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Restrictions of Log Canonical Algebras of General Type

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Abstract. We introduce a diophantine property of a log canonical algebra, and use it to describe the restriction of a log canonical algebra of general type to a log canonical center of codimension one.

0. Introduction

In an inductive study of log canonical algebras, it is important to understand their restrictions to log canonical centers. Let X be a nonsingular complex variety, $Y \subset X$ a nonsingular divisor and $\pi\colon X \to S$ a projective morphism. The adjunction formula $(K+Y)|_Y = K_Y$ induces a homomorphism of graded $\mathcal{O}_S$-algebras

$$\mathcal{R}_{X/S}(K+Y) \to \mathcal{R}_{Y/S}(K_Y),$$

where $\mathcal{R}_{X/S}(K+Y) = \bigoplus_{i=0}^{\infty} \pi_*\mathcal{O}_X(iK+iY)$ is the log canonical algebra of $(X/S, Y)$ and $\mathcal{R}_{Y/S}(K_Y) = \bigoplus_{i=0}^{\infty} \pi_*\mathcal{O}_Y(iK_Y)$ is the log canonical algebra of Y/S. The image of this homomorphism is a graded subalgebra, denoted

$$\mathcal{R}_{X/S}(K+Y)|_Y \subseteq \mathcal{R}_{Y/S}(K_Y). \tag{1}$$

Siu's [17] invariance of plurigenera of varieties of general type can be restated as follows: if S is a smooth curve and Y is a smooth π-fiber of general type, the restricted algebra $\mathcal{R}_{X/S}(K+Y)|_Y$ coincides with $\mathcal{R}_{Y/S}(K_Y)$. Kawamata [9,10] and Nakayama [12, 13] obtained singular versions of this result, and extended it in a different direction: if $K+Y$ is π-big and its relative Iitaka map maps Y birationally onto its image, the restricted algebra $\mathcal{R}_{X/S}(K+Y)|_Y$ coincides with $\mathcal{R}_{Y/S}(K_Y)$ in degrees $i \geq 2$.

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A characterization of the restricted algebra was also expected in the logarithmic case (see Nakayama [12, Theorem 4.9]), but a new point of view was necessary, since it was known that the inclusion in (1) may be strict in all degrees in this case. The new idea, due to Hacon and McKernan, is that the restricted algebra is equivalent with the log canonical algebra of a log structure defined not necessarily on Y, but on a birational model of Y. Hacon and McKernan ([6], Theorem 4.3) obtained this description in the birational log Fano set-up of a prelimiting flip and assuming the validity of the log Minimal Model Program in smaller dimensions. In this paper we sharpen their result, and extend it to log varieties of general type. The other contribution of this paper is a new diophantine property of a log canonical algebra (Lemma 1.5), which is based on ideas of Shokurov and Viehweg. We hope that these two new tools will be useful for bounding log canonical models (see Kollár [11] for basic open problems). We also present two applications in §4.

Our main result is as follows (we work over an algebraically closed field of characteristic zero).

THEOREM 0.1. *Let X be a nonsingular algebraic variety and B an $\mathbb{R}$-divisor on X such that* $\operatorname{Supp}(B)$ *is a simple normal crossings divisor. Assume that Y is a component of B with* $\operatorname{mult}_Y(B) = 1$, $\lfloor B - Y \rfloor = 0$. *Denote $B_Y = (B - Y)|_Y$, so that by adjunction we have $(K + B)|_Y = K_Y + B_Y$. Let $\pi\colon X \to S$ be a projective surjective morphism, and assume*

(a) *$K + B \sim_{\mathbb{Q}} A + C$, where A is a π-ample $\mathbb{R}$-divisor and C is an effective $\mathbb{R}$-divisor with* $\operatorname{mult}_Y(A) = \operatorname{mult}_Y(C) = 0$.

(b) *(Y, B_Y) has canonical singularities in codimension at least two.*

Define $\Theta = \max(B_Y - \lim_{i\to\infty} \frac{(\mathbf{F}\mathrm{ix}(iK+iB)|_Y)_Y}{i}, 0)$ where the maximum is taken componentwise. For every $n \geq 1$, we have a natural inclusion

$$\operatorname{Im}(\pi_*\mathcal{O}_X(nK + nB) \to \pi_*\mathcal{O}_Y(nK_Y + nB_Y)) \subseteq \pi_*\mathcal{O}_Y(nK_Y + n\Theta).$$

The following properties hold:

(1) *The inclusion is an equality if $n \geq 2$ and $\{nB\} \leq B$, or if $n = 1$ and $\pi(Y) \neq \pi(X)$.*

(2) *Assume that B has rational coefficients, and the log canonical divisor $K_Y + \Theta$ has a Zariski decomposition relative to S. Then Θ has rational coefficients, and the graded $\mathcal{O}_S$-algebra $\bigoplus_{n=0}^\infty \pi_* \mathcal{O}_Y(nK_Y + n\Theta)$ is finitely generated.*

In the definition of Θ, $(\mathbf{F}\mathrm{ix}(iK + iB)|_Y)_Y$ is the trace on Y of the restriction to Y of the fixed $\mathbb{R}$-b-divisor of $\pi_* \mathcal{O}_X(iK + iB)$. Precisely, assume that $i(K + B)$ is relatively mobile at Y and choose a birational modification $\mu\colon X' \to X$ such that the mobile part M_i of $\mu^*(iK + iB)$ is relatively free, and the proper transform Y' of Y on X is normal. Then $(\mathbf{F}\mathrm{ix}(iK + iB)|_Y)_Y$ is the push forward of $(\mu^*(iK + iB) - M_i)|_{Y'}$ via the birational map $Y' \to Y$.

The assumption (b) is necessary for the restricted algebra $\mathcal{R}_{X/S}(K + B)|_Y$ to have a presentation as a log canonical algebra on Y. A similar result holds when (Y, B_Y) has only Kawamata log terminal singularities (Theorem 4.1), but one has to pass to a birational model of Y so that Θ takes into account all valuations of Y whose log discrepancy with respect to (Y, B_Y) is less than one (see Section 2 for details).

As for the proof of Theorem 0.1, we recommend that the reader first consults Lemma 2.4, for an argument modulo the log Minimal Model Program in the same dimension. The proof of (1) is based on Siu's idea, with modifications by Kawamata and Nakayama. Siu's idea [17] is to view nK_Y as $K_Y + (n-1)K_Y$, and to pass by induction a property from $(n-1)K_Y$ to nK_Y. This still works in the logarithmic case, provided we replace the given boundary by a canonical sequence of boundaries, satisfying certain arithmetic properties (Lemmas 1.1 and 1.5). The real coefficients of the boundary pose no problem, since Kawamata-Viehweg vanishing is known to hold in the real case. Also, by diophantine approximation, the inequality $\{nB\} \le B$ has infinitely many solutions $n \in \mathbb{N}$.

The proof of (2) is based on a new diophantine property of a log canonical algebra (Lemma 1.5), and a criterion for a real nef divisor to be rational and semiample [2]. The former is a combination of Shokurov's [15] ideas on diophantine properties of graded algebras and Viehweg's [20] method for dealing with pluricanonical sections in his proof of the weak positivity of the push forwards of relative pluricanonical sheaves, and the latter generalizes Kawamata's criterion [7] that log canonical rings of general type are finitely generated if Zariski decomposition exists. Conversely, the log Minimal Model Program (with real boundaries) in the dimension of $\tilde{Y}$ implies the existence of Zariski decomposition for the big log canonical divisor $K_{\tilde{Y}} + \Theta$ in Theorem 0.1 (see Lemma 1.6).

The reader will notice that we make heavy use of Shokurov's new terminology of b-divisors, instead of multiplier ideal sheaves, which are common in this context. The logarithmic implementation of Siu's idea involves taking a log canonical divisor out of the round-up in Lemma 1.5, on sufficiently high birational models of a given variety, and since the log canonical divisor does not have integer coefficients, we work directly with divisors on these high models. B-divisors are a very useful notation for making computations on these high models, finitely many at a time, but the reader may avoid them by simply introducing notation for these models. We are unable to encode this argument on the base variety, in terms of multiplier ideal sheaves. The reader interested in this may consult the arguments of Takayama [19, Theorem 4.1] and Hacon-McKernan [5, Corollary 3.17].

Finally, we expect that the hypothesis (a) in Theorem 0.1 can be weakened to (a'): $K + B \sim_{\mathbb{Q}} C$, where C is effective, π-big and $\mathrm{mult}_Y(C) = 0$. This may follow from an algebrization of the method introduced by Siu for the invariance of plurigenera of manifolds of non-general type (see [18]).

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1. Preliminary

1.A. Boundary arithmetic

LEMMA 1.1. *For $b \in [0, 1]$ and $n \geq 1$ define $b_n = \max(b, \frac{1}{n}\lceil (n-1)b \rceil)$. The following properties hold:*

(1) *$b = b_n$ if and only if $\{nb\} \leq b$.*

(2) *$b \leq b_n \leq b + \frac{1}{n}$.*

(3) *$\lfloor b \rfloor + \lceil (n-1)b_{n-1} \rceil \leq nb_n$ for $n \geq 1$.*

PROOF. Properties (1) and (2) are easy to see. Property (3) is clear if $b = 0$ or 1, so let $b \in (0, 1)$. The claim is then equivalent to

$$(n-1)b \leq \lfloor nb_n \rfloor.$$

Assume first that $\{nb\} \leq b$, that is $b_n = b$. Then $\lfloor nb \rfloor - (n-1)b = b - \{nb\} \geq 0$. Assume now that $\{nb\} \geq b$, that is $nb_n = \lceil (n-1)b \rceil$. Then $\lceil (n-1)b \rceil - (n-1)b = \{(n-1)b\} \geq 0$. $\square$

Lemma 1.2. *Let n be a positive integer and $b, e \in \mathbb{R}_{\geq 0}$ such that $e - bn \in \mathbb{Z}$. Then $\lceil -b + \frac{e}{n} \rceil \leq e$.*

Proof. Let $e - bn = p \in \mathbb{Z}$. For $p \geq 0$, we have

$$e - \lceil -b + \frac{e}{n} \rceil = p - \lceil \frac{p}{n} \rceil + bn \geq 0.$$

If $p \leq 0$, then $\lceil -b + \frac{e}{n} \rceil = \lceil \frac{p}{n} \rceil \leq 0 \leq e$. $\square$

Lemma 1.3. *Let $a, c, d, \gamma \in \mathbb{R}$ such that $a > -1$, $a - \gamma c \geq -1$ and $\gamma > 0$. If n is an integer such that $n \geq 1 + \gamma^{-1}$, then*

$$\lfloor \frac{\lceil a + c + nd \rceil}{n} \rfloor \leq \lceil a + d \rceil.$$

Proof. Since $a + 1 + c \leq (a+1)(1+\gamma^{-1})$ and $\lceil a + 1 + d \rceil - d \geq a + 1$, we obtain $1 + \gamma^{-1} \geq \frac{a+1+c}{\lceil a+1+d \rceil - d}$. Therefore

$$n \geq \frac{a + 1 + c}{\lceil a + 1 + d \rceil - d}.$$

This is equivalent to the conclusion, by a straightforward computation. $\square$

1.B. B-divisors, log pairs, log varieties

We refer the reader to [2] for standard definitions on Shokurov's b-divisors, log pairs and log varieties. Just to fix the notation, recall that a log pair (X, B) is a normal complex variety X endowed with an $\mathbb{R}$-divisor B such that $K + B$ is $\mathbb{R}$-Cartier. A log variety is a log pair whose boundary B is effective. The discrepancy $\mathbb{R}$-b-divisor of a log pair (X, B) is

$$\mathbf{A}(X, B) = \mathbf{K} - \overline{K + B},$$

where $\mathbf{K}$ is the canonical b-divisor of X and $\overline{K + B}$ is the Cartier closure of the log canonical class. If (X, B) has log canonical singularities, let $\mathbf{R}$

be the reduced b-divisor of all prime b-divisors of X which have zero log discrepancy with respect to (X, B). Define $\mathbf{A}(X, B)^* = \mathbf{A}(X, B) + \mathbf{R}$, so that $\lceil \mathbf{A}(X, B)^* \rceil \geq 0$.

Let $\pi \colon X \to S$ be a proper morphism from a normal variety X and let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. We denote by $\mathbf{M}(D)$ the mobile b-divisor of D relative to S, $\mathbf{F}\mathrm{ix}(D) = \overline{D} - \mathbf{M}(D)$, $\mathbf{D}_i(D) = \frac{1}{i}\mathbf{M}(iD)$ and

$$\mathcal{R}_{X/S}(D) = \bigoplus_{i=0}^{\infty} \pi_* \mathcal{O}_X(iD).$$

Note that $\mathrm{Fix}(D)$, the trace of $\mathbf{F}\mathrm{ix}(D)$ on X, is the fixed part of D relative to S in the usual sense. Locally over S,

$$\mathrm{Fix}(D) = \inf\{(a) + D; a \in \pi_* \mathcal{O}_X(D) \setminus 0\}.$$

Lemma 1.4 (Terminal resolution). *Let (X, B) be a log pair with Kawamata log terminal singularities. Then the set of prime b-divisors E of X, having log discrepancy $a(E; X, B)$ less than 1, is finite.*

Proof. We may assume that X is smooth and $\mathrm{Supp}(B)$ has simple normal crossings. Consider the set of pairs of distinct prime divisors on X

$$\mathcal{S} = \{(E_1, E_2); E_1 \cap E_2 \neq \emptyset, a(E_1; X, B) + a(E_2; X, B) \leq 1\}.$$

If $\mathcal{S}$ is empty, it is easy to see that $a(E; X, B) > 1$ for every prime b-divisor E which is exceptional over X. If $\mathcal{S}$ is nonempty, define

$$a = \min_{(E_1, E_2) \in \mathcal{S}} \min(a(E_1; X, B), a(E_2; X, B)).$$

Since (X, B) has Kawamata log terminal singularities, we have $a > 0$ and

$$a(E_1; X, B) + a(E_2; X, B) \geq 2a \text{ for } (E_1, E_2) \in \mathcal{S}.$$

Let $X_1 \to X$ be the composition of the blow-ups of X in $E_1 \cap E_2$, after all $(E_1, E_2) \in \mathcal{S}$, and let $\mu_1^*(K + B) = K_{X_1} + B_{X_1}$. Then X_1 is smooth and $\mathrm{Supp}(B_{X_1})$ has simple normal crossings. Consider the set of distinct prime divisors on X_1

$$\mathcal{S}_1 = \{(E_1, E_2); E_1 \cap E_2 \neq \emptyset, a(E_1; X_1, B_{X_1}) + a(E_2; X_1, B_{X_1}) \leq 1\}.$$

If $\mathcal{S}_1$ is empty, we are done. Otherwise,

$$a(E_1; X_1, B_{X_1}) + a(E_2; X_1, B_{X_1}) \geq 3a \text{ for } (E_1, E_2) \in \mathcal{S}_1,$$

and we repeat the process: let $X_2 \to X_1$ be the composition of the blow-ups of X_1 in $E_1 \cap E_2$, after all $(E_1, E_2) \in \mathcal{S}$, and so on. After n blow-ups, either $\mathcal{S}_n$ is empty, or

$$a(E_1; X_n, B_{X_n}) + a(E_2; X_n, B_{X_n}) \geq (n+1)a \text{ for } (E_1, E_2) \in \mathcal{S}_n.$$

Therefore there exists $n \leq \lceil a^{-1} \rceil$ such that $\mathcal{S}_n = \emptyset$, that is (X_n, B_{X_n}) has terminal singularities in codimension at least two. □

The next lemma is our key tool, a diophantine property of a log canonical algebra. Inspired by Shokurov's notion of asymptotic saturation of a graded algebra [15], it is the logarithmic version of Viehweg's [20] method for dealing with pluricanonical sections, which is similar to Siu's [17]. If X is nonsingular, $B = 0$ and $i = n$, property (2) is implicit in Viehweg's proof of the weak positivity of the push forwards of relative pluricanonical sheaves. If $B = 0$ and K_X is π-semiample, Lemma 1.5 is equivalent to the asymptotic saturation of $\mathcal{R}_{X/S}(K_X)$ with respect to the log variety X. In general, the two notions differ. For example, consider the graded $\mathcal{O}_{\mathbb{C}}$-algebra $\mathcal{R}_{\mathbb{C}/\mathbb{C}}(K_{\mathbb{C}} + b \cdot 0)$, where $b \in (0,1)$ and $\mathbb{C}/\mathbb{C}$ is the identity map. Then Lemma 1.5 says that $\lfloor nb \rfloor \leq \lceil (n-1)b \rceil$ for every $n \geq 1$, with equality if $\{nb\} \leq b$. On the other hand, $\mathcal{R}_{\mathbb{C}/\mathbb{C}}(K_{\mathbb{C}} + b \cdot 0)$ is asymptotically saturated with respect to the log variety $(\mathbb{C}, b \cdot 0)$ if and only if the opposite inequality $\lceil (n-1)b \rceil \leq \lfloor nb \rfloor$ holds for sufficiently large and divisible integers n, that is $b \in \mathbb{Q}$. The reader may consult [2, 3] for more on asymptotically saturated graded algebras. Finally, Lemma 1.5 has an analogue for anti-log canonical algebras, and it seems to be peculiar to these two types of graded algebras.

Lemma 1.5. *Let (X, B) be a log variety with log canonical singularities, and $\pi \colon X \to S$ a proper morphism. Let $\mathbf{R}$ be the reduced b-divisor of all prime b-divisors of X which have zero log discrepancy with respect to (X, B). Note that $\mathbf{R} = 0$ if and only if (X, B) has Kawamata log terminal singularities. Let n be a positive integer such that $\pi_* \mathcal{O}_X(nK + nB) \neq 0$. Then the following properties hold:*

(1) *For every $i \in n\mathbb{N}$ we have an inclusion*

$$\pi_*\mathcal{O}_X(nK+nB) \subseteq \pi_*\mathcal{O}_X(\lceil \mathbf{K}+\mathbf{R}+(n-1)\mathbf{D}_i(K+B)\rceil).$$

The sheaf on the right-hand side is independent of i sufficiently large and divisible.

(2) *Equality holds in (1) if $i=n$, or if $\{nB\} \le B$ and $i \in n\mathbb{N}$.*

Proof. (1) This follows from the case $i=n$ of (2), since $\mathbf{D}_n(K+B) \le \mathbf{D}_i(K+B)$ for $n|i$.

(2a) We show equality holds for $i=n$. For the direct inclusion, let $a \in k(X)^\times$ with $(a)+nK+nB \ge 0$. In particular, $\overline{(a)}+n\mathbf{D}_n(K+B) \ge 0$. Since $\mathbf{D}_n(K+B) \le \overline{K+B}$, we have

$$\mathbf{K}+\mathbf{R}+(n-1)\mathbf{D}_n(K+B) \ge \mathbf{A}(X,B)^* + n\mathbf{D}_n(K+B).$$

Since $\lceil \mathbf{A}(X,B)^* \rceil \ge 0$, we obtain

$$\overline{(a)}+\lceil \mathbf{K}+\mathbf{R}+(n-1)\mathbf{D}_n(K+B)\rceil \ge \lceil \mathbf{A}(X,B)^*+(\overline{(a)}+n\mathbf{D}_n(K+B))\rceil \ge 0.$$

We now consider the opposite inclusion. By Hironaka, there exists a proper birational morphism $\mu\colon X' \to X$ and a Cartier divisor M_n on X' such that

(i) $M_n \le \mu^*(nK+nB)$ is $(\pi\circ\mu)$-free.

(ii) The inclusion $(\pi\circ\mu)_*\mathcal{O}_{X'}(M_n) \subset \pi_*\mathcal{O}_X(nK+nB)$ is an equality.

(iii) X' is nonsingular, $\mathrm{Supp}(M_n)\cup\mathrm{Supp}(B_{X'})$ is a simple normal crossings divisor, where $\mu^*(K+B)=K_{X'}+B_{X'}$ is the log pullback.

Thus $\mathbf{D}_n(K+B)$ is the Cartier closure of $\frac{1}{n}M_n$. Let $R=\sum_{\mathrm{mult}_E(B_{X'})=1} E$. By construction, $F_n=\mu^*(nK+nB)-M_n$ is an effective $\mathbb{R}$-divisor. We have

$$\lceil K_{X'}+R+\frac{n-1}{n}M_n\rceil = M_n+\lceil -B_{X'}+R+\frac{1}{n}F_n\rceil.$$

Since (X,B) has log canonical singularities and B is effective, the divisor $\lceil -B_Y+R\rceil$ is effective and μ-exceptional. Since F_n is effective, we obtain

$$0 \le \lceil -B_{X'}+R+\frac{1}{n}F_n\rceil.$$

On the other hand, $F_n - nB_{X'} = nK_{X'} - M_n$ is an integral divisor. By Lemma 1.2, we obtain

$$\lceil -B_{X'} + R + \frac{1}{n}F_n \rceil \leq F_n + G_n,$$

where $G_n = \sum_{\mathrm{mult}_P(B_{X'})<0} \lceil \mathrm{mult}_P(-B_{X'} + \frac{1}{n}F_n) \rceil \cdot P$, where the sum runs after the prime divisors in X' where $B_{X'}$ is negative. Combining the above inequalities, we obtain

$$M_n \leq \lceil K_{X'} + R + \frac{n-1}{n}M_n \rceil \leq \mu^*(nK + nB) + G_n.$$

The Cartier divisor G_n is effective and μ-exceptional, since B is effective. In particular,

$$\mu_*\mathcal{O}_{X'}(\mu^*(nK + nB) + G_n) = \mathcal{O}_X(nK + nB).$$

Therefore we obtain inclusions

$$\mu_*\mathcal{O}_{X'}(M_n) \subseteq \mu_*\mathcal{O}_{X'}(\lceil K_{X'} + R + \frac{n-1}{n}M_n \rceil) \subseteq \mathcal{O}_X(nK + nB).$$

Since $\pi_*\mu_*\mathcal{O}_{X'}(M_n) = \pi_*\mathcal{O}_X(nK + nB)$, we obtain

$$\pi_*\mathcal{O}_X(nK + nB) = \pi_*\mu_*\mathcal{O}_{X'}(\lceil K_{X'} + R + \frac{n-1}{n}M_n \rceil).$$

(2b) Assume now that $\{nB\} \leq B$ and $i \in n\mathbb{N}$. We have inclusions

$$\begin{aligned} \pi_*\mathcal{O}_X(\lceil \mathbf{K} + \mathbf{R} + (n-1)\mathbf{D}_i(K+B) \rceil) &\subseteq \pi_*\mathcal{O}_X(\lceil \mathbf{K} + \mathbf{R} + (n-1)\overline{K+B} \rceil) \\ &\subseteq \pi_*\mathcal{O}_X(\lceil nK + \lfloor B \rfloor + (n-1)B \rceil) \\ &\subseteq \pi_*\mathcal{O}_X(nK + nB), \end{aligned}$$

where the first inclusion holds by $\mathbf{D}_i(K+B) \leq \overline{K+B}$, and the third by Lemma 1.1. By (2a), all inclusions are equalities. □

1.C. Zariski decomposition

We refer the reader to Nakayama [13] for an excellent introduction to the Zariski decomposition problem. Several higher dimensional versions of the two-dimensional Zariski decomposition have been proposed, but they all coincide for big divisors.

Consider a proper surjective morphism $\pi\colon X \to S$ and a π-big $\mathbb{R}$-divisor D on X. We say that D has a Zariski decomposition, relative to S, if there exists a birational contraction $\mu\colon X' \to X$ and a $(\pi \circ \mu)$-nef and $(\pi \circ \mu)$-big $\mathbb{R}$-divisor P on X' such that

(i) $P \le \mu^* D$;

(ii) $\mathcal{R}_{X'/S}(P) = \mathcal{R}_{X/S}(D)$.

LEMMA 1.6. *Let (X, B) be a log variety with log canonical singularities, and $\pi\colon X \to S$ be a proper surjective morphism such that $K + B$ is π-big. Assume the log Minimal Model Program (with real boundaries) is valid in dimension* $\dim(X)$.

Then $K + B$ has a Zariski decomposition, and P is the pullback to a suitable model of the log canonical class of the log minimal model of (X, B).

PROOF. If log Minimal Model Program holds for $(X/S, B)$, we obtain a birational map to a log minimal model over S

$$\Phi\colon (X, B) \dashrightarrow (X', B').$$

If we consider the normalization of the graph of Φ, this means that we have a Hironaka hut

$$\begin{tikzcd} & X'' \arrow[dl, "\mu"'] \arrow[dr, "\mu'"] & \\ X \arrow[rr, dotted, "\Phi"] & & X' \end{tikzcd}$$

with the following properties:

(i) (X', B') is a log variety with log canonical singularities and $K_{X'} + B'$ is relatively nef and big;

(ii) $F = \mu^*(K + B) - \mu'^*(K_{X'} + B')$ is effective and μ'-exceptional.

Denote $P = \mu'^*(K_{X'} + B')$. By (ii), we have $\mathcal{R}_{X/S}(K + B) = \mathcal{R}_{X''/S}(P)$. Therefore $P \le \mu^*(K + B)$ is a Zariski decomposition. $\square$

REMARK 1.7 (Real logMMP). The largest category in which logMMP is expected to work is that of relative log varieties $(X/S, B)$ with log canonical singularities. The Cone and Contraction Theorems are known ([8, 1]).

The existence of extremal flips is numerical, hence it follows from the existence of flips with rational boundary. The termination of log flips is known in dimension 3 (Shokurov [14]) and is open in dimension at least 4 (see Shokurov [16] for more on termination).

2. The Boundary of the Induced Log Canonical Algebra

Let (X,B) be a log pair with log canonical singularities and $\pi\colon X \to S$ a proper morphism. Assume that Y is a normal prime divisor in X with $\operatorname{mult}_Y(B) = 1$, and there exists a positive integer l such that $lK + lB$ is π-mobile at Y.

By codimension one adjunction, there exists a canonically defined log pair structure (Y, B_Y) on Y such that

$$(K+B)|_Y = K_Y + B_Y.$$

For every $i \in l\mathbb{N}$, let $\mathbf{F}\mathrm{ix}(iK+iB)$ be the fixed $\mathbb{R}$-b-divisor of $iK+iB$ relative to S. By assumption, $\mathbf{F}\mathrm{ix}(iK+iB)$ is b-$\mathbb{R}$-Cartier and it has multiplicity zero at Y. Therefore its restriction $\mathbf{F}\mathrm{ix}(iK+iB)|_Y$ is a well defined b-$\mathbb{R}$-Cartier $\mathbb{R}$-b-divisor of Y. Define

$$\mathbf{\Theta} = \sum_E \max(1 - a(E;Y,B_Y) - \lim_{i\to\infty} \operatorname{mult}_E(\mathbf{F}\mathrm{ix}(iK+iB)|_Y), 0)E,$$

where the sum runs after all prime b-divisors E of Y and $a(E;Y,B_Y)$ denotes the log discrepancy of E with respect to (Y,B_Y). It is clear that $\mathbf{\Theta}$ is supported by the prime b-divisors E with $a(E;Y,B_Y) \in [0,1)$. In particular, $\mathbf{\Theta}$ is a well defined effective $\mathbb{R}$-b-divisor of Y.

Lemma 2.1. *Let $\mu\colon X' \to X$ be a birational contraction and $\mu^*(K+B) = K_{X'} + B_{X'}$ the log pullback.*

(i) *The log pairs (X,B) and $(X',B_{X'})$ induce the same $\mathbf{\Theta}$.*

(ii) *Assume that $B \geq 0$ and $K_{X'} + B'$ is $\mathbb{R}$-Cartier, where $B' = \max(B_{X'}, 0)$. Then the log varieties (X,B) and (X',B') induce the same $\mathbf{\Theta}$.*

Proof. Property (i) is clear. For (ii), we have

$$K_{X'} + B' = \mu^*(K+B) + F,$$

where F is effective and μ-exceptional. In particular,

$$\pi_*\mathcal{O}_X(iK+iB)=(\pi\circ\mu)_*\mathcal{O}_{X'}(iK_{X'}+iB'),\forall i\geq 1,$$

$$\mathbf{F}\mathrm{ix}(iK+iB)=\mathbf{F}\mathrm{ix}(iK_{X'}+iB')+i\overline{F},\forall i\geq 1,$$

and $a(E;Y,B_Y)=a(E;X',B'_Y)+\mathrm{mult}_E(\overline{F|_{Y'}})$ for every prime b-divisor E of Y. The claim follows from the definition of $\mathbf{\Theta}$. □

LEMMA 2.2. *Let $\nu\colon Y''\to Y'$ be a birational contraction of birational models of Y. Then*

$$\nu_*\mathcal{O}_{Y''}(nK_{Y''}+n\mathbf{\Theta}_{Y''})\subseteq\mathcal{O}_{Y'}(nK_{Y'}+n\mathbf{\Theta}_{Y'}),n\geq 1.$$

Equality holds if $(Y',\mathbf{\Theta}_{Y'})$ is a log variety with canonical singularities in codimension at least two.

PROOF. The inclusions are clear, since $K_{Y'}+\mathbf{\Theta}_{Y'}=\nu_*(K_{Y''}+\mathbf{\Theta}_{Y''})$. For the second claim, there exists a ν-exceptional $\mathbb{R}$-divisor E such that

$$K_{Y''}+\mathbf{\Theta}_{Y''}=\nu^*(K_{Y'}+\mathbf{\Theta}_{Y'})+E.$$

If $\mathrm{mult}_P(E)<0$ for some prime divisor P on Y'', then $a(P;Y',\mathbf{\Theta}_{Y'})<1$ since $\mathbf{\Theta}_{Y''}$ is effective. Since P is ν-exceptional, we infer that $(Y',\mathbf{\Theta}_{Y'})$ does not have canonical singularities in codimension at least two. Contradiction! Therefore E is effective and ν-exceptional, which implies the claim. □

LEMMA 2.3. *Let $\nu\colon Y'\to Y$ be a birational contraction. Then*

(1) *For every $n\geq 1$, we have natural inclusions*

$$\begin{aligned}&\mathrm{Im}(\pi_*\mathcal{O}_X(nK+nB)\to\pi_*\mathcal{O}_Y(nK_Y+nB_Y))\\ &\subseteq\pi_*\nu_*\mathcal{O}_{Y'}(nK_{Y'}+n\mathbf{\Theta}_{Y'}).\end{aligned}$$

(2) *The $\mathbb{R}$-divisor $\lim_{i\to\infty}\frac{\mathrm{Fix}(iK_{Y'}+i\mathbf{\Theta}_{Y'})}{i}$ has zero multiplicity at each component of $\mathbf{\Theta}_{Y'}$.*

PROOF. By Lemma 2.2, we may assume that $Y'=Y$. Let $i\in l\mathbb{N}$, so that $iK+iB$ is relatively mobile at Y. By Hironaka, there exists a resolution $\mu_i\colon X_i\to X$ with the following properties:

(i) Let $\mu^*(iK+iB)=M_i+F_i$ is the mobile-fixed decomposition relative to S. Then M_i is relatively free.

(ii) Let $\mu^*(K+B)=K_{X_i}+B_{X_i}$ be the log pullback. Then $\operatorname{Supp}(B_{X_i})\cup\operatorname{Supp}(F_i)$ is a simple normal crossings divisor.

Let Y_i be the proper transform of Y on X_i. Let $\pi_i=\pi\circ\mu$. We have

$$\pi_*\mathcal{O}_X(nK+nB)=\pi_{i*}\mathcal{O}_{X_i}(nK_{X_i}+nB_{X_i}).$$

(1) Fix n and choose $i\in ln\mathbb{N}$. We have

$$\pi_{i*}\mathcal{O}_{X_i}(nK_{X_i}+nB_{X_i}-\frac{n}{i}F_i)=\pi_{i*}\mathcal{O}_{X_i}(nK_{X_i}+nB_{X_i}).$$

Therefore the restriction to Y_i of the right hand side is included in

$$\pi_{i*}\mathcal{O}_{Y_i}(nK_{Y_i}+n(B_{X_i}-Y_i)|_{Y_i}-\frac{n}{i}F_i|_{Y_i}).$$

In turn, this sheaf is included in

$$\pi_*\mathcal{O}_Y(nK_Y+nB_Y-(\mu|_{Y_i})_*(\frac{n}{i}F_i|_{Y_i})).$$

But $B_Y-(\mu|_{Y_i})_*(\frac{F_i}{i}|_{Y_i})\le\mathbf{\Theta}_Y$, hence the claim holds.

(2) For $i\in l\mathbb{N}$, denote $\tilde{B}_i=\max(B_Y-(\mu|_{Y_i})_*(\frac{F_i}{i}|_{Y_i}),0)$. We claim that $iK_Y+i\tilde{B}_i$ is relatively mobile at the components of $\tilde{B}_i$.

Indeed, let $B_i=\max(B_{X_i}-\frac{F_i}{i},0)$. By construction, the mobile part of $iK_{X_i}+iB_i$ is M_i and the fixed part $F_i'=iK_{X_i}+iB_i-M_i$ has no components in common with B_i. Since $\operatorname{Supp}(F_i')\cup\operatorname{Supp}(B_i)$ is a simple normal crossings divisor, we infer that F_i' does not contain any intersection of the components of B_i. In particular, $iK_{Y_i}+i(B_i-Y_i)|_{Y_i}$ is relatively mobile at each component of $(B_i-Y_i)|_Y$. But $iK_Y+i\tilde{B}_i=\nu_*(iK_{Y_i}+i(B_i-Y_i)|_{Y_i})$, hence $iK_Y+i\tilde{B}_i$ is relatively mobile at the components of $\tilde{B}_i$.

By convexity, $\tilde{B}_i\le\mathbf{\Theta}_Y$ and $\lim_{i\to\infty}\tilde{B}_i=\mathbf{\Theta}_Y$. Therefore the components of $\tilde{B}_i$ and $\mathbf{\Theta}_Y$ coincide for $i\gg 1$. We have $i(K_Y+\mathbf{\Theta}_Y)=i(K_Y+\tilde{B}_i)+i(\mathbf{\Theta}_Y-\tilde{B}_i)$. Therefore $\operatorname{Fix}(iK_Y+i\mathbf{\Theta}_Y)\le i(\mathbf{\Theta}_Y-\tilde{B}_i)$ at each component of $\mathbf{\Theta}_Y$. Dividing by i and taking the limit, we obtain the claim. $\square$

Lemma 2.4. *In the above notations, assume that (X,B) is a plt log variety with unique log canonical centre Y, and $K+B\sim_{\mathbb{Q}}A+C$, where*

A is a relatively ample $\mathbb{R}$-divisor and C is an effective $\mathbb{R}$-divisor such that $\mathrm{mult}_Y(A) = \mathrm{mult}_Y(C) = 0$.
If the (real) log Minimal Model Program holds in dimension $\dim(X)$*, there exists a birational contraction $\nu\colon \tilde{Y} \to Y$ such that*

$$\mathrm{Im}(\pi_*\mathcal{O}_X(nK + nB) \to \pi_*\mathcal{O}_Y(nK_Y + nB_Y)) = \pi_*\nu_*\mathcal{O}_{\tilde{Y}}(nK_{\tilde{Y}} + n\mathbf{\Theta}_{\tilde{Y}})$$

for every $n \geq 2$ such that $\{nB\} \leq B$. Moreover, $\mathbf{\Theta}$ is rational if B is rational.

Proof. If the real logMMP holds, (X, B) has a log minimal model

$$\Phi\colon (X, B) \dashrightarrow (X', B').$$

If we consider a resolution of the graph of Φ, this means that we have a Hironaka hut

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\mu"'] \arrow[dr, "\mu'"] & \\ X \arrow[rr, dotted, "\Phi"] & & X' \end{tikzcd}$$

such that $K_{X'} + B'$ is relatively nef and big and the $\mathbb{R}$-Cartier divisor $F = \mu^*(K + B) - \mu'^*(K_{X'} + B')$ is effective and μ'-exceptional. Denote $P = \mu'^*(K_{X'} + B')$. Then $\mu^*(K + B) = P + F$ is a Zariski decomposition:

(i) $P \leq \mu^*(K + B)$ is relatively nef and big;

(ii) $\mathcal{R}_{\tilde{X}/S}(P) = \mathcal{R}_{X/S}(K + B)$.

We may assume that $\tilde{Y}$, the proper transform of Y on $\tilde{X}$, is normal. Since $lK + lB$ is relatively mobile at Y, we have $\mathrm{mult}_{\tilde{Y}}(F) = 0$. We have

$$P|_{\tilde{Y}} = K_{\tilde{Y}} + B_{\tilde{Y}} - F|_{\tilde{Y}}.$$

By definition, $\mathbf{\Theta}_{\tilde{Y}} = \max(B_{\tilde{Y}} - F|_{\tilde{Y}}, 0)$. In particular, $P|_{\tilde{Y}} \leq K_{\tilde{Y}} + \mathbf{\Theta}_{\tilde{Y}}$. We claim that this is a Zariski decomposition:

(iii) $P|_{\tilde{Y}} \leq K_{\tilde{Y}} + \mathbf{\Theta}_{\tilde{Y}}$ is relatively nef and big;

(iv) $\mathcal{R}_{\tilde{Y}/S}(P|_{\tilde{Y}}) = \mathcal{R}_{\tilde{Y}/S}(K_{\tilde{Y}} + \mathbf{\Theta}_{\tilde{Y}})$.

Indeed, by assumption Y is mapped birationally by Φ to a prime divisor Y'. Since (X', B') is also plt, Y' is normal. Let $(Y', B'_{Y'})$ be the log variety structure induced by codimension one adjunction. By assumption, $K_{Y'} + B'_{Y'}$ is relatively nef and big. By adjunction, we have an induced Hironaka hut

$$\begin{tikzcd} & \tilde{Y} \arrow[dl, "\nu"'] \arrow[dr, "\nu'"] & \\ Y \arrow[rr, dashed, "\Phi|_Y"] & & Y'. \end{tikzcd}$$

We have $\nu'^*(K_{Y'} + B'_{Y'}) = K_{\tilde{Y}} + B_{\tilde{Y}} - F|_{\tilde{Y}}$. Since $B'_{Y'}$ is effective, the negative part of $B_{\tilde{Y}} - F|_{\tilde{Y}}$ is ν'-exceptional. Therefore $P|_{\tilde{Y}} \le K_{\tilde{Y}} + \mathbf{\Theta}_{\tilde{Y}}$. is a Zariski decomposition.

Let $n \ge 2$ be an integer and let $i \in nl\mathbb{N}$. We have a commutative diagram

$$\begin{tikzcd} \pi_*\mathcal{O}_X(nK + nB) \arrow[r] \arrow[d] & \pi_*\mathcal{O}_X(\lceil \mathbf{K} + Y + (n-1)\mathbf{D}_i(K+B) \rceil) \arrow[d] \\ \pi_*\mathcal{O}_{\tilde{Y}}(nK_{\tilde{Y}} + n\mathbf{\Theta}_{\tilde{Y}}) \arrow[r] & \pi_*\mathcal{O}_Y(\lceil \mathbf{K} + (n-1)\mathbf{D}_i(K_{\tilde{Y}} + \mathbf{\Theta}_{\tilde{Y}})) \rceil) \end{tikzcd}$$

The horizontal arrows are inclusions. For i sufficiently large and divisible, the right hand side vertical arrow becomes

$$\pi_*\mathcal{O}_X(\lceil \mathbf{K} + Y + (n-1)\overline{P} \rceil) \to \pi_*\mathcal{O}_{\tilde{Y}}(\lceil \mathbf{K} + (n-1)\overline{P|_{\tilde{Y}}} \rceil),$$

which is surjective by Kawamata-Viehweg vanishing. Assume moreover that $\{nB\} \le B$. Then the top horizontal arrow is an equality, by Lemma 1.5. Therefore the bottom horizontal arrow is an equality and the left hand side vertical arrow is surjective.

Finally, assume that B is rational. Then B' is rational, hence F is rational. Therefore $\mathbf{\Theta}$ is rational. $\square$

3. Proof of Theorem 0.1

First of all, by Lemmas 2.1 and 2.2, we may assume that the prime components of B_Y are disjoint. Note that assumption (a) is birational in nature (see [12], Lemma 4.8). Denote $\Theta = \mathbf{\Theta}_Y$.

Let H be a π-very ample divisor with $\operatorname{mult}_Y(H) = 0$ and let $A = \dim(X) \cdot H$. There exists a positive integer l such that $l(K+B) \sim A + C$, where C is an effective $\mathbb{R}$-divisor with $\operatorname{mult}_Y(C) = 0$. For $n \geq 1$, define

$$\begin{aligned}
&B_n = \max(B, \tfrac{1}{n}\lceil (n-1)B \rceil) \\
&B_{n,Y} = (B_n - Y)|_Y \\
&\Theta_n = \max(B_{n,Y} - \lim_{i\to\infty} \tfrac{(\mathbf{F}\mathrm{ix}(i(K+B_n+\frac{1}{n}A))|_Y)_Y}{i}, 0) \\
&C_n = C + l(B_n - B)
\end{aligned}$$

Note that (X, B_n) satisfies the same properties as (X, B).

LEMMA 3.1. *The following properties hold:*

(1) $Y + \lceil (n-1)B_{n-1} \rceil \leq nB_n$ *for* $n \geq 1$.

(2) $\Theta \leq \Theta_n \leq B_{n,Y}$.

(3) $\mathbf{D}_i(nK + nB_n + A)|_Y \leq \mathbf{D}_i(nK_Y + n\Theta_n + A|_Y)$ *for* $i \in l\mathbb{N}$.

PROOF. (1) This follows from Lemma 1.1.

(2) The inequality $\Theta_n \leq B_{n,Y}$ holds by construction. We have

$$K + B_n + \frac{1}{n}A = (K+B) + (B_n - B) + \frac{1}{n}A.$$

Since $B_n - B$ is effective and A is π-free, we obtain

$$\mathbf{F}\mathrm{ix}(i(K + B_n + \frac{1}{n}A)) \leq \mathbf{F}\mathrm{ix}(i(K+B)) + i\overline{(B_n - B)} \ \forall i \in ln\mathbb{N},$$

which is equivalent to

$$\overline{B - Y} - \frac{\mathbf{F}\mathrm{ix}(i(K+B))}{i} \leq \overline{B_n - Y} - \frac{\mathbf{F}\mathrm{ix}(i(K+B_n+\frac{1}{n}A))}{i}.$$

Restricting these $\mathbb{R}$-Cartier $\mathbb{R}$-b-divisors to Y and taking the trace on Y we obtain

$$B_Y - \frac{\mathbf{F}\mathrm{ix}(i(K+B))}{i}|_Y \leq B_{n,Y} - \frac{\mathbf{F}\mathrm{ix}(i(K+B_n+\frac{1}{n}A))}{i}|_Y.$$

Taking the effective part and the limit with respect to i, we obtain $\Theta \leq \Theta_n$.

(3) Let $i \in ln\mathbb{N}$. By definition, the mobile b-divisor of $i(K+B_n+\frac{1}{n}A)$ coincides with the mobile b-divisor of

$$\overline{i(K+B_n+\frac{1}{n}A)}-\mathbf{F}\mathrm{ix}(i(K+B_n+\frac{1}{n}A)).$$

The restriction of this $\mathbb{R}$-Cartier $\mathbb{R}$-b-divisor to Y is

$$\overline{i(K_Y+B_{n,Y}+\frac{1}{n}A|_Y)}-\mathbf{F}\mathrm{ix}(i(K+B_n+\frac{1}{n}A))|_Y,$$

whose mobile b-divisor is at most the mobile b-divisor of

$$i(K_Y+B_{n,Y}+\frac{1}{n}A|_Y)-(\mathbf{F}\mathrm{ix}(i(K+B_n+\frac{1}{n}A))|_Y)_Y.$$

By the definition of Θ_n and the convexity of the sequence of fixed parts, we have

$$B_{n,Y}-\frac{(\mathbf{F}\mathrm{ix}(i(K+B_n+\frac{1}{n}A))|_Y)_Y}{i}\le \Theta_n.$$

Therefore the restriction to Y of the mobile b-divisor of $i(K+B_n+\frac{1}{n}A)$ is at most the mobile b-divisor of $i(K_Y+\Theta_n)$. We obtain the claim by dividing by i. $\square$

Proof of Theorem (0.1)(1).

Step 1. For every $n\ge 0$, there exists $i_n\in l\mathbb{N}$ such that for every $i\in i_n\mathbb{N}$ the following inclusion holds

$$\mathcal{O}_Y(\lceil \mathbf{K}+n\mathbf{D}_i(K_Y+\Theta)+\overline{A|_Y}\rceil)\subseteq \mathcal{O}_Y(\lceil \mathbf{K}+\mathbf{D}_i(n(K+B_n)+A)|_Y\rceil).$$

For $n=0$, set $i_0=l$. Since A is free, we have $\mathbf{D}_i(A)=\overline{A}$ for every i. Therefore the inclusion is an equality.

Let now $n\ge 1$ and assume the inclusion holds for $n-1$, with corresponding index i_{n-1}. Let i_n be the smallest positive integer j satisfying the following properties:

(i) $i_{n-1}|j$,

(ii) $\mathbf{D}_j((n-1)(K+B_{n-1})+A)$ is b-big,

(iii) $\lceil n(K_Y+\Theta_n)+A|_Y\rceil \le \lceil \mathbf{D}_j(n(K+B_n)+A)|_Y\rceil$ at every prime component of Θ.

The existence of a solution for (iii) follows from the inequality $\Theta \le \Theta_n$ and the definition of Θ_n. Let $i \in i_n\mathbb{N}$. Let R be the reduced divisor supported by $B - Y$ with $R|_Y = \lceil \Theta \rceil$. The following inclusions hold:

$$\begin{aligned}
&\pi_*\mathcal{O}_X(\lceil \mathbf{K} + Y + \mathbf{D}_i((n-1)(K + B_{n-1}) + \overline{A})\rceil) \\
&\subseteq \pi_*\mathcal{O}_X(\lceil K + Y + (n-1)(K + B_{n-1}) + A\rceil) \\
&\subseteq \pi_*\mathcal{O}_X(n(K + B_n) + A) \\
&\subseteq \pi_*\mathcal{O}_X(\lceil \mathbf{A}(X, Y + R)^* + \mathbf{D}_i(n(K + B_n) + A)\rceil)
\end{aligned}$$

Indeed, the first inclusion is clear, the second holds by Lemma 3.1.(1), and the third follows from the inequality $\lceil \mathbf{A}(X, Y + R)^* \rceil \ge 0$. By Kawamata-Viehweg vanishing, the restriction to Y of the first sheaf is $\pi_*\mathcal{O}_Y(\lceil \mathbf{K} + \mathbf{D}_i((n-1)(K + B_{n-1}) + A)|_Y\rceil)$. Since $\mathbf{A}(X, Y + R)^*|_Y = \mathbf{A}(Y, \lceil \Theta \rceil)^*$, we obtain by restricting to Y

$$\begin{aligned}
&\pi_*\mathcal{O}_Y(\lceil \mathbf{K} + \mathbf{D}_i((n-1)(K + B_{n-1}) + A)|_Y\rceil) \\
&\subseteq \pi_*\mathcal{O}_Y(\lceil \mathbf{A}(Y, \lceil \Theta \rceil)^* + \mathbf{D}_i(n(K + B_n) + A)|_Y\rceil).
\end{aligned}$$

Combining this with Step 1 for $n - 1$, we obtain

$$\begin{aligned}
&\pi_*\mathcal{O}_Y(\lceil \mathbf{K} + (n-1)\mathbf{D}_i(K_Y + \Theta) + \overline{A|_Y}\rceil) \\
&\subseteq \pi_*\mathcal{O}_Y(\lceil \mathbf{A}(Y, \lceil \Theta \rceil)^* + \mathbf{D}_i(n(K + B_n) + A)|_Y\rceil).
\end{aligned}$$

Since $\mathcal{O}_Y(\lceil \mathbf{K} + (n-1)\mathbf{D}_i(K_Y + \Theta) + \overline{A|_Y}\rceil)$ is $\pi|_Y$-generated ([12], Lemma 3.9), this is equivalent to

$$\begin{aligned}
&\mathcal{O}_Y(\lceil \mathbf{K} + (n-1)\mathbf{D}_i(K_Y + \Theta) + \overline{A|_Y}\rceil) \\
&\subseteq \mathcal{O}_Y(\lceil \mathbf{A}(Y, \lceil \Theta \rceil)^* + \mathbf{D}_i(n(K + B_n) + A)|_Y\rceil).
\end{aligned}$$

In particular, we obtain inclusions

$$\begin{aligned}
&\mathcal{O}_Y(\lceil \mathbf{K} + n\mathbf{D}_i(K_Y + \Theta) + \overline{A|_Y}\rceil) \\
&\subseteq \mathcal{O}_Y(\lceil \mathbf{K} + (n-1)\mathbf{D}_i(K_Y + \Theta) + \overline{A|_Y}\rceil + \overline{K_Y + \lceil \Theta \rceil}) \\
&\subseteq \mathcal{O}_Y(\lceil \mathbf{A}(Y, \lceil \Theta \rceil)^* + \overline{K_Y + \lceil \Theta \rceil} + \mathbf{D}_i(n(K + B_n) + A)|_Y\rceil) \\
&= \mathcal{O}_Y(\lceil \mathbf{K} + \lceil \Theta \rceil + \mathbf{D}_i(n(K + B_n) + A)|_Y\rceil).
\end{aligned}$$

Let $U \subset Y$ be an open set and let a be a nonzero rational function on Y such that

$$\overline{(a)} + \lceil \mathbf{K} + n\mathbf{D}_i(K_Y + \Theta) + \overline{A|_Y}\rceil|_U \ge 0.$$

We can write $\overline{(a)} + \lceil \mathbf{K} + \mathbf{D}_i(n(K + B_n) + A)|_Y \rceil|_U = \mathbf{E}^+ - \mathbf{E}^-$, where $\mathbf{E}^+, \mathbf{E}^-$ are effective b-divisors of U with no common components. From the inclusions above, we have $\lceil \Theta \rceil + \mathbf{E}^+ - \mathbf{E}^- \geq 0$. Therefore $\mathbf{E}^- \leq \lceil \Theta \rceil$. But $\mathbf{E}^-$ has zero multiplicity at the components of Θ, by (iii). Therefore $\mathbf{E}^- = 0$, that is

$$\overline{(a)} + \lceil \mathbf{K} + \mathbf{D}_i(n(K + B_n) + A)|_Y \rceil|_U \geq 0.$$

This shows that

$$\mathcal{O}_Y(\lceil \mathbf{K} + n\mathbf{D}(K_Y + \Theta) + \overline{A|_Y} \rceil) \subseteq \mathcal{O}_Y(\lceil \mathbf{K} + \mathbf{D}(n(K + B_n) + A)|_Y \rceil),$$

which is the desired inclusion for n.

Step 2. For $n \geq 1$ and $i \in i_n\mathbb{N}$ we have

$$\begin{aligned} &\mathcal{O}_Y(\lceil \mathbf{K} + (n-1)\mathbf{D}_i(K_Y + \Theta) + \overline{A|_Y} \rceil) \\ &\subseteq \mathcal{O}_Y(\lceil \mathbf{A}(Y, B_{n,Y}) + \mathbf{D}_i(n(K + B_n) + A)|_Y \rceil). \end{aligned}$$

To see this, note first the inclusions

$$\begin{aligned} &\pi_*\mathcal{O}_X(\lceil \mathbf{K} + Y + \mathbf{D}_i((n-1)(K + B_{n-1}) + \overline{A}) \rceil) \\ &\subseteq \pi_*\mathcal{O}_X(\lceil K + Y + (n-1)(K + B_{n-1}) + A \rceil) \\ &\subseteq \pi_*\mathcal{O}_X(n(K + B_n) + A) \\ &\subseteq \pi_*\mathcal{O}_X(\lceil \mathbf{A}(X, B_n)^* + \mathbf{D}_i(n(K + B_n) + A) \rceil). \end{aligned}$$

The argument is the same as in Step 1, except that for the last inclusion we use $\lceil \mathbf{A}(X, B_n)^* \rceil \geq 0$. By Kawamata-Viehweg vanishing, the restriction to Y of the first sheaf is $\pi_*\mathcal{O}_Y(\lceil \mathbf{K} + (n-1)\mathbf{D}_i(K_Y + \Theta) + \overline{A|_Y} \rceil)$. Since $\mathbf{A}(X, B_n)^*|_Y = \mathbf{A}(Y, B_{n,Y})^*$, we obtain

$$\begin{aligned} &\pi_*\mathcal{O}_Y(\lceil \mathbf{K} + \mathbf{D}_i((n-1)(K + B_{n-1}) + A)|_Y \rceil) \\ &\subseteq \pi_*\mathcal{O}_Y(\lceil \mathbf{A}(Y, B_{n,Y}) + \mathbf{D}_i(n(K + B_n) + A)|_Y \rceil) \end{aligned}$$

Combining this with Step 1, we obtain

$$\begin{aligned} &\pi_*\mathcal{O}_Y(\lceil \mathbf{K} + (n-1)\mathbf{D}_i(K_Y + \Theta) + \overline{A|_Y} \rceil) \\ &\subseteq \pi_*\mathcal{O}_Y(\lceil \mathbf{A}(Y, B_{n,Y}) + \mathbf{D}_i(n(K + B_n) + A)|_Y \rceil) \end{aligned}$$

Since $\mathcal{O}_Y(\lceil \mathbf{K} + (n-1)\mathbf{D}_i(K_Y + \Theta) + \overline{A|_Y}\rceil)$ is $\pi|_Y$-generated ([12], Lemma 3.9), this is equivalent to the claim.

Step 3. For $n \geq 1$ and $i \in i_n\mathbb{N}$ we have

$$\mathcal{O}_Y(\lceil \mathbf{K}+(n-1)\mathbf{D}_i(K_Y+\Theta)\rceil) \subseteq \mathcal{O}_Y(\lceil \mathbf{A}(Y, B_{n,Y})+\overline{C_n|_Y}+n\mathbf{D}_i(K+B_n)|_Y\rceil).$$

Indeed, $l(K + B_n) \sim A + C_n$. Since $l \geq 1$ and A is π-free, we obtain

$$\mathbf{D}_i(n(K + B_n) + A) \leq n\mathbf{D}_i(K + B_n) + \overline{A + C_n}.$$

We obtain the claim by restricting to Y, using Step 2 and canceling $A|_Y$.

Step 4. For $n \geq 1$, there exists $i'_n \in i_n\mathbb{N}$ such that for every $i \in i'_n\mathbb{N}$ we have

$$\pi_*\mathcal{O}_Y(n(K_Y + \Theta)) \subseteq \pi_*\mathcal{O}_Y(\lceil \mathbf{A}(Y, B_{n,Y}) + n\mathbf{D}_i(K + B_n)|_Y\rceil).$$

Fix $n \geq 1$. There exists $\gamma_n > 0$ such that the log variety

$$(Y, B_{n,Y} + \gamma(C|_Y + (\frac{l}{n} + 1)\lceil B_Y\rceil))$$

has Kawamata log terminal singularities. By diophantine approximation [4], there exists an integer $e \geq 1 + \gamma^{-1}$ such that $B_{ne} \leq B_n$. Define $i'_n = i_{ne}$. Let $\mathbf{M}_n$ be the mobile b-divisor of $\pi_*\mathcal{O}_Y(nK_Y + n\Theta)$. By Step 3, for every $i \in i'_n$ we have

$$\mathbf{M}_n \leq \frac{1}{e}\mathbf{M}_{ne} \leq \frac{1}{e}\lceil \mathbf{A}(Y, B_{ne,Y}) + \overline{C_n|_Y} + ne\mathbf{D}_i(K + B_{ne})|_Y\rceil.$$

Since $\mathbf{M}_n$ has integer coefficients, this is equivalent to

$$\mathbf{M}_n \leq \lfloor \frac{1}{e}\lceil \mathbf{A}(Y, B_{n,Y}) + \overline{(B_n - B_{ne} + C_n)|_Y} + ne\mathbf{D}_i(K + B_{ne})|_Y\rceil\rfloor$$

By the choice of γ and Lemma 1.1.(2), we have

$$\lceil \mathbf{A}(Y, B_{n,Y}) - \gamma\overline{(B_n - B_{ne} + C_n)|_Y}\rceil \geq 0.$$

Therefore we may apply Lemma 1.3 at each prime b-divisor of Y and obtain

$$\mathbf{M}_n \leq \lceil \mathbf{A}(Y, B_{n,Y}) + n\mathbf{D}_i(K + B_{ne})|_Y\rceil.$$

We have $\mathbf{D}_i(K+B_{ne}) \le \mathbf{D}_i(K+B_n)$ since $B_{ne} \le B_n$, hence we get the claim.

Step 5. For $n \ge 2$ we have

$$\pi_*\mathcal{O}_Y(nK_Y + n\Theta) \subseteq \mathrm{Im}(\pi_*\mathcal{O}_X(nK+nB_n) \to \pi_*\mathcal{O}_Y(nK_Y + nB_{n,Y})).$$

Indeed, the following inequality holds:

$$\mathbf{A}(Y, B_{n,Y}) + n\mathbf{D}_i(K+B_n)|_Y \le \mathbf{K} + (n-1)\mathbf{D}_i(K+B_n)|_Y.$$

By Step 4, for $n \ge 1$ and $i \gg 1$ we have

$$\pi_*\mathcal{O}_Y(nK_Y + n\Theta) \subseteq \pi_*\mathcal{O}_Y(\lceil \mathbf{K} + (n-1)\mathbf{D}_i(K+B_n)|_Y \rceil).$$

By Kawamata-Viehweg vanishing, $\pi_*\mathcal{O}_Y(\lceil \mathbf{K}+(n-1)\mathbf{D}_i(K+B_n)|_Y)$ lifts to $\pi_*\mathcal{O}_X(\lceil \mathbf{K}+Y+(n-1)\mathbf{D}_i(K+B_{n-1})\rceil)$, which is included in $\pi_*\mathcal{O}_X(\lceil K+Y+(n-1)(K+B_{n-1})\rceil)$. By Lemma 1.1.(3), this is included in $\pi_*\mathcal{O}_X(nK+nB_n)$. This proves the claim.

Step 6. Let $n \ge 2$ such that $\{nB\} \le B$. Then

$$\pi_*\mathcal{O}_Y(nK_Y + n\Theta) = \mathrm{Im}(\pi_*\mathcal{O}_X(nK+nB) \to \pi_*\mathcal{O}_Y(nK_Y + nB_Y)).$$

The second assumption means $B_n = B$, hence Step 5 gives the direct inclusion. The opposite inclusion is clear.

Step 7. Assume that $n = 1$ and $\pi(Y) \ne \pi(X)$. Then

$$\mathrm{Im}(\pi_*\mathcal{O}_X(K+B) \to \pi_*\mathcal{O}_Y(K_Y + B_Y)) = \pi_*\mathcal{O}_Y(K_Y + \Theta).$$

Indeed, Steps 5 and 6 hold for $n = 1$ as well, by Kollár's torsion freeness rather than Kawamata-Viehweg vanishing. $\square$

Proposition 3.2 (Generalized asymptotic saturation). *Let $n \ge 2$ such that $\{nB\} \le B$ and let $i \ge 1$ such that $\pi_*\mathcal{O}_X(iK+iB) \ne 0$. Then*

$$\pi_*\mathcal{O}_Y(\lceil \mathbf{K} + (n-1)\mathbf{D}_i(K_Y+\Theta)\rceil) \subseteq \pi_*\mathcal{O}_Y(nK_Y + n\Theta).$$

PROOF. The log canonical divisor $K_Y + \Theta$ is relatively big, by the assumption (a). By diophantine approximation [4], there exists an integer $i' \geq 2$ such that $i|i'$, $\{i'B\} \leq B$ and $\mathbf{D}_{i'}(K+B)$ is relatively b-nef and b-big. By Theorem 0.1.(1), we have $\mathbf{D}_{i'}(K+B)|_Y = \mathbf{D}_{i'}(K_Y+\Theta)$. In particular,

$$\pi_*\mathcal{O}_Y(\lceil \mathbf{K} + (n-1)\mathbf{D}_{i'}(K_Y+\Theta)\rceil) = \pi_*\mathcal{O}_Y(\lceil \mathbf{K} + (n-1)\mathbf{D}_{i'}(K+B)|_Y\rceil).$$

Since $\mathbf{D}_{i'}(K+B)$ is relatively b-nef and b-big, the right hand side lifts to $\pi_*\mathcal{O}_Y(\lceil \mathbf{K} + Y + (n-1)\mathbf{D}_{i'}(K+B)\rceil)$, by Kawamata-Viehweg vanishing. Since $\{nB\} \leq B$, we have

$$\pi_*\mathcal{O}_Y(\lceil \mathbf{K} + Y + (n-1)\mathbf{D}_i(K+B)\rceil) \subseteq \pi_*\mathcal{O}_X(nK+nB).$$

The restriction to Y of $\pi_*\mathcal{O}_X(nK+nB)$ is $\pi_*\mathcal{O}_Y(nK_Y+n\Theta)$, by Theorem 0.1.(1). Therefore

$$\pi_*\mathcal{O}_Y(\lceil \mathbf{K} + (n-1)\mathbf{D}_{i'}(K_Y+\Theta)\rceil) \subseteq \pi_*\mathcal{O}_Y(nK_Y+n\Theta).$$

The claim follows from $\mathbf{D}_i(K_Y+\Theta) \leq \mathbf{D}_{i'}(K_Y+\Theta)$. $\square$

PROOF OF THEOREM (0.1)(2). Let $I \geq 2$ be an integer such that IB is integral. Assume that the real log divisor $K_Y+\Theta$, which is $\pi|_Y$-big by the assumption (a) in Theorem 0.1, has a Zariski decomposition. Thus there exists a birational contraction $\mu\colon Y' \to Y$ and a $(\pi\circ\mu)$-nef and $(\pi\circ\mu)$-big $\mathbb{R}$-divisor P on Y' such that

(i) $P \leq \mu^*(K_Y+\Theta)$;

(ii) $\mathcal{R}_{Y'/S}(P) = \mathcal{R}_{Y/S}(K_Y+\Theta)$.

Step 1. We claim that P is rational and relatively semiample. In particular, by (ii), the $\mathcal{O}_S$-algebra $\mathcal{R}_{Y/S}(K_Y+\Theta)$ is finitely generated.

Indeed, after possibly blowing up Y' and replacing P by its pullback, we may assume that Y' is smooth and $\mathrm{Supp}(P) \cup \mathrm{Supp}(\Theta_{Y'})$ is a simple normal crossings divisor, where $\mu^*(K_Y+\Theta) = K_{Y'}+\Theta_{Y'}$ is the log pullback. Denote $\pi' = \pi\circ\mu$. By (ii) and the argument of [2], Proposition 3.1, we have $\lim_{i\to\infty}\mathbf{D}_i(K_Y+\Theta) = \overline{P}$ and the inclusions in Lemma 3.2 become

$$\pi_*\mathcal{O}_Y(\lceil \mathbf{K} + (n-1)\overline{P}\rceil) \subseteq \pi_*\mathcal{O}_Y(nK_Y+n\Theta), \forall I|n.$$

Since $\pi'_*\mathcal{O}_{Y'}(P) = \pi_*\mathcal{O}_Y(nK_Y + n\Theta)$ and $\mathrm{Supp}(P)$ has simple normal crossings, this is equivalent to

$$\pi'_*\mathcal{O}_{Y'}(\lceil K_{Y'} + (n-1)P \rceil) \subseteq \pi'_*\mathcal{O}_{Y'}(nP), \forall I|n.$$

If we denote $N = K_{Y'} + \Theta_{Y'} - P \geq 0$, these inclusions become

$$\pi'_*\mathcal{O}_{Y'}(\lceil -(\Theta_{Y'} - N) + nP \rceil) \subseteq \pi'_*\mathcal{O}_{Y'}(nP), \forall I|n.$$

The log pair $(Y', \Theta_{Y'} - N)$ has Kawamata log terminal singularities (the boundary may not be effective), and $2P - (K_{Y'} + \Theta_{Y'} - N) = P$ is a π'-nef and π'-big $\mathbb{R}$-divisor. We have verified the assumptions of [2], Theorem 2.1 for $(Y'/S, \Theta_{Y'} - N)$ and P, hence we conclude that P is rational and π'-semiample.

Step 2. Since P is relatively semiample and big, there exists a birational contraction $\nu\colon Y' \to Y''$, defined over S, and a relatively ample $\mathbb{Q}$-Cartier divisor P'' on Y'' such that $P = \nu^*(P'')$.

$$\begin{tikzcd} & Y' \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ Y \arrow[rr, dotted] & & Y'' \end{tikzcd}$$

Let $\pi''\colon Y'' \to S$ be the induced morphism. Since P'' is relatively ample, there exists a positive integer n such that nP'' is Cartier and the sheaf $\nu_*\mathcal{O}_{Y'}(nN) \otimes \mathcal{O}_{Y''}(nP'')$ is π''-generated. We have $\pi'_*\mathcal{O}_{Y'}(nP) = \pi'_*\mathcal{O}_{Y'}(nP + nN)$, hence

$$\pi''_*\mathcal{O}_{Y'}(nP'') = \pi''_*(\nu_*\mathcal{O}_{Y'}(nN) \otimes \mathcal{O}''_Y(nP'')).$$

The right hand side is generated by global sections, hence

$$\nu_*\mathcal{O}_{Y'}(nN) \otimes \mathcal{O}_{Y''}(nP'') \subseteq \mathcal{O}_{Y'}(nP''),$$

that is $\nu_*\mathcal{O}_{Y'}(nN) \subseteq \mathcal{O}_{Y''}$. Therefore N is ν-exceptional. In particular,

$$P'' = K_{Y''} + \nu_*(\Theta_{Y'}),$$

and $\nu_*(\Theta_{Y'})$ is rational. We have

$$\mu^*(K_Y + \Theta) = \nu^*(K_{Y''} + \nu_*(\Theta_{Y'})) + N.$$

In particular, we have $\mathrm{Fix}(\mu^*(iK_Y + i\Theta)) \geq iN$ for every i, hence $\lim_{i\to\infty} \frac{\mathrm{Fix}(\mu^*(iK_Y+i\Theta))}{i} \geq N$. Pushing this down to Y, we obtain

$$\lim_{i\to\infty} \frac{\mathrm{Fix}(iK_Y + i\Theta)}{i} \geq \mu_* N.$$

By Lemma 2.3.(2), $\mu_* N$ has zero multiplicity at each component of Θ. Combining with the above, we obtain

$$\mathrm{mult}_E(\Theta) = \mathrm{mult}_E(\nu^*(K_{Y''} + \nu_*(\Theta_{Y'})) - K_{Y'})$$

for each prime divisor E in the support of Θ. Therefore Θ is rational. $\square$

4. Applications

Theorem 4.1 generalizes [6], Theorem 4.3, and simplifies its proof. In particular, Shokurov's prelimiting flips [15] exist if big (real) log canonical divisors have a Zariski decomposition in one dimension less. For the convenience of the reader, we compare the two approaches. With different assumptions, Hacon and M^cKernan prove Theorem 4.1.(1) in two steps. The first step proves (1) in the special case $\Theta = B_Y$. In particular, the restricted algebra $\mathcal{R}_{X/S}(K + B)|_Y$ has a representation $\bigoplus_{n=0}^{\infty} \pi_* \mathcal{O}_Y(nK_Y + n\Delta_n)$, where $(\Delta_n)_{n\geq 0}$ is a sequence of boundaries converging to Θ. The second step shows the stabilization $\Delta_n = \Theta$ for suitable n, and here one needs the log Minimal Model Program in the dimension of Y, Shokurov's asymptotic saturation of the restricted algebra, and the log Fano assumption that $-(K + B)$ is π-ample. The log Fano assumption is essential in this second step, in order to apply Shokurov's asymptotic saturation, and it is unclear how to proceed without it. Instead, using the new diophantine ingredients in §1, we obtain (1) directly. As for Theorem 4.1.(2), Shokurov's asymptotic saturation is not enough for rationality and finite generation if $-(K + B)$ is not π-ample. Instead, we introduce a stronger diophantine property (Lemma 1.5), which is preserved by restriction to Y (Proposition 3.2).

Theorem 4.1. *Let (X, B) be a log variety endowed with a proper contraction $\pi\colon X \to S$, satisfying the following properties:*

(a) *(X, B) is pure log terminal, with unique log canonical center $Y \subset X$.*

(b) *$K+B \sim_{\mathbb{Q}} A+C$, where A is a π-ample $\mathbb{R}$-divisor and C is an effective $\mathbb{R}$-divisor with* $\operatorname{mult}_Y(A) = \operatorname{mult}_Y(C) = 0$.

Thus Y is normal, and by codimension one adjunction there exists a canonically defined boundary B_Y such that $(K+B)|_Y = K_Y + B_Y$. Then there exists a birational modification $\mu\colon \tilde{Y} \to Y$ with a natural structure $(\tilde{Y}, \Theta)$ of log variety with Kawamata log terminal singularities, satisfying the following properties:

(1) *For every $n \geq 1$, we have natural inclusions*

$$\operatorname{Im}(\pi_* \mathcal{O}_X(nK+nB) \to \pi_* \mathcal{O}_Y(nK_Y+nB_Y)) \subseteq (\pi \circ \mu)_* \mathcal{O}_{\tilde{Y}}(nK_{\tilde{Y}}+n\Theta).$$

The inclusion is an equality if $n \geq 2$ and $\{nB\} \leq B$.

(2) *Assume that B has rational coefficients and the log canonical divisor $K_{\tilde{Y}}+\Theta$ has a Zariski decomposition relative to S. Then Θ has rational coefficients and $\bigoplus_{n=0}^{\infty} (\pi \circ \mu)_* \mathcal{O}_{\tilde{Y}}(nK_{\tilde{Y}} + n\Theta)$ is a finitely generated $\mathcal{O}_S$-algebra.*

Proof. The log variety (Y, B_Y) has Kawamata log terminal singularities. By Lemma 1.4, it has finitely many valuations with log discrepancy less than one. By Hironaka's resolution of singularities, we may construct a log resolution

$$\begin{tikzcd} \tilde{Y} \arrow[r] \arrow[d, "\nu"'] & \tilde{X} \arrow[d, "\mu"] \\ Y \arrow[r] & X \end{tikzcd}$$

with the following properties:

(1) The exceptional locus $\operatorname{Exc}(\mu)$ is a divisor on $\tilde{X}$ and $\operatorname{Exc}(\mu) \cup \operatorname{Supp}(B_{\tilde{X}})$ has simple normal crossings, where $\mu^*(K+B) = K_{\tilde{X}} + B_{\tilde{X}}$ is the log pullback.

(2) $\tilde{Y}$ is the proper transform of Y on $\tilde{X}$, with $\operatorname{mult}_{\tilde{Y}}(B_{\tilde{X}}) = 1$.

(3) Let $\tilde{B} = \max(B_{\tilde{X}}, 0)$ and $\tilde{B}_{\tilde{Y}} = (\tilde{B} - \tilde{Y})|_{\tilde{Y}}$. Then the components of $\tilde{B}_{\tilde{Y}}$ are disjoint.

Define

$$\begin{aligned}\Theta &= \max(B_{\tilde{Y}} - \lim_{i\to\infty} \frac{(\mathbf{F}\mathrm{ix}(iK+iB)|_{\tilde{Y}})_{\tilde{Y}}}{i}, 0) \\ &= \max(\tilde{B}_{\tilde{Y}} - \lim_{i\to\infty} \frac{(\mathbf{F}\mathrm{ix}(iK_{\tilde{X}}+i\tilde{B})|_{\tilde{Y}})_{\tilde{Y}}}{i}, 0),\end{aligned}$$

where the second equality holds by Lemma 2.1. Let $\tilde{\pi} = \pi \circ \mu$. By Theorem 0.1.(1), we have inclusions

$$\mathrm{Im}(\tilde{\pi}_*\mathcal{O}_{\tilde{X}}(nK_{\tilde{X}} + n\tilde{B}) \to \tilde{\pi}_*\mathcal{O}_{\tilde{Y}}(nK_{\tilde{Y}} + n\tilde{B}_{\tilde{Y}})) \subseteq \tilde{\pi}_*\mathcal{O}_{\tilde{Y}}(nK_{\tilde{Y}} + n\Theta)$$

for every $n \geq 1$, and equality holds if $n \geq 2$ and $\{n\tilde{B}\} \leq \tilde{B}$.

Let now $n \geq 2$ with $\{nB\} \leq B$. In particular,

$$\pi_*\mathcal{O}_X(nK + nB) = \pi_*\mathcal{O}_X(\lceil K + Y + (n-1)(K+B)\rceil).$$

Therefore

$$\pi_*\mathcal{O}_X(nK + nB) = \pi_*\mathcal{O}_X(\lceil \mathbf{K} + Y + (n-1)\mathbf{D}_i(K+B)\rceil), i \gg 1.$$

This is equivalent to

$$\tilde{\pi}_*\mathcal{O}_{\tilde{X}}(nK_{\tilde{X}} + n\tilde{B}) = \tilde{\pi}_*\mathcal{O}_{\tilde{X}}(\lceil \mathbf{K} + \tilde{Y} + (n-1)\mathbf{D}_i(K_{\tilde{X}}+\tilde{B})\rceil), i \gg 1.$$

We have a commutative diagram

$$\begin{array}{ccc} \tilde{\pi}_*\mathcal{O}_{\tilde{X}}(nK_{\tilde{X}} + n\tilde{B}) & \longrightarrow & \tilde{\pi}_*\mathcal{O}_{\tilde{X}}(\lceil \mathbf{K} + \tilde{Y} + (n-1)\mathbf{D}_i(K_{\tilde{X}}+\tilde{B})\rceil) \\ \downarrow & & \downarrow \\ \tilde{\pi}_*\mathcal{O}_{\tilde{Y}}(nK_{\tilde{Y}} + n\Theta) & \longrightarrow & \tilde{\pi}_*\mathcal{O}_{\tilde{Y}}(\lceil \mathbf{K} + (n-1)\mathbf{D}_i(K_{\tilde{Y}}+\Theta))\rceil) \end{array}$$

The horizontal arrows are inclusions. Choose $i \geq 2$ such that $n|i$, $\{i\tilde{B}\} \leq \tilde{B}$ and $\mathbf{D}_i(K_{\tilde{X}} + \tilde{B})$ is relatively b-big. From above, we have $\mathbf{D}_i(K_{\tilde{X}} + \tilde{B})|_{\tilde{Y}} = \mathbf{D}_i(K_{\tilde{Y}} + \Theta)$, hence by Kawamata-Viehweg vanishing we infer that the right hand side vertical arrow is surjective. The top horizontal arrow is an equality from above. This implies that the bottom horizontal arrow is an equality and the left hand side vertical arrow is surjective. This finishes the proof of (1).

For (2), assume that B is rational and $K_{\tilde{Y}} + \Theta$ has a Zariski decomposition relative to S. In particular, $\tilde{B}$ has rational coefficients. By Theorem 0.1 applied to $(\tilde{X}/S, \tilde{B})$, we infer that Θ is rational and the $\mathcal{O}_S$-algebra $\bigoplus_{n=0}^{\infty}(\pi \circ \mu)_* \mathcal{O}_{\tilde{Y}}(nK_{\tilde{Y}} + n\Theta)$ is finitely generated. $\square$

THEOREM 4.2. *Let X be a nonsingular variety, endowed with an $\mathbb{R}$-divisor B such that* $\operatorname{Supp}(B)$ *is a simple normal crossings divisor. Assume that Y is a component of B of multiplicity one and $\lfloor B - Y \rfloor = 0$. Denote $B_Y = (B - Y)|_Y$, so that by adjunction we have $(K + B)|_Y = K_Y + B_Y$. Let $\pi \colon X \to S$ be a projective surjective morphism, H a π-free divisor on X and $n \geq 2$ an integer such that $\{nB\} \leq B$. Assume that*

(a) *$nK + nB + H \sim_{\mathbb{Q}} A + C$, where A is a π-ample $\mathbb{R}$-divisor and C is an effective $\mathbb{R}$-divisor with* $\operatorname{mult}_Y(A) = \operatorname{mult}_Y(C) = 0$.

(b) *(Y, B_Y) has canonical singularities in codimension at least two.*

(c) *$n \geq (1-b)^{-1}$, where b is the maximum multiplicity of the components of B_Y.*

Define $\Theta_n = \max(B_Y - \lim_{i\to\infty} \frac{(\mathbf{F}\mathrm{ix}(i(K+B+\frac{H}{n}))|_Y)_Y}{i}, 0)$. *Then*

$$\begin{aligned}&\operatorname{Im}(\pi_* \mathcal{O}_X(nK + nB + H) \to \pi_* \mathcal{O}_Y(nK_Y + nB_Y + H|_Y)) \\ &= \pi_* \mathcal{O}_Y(nK_Y + n\Theta_n + H|_Y).\end{aligned}$$

PROOF. We may assume that S is affine. Let $D \in |H|$ be a general member such that $D|_Y$ does not contain any prime component of the $\mathbb{R}$-divisor $\lim_{i\to\infty} \frac{(\mathbf{F}\mathrm{ix}(iK+iB)|_Y)_Y}{i}$. Define $B_n = B + \frac{1}{n}D$ and $B_{n,Y} = (B_n - Y)|_Y$. By (c) and since H is π-free, $(Y, B_{n,Y})$ is also canonical in codimension a least two. By construction, $B_{n,Y} = B_Y + \frac{1}{n}D|_Y$. By the choice of D, we have

$$\max(B_{n,Y} - \lim_{i\to\infty} \frac{(\mathbf{F}\mathrm{ix}(i(K + B + \frac{D}{n}))|_Y)_Y}{i}, 0) = \Theta_n + \frac{1}{n}D|_Y.$$

Now apply Theorem 0.1 to (X, B_n) and n. $\square$

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