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Spin Structures on Seiberg-Witten Moduli Spaces

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Abstract. Let M be an oriented closed 4-manifold with a spin^c structure $\mathcal{L}$. In this paper we prove that under a suitable condition for $(M, \mathcal{L})$ the Seiberg-Witten moduli space has a canonical spin structure and its spin bordism class is an invariant of M. We show that the invariant of $M = \#_{j=1}^{l} M_j$ is non-trivial for some spin^c structure when l is 2 or 3 and each M_j is a $K3$ surface or a product of two oriented closed surfaces of odd genus. As a corollary, we obtain the adjunction inequality for the 4-manifold. Moreover we calculate the Yamabe invariant of $M \# N_1$ for some negative definite 4-manifold N_1. We also show that $M \# N_2$ does not admit an Einstein metric for some negative definite 4-manifold N_2.

1. Introduction

Since E. Witten introduced the Seiberg-Witten equations ([W]), the moduli space of solutions to the equations has been applied to 4-dimensional topology. M. Furuta used the Seiberg-Witten equations themselves, rather than the moduli space, to obtain the 10/8 theorem ([F]). Roughly speaking, the Seiberg-Witten moduli space is the zero locus of the map defining the equations, which we call the Seiberg-Witten map, between two Hilbert bundles over the Jacobian torus. Furuta used finite dimensional approximation of the Seiberg-Witten map to prove the 10/8 theorem. Moreover using finite dimensional approximation of the Seiberg-Witten map, S. Bauer and Furuta refined the Seiberg-Witten invariants ([BF]). The refined invariant is more powerful than the usual Seiberg-Witten invariant. There are 4-manifolds for which the usual Seiberg-Witten invariants vanish but the Bauer-Furuta invariants do not ([B, FKM]). It is, however, hard in general to detect the Bauer-Furuta invariants.

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To detect the Bauer-Furuta invariants explicitly, we define new invariants for 4-manifolds. This invariant is weaker than the Bauer-Furuta invariant, but easier to treat, in particular when the first Betti number of the 4-manifold is positive. An outline of the definition of the invariant is as follows.

Let (M, g) be an oriented, closed 4-dimensional Riemannian manifold with $b^+(M) > 1$, and $\mathcal{L}$ a spin^c structure on M. We write $\mathrm{Ind}(D)$ for the index bundle of the Dirac operators parameterized by $T = H^1(M;\mathbb{R})/H^1(M;\mathbb{Z})$ (see §3.1). If $c_1(\mathrm{Ind}(D)) \equiv 0 \mod 2$, then the Seiberg-Witten moduli space allows a spin structure, and a choice of square root of the determinant line bundle $\det \mathrm{Ind}(D)$ determines a spin structure of the moduli space. The spin bordism class of the moduli space is an invariant of M which depends only on $\mathcal{L}$ and the choice of square root of $\mathrm{Ind}(D)$.

We calculate the invariant for $M = \#_{j=1}^{l} M_j$ when M_j is a $K3$ surface or a product of two oriented closed surfaces of odd genus, and l is 2 or 3. We take a spin^c structure on M of the form $\mathcal{L} = \#_{j=1}^{l} \mathcal{L}_j$, where $\mathcal{L}_j$ is a spin^c structure on M_j induced by a complex structure. We show that in this case $c_1(\mathrm{Ind}(D)) \equiv 0 \mod 2$ and our invariant is non-trivial. As an application, we obtain the adjunction inequality for such M, i.e., if an oriented closed surface Σ of positive genus is embedded in M satisfying that its self-intersection number $\Sigma \cdot \Sigma$ is nonnegative, then we have

$$\Sigma \cdot \Sigma \leq \langle c_1(\det \mathcal{L}), \Sigma \rangle + 2g(\Sigma) - 2.$$

Here $\det \mathcal{L}$ is the determinant complex line bundle of $\mathcal{L}$, and $g(\Sigma)$ is the genus of Σ.

As another application, following Ishida and LeBrun's argument in [IL], we compute the Yamabe invariant of $M \# N_1$ when N_1 is an oriented, closed, negative definite 4-manifold admitting a Riemannian metric with scalar curvature nonnegative at each point. We also show that if N_2 is an oriented, closed, negative definite 4-manifold satisfying

$$4l - (2\chi(N_2) + 3\tau(N_2)) \geq \frac{1}{3} \sum_{j=1}^{l} c_1(M_j)^2,$$

then $M \# N_2$ does not admit an Einstein metric, where $\tau(N_2)$ and $\chi(N_2)$ are the signature and the Euler number of N_2 respectively.

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2. Finite Dimensional Approximations of the Seiberg-Witten Map

In this section, we review the definition of the Seiberg-Witten map and its finite dimensional approximation. We refer the readers to [BF] for details.

2.1. The Seiberg-Witten map

Let M be an oriented, closed, connected 4-manifold and let g be a Riemannian metric on M. Assume that $b^+(M) > 1$. Choose a spin^c structure $\mathcal{L}$ on M. We write $S^{\pm}(\mathcal{L})$ for the positive and negative spinor bundles, and $\det \mathcal{L}$ for the determinant line bundle associated with $\mathcal{L}$.

Let k be an integer larger than or equal to 4 and set $\hat{\mathcal{G}} = \{\gamma \in L^2_{k+1}(M, U(1)) | \gamma(x_0) = 1\}$ for a fixed base point $x_0 \in M$. Fix a connection A_0 on $\det \mathcal{L}$, and define $T := (A_0 + i \operatorname{Ker} d)/\hat{\mathcal{G}}$, where $d : L^2_k(T^*M) \to L^2_{k-1}(\Lambda^2 T^*M)$ is the exterior derivative. The action of $\gamma \in \hat{\mathcal{G}}$ on $A \in (A_0 + i \operatorname{Ker} d)$ is defined by

$$\gamma A := A + 2\gamma^{-1} d\gamma. \tag{2.1}$$

Put

$$\tilde{\mathcal{C}}(\mathcal{L}) := L^2_k(S^+(\mathcal{L}) \oplus T^*M),$$
$$\tilde{\mathcal{Y}}(\mathcal{L}) := L^2_{k-1}(S^-(\mathcal{L}) \oplus \Lambda^+ T^*M) \oplus \mathcal{H}^1_g(M) \oplus (L^2_{k-1}(M)/\mathbb{R}),$$

where $\mathbb{R}$ represents the space of constant functions on M and $\mathcal{H}^1_g(M)$ is the space of harmonic 1-forms on M with respect to g. Let $\mathcal{C}(\mathcal{L}) \to T$ and $\mathcal{Y}(\mathcal{L}) \to T$ be Hilbert bundles on T defined by

$$\mathcal{C}(\mathcal{L}) := (A_0 + i \operatorname{Ker} d) \times_{\hat{\mathcal{G}}} \tilde{\mathcal{C}}(\mathcal{L}),$$
$$\mathcal{Y}(\mathcal{L}) := (A_0 + i \operatorname{Ker} d) \times_{\hat{\mathcal{G}}} \tilde{\mathcal{Y}}(\mathcal{L}).$$

The action of $\hat{\mathcal{G}}$ on $(A_0 + i \operatorname{Ker} d)$ is given by (2.1). We define actions of $\hat{\mathcal{G}}$ on $L^2_k(S^+(\mathcal{L}))$ and on $L^2_{k-1}(S^-(\mathcal{L}))$ by fiber-wise scalar products. We define actions of $\hat{\mathcal{G}}$ on the other terms to be trivial. We define $U(1)$-actions on $\mathcal{C}(\mathcal{L})$ and $\mathcal{Y}(\mathcal{L})$ by scalar products on $L^2_k(S^+(\mathcal{L}))$ and $L^2_{k-1}(S^-(\mathcal{L}))$ and set

$$\mathcal{P} := \left\{(g,\eta) \in Riem(M) \times L^2_k(\Lambda^2 T^*M) \,\middle|\, [\eta]^+_g \neq [F_{A_0}]^+_g \right\},$$

where $Riem(M)$ is the space of Riemannian metrics on M, and $[\eta]^+_g$ and $[F_{A_0}]^+_g$ are $\mathcal{H}^+_g(M)$ parts of η and F_{A_0} respectively. For $(g,\eta) \in \mathcal{P}$, we define the Seiberg-Witten map by

$$\begin{array}{rccc} SW_{g,\eta}: & \mathcal{C}(\mathcal{L}) & \longrightarrow & \mathcal{Y}(\mathcal{L}) \\ & (A,\phi,a) & \longmapsto & (A, D_{A+ia}\phi, F^+_{A+ia} - q(\phi) - \eta^+, p(a), d^*a), \end{array}$$

where $q(\phi)$ is a quadratic form of ϕ and $p : L^2_k(T^*M) \to \mathcal{H}^1_g(M)$ is the L^2-projection. The moduli space $\mathcal{M}_M(\mathcal{L},g,\eta)$ of solutions to the Seiberg-Witten equations perturbed by (g,η) is identified with $SW^{-1}_{g,\eta}(0)/U(1)$ naturally.

The following fact is well known.

THEOREM 2.1 ([KM]). *For generic $(g,\eta) \in \mathcal{P}$, $\mathcal{M}_M(\mathcal{L},g,\eta)$ is a compact smooth manifold and an orientation on $\mathcal{H}^1_g(M;\mathbb{R}) \oplus \mathcal{H}^+_g(M;\mathbb{R})$ determines an orientation on $\mathcal{M}_M(\mathcal{L},g,\eta)$.*

2.2. Finite dimensional approximation

We explain finite dimensional approximations of the Seiberg-Witten map briefly.

Let $\mathcal{D} : \mathcal{C}(\mathcal{L}) \to \mathcal{Y}(\mathcal{L})$ be the linear part of the SW map:

$$\begin{array}{rccc} \mathcal{D}: & \mathcal{C}(\mathcal{L}) & \longrightarrow & \mathcal{Y}(\mathcal{L}) \\ & (A,\phi,a) & \longmapsto & (A, D_A\phi, d^+a, p(a), d^*a). \end{array}$$

By Kuiper's theorem [Ku], we have a global trivialization of $\mathcal{Y}(\mathcal{L})$

$$\mathcal{Y}(\mathcal{L}) \cong T \times H,$$

where H is a Hilbert space. We fix a trivialization of $\mathcal{Y}(\mathcal{L})$. Since $\mathcal{Y}(\mathcal{L})$ has the complex part and the real part, H decomposes into the direct sum $H_{\mathbb{C}} \oplus H_{\mathbb{R}}$ of a complex Hilbert space $H_{\mathbb{C}}$ and a real Hilbert space $H_{\mathbb{R}}$.

For a finite dimensional subspace $W \subset H$, let $p_W : \mathcal{Y}(\mathcal{L}) = T \times H \to W$ be the projection. We denote $\mathcal{D}^{-1}(T \times W)$ by $\mathcal{F}(W)$. Then we define $f_W : \mathcal{F}(W) \to W$ by

$$f_W = p_W \circ SW|_{\mathcal{F}(W)} : \mathcal{F}(W) \longrightarrow W.$$

THEOREM 2.2 ([BF]). *Let W^+ and $\mathcal{F}(W)^+$ be the one-point compactifications of W and $\mathcal{F}(W)$. Then $f_W : \mathcal{F}(W) \to W$ induces a $U(1)$-equivariant map $f_W^+ : \mathcal{F}(W)^+ \to W^+$, and there is a finite dimensional subspace $W \subset H$ such that the following conditions are satisfied.*

(1) $\operatorname{Im}\mathcal{D} + (T \times W) = \mathcal{Y}(\mathcal{L})$.

(2) *For all finite dimensional subspace $W' \subset H$ such that $W \subset W'$, the diagram*

$$\begin{array}{ccc} \mathcal{F}(W')^+ & \xrightarrow{\quad f_{W'}^+ \quad} & (W')^+ \\ \Big\| & & \Big\| \\ (\mathcal{F}(W) \oplus \mathcal{F}(U))^+ & \xrightarrow[(f_W \oplus p_U \mathcal{D}|_{\mathcal{F}(U)})^+]{} & (W \oplus U)^+ \end{array}$$

is $U(1)$-equivariant homotopy commutative as pointed maps, where U is the orthogonal complement of W in W'.

DEFINITION 2.3. When $W \subset H$ satisfies (1) and (2), we call $f_W : \mathcal{F}(W) \to W$ a finite dimensional approximation of the Seiberg-Witten map.

3. Spin Structures on Moduli Spaces

In §3.1, by using finite dimensional approximation of the Seiberg-Witten map, we show a sufficient condition for the moduli space to be a spin manifold. In §3.2, we will prove that the spin bordism class of the spin structure on the moduli space is an invariant of M. In §3.3, we give some applications of this invariant.

3.1. A sufficient condition for moduli space to have a spin structure

Let $f = f_W : V = \mathcal{F}(W) \to W$ be a finite dimensional approximation of the Seiberg-Witten map. Note that V has a natural decomposition $V = V_{\mathbb{C}} \oplus V_{\mathbb{R}}$ into the direct sum of a complex vector bundle and a real vector bundle. Similarly decompose W as $W = W_{\mathbb{C}} \oplus W_{\mathbb{R}}$.

If we take a generic $(g, \eta) \in \mathcal{P}$ as in Theorem 2.1, $\mathcal{M}_M(\mathcal{L}, g, \eta)$ does not include reducible monopoles, hence $f^{-1}(0)$ lies in $V_{irr} := (V_{\mathbb{C}} \backslash \{0\}) \times_T V_{\mathbb{R}}$.

Put $\bar{V} := V_{irr}/U(1)$ and $\mathcal{M} := f^{-1}(0)/U(1)$. We define a vector bundle $\bar{E} \to \bar{V}$ by $\bar{E} := V_{irr} \times_{U(1)} W = \bar{E}_{\mathbb{C}} \oplus \bar{E}_{\mathbb{R}}$, where $\bar{E}_{\mathbb{C}} = V_{irr} \times_{U(1)} W_{\mathbb{C}}$, $\bar{E}_{\mathbb{R}} = V_{irr} \times W_{\mathbb{R}}$. Since f is $U(1)$-equivariant, f induces a section $s : \bar{V} \to \bar{E}$. Then $\mathcal{M}$ is the zero locus of s. If necessary, we perturb s on a compact subset in $\bar{V}$ so that s is transverse to the zero section of $\bar{E}$ and $\mathcal{M}$ is a compact smooth submanifold of $\bar{V}$.

We can orient $\mathcal{M}$ by using an orientation on $\mathcal{H}^1_g(X) \oplus \mathcal{H}^+_g(X)$ in the following way. The real part $\mathcal{D}_{\mathbb{R}}$ of $\mathcal{D}$ is independent of $A \in T$ and the cokernel is naturally identified with $\mathcal{H}^+_g(X)$. So $W_{\mathbb{R}}$ has the form $\mathcal{H}^+_g(X) \oplus W'_{\mathbb{R}}$ and $\mathcal{D}_{\mathbb{R}}$ induces the isomorphism between each fiber of $V_{\mathbb{R}}$ and $W'_{\mathbb{R}}$. (Hence $V_{\mathbb{R}}$ is a trivial vector bundle on T.) If we choose orientations on $W'_{\mathbb{R}}$ and $\mathcal{H}^+_g(X)$, we get an orientation on $\bar{E}_{\mathbb{R}}$ and orientations on $V_{\mathbb{R}}$ and $\mathcal{H}^1_g(X)$ compatible with $\mathcal{D}_{\mathbb{R}}$ and $\mathcal{O}$. T is naturally identified with $H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})$, so the tangent bundle $T(T)$ of T has a natural trivialization $T(T) \cong T \times H^1(X;\mathbb{R}) \cong T \times \mathcal{H}^1_g(X)$. The orientation on $\mathcal{H}^1_g(X)$ induces an orientation on $T(T)$. These orientations induce an orientation on the tangent bundle $T\bar{V}$ by Lemma 3.4 below. The derivative of s induces an isomorphism between $\bar{E}|_{\mathcal{M}}$ and the normal bundle $\mathcal{N}$ of $\mathcal{M}$ in $\bar{V}$. The orientation on $\bar{E}$ induces an orientation on $\mathcal{N}$ through this isomorphism, and we have an orientation on $\mathcal{M}$ compatible with the decomposition $T\bar{V}|_{\mathcal{M}} = T\mathcal{M} \oplus \mathcal{N}$. (It is easy to check that this orientation on $\mathcal{M}$ is independent of the choices of the orientations on $W'_{\mathbb{R}}$ and $\mathcal{H}^+_g(X)$.) So we have the following.

LEMMA 3.1. *A choice of orientation on $\mathcal{H}^1_g(X) \oplus \mathcal{H}^+_g(X)$ induces an orientation on $\mathcal{M}$.*

When $T\bar{V}$ and $\bar{E}$ have spin structures, we can equip $\mathcal{M}$ with a spin structure as in the case of orientation. The spin structure on $\bar{E}$ induces a spin structure on $\mathcal{N}$ through the derivative of s. Since $T\bar{V}|_{\mathcal{M}}$ is the direct sum of $T\mathcal{M}$ and $\mathcal{N}$, spin structures on $T\bar{V}$ and $\mathcal{N}$ induce a spin structure on $\mathcal{M}$, from the next well-known lemma.

LEMMA 3.2. *Let X be a smooth manifold, F_1 and F_2 be oriented vector bundles on X. If F_1 and F_2 have spin structures, then spin structures on F_1 and F_2 determine a spin structure on $F_1 \oplus F_2$. If F_1 and $F_1 \oplus F_2$ have spin structures, then spin structures on F_1 and $F_1 \oplus F_2$ determine a spin structure on F_2 naturally.*

Therefore we have shown the following.

LEMMA 3.3. *Let $f : V \to W$ be a finite dimensional approximation of the Seiberg-Witten map. Assume that $T\bar{V}$ and $\bar{E}$ have a spin structure. Choose spin structures $\mathfrak{s}_{\bar{V}}$ and $\mathfrak{s}_{\bar{E}}$ on $T\bar{V}$ and $\bar{E}$. Then $\mathfrak{s}_{\bar{V}}, \mathfrak{s}_{\bar{E}}$ and f induce a spin structure on $\mathcal{M} = f^{-1}(0)/U(1)$.*

We calculate $w_2(T\bar{V})$ and $w_2(\bar{E})$ to know when $T\bar{V}$ and $\bar{E}$ have spin structures.

Let $a \in \mathbb{Z}$ be the index of the Dirac operator, let $\operatorname{Ind} D \in K(T)$ be the index bundle of the Dirac operators $\{D_A\}_{A \in T}$ parameterized by T. Then we have $\operatorname{Ind} D = [V_{\mathbb{C}}] - [\underline{\mathbb{C}}^m] \in K(T)$, $V_{\mathbb{R}} = \underline{\mathbb{R}}^n$, $W_{\mathbb{C}} = \mathbb{C}^m$, $W_{\mathbb{R}} = \mathcal{H}^+_g(X) \oplus W'_{\mathbb{R}}$, $\dim W'_{\mathbb{R}} = n$ for some $m, n \in \mathbb{Z}_{\geq 0}$.

LEMMA 3.4. *Let $\bar{\pi} : \bar{V} \to T$ be the projection and define a complex line bundle $H \to \bar{V}$ by $H := V_{irr} \times_{U(1)} \mathbb{C}$. Then there is a natural isomorphism*

$$T\bar{V} \oplus \underline{\mathbb{R}} \cong \bar{\pi}^* T(T) \oplus (\bar{\pi}^* V_{\mathbb{C}} \otimes_{\mathbb{C}} H) \oplus \bar{\pi}^* V_{\mathbb{R}}.$$

PROOF. Let $\pi_{irr} : V_{irr} \to T$ and $p : V_{irr} \to \bar{V} = V_{irr}/U(1)$ be the projections. Note that we have a $U(1)$-equivariant isomorphism

$$p^*(T\bar{V}) \oplus \underline{\mathbb{R}} \cong TV_{irr} = \pi^*_{irr}(T(T) \oplus V).$$

where $\underline{\mathbb{R}}$ stands for the $U(1)$-orbit direction. Then by dividing by the $U(1)$-actions, we obtain the required isomorphism. □

By Lemma 3.4 and the triviality of $V_{\mathbb{R}}$, we have $w_2(T\bar{V}) \equiv \bar{\pi}^* c_1(V_{\mathbb{C}}) + (m+a)c_1(H) \mod 2$. By (1) in Theorem 2.2, $c_1(V_{\mathbb{C}})$ is equal to $c_1(\operatorname{Ind}(D))$, thus we have

$$w_2(T\bar{V}) \equiv \bar{\pi}^* c_1(\operatorname{Ind}(D)) + (m+a)c_1(H) \mod 2. \tag{3.1}$$

T-J. Li and A. Liu calculated $c_1(\operatorname{Ind}(D))$ in [LL] as follows.

Let $\{\alpha_j\}_{j=1}^{b_1}$ be generators of $H^1(M;\mathbb{Z})$. Then we obtain a natural identification,

$$T \cong H^1(M;\mathbb{R})/H^1(M;\mathbb{Z}) \cong \mathbb{R}^{b_1}/\mathbb{Z}^{b_1} = T^{b_1}.$$

Let Ψ be a map $M \to T^{b_1} \cong T$ given by

$$x \longmapsto \left(\int_{x_0}^{x} \alpha_1, \cdots, \int_{x_0}^{x} \alpha_{b_1} \right).$$

This map is well defined by the Stokes theorem and induces the isomorphism $\Psi^* : H^1(T;\mathbb{Z}) \cong H^1(M;\mathbb{Z})$. Set $\beta_j = (\Psi^*)^{-1}(\alpha_j) \in H^1(T;\mathbb{Z})$.

PROPOSITION 3.5 ([LL]). *Let* $\operatorname{Ind} D \in K(T)$ *be the index bundle of the Dirac operators* $\{D_A\}_{A \in T}$ *parameterized by* T. *Then the first Chern class* $c_1(\operatorname{Ind}(D))$ *of* $\operatorname{Ind}(D)$ *is given by*

$$c_1(\operatorname{Ind}(D)) = \frac{1}{2} \sum_{i<j} \langle c_1(\det \mathcal{L}) \alpha_i \alpha_j, [M] \rangle \, \beta_i \beta_j \in H^2(T;\mathbb{Z}).$$

By the equation (3.1) and Proposition 3.5, we have the following.

LEMMA 3.6. *The second Stiefel-Whiteny class of* $T\bar{V}$ *is given by*

$$w_2(T\bar{V}) \equiv \sum_{i<j} c_{ij} \bar{\pi}^* \beta_i \beta_j + (m+a) c_1(H) \mod 2,$$

where $c_{ij} := \dfrac{1}{2} \langle c_1(\det \mathcal{L}) \alpha_i \alpha_j, [M] \rangle$.

On the other hand, by the definitions of $\bar{E}$ and H, we have $\bar{E} = mH \oplus \underline{\mathbb{R}}^{n+b}$. Hence we obtain the following.

LEMMA 3.7. *The second Stiefel-Whiteny class of* $\bar{E}$ *is given by*

$$w_2(\bar{E}) \equiv m c_1(H) \mod 2.$$

By Lemma 3.3, Lemma 3.6 and Lemma 3.7, we have the following.

PROPOSITION 3.8. *Let* $f : V \to W$ *be a finite dimensional approximation of the Seiberg-Witten map such that* $m = \dim_{\mathbb{C}} W_{\mathbb{C}}$ *is even. Then* $T\bar{V}$ *and* $\bar{E}$ *have a spin structure if the pair* $(M, \mathcal{L})$ *satisfies the following conditions.*

$$(*) \begin{cases} (*)_1 & a \equiv 0 \quad \mod 2 \\ (*)_2 & c_{ij} \equiv 0 \quad \mod 2 \ (\forall i, j). \end{cases}$$

Moreover if we choose spin structures $\mathfrak{s}_{\bar{V}}$ *and* $\mathfrak{s}_{\bar{E}}$ *of* $T\bar{V}$ *and* $\bar{E}$, *then* $\mathfrak{s}_{\bar{V}}$, $\mathfrak{s}_{\bar{E}}$ *and* f *equip* $\mathcal{M}$ *with a spin structure.*

3.2. Invariants for 4-manifolds defined by spin structures on $\mathcal{M}$

An orientation on $\mathcal{H}^1_g(M) \oplus \mathcal{H}^+_g(M)$ determines an orientation on $\mathcal{M}$ (§3.1). We will show that when the condition $(*)$ is satisfied, a certain datum in addition to the orientation on $\mathcal{H}^1_g(M) \oplus \mathcal{H}^+_g(M)$ determines a canonical spin structure on $\mathcal{M}$. The datum is actually a square root of $\det \mathrm{Ind}(D)$. To explain it, we need the following lemma.

LEMMA 3.9. *Let X be a smooth manifold and $F \to X$ be a complex bundle with $c_1(F) \equiv 0 \mod 2$. A choice of complex line bundle $L \to X$ which satisfies $L^{\otimes 2} = \det F$ naturally determines a spin structure on F.*

PROOF. The 2-fold cover of $U(n)$ is given by

$$\{(A,t) \in U(n) \times S^1 |\, \det A = t^2\},$$

which is naturally regarded as a subgroup of $Spin(2n)$. Take an open covering $\{U_j\}_j$ of X such that F and L have trivializations on each U_j. Fix hermitian metrics on F and L compatible with the identification $L^{\otimes 2} = \det F$. We denote transition functions on $U_i \cap U_j$ of F and L by $g_{ij} : U_i \cap U_j \to U(n)$ and $z_{ij} : U_i \cap U_j \to S^1$. Then $\det g_{ij} = z_{ij}^2$, since $\det F = L^{\otimes 2}$. Put $\tilde{g}_{ij} = (g_{ij}, z_{ij}) : U_i \cap U_j \to Spin(2n)$, then $\{\tilde{g}_{ij}\}_{ij}$ satisfies the cocycle condition and determines a spin structure of F. □

When the condition $(*)_2$ is satisfied, then $c_1(\mathrm{Ind}(D)) \equiv 0 \mod 2$. So we can take a complex line bundle $L \to T$ such that $L^{\otimes 2} = \det \mathrm{Ind}(D)$.

PROPOSITION 3.10. *Assume that the pair $(M, \mathcal{L})$ satisfies the conditions $(*)$. Let $f : V \to W$ be a finite dimensional approximation of the Seiberg-Witten map such that $m = \dim_{\mathbb{C}} W_{\mathbb{C}}$ is even. Then the finite dimensional approximation f, an orientation $\mathcal{O}$ of $\mathcal{H}^1_g(M) \oplus \mathcal{H}^+_g(M)$ and a complex line bundle $L \to T$ which satisfies $L^{\otimes 2} = \det \mathrm{Ind}(D)$ determine a canonical spin structure on $\mathcal{M}$.*

PROOF. Suppose that the pair $(M, \mathcal{L})$ satisfies the condition $(*)$. By Lemma 3.3, spin structures on $T\bar{V}$, $\bar{E}$ and a finite dimensional approximation f induce a canonical spin structure on $\mathcal{M}$. So it is sufficient to show that $\mathcal{O}$ and L induce spin structures on $T\bar{V}$ and $\bar{E}$. By Lemma 3.4, we have only to show that the choices of $\mathcal{O}$ and L induce spin structures on $\bar{\pi}^* V_{\mathbb{C}} \otimes H$, $V_{\mathbb{R}}$, $T(T)$ and $\bar{E}$.

Since m is even and condition $(*)_1$ is satisfied, $\bar{\pi}^* L \otimes H^{\otimes \frac{m+a}{2}}$ is a square root of $\det(\bar{\pi}^* V_{\mathbb{C}} \otimes H) = (\bar{\pi}^* \det V_{\mathbb{C}}) \otimes H^{\otimes (m+a)}$. So by Lemma 3.9, we have a spin structure on $\bar{\pi}^* V_{\mathbb{C}} \otimes H$.

Recall that $W_{\mathbb{R}}$ is the direct sum $\mathcal{H}_g^+(X) \oplus W'_{\mathbb{R}}$. We fix orientations on $\mathcal{H}_g^+(X)$ and $W'_{\mathbb{R}}$, then we have orientations on $V_{\mathbb{R}}$ and $\mathcal{H}_g^1(X)$ compatible with $\mathcal{D}_{\mathbb{R}}$ and $\mathcal{O}$. (See §3.1.) Since the real part $\mathcal{D}_{\mathbb{R}}$ of $\mathcal{D}$ is independent of $A \in T$, $V_{\mathbb{R}}$ has a natural trivialization compatible with the orientation. This trivialization equips $V_{\mathbb{R}}$ with a spin structure. The tangent bundle $T(T)$ of T has a natural trivialization $T(T) = T \times \mathcal{H}_g^1(M)$ and the orientation $\mathcal{H}_g^1(X)$ orients $T(T)$. So we have a spin structure on $T(T)$ compatible with this trivialization.

Lastly we consider $\bar{E}$. Let $\bar{E}_{\mathbb{C}}$ be the complex part of $\bar{E}$, i.e. $\bar{E}_{\mathbb{C}} = V_{irr} \times_{U(1)} \mathbb{C}^m$. Since $\det \bar{E}_{\mathbb{C}} = H^{\otimes m}$, $H^{\otimes \frac{m}{2}}$ is a square root of $\det \bar{E}_{\mathbb{C}}$. So by Lemma 3.9, a spin structure of $\bar{E}_{\mathbb{C}}$ is determined. Let $\bar{E}_{\mathbb{R}}$ be the real part of $\bar{E}$. Then $\bar{E}_{\mathbb{R}} = V_{irr} \times W_{\mathbb{R}} = V_{irr} \times (\mathcal{H}_g^+(X) \oplus W'_{\mathbb{R}})$. Hence $\bar{E}_{\mathbb{R}}$ has a natural spin structure induced by the trivialization.

We have seen that f, $\mathcal{O}$ and L determine a spin structure on $\mathcal{M}$ if we choose orientations on $\mathcal{H}_g^+(X)$ and $W'_{\mathbb{R}}$. It is easy to see that this spin structure is independent of the choices of orientations on $\mathcal{H}_g^+(X)$ and $W'_{\mathbb{R}}$. □

Let $\pi : \mathcal{M} \to T$ be the restriction of the projection $\bar{V} \to T$ to $\mathcal{M}$. We show that the class $(\mathcal{M}, \pi) \in \Omega_d^{spin}(T)$ induced by $f, \mathcal{O}, L$ is an invariant of M. Here d is the dimension of $\mathcal{M}$.

THEOREM 3.11. *Assume that the pair $(M, \mathcal{L})$ satisfies the condition $(*)$. The class $(\mathcal{M}, \pi) \in \Omega_d^{spin}(T)$ which is induced by $f, \mathcal{O}, L$ is independent of the perturbation $(g, \eta) \in \mathcal{P}$ and the finite dimensional approximation f.*

PROOF. Fix $(g, \eta) \in \mathcal{P}$, and take different finite dimensional approximations $f_i : V_i \to W_i, (i = 0, 1)$ of the Seiberg-Witten map $SW_{g,\eta}$. Denote $f_i^{-1}(0)/U(1)$ by $\mathcal{M}_i$ and let π_i be the restriction of the projections $\bar{V}_i \to T$ to $\mathcal{M}_i$. By considering a larger finite dimensional approximation $f : V \to W$ with $V_i \subset V$ and $W_i \subset W$, we can assume that $V_0 \subset V_1, W_0 \subset W_1$ without loss of generality.

Let $V_1 = V_0 \oplus V'$ and $W_1 = W_0 \oplus W'$, then $\mathcal{D}|_{V'}$ induces an isomorphism $V' \cong T \times W'$. By Theorem 2.2, the maps

$$(f_1)^+,\ (f_0 \oplus p_{W'} \circ \mathcal{D}|_{V'})^+ : V_1^+ = (V_0 \oplus V')^+ \to W_1^+ = (W_0 \oplus W')^+$$

are $U(1)$-equivariantly homotopic each other as pointed maps. It is clear that the spin structure on $\mathcal{M}_0$ induced by $f_0 \oplus p_{W'} \circ \mathcal{D}|_{V'}$ is equal to one induced by f_0. Let $h : [0,1] \times V_1^+ \to W_1^+$ be a homotopy from $(f_0 \oplus \mathcal{D})^+$ to f_1^+ and set $\widetilde{\mathcal{M}} := h^{-1}(0)/U(1)$. Let $\tilde{\pi}$ be the restriction of the projection $\bar{V}_1 \times [0,1] \to T$ to $\widetilde{\mathcal{M}}$. By using a parallel argument to introduce spin structures on $\mathcal{M}_0$ and $\mathcal{M}_1$, we can equip $\widetilde{\mathcal{M}}$ with a spin structure by using $h, \mathcal{O}$ and L. Then $(\widetilde{\mathcal{M}}, \tilde{\pi})$ is a spin bordism between $(\mathcal{M}_0, \pi_0)$ and $(\mathcal{M}_1, \pi_1)$. This implies that when $(g, \eta) \in \mathcal{P}$ is fixed, the class $(\mathcal{M}, \pi) \in \Omega_d^{spin}(T)$ is independent of a choice of f.

Next choose two elements $(g_0, \eta_0), (g_1, \eta_1) \in \mathcal{P}$. By the assumption $b^+(M) > 1$, $\mathcal{P}$ is path connected, and there is a path $(g(t), \eta(t))_{0 \leq t \leq 1}$ in $\mathcal{P}$ satisfying $(g(i), \eta(i)) = (g_i, \eta_i), (i = 0, 1)$. We define parameterized Seiberg-Witten map

$$\widetilde{SW} : [0,1] \times \mathcal{C}(\mathcal{L}) \to [0,1] \times \mathcal{Y}(\mathcal{L})$$

in the obvious way. Let $\tilde{f} : \widetilde{V} \to \widetilde{W}$ be a finite dimensional approximation of $\widetilde{SW}$. We can endow $\widetilde{\mathcal{M}} = \tilde{f}^{-1}(0)/U(1)$ with a spin structure in the same way as in the case of $\mathcal{M}$. Denote $\widetilde{V}|_{\{i\} \times T}$ and $\widetilde{W}|_{\{i\} \times T}$ by V_i and W_i for $i = 0, 1$. Since $f_i := \tilde{f}|_{V_i} : V_i \to W_i$ is a finite dimensional approximation of SW_{g_i, η_i}, $(\widetilde{\mathcal{M}}, \tilde{\pi})$ is a bordism between $(\mathcal{M}_0, \pi_0)$ and $(\mathcal{M}_1, \pi_1)$. It is showed that the class $(\mathcal{M}, \pi) \in \Omega_d^{spin}(T)$ is independent of a choice of $(g, \eta) \in \mathcal{P}$. □

Definition 3.12. We write $\sigma_M(\mathcal{L}, \mathcal{O}, L)$ for the class in $\Omega_d^{spin}(T)$ represented by the spin structure on $\mathcal{M}$ induced by $f, \mathcal{O}, L$ and the restriction π of the projection $\bar{V} \to T$ to $\mathcal{M}$. Here d is the dimension of $\mathcal{M}$.

3.3. Example

We give an example of calculation of the invariant defined in §3.2. For preparation, we show the following two lemmas.

Lemma 3.13. *Let M_i $(i = 1, 2)$ be an oriented closed 4-manifold with $b^+(M_i) > 1$ and let $\mathcal{L}_i$ be a spin^c structure on M_i. Assume that $(M_1, \mathcal{L}_1)$ and $(M_2, \mathcal{L}_2)$ satisfy the conditions $(*)$, then $(M_1 \# M_2, \mathcal{L}_1 \# \mathcal{L}_2)$ also satisfies the condition $(*)$.*

Proof. The condition $(*)_2$ is satisfied for $(M_1 \# M_2, \mathcal{L}_1 \# \mathcal{L}_2)$ by the definition of c_{ij}. The condition $(*)_1$ is satisfied for $(M_1 \# M_2, \mathcal{L}_1 \# \mathcal{L}_2)$ by the sum formula of the index of the Dirac operator. □

We write Σ_g for an oriented closed surface of genus g.

LEMMA 3.14. *Suppose M is a $K3$ surface or $\Sigma_g \times \Sigma_{g'}$ with g and g' odd. Let $\mathcal{L}$ be a spin^c structure on M which is induced by the complex structure. Then $(M, \mathcal{L})$ satisfies the condition $(*)$.*

PROOF. Note that $c_1(\det \mathcal{L}) = -c_1(K_M)$. Let M be a $K3$ surface. The first Betti number of M is equal to 0, so the condition $(*)_2$ is satisfied. By the index theorem [AS], the index of the Dirac operator is

$$a = \frac{c_1(\det \mathcal{L})^2 - \tau(M)}{8} = \frac{0 - (3 - 19)}{8} = 2 \equiv 0 \mod 2.$$

Hence $(M, \mathcal{L})$ satisfies the condition $(*)$ when M is a K3 surface. Let M be $\Sigma_g \times \Sigma_{g'}$ with g and g' odd. Then we have

$$c_1(\det \mathcal{L}) = -c_1(K_M) = 2(1-g)\alpha + 2(1-g')\alpha'$$

where α and α' are the standard generators of $H^2(\Sigma_g; \mathbb{Z})$ and $H^2(\Sigma_{g'}; \mathbb{Z})$. Since g and g' are odd, we have $c_1(\det \mathcal{L}) \equiv 0 \mod 4$, and then

$$c_{ij} = \frac{1}{2} \langle c_1(\det \mathcal{L})\alpha_i\alpha_j, [M]\rangle \equiv 0 \mod 2,$$

which implies the condition $(*)_2$.

By the index theorem, the index of the Dirac operator is given by

$$a = \frac{c_1(\det \mathcal{L})^2 - \tau(M)}{8} = \frac{c_1(\det \mathcal{L})^2}{8}.$$

Because $c_1(\det \mathcal{L})^2 \equiv 0 \mod 16$, we have $a \equiv 0 \mod 2$. Hence the condition $(*)_1$ is satisfied. □

Let M_j be a $K3$ surface or $\Sigma_g \times \Sigma_{g'}$, where g, g' are odd. By Lemma 3.13 and Lemma 3.14, the pair $(\#_j^l M_j, \#_j^l \mathcal{L}_j)$ satisfies the conditions $(*)$, where $\mathcal{L}_j$ is a spin^c structure on M_j induced by the complex structure. We show that the invariant $\sigma_{\#_{j=1}^l M_j}(\#_{j=1}^l \mathcal{L}_j, \mathcal{O}, L)$ is non-trivial when l is 2 or 3.

THEOREM 3.15. *Let M_j be a $K3$ surface or $\Sigma_g \times \Sigma_{g'}$ with g, g' odd and $\mathcal{L}_j$ be a spin^c structure on M_j which is induced by the complex structure.*

Put $M = \#_{j=1}^{l} M_j$ *and* $\mathcal{L} = \#_{j=1}^{l} \mathcal{L}_j$ *for* $l = 2$ *or* $l = 3$. *Let* $\sigma_M^0(\mathcal{L}, \mathcal{O}, L)$ *be the image of* $\sigma_M(\mathcal{L}, \mathcal{O}, L)$ *under the natural map* $\Omega_{l-1}^{spin}(T) \to \Omega_{l-1}^{spin}(*)$. *Then* $\sigma_M^0(\mathcal{L}, \mathcal{O}, L)$ *is non-trivial in* $\Omega_{l-1}^{spin}(*) \cong \mathbb{Z}_2$.

Proof. Let $L \to T$ be a square root of $\det \mathrm{Ind}(D)$. If $l = 2$, the dimension of the moduli space is one, so the invariant $\sigma_M^0(\mathcal{L}, \mathcal{O}, L)$ is in the one dimensional spin bordism group $\Omega_1^{spin}(*) \cong \mathbb{Z}_2$, and if $l = 3$, the invariant $\sigma_M^0(\mathcal{L}, \mathcal{O}, L)$ is in the two dimensional spin bordism group $\Omega_2^{spin}(*) \cong \mathbb{Z}_2$. We will calculate the invariant for $l = 2$ for simplicity.

Let $f_j : V_j \to W_j$ be a finite dimensional approximation of the Seiberg-Witten map on M_j such that $m_j = \dim W_{j,\mathbb{C}}$ is even, and set $f = f_1 \times f_2 : V = V_1 \times V_2 \to W = W_1 \times W_2$. We make use of Bauer's construction (Theorem 1.1 in [B]). Bauer proved that there is a finite dimensional approximation on M which is $U(1)$-equivariantly homotopic to f.

In general, for a Kähler surface M with $b^+(M) > 1$ and a spin^c structure $\mathcal{L}$ on M induced by the complex structure, the Seiberg-Witten moduli space $\mathcal{M}_M(\mathcal{L}, g, \eta)$ consists of smooth one point, where g is the Kähler metric and η is a suitable 2-form. See, for example, [N]. Thus we may assume that $\mathcal{M}_j = f_j^{-1}(0)/U(1)$ is one point. Hence $f_j^{-1}(0) \cong S^1$ and $\mathcal{M} = f^{-1}(0)/U(1)_d = (f_1 \times f_2)^{-1}(0)/U(1)_d \cong S^1$, where $U(1)_d$ is the diagonal of $U(1) \times U(1)$. For some $t_j \in T_j = H^1(M_j; \mathbb{R})/H^1(M_j; \mathbb{Z})$, $f_j^{-1}(0)$ lies in a fiber V_{j,t_j} of $V_j \to T_j$. Take a small open neighborhood of t_j such that $V_j|_{U_j} \cong U_j \times \mathbb{C}^{m_j + a_j} \times \mathbb{R}_j^n$, where a_j is the index of the Dirac operator associated with $\mathcal{L}_j$. Set $S_j = U_j \times (\mathbb{C}^{m_j + a_j} \backslash \{0\}) \times \mathbb{R}^{n_j}$ and $S = \prod_{j=1}^2 S_j$, then S has a $U(1)_d$-action and a $U(1) \times U(1)$-action. The $U(1)_d$-action is defined by the scalar product on $\prod_{j=1}^2 (\mathbb{C}^{m_j + a_j} \backslash \{0\})$. And for $(\alpha_1, \alpha_2) \in U(1) \times U(1)$, we define the action of (α_1, α_2) on S by the scalar product of α_1 on $(\mathbb{C}^{m_1 + a_1} \backslash \{0\})$ and the scalar product of α_2 on $(\mathbb{C}^{m_2 + a_2} \backslash \{0\})$. Set $\bar{S} = S/U(1)_d$.

We write ξ for a spin structure on $\bar{V} = V_{irr}/U(1)_d$ induced by L. The restriction $\xi|_{\mathcal{M}}$ of ξ to $\mathcal{M}$ is equal to $(\xi|_{\bar{S}})|_{\mathcal{M}}$. Since $H^1(\bar{S}; \mathbb{Z}_2) = 0$, $\bar{S}$ has just one spin structure. So it is sufficient to consider the restriction of the unique spin structure on $\bar{S}$ to $\mathcal{M}$.

Put $U(1)_q = U(1) \times U(1)/U(1)_d \cong U(1)$, then the $U(1) \times U(1)$-action on S induces a free $U(1)_q$-action on $\bar{S}$ and $\bar{S}/U(1)_q = \bar{S}_1 \times \bar{S}_2$, where $\bar{S}_j = S_j/U(1) \cong U_j \times \mathbb{CP}^{m_j + a_j - 1} \times \mathbb{R}_{>0} \times \mathbb{R}^{n_j}$. Moreover this $U(1)_q$-action preserves $\mathcal{M} \subset \bar{S}$ and induces a free $U(1)_q$-action on $\mathcal{M} \cong S^1$. Since $m_j +$

$a_j - 1$ is odd, $T\bar{S}_j$ has a spin structure. So $T(\bar{S}/U(1)_q)$ has a spin structure. Take a spin structure η on $T(\bar{S}/U(1)_q) \oplus \underline{\mathbb{R}}$. Let $p : \bar{S} \to \bar{S}/U(1)_q$ be the projection. Then there is a natural isomorphism $T\bar{S} \cong p^*(T(\bar{S}/U(1)_q) \oplus \underline{\mathbb{R}})$. So $p^*(\eta)$ is the unique spin structure ξ on $T\bar{S}$. Because p is the projection $\bar{S} \to \bar{S}/U(1)_q$, the $U(1)_q$-action on $\bar{S}$ lifts to an action on $\xi = p^*(\eta)$. So the $U(1)_q$-action on $\mathcal{M} \cong S^1$ lifts to an action on restriction of $\xi|_{\mathcal{M}}$. In the same way, we can prove that the $U(1)_q$-action on $\mathcal{M}$ lifts to an action on the spin structure on $\bar{E}|_{\mathcal{M}}$. Since $f|_S = f_1|_{S_1} \times f_2|_{S_2} : S_1 \times S_2 \to W_1 \times W_2$ is $U(1) \times U(1)$-equivariant, the $U(1)_q$-action on $\mathcal{M}$ lifts to an action on the spin structure of $\mathcal{N}$ induced by f and the spin structure on $\bar{E}|_{\mathcal{M}}$. Therefore the $U(1)_q$-action on $\mathcal{M}$ lifts to an action on the spin structure on $\mathcal{M}$ induced by $f, \mathcal{O}$ and L. Such a spin structure determines a non-trivial class in $\Omega_1^{spin}(*) \cong \mathbb{Z}_2$, so $\sigma_M^0(\mathcal{L}, \mathcal{O}, L)$ is non-trivial class in $\Omega_1^{spin}(*)$ (See [K]).

In the case of $l = 3$, $\mathcal{M}$ is the 2-dimensional torus if we perturb the equations suitably. We can show that the spin structure on $\mathcal{M}$ is the Lie group spin structure as in the case of $l = 2$ and the spin bordism class $\sigma_M^0(\mathcal{L}, \mathcal{O}, L)$ is non-trivial in $\Omega_2^{spin}(*) \cong \mathbb{Z}_2$. □

REMARK 3.16. Let l be larger or equal to 4. Then we may assume that the moduli space is a $(l-1)$-dimensional torus T^{l-1}. In the same way as in Theorem 3.15, we can see that the spin structure on $\mathcal{M}$ induced by $f, \mathcal{O}$ and L is equal to the spin structure induced by the Lie group structure of T^{l-1}. Such a spin structure is trivial in $\Omega_{l-1}^{spin}(*)$ if l is larger or equal to 4. Hence $\sigma_M^0(\mathcal{L}, \mathcal{O}, L)$ is trivial in $\Omega_{l-1}^{spin}(*)$ when l is larger than or equal to 4.

By Theorem 3.15, we obtain the adjunction inequality for M. See [KM] for proof.

COROLLARY 3.17. *Let M_j,M and $\mathcal{L}$ be as in Theorem 3.15. Assume that an oriented, closed surface Σ of positive genus is embedded in M and its self intersection number $\Sigma \cdot \Sigma$ is nonnegative. Then*

$$\Sigma \cdot \Sigma \leq \langle c_1(\det \mathcal{L}), [\Sigma] \rangle + 2g(\Sigma) - 2,$$

where $g(\Sigma)$ is the genus of Σ.

There are applications of Theorem 3.15 to computation of the Yamabe invariant and nonexistence of Einstein metric.

DEFINITION 3.18. Let M be an oriented, closed 4-manifold. Then the Yamabe invariant of M is defined by

$$\mathcal{Y}(M) = \sup_{\gamma \in \mathrm{Conf}(M)} \inf_{g \in \gamma} \frac{\int_M s_g d\mu_g}{\left(\int_M d\mu_g\right)^{\frac{1}{2}}}$$

where $\mathrm{Conf}(M)$ is the space of conformal classes of Riemannian metrics on M, s_g is the scalar curvature and $d\mu_g$ is the volume form of g.

THEOREM 3.19. *Let M_j and M be as in Theorem 3.15, and N_1 an oriented, closed, negative definite 4-manifold admitting a Riemannian metric with scalar curvature nonnegative at each point. Then*

$$\mathcal{Y}(M \# N_1) = -4\pi \sqrt{2 \sum_{j=1}^{l} c_1(M_j)^2}.$$

THEOREM 3.20. *Let M_j and M be as in Theorem 3.15. If N_2 be an oriented, closed, negative definite 4-manifold satisfying*

$$4l - (2\chi(N_2) + 3\tau(N_2)) \geq \frac{1}{3} \sum_{j=1}^{l} c_1(M_j)^2, \tag{3.2}$$

then $M \# N_2$ does not admit an Einstein metric.

PROOF OF THEOREM 3.19 AND THEOREM 3.20. In [IL], Ishida and LeBrun showed a similar statement under a somewhat different assumption (Theorem D). The main point of their argument is non-vanishing of the Bauer-Furuta invariant. In our case, the invariant $\sigma_M(\mathcal{L}, \mathcal{O}, L)$ is non-trivial. Hence we can apply their argument to our situation. □

On the other hand, there is a topological obstruction for 4-manifolds to have an Einstein metric ([H]).

THEOREM 3.21 (Hitchin-Thorpe inequality [H]). *Let X be an oriented closed 4-manifold admitting an Einstein metric, then*

$$3|\tau(X)| \leq 2\chi(X). \tag{3.3}$$

Example 3.22. Let $M_i = \Sigma_{g_i} \times \Sigma_{g'_i}$ for positive odd integers g_i, g'_i, let $M = M_1 \# M_2$ and let $N = (\#^r \overline{\mathbb{CP}^2}) \# (\#^s S^1 \times S^3)$. Then $b^+(N) = 0$ and the inequality (3.2) is satisfied if $r \geq \frac{8}{3}G - 4s - 4$, where $G := \sum_{i=1}^{2} (g_i - 1)(g'_i - 1)$. By Theorem 3.20, $X = M \# N$ does not admit an Einstein metric when $r \geq \frac{8}{3}G - 4s - 4$. On the other hand, if $r \leq 8G - 4s - 4$, then X satisfies the Hitchin-Thorpe inequality (3.3). Thus if

$$\frac{8}{3}G - 4s - 4 \leq r \leq 8G - 4s - 4,$$

X satisfies the Hitchin-Thorpe inequality, but does not admit an Einstein metric.

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