator of $\pi_n(S^n) \cong \mathbb{Z}$ represented by the identity map $S^n \to S^n$, and Q_F is the intersection form of F (see [Wal62] or [Wal63]). Thus, we have the following properties.

(1) $\alpha_F(0) = 0$ (Put $\zeta = 0$ in the above formula).

(2) $\alpha_F(-\xi) = -\alpha_F(\xi) + Q_F(\xi, \xi)\partial t_n$ (Put $\zeta = -\xi$ in the above formula).

Hence, the value of $\alpha_F(\xi)$ and the intersection form determine the values of α_F over the multiples of ξ, and consequently, α_F is uniquely determined by their values on the generators of $H_n(F)$, for each fixed intersection form.

Remark 3.3. Given an $(n-1)$-connected closed $(2n+1)$-dimensional manifold M with $n \geq 2$, we define the tangential invariant $\alpha_M : H_n(M) \to \pi_{n-1}(SO(n+1))$ in the same way as we did for α_F, since each element of $H_n(M) \cong \pi_n(M)$ can be represented by an embedded n-sphere, uniquely determined up to isotopy for $n \geq 2$ [Hae61, Wal63]. Note that α_M is always a homomorphism. See [Wal67].

Remark 3.4. If $i_F : F \hookrightarrow M$ is an embedding, then the tubular neighborhood of $i_F(F)$ in M is diffeomorphic to $F \times [0,1]$ and a relationship between the tangential invariants of F and M is given by $i_* \circ \alpha_F = \alpha_M \circ i_{F*}$, where $i_* : \pi_{n-1}(SO(n)) \to \pi_{n-1}(SO(n+1))$ is the homomorphism of (2.1).

3.2. Rational Seifert form

In the following, let us analyze invariants associated with an embedding of a $2n$-dimensional manifold F into a $(2n+1)$-dimensional manifold M.

The following lemma holds, provided that F is a page of some open book structure on M. The proof is easy and is left to the reader.

Lemma 3.5. *If F is the typical page of a simple open book structure on a closed $(2n+1)$-dimensional manifold M, then the homomorphism i_{F*} : $H_n(F) \to H_n(M)$ induced by the inclusion $i_F : F \to M$ is surjective.*

Let us now define the rational Seifert form of a $2n$-dimensional manifold embedded in a $(2n+1)$-dimensional manifold M, which is a generalization of the usual Seifert linking form for Seifert manifolds of knots and links in spheres (see [Dur74, Kat74, Kau74]). In order to define the rational Seifert form, we need the concept of the rational linking number defined as follows.

Let a and b be two disjoint n-cycles in an $(n-1)$-connected closed oriented $(2n+1)$-dimensional manifold M, representing torsion elements of $H_n(M)$. Thus ra vanishes in $H_n(M)$ and bounds some $(n+1)$-chain A in M for some integer $r \neq 0$. We define the (rational) *linking number* of a and b in M as $\mathrm{lk}(a,b) = (1/r)A \cdot b \in \mathbb{Q}$, where $A \cdot b$ represents the intersection number of A and b in M. To see that this is well-defined, let us first note that there exists an integer $s \neq 0$ such that $sb = 0$. If A' is another $(n+1)$-chain in M such that $\partial A' = ra$, then $0 = (A \cup (-A')) \cdot 0 = (A \cup (-A')) \cdot (sb) = s\left((A \cup (-A')) \cdot b\right)$, and since $s \neq 0$, we have $(A \cup (-A')) \cdot b = 0$. Now, since $a \cap b = \emptyset$ and $\partial A = \partial A' = ra$, we have $0 = (A \cup (-A')) \cdot b = A \cdot b - A' \cdot b$, and hence $\mathrm{lk}(a,b)$ does not depend on the choice of A. Similarly, it does not depend on the choice of $r \neq 0$, either.

Observe that the linking pairing $\mathrm{lk}(\cdot,\cdot)$ is $(-1)^{n+1}$-symmetric, i.e. $\mathrm{lk}(a,b) = (-1)^{n+1}\,\mathrm{lk}(b,a)$, which can be checked by using Wall's argument [Wal67].

Now let F be a compact oriented $2n$-dimensional manifold embedded in a closed oriented $(2n+1)$-dimensional manifold M. Note that, for the moment, F may not necessarily be a page of some open book structure on M. As in [Ker65], define $\nu^+ : F \to M - \mathrm{Int}\,F$ and $\nu^- : F \to M - \mathrm{Int}\,F$ as small push-off's in the positive normal and the negative normal directions to F, respectively. Then ν^+_* and ν^-_* are homomorphisms from $H_n(F)$ to $H_n(M - \mathrm{Int}\,F)$. In the case that F is an oriented page of an oriented open book (M, K, φ), ν^+_* and ν^-_* are isomorphisms.

Now we need the following definition.

Definition 3.6 ([KaM79]). Let G be a finitely generated free $\mathbb{Z}$-module and $H \subset G$ a submodule. We define

$$R(H) = \{g \in G \,:\, rg \in H \text{ for some } r \in \mathbb{Z} - \{0\}\}$$

and call it the *radical closure* of H in G. Note that $R(H)$ coincides with the smallest direct summand of G containing H.

DEFINITION 3.7. Let $i_{F*} : H_n(F) \to H_n(M)$ denote the homomorphism induced by the inclusion $i_F : F \hookrightarrow M$ of a compact oriented $2n$-dimensional manifold F embedded in a closed oriented $(2n+1)$-dimensional manifold M. Note that $i_{F*}(R(\operatorname{Ker} i_{F*}))$ is contained in $\tau H_n(M)$, where $\tau H_n(M)$ denotes the torsion part of $H_n(M)$. If ξ and $\eta \in R(\operatorname{Ker} i_{F*}) \subset H_n(F)$ are represented by n-cycles a and b respectively, then $\nu_*^+(a)$ and b are disjoint n-cycles in M representing elements in $\tau H_n(M)$. Then we define the bilinear form

$$\Gamma_F : R(\operatorname{Ker} i_{F*}) \times R(\operatorname{Ker} i_{F*}) \to \mathbb{Q}$$

by $\Gamma_F(\xi, \eta) = \operatorname{lk}(\nu_*^+(a), b)$, where $\nu_*^+(a)$ and b are regarded as cycles in M. We call this form the *rational Seifert form* of F.

To see that the rational Seifert form is well-defined, we have to show that the definition does not depend on the choices of the cycles representing the homology classes in $H_n(F)$. For this, suppose that a and a' are n-cycles representing the same element in $H_n(F)$. Thus a and a' are homologous in F, and there exists an $(n+1)$-chain C in F such that $\partial C = a - a'$. Now, suppose that $r\nu^+(a)$ bounds an $(n+1)$-chain A in M. Then $r\nu^+(a')$ bounds $A - r\nu^+(C)$ and $\operatorname{lk}(\nu^+(a'), b) = (1/r)(A - r\nu^+(C)) \cdot b$, where "$\cdot$" denotes the intersection number in M. Since $C \subset F$, $\nu^+(C)$ does not intersect b and we have $(A - r\nu^+(C)) \cdot b = A \cdot b$. Thus $\operatorname{lk}(\nu^+(a), b) = \operatorname{lk}(\nu^+(a'), b)$. Now suppose that b and b' are n-cycles in F representing the same element in $H_n(F)$. Then b and b' are homologous in F, and there exists an $(n+1)$-chain D in F such that $\partial D = b - b'$. Suppose that $r\nu^+(a)$ bounds an $(n+1)$-chain A in M. By choosing A appropriately, we may assume that $A \cap D$ is a 1-chain in F such that $\partial(A \cap D) = A \cap b - A \cap b'$. Thus we have $A \cdot b = A \cdot b'$ in M, and consequently $\operatorname{lk}(\nu^+(a), b) = \operatorname{lk}(\nu^+(a), b')$. Hence Γ_F is well-defined.

REMARK 3.8. In the case that M is the $(2n+1)$-sphere S^{2n+1}, we have that $R(\operatorname{Ker} i_{F*}) = H_n(F)$ and $\Gamma_F : H_n(F) \times H_n(F) \to \mathbb{Z} \subset \mathbb{Q}$. Thus, the rational Seifert form reduces to the classical Seifert form.

Since ν_*^+ and ν_*^- map $R(\operatorname{Ker} i_{F*})$ to $R(\operatorname{Ker}(i_{M-\operatorname{Int} F})_*)$ homomorphically, where $i_{M-\operatorname{Int} F} : M - \operatorname{Int} F \to M$ is the inclusion map, we can prove several generalizations of results in [Kau74]. Among them is a generalization of [Kau74, Lemma 2.1] as follows.

Lemma 3.9. *Let Γ_F be the rational Seifert form of a compact oriented $2n$-dimensional manifold F embedded in a closed oriented $(2n+1)$-dimensional manifold M. Then we have*

$$\Gamma_F(\xi,\zeta) + (-1)^n \Gamma_F(\zeta,\xi) = Q_F(\xi,\zeta)$$

for all $\xi,\zeta \in R(\operatorname{Ker} i_{F*})$, where Q_F denotes the intersection form on $H_n(F)$.

The proof of the above lemma is similar to that of [Kau74, Lemma 2.1] and is left to the reader. Note that the signs appearing in the formula of the above lemma are slightly different from those of [Kau74, Lemma 2.1]. This is due to our definition of the linking number as in [Wal67], which is slightly different from that of Kauffman [Kau74].

Lemma 3.10. *If Γ_F is the rational Seifert form of a page of an open book, then $\det \Gamma_F = \pm |\tau H_n(M)|^{-1}$, where $\det \Gamma_F \in \mathbb{Q}$ denotes the determinant of the rational Seifert form Γ_F defined as the determinant of an associated matrix, and $|\tau H_n(M)|$ denotes the order of the torsion part $\tau H_n(M)$ of $H_n(M)$.*

Proof. By Lemma 5.6, which will be proved in §5, we have

$$\det \tilde{\nu}_*^+ = \pm |\tau H_n(M)| \det \Gamma_F,$$

where

$$\tilde{\nu}_*^+ = \nu_*^+|_{R(\operatorname{Ker} i_{F*})} : R(\operatorname{Ker} i_{F*}) \to R(\operatorname{Ker}(i_{M-\operatorname{Int} F})_*).$$

Since ν_*^+ is an isomorphism for the case of an open book and

$$(\nu_*^+)^{-1}\left(R(\operatorname{Ker}(i_{M-\operatorname{Int} F})_*)\right) = R(\operatorname{Ker} i_{F*}),$$

we see that $\tilde{\nu}_*^+$ is also an isomorphism. This completes the proof. □

Other important properties of the rational Seifert form involve the invariants of M defined in [Wal67] that are more sophisticated than those used in Lemma 3.10, and require additional concepts as follows.

Definition 3.11. For $n \geq 1$, we define the bilinear form $b_M : \tau H_n(M) \times \tau H_n(M) \to \mathbb{Q}/\mathbb{Z}$ by $b_M(\xi,\zeta) = \operatorname{lk}(a,c) \pmod 1$, where a and

c are disjoint n-cycles in M representing the homology classes ξ and $\zeta \in \tau H_n(M)$ respectively [Wal67]. Note that b_M is a well-defined bilinear form, which is sometimes called the *torsion linking pairing* of M.

REMARK 3.12. For n odd with $n \geq 5, n \neq 7$, we have $\operatorname{Im} \partial \cong \mathbb{Z}_2$ by Lemma 2.9, where $\partial : \pi_n(S^n) \to \pi_{n-1}(SO(n))$ is the boundary homomorphism of (2.1). Since the homotopy exact sequence (2.1) takes the form

$$\begin{cases} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{i_*} \mathbb{Z}_2 \xrightarrow{p_*} 0, & n \equiv 1 \pmod 8, \\ \mathbb{Z} \xrightarrow{\partial} \mathbb{Z}_2 \xrightarrow{i_*} 0 \xrightarrow{p_*} 0, & n \equiv 3, 5, 7 \pmod 8 \end{cases}$$

(see [Ker60, Wal65]), we have a well-defined and natural extension of the isomorphism $\operatorname{Im} \partial \cong \mathbb{Z}_2$, which will be denoted by $\phi : \pi_{n-1}(SO(n)) \to \mathbb{Z}_2$, such that $(\phi, i_*) : \pi_{n-1}(SO(n)) \to \mathbb{Z}_2 \oplus \pi_{n-1}(SO(n+1))$ is an isomorphism (see [Wal65]). Note that ϕ is an epimorphism.

DEFINITION 3.13. For n odd with $n \geq 5, n \neq 7$, define the quadratic form

$$q_M : \tau H_n(M) \to \mathbb{Q}/2\mathbb{Z},$$

introduced by [Wal67], as follows. Let a be a spherical representation of $\xi \in \tau H_n(M)$ uniquely determined up to isotopy, and consider a tubular neighborhood $N(a)$ of a. Then $\partial N(a)$ is an S^n-bundle over $a \cong S^n$ and the tubular neighborhood E of a section of $\partial N(a) \to a$ is a D^n-bundle over a. Denote its characteristic map by $\alpha_1 \in \pi_{n-1}(SO(n))$. For n as above, we can adjust this section so that $\phi(\alpha_1) = 0$ (see Lemma 2.12), where $\phi : \pi_{n-1}(SO(n)) \to \mathbb{Z}_2$ is the epimorphism of Remark 3.12. Define $q_M(\xi)$ as the rational linking number between a and the core of E modulo 2. Then $q_M(\xi)$ is well-defined, and it is a quadratic form associated with the bilinear form $2b_M$, where b_M is the torsion linking pairing of M defined in Definition 3.11, i.e.

$$q_M(\xi + \zeta) - q_M(\xi) - q_M(\zeta) \equiv 2b_M(\xi, \zeta) \pmod 2$$

for all $\xi, \zeta \in \tau H_n(M)$ (see [Wal67]).

The rational Seifert form is compatible with the invariants of M, by the following lemma. Note that F may not necessarily be a page of an open book.

LEMMA 3.14. *For $n \geq 2$ with $n \neq 3, 7$, the rational Seifert form Γ_F has the following properties.*

(1) *The congruence*

$$\Gamma_F(\xi, \zeta) \equiv b_M(i_{F*}(\xi), i_{F*}(\zeta)) \pmod 1$$

holds for all $\xi, \zeta \in R(\mathrm{Ker}\, i_{F*})$, where $b_M : \tau H_n(M) \times \tau H_n(M) \to \mathbb{Q}/\mathbb{Z}$ is the torsion linking pairing of M defined in Definition 3.11.

(2) *If n is odd, then we have*

$$\Gamma_F(\xi, \xi) \equiv q_M(i_{F*}(\xi)) + \phi(\alpha_F(\xi)) \pmod 2$$

for all $\xi \in R(\mathrm{Ker}\, i_{F*})$, where $q_M : \tau H_n(M) \to \mathbb{Q}/2\mathbb{Z}$ is the quadratic form of Definition 3.13 *and $\phi : \pi_{n-1}(SO(n)) \to \mathbb{Z}_2$ is the epimorphism of Remark* 3.12.

PROOF. (1) This follows from the definitions of the Seifert form Γ_F and of the torsion linking pairing b_M.

(2) Let $a \subset F$ be the spherical representation of $\xi \in R(\mathrm{Ker}\, i_{F*})$. The translation in the positive normal direction of F determines a section of $\partial N(a) \to a$, where $N(a)$ is the tubular neighborhood of a in M. Denote the image of the section by $\tilde{a}$ and its tubular neighborhood in $\partial N(a)$ by E. Since E is parallel to $N(a) \cap F$, the characteristic map α_1 of E is equal to $\alpha_F(\xi)$, where $\alpha_F : H_n(F) \to \pi_{n-1}(SO(n))$ is the tangential invariant of F. Since $\Gamma_F(\xi, \xi)$ is the rational linking number between a and its translation in the positive normal direction, $\Gamma_F(\xi, \xi) \equiv q_M(i_{F*}(\xi)) \pmod 2$, provided that $\phi(\alpha_F(\xi)) = 0$.

If $\phi(\alpha_F(\xi)) = \phi(\alpha_1) \neq 0$, then we need to adjust the section so that we have $\phi(\alpha_1) = 0$. Note that the sections of $\partial N(a) \to a$ are in one-to-one correspondence with the unit normal vector fields on a. Choose the normal vector field that differs by t_n from the normal vector field of F restricted to a, where t_n is the generator of $\pi_n(S^n) \cong \mathbb{Z}$, and denote the image of the section determined by this new vector field by a' and the tubular neighborhood of a' in $\partial N(a)$ by E', respectively. Then we have $\partial t_n = \alpha'_1 - \alpha_1$ by Lemma 2.12, where α'_1 is the characteristic map of the bundle E'. Therefore, we have $\phi(\alpha'_1) = \phi(\partial t_n) + \phi(\alpha_1) \in \mathbb{Z}_2$. Recall that $\phi|_{\mathrm{Im}\,\partial} : \mathrm{Im}\,\partial \cong \mathbb{Z}_2 \to \mathbb{Z}_2$ is an isomorphism (see Remark 3.12) and $\partial t_n \neq 0$, since ∂ is a non-vanishing map for our values of n (see Lemma 2.9). This implies that $\phi(\partial t_n) \neq 0$ in $\mathbb{Z}_2$. Since $\phi(\alpha_1) \neq 0$ by our hypothesis, we have $\phi(\partial t_n) = \phi(\alpha_1)$, and consequently, $\phi(\alpha_1') = 0$. Thus $q_M(i_{F*}(\xi))$ coincides with the linking number of a and a' modulo 2.

Since $\mathrm{lk}(a, a')$ and $\mathrm{lk}(a, \tilde{a}) = \pm\Gamma_F(\xi, \xi)$ differs by ± 1, we have

$$q_M(i_{F*}(\xi)) \equiv (\Gamma_F(\xi, \xi) + 1) + (\phi(\alpha_F(\xi)) + 1) \pmod 2,$$

which completes the proof of (2). □

3.3. System of invariants

So far, we have defined several invariants associated with a compact oriented $2n$-dimensional manifold F embedded in a closed oriented $(2n+1)$-dimensional manifold. In order to consider the abstract set of invariant systems for a given ambient manifold M, we need the following definition.

Definition 3.15. Let M be an $(n-1)$-connected closed oriented $(2n+1)$-dimensional manifold with $n \geq 3$. A *system of open book invariants* with respect to M consists of five algebraic objects $\{G, Q_G, \alpha_G, i_G, \Gamma_G\}$ as follows:

(1) a finitely generated free abelian group G,

(2) a $(-1)^n$-symmetric bilinear form $Q_G : G \times G \to \mathbb{Z}$, called an intersection form,

(3) an epimorphism $i_G : G \to H_n(M)$,

(4) a map $\alpha_G : G \to \pi_{n-1}(SO(n))$, called a tangential invariant, such that

  (a) the diagram

$$\begin{array}{ccc} G & \xrightarrow{\alpha_G} & \pi_{n-1}(SO(n)) \\ {\scriptstyle i_G}\downarrow & & \downarrow{\scriptstyle i_*} \\ H_n(M) & \xrightarrow{\alpha_M} & \pi_{n-1}(SO(n+1)) \end{array}$$

  is commutative, where α_M is the tangential invariant of M, and i_* is the homomorphism induced by the natural inclusion $i : SO(n) \to SO(n+1)$,

  (b) $p_*\alpha_G(\xi) = Q_G(\xi, \xi) \in \pi_{n-1}(S^{n-1}) \cong \mathbb{Z}$ for all $\xi \in G$, where $p : SO(n) \to S^{n-1}$ denotes the projection of Lemma 2.10,

(c) $\alpha_G(\xi+\zeta) = \alpha_G(\xi) + \alpha_G(\zeta) + Q_G(\xi,\zeta)\partial t_n$ for all $\xi, \zeta \in G$, where $\partial : \pi_n(S^n) \to \pi_{n-1}(SO(n))$ is the boundary homomorphism appearing in Lemma 2.9 and t_n is the generator of $\pi_n(S^n) \cong \mathbb{Z}$ represented by the identity map of S^n,

(5) a bilinear form, called a rational Seifert form, $\Gamma_G : R(\mathrm{Ker}\, i_G) \times R(\mathrm{Ker}\, i_G) \to \mathbb{Q}$, where $R(\mathrm{Ker}\, i_G)$ is the radical closure of $\mathrm{Ker}\, i_G$, such that

(a) $\det \Gamma_G = \pm|\tau H_n(M)|^{-1}$, where $|\tau H_n(M)|$ denotes the order of the torsion part of $H_n(M)$, and $\det \Gamma_G$ is the determinant of Γ_G,

(b) $\Gamma_G(\xi,\zeta) + (-1)^n \Gamma_G(\zeta,\xi) = Q_G(\xi,\zeta)$ for all $\xi,\zeta \in R(\mathrm{Ker}\, i_G)$,

(c) the diagram

$$\begin{array}{ccc} R(\mathrm{Ker}\, i_G) \times R(\mathrm{Ker}\, i_G) & \xrightarrow{\Gamma_G} & \mathbb{Q} \\ {\scriptstyle i_G \times i_G}\downarrow & & \downarrow{\scriptstyle \pi} \\ \tau H_n(M) \times \tau H_n(M) & \xrightarrow{b_M} & \mathbb{Q}/\mathbb{Z}, \end{array}$$

is commutative, where b_M is the torsion linking pairing of M defined in Definition 3.11, and π is the natural projection,

(d) for n odd with $n \geq 5, n \neq 7$,

$$\Gamma_G(\xi,\xi) \equiv q_M(i_G(\xi)) + \phi(\alpha_G(\xi)) \pmod 2$$

for all $\xi \in R(\mathrm{Ker}\, i_G)$, where q_M is the quadratic form of Definition 3.13 and $\phi : \pi_{n-1}(SO(n)) \to \mathbb{Z}_2$ is the epimorphism of Remark 3.12.

The collection of invariants $\{H_n(F), Q_F, \alpha_F, i_{F*}, \Gamma_F\}$ associated with an open book structure on M with typical page F forms a system of open book invariants with respect to M for $n \geq 3$, since i_{F*} is surjective by Lemma 3.5, the properties of α_G follow from Remark 3.4, Lemma 2.10 and Remark 3.2, and those of the rational Seifert form follow from Lemmas 3.10, 3.9, and 3.14.

REMARK 3.16. If n is even, then the above tangential invariant α_G is uniquely determined by the intersection form Q_G, due to item (4b) of

Definition 3.15 and the injectivity of $p_* : \pi_{n-1}(SO(n)) \to \pi_{n-1}(S^{n-1})$ (see [Ste51]), where p is the projection of Lemma 2.10. Thus, we may sometimes omit the tangential invariant in a system of open book invariants in the case of n being even.

Remark 3.17. In the case that M is a homotopy $(2n+1)$-sphere ($n \geq 3$), there exists a one-to-one correspondence between the set of systems of open book invariants with respect to M and the set of unimodular Seifert forms, which can be seen as follows (see [Mas00] for details).

To see that the system of invariants is uniquely determined by the Seifert form, first note that $i_G = 0$ and Q_G is determined by the Seifert form. In the case that n is even, Q_G determines α_G by Remark 3.16. When $n = 3, 7$, we have that $\alpha_G = 0$, since $\pi_{n-1}(SO(n)) = 0$ (see [Ker60]). In the case that n is odd, $n \neq 3, 7$, note that $\alpha_M = 0$ and we have $\operatorname{Im} \alpha_G \subset \operatorname{Ker} i_* = \operatorname{Im} \partial$, where $\partial : \pi_n(S^n) \to \pi_{n-1}(SO(n))$ is the boundary homomorphism of (2.1). Hence, by item (5d) of Definition 3.15, we see that α_G is determined by the Seifert form.

We show that an arbitrary unimodular Seifert form Γ_G can be completed in a way to form a system of open book invariants. For this, we put $i_G = 0$ and define Q_G by the formula of item (5b) of Definition 3.15, observing that $R(\operatorname{Ker} i_G) = G$. Let us now define α_G in a coherent way. For n even, observe that $Q_G(\xi, \xi)$ is even for all $\xi \in G$, since Q_G is symmetric. Thus $Q_G(\xi, \xi) \in 2\mathbb{Z} = \operatorname{Im}(p_* \circ \partial)$ by Remark 2.11. On the other hand, $p_*|_{\operatorname{Im} \partial} : \operatorname{Im} \partial \to 2\mathbb{Z} \subset \mathbb{Z} \cong \pi_{n-1}(S^{n-1})$ is an isomorphism. Thus, we can define $\alpha_G(\xi) = (p_*|_{\operatorname{Im} \partial})^{-1}(Q_G(\xi, \xi))$. When $n = 3, 7$, we have $\pi_{n-1}(SO(n)) = 0$ (see [Ker60]) and consequently, it is enough to define $\alpha_G = 0$ so that it satisfies the condition of item (4a) of Definition 3.15. For n odd with $n \neq 3, 7$, observe that $\phi|_{\operatorname{Im} \partial} : \operatorname{Im} \partial \to \mathbb{Z}_2$ is an isomorphism, where $\phi : \pi_{n-1}(SO(n)) \to \mathbb{Z}_2$ is the epimorphism of Remark 3.12. Define $\alpha_G(\xi) = (\phi|_{\operatorname{Im} \partial})^{-1} (\Gamma_G(\xi, \xi) \pmod 2)$ so that α_G satisfies the relation of item (5d) of Definition 3.15, since $q_M = 0$. Finally, we can check that the system $\{G, Q_G, \alpha_G, i_G, \Gamma_G\}$ thus constructed satisfies all the required properties as in Definition 3.15.

We define an equivalence relation on the systems of open book invariants.

Definition 3.18. Suppose that $\{G, Q_G, \alpha_G, i_G, \Gamma_G\}$ forms a system of open book invariants with respect to M, and $\{G', Q_{G'}, \alpha_{G'}, i_{G'}, \Gamma_{G'}\}$ forms another system of open book invariants with respect to the same manifold M. We say that the two systems of invariants are *equivalent*, if there exists an isomorphism $\Psi : G \to G'$ such that the following conditions are satisfied.

(1) The isomorphism Ψ is an isometry, i.e., $Q_{G'}(\Psi(\xi), \Psi(\zeta)) = Q_G(\xi, \zeta)$ for all $\xi, \zeta \in G$.

(2) The diagram bellow is commutative.

$$\begin{array}{ccc} G & \xrightarrow{\Psi} & G' \\ {}_{i_G}\searrow & & \swarrow_{i_{G'}} \\ & H_n(M) & \end{array}$$

(3) The isomorphism Ψ preserves the tangential invariants, i.e., $\alpha_{G'}(\Psi(\xi)) = \alpha_G(\xi)$ for all $\xi \in G$.

(4) The isomorphism Ψ preserves the rational Seifert form, i.e.,

$$\Gamma_{G'}(\Psi(\xi), \Psi(\zeta)) = \Gamma_G(\xi, \zeta), \quad \forall \xi, \forall \zeta \in R(\operatorname{Ker} i_G)$$

(note that $\Psi\,(R(\operatorname{Ker} i_G)) = R(\operatorname{Ker} i_{G'})$ by condition (2) above).

Definition 3.19. The set of all equivalence classes of systems of open book invariants with respect to M is denoted by $\mathcal{A}(M)$.

Remark 3.20. In the case that M is a homotopy $(2n+1)$-sphere ($n \geq 3$), the set $\mathcal{A}(M)$ coincides with the set of all congruence classes of unimodular matrices (see Remark 3.17).

Now let us define a geometric equivalence between two open book structures on the same manifold.

Definition 3.21. Let us consider two (simple and oriented) open book structures (K_j, φ_j), $j = 1, 2$, on a closed $(2n+1)$-dimensional manifold M. A *structural isotopy* between (K_1, φ_1) and (K_2, φ_2) is an ambient isotopy

$\Phi = \{\Phi_t\}_{t\in[0,1]}$ of M such that $\Phi_0 = \mathrm{id}$, $\Phi_1(K_1) = K_2$ (preserving the orientations) and the diagram

$$\begin{array}{ccc} M - K_1 & \overset{\Phi_1|_{M-K_1}}{\longrightarrow} & M - K_2 \\ {\scriptstyle \varphi_1} \searrow & & \swarrow {\scriptstyle \varphi_2} \\ & S^1 & \end{array}$$

is commutative. When such a structural isotopy between (K_1, φ_1) and (K_2, φ_2) exists, we say that they are *structurally isotopic* or *isotopic through open books* ([Dur74]).

Remark 3.22. If two open book structures (K_1, φ_1) and (K_2, φ_2) on M are structurally isotopic by an isotopy Φ, then $\Phi_{1*} : H_n(F_1) \to H_n(F_2)$ clearly establishes an equivalence between their systems of open book invariants (see Definition 3.18) for $n \geq 3$, where F_1 and F_2 are the typical pages of (K_1, φ_1) and (K_2, φ_2) respectively.

Definition 3.23. For an open book (M, K, φ), we denote the equivalence class of its associated system of open book invariants by

$$\mathcal{S}(M, K, \varphi) = \{H_n(F), Q_F, \alpha_F, i_{F*}, \Gamma_F\},$$

which represents an element of $\mathcal{A}(M)$, where F is the typical page of (M, K, φ). When M is obvious in the context, the system of open book invariants will be denoted simply by $\mathcal{S}(K, \varphi)$.

Let us denote by $\mathcal{OB}(M)$ the set of all structural isotopy classes of simple and oriented open book structures on a given $(n-1)$-connected closed oriented $(2n+1)$-dimensional manifold M. Then we define the map

$$\mathcal{S} : \mathcal{OB}(M) \to \mathcal{A}(M) \tag{3.1}$$

so that it sends each structural isotopy class of an open book structure (K, φ) on M to the equivalence class $\mathcal{S}(K, \varphi)$ of its system of open book invariants. Note that this is a well-defined map by Remark 3.22.

We will show later that the above map $\mathcal{S}$ is in fact a bijection in our situation.

4. Isotopy Criterion for Open Book Structures

In this section, we prove that two open book structures with equivalent systems of open book invariants are structurally isotopic, i.e. the map $\mathcal{S}$ of (3.1) is injective.

More precisely, the main theorem of this section is the following isotopy criterion. Note that an $(n-1)$-connected closed $(2n+1)$-dimensional manifold M is a *rational homology sphere*, if $H_{n+1}(M) = 0$.

THEOREM 4.1. *Let M be an $(n-1)$-connected closed oriented manifold of dimension $2n+1$ with $n \geq 4, n \neq 7$, or an $(n-1)$-connected oriented rational homology $(2n+1)$-sphere with $n = 3, 7$. If two simple and oriented open book structures on M have equivalent systems of open book invariants, then they are structurally isotopic.*

4.1. Isotopy of pages

In this subsection, we show the following.

LEMMA 4.2. *Let M be an $(n-1)$-connected closed $(2n+1)$-dimensional manifold with $n \geq 4, n \neq 7$, or an $(n-1)$-connected rational homology $(2n+1)$-sphere with $n = 3, 7$. If two open book structures on M have equivalent systems of invariants, then their typical pages are isotopic in M by an isotopy which preserves the orientations of the pages.*

PROOF. Let F and F' be the typical pages of the two open book structures which have equivalent systems of open book invariants. Denote the Seifert form, intersection form, and the tangential invariant of the open book associated with F by Γ_F, Q_F and α_F, respectively, and those associated with F' by $\Gamma_{F'}$, $Q_{F'}$ and $\alpha_{F'}$, respectively. Suppose that $\Psi : H_n(F) \to H_n(F')$ is an isomorphism which establishes an equivalence between the two systems of open book invariants.

Let us consider the decomposition $H_n(F) = R(\operatorname{Ker} i_{F*}) \oplus A$, where $A \cong H_n(F)/R(\operatorname{Ker} i_{F*})$, and take a basis $\{e_1, \ldots, e_r\}$ of $H_n(F)$ associated with this decomposition such that $\{e_1, \ldots, e_s\}$, $s \leq r$, forms a basis of A and $\{e_{s+1}, \ldots, e_r\}$ forms a basis of $R(\operatorname{Ker} i_{F*})$. Take a basis of $H_n(F')$ as $\{e'_1 = \Psi(e_1), \ldots, e'_r = \Psi(e_r)\}$.

Using Lemma 2.3, construct handlebody decompositions $F = D^{2n}_1 \cup h_1 \cup \cdots \cup h_r$ and $F' = D^{2n}_2 \cup h'_1 \cup \cdots \cup h'_r$ of F and F' respectively in M, associated with the above bases, where $h_1, \dots, h_r$ and $h'_1, \dots, h'_r$ are n-handles attached simultaneously to the 0-handles D^{2n}_1 and D^{2n}_2 respectively. Denote the n-disks which represent the cores of h_i and h'_i by c_i and c'_i respectively. In the above decompositions, we may assume that the 0-handles D^{2n}_1 and D^{2n}_2 coincide with each other, including the orientations, and we denote it by D^{2n}.

Since ∂c_i is an $(n-1)$-sphere embedded in ∂D^{2n} which is a $(2n-1)$-sphere with $n > 2$, it bounds an n-disk in ∂D^{2n} by [Hae61]. Pushing the interior of this n-disk to the interior of D^{2n}, we may assume that ∂c_i bounds an n-disk in D^{2n} such that the intersection with ∂D^{2n} is exactly ∂c_i. Attaching this disk to c_i along the boundary and applying the smoothing process, we obtain an embedded n-sphere $\bar{c}_i \subset F$ representing the homology class $e_i \in H_n(F)$. Using the same argument, we obtain an embedded n-sphere $\bar{c}'_i \subset F'$ representing the element $e'_i \in H_n(F')$, for $i = 1, 2, \dots, r$.

Now, we make use of Levine's argument [Lev70] in order to conclude that the ordered links $\{\partial c_1, \dots, \partial c_r\}$ and $\{\partial c'_1, \dots, \partial c'_r\}$ are isotopic in ∂D^{2n} as follows. We have $\mathrm{lk}(\partial c_i, \partial c_j) = Q_F(e_i, e_j)$ and $\mathrm{lk}(\partial c'_i, \partial c'_j) = Q_{F'}(e'_i, e'_j)$ $(i \neq j)$, where lk denotes the linking number in ∂D^{2n}. Since Ψ preserves the intersection form, we have that $Q_F(e_i, e_j) = Q_{F'}(\Psi(e_i), \Psi(e_j)) = Q_{F'}(e'_i, e'_j)$. Thus $\mathrm{lk}(\partial c_i, \partial c_j) = \mathrm{lk}(\partial c'_i, \partial c'_j)$ for all $i \neq j$. Since $n > 2$, we see that $\{\partial c_1, \dots, \partial c_r\}$ and $\{\partial c'_1, \dots, \partial c'_r\}$ are isotopic as ordered links in ∂D^{2n}. Thus, we may assume that $\partial c_i = \partial c'_i$ for all i.

Now we need the following lemma to continue the proof of Lemma 4.2.

Lemma 4.3. *There exists an ambient isotopy of M relative to D^{2n}, carrying c_k to c'_k for all $k = 1, \dots, r$.*

Proof. An isotopy of the cores of the handles is obtained by induction on k.

Suppose that there exists an ambient isotopy of M relative to D^{2n} which carries c_i to c'_i for $i = 1, \dots, k-1$. Then we may assume that $c_i = c'_i$ for all $i < k$. Moreover, we may assume that $c_k \cap c'_k = \partial c_k = \partial c'_k$. Let us try to show that there exists an isotopy carrying c_k to c'_k relative to $D^{2n} \cup c_1 \cup \cdots \cup c_{k-1}$.

First, observe that $\bar{c}_k$ and $\bar{c}'_k$ represent the same homology class in

$H_n(M)$ by our hypothesis (see item (2) of Definition 3.18). Thus, $c_k \cup_{\partial c_k} (-c'_k)$ represents the zero class in $\pi_n(M - \operatorname{Int} D^{2n}) \cong \pi_n(M) \cong H_n(M)$, and consequently, c_k and c'_k are homotopic relative to boundary in $M - \operatorname{Int} D^{2n}$. By the engulfing theorem [HZ66], we may assume that $c_k \cup_{\partial c_k} (-c'_k)$ is contained in a tubular neighborhood of a point. In this way, $c_k \cup_{\partial c_k} (-c'_k)$ can be considered as an embedded n-sphere in a $(2n+1)$-disk contained in $M - \operatorname{Int} D^{2n}$, $n \geq 2$, which implies that $c_k \cup_{\partial c_k} (-c'_k)$ bounds an $(n+1)$-disk D'_k embedded in $M - \operatorname{Int} D^{2n}$.

Using an isotopy relative to boundary, we can modify D'_k so that it intersects $c_1 \cup \cdots \cup c_{k-1}$ transversely at a finite number of points. Since D'_k is a disk with boundary $c_k \cup_{\partial c_k} (-c'_k)$, we can construct an isotopy $H : D^n \times [0,1] \to M$ such that

$$\begin{cases} H(D^n \times [0,1]) = D'_k,\ H(D^n \times \{0\}) = c_k,\ H(D^n \times \{1\}) = c'_k, \\ H(x,t) = H(x,0) \text{ for all } (x,t) \in \partial D^n \times [0,1], \end{cases}$$

and $H|_{\operatorname{Int} D^n \times [0,1]} : \operatorname{Int} D^n \times [0,1] \to M$ is an embedding.

Modifying H if necessary, we may assume that for each t, $H(D^n \times \{t\})$ does not intersect $\bigcup_{i<k} c_i$, or intersects it at a unique point. Moreover, we can modify it so that the intersection of $H(D^n \times \{t\})$ with $\bigcup_{i<k, i \leq s} c_i$ occurs only for $t \in (0, 1/2)$ and the intersection with $\bigcup_{s<i<k} c_i$ occurs only for $t \in (1/2, 1)$. We enumerate the values of t such that $H(D^n \times \{t\}) \cap \left(\bigcup_{i<k} c_i\right) \neq \emptyset$, obtaining $0 < t_1 < \cdots < t_p < 1/2 < t_{p+1} < \cdots < t_q < 1$.

For $n = 3, 7$, we are assuming that M is an $(n-1)$-connected rational homology $(2n+1)$-sphere and it is not necessary to analyze the case $i \leq s$. For the other cases, we need the following.

Lemma 4.4. *For $n \geq 4, n \neq 7$, we can modify H above so that $H(D^n \times I)$ does not intersect c_i for all $i < k$ with $i \leq s$.*

Proof. Since $i \leq s$, e_i is an element of the basis of A. Thus, $i_{F*}(e_i)$ is a primitive element of $H_n(M)$, since $i_{F*}|_A : A \to H_n(M)/\tau H_n(M)$ is an isomorphism by Lemma 3.5, where $\tau H_n(M)$ is the torsion part of $H_n(M)$ and $H_n(M)/\tau H_n(M)$ is the free part of $H_n(M)$. Consequently, by Poincaré duality, there exist $\hat{e}_i \in H_{n+1}(M)$, $1 \leq i \leq s$, such that $\hat{e}_i \cdot i_{F*}(e_j) = \delta_{ij}$ $(1 \leq i \leq s, 1 \leq j \leq r)$, where $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$, and "$\cdot$" denotes the intersection number in M. Since $n \geq 4$, we have that the

Hurewicz homomorphism $\pi_{n+1}(M) \to H_{n+1}(M)$ is surjective (see [Hu59, Chapter X, Theorem 8.1]), and we can represent $\hat{e}_i$ by an embedded $(n+1)$-sphere $\hat{c}_i$ in M [Wal63, Hae61]. Moreover, since $H_{n+1}(M - D^{2n}) \cong H_{n+1}(M)$ by the isomorphism induced by the inclusion map, $\hat{c}_i$ can be chosen in $M - D^{2n}$. Using the Whitney trick [Whi44, Mil65], we may assume that $\hat{c}_i \cap c_j = \emptyset$ for $i \neq j$ $(1 \leq i \leq s, 1 \leq j \leq r)$ and $\hat{c}_i$ intersects transversely with c_i at a unique point.

For each l such that $1 \leq l \leq p$, let i $(i < k, i \leq s)$ be the index such that $H(D^n \times \{t_l\}) \cap c_i \neq \emptyset$, and let γ_l be an embedded curve in c_i which joins the point $H(D^n \times \{t_l\}) \cap c_i$ in $D'_k \cap c_i$ with the point $\hat{c}_i \cap c_i$, such that γ_l intersects $D'_k \cap (\cup_{i<k} c_i)$ at the unique point, the initial point of γ_l. Taking the connected sum of D'_k with $\hat{c}_i$ along γ_l, using an appropriate orientation for $\hat{c}_i$, we can eliminate the intersection of $H(D^n \times \{t_l\})$ with c_i. Using the Whitney trick, we can eliminate the intersections of c_k, c'_k with $\hat{c}_i (i < k)$. Putting $\hat{c}_i$ in a transverse position with respect to D'_k, using an isotopy, we may assume that D'_k and $\hat{c}_i$ intersect along some embedded circles. Moreover, we may assume that their intersections occur on $H\,(D^n \times (0, t_1/2))$. Since $\hat{c}_i$ is an embedded $(n+1)$-sphere which does not intersect $H(D^n \times [t_l - \varepsilon, t_l + \varepsilon])$ for $\varepsilon > 0$ sufficient small, we can modify H on $D^n \times [t_l - \varepsilon, t_l + \varepsilon]$ so that $H(D^n \times [0, 1])$ is the connected sum described above.

Repeating this process for $l = 1, \ldots, p$, we can eliminate the intersections of $H(D^n \times [0, 1])$ with c_i, $i < k, i \leq s$. This completes the proof of Lemma 4.4. □

Let us return to the proof of Lemma 4.3. In the case where $k \leq s$, we already have the desired isotopy.

In the case where $k > s$, we need to eliminate the intersections of $H(D^n \times [0, 1])$ with c_j for $s < j < k$. For this, note that $e_j, e_k \in R(\operatorname{Ker} i_{F*})$ and $e'_j, e'_k \in R(\operatorname{Ker} i_{F'*})$ by our choice of the indices so that we can consider the values of the rational Seifert forms $\Gamma_F(e_j, e_k)$ and $\Gamma_{F'}(e'_j, e'_k)$.

Set $D''_k = H(D^n \times [1/2, 1])$. Then D''_k lies as an embedded $(n+1)$-disk in $M - \operatorname{Int} D^{2n}$. Observe that $\hat{c}_i \cap c_j = \emptyset$ for $i \neq j$, which implies that $H(D^n \times [0, 1/2])$ remains disjoint from c_j. Thus, the intersection number of D''_k with c_j is equal to the intersection number of $H(D^n \times [0, 1])$ with c_j, which is equal to the linking number between $\partial H(D^n \times [0, 1]) = c_k \cup_{\partial c_k} (-c'_k)$ and $\bar{c}_j$, where $\bar{c}_j$ is the embedded n-sphere in F corresponding to c_j. Moreover, we have

$$\begin{aligned}\operatorname{lk}\big(c_k\cup_{\partial c_k}(-c'_k), \bar{c}_j\big) &= \operatorname{lk}(\nu^+\bar{c}_k, \bar{c}_j) - \operatorname{lk}(\nu'^+\bar{c}'_k, \bar{c}'_j) \\ &= \Gamma_F(e_k, e_j) - \Gamma_{F'}(e'_k, e'_j) = 0\end{aligned}$$

by our hypothesis, where $\bar{c}'_j = \bar{c}_j$, $\bar{c}_k$ and $\bar{c}'_k$ are the embedded n-spheres in F and F' corresponding to c_k and c'_k respectively, and ν^+ and ν'^+ are the translations into the positive normal directions of F and F' respectively.

Since the algebraic intersection of D''_k and c_j is zero, we can use the Whitney trick [Whi44, Mil65] to remove the intersections of D''_k with c_j for all j with $s < j < k$, by using an isotopy of D''_k in $\big(M - (D^{2n} \cup c_1 \cup \cdots \cup c_s)\big) \cup \partial c_k$ relative to boundary. Since D''_k is an $(n+1)$-disk, we can modify the isotopy H on $D^n \times [1/2, 1]$ so that $H(D^n \times [1/2, 1]) = D''_k$.

Thus we have an isotopy

$$H : D^n \times [0,1] \to \big(M - (D^{2n} \cup c_1 \cup \cdots \cup c_{k-1})\big) \cup \partial c_k$$

of c_k to c'_k relative to $\partial D^n \times [0,1]$.

Then, by the isotopy extension theorem, we have an ambient isotopy of M, relative to $D^{2n} \cup c_1 \cup \cdots \cup c_{k-1}$, carrying c_k to c'_k.

The successive composition of the above ambient isotopies for $k = 1, \ldots, r$ gives us an ambient isotopy of M relative to D^{2n} which carries c_k to c'_k for all k. This completes the proof of Lemma 4.3. □

Now let us return to the proof of Lemma 4.2. We will prove that there exists an isotopy which carries the handle h_i to h'_i for all i. Since the cores of the handles are isotopic by Lemma 4.3, we may assume that the cores c_i and c'_i of the handles of F and F', respectively, coincide with each other for each i. We may further assume that the handles h_i and h'_i are embedded as subbundles of a disk bundle, which is the tubular neighborhood $N(c_i)$ of c_i in M, associated to a normal vector bundle of c_i in M. Thus, h_i and h'_i are determined by their unit normal positive vector fields v_i and v'_i respectively, along c_i in $N(c_i)$.

Note that the two embeddings h_i and h'_i are isotopic as subbundles relative to $h_i \cap D^{2n} = h'_i \cap D^{2n}$, if and only if v_i and v'_i are homotopic relative to ∂c_i.

Since v_i and v'_i coincide along ∂c_i, we can glue them to obtain a vector field ϑ_i on the n-sphere $c_i \cup_{\partial c_i} (-c_i)$ which is considered to be a formal union.

Given a trivialization of the D^{n+1}-bundle $N(c_i) \cup_{\partial c_i \times D^{n+1}} (-N(c_i))$, the homotopy class of ϑ_i determines and is uniquely determined by an element of $\pi_n(S^n)$. Moreover, v_i is homotopic to v_i' relative to ∂c_i, if and only if ϑ_i vanishes as an element of $\pi_n(S^n)$. Since $\alpha_F(e_i) - \alpha_{F'}(e_i')$ is the characteristic map of the subbundle determined by the normal vector field ϑ_i, we have

$$\partial\vartheta_i = \alpha_F(e_i) - \alpha_{F'}(e_i') = 0 \tag{4.1}$$

by Lemma 2.12 and our hypothesis.

In the case where n is even ($n \geq 4$), we have that $\partial : \pi_n(S^n) \to \pi_{n-1}(SO(n))$ is injective by Lemma 2.9. Thus, we have that $\vartheta_i = 0$, and consequently, h_i and h_i' are isotopic relative to $h_i \cap D^{2n} = h_i' \cap D^{2n}$.

When n is odd ($n \geq 3$), the proof goes as follows. First, observe that

$$\Gamma_F(e_i, e_i) - \Gamma_{F'}(e_i', e_i') = \vartheta_i \in \pi_n(S^n) \cong \mathbb{Z}$$

for all $i > s$ ($e_i \in R(\operatorname{Ker} i_{F*})$, $e_i' \in R(\operatorname{Ker} i_{F'*})$). To check this, recall that $\Gamma_F(e_i, e_i)$ and $\Gamma_{F'}(e_i', e_i')$ are the linking numbers of $\bar{c}_i$ and its translations by $\bar{v}_i$ and by $\bar{v}_i'$ respectively, where $\bar{v}_i$ and $\bar{v}_i'$ are the obvious extensions of v_i and v_i' on $\bar{c}_i = \bar{c}_i'$, respectively. Since $\vartheta_i \in \pi_n(S^n)$ is the difference between these two vector fields, we have that $\Gamma_F(e_i, e_i) - \Gamma_{F'}(e_i', e_i') = \vartheta_i$.

Since the isomorphism $\Psi : H_n(F) \to H_n(F')$ preserves the Seifert forms by our hypothesis, we have that $\vartheta_i = 0$. Thus the normal vector fields v_i and v_i' are homotopic relative to boundary, for $i > s$, and consequently, the handle h_i is isotopic to h_i' relative to $h_i \cap D^{2n} = h_i' \cap D^{2n}$ for these values of i.

For $n = 3, 7$, we are assuming that M is a rational homology sphere, which implies that $H_n(F) = R(\operatorname{Ker} i_{F*})$, and it is not necessary to worry about the case $i \leq s$. Thus, h_i and h_i' are isotopic relative to $h_i \cap D^{2n} = h_i' \cap D^{2n}$ for all i.

Now it remains only the case that n is odd with $n \neq 3, 7$ and $i \leq s$.

In order to prove that the handle h_i is isotopic to h_i' relative to $h_i \cap D^{2n} = h_i' \cap D^{2n}$ for $i \leq s$, recall that $\operatorname{Im}\partial \cong \mathbb{Z}_2$ for n odd with $n \geq 5, n \neq 7$, where $\partial : \pi_n(S^n) \to \pi_{n-1}(SO(n))$ is the boundary homomorphism (see Lemma 2.9). Since $\partial\vartheta_i = 0$ as has been seen in (4.1), $\vartheta_i \in \pi_n(S^n) \cong \mathbb{Z}$ is a multiple of 2. To complete the proof of Lemma 4.2, we need the following.

Lemma 4.5. *When n is odd with $n \geq 5, n \neq 7$, for each $i \leq s$, there exists an ambient isotopy $\Phi = \{\Phi_t\}_{t \in [0,1]}$ of M relative to $D^{2n} \cup h_1 \cup \cdots \cup$*

$h_{i-1} \cup h_{i+1} \cup \cdots \cup h_r$ such that $\Phi_0 = \mathrm{id}$, $c_i = \Phi_1(c_i)$, and the difference between the vector fields associated with h_i and with $\Phi_1(h_i)$, denoted by ϑ_i, represents the element 2 in $\pi_n(S^n) \cong \mathbb{Z}$.

PROOF. Consider the embedded spheres $\bar{c}_i \subset F$ and $\hat{c}_i \subset M - D^{2n} \subset M$ representing the elements $e_i \in H_n(F)$ and the dual $\hat{e}_i \in H_{n+1}(M)$ respectively such that $\hat{c}_i \cap \bar{c}_j = \emptyset$ for $i \neq j$, $1 \leq j \leq r$, and $\hat{c}_i$ transversely intersects c_i at a unique point, as discussed in the proof of Lemma 4.4. We may further assume that $\hat{c}_i \cap h_j = \emptyset$ for $i \neq j$, $1 \leq j \leq r$. We set $c_i \cap \hat{c}_i = \{p\}$.

Consider a sufficiently small tubular neighborhood $N(\hat{c}_i)$ associated with the normal bundle of $\hat{c}_i$ in M, and the fibration $\pi : N(\hat{c}_i) \to \hat{c}_i$. Furthermore, consider a neighborhood $D_p \cong D_1^n \times D^1$ of p in $\hat{c}_i$. Then we can identify $\pi^{-1}(D_p)$ with $D_1^n \times D^1 \times D_2^n \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, where D_1^n, D^1 and D_2^n are the unit disks in $\mathbb{R}^n, \mathbb{R}$ and $\mathbb{R}^n$ respectively, and we may assume that $\pi^{-1}(D_p) \supset c_i \cap N(\hat{c}_i)$ in such a way that $c_i \cap N(\hat{c}_i) = \{0\} \times \{0\} \times D_2^n$ (see the left of Figure 1).

Now we identify D^n with $D_3^n \cup S^{n-1} \times [0,1] \cup S^{n-1} \times [1,2]$, where D_3^n is the unit disk and $\partial D_3^n = S^{n-1} \times \{0\}$. Define the embedding

$$\begin{aligned} \eta : D^n &= D_3^n \cup S^{n-1} \times [0,1] \cup S^{n-1} \times [1,2] \\ &\to D_1^n \times D^1 \times D_2^n = \pi^{-1}(D_p) \subset N(\hat{c}_i) \end{aligned}$$

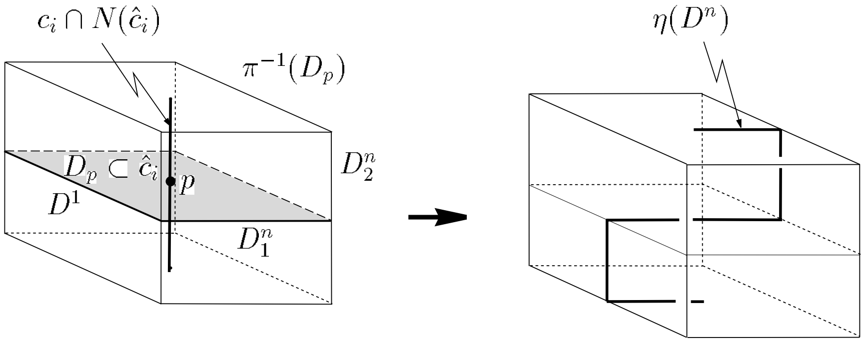

Fig. 1. Modifying $c_i \cap N(\hat{c}_i)$.

by

$$\begin{cases} \eta(x) = (x, 0, 0), & x \in D_3^n, \\ \eta(x,t) = (x, 0, tx), & (x,t) \in S^{n-1} \times [0,1], \\ \eta(x,t) = ((2-t)x, 0, x)\,, & (x,t) \in S^{n-1} \times [1,2] \end{cases}$$

followed by a smoothing. Then η defines an embedding such that $\eta(\partial D^n) = \eta(S^{n-1} \times \{2\}) = \{0\} \times \{0\} \times \partial D_2^n$ (see the right of Figure 1).

If $d = \overline{c_i - (c_i \cap N(\hat{c}_i))} \cup \eta(D^n)$, then d is diffeomorphic to c_i. Moreover, c_i and d are isotopic in M relative to $D^{2n} \cup h_1 \cup \cdots \cup h_{i-1} \cup h_{i+1} \cup \cdots \cup h_r$. Thus we may assume that d is the core of h_i.

Set $\Delta = \eta(D_3^n) = D_1^n \times \{0\} \times \{0\}$, then $\Delta = d \cap \hat{c}_i \subset \hat{c}_i$. Since Δ is a disk of codimension one in $\hat{c}_i \cong S^{n+1}$, there exists a trivial open book structure on $\hat{c}_i$ (see Definition 2.8) such that Δ is the typical page. Denote the one-parameter family of diffeomorphisms of $\hat{c}_i$ associated with this open book by $\{\nu_t\}_{t\in\mathbb{R}}$ (for details, see Definition 2.5). Then $\nu_t : \hat{c}_i \to \hat{c}_i$ satisfies $\nu_0 = \mathrm{id}$ and $\nu_t|_{\partial\Delta} = \mathrm{id}$ for all t. Define $d_t = \overline{(d - \Delta)} \cup \nu_t(\Delta)$ for $t \in \mathbb{R}$. Then d_t, $t \in [0,1]$, determines an isotopy of d in $M - \big(D^{2n} \cup h_1 \cup \cdots \cup h_{i-1} \cup h_{i+1} \cup \cdots \cup h_r\big)$ which can be extended to an ambient isotopy $\Phi : M \times [0,1] \to M$ relative to $D^{2n} \cup h_1 \cup \cdots \cup h_{i-1} \cup h_{i+1} \cup \cdots \cup h_r$, by the isotopy extension theorem.

Now, the isotopy Φ carries the handle h_i with core d to a handle $\tilde{h}_i$ with the same core d. Let us consider the difference between the normal vector fields on d determining $\tilde{h}_i$ and h_i, and denote it by θ. Then θ is trivial outside $N(\hat{c}_i) \cap d$, and we may assume that the difference exists only on $S^{n-1} \times [0,1] = \eta(S^{n-1} \times [0,1]) \subset d$. In this part, θ represents the twisting of the vector field produced by the isotopy Φ. Since the isotopy Φ is obtained as an extension of d_t, Φ produces a rotation along $S^{n-1} \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}$. Therefore, to absorb the twist in $S^{n-1} \times [0,1]$, $\theta|_{S^{n-1}\times[0,1]}$ makes a rotation along $S^{n-1} \times \{t\}$ when we increase the value of $t \in [0,1]$, completing one turn at $t = 1$ (see Figure 2).

Thus, θ is defined as a map

$$\theta : D_4^n \cup S^{n-1} \times [0,1] \cup D_5^n \cong S^n \to S^n \subset \mathbb{R}^n \times \mathbb{R} \times \{0\}$$

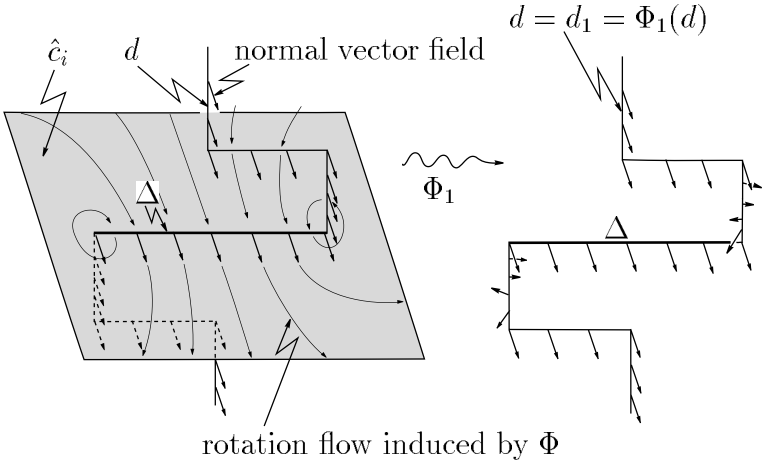

Fig. 2. The change of normal vector field by the action of the isotopy Φ.

by

$$\begin{cases} \theta(u) = (0,1,0) \in \mathbb{R}^n \times \mathbb{R} \times \{0\}, & u \in D_4^n, \\ \theta(v) = (0,1,0) \in \mathbb{R}^n \times \mathbb{R} \times \{0\}, & v \in D_5^n, \\ \theta(x,t) = (\sin(2\pi t)x, \cos(2\pi t), 0) & \\ \qquad\qquad \in \mathbb{R}^n \times \mathbb{R} \times \{0\}, & (x,t) \in S^{n-1} \times [0,1], \end{cases}$$

where $S^{n-1} = \partial\Delta$. Observe that the normal space of $S^{n-1} = \partial\Delta \subset \mathbb{R}^n \times \mathbb{R} \times \{0\}$ in $\hat{c}_i \cong S^{n+1}$ at the point x is determined by $(x, 0, 0)$ and $(0, 1, 0)$.

Now it is an easy exercise to show that the degree of the above map θ is equal to -2 for n odd and 0 for n even. Therefore, we have that the constructed isotopy Φ changes the normal vector field associated with the handle by -2 as an element of $\pi_n(S^n) \cong \mathbb{Z}$. This implies that there also exists an isotopy which changes it by 2. This completes the proof of Lemma 4.5. □

We return to the proof of Lemma 4.2 in the case where n is odd with $n \neq 3, 7$ and $i \leq s$.

By successive applications of the isotopies relative to $D^{2n} \cup h_1 \cup \cdots \cup h_{i-1} \cup h_{i+1} \cup \cdots \cup h_r$ given by Lemma 4.5, we can adjust v_i $(1 \leq i \leq s)$ so that ϑ_i determined as the difference between v_i and v_i' corresponds to the zero element of $\pi_n(S^n)$. Thus, h_i and h_i' are isotopic in

$$\big(M - (D^{2n} \cup h_1 \cup \cdots \cup h_{i-1} \cup h_{i+1} \cup \cdots \cup h_r)\big) \cup (h_i \cap D^{2n})$$

relative to $h_i \cap D^{2n} = h_i' \cap D^{2n}$ for all $i \leq s$. This complete the proof of Lemma 4.2. □

REMARK 4.6. In the proof of Lemma 4.2, F and F' may not necessarily be pages of open books. The necessary requirement is that F and F' are homotopy equivalent to a bouquet of n-spheres, and that i_{F*} and $i_{F'*}$ are surjective. Thus the lemma gives an isotopy criterion for two compact oriented $2n$-dimensional manifolds homotopy equivalent to a bouquet of n-spheres, embedded in an $(n-1)$-connected closed oriented $(2n+1)$-dimensional manifold, such that the inclusions induce epimorphisms in the homology level.

Analyzing the proof of Lemma 4.2, we obtain the following stronger result.

LEMMA 4.7. *Let M be an $(n-1)$-connected closed $(2n+1)$-dimensional manifold with $n \geq 4, n \neq 7$, or an $(n-1)$-connected rational homology $(2n+1)$-sphere with $n = 3, 7$. Suppose that $j_i : F \to M$, $i = 1, 2$, are embeddings of an $(n-1)$-connected compact oriented $2n$-dimensional manifold with non-empty $(n-2)$-connected boundary such that $j_{i*} : H_n(F) \to H_n(M)$ are surjective. If the systems of invariants associated with j_i are equivalent to each other, and if the equivalence is induced by the identity map of F, then the embeddings j_i are isotopic to each other as maps into M.*

PROOF. Let $F = D^{2n} \cup h_1 \cup \cdots \cup h_r$ be the handlebody decomposition of F as in the proof of Lemma 4.2. Then we can easily isotope j_1 so that $j_1|_{D^{2n}} = j_2|_{D^{2n}}$. Then, by Lemma 4.3, we can isotope j_1 relative to D^{2n} so that $j_1(h_k) = j_2(h_k)$, for all k with $1 \leq k \leq r$. In fact, in the proof of Lemma 4.3, we can choose the isotopy H appropriately so that we have $j_1|_{h_k} = j_2|_{h_k}$ at the end of the isotopy. This is possible, since every continuous map $D^{n+1} \to \operatorname{Int} D^{2n+1}$, $n \geq 3$, whose restriction to boundary is a smooth embedding is homotopic to a smooth embedding relative to boundary. Then the rest of the proof of Lemma 4.2 shows that we can further isotope j_1 so that it coincides with j_2 as a map of F into M. This completes the proof of Lemma 4.7. □

The above lemma will be used in §8 for the study of isotopies of certain diffeomorphisms of the $2n$-dimensional manifold F.

4.2. Isotopy of open book structures

Lemma 4.8. *Two open book structures on M with isotopic typical pages are structurally isotopic for $n \geq 3$.*

Proof. The argument of Durfee [Dur74] for simple fibered knots in S^{2n+1} works without problem in our case as well as follows.

By our hypothesis, we may assume that the typical pages F and F' of the two open book structures coincide. Then the bindings also coincide and we denote it by K.

Consider $E = \overline{M - N(K)}$, where $N(K)$ is a tubular neighborhood of K in M as in Definition 2.1, and let $\varphi_1 : E \to S^1$ and $\varphi_2 : E \to S^1$ be the fibrations corresponding to the two open book structures. Note that we can take the same tubular neighborhood $N(K)$ of K for both open book structures, because of the uniqueness of a tubular neighborhood up to ambient isotopy. Furthermore, a trivialization of the tubular neighborhood of K is unique up to isotopy, since K is simply connected, and hence we may assume that $\varphi_1|_{\partial E} = \varphi_2|_{\partial E}$. Denote $F \cap E$ and $F' \cap E$ by F and F' respectively by abuse of notation. Then we have $F = \varphi_1^{-1}(0)$ and $F' = \varphi_2^{-1}(0)$. Consider a closed neighborhood $J \subset S^1 = \mathbb{R}/\mathbb{Z}$ of 0. Then, by the uniqueness of a tubular neighborhood $N(F)$ of $F = F'$ in E, we may assume that $N(F) = \varphi_1^{-1}(J) = \varphi_2^{-1}(J)$ and $\varphi_1|_{N(F)} = \varphi_2|_{N(F)} : N(F) \to J$. Now consider $E' = \overline{E - N(F)}$ and the two fibrations $\varphi_1' = \varphi_1|_{E'} : E' \to \overline{S^1 - J} = [0,1]$ and $\varphi_2' = \varphi_2|_{E'} : E' \to \overline{S^1 - J} = [0,1]$. Note that we have $\varphi_1'|_{\partial E'} = \varphi_2'|_{\partial E'}$.

Observe that all fibrations over $[0,1]$ are trivial, which implies that there exist trivializations $g_1 : E' \to F \times [0,1]$ and $g_2 : E' \to F \times [0,1]$ such that the diagram

$$\begin{array}{ccc} E' & \xrightarrow{g_j} & F \times [0,1] \\ & {\scriptstyle \varphi'_j} \searrow \quad \swarrow {\scriptstyle p} & \\ & [0,1] & \end{array}$$

commutes for $j = 1, 2$, where p denotes the projection to the second factor. Since $\varphi'_1|_{\partial E'} = \varphi'_2|_{\partial E'}$, we may assume that

$$g_1 \circ g_2^{-1}|_{F\times\{0\}} = \mathrm{id} \quad \text{and} \quad g_1 \circ g_2^{-1}|_{\partial F\times[0,1]} = \mathrm{id}\,.$$

Thus $g_1 \circ g_2^{-1}$ gives a pseudo-isotopy of $F \times I$ which is the identity over $(F \times \{0\}) \cup (\partial F \times I)$, where $I = [0, 1]$. Since $n \geq 3$, by the relative version of the pseudo-isotopy theorem of Cerf [Cer70], it is isotopic, as a pseudo-isotopy relative to $(F \times \{0\}) \cup (\partial F \times I)$, to an isotopy which is not necessarily the identity map on $F \times \{1\}$, since $g_1 \circ g_2^{-1}|_{F\times\{1\}}$ is not necessarily the identity. This means that there exists an ambient isotopy $H : (F \times [0, 1]) \times [0, 1] \to F \times [0, 1]$ of $F \times [0, 1]$ such that $H_0 = \mathrm{id}$, $H_1 = g_1 \circ g_2^{-1}$, and $H_t|_{(F\times\{0\})\cup(\partial F\times[0,1])}$ is the identity map for all $t \in [0, 1]$, where $H_t : F \times [0, 1] \to F \times [0, 1], t \in [0, 1]$, is given by $H_t(x, s) = H((x, s), t)$.

Define the map $\mathcal{H} : E' \times I \to E'$ by $\mathcal{H}(x, t) = g_2^{-1}(H(g_2(x), t))$. Then, $\mathcal{H}_0 = \mathrm{id}$ and $\varphi'_2 \circ \mathcal{H}_1 = \varphi'_2 \circ g_2^{-1} \circ g_1 = p \circ g_1 = \varphi'_1$, where $\mathcal{H}_t : E' \to E', t \in [0, 1]$, is given by $\mathcal{H}_t(x) = \mathcal{H}(x, t)$. Moreover, $\mathcal{H}_t$ is the identity over $\overline{\partial E' - g_2^{-1}(F \times \{1\})}$. Since $\mathcal{H}_0 = \mathrm{id}$, we have that $\mathcal{H}_t|_{g_2^{-1}(F\times\{1\})}$ is isotopic to the identity for all t, and consequently, we can extend $\mathcal{H}$ to $E' \cup (F \times [0, 1])$ so that it is the identity on the boundary and that it carries $F \times \{t\}$ to $F \times \{t\}$ for each $t \in [0, 1]$, where $F \times [0, 1]$ is considered to be a collar neighborhood of $g_2^{-1}(F \times \{1\})$ in $\varphi_2^{-1}(J)$. Thus, we can extend $\mathcal{H}$ to an isotopy of E such that it is the identity on ∂E, which allows us to extend it further to an ambient isotopy of M. Due to the properties of the original $\mathcal{H}$, we have that $\varphi_2 \circ \mathcal{H}_1|_E = \varphi_1$, and consequently, $\mathcal{H}$ is an isotopy between φ_1 and φ_2. Thus, the two open book structures are structurally isotopic (or, isotopic through open books). This completes the proof of Lemma 4.8. □

Theorem 4.1 now follows from Lemmas 4.2 and 4.8.

Remark 4.9. The additional condition for M, to be a rational homology $(2n+1)$-sphere for $n = 3, 7$ in Theorem 4.1, has been necessary because of the proof of Lemma 4.2, where we used Lemma 2.9 in order to guarantee that the difference $\vartheta_i \in \pi_n(S^n) \cong \mathbb{Z}$ is a multiple of 2. This argument is not valid for $n = 3, 7$. In the case that $n = 3$, we also have the problem of spherical representations of elements of $H_{n+1}(M)$ used in the proofs of Lemmas 4.4 and 4.5.

We do not know if Theorem 4.1 is valid for $n = 3, 7$ without the additional condition.

5. Realization of Invariants

In this section, we prove that each system of open book invariants can be realized by an open book structure. This means that given a system of invariants $\bar{s} \in \mathcal{A}(M)$, there exists an open book (M, K, φ) such that $\mathcal{S}(M, K, \varphi) = \bar{s}$, where $\mathcal{S}(M, K, \varphi)$ denotes the equivalence class of the system of open book invariants associated with (M, K, φ). This construction will be achieved in two steps. The first one is to construct a submanifold of codimension one which realizes the system of invariants, and the second step is to prove that this submanifold is the page of an open book structure.

5.1. Realization by a codimension one submanifold

In the following, hoping no confusion arises, we will not distinguish between a system of open book invariants and its equivalence class.

Proposition 5.1. *Let $\bar{s} = \{G, Q_G, \alpha_G, i_G, \Gamma_G\} \in \mathcal{A}(M)$ be a system of open book invariants with respect to M, where M is an $(n-1)$-connected closed oriented $(2n+1)$-dimensional manifold with $n \geq 3$. Then there exists an embedding of a compact oriented $2n$-dimensional manifold F in M which is homotopy equivalent to a bouquet of n-spheres such that the system of invariants associated with F coincides with $\bar{s}$.*

Proof. Let $\{e_i\}_{i=1}^r$ with $r = \operatorname{rank} G$ be a basis of G such that $\{e_{s+1}, \ldots, e_r\}$ is a basis of $R(\operatorname{Ker} i_G)$, $0 \leq s \leq r$. Consider the complex $V' = D^{2n} \cup \gamma'_1 \cup \cdots \cup \gamma'_r$, where each $\gamma'_i \cong D^n$ is attached to ∂D^{2n} using the embedding $\partial\gamma'_i \hookrightarrow \partial D^{2n}$ such that $\partial\gamma'_i \cap \partial\gamma'_j = \emptyset$ and $\operatorname{lk}(\partial\gamma'_i, \partial\gamma'_j) = Q_G(e_i, e_j)$ for $i \neq j$, where lk denotes the linking number in $\partial D^{2n} \cong S^{2n-1}$. This is possible, since $n \geq 2$, and we can embed V' in $D^{2n+1} \subset M$.

Observe that each element $i_G(e_i) \in H_n(M) \cong \pi_n(M)$ can be represented by an embedded n-sphere in $M - D^{2n+1}$, since $n \geq 1$ [Hae61, Wal63]. We may assume that the n-spheres are disjoint from each other.

Now, take the connected sum, inside $M - D^{2n}$, of the n-sphere representing $i_G(e_i)$ with γ_i' of V', for each $i = 1, \ldots, r$. Denote the resulting embedded complex by $V = D^{2n} \cup \gamma_1 \cup \cdots \cup \gamma_r$, where $\gamma_i \cong D^n$ corresponds to γ_i' in V'.

Let v_i be a non-vanishing normal vector field on γ_i. Then v_i restricted to $\partial\gamma_i$ represents an element of $\pi_{n-1}(S^n) = 0$, and consequently, it is homotopic to a normal vector field on $\partial\gamma_i$ determined as the restriction of a non-vanishing normal vector field on D^{2n}. Thus, we may assume that v_i is normal to D^{2n} on $\partial\gamma_i$. This normal vector field determines an n-handle $h_i = \gamma_i \times D^n \subset N(\gamma_i)$ with core γ_i, $i = 1, \ldots, r$, so that it is normal to the n-handle, where $N(\gamma_i)$ is a tubular neighborhood of γ_i in M. Thus we have constructed a $2n$-dimensional manifold $F = D^{2n} \cup h_1 \cup \cdots \cup h_r$ embedded in M. Note that F is homotopy equivalent to a bouquet of n-spheres and $H_n(F) = G$ under the identification of $e_i \in G$ with the element of $H_n(F)$ corresponding to γ_i. Using this identification, we have that $i_{F*} = i_G : H_n(F) = G \to H_n(M)$, where $i_F : F \to M$ denotes the inclusion. In the following, we always assume that the intersection form and the Seifert form are represented as matrices with respect to the bases $\{e_i\}_{i=1}^r$ and $\{e_i\}_{i=s+1}^r$ of $G = H_n(F)$ and $R(\operatorname{Ker} i_G) = R(\operatorname{Ker} i_{F*})$ respectively. Note that we have $R(\operatorname{Ker} i_G) = R(\operatorname{Ker} i_{F*})$ under the above identification.

The intersection matrix off its diagonal coincides with the linking matrix of $\{\partial\gamma_i\}_{i=1}^r$ by our construction. In the case that n is odd, the diagonal of the intersection matrix always vanishes, since it is $(-1)^n$-symmetric, which implies that the intersection form of F coincides with Q_G. When n is even, observe that the diagonals of the intersection matrices on G and on $H_n(F)$ are uniquely determined by the tangential invariants on the elements of the basis (see item (4b) of Definition 3.15 and Lemma 2.10). Thus, let us adjust the tangential invariants.

No matter whether n is even or odd, the n-handle h_i of F is uniquely determined by the homotopy class of the normal vector field v_i on the core γ_i, up to isotopy. If we substitute h_i by $\tilde{h}_i$, where $\tilde{h}_i$ is determined by a normal vector field $\tilde{v}_i$ on γ_i such that v_i and $\tilde{v}_i$ coincide on $\partial\gamma_i$, then an element $\vartheta_i \in \pi_n(S^n)$ is determined as the difference between v_i and $\tilde{v}_i$. Note that given any $m \in \pi_n(S^n) \cong \mathbb{Z}$, we can always obtain $\tilde{v}_i$ such that the ϑ_i associated with the difference is represented by m, since the correspondence between the homotopy classes of non-vanishing sections of $D^{n+1} \times S^n \to S^n$ and the elements of $\pi_n(S^n) \cong \mathbb{Z}$ is bijective.

Definition 5.2. A *twist* of h_i by $m \in \mathbb{Z}$ is the replacement of h_i by $\tilde{h}_i$ so that $\vartheta_i = m$ in $\pi_n(S^n) \cong \mathbb{Z}$.

As discussed in the proof of Lemma 4.2, $\partial\vartheta_i$ coincides with the difference between the tangential invariants associated with h_i and $\tilde{h}_i$ by Lemma 2.12, where ∂ is the boundary homomorphism of the homotopy exact sequence (2.1).

Since the tangential invariant α_F of F and the desired tangential invariant α_G are compatible with the tangential invariant of M (see Remark 3.4 and item (4a) of Definition 3.15), the difference $\alpha_G(e_i) - \alpha_F(e_i)$ lies in $\operatorname{Ker} i_* = \operatorname{Im}\partial$. Thus, there exists an element $m \in \pi_n(S^n)$ such that $\partial m = \alpha_G(e_i) - \alpha_F(e_i)$. Then, using the twist of h_i by m, we can eliminate this difference. Consequently, we may assume that $\alpha_F = \alpha_G$ on all the elements of the basis. As previously observed, α_F and α_G determine the diagonals of the intersection forms for n even, and as a consequence, we have $Q_F = Q_G$.

Since α_F and α_G coincide on the elements of a basis and $Q_F = Q_G$, the addition formula (Remark 3.2 for α_F and item (4c) of Definition 3.15 for α_G) implies that $\alpha_F = \alpha_G$ on $H_n(F) = G$.

By the above argument, we may assume that F realizes all the invariants except the Seifert form. Let us now adjust the diagonal of the Seifert matrix. When n is even, we have that $2\Gamma_G(e_i, e_i) = Q_G(e_i, e_i)$ (item (5b) of Definition 3.15) and $2\Gamma_F(e_i, e_i) = Q_F(e_i, e_i)$ (Lemma 3.9) for $s+1 \leq i \leq r$. Since $Q_G = Q_F$, the diagonal of the Seifert matrix of F coincides with that of G.

When $n = 3, 7$, observe that

$$\begin{aligned}\Gamma_G(e_i, e_i) - \Gamma_F(e_i, e_i) &\equiv b_M(i_G(e_i), i_G(e_i)) - b_M(i_{F*}(e_i), i_{F*}(e_i)) \\ &\equiv 0 \pmod 1\end{aligned}$$

for $s+1 \leq i \leq r$, due to item (5c) of Definition 3.15 and item (1) of Lemma 3.14. Consequently, $\Gamma_G(e_i, e_i)$ and $\Gamma_F(e_i, e_i)$ differ by an integer and hence, we can adjust it by using a twist associated with this difference. Since $\pi_{n-1}(SO(n)) = 0$ for $n = 3, 7$ (see Lemma 2.9), we always have $\alpha_G = 0 = \alpha_F$ and consequently, the tangential invariant does not change.

When n is odd ($n \geq 5, n \neq 7$), observe that $R(\mathrm{Ker}\, i_G) = R(\mathrm{Ker}\, i_{F*})$ by the identification $i_{F*} = i_G$. We have $\Gamma_G(e_i, e_i) = q_M(i_G(e_i)) + \phi(\alpha_G(e_i))$ (mod 2) (item (5d) of Definition 3.15) and $\Gamma_F(e_i, e_i) = q_M(i_{F*}(e_i)) + \phi(\alpha_F(e_i))$ (mod 2) (item (2) of Lemma 3.14) for $s+1 \leq i \leq r$, since $n \geq 5, n \neq 7$. Since the tangential invariants coincide, we have that $\Gamma_G(e_i, e_i) - \Gamma_F(e_i, e_i) \equiv 0 \pmod 2$ and as a consequence, $\Gamma_G(e_i, e_i) - \Gamma_F(e_i, e_i) = 2m_i$ for some $m_i \in \mathbb{Z}$. Now, we make the twist of the handle h_i of F corresponding to e_i by $2m_i$. Since $\Gamma_F(e_i, e_i)$ is defined to be the linking number between the translation of an n-cycle E_i representing e_i, in the positive normal direction of F, and E_i, the twist by $2m_i$ changes $\Gamma_F(e_i, e_i)$ by $2m_i$, since it changes the normal vector field exactly by $2m_i$. Thus, after the twist, we may assume that $\Gamma_G(e_i, e_i) = \Gamma_F(e_i, e_i)$. Note that the change of $\alpha_F(e_i)$ by the above mentioned twist is $\partial(2m_i) = 2(\partial m_i) = 0$ in $\mathrm{Im}\,\partial \cong \mathbb{Z}_2$, since n is odd, $n \geq 5, n \neq 7$ (see Lemmas 2.9 and 2.12). Thus the tangential invariant does not change during this process.

In this way, we may assume that the diagonals of the Seifert matrices of F and G coincide, independently of n being even or odd. In order to complete the adjustment of the Seifert form, we use Kervaire's method [Ker65] as follows.

By item (5c) of Definition 3.15 and item (1) of Lemma 3.14, we have that $\Gamma_G - \Gamma_F$ is an integer matrix (with vanishing diagonal). Furthermore, by item (5b) of Definition 3.15 and Lemma 3.9, we have that

$$(\Gamma_G - \Gamma_F) + (-1)^n \cdot {}^t(\Gamma_G - \Gamma_F) = Q_G - Q_F = 0,$$

where ${}^tA$ denotes the transpose of a matrix A. Thus, $\Gamma_G - \Gamma_F = (-1)^{n+1}X + {}^tX$ for some integer matrix $X = (x_{ij})_{s+1 \leq i,j \leq r}$ such that $x_{ii} = 0$.

Since $n \geq 3$, $\partial\gamma_i$ is the trivial knot in ∂D^{2n}, where D^{2n} is the 0-handle of the decomposition $F = D^{2n} \cup h_1 \cup \cdots \cup h_r$. Thus $\partial\gamma_i$ bounds an n-disk in ∂D^{2n}. Attaching this n-disk to γ_i along their boundaries, we obtain an n-cycle in F representing the element $e_i \in H_n(F)$. Denote this n-cycle by $\bar{\gamma}_i$.

For $s+1 \leq j \leq r$, let D_j be a small $(n+1)$-disk in M which intersects γ_j transversely at a unique point in their interiors. Take D_j sufficiently small so that D_j does not intersect γ_i for $i \neq j$, $1 \leq i \leq r$, and that $D_i \cap D_j = \emptyset$ for $i \neq j$, $s+1 \leq i \leq r$. Orient D_j so that $\mathrm{lk}(\bar{\gamma}_i, \partial D_j) = \delta_{ij}$, for $s+1 \leq i, j \leq r$, where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$.

Now consider a trivial D^n-bundle $N(\partial D_j) \cong \partial D_j \times D^n$ which is a tubular neighborhood of ∂D_j in a $2n$-disk containing D_j inside its interior. We can choose $N(\partial D_j)$ small enough so that it does not intersect F.

Now we consider $h_i = \gamma_i \times D^n$, $s+1 \leq i \leq r$, and take an ambient *fiber connected sum* of $N(\partial D_j)$ and h_i as follows.

Let $D'_j \subset \partial D_j$ and $D''_i \subset \gamma_i$ be small n-disks such that the fibrations $N(\partial D_j)$ and h_i restricted to them are trivial. Substitute $h_i = \gamma_i \times D^n$ by

$$\left((\gamma_i - \mathrm{Int}\, D''_i) \times D^n\right) \cup \left([0,1] \times S^{n-1} \times D^n\right) \cup \left((\partial D_j - \mathrm{Int}\, D'_j) \times D^n\right),$$

where the union is made as follows. Consider a small tubular neighborhood $[0,1] \times D^n \times D^n$ of an embedded curve which joins the centers of D'_j and D''_i and which is disjoint from F and $N(\partial D_k)$, $s+1 \leq k \leq r$, except at its end points. We may suppose that $\{0\} \times D^n \times \{0\} = D'_j$ and $\{1\} \times D^n \times \{0\} = D''_i$. Then, we identify $\{0\} \times S^{n-1} \times D^n \subset [0,1] \times S^{n-1} \times D^n$ with $\partial D'_j \times D^n$ and $\{1\} \times S^{n-1} \times D^n \subset [0,1] \times S^{n-1} \times D^n$ with $\partial D''_i \times D^n$ (see Figure 3).

After this fiber connected sum, the n-cycle associated with the new handle corresponding to e_i coincides with the embedded connected sum

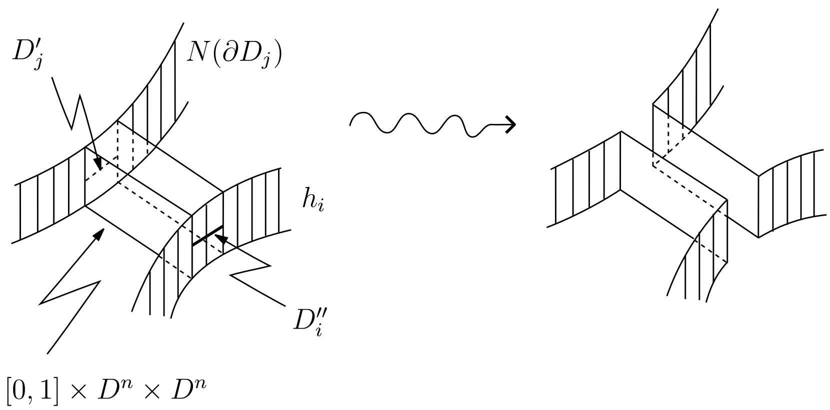

Fig. 3. Fiber connected sum of h_i and $N(\partial D_j)$.

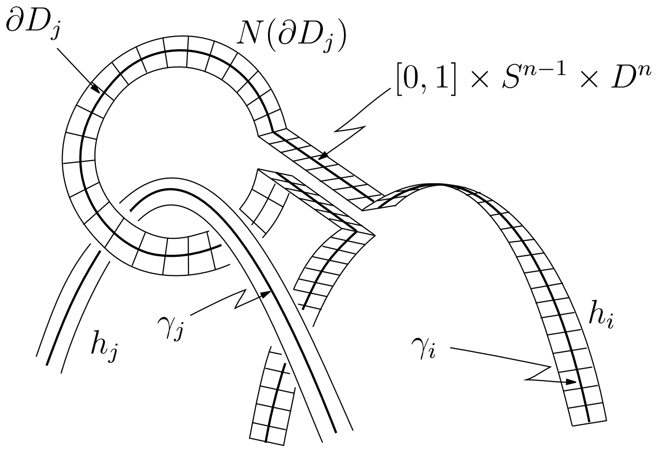

Fig. 4. Adjusting the Seifert form.

$\bar{\gamma}_i \sharp \partial D_j$ (see Figure 4). Note also that the diffeomorphism type of F or the homomorphism i_{F*} do not change under this operation.

When $x_{ij} > 1$, we iterate the above fiber connected sum operations x_{ij} times, using different D_j at each step in a way to avoid the intersection with the previous ones. If $x_{ij} < 0$, then we use $-D_j$, which is D_j with the orientation reversed, in place of D_j, and perform the fiber connected sum operations $|x_{ij}|$ times. When $x_{ij} = 0$, no operation is needed.

After all these operations for $s+1 \leq i, j \leq r$, the n-cycle associated with the core of the new handle $\tilde{h}_i$ corresponding to e_i coincides with the embedded connected sum

$$\tilde{\gamma}_i = \bar{\gamma}_i \mathop{\sharp}_{j=s+1}^{r} x_{ij} \partial D_j.$$

Denote the resulting manifold by $\tilde{F}$. Then, for $s+1 \leq i, j \leq r$, we have that

$$\begin{aligned} \Gamma_{\tilde{F}}(e_i, e_j) &= \operatorname{lk}(\nu^+ \tilde{\gamma}_i, \tilde{\gamma}_j) \\ &= \operatorname{lk}\left(\nu^+\left(\bar{\gamma}_i \mathop{\sharp}_{k=s+1}^{r} x_{ik} \partial D_k\right),\ \bar{\gamma}_j \mathop{\sharp}_{l=s+1}^{r} x_{jl} \partial D_l\right) \end{aligned}$$

$$
\begin{aligned}
&= \operatorname{lk}(\nu^+\bar{\gamma}_i, \bar{\gamma}_j) + \sum_{l=s+1}^{r} x_{jl} \operatorname{lk}(\nu^+\bar{\gamma}_i, \partial D_l) \\
&\quad + \sum_{k=s+1}^{r} x_{ik} \operatorname{lk}(\nu^+\partial D_k, \bar{\gamma}_j) \\
&\quad + \sum_{k=s+1}^{r} \sum_{l=s+1}^{r} x_{ik}x_{jl} \operatorname{lk}(\nu^+\partial D_k, \partial D_l).
\end{aligned}
$$

By our choice of the $(n+1)$-disks D_k, we have $\operatorname{lk}(\partial D_k, \partial D_l) = 0$, and since $\operatorname{lk}(\bar{\gamma}_i, \partial D_k) = \delta_{ik}$, we have

$$
\begin{aligned}
\Gamma_{\tilde{F}}(e_i, e_j) &= \operatorname{lk}(\nu^+\bar{\gamma}_i, \bar{\gamma}_j) + x_{ji} + (-1)^{n+1}x_{ij} \\
&= \Gamma_F(e_i, e_j) + (-1)^{n+1}x_{ij} + x_{ji} = \Gamma_G(e_i, e_j).
\end{aligned}
$$

Observe that i_{F*}, the intersection form, and the fibering structure of h_i are maintained by this operation, and as a consequence, the Seifert form is the only invariant that is changed. Thus, $\tilde{F}$ realizes all the desired invariants. This completes the proof of Proposition 5.1. □

5.2. Realization by an open book structure

In this subsection, we prove that an embedding of a compact $2n$-dimensional manifold homotopy equivalent to a bouquet of n-spheres which realizes a system of open book invariants is in fact a page of an open book structure.

Definition 5.3 ([Qui79]). Let $F \hookrightarrow M$ be an embedding of a compact $2n$-dimensional manifold F with boundary into a $(2n+1)$-dimensional manifold M such that the normal bundle of F in M is trivial. Then the tubular neighborhood $N(F)$ of F can be identified with $F \times I \subset M$, where $I = [0,1]$. We say that $F \subset M$ is a *homotopy page*, if the inclusions of $F \times \{0\}$ and $F \times \{1\}$ into $\overline{M - (F \times I)}$ induce isomorphisms on the homotopy groups.

Our goal is to prove that the F constructed in Proposition 5.1 is always a homotopy page.

We put $W = \overline{M - (F \times I)}$ for simplicity, where $F \times I$ is a tubular neighborhood of F in M. Note that the inclusion of $F \times \{1\}$ into W can be identified (homotopically) with a small translation in the positive normal direction of F, which is denoted by $\nu^+ : F \to M - \operatorname{Int} F$. In the same manner, the inclusion of $F \times \{0\}$ into W can be identified with the translation in the negative normal direction, denoted by $\nu^- : F \to M - \operatorname{Int} F$.

By our construction, F is obtained by attaching simultaneously some n-handles to the 0-handle, and since $n \geq 3$, ∂F is $(n-2)$-connected by Lemma 2.3.

LEMMA 5.4. *Let F be an $(n-1)$-connected compact $2n$-dimensional manifold with $(n-2)$-connected boundary $\partial F \neq \emptyset$ embedded in an $(n-1)$-connected closed $(2n+1)$-dimensional manifold M, such that $i_{F*} : H_n(F) \to H_n(M)$ is surjective, with $n \geq 3$. Then $W = \overline{M - (F \times I)}$ is $(n-1)$-connected and $H_*(W) \cong H_*(F)$.*

PROOF. Since $n \geq 3$, by the conditions on F and ∂F, and by the Seifert-van Kampen theorem, we have $\pi_1(\partial(F \times I)) = \{1\}$. Since M is simply connected, the Seifert-van Kampen theorem again implies that W is also simply connected. Thus we have only to prove $H_i(W) = 0$ for $i \geq 2, i \neq n$, and $H_n(W) \cong H_n(F)$.

Consider the homology exact sequence

$$\cdots \to H_{i+1}(M, W) \to H_i(W) \to H_i(M) \to \cdots$$

of the pair (M, W) and observe that $H_{i+1}(M, W) \cong H_{i+1}(F \times I, \partial(F \times I)) \cong H_i(F, \partial F)$ by excision and the Künneth theorem. Thus the sequence becomes

$$\cdots \to H_i(F, \partial F) \to H_i(W) \to H_i(M) \to \cdots$$

and hence we have $H_i(W) = 0$ for $i = 2, \ldots, n-1$, since $H_i(F, \partial F) = 0 = H_i(M)$ for these values of i. Consequently, W is $(n-1)$-connected.

Let us consider the homology exact sequence of the pair $(M, F \times I)$

$$\cdots \to H_i(F \times I) \to H_i(M) \to H_i(M, F \times I) \to H_{i-1}(F \times I) \to \cdots .$$

Since $H_i(M, F \times I) \cong H_i(W, \partial W)$ by excision, the sequence becomes

$$\cdots \to H_i(F \times I) \to H_i(M) \to H_i(W, \partial W) \to H_{i-1}(F \times I) \to \cdots$$

and hence we have that $H_i(W, \partial W) = 0$ for $i = 0, 1, \ldots, n-1$ due to the high connectivities of F and M.

For dimension n, the above sequence becomes

$$\cdots \to H_n(F \times I) \to H_n(M) \to H_n(W, \partial W) \to 0.$$

Since $i_{F*} : H_n(F) \to H_n(M)$ is surjective by hypothesis, we have $H_n(W, \partial W) = 0$.

Now, by Poincaré-Lefschetz duality and the universal coefficient theorem for cohomology, we have

$$\begin{aligned} H_i(W) &\cong H^{2n+1-i}(W, \partial W) \\ &\cong \operatorname{Hom}(H_{2n+1-i}(W, \partial W), \mathbb{Z}) \oplus \operatorname{Ext}(H_{2n-i}(W, \partial W), \mathbb{Z}) = 0, \end{aligned}$$

for $i > n$, since $H_{2n+1-i}(W, \partial W) = 0$ for these values of i.

For $i = n$, we have

$$\begin{aligned} H_n(W) &\cong H^{n+1}(W, \partial W) \\ &\cong \operatorname{Hom}(H_{n+1}(W, \partial W), \mathbb{Z}) \oplus \operatorname{Ext}(H_n(W, \partial W), \mathbb{Z}) \\ &\cong \operatorname{Hom}(H_{n+1}(W, \partial W), \mathbb{Z}) \end{aligned}$$

and hence $H_n(W)$ is free over $\mathbb{Z}$. Since $H_n(W)$ and $H_n(F)$ are free $\mathbb{Z}$-modules, they are isomorphic, if and only if their ranks coincide.

Consider the homology exact sequences

$$0 \to H_{n+1}(F \times I, \partial(F \times I)) \to H_n(\partial(F \times I)) \to H_n(F \times I) \to 0$$

associated with the pair $(F \times I, \partial(F \times I))$ and

$$0 \to H_{n+1}(W, \partial W) \to H_n(\partial W) \to H_n(W) \to 0$$

associated with the pair $(W, \partial W)$. Using the Poincaré-Lefschetz duality, and the fact that $H_n(F)$ and $H_n(W)$ are free $\mathbb{Z}$-modules, we have

$$H_n(\partial(F \times I)) \cong H_n(F) \oplus H^n(F) \text{ and } H_n(\partial W) \cong H_n(W) \oplus H^n(W).$$

Since $\partial(F \times I) = \partial W$, we have

$$\begin{aligned} H_n(F) \oplus H_n(F) &\cong H_n(F) \oplus H^n(F) \cong H_n(W) \oplus H^n(W) \\ &\cong H_n(W) \oplus H_n(W). \end{aligned}$$

Thus, we have $\operatorname{rank} H_n(F) = \operatorname{rank} H_n(W)$ as desired. □

Set $R = R(\operatorname{Ker} i_{F*})$ and $R' = R(\operatorname{Ker} i_{W*})$, where $R(\operatorname{Ker} i_{F*})$ and $R(\operatorname{Ker} i_{W*})$ are the radical closures of $\operatorname{Ker} i_{F*}$ and $\operatorname{Ker} i_{W*}$ respectively (see Definition 3.6), and $i_F : F \to M$ and $i_W : W \to M$ are the inclusions. Since $i_W \circ \nu^+ : F \to M$ and $i_F : F \to M$ are isotopic, the following diagram is commutative:

$$(5.1) \qquad \begin{array}{ccc} H_n(F) & \xrightarrow{i_{F*}} & H_n(M) \longrightarrow 0 \\ {\scriptstyle \nu_*^+}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\ H_n(W) & \xrightarrow{i_{W*}} & H_n(M) \longrightarrow 0. \end{array}$$

Thus we have $\nu_*^+(\operatorname{Ker} i_{F*}) \subset \operatorname{Ker} i_{W*}$, which implies that $\nu_*^+(R) \subset R'$. Thus, the homomorphisms $\bar{\nu}_*^+ : H_n(F)/R \to H_n(W)/R'$ and $\tilde{\nu}_*^+ = \nu_*^+|_R : R \to R'$ induced by $\nu_*^+ : H_n(F) \to H_n(W)$ are well-defined. Moreover, ν_*^+ is an isomorphism if and only if both $\bar{\nu}_*^+$ and $\tilde{\nu}_*^+$ are isomorphisms, since R and R' are direct summands of $H_n(F)$ and $H_n(W)$ respectively.

Lemma 5.5. *Let F be an $(n-1)$-connected compact $2n$-dimensional manifold with $(n-2)$-connected boundary $\partial F \neq \emptyset$ embedded in an $(n-1)$-connected closed $(2n+1)$-dimensional manifold M, such that $i_{F*} : H_n(F) \to H_n(M)$ is surjective, with $n \geq 3$. Then, $\bar{\nu}_*^+ : H_n(F)/R \to H_n(W)/R'$ is an isomorphism.*

Proof. By excision and Poincaré-Lefschetz duality, we have

$$H_n(M, W) \cong H_n(F \times I, \partial(F \times I)) \cong H^{n+1}(F \times I) \cong H^{n+1}(F) = 0.$$

Then the homology exact sequence of the pair (M, W) gives the exact sequence

$$(5.2) \qquad \begin{aligned} H_{n+1}(M) \longrightarrow\ & H_{n+1}(F \times I, \partial(F \times I)) \\ & \xrightarrow{\partial} H_n(W) \xrightarrow{i_{W*}} H_n(M) \longrightarrow 0, \end{aligned}$$

and consequently, i_{W*} is surjective.

By the commutative diagram (5.1) and the fact that $\nu_*^+(R) \subset R'$, we have that the diagram

$$\begin{array}{ccc} H_n(F)/R & \xrightarrow{\bar{i}_{F*}} & H_n(M)/\tau H_n(M) \\ {\scriptstyle \bar{\nu}_*^+}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\ H_n(W)/R' & \xrightarrow{\bar{i}_{W*}} & H_n(M)/\tau H_n(M) \end{array}$$

is commutative, where $\bar{i}_{F*}$ and $\bar{i}_{W*}$ are the homomorphisms induced by i_{F*} and i_{W*}, respectively. Since i_{F*} and i_{W*} are surjective, $\bar{i}_{F*}$ and $\bar{i}_{W*}$ are isomorphisms. Thus $\bar{\nu}_*^+$ is also an isomorphism. □

LEMMA 5.6. *Let F be an $(n-1)$-connected compact $2n$-dimensional manifold with $(n-2)$-connected boundary $\partial F \neq \emptyset$ embedded in an $(n-1)$-connected closed $(2n+1)$-dimensional manifold M, such that $i_{F*} : H_n(F) \to H_n(M)$ is surjective, with $n \geq 3$. Then we have that*

$$\det \tilde{\nu}_*^+ = \pm |\tau H_n(M)| \det \Gamma_F,$$

where $\det \tilde{\nu}_*^+$ and $\det \Gamma_F$ are the determinants of $\tilde{\nu}_*^+$ and Γ_F respectively, and $|\tau H_n(M)|$ is the order of the torsion part $\tau H_n(M)$ of $H_n(M)$.

PROOF. In order to prove the lemma, define the homomorphism

$$\psi : H_n(F, \partial F) \to H_{n+1}(N(F), \partial N(F)) \cong H_n(F, \partial F) \otimes H_1(I, \partial I)$$

by $\psi(A) = A \otimes [(I, \partial I)]$ for $A \in H_n(F, \partial F)$, where $N(F)$ is the tubular neighborhood of F in M, $I = [0,1]$, and $[(I, \partial I)] \in H_1(I, \partial I) \cong \mathbb{Z}$ is the canonical generator. Set $\Phi = \partial \circ \psi$, where $\partial : H_{n+1}(N(F), \partial N(F)) \to H_n(W)$ is the boundary homomorphism of the exact sequence (5.2) in the proof of Lemma 5.5.

Then, $\Phi(H_n(F, \partial F)) \subset \operatorname{Ker} i_{W*}$ and the following diagram is commutative:

$$\begin{array}{ccc} H_n(F, \partial F) & & \\ \psi \big\downarrow & \searrow^{\Phi} & \\ H_{n+1}(N(F), \partial N(F)) & \xrightarrow{\partial} & \operatorname{Ker} i_{W*}. \end{array}$$

Note that ψ is an isomorphism. We also have that ∂ is surjective by the exact sequence (5.2) and as a consequence, Φ is surjective.

We now make use of the following notion of orthogonal submodules.

DEFINITION 5.7. Consider finitely generated free $\mathbb{Z}$-modules $A = A_1 \oplus A_2$ and B, and a bilinear form $\xi : A \times B \to \mathbb{Z}$. For $i = 1, 2$, define the submodule $A_i^\perp$ of B, called the *orthogonal complement* of A_i with respect to ξ, by

$$A_i^\perp = \{b \in B : \xi(a, b) = 0 \text{ for all } a \in A_i\}.$$

LEMMA 5.8. *If $\xi : A \times B \to \mathbb{Z}$ is a unimodular bilinear form, where $A = A_1 \oplus A_2$ and B are finitely generated free $\mathbb{Z}$-modules of the same rank r, then we have that $B = A_1^\perp \oplus A_2^\perp$.*

PROOF. Since ξ is unimodular, there exist bases $\{a_i\}_{i=1}^r$ and $\{b_i\}_{i=1}^r$ of A and B respectively such that

$$\begin{cases} \{a_1, \dots, a_s\} \text{ is a basis for } A_1, \\ \{a_{s+1}, \dots, a_r\} \text{ is a basis for } A_2, \\ \xi(a_i, b_j) = \delta_{ij}, \end{cases}$$

where $s = \operatorname{rank} A_1$, $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$. Then we see easily that $\{b_1, \dots, b_s\}$ and $\{b_{s+1}, \dots, b_r\}$ are bases for $A_2^\perp$ and $A_1^\perp$ respectively. Consequently, we have $B = A_1^\perp \oplus A_2^\perp$. $\square$

REMARK 5.9. Observe that $\operatorname{rank} A_1^\perp = \operatorname{rank} A_2$ and $\operatorname{rank} A_2^\perp = \operatorname{rank} A_1$.

Let us return to the proof of Lemma 5.6. Since $R \subset H_n(F)$ is a direct summand and the intersection form

$$\langle\, , \rangle : H_n(F) \times H_n(F, \partial F) \to \mathbb{Z} \tag{5.3}$$

is unimodular due to Poincaré-Lefschetz duality,

$$R^\perp = \{a \in H_n(F, \partial F) : \langle b, a \rangle = 0 \text{ for all } b \in R \subset H_n(F)\}$$

is a direct summand of $H_n(F, \partial F)$ by Lemma 5.8 above, where $\langle b, a \rangle$ denotes the intersection number of b and a in F.

LEMMA 5.10. *We have $R^\perp \supset \operatorname{Ker} \Phi$.*

PROOF. Consider the commutative diagram

$$\begin{array}{ccccc} H_{n+1}(M) & \xrightarrow{\delta} & H_n(F, \partial F) & & \\ \| & & \psi \big\downarrow & \searrow^{\Phi} & \\ H_{n+1}(M) & \xrightarrow{\delta'} & H_{n+1}(N(F), \partial N(F)) & \xrightarrow{\partial} & \operatorname{Ker} i_{W*}, \end{array}$$

where

$$H_{n+1}(M) \xrightarrow{\delta} H_n(F, \partial F) \quad \text{and} \quad H_{n+1}(M) \xrightarrow{\delta'} H_{n+1}(N(F), \partial N(F))$$

are defined by Poincaré-Lefschetz duality and by the homomorphisms

$$H^n(M) \xrightarrow{i_F^*} H^n(F) \quad \text{and} \quad H^n(M) \xrightarrow{i_{N(F)}^*} H^n(N(F))$$

induced by the inclusions $i_F : F \to M$ and $i_{N(F)} : N(F) \to M$, respectively. Observe that the last sequence of the above diagram is a part of the exact sequence (5.2).

Since ψ is an isomorphism and the last sequence of the diagram is exact, we have that $\operatorname{Ker}\Phi = \operatorname{Im}\delta$. Thus, for all $b \in \operatorname{Ker}\Phi$, there exists a $b' \in H_{n+1}(M)$ such that $b = \delta(b')$. Hence, for an arbitrary $a \in R$, we have that $\langle a, b\rangle = \langle a, \delta(b')\rangle = i_{F*}(a) \cdot b' = 0$, where the first two intersections are in F and the last intersection is in M. The last equality holds, since $i_{F*}(a)$ is a torsion element of $H_n(M)$. Thus, $b \in R^\perp$ and as a consequence, we have $\operatorname{Ker}\Phi \subset R^\perp$. □

LEMMA 5.11. *We have* $R^\perp = \operatorname{Ker}\Phi$*, and the homomorphism*

$$\bar{\Phi} : H_n(F, \partial F)/R^\perp \to \operatorname{Ker} i_{W*}$$

induced by Φ *is an isomorphism.*

PROOF. Since $\Phi : H_n(F, \partial F) \to \operatorname{Ker} i_{W*}$ is surjective,

$$\bar{\Phi} : H_n(F, \partial F)/\operatorname{Ker}\Phi \to \operatorname{Ker} i_{W*}$$

induced by Φ is an isomorphism. Thus, it is enough to show that $R^\perp = \operatorname{Ker}\Phi$.

By Lemma 5.10, we have an epimorphism

$$p : H_n(F, \partial F)/\operatorname{Ker}\Phi \to H_n(F, \partial F)/R^\perp$$

defined as the natural projection. Now consider $p \circ \bar{\Phi}^{-1} : \operatorname{Ker} i_{W*} \to H_n(F, \partial F)/R^\perp$, which is also an epimorphism.

Observe that $\operatorname{rank}(\operatorname{Ker} i_{W*}) = \operatorname{rank}(\operatorname{Ker} i_{F*})$ by Lemma 5.4 and the diagram (5.1). Then we have $\operatorname{rank}(\operatorname{Ker} i_{W*}) = \operatorname{rank} R = \operatorname{rank}(H_n(F, \partial F)/R^\perp)$, where the last equality is due to Remark 5.9.

Since $\operatorname{Ker} i_{W*}$ and $H_n(F, \partial F)/R^{\perp}$ are free $\mathbb{Z}$-modules and any epimorphism between two finitely generated free $\mathbb{Z}$-modules of the same rank is an isomorphism, we have that $p \circ \bar{\Phi}^{-1}$ is an isomorphism, and consequently, p is an isomorphism. Thus, we have that $\operatorname{Ker} \Phi = R^{\perp}$, which implies that $H_n(F, \partial F)/\operatorname{Ker} \Phi = H_n(F, \partial F)/R^{\perp}$. This completes the proof of Lemma 5.11. □

Now let us return to the situation of Lemma 5.8. Define $\xi_1 : A_1 \times (B/A_1^{\perp}) \to \mathbb{Z}$ by $\xi_1(a, [b]) = \xi(a, b)$, where $[b]$ denotes the coset of b. Then we see that ξ_1 is a well-defined bilinear form by the definition of $A_1^{\perp}$.

Lemma 5.12. *The bilinear form ξ_1 is unimodular.*

Proof. By the proof of Lemma 5.8, we have that $B/A_1^{\perp} \cong A_2^{\perp}$ and there exist bases $\{[b_1], \dots, [b_s]\}$ of $B/A_1^{\perp}$ and $\{a_1, \dots, a_s\}$ of A_1 respectively such that $\xi_1(a_i, [b_j]) = \xi(a_i, b_j) = \delta_{ij}$ (see the proof of Lemma 5.8). This completes the proof of Lemma 5.12. □

Let us return finally to the proof of Lemma 5.6. Consider the submodule $\operatorname{Ker} i_{W*}$ of the finitely generated free $\mathbb{Z}$-module $R(\operatorname{Ker} i_{W*})$. By [KaM79, Theorem 8.1.1], there exists a basis $\{\tilde{a}_1, \dots, \tilde{a}_{r-s}\}$ of $R' = R(\operatorname{Ker} i_{W*})$ such that $\{\tilde{A}_i = r_i \tilde{a}_i\}_{i=1}^{r-s}$ is a basis of $\operatorname{Ker} i_{W*}$, obtained by diagonalizing the matrix associated with L in the exact sequence

$$0 \longrightarrow \operatorname{Ker} i_{W*} \xrightarrow{L} R(\operatorname{Ker} i_{W*}) \longrightarrow R(\operatorname{Ker} i_{W*})/\operatorname{Ker} i_{W*} \longrightarrow 0,$$

where $r - s = \operatorname{rank}(\operatorname{Ker} i_{F*}) = \operatorname{rank}(\operatorname{Ker} i_{W*})$ and $r_1, \dots, r_{r-s}$ are positive integers.

Since $\bar{\Phi}^{-1} : \operatorname{Ker} i_{W*} \to H_n(F, \partial F)/R^{\perp}$ is an isomorphism, the basis $\{A_i = \bar{\Phi}^{-1}(\tilde{A}_i)\}_{i=1}^{r-s}$ of $H_n(F, \partial F)/R^{\perp}$ is well-defined. Since R and $H_n(F, \partial F)/R^{\perp}$ are related by duality (Lemma 5.12), we can choose a basis $\{a_i\}_{i=1}^{r-s}$ of R, which is dual to the basis $\{A_i\}_{i=1}^{r-s}$ of $H_n(F, \partial F)/R^{\perp}$.

In order to analyze the matrix of $\tilde{\nu}_*^+ : R \to R'$ with respect to these special bases, we observe that

$$\Gamma_F(a_i, a_j) = \operatorname{lk}(\tilde{\nu}_*^+ a_i, a_j) = \operatorname{lk}\left(\sum_{k=1}^{r-s} [\tilde{\nu}_*^+]_{ki} \tilde{a}_k, a_j\right)$$

$$= \sum_{k=1}^{r-s} [\tilde{\nu}_*^+]_{ki} \operatorname{lk}(\tilde{a}_k, a_j) = \sum_{k=1}^{r-s} \frac{1}{r_k} [\tilde{\nu}_*^+]_{ki} \operatorname{lk}(\tilde{A}_k, a_j),$$

where $\tilde{\nu}_*^+ a_i = \sum_{k=1}^{r-s} [\tilde{\nu}_*^+]_{ki} \tilde{a}_k$ with $[\tilde{\nu}_*^+]_{ki} \in \mathbb{Z}$. By the definition of the linking form lk, the last expression is equal to

$$\sum_{k=1}^{r-s} \frac{1}{r_k} [\tilde{\nu}_*^+]_{ki} \operatorname{lk}(\tilde{A}_k, a_j) = \sum_{k=1}^{r-s} \frac{1}{r_k} [\tilde{\nu}_*^+]_{ki} \operatorname{lk}(\bar{\Phi}(A_k), a_j)$$
$$= \pm \sum_{k=1}^{r-s} \frac{1}{r_k} [\tilde{\nu}_*^+]_{ki} \langle a_j, A_k \rangle = \pm \sum_{k=1}^{r-s} \frac{1}{r_k} [\tilde{\nu}_*^+]_{ki} \delta_{jk} = \pm \frac{1}{r_j} [\tilde{\nu}_*^+]_{ji},$$

where $a_j \in R \subset H_n(F)$, $A_k \in H_n(F, \partial F)/R^\perp$, the intersection $\langle a_j, A_k \rangle$ is induced from the usual intersection form of F as in Lemma 5.12, and $\langle a_j, A_k \rangle = \delta_{jk} \in \mathbb{Z}$ by the choice of our dual bases.

Observing that $|\tau H_n(M)| = r_1 \cdots r_{r-s}$, since

$$\tau H_n(M) \cong R(\operatorname{Ker} i_{F*}) / \operatorname{Ker} i_{F*} \cong R(\operatorname{Ker} i_{W*}) / \operatorname{Ker} i_{W*},$$

we have

$$\det \Gamma_F = \frac{\pm 1}{r_1 \cdots r_{r-s}} \det \tilde{\nu}_*^+ = \frac{\pm 1}{|\tau H_n(M)|} \det \tilde{\nu}_*^+.$$

This completes the proof of Lemma 5.6. □

Proposition 5.13. *Let F be an $(n-1)$-connected compact $2n$-dimensional manifold with $(n-2)$-connected boundary $\partial F \neq \emptyset$ embedded in an $(n-1)$-connected closed $(2n+1)$-dimensional manifold M, such that $i_{F*} : H_n(F) \to H_n(M)$ is surjective, with $n \geq 3$. If $\det \Gamma_F = |\tau H_n(M)|^{-1}$, then F is the page of some open book structure on M.*

Proof. By Lemmas 5.5 and 5.6, $\nu_*^+ : H_n(F) \to H_n(W)$ is an isomorphism and an analogous argument shows that $\nu_*^- : H_n(F) \to H_n(W)$ is also an isomorphism, where $W = \overline{M - (F \times [0,1])}$. Since W is homotopy equivalent to a bouquet of n-spheres by Lemma 5.4, the theorem of Whitehead [Spa66, Chapter 7, §5, Theorem 9] implies that ν^+ and ν^- induce homotopy equivalences. Thus $F \subset M$ is a homotopy page. Since $n \geq 3$, the

h-cobordism theorem ([Sma62]) implies that $W = \overline{M - (F \times I)} \cong F \times I$ and as a consequence, F is the page of some open book structure on M. □

The above proposition, item (5a) of Definition 3.15 and Proposition 5.1 give the following.

THEOREM 5.14. *Let M be an $(n-1)$-connected closed oriented $(2n+1)$-dimensional manifold with $n \geq 3$. Then for each equivalence class of a system of open book invariants $\bar{s} \in \mathcal{A}(M)$, there exists a simple and oriented open book structure (K, φ) on M such that $\mathcal{S}(M, K, \varphi) = \bar{s}$, where $\mathcal{S}(M, K, \varphi)$ is the equivalence class of the system of open book invariants associated with (K, φ).*

Now Theorems 5.14 and 4.1 give the following classification theorem presented in the introduction.

THEOREM 5.15. *Let M be an $(n-1)$-connected closed oriented $(2n+1)$-dimensional manifold with $n \geq 4, n \neq 7$, or an $(n-1)$-connected oriented rational homology $(2n+1)$-sphere with $n = 3, 7$. Then the map $\mathcal{S} : \mathcal{OB}(M) \to \mathcal{A}(M)$ defined by sending each structural isotopy class of a simple and oriented open book structure (K, φ) on M to the equivalence class $\mathcal{S}(K, \varphi)$ of its system of open book invariants establishes a one-to-one correspondence between the set $\mathcal{OB}(M)$ of all structural isotopy classes of simple and oriented open book structures on M and the set $\mathcal{A}(M)$ of all equivalence classes of systems of open book invariants with respect to M.*

REMARK 5.16. In the case that $M \cong S^{2n+1}$ with $n \geq 3$, the above result is well-known, where $\mathcal{A}(M)$ can be identified with the set of all congruence classes of unimodular matrices (see Remarks 3.17 and 3.20, and [Dur74, Kat74]). However, the result is not valid for $n = 2$, even when $M \cong S^{2n+1}$ (see [Sae87]).

6. Bindings

In this section, we focus on the isotopy class of the binding of an open book structure as an embedded submanifold of codimension two, obtaining two important results. The first one guarantees the uniqueness of an open book structure for a given binding, which is equivalent to saying that the system of open book invariants, studied in §3, is in fact an invariant of the binding. The second one gives necessary and sufficient conditions for a codimension two embedding to be a binding of some open book structure.

6.1. Open book structure is determined by its binding

LEMMA 6.1. *Consider two simple and oriented open book structures with common binding K, on an $(n-1)$-connected closed $(2n+1)$-dimensional manifold M with $n \geq 3$, with typical pages F_1 and F_2 respectively. Then there exists an orientation preserving diffeomorphism $\Phi : M \to M$ carrying F_1 to F_2, such that $\Phi \circ \nu_1^+ = \nu_2^+ \circ \Phi|_{F_1}$ and $\Phi_* : H_n(M) \to H_n(M)$ is the identity, where $\nu_1^+ : F_1 \to M - \mathrm{Int}\, F_1$ and $\nu_2^+ : F_2 \to M - \mathrm{Int}\, F_2$ are small translations in the positive normal directions of F_1 and F_2 respectively.*

PROOF. The diffeomorphism Φ can be obtained by an argument similar to that used in [Sae99, Lemmas 2.4 and 2.5] as follows.

Set $E = \overline{M - N(K)}$, where $N(K)$ is a tubular neighborhood of K in M, and we denote the fibration $E \to S^1$ associated with the two open book structures by φ_1 and φ_2, and the typical pages by $F_1 = \varphi_1^{-1}(0)$ and $F_2 = \varphi_2^{-1}(0)$ respectively, where $0 \in S^1 = \mathbb{R}/\mathbb{Z}$. Since $H^1(K) = 0$, the trivialization of $N(K)$ is uniquely determined up to homotopy. Then we may assume that $\varphi_1|_{\partial N(K)} = \varphi_2|_{\partial N(K)}$. Consider the universal cover $\tilde{E}$ of E. By the uniqueness of the universal cover, $\tilde{E} = F_1 \times \mathbb{R}$ is diffeomorphic to $F_2 \times \mathbb{R}$. Thus, there exists a diffeomorphism $g : F_2 \times \mathbb{R} \to F_1 \times \mathbb{R}$ such that the diagram

$$\begin{array}{ccc} F_2 \times \mathbb{R} & \xrightarrow{g} & \tilde{E} = F_1 \times \mathbb{R} \\ {\scriptstyle\rho_2} \searrow & & \swarrow {\scriptstyle\rho_1} \\ & E & \end{array} \tag{6.1}$$

commutes, where ρ_1 and ρ_2 are the projections of the universal covers associated with φ_1 and φ_2 respectively. Since F_2 is compact, we may assume that $\tilde{F_2} = g(F_2 \times \{0\}) \subset F_1 \times (0, r)$ for some positive integer r. Then $\tilde{F_1} = F_1 \times \{0\}$ and $\tilde{F_2}$ bound a compact manifold $W_1 \subset F_1 \times [0, r]$, and $\tilde{F_2}$ and $F_1 \times \{r\}$ bound another compact manifold $W_2 \subset F_1 \times [0, r]$. Since φ_1 coincides with φ_2 on the boundary of E, W_1 and W_2 form invertible cobordisms relative to boundary (see [Sie68]). Thus, W_1 is an h-cobordism relative to boundary [Kin78, fact 3]. Since $n \geq 3$, W_1 is diffeomorphic to $F_1 \times [0,1]$.

In order to obtain an embedding of W_1 in $E \times I$ with $I = [0,1]$, we make use of Wall's construction ([Wal70, p. 140], [Lau76], [Sae99, Lemma 2.5]) as follows. For simplicity, we identify $\tilde{E}$ with $F_1 \times \mathbb{R}$. Let $p_2 : \tilde{E} = F_1 \times \mathbb{R} \to \mathbb{R}$ be the projection to the second factor and consider the embedding $(i, p_2/r) : W_1 \to \tilde{E} \times I$, where $i : W_1 \to \tilde{E}$ is the inclusion map, $(p_2/r)(x) = p_2(x)/r$, and $I = [0,1]$. Attaching a compact submanifold of $\tilde{F}_2 \times I$ bounded by $(i, p_2/r)(\tilde{F}_2)$ and $\tilde{F}_2 \times \{1\}$ and smoothing, we obtain an embedding $\tilde{e} : W_1 \to \tilde{E} \times I$. Denote the image of this embedding by W. Then W is a submanifold of $\tilde{E} \times I$ whose boundary relative to $\partial\tilde{E} \times I$ consists of $\tilde{F}_1 \times \{0\}$ and $\tilde{F}_2 \times \{1\}$; i.e. $\overline{\partial W - (\partial\tilde{E} \times I)} = \tilde{F}_1 \times \{0\} \cup \tilde{F}_2 \times \{1\}$ (see the left of Figure 5). Using the projection $\rho_1 \times \mathrm{id} : \tilde{E} \times I \to E \times I$, where ρ_1 is the projection in the diagram (6.1), we define

$$\bar{e} = (\rho_1 \times \mathrm{id}) \circ \tilde{e} : W_1 \to E \times I.$$

It is not difficult to check that it is an embedding, with image $\hat{W} = (\rho_1 \times \mathrm{id})(W)$ bounded by $F_1 \times \{0\}$ and $F_2 \times \{1\}$ relative to $\partial E \times I$; i.e. $\overline{\partial\hat{W} - (\partial E \times I)} = F_1 \times \{0\} \cup F_2 \times \{1\}$ (see the right of Figure 5).

Cutting $E \times I$ along $\hat{W} = (\rho_1 \times \mathrm{id})(W)$, we obtain a compact $(2n+2)$-dimensional manifold, denoted by X. Considering the covering translation $\tau : \tilde{E} \to \tilde{E}$, we see that the compact manifold in $\tilde{E} \times I$ bounded by W

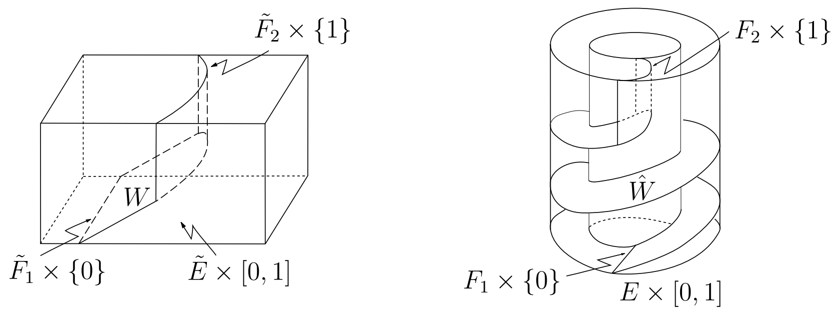

Fig. 5. Embeddings of W_1 in $\tilde{E} \times I$ and in $E \times I$.

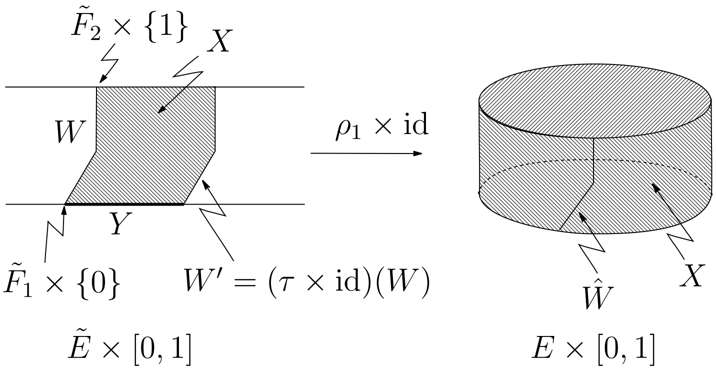

Fig. 6. Cutting $E \times I$ along $\hat{W}$.

and $W' = (\tau \times \mathrm{id})(W)$ can be identified with X by the covering projection $\rho_1 \times \mathrm{id} : \tilde{E} \times I \to E \times I$ (see Figure 6).

LEMMA 6.2. *The manifold X gives an h-cobordism between W and W' relative to boundary.*

PROOF. Cutting $\tilde{E} \times I$ along W, we have two non-compact manifolds whose intersection coincides with W. Since W and $\tilde{E} \times I$ are both connected, these two manifolds are simply connected by the Seifert-van Kampen theorem. We take the one that contains W', and cutting it along W', we obtain two new manifolds, where one of them is X. Since W' is simply connected, both of the resulting manifolds are necessarily simply connected, which implies that X is also simply connected (see Figure 6).

Since W, W' and X are all simply connected, we have only to prove that $H_*(X, W) = 0$. For this, consider $\tilde{E} = F_1 \times \mathbb{R}$, and set $Y = (F_1 \times I) \times \{0\}$. Then the inclusion $W \hookrightarrow W \cup Y \cup W'$ is a homotopy equivalence, and as a consequence, we have by excision

$$H_*(X, W) \cong H_*(X, W \cup Y \cup W') \cong H_*(E \times I, \hat{W} \cup (E \times \{0\})).$$

Since $E \times \{0\} \hookrightarrow \hat{W} \cup (E \times \{0\})$ is a homotopy equivalence, we have that

$$H_*(X, W) \cong H_*(E \times I, \hat{W} \cup (E \times \{0\})) \cong H_*(E \times I, E \times \{0\}) = 0.$$

Similarly, we have $H_*(X, W') = 0$. Since $n \geq 3$, we have that $(X; W, W')$ is an h-cobordism relative to boundary [Mil65]. □

Thus, there exists a diffeomorphism $\Theta : W \times I \to X$ such that $\Theta|_{W\times\{0\}} = \mathrm{id} : W \times \{0\} \to W$. Consider the diffeomorphism $\Theta_1 = \Theta|_{W\times\{1\}} : W \to W'$. Then $\tilde{\Phi} = \Theta_1^{-1} \circ ((\tau \times \mathrm{id})|_W) : W \to W$ is a diffeomorphism and we have $\tilde{\Phi}(F_1) = F_1$ and $\tilde{\Phi}(F_2) = F_2$, where the boundary components $\tilde{F}_1 \times \{0\}$ and $\tilde{F}_2 \times \{1\}$ of W (relative to $\partial\tilde{E} \times [0, 1]$) are naturally identified with F_1 and F_2, respectively.

Since X is an h-cobordism relative to boundary, we have that

$$\tilde{\Phi}|_{F_1} = h_1,\ \tilde{\Phi}|_{F_2} = h_2,\ \tilde{\Phi}|_{\overline{\partial W - (F_1 \cup F_2)}} = \mathrm{id},$$

where h_1 and h_2 are the monodromy maps of φ_1 and φ_2 respectively (see Definition 2.5). On the other hand, since $W_1 \cong W$ is diffeomorphic to $F_1 \times [0, 1]$, there exists a diffeomorphism $\lambda : F_1 \times I \to W$ such that $\lambda|_{F_1\times\{0\}} = \mathrm{id}$. Consider the diffeomorphism $\lambda_1 = \lambda|_{F_1\times\{1\}} : F_1 \to F_2$ induced by λ. Then, h_1 and $\lambda_1^{-1} \circ h_2 \circ \lambda_1$ are pseudo-isotopic relative to boundary by the pseudo-isotopy $\lambda^{-1} \circ \tilde{\Phi} \circ \lambda$.

Now define the diffeomorphism

$$\Phi' : E = F_1 \times I/(x, 1) \sim (h_1(x), 0) \to E = F_2 \times I/(y, 1) \sim (h_2(y), 0)$$

by $\Phi' = (\lambda_1 \times \mathrm{id}) \circ \lambda^{-1} \circ \tilde{\Phi}^{-1} \circ \lambda$.

Note that we have

$$\begin{aligned}\Phi'(x, 1) &= (\lambda_1 \times \mathrm{id}) \circ \lambda^{-1} \circ \tilde{\Phi}^{-1}(\lambda_1(x), 1)\\ &= (\lambda_1 \times \mathrm{id}) \circ \lambda^{-1}\left(h_2^{-1} \circ \lambda_1(x), 1\right)\\ &= (\lambda_1 \times \mathrm{id})\left(\lambda_1^{-1} \circ h_2^{-1} \circ \lambda_1(x), 1\right)\\ &= (h_2^{-1} \circ \lambda_1(x), 1)\end{aligned}$$

and

$$\begin{aligned}\Phi'(h_1(x), 0) &= (\lambda_1 \times \mathrm{id}) \circ \lambda^{-1} \circ \tilde{\Phi}^{-1}(h_1(x), 0) = (\lambda_1 \times \mathrm{id})(x, 0)\\ &= (\lambda_1(x), 0).\end{aligned}$$

Hence, Φ' is well-defined and is an orientation preserving diffeomorphism. Moreover, we have the following.

Lemma 6.3. *The induced homomorphism $\Phi'_* : H_n(E) \to H_n(E)$ is the identity.*

Proof. Since $\overline{E - N(F_1)} \cong F_1 \times I$, where $I = [0,1]$ and $N(F_1)$ is a tubular neighborhood of F_1 in E, we have that

$$\begin{aligned} H_n(E, F_1) &\cong H_n(E, N(F_1)) \cong H_n(F_1 \times I, F_1 \times \{0,1\}) \\ &\cong H^{n+1}(F_1 \times I, \partial F_1 \times I) \cong H^{n+1}(F_1, \partial F_1) \\ &\cong H_{n-1}(F_1) = 0 \end{aligned}$$

by excision and Poincaré-Lefschetz duality. Then the homology exact sequence

$$H_n(F_1) \xrightarrow{i'_{F_1 *}} H_n(E) \longrightarrow H_n(E, F_1)$$

of the pair (E, F_1) implies that $i'_{F_1 *} : H_n(F_1) \to H_n(E)$ is surjective, where $i'_{F_1} : F_1 \to E$ is the inclusion given by $i'_{F_1}(x) = x \times \{1\}$ and $x \times \{1\}$ is in $F_1 \times \{1\} \subset E = F_1 \times I/(x,1) \sim (h_1(x), 0)$.

Thus, an arbitrary $\xi \in H_n(E)$ can be represented by an n-cycle $a \times \{1\} \subset E = F_1 \times I/(x,1) \sim (h_1(x), 0)$ for some n-cycle a in F_1.

Denote the homology class represented by $a \times \{1\}$ in $H_n(E)$ by $[a \times \{1\}]$. Then we have

$$\Phi'_*(\xi) = \Phi'_*([a \times \{1\}]) = [\Phi'(a \times \{1\})] = [h_2^{-1} \circ \lambda_1(a) \times \{1\}].$$

Since h_2 is isotopic to the identity of F_2 in E by a one-parameter family of diffeomorphisms determined by the second open book structure, we have that $\Phi'_*(\xi) = [\lambda_1(a) \times \{1\}]$.

Now, we observe that $\lambda_1(a)$ is isotopic to a in $\tilde{E}$ and hence in E. As a consequence, we have $[\lambda_1(a) \times \{1\}] = [a \times \{1\}]$. Thus, we have $\Phi'_*(\xi) = [a \times \{1\}] = \xi$. This completes the proof of Lemma 6.3. □

Since $\Phi' : E \to E$ is the identity on ∂E, there exists a natural extension $\Phi : M \to M$ of Φ' which preserves the orientations. Then the diagram

$$\begin{array}{ccc} H_n(E) & \xrightarrow{\Phi'_*} & H_n(E) \\ {\scriptstyle i_{E*}}\downarrow & & {\scriptstyle i_{E*}}\downarrow \\ H_n(M) & \xrightarrow{\Phi_*} & H_n(M) \end{array}$$

commutes, where $i_E : E \to M$ is the inclusion. Observing that $K = \partial F_1$ is $(n-2)$-connected, we have, by excision and Poincaré-Lefschetz duality,

$$\begin{aligned} H_n(M, E) &\cong H_n(K \times D^2, \partial(K \times D^2)) \cong H^{n+1}(K \times D^2) \\ &\cong H^{n+1}(K) = H_{n-2}(K) = 0. \end{aligned}$$

Then by the homology exact sequence of the pair (M, E), we have

$$\cdots \longrightarrow H_n(E) \xrightarrow{i_{E*}} H_n(M) \longrightarrow 0$$

and i_{E*} is surjective. Thus, by Lemma 6.3 and the commutativity of the above diagram, we see that $\Phi_* : H_n(M) \to H_n(M)$ is the identity. This completes the proof of Lemma 6.1. □

THEOREM 6.4. *Suppose that K is an $(n-2)$-connected closed oriented $(2n-1)$-dimensional manifold embedded in an $(n-1)$-connected closed oriented $(2n+1)$-dimensional manifold M with $n \geq 4, n \neq 7$, or in an $(n-1)$-connected oriented rational homology $(2n+1)$-sphere with $n = 3, 7$. Then all simple and oriented open book structures on M with binding K are isotopic through open books.*

PROOF. Let (K, φ_1) and (K, φ_2) be open book structures in question and denote the Seifert form and the typical page of (K, φ_1) by Γ_1 and F_1 respectively, and the Seifert form and the typical page of (K, φ_2) by Γ_2 and F_2 respectively. We will show that the isomorphism $H_n(F_1) \to H_n(F_2)$ induced by $\Phi|_{F_1} : F_1 \to F_2$ establishes an equivalence between the two systems of open book invariants, where Φ is the diffeomorphism constructed in Lemma 6.1.

By definition of Φ' in the proof of Lemma 6.1, we have that $\Phi'|_{F_1} : F_1 \times \{0\} \to F_2 \times \{0\}$ is an orientation preserving diffeomorphism. Thus $\Phi|_{F_1} : F_1 \to F_2$ is a diffeomorphism, and induces an isomorphism $(\Phi|_{F_1})_* : H_n(F_1) \to H_n(F_2)$ which preserves the tangential invariants and the intersection forms.

Since Φ_* is the identity on $H_n(M)$, the following diagram commutes:

$$\begin{array}{ccc} H_n(F_1) & \xrightarrow{(\Phi|_{F_1})_*} & H_n(F_2) \\ & {}_{i_{F_1 *}}\searrow \quad \swarrow {}_{i_{F_2 *}} & \\ & H_n(M), & \end{array}$$

where $i_{F_j} : F_j \to M$, $j = 1, 2$, are the inclusion maps. Recall that $\Phi \circ \nu_1^+ = \nu_2^+ \circ \Phi|_{F_1}$, where ν_1^+ and ν_2^+ are small translations in the positive normal directions of F_1 and F_2 respectively. Therefore, we have $\mathrm{lk}(\nu_2^+(\Phi(\xi)), \Phi(\zeta)) = \mathrm{lk}(\Phi(\nu_1^+(\xi)), \Phi(\zeta))$ for all n-cycles ξ and ζ in F_1 representing elements of $R(\mathrm{Ker}\, i_{F_1 *})$. If $r\nu_1^+(\xi)$ bounds an $(n+1)$-chain $\tilde{\xi}$ in M, then $r\Phi(\nu_1^+(\xi))$ bounds $\Phi(\tilde{\xi})$, and as a consequence, we have that $\mathrm{lk}(\Phi(\nu_1^+(\xi)), \Phi(\zeta)) = \mathrm{lk}(\nu_1^+(\xi), \zeta)$, since Φ preserves the orientations. Thus, $\Gamma_2((\Phi|_{F_1})_*(\xi), (\Phi|_{F_1})_*(\zeta)) = \Gamma_1(\xi, \zeta)$ for all ξ and ζ in $R(\mathrm{Ker}\, i_{F_1 *})$.

In this manner, $(\Phi|_{F_1})_*$ gives an equivalence between the systems of open book invariants associated with (K, φ_1) and (K, φ_2). Thus, by Theorem 4.1, the two open book structures are structurally isotopic. □

Remark 6.5. By Theorem 6.4, an open book structure on an $(n-1)$-connected closed $(2n+1)$-dimensional manifold M with $n \geq 4, n \neq 7$, or on an $(n-1)$-connected rational homology $(2n+1)$-sphere M with $n = 3, 7$, is determined uniquely by the isotopy class of the oriented embedding of the binding K in M. In particular, the system of open book invariants is an invariant of the embedding of the binding.

Since the embedding of the binding determines the open book structure, eventually we denote the open book (M, K, φ) simply by (M, K), for $n \geq 4, n \neq 7$, or when $n = 3, 7$ and M is a rational homology sphere.

6.2. Fibering criterion

In this subsection, we give necessary and sufficient conditions for a codimension two submanifold to be a binding of some open book structure.

Theorem 6.6. *Let K be an $(n-2)$-connected closed $(2n-1)$-dimensional manifold embedded in an $(n-1)$-connected closed $(2n+1)$-dimensional manifold M with $n \geq 3$. Then we have the following.*

(1) *K is the binding of some open book structure (which is not necessarily simple) on M with simply connected page, if and only if the normal bundle of K in M is trivial (or equivalently, the tubular neighborhood $N(K)$ of K is trivial), $\pi_1(E) \cong \mathbb{Z}$, and $\pi_i(E)$ are finitely generated for all i, where $E = \overline{M - N(K)}$.*

(2) *The above open book is simple, if and only if $\pi_i(E) = 0$ for $i = 2, 3, \ldots, n-1$.*

PROOF. (1) Suppose that K is the binding of an open book structure (which is not necessarily simple) with simply connected typical page F. Then $N(K)$ is trivial by the very definition of an open book structure.

By the homotopy exact sequence

$$\cdots \to \pi_i(F) \to \pi_i(E) \to \pi_i(S^1) \to \cdots$$

associated with the fibration $F \to E \to S^1$, and by the fact that F is simply connected, we have that $\pi_1(E) \cong \mathbb{Z}$.

Now we note that the above sequence gives that $\pi_i(E) \cong \pi_i(F)$ for $i > 1$. Since F is simply connected, Corollary 16 of [Spa66, Chapter 9, Section 6] implies that $\pi_i(F)$ are finitely generated for all i. Thus, $\pi_i(E)$ are finitely generated for all i.

Now suppose conversely that $N(K)$ is trivial, $\pi_1(E) \cong \mathbb{Z}$ and $\pi_i(E)$ are finitely generated for all i. We will prove that there exists an open book structure (not necessarily simple) with binding K and with simply connected page.

Since $N(K)$ is trivial, $\partial E = \partial N(K)$ can be identified with $K \times S^1$ and as a consequence, we have the trivial fibration $\partial E \to S^1$ defined as the projection to the second factor. We will verify that the fibration extends to E, using the fibration theorem of Browder-Levine [BL66]. For this, we recall some concepts.

Let E be a compact manifold with boundary and suppose that there exists an orientable fibration $f : \partial E \to S^1$. We also suppose that $\dim M > 5$, $\pi_1(E) \cong \mathbb{Z}$, and that $\pi_i(E)$ are finitely generated for all $i \geq 2$. Let $v \in H^1(S^1)$ be the generator corresponding to the orientation class of S^1. Then the class $\vartheta(f) = f^*(v) \in H^1(\partial E)$ is defined. On the other hand, the inclusion map $i : \partial E \to E$ induces a restriction map $i^* : H^1(E) \to H^1(\partial E)$. Then, by Browder-Levine [BL66], if there exists a non-vanishing class $\vartheta \in H^1(E)$ such that $i^*\vartheta = \vartheta(f)$, then f extends to a fibration $\tilde{f} : E \to S^1$.

In our case, almost all conditions of the above result are already verified, except the existence of ϑ, where $f : \partial E = K \times S^1 \to S^1$ is the projection to the second factor of $\partial E = K \times S^1$. Since K is simply connected, we have $H^1(K) = 0$. It follows from the Künneth theorem that $H^1(K \times S^1) \cong H^1(S^1)$ which is induced by the trivial inclusion of S^1 in $K \times S^1$. Then $f^* : H^1(S^1) \to H^1(K \times S^1)$ is an isomorphism and $\vartheta(f) = f^*(v)$ is a generator of $H^1(K \times S^1) \cong \mathbb{Z}$. Thus, if $i^* : H^1(E) \to H^1(\partial E)$ induced by the inclusion $i : \partial E \to E$ is surjective, then there exists a non-vanishing element $\vartheta \in H^1(E)$ such that $i^*\vartheta = \vartheta(f)$.

Consider the cohomology exact sequence associated with the pair $(E, \partial E)$:

$$\cdots \longrightarrow H^1(E) \xrightarrow{i^*} H^1(\partial E) \longrightarrow H^2(E, \partial E) \longrightarrow \cdots .$$

Since we have $H^2(E, \partial E) \cong H_{2n-1}(E)$ by Poincaré-Lefschetz duality, the sequence becomes

$$\cdots \longrightarrow H^1(E) \xrightarrow{i^*} H^1(\partial E) \longrightarrow H_{2n-1}(E) \longrightarrow \cdots .$$

Thus, i^* is surjective, if $H_{2n-1}(E) = 0$. Let us consider the Mayer-Vietoris exact sequence associated with $\{E, N(K)\}$:

$$H_{2n}(M) \to H_{2n-1}(\partial E) \to H_{2n-1}(E) \oplus H_{2n-1}(N(K)) \to H_{2n-1}(M).$$

We have $H_{2n}(M) = H_{2n-1}(M) = 0$ by our connectivity condition, and also $H_{2n-1}(N(K)) \cong H_{2n-1}(K)$, since $N(K) \cong K \times D^2$. Thus, the sequence becomes

$$0 \longrightarrow H_{2n-1}(\partial E) \longrightarrow H_{2n-1}(E) \oplus H_{2n-1}(K) \longrightarrow 0.$$

We have $H_{2n-1}(\partial E) \cong H_{2n-1}(K \times S^1) \cong H^1(K \times S^1) \cong H^1(S^1) \cong \mathbb{Z}$, and $H_{2n-1}(K) \cong \mathbb{Z}$, since K is closed and orientable. Therefore, the sequence reduces to

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_{2n-1}(E) \oplus \mathbb{Z} \longrightarrow 0.$$

This implies that $H_{2n-1}(E) = 0$, and as a consequence, i^* is surjective. Thus, there exists a desired cohomology class ϑ and by the result of Browder-Levine mentioned above, the fibration $f : \partial E \to S^1$ extends to a fibration $\tilde{f} : E \to S^1$.

To verify that the page is simply connected, observe that the homotopy exact sequence associated with the fibration $F \to E \to S^1$,

$$\pi_2(S^1) \to \pi_1(F) \to \pi_1(E) \to \pi_1(S^1),$$

reduces to

$$0 \longrightarrow \pi_1(F) \longrightarrow \pi_1(E) \xrightarrow{\tilde{f}_*} \pi_1(S^1). \tag{6.2}$$

Observe also that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(\partial E) \cong \pi_1(K \times S^1) & \xrightarrow{i_*} & \pi_1(E) \\ {\scriptstyle f_*} \searrow & & \swarrow {\scriptstyle \tilde{f}_*} \\ & \pi_1(S^1). & \end{array}$$

Since f is the projection to the second factor, f_* is surjective, and consequently $\tilde{f}_*$ is also surjective. On the other hand, we have $\pi_1(E) \cong \mathbb{Z}$. Then $\tilde{f}_*$ is an isomorphism, and this implies that $\pi_1(F) = \{1\}$ by the exact sequence (6.2).

In order to verify that F is connected, observe that the homotopy exact sequence of the fibration $F \to E \to S^1$ in the zero level is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\tilde{f}_*} \mathbb{Z} \longrightarrow \pi_0(F) \longrightarrow 0,$$

since E is connected. Since $\tilde{f}_*$ is an isomorphism, we have that $\pi_0(F) = 0$, which implies that F is connected. This completes the proof of part (1).

(2) Consider the homotopy exact sequence

$$\cdots \to \pi_{i+1}(S^1) \to \pi_i(F) \to \pi_i(E) \to \cdots$$

associated with the fibration $F \to E \to S^1$. This implies that $\pi_i(F) \cong \pi_i(E)$ for $i = 2, \ldots, n-1$. Since F is simply connected, it is $(n-1)$-connected, if and only if $\pi_i(E) = 0$ for $i = 2, \ldots, n-1$. Since K is $(n-2)$-connected, we have the desired result. This completes the proof of Theorem 6.6. □

As a consequence of Theorems 5.15, 6.4 and 6.6, we have the following classification theorem of certain codimension two embeddings up to isotopy.

Corollary 6.7. *Suppose that M is an $(n-1)$-connected closed oriented $(2n+1)$-dimensional manifold with $n \geq 4, n \neq 7$, or an $(n-1)$-connected oriented rational homology $(2n+1)$-sphere with $n = 3, 7$. Then there exists a one-to-one correspondence between the set $\mathcal{A}(M)$ of all equivalence classes of systems of open book invariants with respect to M and the set of all isotopy classes of oriented submanifolds of codimension two which satisfy the conditions of Theorem* 6.6.

Remark 6.8. Levine [Lev70] has classified codimension two embeddings of homotopy $(2n-1)$-spheres into S^{2n+1}, up to isotopy, whose complements have the homotopy type of S^1 up to the dimension $n-1$. Such embeddings are called *simple knots*. In his classification, Seifert matrices of such simple knots have played an essential role. We do not know if the above corollary can be generalized to include codimension two submanifolds whose complements have the homotopy type of S^1 up to the dimension $n-1$, but may not necessarily fiber over the circle.

7. Decomposition of Open Books

In this section, we use our classification theorem of open book structures (Theorem 5.15) to study the decomposition of open books with respect to "connected sum". We also introduce the notion of a minimal open book structure for a given ambient manifold and prove its existence.

7.1. Connected sum of open books

DEFINITION 7.1. Let K_i be the oriented binding of an oriented open book structure (which may not necessarily be simple) on an oriented $(2n+1)$-dimensional manifold M_i, $i = 1, 2$. We assume that K_i are connected. Let us take a sufficiently small tubular neighborhood $D_i \cong D^{2n+1}$ in M_i of a point of K_i. We may assume that $(D_i, D_i \cap K_i)$ is diffeomorphic to the standard disk pair (D^{2n+1}, D^{2n-1}) and that the fibration $D_i - (D_i \cap K_i) \to S^1$ induced from the open book structure corresponds to the canonical fibration, which is trivial. We take the connected sum $M_1 \sharp M_2 = (M_1 - \operatorname{Int} D_1) \cup (M_2 - \operatorname{Int} D_2)$ obtained by a natural identification between ∂D_1 and ∂D_2 such that it reverses their orientations, that $\partial D_1 \cap K_1$ corresponds to $\partial D_2 \cap K_2$ orientation reversingly, and that the identification respects the fibrations on the boundaries of the disks. In this process, we can glue $K_1 - (\operatorname{Int} D_1 \cap K_1)$ and $K_2 - (\operatorname{Int} D_2 \cap K_2)$ along their boundaries to obtain the connected sum $K_1 \sharp K_2$.

Then the open book structures on M_1 and M_2 naturally induce an open book structure on the manifold $M_1 \sharp M_2$, with binding $K_1 \sharp K_2$ and page $F_1 \natural F_2$, where $F_1 \natural F_2$ denotes the boundary connected sum of F_1 and F_2. This new open book is called a *book connected sum*, or simply a *connected sum*, if there is no confusion, and is denoted by $(M_1, K_1) \sharp_b (M_2, K_2)$, or simply by $M_1 \sharp_b M_2$. Note that the resulting open book is oriented and that it does not depend on the choice of the points on K_i, since K_i are connected, $i = 1, 2$.

Note that if (M_i, K_i) are simple, then so is $(M_1, K_1)\sharp_b(M_2, K_2)$.

In what follows, we always assume that open books used in book connected sums are simple.

For $n \geq 3$, let $\mathcal{A}_{2n+1}$ be the union of all $\mathcal{A}(M)$, where M runs through all $(n-1)$-connected closed oriented $(2n+1)$-dimensional manifolds. Then we define the sum in $\mathcal{A}_{2n+1}$ as follows.

DEFINITION 7.2. Take $s_1, s_2 \in \mathcal{A}_{2n+1}$. Suppose that $s_1 \in \mathcal{A}(M_1)$ and $s_2 \in \mathcal{A}(M_2)$ for $(n-1)$-connected closed oriented $(2n+1)$-dimensional manifolds M_1 and M_2. Let $\{G_1, Q_{G_1}, \alpha_{G_1}, i_{G_1}, \Gamma_{G_1}\}$ and $\{G_2, Q_{G_2}, \alpha_{G_2}, i_{G_2}, \Gamma_{G_2}\}$ be representatives of s_1 and s_2 respectively. Then we define the sum as

$$s_1 + s_2 = [\{G_1 \oplus G_2, Q_{G_1} \oplus Q_{G_2}, \alpha_{G_1} + \alpha_{G_2}, i_{G_1} \oplus i_{G_2}, \Gamma_{G_1} \oplus \Gamma_{G_2}\}],$$

where $[*]$ denotes the equivalence class of $*$, $G_1 \oplus G_2$, $Q_{G_1} \oplus Q_{G_2}$ are direct sums, $\Gamma_{G_1} \oplus \Gamma_{G_2}$ is the direct sum with respect to $R(\mathrm{Ker}(i_{G_1} \oplus i_{G_2})) = R(\mathrm{Ker}\, i_{G_1}) \oplus R(\mathrm{Ker}\, i_{G_2})$, $i_{G_1} \oplus i_{G_2} : G_1 \oplus G_2 \to H_n(M_1) \oplus H_n(M_2)$ is defined naturally, and $\alpha_{G_1} + \alpha_{G_2} : G_1 \oplus G_2 \to \pi_{n-1}(SO(n))$ is defined by $(\alpha_{G_1} + \alpha_{G_2})(\xi \oplus \zeta) = \alpha_{G_1}(\xi) + \alpha_{G_2}(\zeta)$ for all $\xi \in G_1$ and $\zeta \in G_2$.

We observe that the operation does not depend on the choice of the representatives and is well-defined. Furthermore, we can easily prove the following.

PROPOSITION 7.3. *Let M_1 and M_2 be $(n-1)$-connected closed oriented $(2n+1)$-dimensional manifolds with $n \geq 3$. If (M_1, K_1) and (M_2, K_2) are open book structures on M_1 and M_2 respectively, then we have that*

$$\mathcal{S}((M_1, K_1)\sharp_b(M_2, K_2)) = \mathcal{S}(M_1, K_1) + \mathcal{S}(M_1, K_1),$$

where $\mathcal{S}$ denotes the equivalence class of the system of open book invariants associated with an open book.

Note that the tangential invariant is not a homomorphism in general; instead, we can use the addition rule mentioned in Remark 3.2. The proof of the above proposition is then straightforward (for details, see [Mas00]).

7.2. Trivial open books

In this subsection, we study trivial open books (see Definition 2.8). This is necessary in the study of decomposition of open books, since a book connected sum with a trivial open book hardly produces a new open book, as we will see later in this section.

The following lemma can be proved by using the generalized Poincaré conjecture (see [Sma61] and [Mil65]). The details are left to the reader.

LEMMA 7.4. *If an open book (M, K, φ) is trivial with $n \geq 3$, then*

(1) *M is a homotopy $(2n+1)$-sphere, and*

(2) *the typical page F is diffeomorphic to D^{2n} and the binding $K = \partial F$ is diffeomorphic to S^{2n-1}.*

The following is an immediate consequence of Theorem 5.15.

PROPOSITION 7.5. *Let M be a homotopy $(2n+1)$-sphere with $n \geq 3$. Then there exists a unique trivial open book structure on M up to structural isotopy.*

REMARK 7.6. Consider a (simple) open book structure (M, K) on an $(n-1)$-connected closed $(2n+1)$-dimensional manifold M with $n \geq 3$. Suppose that (S^{2n+1}, K_0) is a trivial open book. Since the trivial open book is associated with the trivial embedding of S^{2n-1}, the book connected sum with (S^{2n+1}, K_0) does not change the open book structure. Thus $(M, K)\sharp_b(S^{2n+1}, K_0)$ is always structurally isotopic to (M, K) under the natural identification $M \sharp S^{2n+1} = M$.

On the other hand, if (Σ, K_Σ) is a trivial open book, where Σ is a homotopy $(2n+1)$-sphere which is not diffeomorphic to S^{2n+1}, then $(M, K)\sharp_b(\Sigma, K_\Sigma)$ may be different from (M, K), since M may not be diffeomorphic to $M \sharp \Sigma$.

REMARK 7.7. In the above remark, in order to get the same conclusion for (Σ, K_Σ) as well, we need some additional conditions as follows.

Consider a (simple) open book structure (M, K) on an $(n-1)$-connected closed $(2n+1)$-dimensional manifold M with $n \geq 4, n \neq 7$, or on an $(n-1)$-connected rational homology $(2n+1)$-sphere M with $n = 3, 7$. Suppose that (Σ, K_Σ) is a trivial open book, where Σ is a homotopy $(2n+1)$-sphere which may not necessarily be diffeomorphic to S^{2n+1} (see Definition 2.8 and Proposition 7.5). Then we have that the system of open book invariants associated with (M, K) corresponds to that of $(M, K)\sharp_b(\Sigma, K_\Sigma)$: however, we have to be careful, since $\mathcal{S}(M, K) \in \mathcal{A}(M)$, while $\mathcal{S}((M, K)\sharp_b(\Sigma, K_\Sigma)) \in \mathcal{A}(M\sharp\Sigma)$.

Thus, if $M \cong M\sharp\Sigma$ by a diffeomorphism which preserves the orientations and which induces the "identity" on homology, then (M, K) and $(M, K)\sharp_b(\Sigma, K_\Sigma)$ are "structurally isotopic" by Theorem 4.1 if we identify M and $M\sharp\Sigma$ by the above diffeomorphism.

7.3. Decomposition of open books

Definition 7.8. An open book (M, K) is said to be *decomposable in the weak sense*, if M is orientation preservingly diffeomorphic to $M_1\sharp M_2$ and if (M, K) is structurally isotopic to $(M_1, K_1)\sharp_b(M_2, K_2)$ under the above diffeomorphism, for some non-trivial open books (M_i, K_i), $i = 1, 2$.

The following lemma is a consequence of the above definition together with Proposition 7.3.

Lemma 7.9. *Let (M, K) be an open book structure on an $(n-1)$-connected closed $(2n+1)$-dimensional manifold M with $n \geq 3$. If (M, K) is decomposable in the weak sense, then the system of open book invariants associated with (M, K) decomposes as*

$$\mathcal{S}(M, K) = [\{G_1, Q_{G_1}, \alpha_{G_1}, i_{G_1}, \Gamma_{G_1}\}] + [\{G_2, Q_{G_2}, \alpha_{G_2}, i_{G_2}, \Gamma_{G_2}\}]$$

for some $[\{G_1, Q_{G_1}, \alpha_{G_1}, i_{G_1}, \Gamma_{G_1}\}]$ and $[\{G_2, Q_{G_2}, \alpha_{G_2}, i_{G_2}, \Gamma_{G_2}\}] \in \mathcal{A}_{2n+1}$ such that $G_1 \neq 0$, $G_2 \neq 0$ and $H_n(M) = \operatorname{Im} i_{G_1} \oplus \operatorname{Im} i_{G_2}$.

The following proposition is a consequence of Theorem 5.15.

Proposition 7.10. *Suppose that (M, K) is an open book and $M = M_1\sharp M_2$, where M, M_1 and M_2 are $(n-1)$-connected closed $(2n+1)$-dimensional manifolds, with $n \geq 4, n \neq 7$, or M, M_1 and M_2 are $(n-1)$-connected rational homology $(2n+1)$-spheres with $n = 3, 7$. Then (M, K) is structurally isotopic to the book connected sum $(M_1, K_1)\sharp_b(M_2, K_2)$ of some open* *books* (M_1, K_1) *and* (M_2, K_2)*, if and only if the system of open book invariants associated with* (M, K) *decomposes as a direct sum with respect to the decomposition* $H_n(M) = H_n(M_1) \oplus H_n(M_2)$ *as in Definition* 7.2*.*

In the above proposition, we have supposed that $M = M_1 \sharp M_2$. However, if $\alpha_M = 0$, then we have a similar result without assuming this hypothesis as follows.

PROPOSITION 7.11. *Let* (M, K) *be a simple open book structure on an* $(n-1)$*-connected closed* $(2n+1)$*-dimensional manifold* M *such that* $\alpha_M = 0$ *and* $n \geq 4, n \neq 7$*. If the system of open book invariants associated with* (M, K) *decomposes as*

$$\mathcal{S}(M, K) = [\{G_1, Q_{G_1}, \alpha_{G_1}, i_{G_1}, \Gamma_{G_1}\}] + [\{G_2, Q_{G_2}, \alpha_{G_2}, i_{G_2}, \Gamma_{G_2}\}]$$

for some $[\{G_1, Q_{G_1}, \alpha_{G_1}, i_{G_1}, \Gamma_{G_1}\}]$ *and* $[\{G_2, Q_{G_2}, \alpha_{G_2}, i_{G_2}, \Gamma_{G_2}\}] \in \mathcal{A}_{2n+1}$ *such that* $G_1 \neq 0$*,* $G_2 \neq 0$ *and* $H_n(M) = H_1 \oplus H_2$*, where* $H_1 = \operatorname{Im} i_{G_1}$ *and* $H_2 = \operatorname{Im} i_{G_2}$*, then there exist non-trivial simple open books* (M_1, K_1) *and* (M_2, K_2) *such that* M *is diffeomorphic to* $M_1 \sharp M_2$ *and that* (M, K) *is structurally isotopic to* $(M_1, K_1) \sharp_b (M_2, K_2)$*. In other words,* (M, K) *is decomposable in the weak sense.*

PROOF. By item (5c) of Definition 3.15, the decomposition $H_n(M) = H_1 \oplus H_2$ is orthogonal with respect to $b_M : \tau H_n(M) \times \tau H_n(M) \to \mathbb{Q}/\mathbb{Z}$. Thus, by [Wal67, Corollary, p. 285], there exists a decomposition $M - \operatorname{Int} D^{2n+1} = M'_1 \natural M'_2$ such that $H_n(M'_i) = H_i$, $i = 1, 2$. Since $\alpha_M = 0$ by our hypothesis, we have that $\alpha_{M'_1} = \alpha_{M'_2} = 0$. Then, by [Wal67, Theorem 8, p. 285], $\partial M'_1 \cong S^{2n} \cong \partial M'_2$. Therefore, we can consider $M_1 = M'_1 \cup D^{2n+1}$ and $M_2 = M'_2 \cup D^{2n+1}$, where the union is taken by identifying the boundaries and we have that $M = M_1 \sharp M_2$.

Then, by our hypothesis and by Proposition 7.10, (M, K) is structurally isotopic to the book connected sum of (M_1, K_1) and (M_2, K_2) for some open books (M_1, K_1) and (M_2, K_2). □

REMARK 7.12. (1) If M is a homotopy sphere, or more generally, if M is stably parallelizable, then $\alpha_M = 0$.

(2) The condition $\alpha_M = 0$ in the above proposition is used to guarantee that $\alpha_{M'_1} = \alpha_{M'_2} = 0$, which implies that $\partial M'_1$ and $\partial M'_2$ are diffeomorphic

to S^{2n}. If we have $\alpha_M \neq 0$, then at least one of $\alpha_{M_1'}$ or $\alpha_{M_2'}$ is non-trivial, which implies that $\partial M_1'$ or $\partial M_2'$ may be an exotic sphere.

(3) In Proposition 7.11, the decomposition $(M_1, K_1)\sharp_b(M_2, K_2)$ is not unique in general. Note that if $M = M_1\sharp M_2$, then we have that $M = (M_1\sharp\Sigma)\sharp(M_2\sharp(-\Sigma))$ for any homotopy sphere Σ of dimension $2n+1$.

So far, we have been interested in the decompositions of open books in the weak sense as defined in Definition 7.8. Since the ambient manifold M may not be the sphere, we have another definition of decomposability for open book structures as follows.

Definition 7.13. An open book structure (M, K) on a manifold M is said to be *decomposable in the strong sense*, if it is structurally isotopic to the connected sum $(M, K_1)\sharp(S^{2n+1}, K_2)$ for some non-trivial open books (M, K_1) and (S^{2n+1}, K_2).

Note that if an open book (M, K) is decomposable in the strong sense, then it is also decomposable in the weak sense.

Definition 7.14. When an open book is not decomposable in the strong (resp. weak) sense, we say that it is *indecomposable in the strong* (resp. *weak*) *sense.*

7.4. Minimal open book structures

In order to give examples of open book structures which are indecomposable in the strong sense, we introduce the following notion.

Definition 7.15. Let F be the typical page of a simple open book structure (K, φ) on a $(2n+1)$-dimensional manifold M. If $\operatorname{rank} H_n(F)$ coincides with the minimum number of generators of $H_n(M)$ over $\mathbb{Z}$, then we say that (K, φ) is *minimal*.

We have the following existence theorem of minimal open book structures as a consequence of our realization theorem of systems of open book invariants.

Theorem 7.16. *If M is an $(n-1)$-connected closed $(2n+1)$-dimensional manifold with $n \geq 3$ such that $H_n(M)$ is torsion free, i.e. $\tau H_n(M) = 0$, then there exists a simple and minimal open book structure on M.*

PROOF. By Theorem 5.14, it suffices to construct a system of open book invariants with respect to M with minimal "rank".

Since $H_n(M)$ is free, we can take $G = H_n(M)$ and $i_G = \mathrm{id}$. Note that $R(\operatorname{Ker} i_G) = 0$, which implies that the Seifert form vanishes. Thus, we have only to construct the tangential invariant α_G and the intersection form Q_G.

In order to construct these invariants, recall that α_M is a homomorphism (see Remark 3.3), while α_G may not.

Let $\{e_1, \dots, e_r\}$ be a basis of $G = H_n(M)$ and choose values for $\alpha_G(e_j) \in \pi_{n-1}(SO(n))$ such that $i_*\alpha_G(e_j) = \alpha_M(e_j)$, $j = 1, \dots, r$. This is possible, since $i_* : \pi_{n-1}(SO(n)) \to \pi_{n-1}(SO(n+1))$ is an epimorphism by the homotopy exact sequence (2.1) together with the fact that $\pi_{n-1}(S^n) = 0$.

When n is odd, $\alpha_G(e_j) \in \pi_{n-1}(SO(n)) \cong 0$, $\mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (see [Wal65]) and $p_*\alpha_G(e_j) = 0 \in \pi_{n-1}(S^{n-1}) \cong \mathbb{Z}$, where

$$p_* : \pi_{n-1}(SO(n)) \to \pi_{n-1}(S^{n-1})$$

is the homomorphism of Remark 2.11 which vanishes for n odd. When n is even, we have $p_*\alpha_G(e_j) \in \pi_{n-1}(S^{n-1}) \cong \mathbb{Z}$. Then we define the intersection form by

$$Q_G(e_j, e_k) = \begin{cases} p_*\alpha_G(e_j) \in \mathbb{Z} & (j = k), \\ 0 & (j \neq k). \end{cases}$$

Observe that it is $(-1)^n$-symmetric.

Now, define the values of $\alpha_G : G \to \pi_{n-1}(SO(n))$ by

$$\alpha_G\left(\sum_{j=1}^{r} k_j e_j\right) = \sum_{j=1}^{r} \left(k_j \alpha_G(e_j) + \frac{k_j(k_j - 1)}{2} Q_G(e_j, e_j) \partial t_n\right) \in \pi_{n-1}(SO(n)),$$

where $r = \operatorname{rank} G$, t_n is the generator of $\pi_n(S^n)$ represented by the identity map $S^n \to S^n$ (see Remark 3.2) and ∂ is the boundary homomorphism of the exact sequence (2.1) (see Lemma 2.9).

In order to show that $\{G, Q_G, \alpha_G, i_G, \Gamma_G\}$ constitutes a system of open book invariants with respect to M, it is enough to prove the following.

LEMMA 7.17. *The map α_G satisfies the conditions of items* (4a), (4b) *and* (4c) *of Definition* 3.15.

PROOF. Let us first check that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\alpha_G} & \pi_{n-1}(SO(n)) \\ || & & \Big\downarrow{\scriptstyle i_*} \\ H_n(M) & \xrightarrow{\alpha_M} & \pi_{n-1}(SO(n+1)). \end{array}$$

We have that

$$i_*\alpha_G\left(\sum_{j=1}^{r} k_j e_j\right) = \alpha_M\left(\sum_{j=1}^{r} k_j e_j\right),$$

which follows from our definition of α_G together with the fact that $i_*\partial t_n = 0$, definition of $\alpha_G(e_j)$, and the fact that α_M is a homomorphism. Thus, we have $i_* \circ \alpha_G = \alpha_M$ on $H_n(M)$.

For n even, we have

$$p_*\alpha_G\left(\sum_{j=1}^{r} k_j e_j\right) = Q_G\left(\sum_{j=1}^{r} k_j e_j, \sum_{j=1}^{r} k_j e_j\right),$$

which follows from the fact that $p_*\partial t_n = 2$ [Ste51, §23.4] (see also Remark 2.11). For n odd, since both $p_* \circ \alpha_G$ and Q_G vanish, we have that

$$p_*\alpha_G\left(\sum_{j=1}^{r} k_j e_j\right) = 0 = Q_G\left(\sum_{j=1}^{r} k_j e_j, \sum_{j=1}^{r} k_j e_j\right).$$

Finally, let us verify that $\alpha_G(\xi + \zeta) = \alpha_G(\xi) + \alpha_G(\zeta) + Q_G(\xi, \zeta)\partial t_n$ for all $\xi, \zeta \in G$. We have that

$$\begin{aligned} (7.1) \qquad & \alpha_G\left(\sum_{j=1}^{r} k_j e_j + \sum_{j=1}^{r} l_j e_j\right) \\ &= \sum_{j=1}^{r}\left((k_j + l_j)\alpha_G(e_j) + \frac{(k_j + l_j)(k_j + l_j - 1)}{2} Q_G(e_j, e_j)\partial t_n\right) \end{aligned}$$

by our definition of α_G. On the other hand, we have

$$\begin{aligned}
&\alpha_G\left(\sum_{j=1}^r k_j e_j\right) + \alpha_G\left(\sum_{j=1}^r l_j e_j\right) + Q_G\left(\sum_{j=1}^r k_j e_j, \sum_{j=1}^r l_j e_j\right)\partial t_n \\
=&\sum_{j=1}^r\left(k_j\alpha_G(e_j) + \frac{k_j(k_j-1)}{2}Q_G(e_j,e_j)\partial t_n\right) \\
&+\sum_{j=1}^r\left(l_j\alpha_G(e_j) + \frac{l_j(l_j-1)}{2}Q_G(e_j,e_j)\partial t_n\right) \\
&+\left(\sum_{j=1}^r k_j l_j Q_G(e_j,e_j)\right)\partial t_n,
\end{aligned}$$

which coincides with the right hand side of (7.1). This completes the proof of Lemma 7.17. □

Thus, we obtain a system of open book invariants with respect to M. By the realization theorem (Theorem 5.14), we see that there exists an open book structure on M which realizes the above system of open book invariants. This completes the proof of Theorem 7.16. □

We do not know if the condition $\tau H_n(M) = 0$ is essential in Theorem 7.16. Our conjecture is the following.

CONJECTURE 7.18. *If M is an $(n-1)$-connected closed $(2n+1)$-dimensional manifold with $n \geq 3$, then there exists a minimal open book structure on M.*

REMARK 7.19. Note that the unique minimal open book structure on S^{n+1} with $n \geq 3$ is the trivial one. However, for general manifolds, we do not have the uniqueness of minimal open book structures, as shown by the following example.

Example 7.20. For $n \geq 4, n \neq 7$, set $M = S^n \times S^{n+1}$. Since $H_n(M) \cong \mathbb{Z}$ is torsion free, the Seifert form of a minimal open book structure on M should be trivial, and as a consequence, a minimal open book structure on M is uniquely determined by the intersection form and the tangential invariant. Note that the tangential invariant α_G is compatible with the tangential invariant α_M of M, if and only if $\operatorname{Im}\alpha_G \subset \operatorname{Ker} i_* = \operatorname{Im}\partial$, since $\alpha_M = 0$, where $i_* : \pi_{n-1}(SO(n)) \to \pi_{n-1}(SO(n+1))$ is induced by the inclusion and $\partial : \pi_n(S^n) \to \pi_{n-1}(SO(n))$ is the boundary homomorphism of (2.1).

For n even, $\operatorname{Im}\partial \cong \mathbb{Z}$ (see Lemma 2.9) and we can choose α_G corresponding to each integer. Since α_G uniquely determines the intersection form, we have that the set of structural isotopy classes of minimal open book structures on $M = S^n \times S^{n+1}$ is in one-to-one correspondence with the set of integers, due to our classification theorem (Theorem 5.15).

For n odd, $\operatorname{Im}\partial \cong \mathbb{Z}_2$ by Lemma 2.9. Thus there exist exactly two choices for α_G, and the intersection form always vanishes. Therefore, we have exactly two structural isotopy classes of minimal open book structures on $M = S^n \times S^{n+1}$.

REMARK 7.21. Note that a minimal open book structure is always indecomposable in the strong sense. In fact, if there exists a decomposition $(M, K) = (M, K_1)\sharp_b(S^{2n+1}, K_2)$ with $H_n(F_1) \neq 0$ and $H_n(F_2) \neq 0$, where F_1 is the typical page of (M, K_1) and F_2 is the typical page of (S^{2n+1}, K_2), then the open book (M, K_1) satisfies $\operatorname{rank} H_n(F_1) < \operatorname{rank} H_n(F)$, where $F = F_1 \natural F_2$ is the typical page of (M, K). Consequently, (M, K) is not minimal.

Thus, Theorem 7.16 implies that there exists at least one indecomposable open book structure in the strong sense, on any $(n-1)$-connected closed $(2n+1)$-dimensional manifold M such that $\tau H_n(M) = 0$, when $n \geq 3$.

However, a minimal open book is not always indecomposable in the weak sense. For example, consider the book connected sum of two non-trivial minimal open book structures on manifolds with torsion free homologies. Then it is minimal, but is decomposable in the weak sense.

The above remark and Conjecture 7.18 suggest the following.

CONJECTURE 7.22. *Let M be an $(n-1)$-connected closed $(2n+1)$-dimensional manifold with $n \geq 3$. Then there exists an indecomposable open book structure in the strong sense on M.*

An indecomposable open book structure in the strong sense may not necessarily be minimal, as shown by the following example.

Example 7.23. Let us consider $M = S^n \times S^{n+1}$ with n odd, $n \geq 3$, and $G = \mathbb{Z} \oplus \mathbb{Z}$. Then for any epimorphism $i_G : G \to H_n(M)$, we have $R(\operatorname{Ker} i_G) \cong \mathbb{Z}$, and the Seifert matrix is given by $\Gamma_G = \pm 1$ (see item (5a) of Definition 3.15). In this case, the Seifert form determines only the self-intersection of the generator, which is zero, since n is odd. Let us define the intersection form so that it is non-zero. Now, we define the tangential invariant α_G. Let $\{e_1, e_2\}$ be a basis of G such that e_2 is a generator of $R(\operatorname{Ker} i_G)$. Define $\alpha_G : G \to \pi_{n-1}(SO(n))$ by

$$\alpha_G(ke_1 + le_2) = (k + l + klQ_G(e_1, e_2))\partial t_n \in \pi_{n-1}(SO(n)).$$

Since $\alpha_{S^n \times S^{n+1}} = 0$, the condition of item (4a) of Definition 3.15 is satisfied.

For the condition of item (4b) of Definition 3.15, note that

$$p_*\alpha_G(ke_1 + le_2) = (k + l + klQ_G(e_1, e_2))p_*\partial t_n = 0,$$

since n is odd. Furthermore, we have $Q_G(ke_1 + le_2, ke_1 + le_2) = 0$, since n is odd. This verifies the condition.

Now we verify the condition of item (4c) of Definition 3.15. We have that

$$\begin{aligned} (7.2) \qquad & \alpha_G\left((ke_1 + le_2) + (k'e_1 + l'e_2)\right) = \alpha_G\left((k + k')e_1 + (l + l')e_2\right) \\ & = \left(k + k' + l + l' + (k + k')(l + l')Q_G(e_1, e_2)\right)\partial t_n. \end{aligned}$$

On the other hand, we have that

$$\begin{aligned} & \alpha_G(ke_1 + le_2) + \alpha_G(k'e_1 + l'e_2) + Q_G(ke_1 + le_2, k'e_1 + l'e_2)\partial t_n \\ = & (k + l + klQ_G(e_1, e_2))\partial t_n + (k' + l' + k'l'Q_G(e_1, e_2))\partial t_n \\ & + (kl' - lk')Q_G(e_1, e_2)\partial t_n, \end{aligned}$$

which coincides with the right hand side of (7.2), since ∂t_n is of order two.

Finally, as to item (5d) of Definition 3.15, we have that $\Gamma_G(le_2, le_2) = l^2\Gamma_G(e_2, e_2) = \pm l^2$ and

$$\begin{aligned} q_M(i_G(le_2)) + \phi(\alpha_G(le_2)) &= q_M(0) + \phi(l\partial t_n) \\ &= l\phi(\partial t_n) = l \pmod 2, \end{aligned}$$

which verifies the condition.

Thus, there exists an open book structure on M associated with the system of open book invariants constructed above, by our Theorem 5.14. This open book structure is not minimal and is indecomposable in the strong sense, or even in the weak sense. In fact, if it were decomposable in the weak sense, then the intersection form would be in a diagonal form, and since n is odd, it should vanish, which contradicts the construction. Thus, we have constructed an indecomposable open book structure in the strong sense which is not minimal.

In the case that M is the sphere S^{2n+1}, it is easy to see that there exists an indecomposable open book structure on $M = S^{2n+1}$ which is not minimal, due to the existence of Seifert matrices which do not decompose as a non-trivial direct sum and the classification theorem for simple fibered knots in the spheres [Lev70, Dur74, Kat74].

Minimal open book structures on decomposable manifolds may be indecomposable in the weak sense as the following example shows.

Example 7.24. Set $M = (S^n \times S^{n+1})\sharp(S^n \times S^{n+1})$, $n \geq 3$. Since $H_n(M) \cong \mathbb{Z} \oplus \mathbb{Z}$, the Seifert form of a minimal open book structure must be trivial, and hence, a minimal open book structure is uniquely determined by the intersection form and the tangential invariant. Let $\{e_1, e_2\}$ be a basis of $G = H_n(M)$. Now we define the intersection form Q_G so that the matrix in the basis $\{e_1, e_2\}$ is given by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ when n is odd, and by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ when n is even.

Now we define α_G by $\alpha_G(ke_1 + le_2) = klQ_G(e_1, e_2)\partial t_n = kl\partial t_n$. Then we can verify that the intersection form and the tangential invariant satisfy the conditions of Definition 3.15, using an argument similar to that in the above examples. Therefore, there exists an open book structure associated with the system of open book invariants constructed above by our realization theorem (Theorem 5.14). This open book structure is clearly minimal. Furthermore, it is indecomposable in the strong sense, or even in the weak sense, since the intersection form does not decompose as a direct sum.

Using the existence of minimal open book structures, we can show that there exists an injection from the set of structural isotopy classes of open book structures on the sphere S^{2n+1} to that on a given $(n-1)$-connected closed $(2n+1)$-dimensional manifold M with $n \geq 3$ such that $H_n(M)$ is torsion free. For this, we first show the following cancellation lemma for minimal open books.

LEMMA 7.25. *Let (M, K) be a minimal open book structure on an $(n-1)$-connected closed $(2n+1)$-dimensional manifold M with $n \geq 3$ such that $\tau H_n(M) = 0$ and let N be an $(n-1)$-connected rational homology $(2n+1)$-sphere. Suppose that (N, K_1) and (N, K_2) are two open books such that $(M, K)\sharp_b(N, K_1)$ and $(M, K)\sharp_b(N, K_2)$ are structurally isotopic to each other. Then (N, K_1) and (N, K_2) are structurally isotopic.*

PROOF. Let F be the typical page of (M, K), and F_1 and F_2 the typical pages of (N, K_1) and (N, K_2) respectively. Then the typical page of $(M, K)\sharp_b(N, K_j)$ is the boundary connected sum $F \natural F_j$, $j = 1, 2$, and we have the natural decomposition $H_n(F \natural F_j) = H_n(F) \oplus H_n(F_j)$. For this decomposition, we have that $R(\operatorname{Ker} i_{F\natural F_j *}) = H_n(F_j)$, $j = 1, 2$, since F is the typical page of a minimal open book structure, $H_n(M)$ is torsion free, and $H_n(N) \subset H_n(M \sharp N)$ is exactly the torsion part, where $i_{F\natural F_j} : F \natural F_j \to M$, $j = 1, 2$, are the inclusion maps.

By our assumption, there exists a structural isotopy $\Phi = \{\Phi_t\}_{t\in[0,1]}$ between $(M, K)\sharp_b(N, K_1)$ and $(M, K)\sharp_b(N, K_2)$. Then $\Phi_{1*} : H_n(F \natural F_1) \to H_n(F \natural F_2)$ is an isomorphism which establishes an equivalence between the systems of open book invariants of $(M, K)\sharp_b(N, K_1)$ and $(M, K)\sharp_b(N, K_2)$ (see Remark 3.22). Since Φ satisfies the commutative diagram

$$\begin{array}{ccc} H_n(F\natural F_1) & \xrightarrow{\Phi_{1*}} & H_n(F\natural F_2) \\ & {\scriptstyle i_{F\natural F_1 *}} \searrow \quad \swarrow {\scriptstyle i_{F\natural F_2 *}} & \\ & H_n(M), & \end{array}$$

Φ_{1*} maps $\operatorname{Ker} i_{F\natural F_1 *}$ to $\operatorname{Ker} i_{F\natural F_2 *}$ and induces an isomorphism $\bar{\Phi} : H_n(F_1) \to H_n(F_2)$.

Since Φ_{1*} preserves the systems of open book invariants and $\bar{\Phi}$ is its restriction, it also preserves the systems of open book invariants, giving an equivalence between the systems of open book invariants of (N, K_1) and (N, K_2). Thus (N, K_1) is structurally isotopic to (N, K_2) by Theorem 4.1. □

By the above lemma, we immediately have the following two results.

Proposition 7.26. *Let (M, K) be a minimal open book structure on an $(n-1)$-connected closed $(2n+1)$-dimensional manifold M with $n \geq 3$ such that $\tau H_n(M) = 0$. Suppose that (S^{2n+1}, K_1) and (S^{2n+1}, K_2) are two open book structures on the sphere such that $(M, K)\sharp_b(S^{2n+1}, K_1)$ and $(M, K)\sharp_b(S^{2n+1}, K_2)$ are structurally isotopic to each other. Then (S^{2n+1}, K_1) is structurally isotopic to (S^{2n+1}, K_2).*

Corollary 7.27. *Let M be an $(n-1)$-connected closed $(2n+1)$-dimensional manifold such that $H_n(M)$ is torsion free with $n \geq 3$. Then we have an injective map of the set of all structural isotopy classes of open book structures on S^{2n+1}, to the set of all structural isotopy classes of open book structures on M, defined by sending (S^{2n+1}, K') to $(M, K)\sharp_b(S^{2n+1}, K')$, where (M, K) is a minimal open book structure on M, whose existence is guaranteed by Theorem* 7.16.

When $H_n(M)$ is not torsion free, we do not know if the above results are valid or not. Our conjecture is the following.

Conjecture 7.28. *Let M be an $(n-1)$-connected closed $(2n+1)$-dimensional manifold with $n \geq 3$. Then we have an injective map from the set of all structural isotopy classes of open book structures on S^{2n+1}, to the set of all structural isotopy classes of open book structures on M, defined by sending (S^{2n+1}, K') to $(M, K)\sharp_b(S^{2n+1}, K')$, for some open book structure (M, K) on M.*

8. Variation Map and Its Application

In this section, we introduce the notion of a variation map, which is a homomorphism induced in the homology level, for a self-diffeomorphism of a manifold with boundary. Applying this to the monodromy of an open book, we obtain the variation map associated with an open book. When the ambient manifold is the sphere, the variation map is always an isomorphism, and it has been known that giving the Seifert form is equivalent to giving the variation map, so that the variation map is a very important invariant in this case (see, for example, [Kau74]). We will see that this is also the case in the general (possibly non-spherical) case as well. Furthermore, as an application of the variation map, together with our classification theorem of open book structures, we study isotopy of certain diffeomorphisms of $(n-1)$-connected compact $2n$-dimensional manifolds with boundary.

8.1. Variation map

DEFINITION 8.1. Let $h : F \to F$ be a self-diffeomorphism of an $(n-1)$-connected compact $2n$-dimensional manifold F such that $h|_{\partial F} = \mathrm{id}$. Given an element $\xi \in H_n(F, \partial F)$, let x be an n-chain of $(F, \partial F)$ representing ξ. We define $\Delta(\xi) = [x - h(x)]$, where $[x - h(x)]$ denotes the homology class represented by the n-cycle $x - h(x)$ in $H_n(F)$. Then $\Delta : H_n(F, \partial F) \to H_n(F)$ is a well-defined homomorphism, which is called the *variation map* associated with h. Note that Δ depends only on the homotopy class of h relative to boundary.

Suppose that h_1 and $h_2 : F \to F$ are diffeomorphisms such that $h_1|_{\partial F} = h_2|_{\partial F} = \mathrm{id}$. It is easy to see that if they are homotopic relative to boundary, then $\Delta_1 = \Delta_2$, where Δ_1 and Δ_2 are the variation maps associated with h_1 and h_2 respectively. In particular, if they are isotopic relative to boundary, then we have $\Delta_1 = \Delta_2$.

DEFINITION 8.2. Let (M, K) be an open book with typical page F. As we have mentioned in Definition 2.5, we have the characteristic map (or the monodromy) $h : F \to F$ of the fibration over the circle. Note that h satisfies $h|_{\partial F} = \mathrm{id}$ and is uniquely determined up to isotopy relative to boundary. Thus the variation map $\Delta : H_n(F, \partial F) \to H_n(F)$ of h is defined and does not depend on the choice of h. We call Δ the *variation map* of the open book (M, K).

Let (M, K) be a (simple) open book structure on an $(n-1)$-connected closed $(2n+1)$-dimensional manifold M with $n \geq 3$. We denote its typical page by F. Then we have the exact sequence

$$0 \longrightarrow H_{n+1}(M) \xrightarrow{i_{M*}} H_{n+1}(M, F \times I) \xrightarrow{\partial} H_n(F \times I) \xrightarrow{i_{F*}} H_n(M) \longrightarrow 0$$

of the pair $(M, F \times I)$, where $I = [0, 1]$, $F \times I$ is a tubular neighborhood of F in M, $i_M : M \to (M, F \times I)$ is the inclusion, and i_{F*} can be identified with the epimorphism induced by the inclusion $i_F : F \to M$. Note that we have

$$H_{n+1}(M, F \times I) \cong H_{n+1}(\overline{M - (F \times I)}, \partial(F \times I)) \\ \cong H_{n+1}(F \times I, \partial(F \times I)) \cong H_n(F, \partial F)$$

by excision, by the fact that $\overline{M - (F \times I)} \cong F \times I$, and by the Künneth theorem, respectively. Analyzing carefully theses identifications, we obtain the exact sequence

$$(8.1) \qquad 0 \longrightarrow H_{n+1}(M) \xrightarrow{\delta} H_n(F, \partial F) \xrightarrow{\Delta} H_n(F) \xrightarrow{i_{F*}} H_n(M) \longrightarrow 0,$$

where δ is induced by the restriction map and Δ is the variation map of the open book.

It is a well-known fact that for simple open book structures on the spheres, the variation map is always an isomorphism. Furthermore, the variation map determines and is determined by the Seifert form (see [Kau74, Lemma 2.7]).

For an open book structure on a general manifold, the variation map may not necessarily be an isomorphism. In order to get a monomorphism, let us consider the orthogonal complement $R^\perp$ of $R = R(\operatorname{Ker} i_{F*})$ with respect to the intersection form (5.3) defined in Definition 5.7. Then we have the following.

Lemma 8.3. *We have* $\operatorname{Ker} \Delta = R^\perp$.

Proof. Let x be an arbitrary n-cycle representing an element of $H_n(F, \partial F)$. Since $\nu^+ : F \to M - \operatorname{Int} F$ and $\nu^- : F \to M - \operatorname{Int} F$ can be identified with the positive and the negative "half turns" with respect to the fibration $M - K \to S^1$ respectively, we have

$$\nu_*^- \circ \Delta([x]) = \nu_*^-[x - h(x)] = [\nu^-(x) - \nu^- \circ h(x)] = [\nu^-(x) - \nu^+(x)].$$

Recall the homomorphism $\Phi : H_n(F, \partial F) \to \operatorname{Ker} i_{W*}$ defined in the proof of Lemma 5.6, where $W = \overline{M - (F \times I)}$ and $i_W : W \to M$ is the inclusion map. Then, regarding $\nu^\pm$ as maps of F into W, we obtain $[\nu^-(x) - \nu^+(x)] = -[\nu^+(x) - \nu^-(x)] = -\Phi([x])$. Since $\nu_*^- : H_n(F) \to H_n(W)$ is an isomorphism, we obtain $\operatorname{Ker} \Delta = \operatorname{Ker} \Phi = R^\perp$ by Lemma 5.11. $\square$

By the above lemma, Δ induces the homomorphism

$$\bar{\Delta} : H_n(F, \partial F)/R^{\perp} \to H_n(F).$$

Thus we obtain the exact sequence

$$0 \longrightarrow H_n(F, \partial F)/R^{\perp} \xrightarrow{\bar{\Delta}} H_n(F) \xrightarrow{i_{F*}} H_n(M) \longrightarrow 0,$$

and hence

$$0 \longrightarrow H_n(F, \partial F)/R^{\perp} \xrightarrow{\bar{\Delta}} R \xrightarrow{i_{F*}|_R} \tau H_n(M) \longrightarrow 0,$$

is exact. Then, we have the following generalization of [Kau74, Lemma 2.7].

LEMMA 8.4. *We have* $\Gamma_F(a, \bar{\Delta}B) = \langle a, B \rangle$ *for all* $a \in R$ *and* $B \in H_n(F, \partial F)/R^{\perp}$*, where* $\langle\, , \,\rangle : R \times H_n(F, \partial F)/R^{\perp} \to \mathbb{Z}$ *is the unimodular bilinear form induced from the restricted intersection form* $R \times H_n(F, \partial F) \to \mathbb{Z}$ *of* F *as in Lemma* 5.12.

PROOF. The lemma follows from the following:

$$\begin{aligned}\Gamma_F(a, \bar{\Delta}B) &= \operatorname{lk}(\nu_*^+ a, \bar{\Delta}B) = \operatorname{lk}(a, \nu_*^- \circ \bar{\Delta}B) = \operatorname{lk}(a, -\Phi(B)) \\ &= -\operatorname{lk}(a, \Phi(B)) = -(-1)^{n+1} \operatorname{lk}(\Phi(B), a) \\ &= (-1)^n (I \times B) \cdot a = \langle a, B \rangle,\end{aligned}$$

where $(I \times B) \cdot a$ denotes the intersection number in M. Note that

$$[\partial(I \times x)] = [\nu^+(x) - \nu^-(x)] = \Phi(B)$$

for a cycle x representing B. □

Note that the sign appearing in the formula of the above proposition is slightly different from that of [Kau74, Lemma 2.7]. This is due to our definition of the linking number as in [Wal67], which is slightly different from that of Kauffman [Kau74].

The above lemma suggests the following notion of an algebraic variation map associated with a system of open book invariants.

Definition 8.5. Let M be an $(n-1)$-connected closed $(2n+1)$-dimensional manifold and $s = \{G, Q_G, \alpha_G, i_G, \Gamma_G\}$ a system of open book invariants with respect to M. Then the *algebraic variation map* $\bar{\Delta} : G^*/R^{\perp} \to R$ is defined by the formula

$$\Gamma_G(a, \bar{\Delta}B) = \langle a, B \rangle \tag{8.2}$$

for all $a \in R$ and $B \in G^*/R^{\perp}$, where G^* is the dual of G and $R^{\perp}$ is the orthogonal complement of R with respect to the restriction of the natural paring $\langle \, , \, \rangle : G \times G^* \to \mathbb{Z}$. The map $\bar{\Delta}$ is a well-defined homomorphism as the following lemma shows.

Lemma 8.6. *The map $\bar{\Delta} : G^*/R^{\perp} \to R$ is a well-defined homomorphism and is uniquely determined by the formula* (8.2)*.*

Proof. By Proposition 5.1, the given system of open book invariants

$$s = \{G, Q_G, \alpha_G, i_G, \Gamma_G\}$$

can be realized by an $(n-1)$-connected compact $2n$-dimensional manifold F embedded in M. Then by the last part of the proof of Lemma 5.6, we see that the matrix $(\Gamma_G(a_i, r_j a_j))$ is unimodular for some basis $\{a_i\}$ of R and some integers r_j. Then the result follows immediately. $\square$

The following lemma is a consequence of the above definition.

Lemma 8.7. *We have* $\det \bar{\Delta} = \pm|\tau H_n(M)|$*.*

Proposition 8.8. *There exists an exact sequence*

$$0 \longrightarrow G^*/R^{\perp} \xrightarrow{\bar{\Delta}} R \xrightarrow{i_G|_R} \tau H_n(M) \longrightarrow 0.$$

Proof. Since $\det \bar{\Delta} = \pm|\tau H_n(M)|$, $\bar{\Delta}$ is injective. Furthermore, since $i_G : G \to H_n(M)$ is surjective, $i_G|_R : R \to \tau H_n(M)$ is also surjective. Thus, we have only to prove that $i_G \circ \bar{\Delta} = 0$.

Let $B \in G^*/R^{\perp}$ be an arbitrary element. Item (5c) of Definition 3.15 implies

$$\Gamma_G(a, \bar{\Delta}B) \equiv b_M(i_G(a), i_G(\bar{\Delta}B)) \pmod{\mathbb{Z}}$$

for all $a \in R$. By our definition of $\bar{\Delta}$, we have $\Gamma_G(a, \bar{\Delta}B) = \langle a, B\rangle \in \mathbb{Z}$, and hence we have $b_M(i_G(a), i_G(\bar{\Delta}B)) = 0 \in \mathbb{Q}/\mathbb{Z}$ for all $a \in R$. Since $i_G : R \to \tau H_n(M)$ is surjective and b_M is non-singular by [Wal67], we have $i_G(\bar{\Delta}B) = 0$. This completes the proof of Proposition 8.8. □

Note that the Seifert form determines, and is determined by the algebraic variation map.

8.2. Application to diffeomorphisms of manifolds with boundary

In this subsection, we use variation maps and the previous results in order to study isotopy of diffeomorphisms of $(n-1)$-connected compact $2n$-dimensional manifolds with boundary.

Recall the open book construction studied in Definition 2.6.

PROPOSITION 8.9. *Let F be an $(n-1)$-connected compact $2n$-dimensional manifold with $(n-2)$-connected and non-empty boundary with $n \equiv 2, 5, 6 \pmod 8$ and $n \geq 4$. Furthermore, let h_1 and $h_2 : F \to F$ be orientation preserving diffeomorphisms which are the identity on the boundary. We suppose that $\mathrm{Coker}(\Delta_1) \otimes \mathbb{Z}_2 = \mathrm{Coker}(\Delta_2) \otimes \mathbb{Z}_2 = 0$, where Δ_1 and Δ_2 are the variation maps associated with h_1 and h_2 respectively. If $\Delta_1 = \Delta_2$, then the manifolds obtained by the open book constructions with respect to h_1 and h_2 are diffeomorphic to each other up to taking connected sum with some homotopy $(2n+1)$-sphere.*

PROOF. Let M_1 and M_2 be the manifolds obtained by the open book constructions with respect to h_1 and h_2 respectively (see Definition 2.6). We denote the respective bindings by K_1 and K_2. Then the manifolds M_1 and M_2 are diffeomorphic to each other up to taking connected sum with a homotopy sphere, if and only if their *systems of invariants* defined by Wall [Wal67] are equivalent.

Consider the homology exact sequence

$$0 \longrightarrow H_{n+1}(M_j) \longrightarrow H_n(F_j, \partial F_j) \xrightarrow{\Delta_j} H_n(F_j) \xrightarrow{i_{F_j *}} H_n(M_j) \longrightarrow 0$$

associated with the open book (M_j, K_j), $j = 1, 2$, where F_j is the typical page (see (8.1)). Since $F_1 = F = F_2$ and $\Delta_1 = \Delta_2$ by our assumption, we have that $H = H_n(M_1) = \mathrm{Coker}(\Delta_1) = \mathrm{Coker}(\Delta_2) = H_n(M_2)$.

Since $F_1 = F = F_2$, we can identify $H_n(F_1)$ with $H_n(F_2)$, and since $\Delta_1 = \Delta_2$, we have that $\Gamma_{F_1} = \Gamma_{F_2}$ by Lemma 8.4. Consequently, by item (5c) of Definition 3.15, we have $b_{M_1} = b_{M_2}$.

By Remark 3.4, we have that $i_* \circ \alpha_F = \alpha_{M_j} \circ i_{F_j *}$, $j = 1, 2$, where $i_* : \pi_{n-1}(SO(n)) \to \pi_{n-1}(SO(n+1))$ is the homomorphism of (2.1). Therefore, $\alpha_{M_1} = \alpha_{M_2}$ holds under the above identification $H_n(M_1) = H_n(M_2)$, since $i_{F_j *}$ are surjective.

Then by item (5d) of Definition 3.15, we have $q_{M_1} = q_{M_2}$ when n is odd.

By our hypothesis, we have $H \otimes \mathbb{Z}_2 = \mathrm{Coker}(\Delta_1) \otimes \mathbb{Z}_2 = \mathrm{Coker}(\Delta_2) \otimes \mathbb{Z}_2 = 0$. Hence, for n even with $n \neq 4, 8$, the invariants $\hat{\phi}(M_1) \in H^{n+1}(M_1; \mathbb{Z}_2) \cong H \otimes \mathbb{Z}_2 = 0$ and $\hat{\phi}(M_2) \in H^{n+1}(M_2; \mathbb{Z}_2) \cong H \otimes \mathbb{Z}_2 = 0$ vanish.

Since $n \equiv 2, 5, 6 \pmod 8$, $\pi_n(SO) \cong \pi_n(SO(n+2))$ vanishes (see [Wal65]). Thus the invariant $\hat{\beta} \in H \otimes \pi_n(SO)$ also vanishes for M_1 and M_2. Note that no other invariants are necessary in Wall's classification theorem [Wal67, Theorem 7], since $n \not\equiv 0, 1, 4 \pmod 8$.

Thus, $M_1 - \mathrm{Int}\, D^{2n+1} \cong M_2 - \mathrm{Int}\, D^{2n+1}$ by the classification theorem of $(n-1)$-connected almost closed $(2n+1)$-dimensional manifolds [Wal67]. Therefore, there exists a homotopy $(2n+1)$-sphere Σ such that $M_1 \sharp \Sigma \cong M_2$. This completes the proof of Proposition 8.9. □

REMARK 8.10. As Wall's result shows, the diffeomorphism $M_1 \sharp \Sigma \cong M_2$ can be chosen so that the induced isomorphism $H_n(M_1) = H_n(M_1 \sharp \Sigma) \to H_n(M_2)$ coincides with the identification $H_n(M_1) = H_n(M_2)$ mentioned above.

THEOREM 8.11. *Under the same assumption as in Proposition* 8.9, *we have the following.*

(1) *The diffeomorphisms* $h_i \natural h_i^{-1} : F \natural (-F) \to F \natural (-F)$, $i = 1, 2$, *are isotopic to each other relative to boundary.*

(2) *There exists a positive integer* k *such that*

$$\underbrace{h_i \natural \cdots \natural h_i}_{k \text{ times}} : \underbrace{F \natural \cdots \natural F}_{k \text{ times}} \to \underbrace{F \natural \cdots \natural F}_{k \text{ times}}, \quad i = 1, 2,$$

are isotopic to each other relative to boundary.

(3) *There exists a positive integer k such that $h_i^k : F \to F$, $i = 1, 2$, are isotopic to each other relative to boundary.*

(4) *We can modify h_1 homotopically on a disk D^{2n} embedded in the interior of F fixing ∂D^{2n} so that it is isotopic to h_2 relative to boundary. In particular, h_1 and h_2 are homotopic relative to boundary.*

PROOF. We use the same notation as in the proof of Proposition 8.9. Note that we have $M_2 \cong M_1 \sharp \Sigma$ for some homotopy sphere Σ.

(1) We have $M_2 \sharp (-M_2) \cong M_1 \sharp (-M_1)$, since $\Sigma \sharp (-\Sigma)$ is always diffeomorphic to S^{2n+1}. Furthermore, by Remark 8.10 and the proof of Proposition 8.9, the diffeomorphism can be chosen so that it preserves the systems of open book invariants associated with $M_1 \sharp_b (-M_1) = (M_1, \partial F) \sharp_b (-(M_1, \partial F))$ and $M_2 \sharp_b (-M_2) = (M_2, \partial F) \sharp_b (-(M_2, \partial F))$, since systems of open book invariants and invariant systems of $(n-1)$-connected closed $(2n+1)$-dimensional manifolds behave well under book connected sum. Hence, under the above identification, they are structurally isotopic by Theorem 4.1. Note that the typical page of $-(M_i, \partial F)$ is identified with $-F$ and that its geometric monodromy is identified with h_i^{-1}, $i = 1, 2$.

Let us denote the structural isotopy between the open books $M_1 \sharp_b (-M_1)$ and $M_2 \sharp_b (-M_2)$ by $\Phi = \{\Phi_t\}_{t \in [0,1]}$. Then by Lemma 4.7, we may assume that $\Phi_1 : F \natural (-F) = F_1 \natural (-F_1) \to F_2 \natural (-F_2) = F \natural (-F)$ is the identity map, where F_1 and F_2 are the typical pages of $(M_1, \partial F)$ and $(M_2, \partial F)$ respectively, and $F_1 \natural (-F_1)$ and $F_2 \natural (-F_2)$ are the typical pages of $M_1 \sharp_b (-M_1)$ and $M_2 \sharp_b (-M_2)$ respectively. Therefore, their geometric monodromies $h_1 \natural h_1^{-1}$ and $h_2 \natural h_2^{-1}$ are isotopic relative to boundary.

(2) Note that the h-cobordism group θ^{2n+1} of homotopy $(2n+1)$-spheres is finite for $n \neq 1$ (see [KeM63]). Let k be the order of Σ in θ^{2n+1}. Then we have,

$$\underbrace{M_2 \sharp \cdots \sharp M_2}_{k \text{ times}} = \underbrace{M_1 \sharp \cdots \sharp M_1}_{k \text{ times}}. \tag{8.3}$$

Furthermore, as in (1), by Theorem 4.1, we see that the open books

$$\underbrace{M_1 \sharp_b \cdots \sharp_b M_1}_{k \text{ times}} \quad \text{and} \quad \underbrace{M_2 \sharp_b \cdots \sharp_b M_2}_{k \text{ times}}$$

are structurally isotopic under the above identification (8.3). Consequently, their monodromy diffeomorphisms

$$\underbrace{h_1 \natural \cdots \natural h_1}_{k \text{ times}} \text{ and } \underbrace{h_2 \natural \cdots \natural h_2}_{k \text{ times}} : \underbrace{F \natural \cdots \natural F}_{k \text{ times}} \to \underbrace{F \natural \cdots \natural F}_{k \text{ times}}$$

are isotopic to each other relative to boundary.

(3) Let us consider the diffeomorphism $\bar{h}_1 = h_1 \natural h_\Sigma^{-1} : F \natural D^{2n} \to F \natural D^{2n}$, where $h_\Sigma : D^{2n} \to D^{2n}$ is the monodromy of the trivial open book structure (Σ, K_Σ) on the homotopy sphere Σ (see Definition 2.8 and Proposition 7.5). Let $\bar{M}_1$ be the manifold obtained by the open book construction with respect to $\bar{h}_1$. Then $\bar{M}_1$ is diffeomorphic to $M_1 \sharp (-\Sigma) \cong M_2 \sharp \Sigma \sharp (-\Sigma) \cong M_2$. Furthermore, under an appropriate identification of $\bar{M}_1$ and M_2, the systems of open book invariants associated with the open books determined by $\bar{h}_1$ and h_2 coincide with each other. Hence, the open book structures are structurally isotopic by Theorem 4.1. Thus, $\bar{h}_1$ and h_2 are isotopic relative to boundary as in (1).

Let k be the order of Σ in the h-cobordism group θ^{2n+1} of homotopy $(2n+1)$-spheres. Note that we have $\bar{h}_1^k = h_1^k \natural h_\Sigma^{-k}$. Since h_Σ^{-k} is isotopic to the identity of D^{2n} relative to boundary, we see that $\bar{h}_1^k$ is isotopic to h_1^k relative to boundary. Thus, h_1^k and h_2^k are isotopic relative to boundary.

(4) This follows from the proof of (3). □

We can weaken the condition of Theorem 8.11 as follows.

Corollary 8.12. *Let F be an $(n-1)$-connected compact $2n$-dimensional manifold with $(n-2)$-connected and non-empty boundary with $n \equiv 2, 5, 6 \pmod 8$ and $n \geq 4$. We suppose that there exists a diffeomorphism $h : F \to F$ with $h|_{\partial F} = \mathrm{id}$ such that $\mathrm{Coker}(\Delta_h) \otimes \mathbb{Z}_2 = 0$, where Δ_h is the variation map associated with h. Let h_1 and $h_2 : F \to F$ be orientation preserving diffeomorphisms which are the identity on the boundary. If the variation maps associated with h_1 and h_2 coincide with each other, then we have the following.*

(1) *The diffeomorphisms $h_i \natural h_i^{-1} : F \natural (-F) \to F \natural (-F)$, $i = 1, 2$, are isotopic to each other relative to boundary.*

(2) *There exists a positive integer k such that*

$$\underbrace{h_i\natural\cdots\natural h_i}_{k\ times} : \underbrace{F\natural\cdots\natural F}_{k\ times} \to \underbrace{F\natural\cdots\natural F}_{k\ times}, \quad i = 1, 2,$$

are isotopic to each other relative to boundary.

(3) *There exists a positive integer k such that $h_i^k : F \to F$, $i = 1, 2$, are isotopic to each other relative to boundary.*

(4) *We can modify h_1 homotopically on a disk D^{2n} embedded in the interior of F fixing ∂D^{2n} so that it is isotopic to h_2 relative to boundary. In particular, h_1 and h_2 are homotopic relative to boundary.*

Proof. Let g_1 and $g_2 : F \to F$ be the diffeomorphisms defined by $g_1 = (h_1 \circ h_2^{-1}) \circ h$ and $g_2 = h$. Then g_1 and g_2 satisfy the conditions of Theorem 8.11. Hence item (4) holds for g_1 and g_2, and hence for $h_1 \circ h_2^{-1}$ and id. This implies that item (4) holds for h_1 and h_2. Then the other items (1)–(3) follow immediately. □

Remark 8.13. In the above corollary, the condition that there exists a diffeomorphism $h : F \to F$ with $h|_{\partial F} = \mathrm{id}$ such that $\mathrm{Coker}(\Delta_h) \otimes \mathbb{Z}_2 = 0$ is equivalent to that F can be embedded in a $\mathbb{Z}_2$-homology $(2n+1)$-sphere as a page of an open book structure. In particular, if F is a page of an open book structure on S^{2n+1} — i.e., if F is a fiber of a fibered knot — then two diffeomorphisms h_1 and $h_2 : F \to F$ with $h_1|_{\partial F} = h_2|_{\partial F} = \mathrm{id}$ satisfy the condition (4) of the above corollary, if and only if their associated variation maps coincide with each other.

Remark 8.14. In [Sae99], [Sae02], the third author has studied simple open book structures on simply connected rational homology 5-spheres and has obtained results similar to Theorem 8.11 and Corollary 8.12 for $n = 2$.

References

[A'C72] A'Campo, N., Feuilletages de codimension 1 sur les variétés simplement connexes de dimension 5, Comment. Math. Helv. **47** (1972), 514–525.

[Ale23] Alexander, J. W., A lemma on systems of knotted curves, Proc. Nat. Acad. Sci. USA **9** (1923), 93–95.

[BL66] Browder, W. and J. Levine, Fibering manifolds over a circle, Comment. Math. Helv. **40** (1966), 153–160.

[Cer70] Cerf, J., La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie, Inst. Hautes Études Sci. Publ. Math. **39** (1970), 185–353.

[Dur74] Durfee, A. H., Fibered knots and algebraic singularities, Topology **13** (1974), 47–59.

[Hae61] Haefliger, A., Differentiable imbeddings, Bull. Amer. Math. Soc. **67** (1961), 109–112.

[Ham71] Hamm, H. A., Lokale topologische Eigenschaften Komplexer Räume, Math. Ann. **191** (1971), 235–252.

[HZ66] Hirsch, M. W. and E. C. Zeeman, Engulfing, Bull. Amer. Math. Soc. **72** (1966), 113–115.

[Hu59] Sze-tsen Hu, *Homotopy theory*, Pure and Applied Mathematics, vol. VIII, Academic Press, New York, London, 1959.

[KaM79] Kargapolov, M. I. and J. I. Merzljakov, *Fundamentals of the theory of groups*, Graduate Texts in Mathematics, vol. 62, Springer-Verlag, New York, Berlin, 1979.

[Kat74] Kato, M., Classification of simple spinnable structures on a 1-connected Alexander manifold, J. Math. Soc. Japan **26** (1974), 454–463.

[Kau74] Kauffman, L. H., Branched coverings, open books and knot periodicity, Topology **13** (1974), 143–160.

[Ker60] Kervaire, M. A., Some nonstable homotopy groups of Lie groups, Illinois J. Math. **4** (1960), 161–169.

[Ker65] Kervaire, M. A., Les nœuds de dimensions supérieures, Bull. Soc. Math. France **93** (1965), 225–271.

[KeM63] Kervaire, M. A. and J. W. Milnor, Groups of homotopy spheres: I, Ann. of Math. **77** (1963), 504–537.

[Kin78] King, H. C., Topological type of isolated critical points, Ann. of Math. **107** (1978), 385–397.

[Kin97] King, H. C., Topology of isolated critical points of functions on singular spaces, Stratifications, singularities and differential equations, II (Marseille, 1990; Honolulu, HI, 1990), Travaux en Cours **55**, Hermann, Paris, 1997, pp. 63–72.

[Lau76] Laudenbach, F., Submersions sur le cercle, Bull. Soc. Math. France **104** (1976), 417–431.

[Law78] Lawson, T., Open book decompositions for odd dimensional manifolds, Topology **17** (1978), 189–192.

[Lê92] Lê Dũng Tráng, Complex analytic functions with isolated singularities, J. of Algebraic Geometry **1** (1992), 83–100.

[Lev70] Levine, J., An algebraic classification of some knots of codimension two, Comment. Math. Helv. **45** (1970), 185–198.

[Mas00] Massago, S., *Estrutura "Open Book" em variedades* $(n-1)$*-conexas de dimensão* $2n+1$ (in Portuguese), PhD Thesis, ICMC, University of São Paulo, 2000.

[Mil65] Milnor, J., *Lectures on the h-cobordism theorem*, Princeton Univ. Press, Princeton, N.J., 1965.

[Mil68] Milnor, J., *Singular points of complex hypersurfaces*, Ann. Math. Stud., vol. 61, Princeton Univ. Press, Princeton, N.J., 1968.

[Qui79] Quinn, F., Open book decompositions, and the bordism of automorphisms, Topology **18** (1979), 55–73.

[Sae87] Saeki, O., On simple fibered knots in S^5 and the existence of decomposable algebraic 3-knots, Comment. Math. Helv. **62** (1987), 587–601.

[Sae99] Saeki, O., Theory of fibered 3-knots in S^5 and its applications, J. Math. Sci. Univ. Tokyo **6** (1999), 691–756.

[Sae02] Saeki, O., Open books on 5-dimensional manifolds, Hiroshima Math. J. **32** (2002), 189–205.

[Sie68] Siebenmann, L. C., Pseudo-annuli and invertible cobordisms, Arch. der Math. **19** (1968), 528–535.

[Sma61] Smale, S., Generalized Poincaré's conjecture in dimensions greater than four, Ann. of Math. **74** (1961), 391–406.

[Sma62] Smale, S., On the structure of manifolds, Amer. J. Math. **84** (1962), 387–399.

[Spa66] Spanier, E. H., *Algebraic topology*, McGraw-Hill, New York, Toronto, London, 1966.

[Ste51] Steenrod, N., *Topology of fibre bundles*, Princeton Univ. Press, Princeton, N.J., 1951.

[Tam73] Tamura, I., Foliations and spinnable structures on manifolds, Ann. Inst. Fourier **23** (1973), 197–214.

[Tam93] Tamura, I., Fundamental theorems in global knot theory, J. Fac. Sci. Univ. Tokyo, Sect. IA Math. **40** (1993), 561–606.

[Wal62] Wall, C. T. C., Classification of $(n-1)$-connected $2n$-manifolds, Ann. of Math. **75** (1962), 163–189.

[Wal63] Wall, C. T. C., Classification problems in differential topology - I: Classification of handlebodies, Topology **2** (1963), 253–261.

[Wal65] Wall, C. T. C., Classification problems in differential topology - III: Application to special cases, Topology **3** (1965), 291–304.

[Wal67] Wall, C. T. C., Classification problems in differential topology - VI: Classification of $(s-1)$-connected $(2s+1)$-manifolds, Topology **6** (1967), 273–296.

[Wal70] Wall, C. T. C., *Surgery on compact manifolds*, Academic Press, London, New York, 1970.

[Whi44] Whitney, H., The self-intersection of a smooth n-manifold in $2n$-space, Ann. of Math. **45** (1944), 220–246.

[Win73] Winkelnkemper, H. E., Manifolds as open books, Bull. Amer. Math. Soc. **79** (1973), 45–51.

(Received October 30, 2003)

Oziride Manzoli Neto
Instituto de Ciências Matemáticas e de Computação
Universidade de São Paulo
Caixa Postal 668, CEP 13560-970
São Carlos - SP, Brazil
E-mail: ozimneto@icmc.usp.br
URL: http://www.icmc.usp.br/˜ozimneto/

Sadao Massago
Universidade Federal de São Carlos
Departamento de Matemática
Caixa Postal 676, CEP 13565-905
São Carlos - SP, Brazil
E-mail: sadao@dm.ufscar.br
URL: http://www2.dm.ufscar.br/˜sadao/

Osamu Saeki
Faculty of Mathematics
Kyushu University
Hakozaki, Fukuoka 812-8581, Japan
E-mail: saeki@math.kyushu-u.ac.jp
URL: http://www.math.kyushu-u.ac.jp/˜saeki/