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The Bergman Kernel on Tube Domains of Finite Type

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Abstract. In this paper, asymptotic expansions of the Bergman kernel and the Szegö kernel are computed for pseudoconvex tube domains of finite type in $\mathbb{C}^{n+1}$ ($n \geq 1$).

Contents

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1. Introduction

In the function theory of several complex variables, it is a very important thema to understand the boundary behavior of the Bergman kernel $B(z)$ and there are many interesting studies about this behavior. In particular, the strongly pseudoconvex case is quite well understood. Let Ω be a C^∞-smoothly bounded strongly pseudoconvex domain in $\mathbb{C}^{n+1}$ $(n \geq 1)$. Hörmander [19] and Diederich [9],[10] showed the limit of $B(z)d(z)^{n+2}$ at a boundary point z_0 equals the determinant of the Levi form at z_0 times $(n+1)!/4\pi^{n+1}$. Here $d(z)$ is the Euclidean distance from z to the boundary. Later C. Fefferman [14] obtained the following very strong result about the asymptotic expansion:

$$B(z) = \varphi(z)r(z)^{-n-2} + \psi(z)\log r(z), \tag{1.1}$$

where $-r$ is a defining function of Ω and φ, ψ are C^∞-functions on $\bar{\Omega}$.

In this paper, we are interested in the case of weakly pseudoconvex domains of finite type. In this case, many detailed results have been obtained in estimating the size of the Bergman kernel (see the reference in [2],[24], etc.). More precisely, Boas, Straube and Yu [2] (see also [12]) obtained a result about the *boundary limit* in the sense of Hörmander for some large class of finite type domains. Indeed, they showed that if Ω is a bounded pseudoconvex domain in $\mathbb{C}^{n+1}$ and the boundary point z_0 is semiregular (which is also called h-extendible) with multitype $(1, 2m_1, \ldots, 2m_n)$, then the nontangential limit of $B(z)d(z)^{\sum_{j=1}^n 1/m_j+2}$ at z_0 equals some positive number which is determined by local model only. But, there seems very few study about asymptotic expansions like (1.1) in the weakly pseudoconvex case. The author [22] has computed an asymptotic expansion of the Bergman kernel for two-dimensional pseudoconvex tube domains of finite type. The purpose of this paper is to give an asymptotic expansion of the Bergman kernel in the general dimensional case.

Let us explain our analysis for the Bergman kernel. For tube domains, it is known that the Bergman kernel can be expressed by using simple integrals

(Section 4). Our analysis is based on this integral expression. From this expression, the integral of the form:

$$F(x) = \int_{\mathbb{R}^n} e^{-2[f(w)-(x|w)]} dw \qquad (x \in \mathbb{R}^n)$$

appears and its analysis is important. Here the function f locally defines the base of the tube domain. The finite type condition implies that f can be locally approximated by a convex quasihomogeneous polynomial P (Section 2). The tube domain defined by this polynomial P can be considered as an appropriate model and we analyze the singularity of the Bergman kernel for this model domain (Section 5). On the other hand, the singularity of the Bergman kernel is completely determined by local geometry of the boundary in our case (Section 6). By using this localization, general domains can be considered as perturbations of model domains. Some computation in Section 7 implies that this perturbation reflects the many terms of the asymptotic expansion of the Bergman kernel. In the computation, the precise analysis of the integral F is necessary. Roughly speaking, we give some estimates for the derivatives of F by using F itself in Lemma 7.5.

Last let us explain an important geometrical idea in our computation. Let z_0 be a weakly pseudoconvex point on the boundary. Generally, the geometrical situation of the boundary around z_0 is complicated. Indeed, D'Angelo's variety type and Catlin's multitype are not always uniform around z_0. This fact gives a serious influence to the singularity of the Bergman kernel. It is a natural phenomenon that its behavior from the tangential direction becomes complicated. But in the case of tube domains, the domains can be approximated by quasihomogeneous domains whose boundaries have relatively simple stratification structures from the viewpoint of the multitype. (More generally, the class of semiregular domains has the same properties, see [11],[36].) From this geometrical property, we introduce new variables which induce a real blowing up at z_0 (Section 2.2). By using these variables, the singularity can be stratified in a clear form. We express the singularity from the vertical direction in the form of an asymptotic expansion. In the weakly pseudoconvex case, several variables are necessary to express the singularity. In this respect, the weakly pseudoconvex case differs from the strongly pseudoconvex case.

In Section 8, an analogous result about the Szegö kernel is given.

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2. Results

Let Ω be a domain in $\mathbb{C}^{n+1}$ $(n \geq 1)$ and denote by $A^2(\Omega)$ the subspace of $L^2(\Omega)$ consisting of holomorphic functions. The Bergman kernel $B(z)$ of Ω (on the diagonal) is defined by

$$B(z) = \sum_j |\varphi_j(z)|^2,$$

where $\{\varphi_j\}_j$ is a complete orthonormal basis of $A^2(\Omega)$. The above sum is uniformly convergent on any compact set in Ω. This implies that $B(z)$ is real analytic on Ω.

In this paper, we consider the following class of domains. Given a domain ω in $\mathbb{R}^{n+1}$. The *tube domain* over the base ω is defined by

$$\Omega = \mathbb{R}^{n+1} + i\omega = \{z = x + iy \in \mathbb{C}^{n+1}; x \in \mathbb{R}^{n+1}, y \in \omega\}.$$

Here we set $z = (z', z_{n+1}) = (z_1, \dots, z_n, z_{n+1}) \in \mathbb{C}^{n+1}$ with $z_j = x_j + iy_j$, $x = (x', x_{n+1}) = (x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$ and $y = (y', y_{n+1}) = (y_1, \dots, y_n, y_{n+1}) \in \mathbb{R}^{n+1}$. A projection Π from $\mathbb{C}^{n+1}$ to $\mathbb{R}^{n+1}$ is defined by $\Pi(z) = \Im(z) = y$. It is well known that the pseudoconvexity of $\Omega = \mathbb{R}^{n+1} + i\omega$ is equivalent to the convexity of the base ω.

2.1. Assumptions

Throughout this section, we give the following assumptions on a tube domain $\Omega = \mathbb{R}^{n+1} + i\omega$ and its boundary point z_0. The base domain ω is a convex domain in $\mathbb{R}^{n+1}$ with C^∞-smooth boundary and z_0 is a point of finite type, in the sense of D'Angelo. Moreover, Catlin's multitype of z_0 is $(m_1(\partial\Omega, z_0), \dots, m_{n+1}(\partial\Omega, z_0))$. Note that Lemma 3.1, below, implies that $y_0 = \Pi(z_0) \in \partial\omega$ is of finite type in the sense of Bruna-Nagel-Wainger. (In Section 3.1, we will explain the concepts of these "types" and this type for y_0 is called $\mathbb{R}$-finite type.)

2.2. Appropriate coordinates

From the convexity of ω and the finite type condition of y_0, we can choose a coordinate in $\mathbb{R}^{n+1}$, where the base is contained, so that:

(1) The point y_0 is the origin.

(2) The $y_1, \dots, y_n$ directions give the tangent plane to $\partial\omega$ at y_0.

(3) The y_{n+1} direction gives the normal (in the case of a bounded ω) or it gives a half line which is contained in $\omega \cup \{y_0\}$ (in the case of an unbounded ω).

For an unbounded ω, there are a domain A in $\mathbb{R}^n$ (possibly, $A = \mathbb{R}^n$) containing the origin and a C^∞-function f on A such that $f(0) = |\nabla f(0)| = 0$ and

$$\omega = \{y \in \mathbb{R}^{n+1}; y_{n+1} > f(y_1, \dots, y_n) = f(y') \quad (y' \in A)\}.$$

For a bounded domain ω, there are a domain A in $\mathbb{R}^n$ containing the origin and C^∞-functions f and $\tilde{f}$ on A such that

$$\omega = \{y \in \mathbb{R}^{n+1}; f(y') < y_{n+1} < \tilde{f}(y') \quad (y' \in A)\}.$$

According to the following result of Schulz [32], the finite type condition implies that the function $f(y')$ can be decomposed into a quasihomogeneous convex polynomial and a remainder term as follows.

Lemma 2.1 ([32]). *There exists a rotation L in $\mathbb{R}^n$ so that the function $f(y')$ can be expressed near the origin as follows:*

$$f(Ly') = P(y') + R(y').$$

Here P and R satisfy the following properties. Set $m_j = m_j(\partial\Omega, P)/2$ $(j = 1, \dots, n)$.

(i) *$P(y')$ is a convex polynomial having the quasihomogeneity:*

$$P(t^{1/2m_1} y_1, \dots, t^{1/2m_n} y_n) = tP(y_1, \dots, y_n) \quad \textit{for all } t > 0$$

and $P(y') > 0$ if $y' \neq 0$.

(ii) *There exist constants* $C > 0$ *and* $\gamma \in (0,1]$ *such that* $|R(y')| \leq C\sigma(y')^{1+\gamma}$, *where* $\sigma(y') := \sum_{j=1}^{n} y_j^{2m_j}$ *near the origin.*

From the above lemma, we will consider the domain $\omega_P = \{y \in \mathbb{R}^{n+1}; y_{n+1} > P(y')\}$ as an appropriate model for the analysis on the domain ω near the origin. Hereafter we choose the coordinates $y' = (y_1, \dots, y_n)$ so that f is divided as in the above lemma.

2.3. Real blowing up

Next let us introduce a mapping "real blowing up" at $y_0 \in \partial\omega$.

For $\delta > 0$, let $\omega_\delta = \{y \in \omega; y_{n+1} < \delta\}$. Let m be the least common multiplicity of $m_1, \dots, m_n$ and let $l_j = m/m_j$. Let $\tilde{\pi}$ be a mapping from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$ such that $\tilde{\pi}(\tau_1, \dots, \tau_n, \rho) = \tilde{\pi}(\tau, \rho) = (y_1, \dots, y_n, y_{n+1})$, where

$$\begin{cases} y_j = \tau_j \rho^{l_j} & (j = 1, \dots, n) \\ y_{n+1} = \rho^{2m}. \end{cases} \tag{2.1}$$

We set $U = \tilde{\pi}^{-1}(\omega)$ and $U_\delta = \tilde{\pi}^{-1}(\omega_\delta)$. It is easy to see $\tilde{\pi}^{-1}(\omega_P) = \Delta_P \times (0, \infty)$, where

$$\Delta_P = \{\tau \in \mathbb{R}^n; P(\tau) < 1\}.$$

Let π be the restriction of the mapping $\tilde{\pi}$ on the set $\bar{U}$. Note that π is a diffeomorphic mapping from U to ω and $\pi^{-1}(0) = \overline{\Delta_P} \times \{0\}$. This fact means that π is a *real blowing up* at 0.

2.4. Asymptotic expansion

Let D be a set in $\mathbb{R}^p$, not necessarily open. We say that f is a C^∞-function on D if f is C^∞-smooth in the interior of D and all partial derivatives of f can be continuously extended to the boundary. For $\delta > 0$, we define $\Gamma_\delta = \{(\tau, \rho) \in \Delta_P \times [0, \delta); P(\tau) + C\rho^{2m\gamma}\sigma(\tau)^{1+\gamma} < 1\}$, where C, γ are positive numbers as in Lemma 2.1. The following is a main result of this paper.

THEOREM 2.2. *The Bergman kernel* $B(z)$ *of a tube domain* $\Omega = \mathbb{R}^{n+1} + i\omega$ *has the form near* $z_0 \in \partial\Omega$:

$$B(z) = \frac{\Phi(\tau, \rho)}{\rho^{2m(\nu+2)}} + \tilde{\Phi}(\tau, \rho) \log \rho, \tag{2.2}$$

where $\nu = \sum_{j=1}^{n} 1/m_j$ *and* $\Phi(\tau,\rho)$, $\tilde{\Phi}(\tau,\rho)$ *are* C^∞*-functions on the set* U_δ*, with some small positive number* δ*, satisfying the following properties.*

(i) $\Phi(\tau,\rho)$ *can be extended to be a* C^∞*-function on* $U_\delta \cup (\Delta_P \times \{0\})$*. More precisely,* $\Phi(\tau,\rho)$ *admits the following asymptotic expansion with respect to* ρ*: for any* $N \in \mathbb{N}$*,*

$$\Phi(\tau,\rho) = \sum_{k=0}^{N} \Phi_k(\tau)\rho^k + R_N(\tau,\rho)\rho^{N+1} + \tilde{\tilde{\Phi}}(\tau,\rho)\rho^{2m(\nu+2)},$$

where each coefficients $\Phi_k(\tau)$ *are* C^∞*-functions on* Δ_P*,* $R_N(\tau,\rho)$ *is continuous on* Γ_δ *and* $\tilde{\tilde{\Phi}}(\tau,\rho)$ *is a* C^∞*-function on* $\overline{U_\delta}$*. In particular, the first coefficient* $\Phi(\tau,0) = \Phi_0(\tau)$ *is* $\Phi(\tau)$ *as in (5.3) in Section 5, which is positive on* Δ_P *and is unbounded as* τ *approaches the boundary of* Δ_P*.*

(ii) $\tilde{\Phi}(\tau,\rho)$ *can be extended to be a* C^∞*-function on* $\overline{U_\delta}$*.*

Remark 2.3. From the theorem, we obtain a result about the boundary limit as in the Introduction. The nontangential limit of $B(z)\rho^{2m(\nu+2)}$, as $z \to z_0 \in \partial\Omega$, equals

$$\Phi(0) = \Phi_0(0) = \frac{1}{2^{n+\nu+2}\pi^{n+1}} \int_{\mathbb{R}^n} \frac{d\zeta}{\int_{\mathbb{R}^n} e^{-2[P(\mu)-(\zeta|\mu)]}d\mu}$$

(see Section 5). This value is determined by the function P only. More precisely, if the Bergman kernel is restricted to the set $\{y \in \mathbb{R}^{n+1}; y_{n+1} > P(y')^{1+\epsilon}\}$ $(\epsilon > 0)$, then the coefficient of ρ^k equals the constant $\Phi_k(0)$ for $k \geq 0$. We will discuss about the coefficients in more detail in Section 7.4.

Remark 2.4. In this paper, we do not discuss about the singularities of the coefficient functions $\Phi_k(\tau)$ at $\partial\Delta_P$ in detail. Roughly speaking, the singularity with respect to τ concerns with the singularity from the tangential direction. In the two-dimensional case, their singularities are computed in [22] (see Remark 7.20). But, the geometrical situation of the boundary around z_0 is very complicated in the general dimensional case. Therefore the singularity of $\Phi_k(\tau)$ also becomes complicated and it must be expressed by using several variables. The singularity from the vertical direction is essentially important and it can be seen as in the theorem.

Remark 2.5. Our asymptotic expansion, with respect to ρ, has a similar form to (1.1) in the strongly pseudoconvex case. The essential difference appears in the expansion variable. That is to say, in the strongly or the weakly pseudoconvex case, the asymptotic expansion takes the Taylor series type or the Puiseux series type, respectively. In [24], a similar asymptotic expansion is computed for another class of domains of semiregular. From these observations, we may conjecture that the Bergman kernel always admits an asymptotic expansion like (2.2) for the class of pseudoconvex domains of semiregular. But it is known that this type of expansion cannot be generalized to the general finite type domains (see [17],[24]).

3. Finite Type Conditions

3.1. Four kinds of type

The concepts of many kinds of *types*, which are introduced in [6],[32],[21],[8],[27],[7], are very important for precise analysis on degenerate hypersurfaces in real or complex spaces. Here let us recall the definitions of four kinds of *type* at boundary points of domains in $\mathbb{R}^{n+1}$ or $\mathbb{C}^{n+1}$. Let $\mathbb{Z}_+$ be the set of nonnegative integers.

3.1.1 Real line type and real multitype

These types were introduced by Bruna-Nagel-Wainger [6], Schulz [32], Iosevich-Sawyer-Seeger [21]. Let ω be a domain in $\mathbb{R}^{n+1}$ with C^∞-smooth boundary S and $\rho \in C^\infty(\mathbb{R}^{n+1})$ a defining function of ω, i.e., $\nabla\rho(x) \neq 0$ when $\rho(x) = 0$ and

$$\omega = \{x \in \mathbb{R}^{n+1}; \rho(x) < 0\} \quad \text{and} \quad S = \{x \in \mathbb{R}^{n+1}; \rho(x) = 0\}.$$

For each $\eta = (\eta_1, \dots, \eta_{n+1}) \in \mathbb{R}^{n+1}$, let $\langle \eta, \nabla \rangle = \sum_{j=1}^{n+1} \eta_j \partial/\partial x_j$ be the directional derivative in direction η and let $\langle \eta, \nabla \rangle^j$ denote the j^{th} power of this derivative. Let T_x be the affine tangent plane to S at x, i.e. $T_x = \{\eta \in \mathbb{R}^{n+1}; \langle \eta, \nabla \rangle \rho(x) = 0\}$. We suppose that $x \in S$ is a convex point, i.e.

$$\langle \eta, \nabla \rangle^2 \rho(x) = \sum_{j,k=1}^{n+1} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(x) \eta_j \eta_k \geq 0 \quad \text{for all } \eta \in T_x.$$

Following [32],[21], let us define the real line type and the real multitype of S at x. For $x \in S$ and $m \in \mathbb{Z}_+$, we define the sets

$$S_m = S_m(x) = \left\{ \eta \in \mathbb{R}^{n+1}; \sum_{j=0}^{m} |\langle \eta, \nabla \rangle^j \rho(x)| = 0 \right\}.$$

It is clear that $S_0 = \mathbb{R}^{n+1}$ and $S_1 = T_x$. If $x \in S$ is a strongly convex point, i.e. $\langle \eta, \nabla \rangle^2 \rho(x) > 0$ for all nonzero $\eta \in T_x$, then $S_2 = \{0\}$. If $j < k$, then $S_j \supset S_k$. As was shown in [32], the convexity implies that S_m are linear subspaces in T_x. Now if there exists an integer m such that $S_m = \{0\}$, then we say that $x \in S$ is a point of $\mathbb{R}$-finite type. From now on, we assume the $\mathbb{R}$-finite type condition on x. Then there are integers $a_1, \dots, a_k$ such that $1 = a_1 < \cdots < a_k$, $2 \le k \le n$ and

$$\{0\} = S_{a_k} \subsetneq \cdots \subsetneq S_{a_2} \subsetneq S_{a_1} = S_1 = T_x \subsetneq S_0 = \mathbb{R}^{n+1}$$

and the sequence is maximal, i.e.,

$$S_{a_j} = S_{a_j+1} = \cdots = S_{a_{j+1}-1}, \quad 1 \le j \le k-1.$$

The largest number a_k is called as the *real line type* of S at x, which is denoted by $\mathbb{R}L(S, x)$. This number means the maximal order of contact of real lines with S at x. Let $d_j = \dim S_{a_j}$. In particular, $d_0 = n+1$, $d_1 = n$, $d_k = 0$. For $j = 1, \dots, n$, let

$$\tilde{m}_j(S, x) = a_l \quad \text{if } n+1-d_{l-1} < j \le n+1-d_l, \;\; l = 1, \dots, k.$$

Then $(n+1)$-tuple $\mathbb{R}\mathcal{M}(S, x) = (\tilde{m}_1(S, x), \dots, \tilde{m}_{n+1}(S, x)) \in \mathbb{N}^{n+1}$ is called the *real multitype* of S at x. The definitions of $\mathbb{R}L(S, x)$ and $\mathbb{R}\mathcal{M}(S, x)$ are independent of the linear coordinate of $\mathbb{R}^{n+1}$.

Now, the next three types are defined on the boundary of complex domains. Let Ω be a domain in $\mathbb{C}^{n+1}$ with C^∞-smooth boundary M and let z_0 lie on M. Let $r \in C^\infty(\mathbb{C}^{n+1})$ be a defining function of Ω, i.e., $|\nabla r(z)| \ne 0$ when $r(z) = 0$ and

$$\Omega = \{z \in \mathbb{C}^{n+1}; r(z) < 0\} \quad \text{and} \quad M = \{z \in \mathbb{C}^{n+1}; r(z) = 0\}.$$

3.1.2 Variety type

This type was introduced by D'Angelo [8]. The *variety (1-)type* of M at z_0 is defined by

$$\Delta_1(M, z_0) = \sup \left\{ \frac{\nu(z^* r)}{\nu(z - z_0)} \right\}, \tag{3.1}$$

where the supremum is taken over all germs of nontrivial one-dimensional complex varieties $z : D \to \mathbb{C}^{n+1}$ with $z(0) = z_0$. Here D is the unit disk in

$\mathbb{C}$, $\nu(f)$ denotes the order of vanishing of the function f at 0 and $z^*r = r \circ z$. In this paper, we say that z_0 is a point of finite type if $\Delta_1(M, z_0) < \infty$.

More generally, one can define the q-type of M at z_0, $q \geq 1$:

$$\Delta_q(M, z_0) = \inf_S \Delta_1(M \cap S, z_0) \qquad 1 \leq q \leq n+1.$$

Here S runs over all $(n-q+2)$-dimensional complex hyperplanes passing through z_0 and $\Delta_1(M \cap S, z_0)$ denotes the 1-type of the domain $\Omega \cap S$ (considered as a domain in S) at z_0.

3.1.3 Complex line type

This type was introduced by McNeal [27]. The *complex line type* $\mathbb{C}L(M, z_0)$ of M at z_0 is defined in the same way as in (3.1) by considering, instead of complex varieties, only affine complex lines through z_0, i.e.,

$$\mathbb{C}L(M, z_0) = \sup_l \nu(l^*r),$$

for l is a parameterization of a complex line with $l(0) = z_0$.

3.1.4 Complex multitype

This type was introduced by Catlin [7]. Let Γ_{n+1} denote the set of all $(n+1)$-tuples of numbers $\lambda = (\lambda_1, \dots, \lambda_{n+1})$ with $0 \leq \lambda_j \leq \infty$ such that

(i) $\lambda_1 \leq \cdots \leq \lambda_{n+1}$.

(ii) For each k, either $\lambda_k = +\infty$ or there is a set of nonnegative integers $a_1, \dots, a_k$, with $a_k > 0$, such that $\sum_{j=1}^k a_j/\lambda_j = 1$.

An element of Γ_{n+1} will be referred to as a *weight*. The set of weights can be ordered lexicographically; i.e., if $\Lambda' = (\lambda'_1, \dots, \lambda'_{n+1})$ and $\Lambda'' = (\lambda''_1, \dots, \lambda''_{n+1})$, then $\Lambda' < \Lambda''$ if for some k, $\lambda'_j = \lambda''_j$ for all $j < k$, but $\lambda'_k < \lambda''_k$. A weight $\Lambda \in \Gamma_{n+1}$ is said to be *distinguished* if there exist holomorphic coordinates $(z_1, \dots, z_{n+1})$ about z_0 with z_0 mapped to the origin such that $D^\alpha \bar{D}^\beta r(z_0) = 0$ whenever $\sum_{j=1}^{n+1} (\alpha_j + \beta_j)/\lambda_j < 1$, where

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_{n+1}^{\alpha_{n+1}}} \quad \text{and} \quad \bar{D}^\beta := \frac{\partial^{|\beta|}}{\partial \bar{z}_1^{\beta_1} \cdots \partial \bar{z}_{n+1}^{\beta_{n+1}}}.$$

The *complex multitype* $\mathbb{C}\mathcal{M}(M, z_0)$ of M at z_0 is defined to be the smallest weight $\mathbb{C}\mathcal{M}(M, z_0) = (m_1(M, z_0), \dots, m_{n+1}(M, z_0))$ in Γ_{n+1} (smallest in the lexicographic sense) such that $\mathbb{C}\mathcal{M}(M, z_0) \geq \Lambda$ for every distinguished weight Λ. Note that $m_1(M, z_0) = 1$

3.2. On tube domains

Let us consider the relation among the above types in the case of tube domains. Under the assumption of the convexity on the base, these types have the following relations.

LEMMA 3.1. *If $z_0 \in \partial\Omega$ is a point of finite type, then $y_0 = \Pi(z_0) \in \partial\omega$ is a point of $\mathbb{R}$-finite type. More precisely, we have*

$$\Delta_q(\partial\Omega, z_0) = m_{n+2-q}(\partial\Omega, z_0) = \tilde{m}_{n+2-q}(\partial\omega, y_0) \qquad 1 \le q \le n+1.$$

and

$$\Delta_1(\partial\Omega, z_0) = \mathbb{C}L(\partial\Omega, z_0) = \mathbb{R}L(\partial\omega, y_0).$$

Moreover the above numbers are even integers, if $q \neq n+1$.

PROOF. From the convexity, the equalities about the types for complex domains are shown by McNeal [27], Boas and Straube [1] and Yu [35]. Next, it is shown in Proposition 5 in [35] that the complex multitype equals the weight Λ_0, which is defined in [35]. We can also obtain other equalities by restricting the concept of Λ_0 to the real space. It is easy to know that the above numbers are even from the convexity. □

4. Integral Formula

For $\zeta = (\zeta_1, \ldots, \zeta_N)$, $\eta = (\eta_1, \ldots, \eta_N)$ in $\mathbb{R}^N$, we set $d\zeta = d\zeta_1 \cdots d\zeta_N$ and $(\zeta|\eta) = \zeta_1\eta_1 + \cdots + \zeta_N\eta_N$.

It is known in [25],[34],[30],[4],[13] that the Bergman kernel of a tube domain $\Omega = \mathbb{R}^{n+1} + i\omega$ is expressed as

$$B(z) = \frac{1}{(2\pi)^{n+1}} \int_{\Lambda^*} e^{-2(y|u)} \frac{1}{\varphi(u)} du, \tag{4.1}$$

where

$$\varphi(u) = \int_\omega e^{-2(u|w)} dw$$

and $\Lambda^* = \{u \in \mathbb{R}^{n+1}; \varphi(u) < \infty\}$. Since the Bergman kernel $B(z)$ is a function of y only in the tube case, we denote $B(z)$ by $K(y)$ hereafter.

Let us see a precise shape of the set Λ^*. For a convex set $\omega \subset \mathbb{R}^{n+1}$, its *recession cone* Λ_ω is defined to be $\Lambda_\omega = \{y \in \mathbb{R}^{n+1}; v + ty \in \omega$ for all $v \in \omega$,

$t \geq 0\}$. The recession cone of a convex set is the maximal element in the family of those cones whose shifts are contained in this set (c.f. [29]). If ω is bounded, then $\Lambda_\omega = \{0\}$ and $\Lambda^* = \mathbb{R}^{n+1}$. In the case of unbounded ω, the integrable condition of $\varphi(u)$ implies that $u \in \Lambda^*$ if and only if $(u|y) > 0$ for all $y \in \omega$. This means that Λ^* is the dual cone of Λ_ω, i.e.,

$$\Lambda^* = \{u \in \mathbb{R}^{n+1}; (u|y) > 0 \text{ for } y \in \Lambda_\omega\}.$$

More precisely, we consider the recession cone in the coordinates which was introduced in Section 2.2. For $R > 0$, set $\tilde{B}_R = \{y' \in \mathbb{R}^n; f(y') < R\}$ and $B_R = \{(y_1/R, \ldots, y_n/R) \in \mathbb{R}^n; y' \in \tilde{B}_R\}$. Let B be the intersection of B_R for all $R > 0$. It is easy to see that B is a nonempty set and that the recession cone of ω is

$$\Lambda_\omega = \{(t\hat{y}', t) \in \mathbb{R}^{n+1}; t \geq 0, \hat{y}' \in B\}.$$

The definition of the dual cone leads to

$$\Lambda^* = \{(-s\hat{u}', s) \in \mathbb{R}^{n+1}; s > 0, \hat{u}' \in B^*\},$$

where

$$B^* = \{\hat{u}' \in \mathbb{R}^n; (\hat{u}'|\hat{y}') < 1 \text{ for } \hat{y}' \in B\}.$$

Note that B^* contains the origin. For example, if there are positive numbers C and ϵ such that $f(y') \geq C|y'|^{1+\epsilon}$ when $|y'| \geq 1$, then $B = \{0\}$ and $B^* = \mathbb{R}^n$. In the bounded case, we set $B^* = \mathbb{R}^n$.

For the convenience for the computation later, we rewrite the integral representation (4.1) by changing the integral variables ($u_j = -s\hat{u}_j$ ($j = 1, \ldots, n$), $u_{n+1} = s$). In the unbounded case, we have

$$\begin{aligned} K(y) &= \frac{2}{(2\pi)^{n+1}} \int_0^\infty e^{-2y_{n+1}s} F(y'; s) s^{n+1} ds, \\ F(y'; s) &= \int_{B^*} e^{2s(y'|\hat{u}')} \frac{1}{D(\hat{u}'; s)} d\hat{u}', \\ D(\hat{u}'; s) &= \int_A e^{-2s[f(w') - (\hat{u}'|w')]} dw'. \end{aligned} \tag{4.2}$$

5. Analysis in the Model Case

Throughout this paper, the following terminology and notation are used. For a weight $\Gamma := (p_1, \dots, p_N)$ ($p_j > 0$) and a positive number α, we say that a polynomial $P(x_1, \dots, x_N)$ is $(\Gamma : \alpha)$-homogeneous, if P satisfies

$$P(t^{p_1}x_1, \dots, t^{p_N}x_N) = t^\alpha P(x_1, \dots, x_N) \quad \text{for all } t > 0.$$

Note that the case $\Gamma = \Gamma_m := (1/2m_1, \dots, 1/2m_n)$ will often appear later, where $(1, 2m_1, \dots, 2m_n)$ is the multitype at a boundary point. The following symbols are useful in our computation below. For $t > 0$ and $x \in \mathbb{R}^n$, define $\delta_t(x)$ and $\delta_t^*(x)$ by

$$\begin{aligned} \delta_t(x) &= (t^{1/2m_1}x_1, \dots, t^{1/2m_n}x_n), \\ \delta_t^*(x) &= (t^{1-1/2m_1}x_1, \dots, t^{1-1/2m_n}x_n). \end{aligned}$$

Let us consider the model case of our analysis. Let P be a convex polynomial of $x \in \mathbb{R}^n$ satisfying the property (i) as in Lemma 2.1, that is, P is $(\Gamma_m : 1)$-homogeneous and satisfies $P(x) > 0$ if $x \neq 0$. Note that the above conditions imply $P(0) = |\nabla P(0)| = 0$. In this section, we consider the tube domain $\Omega_P = \mathbb{R}^{n+1} + i\omega_P$, where ω_P is defined by

$$\omega_P = \{y = (y', y_{n+1}) \in \mathbb{R}^{n+1}; y_{n+1} > P(y') = P(y_1, \dots, y_n)\}.$$

This class of domains is considered as appropriate models for our analysis on general tube domains of finite type.

By using the mapping π as in Section 2.2, the singularity of the Bergman kernel can be expressed in the following clear form in the model case. Recall that $\tau_j = y_j/y_{n+1}^{1/2m_j}$ $(j = 1, \dots, n)$ and $\rho = y_{n+1}^{1/2m}$.

Proposition 5.1. *The Bergman kernel $K(y)$ has the form*

$$K(y) = \Phi(\tau)\rho^{-2m(\nu+2)}, \tag{5.1}$$

where $\nu = \sum_{j=1}^n 1/m_j$ and $\Phi(\tau)$ is a C^∞-function on Δ_P and is unbounded as τ approaches $\partial\Delta_P$.

Proof. In the model case, $B^* = \mathbb{R}^n$ and $A = \mathbb{R}^n$ in (4.2). By changing the integral variables ($\mu_j = s^{1/2m_j}w_j$, $v_j = s^{1-1/2m_j}\hat{u}_j$ $(j = 1, \dots, n)$), we obtain

$$
(5.2)\qquad
\begin{aligned}
K(y) &= \frac{2}{(2\pi)^{n+1}} \int_0^\infty e^{-2sy_{n+1}} G(\delta_s(y')) s^{\nu+1} ds, \\
G(X) &= \int_{\mathbb{R}^n} e^{2(X|v)} \frac{1}{E(v)} dv, \\
E(v) &= \int_{\mathbb{R}^n} e^{-2[P(\mu)-(v|\mu)]} d\mu.
\end{aligned}
$$

Moreover, by changing the integral variables ($s \leftrightarrow sy_{n+1}$) in the above integral, the equation (5.1) can be obtained, where

$$
(5.3)\qquad \Phi(\tau) = \frac{2}{(2\pi)^{n+1}} \int_0^\infty e^{-2s} G(\delta_s(\tau)) s^{\nu+1} ds.
$$

It is easy to see the regularity of $\Phi(\tau)$ on Δ_P. Suppose that $\rho = 1$. If τ tends to $\partial\Delta_P$, then z tends to $\partial\Omega_P$. Since the Bergman kernel of Ω_P is unbounded as $z \to \partial\Omega_P$, then (5.1) implies that $\Phi(\tau)$ is also unbounded as $\tau \to \partial\Delta_P$. $\square$

Let q be the rank of the Levi form at the origin and let $p = n - q$. In this case, the number m_j such that $m_j > 1$ is p. Set $\Gamma_{\hat{m}} = (1/2m_1, \ldots, 1/2m_p) \in \mathbb{N}^p$. By using the following lemma from [21], the singularity of the Bergman kernel can be essentially expressed by using the $(p+1)$-variables.

LEMMA 5.2 ([21]). *The rotation L in Lemma 2.1 can be chosen so that*

$$
P(y') = \hat{P}(y_1, \ldots, y_p) + c_{p+1} y_{p+1}^2 + \cdots + c_n y_n^2,
$$

where $\hat{P}$ is $(\Gamma_{\hat{m}} : 1)$-homogeneous and c_j are positive numbers.

Now set $\Delta_{\hat{P}} = \{\hat{\tau} \in \mathbb{R}^p; \hat{P}(\hat{\tau}) < 1\}$. Let us introduce the variables $(\hat{\tau}_1, \ldots, \hat{\tau}_p, \hat{\rho}) = (\hat{\tau}, \hat{\rho})$ as

$$
\begin{cases}
\hat{\tau}_j = y_j [y_{n+1} - \sum_{k=p+1}^n c_k y_k^2]^{-1/2m_j} & (j = 1, \ldots, p) \\
\hat{\rho} = [y_{n+1} - \sum_{k=p+1}^n c_k y_k^2]^{1/2m}.
\end{cases}
$$

The above relations induce a mapping from ω to $\Delta_{\hat{P}} \times (0, \infty)$. In the model case, the singularity of the Bergman kernel is essentially expressed by $(\hat{\tau}, \hat{\rho}) \in \Delta_{\hat{P}} \times (0, \infty)$.

PROPOSITION 5.3. *The Bergman kernel $K(y)$ has the form*

$$K(y) = \hat{\Phi}(\hat{\tau})\hat{\rho}^{-2m(\hat{\nu}+q+2)},$$

where $\hat{\nu} = \sum_{j=1}^{p} 1/m_j$ and $\hat{\Phi}(\hat{\tau})$ is a C^∞-function on $\Delta_{\hat{P}}$ and is unbounded as $\hat{\tau}$ approaches $\partial\Delta_{\hat{P}}$.

PROOF. Set $\hat{\mu} = (\mu_1, \dots, \mu_p)$, $\hat{v} = (v_1, \dots, v_p)$, $\hat{X} = (X_1, \dots, X_p)$, $\hat{y} = (y_1, \dots, y_p)$. We define

$$\hat{G}(\hat{X}) = \int_{\mathbb{R}^p} e^{2(\hat{X}|\hat{v})}\frac{1}{\hat{E}(\hat{v})}d\hat{v} \quad \text{and} \quad \hat{E}(\hat{v}) = \int_{\mathbb{R}^p} e^{-2[\hat{P}(\hat{\mu})-(\hat{v}|\hat{\mu})]}d\hat{\mu}.$$

By Lemma 5.2,

$$\begin{aligned}
E(v) &= \int_{\mathbb{R}^n} e^{-2[P(\mu)-(v|\mu)]}d\mu \\
&= \hat{E}(\hat{v}) \prod_{j=p+1}^{n} \int_{\mathbb{R}} e^{-2(c_j\mu_j^2 - v_j\mu_j)}d\mu_j \\
&= \left(\frac{\pi}{2}\right)^{\frac{n-p}{2}} \left(\prod_{j=p+1}^{n} c_j^{-1/2}\right) e^{\frac{1}{2}\sum_{k=p+1}^{n}\frac{1}{c_k}v_k^2}\hat{E}(\hat{v}).
\end{aligned}$$

By substituting the above into the integral $G(X)$,

$$\begin{aligned}
G(X) &= \int_{\mathbb{R}^n} e^{(X|v)}\frac{1}{E(v)}dv \\
&= \left(\frac{2}{\pi}\right)^{\frac{n-p}{2}} \left(\prod_{j=p+1}^{n} c_j^{1/2}\right) \hat{G}(\hat{X}) \prod_{j=p+1}^{n} \int_{\mathbb{R}} e^{-\frac{1}{2}\frac{1}{c_j}v_j^2 + 2X_jv_j}dv_j \\
&= 2^{n-p}\left(\prod_{j=p+1}^{n} c_j\right) \hat{G}(\hat{X}) \prod_{j=1}^{n} e^{2c_jX_j^2}.
\end{aligned}$$

Moreover, by substituting the above into the integral $K(y)$,

$$\begin{aligned}
&K(y) \\
&= \frac{1}{2^p\pi^{n+1}}\left(\prod_{j=p+1}^{n} c_j\right)\int_0^\infty e^{-2s[y_{n+1}-\sum_{k=p+1}^{n}c_ky_k^2]}\hat{G}(\hat{\delta}_s(\hat{y}))s^{q+\hat{\nu}+1}ds \\
&= \hat{\Phi}(\hat{\tau})\hat{\rho}^{-q-\hat{\nu}-2},
\end{aligned}$$

where

$$\hat{\Phi}(\hat{\tau}) = \frac{1}{2^p \pi^{n+1}} \left(\prod_{j=p+1}^{n} c_j \right) \int_0^\infty e^{-2s} \hat{G}(\hat{\delta}_s(\hat{\tau})) s^{q+\hat{\nu}+1} ds$$

and $\hat{\delta}_s(\hat{\tau}) = (s^{1/2m_1}\tau_1, \dots, s^{1/2m_p}\tau_p)$. It is easy to see the regularity of $\hat{\Phi}$ on $\Delta_{\hat{P}}$. Suppose that $\hat{\rho} = 1$. If $\hat{\tau}$ tends to $\partial\Delta_P$, then z tends to $\partial\Omega_P$. In a similar fashion as in Proposition 5.1, $\hat{\Phi}(\hat{\tau})$ is also unbounded as $\hat{\tau} \to \partial\Delta_{\hat{P}}$. □

6. Localization

In this section, we show that the singularity of the Bergman kernel is completely determined by local geometry of the boundary under some assumption. Similar types of localization lemmas have been obtained in [22],[23],[15], but our localization is concerned with slightly more general case. Actually, our proof of Theorem 2.2 needs the localization lemmas, below. For $k \in \mathbb{N}$ and $R > 0$, let $B_k(R)$ be the k-dimensional ball of radius R. In this paper, we sometimes use c, c_j, C etc. for various constants without further comments.

Suppose that a domain $\omega \subset \mathbb{R}^{n+1}$ satisfies the following hypotheses. There exist a neighborhood V_0 of the origin in $\mathbb{R}^n$, a C^1-function f on V_0 and positive numbers δ_0, C_0 such that

(1) $f(x) > 0$ if $x \neq 0$ and $f(0) = 0$,

(2) $\omega \cap [V_0 \times (0, \delta_0)] = \{y \in V_0 \times (0, \delta_0); y_{n+1} > f(y')\}$,

(3) $\omega \setminus [V_0 \times (0, \delta_0)] \subset \{y \in \mathbb{R}^{n+1}; y_{n+1} > C_0|y'|\}$.

If ω is a convex domain with C^∞-smooth boundary $\partial\omega$ containing the origin and the origin is of $\mathbb{R}$-finite type, then ω essentially satisfies the above hypotheses.

Let us consider the Bergman kernel $K(y)$ for a tube domain $\mathbb{R}^{n+1} + i\omega$. For a set W in $\mathbb{R}^n$, let $\mathcal{N}(W)$ be the set of open sets in W containing the origin. For $U \in \mathcal{N}(B^*)$, $V \in \mathcal{N}(\mathbb{R}^n)$ and $\delta > 0$, define the integral:

$$K(y; U, V, \delta) = \frac{1}{(2\pi)^{n+1}} \int_{\Lambda^*(U)} e^{-2(y|u)} \frac{1}{\varphi(u; V, \delta)} du,$$

$$\varphi(u; V, \delta) = \int_{\omega \cap [V \times (0,\delta)]} e^{-2(u|w)} dw,$$

where $\Lambda^*(U) := \{(t\hat{u}', t) \in \mathbb{R}^{n+1}; t > 0, \hat{u}' \in U\}$.

PROPOSITION 6.1. *For any $U \in \mathcal{N}(B^*)$, $K - K(\cdot; U, \mathbb{R}^n, \infty)$ is real analytic near the origin.*

PROOF. The hypothesis (1) on f implies that $f(x) = o(|x|)$ as $x \to 0$. Therefore, for a small positive number ϵ, $[\Lambda^* \setminus \Lambda^*(U)] \cap \partial B_{n+1}(1)$ can be divided into finitely many sets $\{U_j\}$ such that, for each j, there exists nonempty set

$$\omega_j = \{w \in \omega; w \in B_{n+1}(1) \text{ and } (w|\tilde{u}) < -\epsilon \text{ for } \tilde{u} \in U_j\},$$

whose volume (denoted by $Vol(\omega_j)$) is positive. If $\tilde{u} \in U_j$ and $|y| < \epsilon/2$, then $-2(\tilde{u}|w - y) > 2(\epsilon - \epsilon/2) = \epsilon$. For $u \in \Lambda_j := \{t\tilde{u}; t > 0, \tilde{u} \in U_j\}$,

$$e^{2(y|u)}\varphi(u) = \int_\omega e^{-2(u|w-y)}dw \geq \int_{\omega_j} e^{-2(u|w-y)}dw \geq Vol(\omega_j)e^{\epsilon|u|}. \tag{6.1}$$

Here we divide the integral as follows.

$$\begin{aligned} K(y) - K(y; U, \mathbb{R}^n, \infty) &= \frac{1}{(2\pi)^{n+1}} \int_{\Lambda^* \setminus \Lambda^*(U)} e^{-2(y|u)} \frac{1}{\varphi(u)} du \\ = \frac{1}{(2\pi)^{n+1}} \sum_j & \int_{\Lambda_j \cap [\Lambda^* \setminus \Lambda^*(U)]} e^{-2(y|u)} \frac{1}{\varphi(u)} du. \end{aligned}$$

The inequality (6.1) implies that each integral in the above sum is real analytic in $B_{n+1}(\epsilon/2)$, so the lemma can be obtained. □

PROPOSITION 6.2. *For any $V \in \mathcal{N}(V_0)$, there exists $U_1 \in \mathcal{N}(B^*)$ such that if $U \in \mathcal{N}(U_1)$, then $K(\cdot; U, \mathbb{R}^n, \infty) - K(\cdot; U, V, \delta_0)$ is real analytic near the origin.*

PROOF. By simple computation,

$$\begin{aligned} &|K(y; U, \mathbb{R}^n, \infty) - K(y; U, V, \delta)| \\ &\leq \frac{1}{(2\pi)^{n+1}} \int_{\Lambda^*(U)} e^{-2(y|u)} \left| \frac{1}{\varphi(u)} - \frac{1}{\varphi(u; V, \delta)} \right| du \\ &\leq \frac{1}{(2\pi)^{n+1}} \int_{\Lambda^*(U)} e^{-2(y|u)} \frac{|\varphi(u) - \varphi(u; V, \delta)|}{|\varphi(u)||\varphi(u; V, \delta)|} du. \end{aligned}$$

From the above inequalities, the proposition can be shown by using the two lemmas below. $\square$

LEMMA 6.3. *For any $V \in \mathcal{N}(V_0)$, there exist $U_2 \in \mathcal{N}(B^*)$ and positive numbers A_1, A_2 such that if $U \in \mathcal{N}(U_2)$, then*

$$|\varphi(u) - \varphi(u; V, \delta_0)| \leq A_1 u_{n+1}^{-1} e^{-A_2 u_{n+1}} \qquad \text{for } u \in \Lambda^*(U) \setminus B_{n+1}(1).$$

PROOF. We divide the integral and define $\psi_1(u), \psi_2(u)$ as follows.

$$\begin{aligned} &\varphi(u) - \varphi(u; V, \delta_0) \\ &= \int_{\omega \cap [V \times (\delta_0, \infty)]} e^{-2(u|w)} dw + \int_{\omega \setminus [V \times (0,\infty)]} e^{-2(u|w)} dw \\ &=: \psi_1(u) + \psi_2(u). \end{aligned} \tag{6.2}$$

First, for $V \in \mathcal{N}(V_0)$, there is a positive number η_1 such that $V \subset B_{n+1}(\eta_1)$. Then

$$\begin{aligned} \psi_1(u) &\leq \int_{V \times (\delta_0, \infty)} e^{-2(u|w)} dw \\ &\leq \frac{1}{2u_{n+1}} e^{-2\delta_0 u_{n+1}} \int_{|w'| < \eta_1} e^{2|u'||w'|} dw' \quad \leq \frac{C}{u_{n+1}} e^{-2[\delta_0 u_{n+1} - \eta_1 |u'|]}. \end{aligned} \tag{6.3}$$

Second, for $V \in \mathcal{N}(V_0)$, there are positive numbers C_1, η_2 such that the set $\Gamma_1 := \{w \in \mathbb{R}^{n+1}; w_{n+1} > C_1 |w'| \text{ and } |w'| > \eta_2\}$ contains the set $\omega \setminus [V \times (0, \infty)]$. Then

$$\begin{aligned} \psi_2(u) &\leq \int_{\Gamma_1} e^{-2(u|w)} dw \leq \frac{1}{2u_{n+1}} \int_{|w'| > \eta_2} e^{-2|w'|[C_1 u_{n+1} - |u'|]} dw' \\ &\leq \frac{C}{u_{n+1}(C_1 u_{n+1} - |u'|)^n} e^{-2\eta_2 [C_1 u_{n+1} - |u'|]}. \end{aligned} \tag{6.4}$$

Putting (6.2),(6.3),(6.4) together, we can find a positive number M such that if $|u'| < M u_{n+1}$, then the inequality in the lemma holds. It is enough to set $U_2 = B_n(M)$. $\square$

Lemma 6.4. *For any $V \in \mathcal{N}(V_0)$, there are $U_3 \in \mathcal{N}(B^*)$ and a positive number A_3 such that if $U \in \mathcal{N}(U_3)$, then*

$$\varphi(u) \geq \varphi(u; V, \delta_0) \geq \frac{A_3}{u_{n+1}^{n+1}} \qquad \text{for } u \in \Lambda^*(U) \setminus B_{n+1}(1).$$

Proof. There exist positive numbers η_3, C_2 such that the set $\Gamma_2 := \{w \in \mathbb{R}^{n+1}; |w'| < \eta_3 \text{ and } C_2|w'| < w_{n+1} < \delta_0\}$ is contained in $\omega \cap [V \times (0, \delta_0)]$. Therefore, if $u \in \Lambda^*(U) \setminus B_{n+1}(1)$, then

$$\varphi(u; V, \delta_0) \geq \int_{\Gamma_2} e^{-2(u|w)} dw = \frac{C}{2u_{n+1}} \frac{1 - e^{-2\eta_3[C_2 u_{n+1} + |u'|]}}{(2[C_2 u_{n+1} + |u'|])^n} \geq \frac{C}{u_{n+1}^{n+1}}. \ \square$$

Let ω_1, ω_2 be domains satisfying the hypotheses in the beginning of this section and satisfying $\omega_1 \cap [V_0 \times (0, \delta_0)] = \omega_2 \cap [V_0 \times (0, \delta_0)]$. For $j = 1, 2$, let $K^{(j)}(y)$ be the Bergman kernel of the domain $\Omega_j = \mathbb{R}^{n+1} + i\omega_j$, respectively. As a corollary of Propositions 6.1 and 6.2, we can get the following.

Proposition 6.5. *$K^{(1)} - K^{(2)}$ is real analytic near the origin.*

7. Proof of Theorem 2.2

7.1. Preparation

We introduce a coordinate into $\mathbb{R}^{n+1}$ containing the base ω as in Section 2.2. By using Lemma 2.1 in Section 2 and Proposition 6.5 in Section 6, in order to analyze the singularities of the Bergman kernel, it is sufficient to consider a domain whose base is $\omega = \{y \in \mathbb{R}^{n+1}; y_{n+1} > f(y')\}$, where f is a convex C^∞-function on $\mathbb{R}^n$ such that f takes the form $f(y') = P(y') + R(y')$ near the origin, where P, R are as in Lemma 2.1, and f satisfies $f(y') \geq C|y'|^{1+\epsilon}$ for $|y'| \geq 1$ where C, ϵ are some positive numbers.

Here recall some notation and symbol. Let m be the least common multiplicity of $m_1, \ldots, m_n$ and let $l_j = m/m_j$. Let $\tilde{\pi}$ be a mapping from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$ such that $\tilde{\pi}(\tau_1, \ldots, \tau_n, \rho) = \tilde{\pi}(\tau, \rho) = (y_1, \ldots, y_n, y_{n+1})$, where

$$\left\{ \begin{array}{ll} y_j = \tau_j \rho^{l_j} & (j = 1, \ldots, n) \\ y_{n+1} = \rho^{2m}. & \end{array} \right. \tag{7.1}$$

We set $U = \tilde{\pi}^{-1}(\omega)$ and $\pi = \tilde{\pi}|_{\bar{U}}$.

By changing the integral variables ($s^{1/2m_j} w_j \leftrightarrow w_j$ $(j = 1, \dots, n)$, $s y_{n+1} \leftrightarrow s$), (4.2) in Section 4 can be rewritten as

$$
(7.2)\qquad
\begin{aligned}
K(y) &= \frac{2}{(2\pi)^{n+1}} \frac{1}{\rho^{2m(\nu+2)}} \int_0^\infty e^{-2s} G(\delta_s(\tau); \rho s^{-1/2m}) s^{\nu+1} ds, \\
G(X;\xi) &= \int_{\mathbb{R}^n} e^{2(X|v)} \frac{1}{E(v;\xi)} dv \qquad (X \in \mathbb{R}^n, \xi \in (0,1]), \\
E(v;\xi) &= \int_{\mathbb{R}^n} e^{-2[P(w)+a(w;\xi)-(v|w)]} dw, \\
&\quad a(w;\xi) = \xi^{-2m} R(\delta_{\xi^{2m}}(w)).
\end{aligned}
$$

We divide the integral in (7.2) and define the integrals K_1, K_2, K_3 as follows:

$$
\begin{aligned}
K(y) &= \frac{2}{(2\pi)^{n+1}} \frac{1}{\rho^{2m(\nu+2)}} \times \\
&\quad \left\{ \int_1^\infty + \int_{\rho^{2m}}^1 + \int_0^{\rho^{2m}} \right\} e^{-2s} G(\delta_s(\tau); \rho s^{-1/2m}) s^{\nu+1} ds \\
&=: \frac{2}{(2\pi)^{n+1}} \frac{1}{\rho^{2m(\nu+2)}} \{K_1(\tau,\rho) + K_2(\tau,\rho) + K_3(\tau,\rho)\}.
\end{aligned}
$$

It is easy to see that $\rho^{-2m(\nu+2)} K_3(\tau,\rho)$ is real analytic on a neighborhood of $\bar{U}$. We will analyze the behaviors of the functions $K_1(\tau,\rho)$ and $K_2(\tau,\rho)$ at $\partial\Delta_P \times \{0\}$. Owing to the mapping π, we can decompose their singularities clearly and investigate them with respect to each variables. Roughly speaking, it will be shown that $K_1(\tau,\rho)$ can be smoothly extended when ρ tends to 0 but is unbounded as τ tends to $\partial\Delta_P$, while $K_2(\tau,\rho)$ has the logarithmic singularities at $\rho = 0$ but is smooth for $\tau \in \overline{\Delta_P}$.

7.2. Analysis of $K_1(\tau,\rho)$

From (7.2), by changing the integral variables ($w_j \leftrightarrow s^{1/2m_j} w_j$ $(j = 1, \dots, n)$), we have

$$
(7.3)\qquad
\begin{aligned}
K_1(\tau,\rho) &= \int_1^\infty e^{-2s} \tilde{G}(\tau,\rho;s) s^{n+1} ds, \\
\tilde{G}(\tau,\rho;s) &= \int_{\mathbb{R}^n} e^{2s(\tau|v)} \frac{1}{\tilde{E}(v;\rho;s)} dv, \\
\tilde{E}(v;\rho;s) &= \int_{\mathbb{R}^n} e^{-2s[P(w)+a(w;\rho)-(v|w)]} dw.
\end{aligned}
$$

This expression is useful to analyze $K_1(\tau,\rho)$.

7.2.1 Localization

For $\eta > 0$, set $S_\eta := \{v \in \mathbb{R}^n; |v| = \eta\}$. In particular we write $S = S_1$. In order to localize the singularities, let us introduce the following two functions χ_c and R_d.

We express the variable v in the integral by using a polar coordinate: $v = \delta_u^*(\hat{v})$ for $u > 0$, $\hat{v} \in S_\eta$. Here η is a small positive number, which will be determined later in the proof of Lemma 7.5. For $c > 0$, $\chi_c(v,\rho) \in C^\infty(\mathbb{R}^n \times (0,\infty))$ is a cut-off function such that

- $\chi_c(v,\rho)$ is independent of $\hat{v} \in S_\eta$.
- $\chi_c(v,\rho) = 1$ if $0 \le u \le c\rho^{-2m}/2$ and $\chi_c(v,\rho) = 0$ if $u \ge c\rho^{-2m}$.
- $0 \le \chi_c(v,\rho) \le 1$ for $v \in \mathbb{R}^n, \rho > 0$.

For $R > 0$, define $D(R) = \{w = \delta_{u^{2m}}(\tilde{w}) \in \mathbb{R}^n; 0 \le u \le R, \tilde{w} \in S\}$. For $d > 0$, let R_d be a C^∞-function in $\mathbb{R}^n$ such that

- $R_d(w) = R(w)$ for $w \in D(d/2)$ and $|R_d(w)| \le |R(w)|$ for $w \in \mathbb{R}^n$.
- The support of R_d is contained in the set $D(d)$.
- There exists a positive number C_0 such that $P(w) + R_d(w) \ge C_0|w|$ if $w \notin D(d/2)$.

Here let us introduce the functions χ_c, R_d into the integrals (7.3) and define the integrals with the parameters c, d as follows.

$$
\begin{aligned}
K_1(\tau,\rho;c,d) &= \int_1^\infty e^{-2s}\tilde{G}(\tau,\rho;s;c,d)s^{n+1}ds,\\
\tilde{G}(\tau,\rho;s;c,d) &= \int_{\mathbb{R}^n} e^{2s(\tau|v)}\frac{\chi_c(v,\rho)}{\tilde{E}(v;\rho;s;d)}dv,\\
\tilde{E}(v;\rho;s;d) &= \int_{\mathbb{R}^n} e^{-2s[P(w)+a_d(w;\rho)-(v|w)]}dw,\\
a_d(w;\rho) &= \rho^{-2m}R_d(\delta_{\rho^{2m}}(w)).
\end{aligned}
\tag{7.4}
$$

Note that the above integrals formally tend to K_1, $\tilde{G}$, $\tilde{E}$ in (7.3), as $c, d \to \infty$, respectively.

The following lemma implies that it suffices to analyze the function $K_1(\tau,\rho;c,d)$, with some small c,d, to see the singularities $K_1(\tau,\rho)$.

Lemma 7.1. *For any $d>0$, there exists $c_0>0$ such that if $c\in(0,c_0)$, then $K_1(\cdot,\cdot;c,d)-K_1(\cdot,\cdot)$ is real analytic near an open neighborhood of $\overline{U_\delta}$.*

Proof. First we regard $K_1(\tau,\rho;c,d)-K_1(\tau,\rho)$ as a function of y. By changing the integral variables from (4.2) into (7.3), the integral regions B^*,A are changed into $\delta^*_{1/\rho^{2m}}(B^*),\delta_{1/\rho^{2m}}(A)$, respectively. The regions B^*,A can be localized as in Propositions 6.1 and 6.2. Corresponding the supports of χ_c,R_d to the sets $\delta^*_{1/\rho^{2m}}(U),\delta_{1/\rho^{2m}}(V)$ respectively, where U,V are as in Section 6, we can see the real analyticity of the above function with respect to y near the origin. Moreover, the property (7.1) of π implies the real analyticity with respect to (τ,ρ). $\square$

7.2.2 Properties for $a_d(w;X)$

The difference between integrals (7.2) and (5.2) in Section 5 shows that the general case of finite type domains can be considered as some kind of perturbation of the model case as in Section 5 and the information of the perturbation is concentrated in the term $a(w;\rho)$. The localization lemma (Lemma 7.1) is necessary to investigate the function $a_d(w;X)$ for $(w;X)\in\mathbb{R}^n\times[0,\infty)$.

Since $|R(w)|\le C\sigma(w)^{1+\gamma}$ for small $|w|$, it is easy to see that $a_d(w;X)$ is a C^∞-function on $\mathbb{R}^n\times[0,\infty)$. The following two lemmas are used to estimate the integral $\tilde{E}(v;\rho;s;d)$ in the next subsection.

Lemma 7.2. *For any $\epsilon>0$, there exists a positive number d_0 such that if $d\in(0,d_0]$, then*

$$|a_d(w;X)|\le\epsilon\sigma(w)\quad \textit{for } (w;X)\in\mathbb{R}^n\times[0,\infty).$$

Proof. From Lemma 2.1 in Section 2, if $|w|$ is small, then $|R(w)|\le C\sigma(w)^{1+\gamma}$. Therefore, if $d>0$ is a sufficiently small and if $\delta_{X^{2m}}(w)\in D(d)$, then we have

$$\begin{aligned}|a_d(w;X)|&=|X^{-2m}R_d(\delta_{X^{2m}}(w))|\le|X^{-2m}R(\delta_{X^{2m}}(w))|\\&\le CX^{2m\gamma}\sigma(w)^{1+\gamma}\le C(X^{2m}\sigma(w))^\gamma\cdot\sigma(w)\\&=C\sigma(\delta_{X^{2m}}(w))^\gamma\cdot\sigma(w).\end{aligned}$$

Note that C is independent of d. Of course, $a_d(w, X) = 0$, if $\delta_{X^{2m}}(w) \notin D(d)$. The lemma can be shown by using the above estimates. $\square$

Next the following lemma shows that the partial derivatives of a_d with respect to X can be uniformly estimated by using $\sigma(w)$.

LEMMA 7.3. *For any $k \in \mathbb{Z}_+$, there exists a positive number C_k such that*

$$\left| \frac{\partial^k}{\partial X^k} a_d(w; X) \right| \leq C_k \sigma(w)^{1+\frac{k}{2m}} \quad \textit{for } (w; X) \in \mathbb{R}^n \times [0, \infty).$$

PROOF. Let S_1, S_2 be C^∞-functions on $\mathbb{R}^n$ such that $R_d(w) = S_1(w)S_2(w)$ and $S_2(w)$ equals to 1 on $D(d/2)$ and its support is contained in $D(R)$. For any $l \geq 1$, Taylor's formula implies that there exist C^∞-functions σ_j $(j = 1, \ldots, l-1)$ and $\mathcal{R}_l$ such that

$$S_1(w) = \sigma_1(w) + \cdots + \sigma_{l-1}(w) + \mathcal{R}_l(w)$$

and each σ_j is a polynomial with $(\Gamma_m : 1 + \frac{j}{2m})$-homogeneity and $\mathcal{R}_l$ satisfies the estimate: $|\mathcal{R}_l(w)| \leq C_l \sigma(w)^{1+\frac{l}{2m}}$ on $D(R)$ where C_l is a positive constant.

Since each σ_j has the above homogeneity, we have

$$S_1(\delta_{X^{2m}}(w)) = X^{2m+1}\sigma_1(w) + \cdots + X^{2m+l-1}\sigma_{l-1}(w) + \mathcal{R}_l(\delta_{X^{2m}}(w)).$$

By using the above equation, a simple computation implies that there exists the C^∞-function $\tilde{\mathcal{R}}_l(a) = \tilde{\mathcal{R}}_l(a_1, \ldots, a_n)$ on $\mathbb{R}^n$, with $a_j = X^{l_j} w_j$, such that

$$\frac{\partial^l}{\partial X^l} \left(\frac{1}{X^{2m}} S_1(\delta_{X^{2m}}(w)) \right) = \frac{1}{X^{2m+l}} \tilde{\mathcal{R}}_l(a).$$

Here $\tilde{\mathcal{R}}_l(a)$ satisfies $|\tilde{\mathcal{R}}_l(a)| \leq C_l \sigma(a)^{1+\frac{l}{2m}}$ where C_l is a positive constant. Since the equation $\sigma(a) = X^{2m}\sigma(w)$ holds, we can get

$$\begin{aligned} &\left| \frac{\partial^l}{\partial X^l} \left(\frac{1}{X^{2m}} S_1(\delta_{X^{2m}}(w)) \right) \right| \\ &\qquad \leq \frac{|\tilde{\mathcal{R}}_l(a)|}{\sigma(a)^{1+\frac{l}{2m}}} \sigma(w)^{1+\frac{l}{2m}} \leq C_l \sigma(w)^{1+\frac{l}{2m}}. \end{aligned} \tag{7.5}$$

It is easy to see that the above inequalities hold when $l=0$.

On the other hand, a simple computation implies that there exists $\tilde{\tilde{\mathcal{R}}}_l \in C_0^\infty(\mathbb{R}^n)$ such that

$$\frac{\partial^l}{\partial X^l} S_2(\delta_{X^{2m}}(w)) = \frac{1}{X^l}\tilde{\tilde{\mathcal{R}}}_l(a).$$

When $l \geq 1$, since $\tilde{\tilde{\mathcal{R}}}_l(a)$ is identically zero on $D(d/2)$, we have

$$\left|\frac{\partial^l}{\partial X^l} S_2(\delta_{X^{2m}}(w))\right| \leq \frac{|\tilde{\tilde{\mathcal{R}}}_l(a)|}{\sigma(a)^{\frac{l}{2m}}}\sigma(w)^{\frac{l}{2m}} \leq C_l \sigma(w)^{\frac{l}{2m}}. \tag{7.6}$$

Of course, when $l=0$, the above inequalities hold. Putting (7.5),(7.6) together, we can obtain the inequality in the lemma as follows: for $k \in \mathbb{Z}_+$,

$$\begin{aligned}
\left|\frac{\partial^k}{\partial X^k} a(w;X)\right| &= \left|\frac{\partial^k}{\partial X^k}\left(\frac{1}{X^{2m}} S_1(\delta_{X^{2m}}(w)) \cdot S_2(\delta_{X^{2m}}(w))\right)\right| \\
&\leq \sum_{j=0}^{k} C_j \left|\frac{\partial^j}{\partial X^j}\left(\frac{1}{X^{2m}} S_1(\delta_{X^{2m}}(w))\right)\right| \cdot \left|\frac{\partial^{k-j}}{\partial X^{k-j}} S_2(\delta_{X^{2m}}(w))\right| \\
&\leq C_k \sigma(w)^{1+\frac{k}{2m}}.
\end{aligned}$$

This completes the proof of the lemma. □

By using the above argument, the limit of the partial derivatives of a_d can be computed. For $k \in \mathbb{N}$, let $\sigma_k(w)$ be the $(\Gamma_m : 1 + \frac{k}{2m})$-homogeneous polynomial as in the proof of Lemma 7.3.

LEMMA 7.4. *For any* $k \in \mathbb{N}$,

$$\lim_{X \to 0} \frac{\partial^k}{\partial X^k} a_d(w;X) = k!\sigma_k(w) \quad \textit{for each } w \in \mathbb{R}^n.$$

PROOF. We use the same symbols as in the proof of the previous lemma. As in the proof of Lemma 7.3, we have

$$\begin{aligned}
&\frac{\partial^l}{\partial X^l}\left(\frac{1}{X^{2m}}S_1(\delta_{X^{2m}}(w))\right)\\
&=\frac{\partial^l}{\partial X^l}\left\{X\sigma_1(w)+\cdots+X^l\sigma_l(w)+\frac{1}{X^{2m}}\mathcal{R}_{l+1}(\delta_{X^{2m}}(w))\right\}\\
&=l!\sigma_l(w)+\frac{\partial^l}{\partial X^l}\left\{\frac{1}{X^{2m}}\mathcal{R}_{l+1}(\delta_{X^{2m}}(w))\right\}\\
&=l!\sigma_l(w)+\frac{1}{X^{2m+l}}\tilde{\mathcal{R}}_{l+1}(a).
\end{aligned}$$

Since the estimate $|X^{-2m-l}\tilde{\mathcal{R}}_{l+1}(a)|\le C_{l+1}\sigma(w)^{1+\frac{l+1}{2m}}X$ holds, we have

$$\lim_{X\to 0}\frac{\partial^l}{\partial X^l}\left(\frac{1}{X^{2m}}S_1(\delta_{X^{2m}}(w))\right)=l!\sigma_l(w)\quad\text{for each } w\in\mathbb{R}^n.$$

On the other hand, for $l\in\mathbb{N}$,

$$\lim_{X\to 0}\frac{\partial^l}{\partial X^l}\left(S_2(\delta_{X^{2m}}(w))\right)=0\quad\text{for each } w\in\mathbb{R}^n.$$

Therefore, for each $w\in\mathbb{R}^n$,

$$\begin{aligned}
&\frac{\partial^k}{\partial X^k}a_d(w;X)\\
&=\sum_{j=0}^{k}C_j\frac{\partial^j}{\partial X^j}\left(\frac{1}{X^{2m}}S_1(\delta_{X^{2m}}(w))\right)\cdot\frac{\partial^{k-j}}{\partial X^{k-j}}S_2(\delta_{X^{2m}}(w))\\
&\longrightarrow k!\sigma_k(w)\qquad\text{as } X\to 0.
\end{aligned}$$

This completes the proof of the lemma. □

7.2.3 Estimates for $\tilde{E}(v;\rho;s;d)$

Let us show the following lemma. Its proof is technically complicated, so essential ideas will be explained in Remark 7.6 after the proof.

LEMMA 7.5. *We can set $d=d_1>0$ as follows. For any $k\in\mathbb{N}$, there exists a positive constant C_k such that*

$$\left|\frac{\partial^k}{\partial\rho^k}\tilde{E}(v;\rho;s;d_1)\right|\le C_k s^k u^{(1+\frac{1}{2m})k}\tilde{E}(v;\rho;s;d_1)$$

for $u \geq 1,\ s \geq 1.$

PROOF. For the computation below, the integral variables in the integral $\tilde{E}(v;\rho;s;d)$ in (7.4) are changed as follows:

$$\tilde{E}(v;\rho;s;d) = u^{\nu/2}\int_{\mathbb{R}^n} e^{-2su[P(w)+a_d(w;X)-(\hat{v}|w)]}dw \quad \text{with } X = u^{1/2m}\rho$$

$(u > 0,\ |\hat{v}| = \eta)$. Note that the above integral variable w is not the same as that in (7.4).

Hereafter, in this proof, we always assume that $u \geq 1$ and $s \geq 1$. Let Σ denote the set $\{w \in \mathbb{R}^{n+1}; P(w) < 1\}$ and define three integrals $\tilde{E}_1, \tilde{E}_2, \tilde{E}_3$ by

$$\begin{aligned}
\tilde{E}_1(v;\rho;s;d) &= u^{\nu/2}\int_{\Sigma} \frac{\partial^k}{\partial X^k} e^{-2su[P(w)+a_d(w;X)-(\hat{v}|w)]}dw,\\
\tilde{E}_2(v;\rho;s;d) &= u^{\nu/2}\int_{\mathbb{R}^n\setminus\Sigma} \frac{\partial^k}{\partial X^k} e^{-2su[P(w)+a_d(w;X)-(\hat{v}|w)]}dw,\\
\tilde{E}_3(v;\rho;s;d) &= u^{\nu/2}\int_{\Sigma} e^{-2su[P(w)+a_d(w;X)-(\hat{v}|w)]}dw.
\end{aligned}$$

The convergence of the second integral will be shown soon later.

[*An estimate of the integrand*] We give some estimate for the integrand of the above integrals. By a direct computation, we have

$$\begin{aligned}
&\frac{\partial^k}{\partial X^k} e^{-2su[P(w)+a_d(w;X)-(\hat{v}|w)]}\\
&= \left(\sum_{l=1}^{k}(-2su)^l B_{kl}(w;X)\right) e^{-2su[P(w)+a_d(w;X)-(\hat{v}|w)]}.
\end{aligned} \tag{7.7}$$

Here each $B_{kl}(w;X)$ can be written as some linear combination of the products of $A_j(w;X) := \frac{\partial^j}{\partial X^j}a_d(w;X)$ $(1 \leq j \leq k)$. In fact,

$$B_{kl}(w;X) = \sum_{\alpha} C_\alpha \prod_{j=1}^{l} A_j(w;X)^{\alpha_j} \qquad (C_\alpha \in \mathbb{R}),$$

where the above summation is taken over all $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{Z}_+^l$ with $\alpha_1 \leq \cdots \leq \alpha_l$ and $\alpha_1 + \cdots + \alpha_l = k$. From Lemma 7.3, we obtain

$|B_{kl}(w;X)| \le C_{kl}\sigma(w)^{l+\frac{k}{2m}}$ for a positive constant C_{kl}. Therefore, there is a positive constant C_k such that

$$\begin{aligned}&\left|\frac{\partial^k}{\partial X^k}e^{-2su[P(w)+a_d(w;X)-(\hat{v}|w)]}\right| \\ &\qquad \le C_k\left(\sum_{l=1}^{k}(su)^l\sigma(w)^{l+\frac{k}{2m}}\right)e^{-2su[P(w)+a_d(w;X)-(\hat{v}|w)]}.\end{aligned}$$

[*An estimate for* $\tilde{E}_1(v;\rho;s;d)$] The above inequality gives the following estimate.

$$\begin{aligned}|\tilde{E}_1(v;\rho;s;d)| &\le u^{\frac{\nu}{2}}\int_{\Sigma}\left|\frac{\partial^k}{\partial X^k}e^{-2su[P(w)+a_d(w;X)-(\hat{v}|w)]}\right|dw \\ &\le C(su)^k u^{\frac{\nu}{2}}\int_{\Sigma}\sigma(w)^{1+\frac{k}{2m}}e^{-2su[P(w)+a_d(w;X)-(\hat{v}|w)]}dw \\ &\le Cs^k u^k\tilde{E}(v;\rho;s;d).\end{aligned} \tag{7.8}$$

[*An estimate for* $\tilde{E}_2(v;\rho;s;d)$] Before considering $\tilde{E}_2(v;\rho;s;d)$, we estimate the phase function in the integral. Let κ be the maximum of $\sigma(w)$ on $\partial\Sigma = \{w; P(w) = 1\}$ and set $\epsilon_1 = 1/(2\kappa)$. From Lemma 7.2, we can set $d = d_1 > 0$ such that $|a_{d_1}(w;X)| \le \epsilon_1\sigma(w)$. Moreover the homogeneity of $P(w) - \epsilon_1\sigma(w)$ implies

$$P(w) + a_{d_1}(w,X) \ge P(w) - \epsilon_1\sigma(w) \ge \frac{1}{2}P(w) \quad \text{on } \mathbb{R}^n. \tag{7.9}$$

We use the polar coordinate: $w = \delta_t(\hat{w})$ $(t > 0, \hat{w} \in \partial\Sigma)$. The value $\eta = |\hat{v}|$ can be chosen so small that, if $t \ge 1$,

$$\begin{aligned}\frac{1}{2}P(w) - (\hat{v}|w) &= \frac{t}{2} - \sum_{j=1}^{n}\hat{v}_j\hat{w}_j t^{\frac{1}{2m_j}} \\ &\ge \frac{t}{2}\left(1 - 2\sum_{j=1}^{n}|\hat{v}_j||\hat{w}_j|t^{\frac{1}{2m_j}-1}\right) \ge \frac{t}{4}.\end{aligned} \tag{7.10}$$

Here we set the above value depending only on $\partial\Sigma$. By using inequalities

(7.9),(7.10), $\tilde{E}_2(v;\rho;s;d_1)$ can be estimated as follows:

$$
\begin{aligned}
|\tilde{E}_2(v;\rho;s;d_1)| &\le u^{\frac{\nu}{2}} \int_{\mathbb{R}^n\setminus\Sigma} \left| \frac{\partial^k}{\partial X^k} e^{-2su[P(w)+a_{d_1}(w;X)-(\hat{v}|w)]} \right| dw \\
&\le C_k s^k u^{k+\frac{\nu}{2}} \int_{\mathbb{R}^n\setminus\Sigma} \sigma(w)^{k+\frac{k}{2m}} e^{-2su[P(w)+a_{d_1}(w;X)-(\hat{v}|w)]} dw \\
&\le C_k s^k u^{k+\frac{\nu}{2}} \int_{\mathbb{R}^n\setminus\Sigma} \sigma(w)^{k+\frac{k}{2m}} e^{-2su[\frac{1}{2}P(w)-(\hat{v}|w)]} dw \\
&= C_k s^k u^{k+\frac{\nu}{2}} \int_1^{\infty} t^{\frac{k}{2m}+\frac{\nu}{2}+k-1} e^{-\frac{1}{2}sut} dt \\
&\le C_k s^{-\frac{\nu}{2}-\frac{k}{2m}} u^{-\frac{k}{2m}} e^{-\frac{1}{2}su}.
\end{aligned}
\tag{7.11}
$$

[*An estimate for* $\tilde{E}_3(v;\rho;s;d)$] By a similar computation to the case of $\tilde{E}_2$, the phase can be estimated as follows. If $0 \le t \le 1$, then

$$
P(w) + a_{d_1}(w,X) - (\hat{v}|w) \le \frac{3}{2}P(w) - (\hat{v}|w) \le \frac{3}{2}t + \frac{1}{4}t^{\frac{1}{2M}} \le \frac{7}{4}t^{\frac{1}{2M}},
$$

where M is the maximum of $m_1, \dots, m_n$. The above estimate implies

$$
\begin{aligned}
\tilde{E}_3(v;\rho;s;d_1) &= u^{\frac{\nu}{2}} \int_{\Sigma} e^{-2su[P(w)+a_{d_1}(w,X)-(\hat{v}|w)]} dw \\
&\ge u^{\frac{\nu}{2}} \int_0^1 e^{-\frac{7}{2}sut^{\frac{1}{2M}}} t^{\frac{\nu}{2}-1} dt \\
&\ge C u^{(\frac{1}{2}-M)\nu} s^{-M\nu}.
\end{aligned}
\tag{7.12}
$$

[*The estimate in the lemma*] From (7.11),(7.12),

$$
|\tilde{E}_2(v;\rho;s;d_1)| \le C\tilde{E}_3(v;\rho;s;d_1) \le C\tilde{E}(v;\rho;s;d_1). \tag{7.13}
$$

By putting (7.8),(7.13) together,

$$
\begin{aligned}
\left| \frac{\partial^k}{\partial \rho^k} \tilde{E}(v;\rho;s;d_1) \right| &= u^{\frac{k}{2m}} \left| \frac{\partial^k}{\partial X^k} \tilde{E}(v;\rho;s;d_1) \right| \\
&\le u^{\frac{k}{2m}} \{ |\tilde{E}_1(v;\rho;s;d_1)| + |\tilde{E}_2(v;\rho;s;d_1)| \} \\
&\le C s^k u^{(1+\frac{1}{2m})k} \tilde{E}(v;\rho;s;d_1).
\end{aligned}
$$

This completes the proof of the lemma. □

REMARK 7.6. Here we explain essential ideas of the above proof. By applying "the chain rule", the derivatives of $\tilde{E}$ with respect to ρ reflect some powers of s, u times some kind of derivatives of the function $a_d(w; X)$. By using Lemma 7.3, these derivatives can be estimated by using $s, u, \sigma(w)$ only. Note that this estimate is independent of ρ. Moreover, we intrinsically apply an idea of "stationary phase method" (c.f. Chapter VIII in [33]). By dividing the integral, we can see an essentially strong part in $\tilde{E}_1$. Indeed the critical point of the phase is contained in the set Σ, and so $\tilde{E}_2$ can be considered as a small term when s, u are large. The restriction of $\sigma(w)$ on Σ is bounded and, as a result, the derivatives of $\tilde{E}$ with respect to ρ can be estimated by $\tilde{E}$ times some powers of s, u.

LEMMA 7.7. *For any $k \in \mathbb{Z}_+$, there exists a positive number C_k such that*

$$\left|\frac{\partial^k}{\partial\rho^k}\frac{1}{\tilde{E}(v;\rho;s;d_1)}\right| \le C_k s^k u^{(1+\frac{1}{2m})k}\frac{1}{\tilde{E}(v;\rho;s;d_1)} \qquad \text{for } u \ge 1, s \ge 1.$$

PROOF. A simple computation implies

$$\frac{\partial^k}{\partial\rho^k}\frac{1}{\tilde{E}(v;\rho;s;d_1)} = \left(\sum_{\beta} C_\beta \prod_{|\beta|=k}\frac{\tilde{E}^{(\beta_j)}(v;\rho;s;d_1)}{\tilde{E}(v;\rho;s;d_1)}\right)\frac{1}{\tilde{E}(v;\rho;s;d_1)}, \tag{7.14}$$

where $|\beta| := \beta_1 + \cdots + \beta_k$, the above summation is taken over all $\beta := (\beta_1, \ldots, \beta_k) \in \mathbb{Z}_+^k$ with $0 \le \beta_1 \le \cdots \le \beta_k \le k$, $\tilde{E}^{(k)}$ denotes the k-th partial derivatives of $\tilde{E}$ with respect to ρ and $C_\beta \in \mathbb{R}$ are constants. By applying Lemma 7.6 to the above equation, we can obtain the estimate in the lemma. □

Next we consider the limit of the derivatives of $\tilde{E}$.

LEMMA 7.8. *For any $k \in \mathbb{N}$, there exist $(\Gamma_m : l + \frac{k}{2m})$-homogeneous* *polynomials* $B_{kl}(w)$ $(l = 1, \ldots, k)$ *such that*

$$\lim_{\rho\to 0} \frac{\partial^k}{\partial\rho^k} \tilde{E}(v;\rho;s;d_1) = \sum_{l=1}^{k} (-2s)^l \int_{\mathbb{R}^n} B_{kl}(w) e^{-2s[P(w)-(v|w)]} dw =: \tilde{E}_k(v;s)$$

for each $v \in \mathbb{R}^n$*,* $s > 0$*. When* $k = 0$*, we have*

$$\lim_{\rho\to 0} \tilde{E}(v;\rho;s;d_1) = \int_{\mathbb{R}^n} e^{-2s[P(w)-(v|w)]} dw =: \tilde{E}_0(v;s).$$

Proof. First we consider the case $k \geq 1$. In the same argument as in the proof of Lemma 7.6, we have

$$\left| \frac{\partial^k}{\partial\rho^k} e^{-2s[P(w)+a_{d_1}(w;\rho)-(v|w)]} \right| \leq C(2s)^k \sigma(w)^{k+\frac{k}{2m}} e^{-2s[\frac{1}{2}P(w)-(v|w)]}.$$

Since the right hand side of the above is integrable with respect to w on $\mathbb{R}^n$, Lebesgue's convergence theorem implies

$$\lim_{\rho\to 0} \frac{\partial^k}{\partial\rho^k} \tilde{E}(v;\rho;s;d_1) = \int_{\mathbb{R}^n} \lim_{\rho\to 0} \frac{\partial^k}{\partial\rho^k} e^{-2s[P(w)+a_{d_1}(w;\rho)-(v|w)]} dw.$$

From (7.7),

$$\frac{\partial^k}{\partial\rho^k} e^{-2s[P(w)+a_{d_1}(w;\rho)-(v|w)]} = \sum_{l=1}^{k} (-2s)^l B_{kl}(w;\rho) e^{-2s[P(w)+a_{d_1}(w;\rho)-(v|w)]}.$$

By using Lemma 7.4,

$$\lim_{\rho\to 0} B_{kl}(w;\rho) = \lim_{\rho\to 0} \sum_{\alpha} C_\alpha \prod_{j=1}^{l} A_j(w;\rho)^{\alpha_j} = \sum_{\alpha} C_\alpha \prod_{j=1}^{l} (j!\sigma_j(w))^{\alpha_j}.$$

Let $B_{kl}(w)$ denote the above limit, then it is easy to check that the function $B_{kl}(w)$ has the $(\Gamma_m : l + \frac{k}{2m})$-homogeneity. By summarizing the above equations,

$$\lim_{\rho\to 0} \frac{\partial^k}{\partial\rho^k} e^{-2s[P(w)+a_{d_1}(w;\rho)-(v|w)]} = \sum_{l=1}^{k} (-2s)^l B_{kl}(w) e^{-2s[P(w)-(v|w)]}.$$

The lemma has been shown when $k \in \mathbb{N}$.

When $k = 1$, it is easy to get the equation in the lemma from the same argument. $\square$

7.2.4 Estimates for $\tilde{G}(\tau, \rho; s; c, d)$

We choose $c = c_1$ in the integral $\tilde{G}$ such that the real analyticity of $K_1(\cdot, \cdot; c_1, d_1) - K_1(\cdot, \cdot)$ holds as in Lemma 7.1. Let us introduce the set Γ_δ and auxiliary integrals: Ψ_μ, $\tilde{G}_*$, $\tilde{E}_*$. For $\delta > 0$, define

$$\Gamma_\delta = \{(\tau, \rho) \in \Delta_P \times [0, \delta); P(\tau) + C\rho^{2m\gamma}\sigma(\tau)^{1+\gamma} < 1\},$$

where C and γ are as in Lemma 2.1. For $\mu \in \mathbb{Z}_+$, define the integrals:

$$\begin{aligned}
\Psi_\mu(\tau, \rho) &= \int_1^\infty e^{-2s}\tilde{G}_*(\tau, \rho; s)s^{n+\mu+1}ds, \\
\tilde{G}_*(\tau, \rho; s) &= \int_{\mathbb{R}^n} e^{2s(\tau|v)}\frac{1}{\tilde{E}_*(v; \rho; s)}dv, \\
\tilde{E}_*(v; \rho; s) &= \int_{\mathbb{R}^n} e^{-2s[P(w)+C\rho^{2m\gamma}\sigma(w)^{1+\gamma}-(v|w)]}dw.
\end{aligned}$$

The convergence of the integral Ψ_μ for each (τ, ρ) in the interior of Γ_δ is shown by considering the Bergman kernel of the tube domain whose base is $\{y \in \mathbb{C}^{n+\mu+1}; y_{n+\mu+1} > P(y') + C\sigma(y')^{1+\gamma} + \sum_{j=n+1}^{n+\mu} y_j^2\}$ $(y' = (y_1, \dots, y_n))$ and by similar calculation as in the proof of Proposition 5.3.

Let τ_0 be an arbitrary point in Δ_P and ϵ_0 the positive number defined by $\frac{1}{2}$ times the distance from τ_0 to the boundary of Δ_P. Then the set $\Delta_{\epsilon_0}(\tau_0) := \{\tau \in \Delta_P; |\tau - \tau_0| < \epsilon_0\}$ is contained in Δ_P.

LEMMA 7.9. *If $p \in C(\mathbb{R}^n)$ has at most polynomial growth, then there exist a point (τ_*, ρ_*) in Γ_δ (which depends on τ_0) and a positive number C such that*

$$\int_{\mathbb{R}^n} e^{2s(\tau|v)}\frac{p(v)}{\tilde{E}_*(v; \rho; s)}dv \le C\tilde{G}_*(\tau_*, \rho_*; s)$$

for $(\tau, \rho) \in \Delta_{\epsilon_0}(\tau_0) \times [0, \rho_*]$.

PROOF. The set $S = \{v; |v| = 1\}$ can be divided into finitely many sets $\{U_j\}$ as follows. For each j, there exist a point $\tau_*^{(j)}$ in Δ_P and a positive small number ϵ such that $(\tau|\tilde{v}) + \epsilon \le (\tau_*^{(j)}|\tilde{v})$ for all $\tau \in \Delta_{\epsilon_0}(\tau_0)$ and $\tilde{v} \in U_j$.

Then there exists a positive constant C_j such that $e^{2s(\tau|v)}p(v) \le C_j e^{2s(\tau_*^{(j)}|v)}$ for $s \ge 1, v \in \Lambda_j := \{t\tilde{v}; t > 0, \tilde{v} \in U_j\}$. Let ρ_* be a positive number such that $(\tau_*^{(j)}, \rho_*) \in \Gamma_\delta$ for all j. From these facts, if $\tau \in \Delta_\epsilon(\tau_0)$, then

$$\begin{aligned}
&\int_{\Lambda_j} e^{2s(\tau|v)} \frac{p(v)}{\tilde{E}_*(v;\rho;s)} dv \le C_j \int_{\Lambda_j} e^{2s(\tau_*^{(j)}|v)} \frac{1}{\tilde{E}_*(v;\rho;s)} dv \\
&\le C_j \int_{\mathbb{R}^n} e^{2s(\tau_*^{(j)}|v)} \frac{1}{\tilde{E}_*(v;\rho_*;s)} dv = C_j \tilde{G}_*(\tau_*^{(j)}, \rho_*; s).
\end{aligned} \tag{7.15}$$

Here we set $\tau_* = \tau_*^{(j)}$ such that $\tilde{G}_*(\tau_*^{(j)}, \rho_*; s)$ takes the largest value for all j. Then the inequality in the lemma can be obtained by summing (7.15) over j. $\square$

LEMMA 7.10. *For any $\alpha \in \mathbb{Z}_+^k$ and $k \in \mathbb{Z}_+$, there exists a positive number $C_{\alpha,k}$ such that if $(\tau, \rho) \in \Delta_{\epsilon_0}(\tau_0) \times [0, \rho_*]$ and $s \ge 1$, then*

$$\left| \frac{\partial^\alpha}{\partial \tau^\alpha} \frac{\partial^k}{\partial \rho^k} \tilde{G}(\tau, \rho; s; c_1, d_1) \right| \le C_{\alpha,k} s^{|\alpha|+k} \tilde{G}_*(\tau_*, \rho_*, s).$$

PROOF. By Leibniz rule, we have

$$\frac{\partial^k}{\partial \rho^k} \left(\frac{\chi_{c_1}(v,\rho)}{\tilde{E}(v;\rho;s;d_1)} \right) = \sum_{j=0}^k a_j \chi_{c_1}^{\langle j \rangle}(v,\rho) \cdot \frac{\partial^j}{\partial \rho^j} \left(\frac{1}{\tilde{E}(v;\rho;s;d_1)} \right),$$

where $\chi_{c_1}^{\langle j \rangle}(v,\rho) := \frac{\partial^{k-j}}{\partial \rho^{k-j}} \chi_{c_1}(v,\rho)$ and a_j are natural numbers. Since $|\chi_{c_1}^{\langle j \rangle}(v,\rho)|$ $(0 \le j \le k)$ are bounded, Lemma 7.7 implies

$$\begin{aligned}
\left| \frac{\partial^k}{\partial \rho^k} \left(\frac{\chi_{a_1}(v,\rho)}{\tilde{E}(v;\rho;s;d_1)} \right) \right| &\le \sum_{j=0}^k |a_j| |\chi_{c_1}^{\langle j \rangle}(v,\rho)| \cdot \left| \frac{\partial^j}{\partial \rho^j} \left(\frac{1}{\tilde{E}(v;\rho;s;d_1)} \right) \right| \\
&\le C s^k u^{(1+\frac{1}{2m})k} \frac{1}{\tilde{E}(v;\rho;s;d_1)}.
\end{aligned}$$

From the above inequality and the definition of $\tilde{E}_*$, if (τ, ρ) is in Γ_δ, then

$$\begin{aligned}
\frac{\partial^\alpha}{\partial \tau^\alpha} \frac{\partial^k}{\partial \rho^k} \left(e^{2s(\tau|v)} \frac{\chi_{c_1}(v,\rho)}{\tilde{E}(v;\rho;s;d_1)} \right) &\le C s^{|\alpha|+k} e^{2s(\tau|v)} \frac{p_{\alpha,k}(v)}{\tilde{E}(v;\rho;s;d_1)} \\
&\le C s^{|\alpha|+k} e^{2s(\tau|v)} \frac{p_{\alpha,k}(v)}{\tilde{E}_*(v;\rho;s)}.
\end{aligned}$$

From Lemma 7.9, if $(\tau,\rho) \in \Delta_{\epsilon_0}(\tau_0) \times [0,\rho_*]$, then the inequality in the lemma can be obtained. □

REMARK 7.11. From the above argument, $p_{\alpha,k}(v)$ $(v = \delta_u^*(\hat{v}))$ can be estimated as follows:

$$p_{\alpha,k}(v) \le Cu^{1+\frac{k}{2m}+\sum_{j=1}^{n}\alpha_j(1-\frac{1}{2m_j})} \quad \text{for } u \ge 1.$$

Last let us see the specific value of the limit of the derivatives of $\tilde{G}$. For $\alpha \in \mathbb{Z}_+^n$, $k \in \mathbb{Z}_+$, define the integrals:

$$\tag{7.16} \tilde{G}_{\alpha,k}(\tau;s) = \int_{\mathbb{R}^n} e^{2s(\tau|v)}\tilde{F}_k(v;s)\left(\prod_{j=1}^{n} v_j^{\alpha_j}\right)dv,$$

$$\tilde{F}_k(v;s) = \left(\sum_{\beta} C_\beta \prod_{|\beta|=k} \frac{\tilde{E}_{\beta_j}(v;s)}{\tilde{E}_0(v;s)}\right)\frac{1}{\tilde{E}_0(v;s)},$$

where $\tilde{E}_k(v;s)$ are as in Lemma 7.8 and the above summation and the constants are the same as in (7.14).

LEMMA 7.12. *For any $\alpha \in \mathbb{Z}_+^n$, $k \in \mathbb{Z}_+$, $s \ge 1$, $\tau_0 \in \Delta_P$,*

$$\lim_{(\tau,\rho)\to(\tau_0,0)} \frac{\partial^\alpha}{\partial\tau^\alpha}\frac{\partial^k}{\partial\rho^k}\tilde{G}(\tau,\rho;s;c,d) = (-1)^k(2s)^{|\alpha|+k}\tilde{G}_{\alpha,k}(\tau_0;s).$$

PROOF. From Lemmas 7.9 and 7.10, Lebesgue's convergence theorem implies the equation in the lemma. □

7.2.5 Smoothness of $K_1(\tau,\rho)$

Last we show that all partial derivatives of $K_1(\tau,\rho)$ admit a continuous extension to the set $\Delta_P \times \{0\}$.

For $\alpha \in \mathbb{Z}_+^n$, $k \in \mathbb{Z}_+$, define

$$\tag{7.17} \tilde{K}_{\alpha,k}(\tau) = \int_1^\infty e^{-2s}\tilde{G}_{\alpha,k}(\tau;s)s^{n+|\alpha|+k+1}ds.$$

PROPOSITION 7.13. *$K_1(\tau, \rho; c_1, d_1)$ can be extended to be C^∞-smooth in (τ, ρ) on the set $U_\delta \cup (\Delta_P \times \{0\})$. More precisely, for any $\alpha \in \mathbb{Z}_+^n$, $k \in \mathbb{Z}_+$, $\tau_0 \in \Delta_P$,*

$$\lim_{(\tau,\rho)\to(\tau_0,0)} \frac{\partial^\alpha}{\partial\tau^\alpha}\frac{\partial^k}{\partial\rho^k} K_1(\tau, \rho; c_1, d_1) = \tilde{K}_{\alpha,k}(\tau_0).$$

PROOF. Let $\epsilon_0, \tau_*, \rho_*$ be as in Lemma 7.8. Suppose $(\tau, \rho) \in \Delta_{\epsilon_0}(\tau_0) \times [0, \rho_*]$. Then, Lemma 7.10 implies

$$\begin{aligned}
&\int_1^\infty e^{-2s} \left| \frac{\partial^\alpha}{\partial\tau^\alpha}\frac{\partial^k}{\partial\rho^k} \tilde{G}(\tau, \rho; s; c_1, d_1) \right| s^{n+1} ds \\
&\leq C_{\alpha,k} \int_1^\infty e^{-2s} \tilde{G}_*(\tau_*, \rho_*; s) s^{n+|\alpha|+k+1} ds \\
&= C_{\alpha,k} \Psi_{|\alpha|+k}(\tau_*, \rho_*) < \infty.
\end{aligned}$$

Therefore we have

$$\frac{\partial^\alpha}{\partial\tau^\alpha}\frac{\partial^k}{\partial\rho^k} K_1(\tau, \rho; c_1, d_1) = \int_1^\infty e^{-2s} \left(\frac{\partial^\alpha}{\partial\tau^\alpha}\frac{\partial^k}{\partial\rho^k} \tilde{G}(\tau, \rho; s; c_1, d_1) \right) s^{n+1} ds.$$

Moreover, Lebesgue's convergence theorem implies the continuity of the above function on $\Delta_{\epsilon_0}(\tau_0) \times [0, \rho_*]$. The limit in the lemma can be given by using Lemma 7.12. □

7.3. Analysis of $K_2(\tau, \rho)$

The following proposition shows precise situation of the singularities of $K_2(\tau, \rho)$.

PROPOSITION 7.14. *There exist C^∞-functions $\Phi_2(\tau, \rho)$ and $\tilde{\Phi}_2(\tau, \rho)$ on the set $\overline{U_\delta}$ such that*

$$K_2(\tau, \rho) = \Phi_2(\tau, \rho) + \tilde{\Phi}_2(\tau, \rho)\rho^{2m(\nu+2)} \log \rho.$$

PROOF. A simple computation implies that

$$\begin{aligned}
K_2(\tau, \rho) &= \int_{\rho^{2m}}^1 e^{-2s} G(\delta_s(\tau); \rho s^{-1/2m}) s^{\nu+1} ds \\
&= 2m \int_\rho^1 e^{-2u^{2m}} G(\delta_{u^{2m}}(\tau); \rho u^{-1}) u^{2m(\nu+2)-1} du.
\end{aligned}$$

Since $G(X;\xi)$ is C^∞-smooth in X on $\mathbb{R}^n$ and the above integral interval is finite, it is easy to see the differentiability with respect to τ. Thus, to obtain the proposition, it suffices to show the following lemma. □

LEMMA 7.15. *Let $f(u,\xi)$ be a C^∞-function on $[0,1)\times[0,1)$ and $\kappa\in\mathbb{Z}_+$. Then there exist C^∞-functions φ and ψ on $[0,1)$ so that*

$$F(r):=\int_r^1 f(u,ru^{-1})u^{\kappa-1}du=\varphi(r)+\psi(r)r^\kappa\log r.$$

PROOF. Taylor's formula implies that for any $N\in\mathbb{Z}_+$,

$$f(u,\xi)=\sum_{j=0}^{\kappa+N} f_j(u)\xi^j+R_{N+1}(u,\xi)\xi^{\kappa+N+1},$$

where $f_j\in C^\infty([0,1))$ and $R_{N+1}(u,\xi)\in C^\infty([0,1)\times[0,1))$. Substituting the above equation into the integral in the lemma,

$$F(r)=\sum_{j=0}^{\kappa+N} F_j(r)r^j+r^{\kappa+N+1}\int_r^1 R_{N+1}(u,ru^{-1})u^{-N-2}du,$$

where $F_j(r)=\int_r^1 f_j(u)u^{\kappa-j-1}du$. Here it is easy to see that if $0\le j\le\kappa-1$, then $F_j\in C^\infty([0,1))$ and if $\kappa\le j\le\kappa+N$, then $F_j(r)r^j=c_{j-\kappa}r^j\log r+\phi_j(r)$ where ϕ_j are C^∞-functions on $[0,1)$ and

$$c_j=\frac{-1}{j!}f^{(j)}_{\kappa+j}(0)=\frac{-1}{j!(\kappa+j)!}\frac{\partial^j}{\partial u^j}\frac{\partial^{\kappa+j}}{\partial\xi^{\kappa+j}}f(u,\xi)\bigg|_{(u,\xi)=(0,0)}. \tag{7.18}$$

On the other hand, it is easy to check that the integral

$$\tilde{R}_{N+1}(r):=r^{N+1}\int_r^1 R_{N+1}(u,ru^{-1})u^{-N-2}du=\int_r^1 R_{N+1}(rv^{-1},v)v^N dv$$

is a C^N-function on $[0,1)$. From these facts, we have

$$F(r)=\sum_{j=0}^{\kappa-1}F_j(r)r^j+r^\kappa\left\{\sum_{j=0}^{N}c_jr^j\log r+\tilde{R}_{N+1}(r)\right\}+\sum_{j=\kappa}^{\kappa+N}\phi_j(r). \tag{7.19}$$

Now let us take a C^∞-function φ on $(0,1)$ whose asymptotic expansion as $r \to 0$ is $\varphi(r) \sim \sum_{j=0}^{\infty} c_j r^j$. Then there is a C^∞-function $\tilde{\tilde{R}}_N$ on $[0,1)$ such that $\varphi(r) - \sum_{j=0}^{N} c_j r^j = \tilde{\tilde{R}}_{N+1}(r) r^{N+1}$. From (7.19), we have

$$F(r) = \varphi(r) r^\kappa \log r + \tilde{\tilde{R}}_{N+1}(r) r^{N+1} \log r + \tilde{R}_{N+1}(r) + C^\infty\text{-functions}.$$

Here since $\tilde{\tilde{R}}_{N+1}(r) r^{N+1} \log r + \tilde{R}_{N+1}(r)$ is C^N-function on $[0,1)$ and N was any nonnegative integer, the lemma can be shown. □

Remark 7.16. From the above argument, the asymptotic expansion of $\Phi_2(\tau,\rho)$ can be obtained as follows:

$$\Phi_2(\tau,\rho) = \sum_{k=0}^{2m(\nu+2)-1} \tilde{\tilde{K}}_{0,k}(\tau)\rho^k + \tilde{\tilde{R}}(\tau,\rho)\rho^{2m(\nu+2)},$$

where

$$\tilde{\tilde{K}}_{0,k}(\tau) = \frac{1}{k!}\frac{2}{(2\pi)^{n+1}} \int_0^1 e^{-2s} \tilde{G}_{0,k}(\tau; s) s^{n+k+1} ds \tag{7.20}$$

and $\tilde{\tilde{R}}(\tau,\rho)$ is C^∞-smooth on the set $\overline{U_\delta}$.

7.4. Asymptotic expansion

From the analysis of K_1 and K_2 in Sections 7.2 and 7.3, we can obtain

$$K(y) = \frac{\Phi(\tau,\rho)}{\rho^{2m(\nu+2)}} + \tilde{\Phi}(\tau,\rho)\log\rho + \tilde{\tilde{\Phi}}(\tau,\rho),$$

near z_0, where $\Phi(\tau,\rho) \in C^\infty(U_\delta \cup (\Delta_P \times \{0\}))$ and $\tilde{\Phi}(\tau,\rho), \tilde{\tilde{\Phi}}(\tau,\rho) \in C^\infty(\overline{U_\delta})$, with some $\delta > 0$. More precisely, let us see the asymptotic expansion of the functions $\Phi(\tau,\rho), \tilde{\Phi}(\tau,\rho)$ with respect to ρ. We define

$$G_k(X) = \left.\frac{\partial^k}{\partial \xi^k} G(X;\xi)\right|_{\xi=0} = \int_{\mathbb{R}^n} e^{-2(X|v)} F_k(v) dv,$$

$$F_k(v) = \left.\frac{\partial^k}{\partial \xi^k} \frac{1}{\tilde{E}(v;\xi)}\right|_{\xi=0} = \left(\sum_\beta C_\beta \prod_{|\beta|=k} \frac{E_{\beta_j}(v)}{E_0(v)}\right) \frac{1}{E_0(v)},$$

where $E_k(v) = \tilde{E}_k(v;1)$ (see Lemma 7.8) and the summation and the constants are the same as in (7.14).

PROPOSITION 7.17. (i) *$\Phi(\tau,\rho)$ admits the asymptotic expansion: for any $N \in \mathbb{Z}_+$,*

$$\Phi(\tau,\rho) = \sum_{k=0}^{N} \Phi_k(\tau)\rho^k + R_N(\tau,\rho)\rho^{N+1},$$

where $R_N(\tau,\rho)$ is continuous on the set Γ_δ. More precisely, each coefficient $\Phi_k(\tau)$ is a C^∞-function on Δ_P having the form:

$$\Phi_k(\tau) = \frac{1}{k!}\frac{2}{(2\pi)^{n+1}}\int_0^\infty e^{-2s}\tilde{G}_{0,k}(\tau;s)s^{n+k+1}ds \tag{7.21}$$

$$= \frac{1}{k!}\frac{2}{(2\pi)^{n+1}}\int_0^\infty e^{-2s}G_k(\delta_s(\tau))s^{\nu+1-\frac{k}{2m}}ds \tag{7.22}$$

for $1 \le k \le 2m(\nu+2)-1$ where $\tilde{G}_{0,k}$ is as in (7.16). In particular, $\Phi_0(\tau) = \Phi(\tau)$ as in (5.3) in Section 5.

(ii) *$\tilde{\Phi}(\tau,\rho)$ admits the asymptotic expansion: for any $N \in \mathbb{Z}_+$,*

$$\tilde{\Phi}(\tau,\rho) = \sum_{k=0}^{N} \tilde{\Phi}_k(\tau)\rho^k + \tilde{R}_N(\tau,\rho)\rho^{N+1},$$

where $\tilde{R}_N(\tau,\rho)$ is continuous on the set $\overline{U_\delta}$ and each coefficient $\tilde{\Phi}_k(\tau)$ is a C^∞-function on $\overline{\Delta_P}$ having the form

$$\tilde{\Phi}_k(\tau) = C_k \frac{\partial^k}{\partial u^k}\left(e^{-2u^{2m}}G_{2m(\nu+2)+k}(\delta_{u^{2m}}(\tau))\right)\Big|_{u=0}, \tag{7.23}$$

where C_k is a nonzero constant (see (7.24) below).

PROOF. (i) From the computation of K_1 and K_2, the coefficient of ρ^k is $\tilde{K}_{0,k}(\tau) + \tilde{\tilde{K}}_{0,k}(\tau)$ for $1 \le k \le 2m(\nu+2)-1$. From (7.17), (7.20), we get the form (7.21) of $\Phi_k(\tau)$. By changing the integral variables ($w_j \leftrightarrow s^{1/2m_j}w_j$ ($j = 1,\dots,n$)), we can get another expression (7.22).

(ii) In order to obtain Proposition 7.14 from Lemma 7.15, it is sufficient to compare $K_2(\tau,\rho)$ with $F(r)$. In fact, if we put

$$f(u,\xi) = \frac{2\cdot 2m}{(2\pi)^{n+1}}e^{-2u^{2m}}G(\delta_{u^{2m}}(\tau);\xi)$$

and $\kappa = 2m(\nu+2)$, then the equation (7.23) is obtained from (7.18), where

$$C_k = \frac{-1}{k!}\frac{1}{(2m(\nu+2)+k)!}\frac{2\cdot 2m}{(2\pi)^{n+1}}. \tag{7.24}$$

The other parts of the proposition can be directly shown from the analysis in the previous subsections. □

In the above asymptotic expansion, it is very difficult to compute the coefficients in clear form. In order to see the essential quantities, we restrict the Bergman kernel to the vertical line. Then the coefficients of the singular part can be expressed by using $G_j(0)$ ($j \in \mathbb{Z}_+$). Let $r = y_{n+1}$.

Corollary 7.18. *If $K(y) = K(y', y_{n+1})$ is restricted to the set $\{y; y' = 0\}$, then*

$$K(0,r) = \frac{\Psi(r)}{r^{\nu+2}} + \tilde{\Psi}(r)\log r,$$

where

$$\Psi(r) = \sum_{j=0}^{2m(\nu+2)-1} c_j G_j(0) r^{j/(2m)} + O(r^{\nu+2}),$$

$$\tilde{\Psi}(r) = \sum_{k=0}^{N} \tilde{c}_k G_{2m(\nu+2+k)}(0) r^k + O(r^{N+1}),$$

where N is any natural number and $c_j, \tilde{c}_k$ are nonzero constants depending only on j, k and $(m_1, \ldots, m_n)$.

In the strongly pseudoconvex case, there are many studies about the computation of coefficients by using boundary invariants. The following question is analogous to the studies [3],[16],[18],[28],[26] about so-called " Ramadanov Conjecture" : Let Ω be a bounded strongly pseudoconvex domain of $\mathbb{C}^n$. Assume that the logarithmic term does not appear in the asymptotic expansion of the Bergman kernel, then Ω is biholomorphic equivalent to the unit ball of $\mathbb{C}^n$.

Question 7.19. Let Ω be a tube domain and z_0 a boundary point as in Section 2. Assume that the logarithmic term does not appear in the asymptotic expansion of the Bergman kernel of Ω at z_0, then does the boundary of Ω contact with the boundary of its model domain (as in Section 5) at z_0 infinitely ?

In order to give an affirmative answer to the above question, it is enough to show the following: If $G_{2mk}(0) = 0$ for $k \geq \nu + 2$, then $R(y') \sim 0$ in the sense of the Taylor expansion at 0, where R is as in Lemma 2.1. But it is not easy to compute the value of $G_j(0)$ clearly and at present we cannot answer the above question.

Remark 7.20. In this paper, we do not discuss about the singularities of the coefficients Φ_k at $\partial\Delta_P$. Generally, it is difficult to compute asymptotic expansions of Φ_k at $\partial\Delta_P$. But, in the two-dimensional case, the coefficients Φ_k can be expressed in the following interesting form (see [22]). In this case, $\Delta_P = (-1, 1)$, $m = m_1$ (≥ 2), $\tau = \tau_1 = y_1/y_2^{1/(2m)}$ and

$$\Phi_k(\tau) = \frac{\varphi_k(\tau)}{(1-\tau^2)^{3+k}} + \psi_k(\tau)\log(1-\tau^2),$$

where φ_k, ψ_k are C^∞-functions on $[-1, 1]$. Note that $\varphi_0(\tau) > 0$ and $\Phi_k(\tau) \equiv 0$ if k is odd. This singularity can be considered to be of strongly pseudoconvex type (1.1). In the general dimensional case, the singularities of the coefficients are more complicated.

8. The Szegö Kernel

Let Ω be a domain with C^∞-smooth boundary in $\mathbb{C}^{n+1}$ and $d\sigma$ a surface element on $\partial\Omega$. Let $H^2(\partial\Omega, d\sigma)$ be the closed subspace of $L^2(\partial\Omega, d\sigma) = \{f; \int_{\partial\Omega} |f|^2 d\sigma < \infty\}$ consisting of those functions that extend holomorphically to Ω. Let $\{\phi_j(z)\}$ is a complete orthonormal basis of $H^2(\partial\Omega, d\sigma)$. Here each ϕ_j can be considered as a holomorphic function on Ω. The Szegö kernel of Ω (on the diagonal) is defined by

$$S(z) = \sum_j |\phi_j(z)|^2.$$

The sum is uniformly convergent on any compact set in Ω.

In the case of the Szegö kernel, the strongly pseudoconvex case is also well understood. Boutet de Monvel and Sjöstrand [5] obtained the asymptotic expansion of the Szegö kernel:

$$S(z) = \tilde{\varphi}(z) r(z)^{-n-1} + \tilde{\psi}(z) \log r(z),$$

where $-r(z)$ is a defining function of Ω and $\tilde{\varphi}, \tilde{\psi}$ are C^∞-functions on $\bar{\Omega}$ and $\tilde{\varphi}(z) > 0$ on the boundary.

Let us consider the Szegö kernel for a tube domain $\Omega = \mathbb{R}^{n+1} + i\omega$ in $\mathbb{C}^{n+1}$. In the case of tube domains, the surface element $d\sigma$ on $\partial\Omega$ can be expressed by $d\sigma = dx \wedge d\mu$, where $dx = dx_1 \wedge \cdots \wedge dx_{n+1}$ $(z_j = x_j + iy_j)$ and $d\mu$ is a surface element on $\partial\omega$. For a tube domain Ω, the Szegö kernel also has an integral representation (see [31] and compare with (4.1)).

$$S(z) = \frac{1}{(2\pi)^{n+1}} \int_{\Lambda^*} e^{-2(y|u)} \frac{1}{\tilde{\varphi}(u)} du,$$

where

$$\tilde{\varphi}(u) = \int_{\partial\omega} e^{-2(u|w)} d\mu(w)$$

and $\Lambda^* = \{u \in \mathbb{R}^{n+1}; \tilde{\varphi}(u) < \infty\}$.

Now we give the assumptions on $\Omega = \mathbb{R}^{n+1} + i\omega$ and $z_0 \in \partial\Omega$ as in Section 2.1, i.e., ω is a convex domain with C^∞-smooth boundary and z_0 is a point of finite type. We introduce a coordinate into the space $\mathbb{R}^{n+1}$ containing the base as in Section 2.2. From the definition, the Szegö kernel depends on the surface element on the boundary. In this paper, we introduce the following surface element $d\mu$ on $\partial\omega$. For unbounded base ω, we take $d\mu = dw_1 \wedge \cdots \wedge dw_n$. For bounded base ω, a surface element $d\mu$ satisfies that $\int_{\partial\omega} g d\mu \geq 0$ for $g \geq 0$ and takes the form:

$$d\mu = \sum_{j=1}^{n+1} \alpha_j(w) dw_1 \wedge \cdots \wedge \widehat{dw_j} \wedge \cdots \wedge dw_{n+1},$$

($\widehat{dw_j}$ indicates that dw_j is removed) where $\alpha_j \in C^\infty(\bar{\omega})$ satisfy $\alpha_j(w) = 0$ $(1 \leq j \leq n)$ and $\alpha_{n+1}(w) = 1$ on some neighborhood of the origin. The symbols $\tau, \rho, \nu, m, U_\delta, \Delta_P, \Gamma_\delta$ are the same as in Theorem 2.2.

Theorem 8.1. *The Szegö kernel $S(z)$ of a tube domain $\Omega = \mathbb{R}^{n+1} + i\omega$ has the form near $z_0 \in \partial\Omega$:*

$$S(z) = \frac{\Phi^S(\tau, \rho)}{\rho^{2m(\nu+1)}} + \tilde{\Phi}^S(\tau, \rho) \log \rho,$$

where $\Phi^S(\tau, \rho)$, $\tilde{\Phi}^S(\tau, \rho)$ are C^∞-functions on the set U_δ, with some small positive number δ, satisfying the following properties.

(i) *$\Phi^S(\tau,\rho)$ can be extended to be a C^∞-function on $U_\delta \cup (\Delta_P \times \{0\})$. More precisely, $\Phi^S(\tau,\rho)$ admits the following asymptotic expansion with respect to ρ: for any $N \in \mathbb{N}$,*

$$\Phi^S(\tau,\rho) = \sum_{k=0}^{N} \Phi_k^S(\tau)\rho^k + R_N^S(\tau,\rho)\rho^{N+1} + \tilde{\tilde{\Phi}}^S(\tau,\rho)\rho^{2m(\nu+2)},$$

where each coefficients $\Phi_k^S(\tau)$ are C^∞-functions on Δ_P, $R_N^S(\tau,\rho)$ is continuous on Γ_δ and $\tilde{\tilde{\Phi}}^S(\tau,\rho)$ is a C^∞-function on $\overline{U_\delta}$. In particular, the first coefficient $\Phi_0^S(\tau)$ can be written as

$$\Phi_0^S(\tau) = \frac{1}{(2\pi)^{n+1}} \int_0^\infty e^{-2s} G(\delta_s(\tau)) s^\nu ds,$$

where $G(\delta_s(\tau))$ is as in Section 5. Moreover, $\Phi_0^S(\tau)$ is positive on Δ_P and is unbounded as τ approaches the boundary of Δ_P.

(ii) *$\tilde{\Phi}^S(\tau,\rho)$ can be extended to be a C^∞-function on $\overline{U_\delta}$.*

Proof. By a similar argument in Section 6, an analogous localization lemma can be obtained in the case of the Szegö kernel. Therefore it suffices to consider a tube domain as in the Section 7.1. For this tube domain, the Szegö kernel can be written as

$$S(z) = \frac{1}{(2\pi)^{n+1}} \int_0^\infty e^{-2y_{n+1}s} F(y';s) s^n ds,$$

where F is as in (4.2) by changing the integral variables as in the case of the Bergman kernel. Since there is no essential difference between the above integral and (4.2) in the case of the Bergman kernel, we can show the theorem by the same computation. □

Remark 8.2. From the above proof, we see that the coefficients of the asymptotic expansion of the Szegö kernel (with respect to the above surface element) are very similar to those in the case of the Bergman kernel. Actually, the Question 7.19 is equivalent to an analogous question in the case of Szegö kernel.

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