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# *The $L^p$ Boundedness of Wave Operators for Schrödinger Operators with Threshold Singularities II. Even Dimensional Case*

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**Abstract.** Let $H_0 = -\Delta$ and $H = -\Delta + V(x)$ be Schrödinger operators on $\mathbf{R}^m$ and $m \geq 6$ be even. We assume that $\mathcal{F}(\langle x \rangle^{-2\sigma} V) \in L^{m_*}(\mathbf{R}^m)$ for some $\sigma > \frac{1}{m_*}$, $m_* = \frac{m-1}{m-2}$ and $|V(x)| \leq C\langle x \rangle^{-\delta}$ for some $\delta > m+2$, so that the wave operators $W_\pm = \lim_{t \to \pm\infty} e^{itH} e^{-itH_0}$ exist. We show the following mapping properties of $W_\pm$: (1) If 0 is not an eigenvalue of $H$, $W_\pm$ are bounded in Sobolev spaces $W^{k,p}(\mathbf{R}^m)$ for all $0 \leq k \leq 2$ and $1 < p < \infty$ and also in $L^1(\mathbf{R}^m)$ and $L^\infty(\mathbf{R}^m)$; (2) if 0 is an eigenvalue of $H$ and if $V$ satisfies stronger decay condition $|V(x)| \leq C\langle x \rangle^{-\delta}$, $\delta > m+4$ if $m = 6$ and $\delta > m+3$ if $m \geq 8$, $W_\pm$ are bounded in $W^{k,p}(\mathbf{R}^m)$ for all $0 \leq k \leq 2$ and $\frac{m}{m-2} < p < \frac{m}{2}$; (3) the same holds in Sobolev spaces of higher orders if derivatives of $V(x)$ satisfy suitable boundedness conditions. This paper is a continuation of the one with the same title, part one, where odd dimensional cases $m \geq 3$ are treated, however, it can mostly be read independently.

## 1. Introduction

In this paper we study the continuity property in Sobolev or Lebesgue spaces of wave operators for $m$-dimensional Schrödinger operators $H = -\Delta + V(x)$ when it may have spectral singularities at the bottom of the continuous spectrum and when the spatial dimension $m \geq 6$ is even. This is a continuation of the previous paper [26] with the same title, part one, where the case $m \geq 3$ is odd is treated, however, we have tried to make the paper as independently readable as possible though we refer to it for some details. The paper [26] will be referred to as [I] in what follows.

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We assume as in [I] that potential $V(x)$ are real valued and $|V(x)| \leq C\langle x\rangle^{-\delta}$ for some $\delta > 2$, $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$, so that $H$ is selfadjoint in the Hilbert space $\mathcal{H} = L^2(\mathbf{R}^m)$ with domain $D(H) = W^{2,2}(\mathbf{R}^m)$ and $C_0^\infty(\mathbf{R}^m)$ is a core; the spectrum $\sigma(H)$ of $H$ consists of absolutely continuous part $[0,\infty)$ and a finite number of non-positive eigenvalues $\{\lambda_j\}$ of finite multiplicities; the singular continuous spectrum and positive eigenvalues are absent from $H$. We denote the point and the absolutely continuous spectral subspaces for $H$ by $\mathcal{H}_{\mathrm{p}}$ and $\mathcal{H}_{\mathrm{ac}}$ respectively, and the orthogonal projections onto the respective subspaces by $P_{\mathrm{p}}$ and $P_{\mathrm{ac}}$; $H_0 = -\Delta$ is the free Schrödinger operator.

The wave operators $W_\pm$ are defined by the following strong limits in $\mathcal{H}$:

$$W_\pm = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}.$$

It is well known that the limits exist, $W_\pm$ are complete in the sense that Image $W_\pm = \mathcal{H}_{\mathrm{ac}}$ and that they satisfy the intertwining property: For Borel functions $f$ on $\mathbf{R}$, we have

$$f(H)P_{\mathrm{ac}}(H) = W_\pm f(H_0)W_\pm^* \tag{1.1}$$

and, in particular, the absolutely continuous part of $H$ is unitarily equivalent to $H_0$ via $W_\pm$. It follows that the mapping properties of $f(H)P_{\mathrm{ac}}(H)$ may be deduced from those of $f(H_0)$ once corresponding properties of $W_\pm$ are known. In this paper we shall prove the following theorem. We say that $H$ is of *exceptional type* if there exist non-trivial solutions of $-\Delta\phi + V(x)\phi = 0$ which satisfy $|\phi(x)| \leq C\langle x\rangle^{2-m}$; $H$ is of *generic type* otherwise (see Definition 3.3 for an equivalent definition). We write $\mathcal{F}$ for the Fourier transform. Throughout this paper, we assume that $V$ satisfies the following condition:

$$\mathcal{F}(\langle x\rangle^{2\sigma}V) \in L^{m_*} \quad \text{for } \sigma > \frac{1}{m_*} \equiv \frac{m-2}{m-1}. \tag{1.2}$$

For integers $k \geq 0$, $W^{k,p}(\mathbf{R}^m)$ is the Sobolev space of order $k$.

THEOREM 1.1. *Let $m \geq 6$ be even and $V$ satisfy* (1.2).
(1) *Suppose, in addition, that $|V(x)| \leq C\langle x\rangle^{-(m+2+\varepsilon)}$ for some $C, \varepsilon > 0$ and that $H$ is of generic type. Then, for all $1 \leq p \leq \infty$, $W_\pm$ extend to bounded operators in $L^p(\mathbf{R}^m)$:*

$$\|W_\pm u\|_{L^p} \leq C_p\|u\|_{L^p}, \quad u \in L^p(\mathbf{R}^m) \cap L^2(\mathbf{R}^m). \tag{1.3}$$

*For* $1 < p < \infty$, $W_\pm$ *actually are bounded in* $W^{k,p}(\mathbf{R}^m)$ *for* $0 \le k \le 2$. *If derivatives* $\partial^\alpha V(x)$ *are bounded for* $|\alpha| \le \ell$, *in addition, then* $W_\pm$ *are bounded in* $W^{k,p}(\mathbf{R}^m)$ *for all* $0 \le k \le \ell + 2$ *and* $1 < p < \infty$. *For* $p = 1, \infty$, $W_\pm$ *are bounded in* $W^{k,p}(\mathbf{R}^m)$ *if all* $\partial^\alpha V(x)$, $|\alpha| \le k$, *satisfy* (1.2) *and* $|\partial^\alpha V(x)| \le C\langle x\rangle^{-(m+2+\varepsilon)}$ *for some* $C, \varepsilon > 0$, $k = 0, 1, \ldots$.
(2) *Suppose, in addition, that* $|V(x)| \le C\langle x\rangle^{-(m+4+\varepsilon)}$ *if* $m = 6$, *and* $|V(x)| \le C\langle x\rangle^{-(m+3+\varepsilon)}$ *if* $m \ge 8$ *for some* $C, \varepsilon > 0$, *and that* $H$ *is of exceptional type. Then, for* $\frac{m}{m-2} < p < \frac{m}{2}$ *and* $0 \le k \le 2$, $W_\pm$ *extend to bounded operators in* $W^{k,p}(\mathbf{R}^m)$*:*

$$\|W_\pm u\|_{W^{k,p}} \le C_p \|u\|_{W^{k,p}}, \qquad u \in W^{k,p}(\mathbf{R}^m) \cap L^2(\mathbf{R}^m). \tag{1.4}$$

*If* $\partial_x^\alpha V(x)$ *are bounded for* $|\alpha| \le \ell$ *in addition, then* (1.4) *holds for* $0 \le k \le \ell + 2$ *and* $W_\pm$ *are bounded in* $W^{k,p}(\mathbf{R}^m)$ *for all* $0 \le k \le \ell + 2$, $\ell = 0, 1, \ldots$.

Some remarks are in order.

Remark 1.2. Some condition like (1.2) is necessary for $W_\pm$ to be bounded in $L^p(\mathbf{R}^m)$ for all $1 \le p \le \infty$ because of the counter example due to [10] to the dispersive estimates for the corresponding time dependent Schrödinger equation, see below.

Remark 1.3. When $m \ge 3$ is odd, it is proved in [1] that $W_\pm$ are bounded in $L^p(\mathbf{R}^m)$ for all $1 \le p \le \infty$ if $V$ satisfies (1.2) and $|V(x)| \le C\langle x\rangle^{-(m+2+\varepsilon)}$ and if $H$ is of generic type; for $p$ between $\frac{m}{m-2}$ and $\frac{m}{2}$ if $V$ satisfies (1.2) and $|V(x)| \le C\langle x\rangle^{-(m+3+\varepsilon)}$ and if $H$ is of exceptional type. The argument in Section 7 below implies $W_\pm$ are actually bounded in $W^{k,p}(R^m)$ as in Theorem 1.1 under the same condition. In [4] an extension for some non-selfadjoint cases in $m = 3$ and its application to nonlinear equations are presented. When $m = 1$, it is recently shown ([6]) that $W_\pm$ are bounded in $L^p$ for all $1 < p < \infty$ (but not for $p = 1$ or $p = \infty$) if $\int_{\mathbf{R}} \langle x\rangle |V(x)| dx < \infty$ and $H$ is of generic type, or if $\int_{\mathbf{R}} \langle x\rangle^2 |V(x)| dx < \infty$ and $H$ is of exceptional type (see [21], [2] for earlier results).

Remark 1.4. When $m \ge 4$ is even, it is long known ([23]) that (1.3) is satisfied for all $1 \le p \le \infty$ if $V$ satisfies

$$\sum_{|\alpha| \le k + (m-2)/2} \left( \int_{|x-y| \le 1} |\partial_y^\alpha V(y)|^{p_0} dy \right)^{\frac{1}{p_0}} \le C\langle x\rangle^{-\left(\frac{3m}{2}+1+\varepsilon\right)} \tag{1.5}$$

for some $p_0 > \frac{m}{2}$ and $\varepsilon > 0$ and if $H$ *is of generic type.* If $m \ge 6$, condition (1.5) implies that $\partial^\alpha V$, $|\alpha| \le k$, satisfy both (1.2) and $|\partial_x^\alpha V(x)| \le C\langle x\rangle^{-(m+2+\varepsilon)}$ and Theorem 1.1 (1) improves the result of [23] for $m \ge 6$. When $m = 2$, it is known ([24], [13]) that $W_\pm$ are bounded in $L^p(\mathbf{R}^2)$ for $1 < p < \infty$ if $V$ satisfies $|V(x)| \le C\langle x\rangle^{-6-\varepsilon}$ and if $H$ is of generic type.

REMARK 1.5. If $m \ge 4$ and if $H$ is of exceptional type, $W_\pm$ are in general not bounded in $L^p(\mathbf{R}^m)$ when $\frac{m}{2} < p \le \infty$ because this would contradict the well-known result on the decay in time in weighted $L^2$ spaces of solutions $e^{-itH}u$ of the corresponding time dependent Schrödinger equation ([12], [17]). We strongly believe the same is true for $1 \le p < \frac{m}{m-2}$ though the proof is missing. Notice that when $m = 4$, $\frac{m}{m-2} = \frac{m}{2} = 2$.

REMARK 1.6. By interpolating (1.3) for different $k's$ by the real interpolation method ([3]), estimates of Theorem 1.1 can be extended to the ones between Besov spaces.

When $f(\lambda) = e^{-it\lambda}$, (1.1) and (1.3) imply the so called $L^p$-$L^q$ estimates for the propagator of the corresponding time dependent Schrödinger equation:

$$\|e^{-itH}P_c u\|_p \le C|t|^{-m\left(\frac{1}{2}-\frac{1}{p}\right)}\|u\|_q \tag{1.6}$$

where $p, q$ are dual exponents of each other, viz. $1/p + 1/q = 1$, and $2 \le p \le \infty$ if $H$ is of generic type and $2 \le p < m/2$ if $H$ is of exceptional type. When $1 \le m \le 3$ and if $H$ is of generic type, the $L^p$-$L^q$ estimate has been proven for $2 \le p \le \infty$ for a much wider class of potentials by more direct methods ([9], [18], [8]); when $m = 3$ and $H$ is of exceptional type it is proved that (1.6) holds for $2 \le p < 3$ and

$$\|e^{-itH}P_c u\|_{L^{3,\infty}} \le C_p t^{-\frac{1}{2}}\|u\|_{L^{\frac{3}{2},1}}$$

replaces (1.6) at the end point where $L^{p,q}$ are Lorentz spaces ([7], [25]). However, when $m \ge 4$, the result obtained by using wave operators via Theorem 1.1 (1) or [23] gives the best estimates so far as far as the decay and smoothness assumptions on the potentials are concerned. We should also emphasize that the $L^p$-$L^q$ estimate (1.6) is proven for the first time when $m \ge 6$ is even and $H$ is of exceptional type.

The intertwining property and the boundedness results, (1.1) and (1.3), may be applied for various other functions $f(H)P_c$ and can provide useful estimates. We refer the readers to [I] as well as [22] and [23] for some more applications, and we shall be devoted to the proof of Theorem 1.1 in the rest of the paper. Thus, we assume $m \geq 6$ is even in what follows unless otherwise stated.

We explain here the basic idea of the proof and the plan of the paper introducing some notation. We prove Theorem 1.1 only for $W_-$, which we denote by $W$ for brevity. We shall mainly discuss the $L^p$ boundedness, as the extension to Sobolev spaces is immediate as will be shown in Section 7. We write $R(z) = (H - z)^{-1}$ and $R_0(z) = (H_0 - z)^{-1}$ for resolvents; $\mathcal{H}_\gamma = L^2(\mathbf{R}^m, \langle x \rangle^{2\gamma} dx)$ is the weighted $L^2$ space. We parameterize $z \in \mathbf{C} \setminus [0, \infty)$ by $z = \lambda^2$ by $\lambda \in \mathbf{C}^+ = \{z \in \mathbf{C} : \Im z > 0\}$ and define $G(\lambda) = R(\lambda^2)$ and $G_0(\lambda) = R_0(\lambda^2)$ for $\lambda \in \mathbf{C}^+$. They are $\mathbf{B}(\mathcal{H})$-valued meromorphic functions of $\lambda \in \mathbf{C}^+$ and the limiting absorption principle (LAP for short) asserts that $G(\lambda)$ and $G_0(\lambda)$ when considered as $\mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\tau})$-valued functions, $\sigma, \tau > \frac{1}{2}$ and $\sigma + \tau > 2$, have continuous extensions to $\overline{\mathbf{C}}^+ = \{z : \Im z \geq 0\}$, the closure of $\mathbf{C}^+$, possibly except $\lambda = 0$. The proof is based on the stationary representation of wave operators which expresses $W$ via boundary values of the resolvents on the reals (cf. [15], [16]):

$$Wu = u - \frac{1}{\pi i} \int_0^\infty G(\lambda) V (G_0(\lambda) - G_0(-\lambda)) u \lambda d\lambda. \tag{1.7}$$

As in odd dimensions ([26]), we decompose $W$ into the high and the low energy parts $W = W_> + W_< \equiv W\Psi(H_0)^2 + W\Phi(H_0)^2$, by using cut off functions $\Phi(\lambda)$ and $\Psi(\lambda)$ such that $\Phi(\lambda)^2 + \Psi(\lambda)^2 \equiv 1$, $\Phi(\lambda) = 1$ near $\lambda = 0$ and $\Phi(\lambda^2) = 0$ for $|\lambda| > \lambda_0$ for a small constant $\lambda_0 > 0$ to be specified below. By virtue of the intertwining property we have $W_> = \Psi(H) W \Psi(H_0)$ and $W_< = \Phi(H) W \Phi(H_0)$ and, combining this with (1.7)

$$\begin{aligned} W_< = {} & \Phi(H)\Phi(H_0) \\ & - \int_0^\infty \Phi(H) G(\lambda) V (G_0(\lambda) - G_0(-\lambda)) \Phi(H_0) \lambda \frac{d\lambda}{\pi i}, \end{aligned} \tag{1.8}$$

$$\begin{aligned} W_> = {} & \Psi(H)\Psi(H_0) \\ & - \int_0^\infty \Psi(H) G(\lambda) V (G_0(\lambda) - G_0(-\lambda)) \Psi(H_0) \lambda \frac{d\lambda}{\pi i}. \end{aligned} \tag{1.9}$$

Operators $\Phi(H)$ and $\Phi(H_0)$ have continuous integral kernels bounded by $C_N\langle x-y\rangle^{-N}$ for any $N$ and they are bounded in $L^p(\mathbf{R}^m)$ for any $1 \le p \le \infty$. We study the operators defined by the integrals in (1.8) and (1.9) separately.

We use the following terminology.

DEFINITION 1.7. We say that the integral kernel $K(x,y)$ is admissible if

$$\sup_x \int_{\mathbf{R}^m} |K(x,y)|dy + \sup_y \int_{\mathbf{R}^m} |K(x,y)|dx < \infty. \tag{1.10}$$

It is well known that integral operators with admissible integral kernels are bounded in $L^p$ for any $1 \le p \le \infty$. In what follows $\langle x\rangle^\sigma$ denotes the multiplication operator with the function $\langle x\rangle^\sigma$.

DEFINITION 1.8. For $\rho > 0$, the operator valued function $K(\lambda)$ defined on an open interval $(-\lambda_0, \lambda_0)$ is said to satisfy property $(K)_\rho$ if the following two conditions are met:

(1) For $\gamma = 0, \dots, \frac{m-2}{2}$, $\lambda \mapsto \langle x\rangle^{\rho-\gamma}K(\lambda)\langle x\rangle^{\rho-\gamma} \in \mathbf{B}(\mathcal{H})$ is of class $C^\gamma$.

(2) For $\gamma = \frac{m}{2}, \frac{m+2}{2}$, it is of class $C^\gamma$ for $\lambda \neq 0$ and, for some $C > 0$ and $N > 0$,

$$\|\langle x\rangle^{\rho-\frac{m}{2}}K^{(\frac{m}{2})}(\lambda)\langle x\rangle^{\rho-\frac{m}{2}}\|_{\mathbf{B}(\mathcal{H})} \le C\langle\log\lambda\rangle^N, \tag{1.11}$$

$$\|\langle x\rangle^{\rho-\frac{m+2}{2}}K^{(\frac{m+2}{2})}(\lambda)\langle x\rangle^{\rho-\frac{m+2}{2}}\|_{\mathbf{B}(\mathcal{H})} \le C|\lambda|^{-1}\langle\log\lambda\rangle^N. \tag{1.12}$$

These definition are introduced to formulate Proposition 4.2 of Section 4: If $K(\lambda)$ satisfies property $(K)_\rho$ for some $\rho > m+1$,

$$\Omega = \int_0^\infty \Phi(H)G_0(\lambda)K(\lambda)(G_0(\lambda) - G_0(-\lambda))\Phi(H_0)\lambda\tilde{\Phi}(\lambda)d\lambda \tag{1.13}$$

defines an integral operator with admissible kernel, where $\tilde{\Phi} \in C_0^\infty(\mathbf{R})$ is an additional cut-off function such that $\tilde{\Phi}(\lambda)\Phi(\lambda) = \Phi(\lambda)$.

Section 2 is preparatory in nature. We write $\nu = (m-2)/2$; the branch of $z^{\frac{1}{2}}$ is always the one such that $z^{\frac{1}{2}} > 0$ when $z > 0$. When $m$ is even, the free resolvent $G_0(\lambda)$, $\Im\lambda \ge 0$, is the convolution with the kernel

$$G_0(\lambda, x) = \frac{e^{i\lambda|x|}}{2(2\pi)^{\nu+\frac{1}{2}}\Gamma(\nu+\frac{1}{2})|x|^{m-2}} \times \int_0^\infty e^{-t}t^{\nu-\frac{1}{2}}\left(\frac{t}{2} - i\lambda|x|\right)^{\nu-\frac{1}{2}} dt \tag{1.14}$$

and the $(m-2)$-nd derivative with respect to $\lambda$ of $G_0(\lambda)$ becomes logarithmically singular at $\lambda = 0$ if $m$ is even. In subsection 2.1, we shall study the smoothness properties of the operator valued function $G_0(\lambda)$ near $\lambda = 0$ in detail (see Proposition 2.6). In subsection 2.2, we recall from [22] the result on the $L^p$ boundedness of Born approximations of wave operators.

In Section 3 we study $(1+G_0(\lambda)V)^{-1}$ and show the followings where $P_0$ is the orthogonal projection onto the 0 eigenspace of $H$:

(a) If $H$ is of generic type, $V(1 + G_0(\lambda)V)^{-1}$ satisfies the property $(K)_\rho$ for any $\rho < \delta - 1$;

(b) If $H$ is of exceptional type, there exist finite rank operators $D_{jk}$ and an operator valued function $R_r(\lambda)$ such that $VR_r(\lambda)$ satisfies the condition $(K)_\rho$ for any $\rho < \delta - 3$ if $m = 6$ and $\rho < \delta - 2$ if $m \ge 8$ and such that

$$(1 + G_0(\lambda)V)^{-1} = \frac{P_0V}{\lambda^2} + \sum_{j=0}^{2}\sum_{k=1}^{2}\lambda^j(\log\lambda)^k D_{jk} + I + R_r(\lambda), \tag{1.15}$$

In Section 4 and Section 5 we study the low energy part $W_<$ respectively when $H$ is of generic type and of exceptional type. For proving this we substitute $G_0(\lambda)V(1 + G_0(\lambda)V)^{-1}$ for $G(\lambda)V$ on the right of (1.8). Then, the property (a) above and Proposition 4.2 imply that $W_<$ has an admissible kernel if $H$ is of generic type. Thus major task in Section 4 is the proof of Proposition 4.2 which is a result of rather surprizing cancellation.

For studying $W_<$ when $H$ is of exceptional type, we substitute (1.15) for $(1+G_0(\lambda)V)^{-1}$. Then, the identity $I$ produces the first Born approximation, which is bounded in $L^p$ for all $1 \le p \le \infty$ (Lemma 2.7); the property (b) above and Proposition 4.2 imply that $R_r(\lambda)$ produces an integral operator with admissible integral kernel; and we need study operators produced by the singular terms $\lambda^{-2}P_0V + \sum_{j=0}^{2}\sum_{k=1}^{2}\lambda^j(\log\lambda)^k D_{jk}$. We shall deal with

$$W_{s,m} = \int_0^\infty G_0(\lambda)VP_0V(G_0(\lambda) - G_0(-\lambda))\lambda^{-1}\Phi(\lambda)d\lambda,$$

in Subsection 5.1, the one produced by the most singular term $\lambda^{-2}P_0V$, basically following the idea used for odd dimensional cases ([26]): If $P_0 = \sum \phi_j \otimes \phi_j$, $W_{s,m}$ is a linear combination of

$$Z_j u(x) = \int_{\mathbf{R}^m} \frac{(V\phi_j)(x)(F_j u)(|x-y|)}{|x-y|^{m-2}}dy$$

where, with spherical average

$$M_j u(r) = |\Sigma|^{-1}\int_\Sigma (V\phi_j * \check{u})(r\omega)d\omega, \quad \check{u}(x) = u(-x)$$

and $\Sigma = S^{m-1}$ being the unit sphere of $\mathbf{R}^m$, $F_j u(\rho)$ is given by

$$\begin{aligned} F_j u(\rho) &= C\int_0^\infty\int_0^\infty e^{-(t+s)}(ts)^{\frac{m-3}{2}}dtds \\ &\times\left\{\int_0^\infty e^{i\lambda\rho}(s-2i\lambda\rho)^{\frac{m-3}{2}}\left(\int_{\mathbf{R}} e^{-i\lambda r}(t+2i\lambda r)^{\frac{m-3}{2}} rM_j u(r)dr\right)d\lambda\right\}. \end{aligned}$$

Observing that $F_j u(\rho)$ and $M_j u(r)$ are one dimensional objects, we apply the results of one dimensional harmonic analysis, weighted inequalities for the Hilbert transform $\tilde{\mathcal{H}}$ and for the Hardy-Littlewood maximal operator $\mathcal{M}$. However, as the comparison of formulae above with those in the odd dimensional case [26] suggests, the analysis in even dimensions becomes much more intricate. In Subsection 5.2 we shall indicate how to modify the argument in subsection 5.1 for dealing with the operators produced by $\lambda^j(\log\lambda)^k D_{jk}$.

In Section 6, we prove that the high energy part $W_>$ is bounded in $L^p$ for any $1 \le p \le \infty$. As the high energy part is insensitive to the low energy singularities and as the argument used for the same purpose in [26] for odd dimensions applies, we shall only very briefly sketch the proof. In Section 7, we show the continuity of $W$ in Sobolev spaces and complete the proof of Theorem 1.1. For $1 < p < \infty$, this follows from the intertwining property

$W = (H-z)^{-j}W(H_0-z)^j$ and the well known mapping property of the resolvent. For $p = 1$ and $p = \infty$, we may adopt the commutator argument as in [22] and we omit the discussion.

We use the same notation and conventions as in [I]: For $u \in \mathcal{H}_{-\gamma}$ and $v \in \mathcal{H}_\gamma$, $\langle u, v\rangle = \int_{\mathbf{R}^n} \overline{u(x)}v(x)dx$ is the standard coupling of functions; $|u\rangle\langle v| = u \otimes v$ will be interchangeably used to denote the rank 1 operator $\phi \mapsto \langle v, \phi\rangle u$. For Banach spaces $X$ and $Y$, $\mathbf{B}(X,Y)$ (resp. $\mathbf{B}_\infty(X,Y)$) is the Banach space of bounded (resp. compact) operators from $X$ to $Y$, $\mathbf{B}(X) = \mathbf{B}(X,X)$ (resp. $\mathbf{B}_\infty(X) = \mathbf{B}_\infty(X,X)$). The identity operator is denoted by 1. The norm of $L^p$-spaces, $1 \le p \le \infty$, is denoted by $\|u\|_p = \|u\|_{L^p}$. We write $\mathcal{S}(\mathbf{R}^m)$ for the space of rapidly decreasing functions. The Fourier transform is defined by

$$\hat{u}(\xi) = \mathcal{F}u(\xi) = \frac{1}{(2\pi)^{m/2}}\int_{\mathbf{R}^m} e^{-ix\xi}u(x)dx$$

and $\mathcal{F}^*u(\xi) = \mathcal{F}u(-\xi)$ is the conjugate Fourier transform. For $\sigma \in \mathbf{R}$, we define $H^\sigma(\mathbf{R}^m) = \mathcal{F}\mathcal{H}_\sigma(\mathbf{R}^m)$; if $\sigma \ge 0$ is an integer we have $H^\sigma(\mathbf{R}^m) = W^{\sigma,2}(\mathbf{R}^m)$. For functions $f$ on the line $f^{(j)}$ is the $j$-th derivative of $f$, $j = 1, 2, \dots$. For $a \in \mathbf{R}$, $a_+$ or $a_-$ is an arbitrary number larger or smaller than $a$ respectively; $[a]$ is the largest integer not larger than $a$. For a function on an open interval, $f \in C^s$ means that $f$ is $[s]$ times continuously differentiable and $f^{([s])}$ is locally Hölder continuous of order $s - [s]$. When $I = [0, a)$ or $I = (-a, a)$ with $0 < a \le \infty$, $C^s_{0*}(I)$ is the set of functions of order $C^s$ on $I$ which vanishes at $\lambda = 0$ along with the derivatives upto the order $[s]$. We sometimes say that $u$ is of class $C^s_{0*}$ on $I$ when $u \in C^s_{0*}(I)$.

## 2. Preliminaries

### 2.1. Free resolvent

As is well-known (see e.g. [16]), the mapping

$$\tilde{\Gamma}\colon H^\gamma(\mathbf{R}^m) \ni u \mapsto \lambda^{\frac{m-1}{2}}u(\lambda\cdot) \in H^\gamma_0((0,\infty), L^2(\Sigma)) \tag{2.1}$$

is bounded if $0 \le \gamma < \frac{m}{2}$. The upper bound for $\gamma$, however, is relevant only at $\lambda = 0$ and, for any $\varepsilon > 0$, the map (2.1) is bounded for any $0 \le \gamma$ if $H^\gamma_0((0,\infty), L^2(\Sigma))$ is replaces by $H^\gamma((\varepsilon,\infty), L^2(\Sigma))$. It follows by the

Sobolev embedding theorem that the $\mathbf{B}(\mathcal{H}_\gamma, L^2(\Sigma))$-valued function defined by

$$\Gamma(\lambda) \colon \mathcal{H}_\gamma \ni u \mapsto \lambda^{(m-1)/2}\hat{u}(\lambda\cdot) \in L^2(\Sigma),$$

is of class $C_{0*}^{\gamma-\frac{1}{2}}([0,\infty))$ if $\frac{1}{2} < \gamma < \frac{m}{2}$ and if $\gamma - \frac{1}{2}$ is not an integer; and it is of class $C^{\gamma-\frac{1}{2}}$ over $(\varepsilon,\infty)$ for any $\frac{1}{2} < \gamma$ and $\varepsilon > 0$ if $\gamma - \frac{1}{2}$ is not an integer. It is well known that the free resolvent $G_0(\lambda)$ may be expressed in the following form in terms of $\Gamma(\lambda)$:

$$G_0(\lambda) = \int_0^\infty \frac{\Gamma(\mu)^*\Gamma(\mu)}{\mu^2-\lambda^2} d\mu \tag{2.2}$$

We shall use the following well known lemma on the division in Sobolev spaces. The lemma is a result of repeated application of Hardy's inequality when $s$ is an integer and of the complex interpolation theory when $s$ is not an integer.

LEMMA 2.1. *For any $s > 0$, the operator $f(x) \mapsto x^{-s}f(x)$ is bounded from $H_0^\gamma(\mathbf{R}^+, L^2(\Sigma))$ to $H_0^{\gamma-s}(\mathbf{R}^+, L^2(\Sigma))$.*

We define operator valued function $A(\lambda)$ for $\lambda \in \mathbf{R}$ by

$$A(\lambda)u(x) = \frac{1}{(2\pi)^m}\int_\Sigma \int_{\mathbf{R}^m} e^{i\lambda\omega(x-y)}u(y)dyd\omega, \quad x \in \mathbf{R}^m. \tag{2.3}$$

It is clear that $A(\lambda)$ is an even function of $\lambda \in \mathbf{R}$ and $\mu^{m-1}A(\mu) = \Gamma(\mu)^*\Gamma(\mu)$. We shall use the following expressions for $G_0(\lambda)$, $\lambda \in \mathbf{C}^+$.

$$G_0(\lambda) = \frac{1}{2\lambda}\left(\int_0^\infty \frac{\Gamma(\mu)^*\Gamma(\mu)}{\mu-\lambda}d\mu - \int_0^\infty \frac{\Gamma(\mu)^*\Gamma(\mu)}{\mu+\lambda}d\mu\right) \tag{2.4}$$

$$= \int_0^\infty \frac{\mu^{m-1}A(\mu)}{\mu^2-\lambda^2}d\mu = \frac{1}{2}\int_{-\infty}^\infty \frac{\mu^{m-2}\mathrm{sign}\mu\ A(\mu)}{\mu-\lambda}d\mu. \tag{2.5}$$

It can be see from the last expression that $G_0(\lambda)$ becomes logarithmically singular when it is differentiated by $\lambda$ more than $m-3$ times. The following lemmas are basic to the following analysis. We let $D_1$, $D_2$ and $D_3$ be the closed domains in the first quadrant of $(k,\ell)$ plane defined by

$$D_1 = \{(k,\ell) : k,\ell \ge 0,\ k+\ell \le m-1, \ell \le k\},$$

$$D_2 = \{(k, \ell) : k, \ell \geq 0,\, k \leq \frac{m-1}{2}, \ell \geq k\},$$

$$D_3 = \{(k, \ell) : k, \ell \geq 0,\, k + \ell \geq m - 1, \frac{m-1}{2} \leq k \leq m - 1\}.$$

They have disjoint interiors and $D_1 \cup D_2 \cup D_3 = \{(k, \ell) : 0 \leq k \leq m-1, 0 \leq \ell\}$. Define the function $\sigma_0(k, \ell)$ for $0 \leq k \leq m - 1$ and $0 \leq \ell$ by

$$\sigma_0(k, \ell) = \begin{cases} \frac{k+\ell+1}{2}, & (k, \ell) \in D_1 \\ \ell + \frac{1}{2}, & (k, \ell) \in D_2, \\ k + \ell - \frac{m-2}{2}, & (k, \ell) \in D_3. \end{cases} \tag{2.6}$$

The function $\sigma_0(k, \ell)$ is continuous, separately increasing with respect to $k$ and $\ell$ and, on lines $k + \ell = c$ with fixed $c$, decreases with $k$.

LEMMA 2.2. *Let $\ell \geq 0$ be an integer and let $0 \leq k \leq m - 1$. Let $\sigma_0 = \sigma_0(k, \ell)$ be as above and $\sigma > \sigma_0$. Then, $\lambda^{m-1-k} A^{(\ell)}(\lambda)$ is a $\mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})$ valued function of $\lambda \in \mathbf{R}$ of class $C^{\sigma-\sigma_0}$.*

PROOF. Define $\rho(\lambda)u(\omega) = \hat{u}(\lambda\omega)$ for $\lambda \in \mathbf{R}$ and $\omega \in \Sigma$ and write $\Gamma_\lambda = \Gamma(\lambda)$ and $\rho_\lambda = \rho(\lambda)$ for shortening formulae. We also write $\mathcal{X}_\sigma \equiv \mathbf{B}(\mathcal{H}_\sigma, L^2(\Sigma))$. We have $(A(\lambda)u, v) = \langle \rho_\lambda u, \rho_\lambda v \rangle$. By differentiation,

$$\rho_\lambda^{(k)} u(\omega) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbf{R}^m} (-i\omega x)^k e^{-i\lambda\omega x} u(x) dx = \sum_{|\alpha|=k} C_\alpha \omega^\alpha \rho_\lambda(x^\alpha u)(\omega).$$

It follows by Leibniz' rule that

$$(A^{(\ell)}(\lambda)u, v) = \sum_{|\alpha|+|\beta|=\ell} C_{\alpha\beta} \langle \omega^\alpha \rho_\lambda(x^\alpha u), \omega^\beta \rho_\lambda(x^\beta u) \rangle.$$

In terms of $\Gamma_\lambda$ we may write this in the form

$$\lambda^{m-1-k}(A^{(\ell)}(\lambda)u, v) = \lambda^{-k} \sum_{|\alpha|+|\beta|=\ell} C_{\alpha\beta} \langle \omega^\alpha \Gamma_\lambda(x^\alpha u), \omega^\beta \Gamma_\lambda(x^\beta u) \rangle$$

It is an elementary to check that $\sigma_0 = \sigma_0(k, \ell)$ is equal to

$$\max_{|\alpha|+|\beta|=\ell} \min\{\max(a + |\alpha| + \tfrac{1}{2}, b + |\beta| + \tfrac{1}{2}) : 0 \leq a, b \leq \tfrac{m-1}{2}, a + b = k\}.$$

It follows that, if $\sigma > \sigma_0$ then for any $\alpha, \beta$ such that $|\alpha| + |\beta| = \ell$ we can find $0 \le a, b \le \frac{m-1}{2}$ such that

$$a + b = k, \quad a < \sigma - |\alpha| - \frac{1}{2}, \text{ and } b < \sigma - |\beta| - \frac{1}{2}. \tag{2.7}$$

For these $a, b$, $\lambda^{-a}\Gamma_\lambda \langle x\rangle^{|\alpha|}$ and $\lambda^{-b}\Gamma_\lambda \langle x\rangle^{|\beta|}$ are $\mathcal{X}_\sigma$-valued continuous. Indeed, if $a = \frac{m-1}{2}$, then, $\lambda^{-a}\Gamma_\lambda \langle x\rangle^{|\alpha|} = \rho_\lambda \langle x\rangle^{|\alpha|}$ is a $\mathcal{X}_\sigma$-valued function of class $C^{\sigma-|\alpha|-\frac{m}{2}}$ by virtue of Sobolev embedding theorem because $\sigma - |\alpha| > \frac{m}{2}$; if $a < \frac{m-1}{2}$, then, $\Gamma_\lambda \langle x\rangle^{|\alpha|}$ is $\mathcal{X}_\sigma$-valued function of class $C^{\gamma-}_{0*}([0,\infty))$, $\gamma = \min(\frac{m}{2}, \sigma - |\alpha| - \frac{1}{2})$, and $\lambda^{-a}\Gamma_\lambda \langle x\rangle^{|\alpha|}$ is of class $C^{(\gamma-a)-}([0,\infty))$ as a $\mathcal{X}_\sigma$-valued function. A similar proof applies to $\lambda^{-b}\Gamma_\lambda\langle x\rangle^{|\beta|}$. This and the identity

$$\lambda^{-k}\langle \omega^\alpha \Gamma_\lambda(x^\alpha u), \omega^\beta \Gamma_\lambda(x^\beta u)\rangle = \langle \omega^\alpha \lambda^{-a}\Gamma_\lambda(x^\alpha u), \omega^\beta \lambda^{-b}\Gamma_\lambda(x^\beta u)\rangle$$

imply that $\lambda \mapsto \lambda^{m-1-k}A^{(\ell)}(\lambda)$ is $\mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})$ valued continuous on $\mathbf{R}$. To conclude the proof, we need show that it is of class $C^{(\sigma-\sigma_0)-}$. Since

$$(d/d\lambda)\lambda^{m-1-k}A^{(\ell)}(\lambda) = (m-1-k)\lambda^{m-2-k}A^{(\ell)}(\lambda) + \lambda^{m-1-k}A^{(\ell+1)}(\lambda),$$

and $\sigma(k+1, \ell), \sigma(k, \ell+1) \le \sigma(k, \ell) + 1$, $\lambda^{m-1-k}A^{(\ell)}(\lambda)$ is continuously differentiable as an $\mathcal{X}_{\sigma_0+1+\varepsilon}$-valued function for any $\varepsilon > 0$. It follows that we have only to show this when $0 < \sigma - \sigma_0 \le 1$ and that

$$\|\lambda^{m-1-k}A^{(\ell)}(\lambda) - \mu^{m-1-k}A^{(\ell)}(\mu)\|_{\mathcal{X}_{\sigma_0+1+\varepsilon}} \le C|\lambda - \mu|$$

whenever $\lambda, \mu$ stay in bounded sets. Since $\|\lambda^{m-1-k}A^{(\ell)}(\lambda)\|_{\mathcal{X}_{\sigma_0+\varepsilon}}$ is locally bounded we have for $0 < \rho < 1$ that

$$\|\lambda^{m-1-k}A^{(\ell)}(\lambda) - \mu^{m-1-k}A^{(\ell)}(\mu)\|_{\mathcal{X}_{\sigma_0+\rho+\varepsilon}} \le C|\lambda - \mu|^\rho$$

on bounded sets by interpolation and the lemma follows. □

Corollary 2.3. *Let $0 \le a \le m-1$ and $b \ge 0$. Let $j \ge 0$ and $\sigma > \sigma_0(a, b+j)$. Then, $\lambda^{m-1-a}A^{(b)}(\lambda)$ is a $\mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})$-valued continuous function of $\lambda \in \mathbf{R}$ of class $C^{j+(\sigma-\sigma_0)-}$.*

Proof. It suffices to show that $\lambda \mapsto \lambda^{m-1-a-a'}A^{(b+b')}(\lambda) \in \mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})$ are continuous if $a' + b' \le j$ and $a + a' \le m-1$. This follows from Lemma 2.2 since, on the segment $\{(k, \ell) : k + \ell = a + b + j', 0 \le$

$a \le k \le m-1\}$, $\sigma_0$ attains its maximum at $(a, b+j')$ and $\sigma_0(a, b+j')$ increases with $0 \le j'$. $\square$

The following Lemma 2.4 is a slight improvement of Corollary 2.3 for small $\sigma$. We omit the proof as it is identical with that of Lemma 2.1 of [I].

Lemma 2.4. *Let $\frac{1}{2} < \sigma, \tau < \frac{3}{2}$ be such that $\sigma + \tau > 2$ and define $\rho_0 = \tau + \sigma - 2$. Then, as a $B(\mathcal{H}_\sigma, \mathcal{H}_{-\tau})$-valued function, $\lambda^{m-2}A(\lambda)$ is of class $C^\rho$ for any $\rho < \rho_0$ in $\mathbf{R}$ and of class $C^{\min(\sigma-\frac{1}{2}, \tau-\frac{1}{2})}$ in $\mathbf{R} \setminus \{0\}$.*

Lemma 2.5. (1) *Let $1/2 < \sigma$. Then, $G_0(\lambda)$ is a $\mathbf{B}_\infty(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})$ valued function of $\lambda \in \overline{\mathbf{C}}^+ \setminus \{0\}$ of class $C^{(\sigma-\frac{1}{2})-}$. For non-negative integers $j < \sigma - \frac{1}{2}$,*

$$\|G_0^{(j)}(\lambda)\|_{\mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})} \le C_{j\sigma}|\lambda|^{-1}, \quad |\lambda| \ge 1. \tag{2.8}$$

(2) *Let $\frac{1}{2} < \sigma, \tau < m - \frac{3}{2}$ satisfy $\sigma + \tau > 2$. Then, $G_0(\lambda)$ is a $\mathbf{B}_\infty(\mathcal{H}_\sigma, \mathcal{H}_{-\tau})$-valued function of $\lambda \in \overline{\mathbf{C}}^+$ of class $C^{\rho_*-}$, $\rho_* = \min(\tau+\sigma-2, \tau-1/2, \sigma-1/2)$.*

Proof. The first statement is well known and follows immediately from (2.4) and the property of $\Gamma(\lambda)$ stated at the beginning of this subsection. By virtue of Corollary 2.3 and Lemma 2.4, $\operatorname{sign}\mu\ \mu^{m-2}A(\mu)$ is a $\mathbf{B}_\infty(\mathcal{H}_\sigma, \mathcal{H}_{-\tau})$-valued function of $\mu \in \mathbf{R}$ since $\rho_* < m-2$. We apply the Privaloff theorem to the last expression of (2.5). The second statement follows. $\square$

The $(m-2)$-th derivative of $\operatorname{sign}\mu\ \mu^{m-2}A(\mu)$ contains Heaviside type singularity at $\mu = 0$ and, for any large $\sigma$, $\lambda \mapsto \langle x\rangle^{-\sigma} G_0^{(m-2)}(\lambda)\langle x\rangle^{-\sigma} \in \mathbf{B}(\mathcal{H})$ is not continuous at $\lambda = 0$. We now examine this singularity. Let

$$J_k(\lambda) = \frac{1}{\lambda^k}\left(G_0(\lambda) - G_0(0) - \cdots - G_0^{(k-1)}(0)\frac{\lambda^{k-1}}{(k-1)!}\right).$$

Proposition 2.6. *Let $m \ge 4$ be even. Then:*

(1) *Let $k = 0, 1, \ldots, m-3$ and $0 < \rho < m-2-k$. Let $\sigma > \sigma_0(k+1, \rho)$. Then $J_k(\lambda)$ is a $\mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})$-valued function of $\lambda \in \mathbf{R}$ of class $C^\rho$.*

(2) *For* $0 < \pm\lambda < \frac{1}{8}$, $G_0(\lambda)$ *has the following expression as an operator from* $\mathcal{S}(\mathbf{R}^m)$ *to* $\mathcal{S}'(\mathbf{R}^m)$*:*

$$\sum_{j=0}^{\frac{m-4}{2}} \lambda^{2j}(-\Delta)^{-j-1} + \lambda^{m-2}\Big(\pm\frac{i\pi}{2}A(\lambda) - \log|\lambda|\,A(\lambda)\Big) + \lambda^{m-2}F(\lambda),$$

*where* $\lambda \mapsto F(\lambda)$ *is even and, for* $k = 0, \dots, m-1$, $\lambda^{m-1-k}F(\lambda)$ *satisfies the same smoothness property as* $\lambda^{m-1-k}A(\lambda)$ *as stated in Corollary* 2.3 *and Lemma* 2.4*.*

Remark that proof of Lemma 2.2 implies that for $\lambda \mapsto \sum_{j=0}^{\frac{m-4}{2}} \lambda^{2j}(-\Delta)^{-j-1}$ is a $\mathbf{B}(\mathcal{H}_{(\frac{m-1}{2})_+}, \mathcal{H}_{-(\frac{m-1}{2})_+})$ valued polynomial and hence is analytic.

PROOF. If $k = 0$, statement (1) is contained in Lemma 2.5 (2). Let $k > 0$. Substituting $\sum_{j=0}^{k-1} \lambda^j \mu^{-j-1} + (\mu-\lambda)^{-1}\lambda^k\mu^{-k}$ for $(\mu-\lambda)^{-1}$ in the second equation of (2.5), we have for $\lambda \in \mathbf{C}^+$ that

$$G_0(\lambda) = \sum_{j=0}^{k-1} \frac{\lambda^j}{2}\int_{-\infty}^{\infty} \mu^{m-j-3}\mathrm{sign}\mu\ A(\mu)d\mu + \lambda^k \int_{-\infty}^{\infty} \frac{\mu^{m-1}\mathrm{sign}\mu\ A(\mu)ud\mu}{2\mu^{k+1}(\mu-\lambda)}.$$

Since $A(\mu)$ is even, the integrals in the sum vanish for odd $j$ and for even $j$

$$\frac{1}{2}\int_{-\infty}^{\infty} \mu^{m-j-3}\mathrm{sign}\mu\ A(\mu)d\mu = \int_0^{\infty} \frac{\mu^{m-1}A(\mu)}{\mu^{j+2}}d\mu = (-\Delta)^{-(\frac{j+2}{2})}\lambda^j.$$

Thus, we have for $\lambda \in \mathbf{C}^+$

$$J_k(\lambda) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\mu^{m-2-k}\mathrm{sign}\mu\ A(\mu)}{\mu-\lambda}d\mu. \tag{2.9}$$

If $\sigma > \sigma_0(k+1,\rho)$ and $\rho < m-2-k$, $\mu \mapsto \mu^{m-2-k}\mathrm{sign}\mu\ A(\mu) \in \mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})$ is of class $C^\rho$ by virtue of Corollary 2.3 and (1) follows by Privaloff's theorem.

(2) We substitute $\mu^{m-2} = (\mu^2-\lambda^2)(\mu^{m-4} + \lambda^2\mu^{m-6} + \cdots + \lambda^{m-4}) + \lambda^{m-2}$ in the first of (2.5). The result is:

$$G_0(\lambda) = \sum_{j=0}^{\frac{m-4}{2}} \lambda^{2j}(-\Delta)^{-j-1} + \lambda^{m-2}\int_0^{\infty} \frac{\mu A(\mu)}{\mu^2-\lambda^2}d\mu, \quad \lambda \in \mathbf{C}^+. \tag{2.10}$$

We rewrite the integral in (2.10). Take an even function $\chi \in C_0^\infty(\mathbf{R})$ such that $\chi(\mu) = 1$ for $|\mu| \le 1/4$ and $\chi(\mu) = 0$ for $|\mu| \ge 1/2$, and split it as

$$\int_0^\infty \frac{\mu A(\mu)}{\mu^2 - \lambda^2} d\mu = \int_0^\infty \frac{\mu \chi(\mu) A(\mu)}{\mu^2 - \lambda^2} d\mu + \int_0^\infty \frac{\mu(1-\chi(\mu))A(\mu)}{\mu^2 - \lambda^2} d\mu. \tag{2.11}$$

We denote $B(\mu) = \chi(\mu)A(\mu)$, write the first integral in the form

$$\frac{1}{2}\left(\int_0^\infty \frac{B(\mu)}{\mu + \lambda} d\mu + \int_0^\infty \frac{B(\mu)}{\mu - \lambda} d\mu\right), \tag{2.12}$$

and take the boundary values at $-\frac{1}{8} < \lambda < \frac{1}{8}$:

$$\frac{1}{2}\left(\int_0^1 \frac{B(\mu)}{\mu + |\lambda|} d\mu \pm i\pi B(\lambda) + \text{p.v.} \int_0^\infty \frac{B(\mu)}{\mu - |\lambda|} d\mu\right), \quad 0 < \pm\lambda < \tfrac{1}{8}. \tag{2.13}$$

To fix the idea we let $0 < \lambda < \frac{1}{8}$. We split the domain of integral of the second integral: $[0,\infty) = [0, 2\lambda) \cup [2\lambda, \infty)$. The integral over $[2\lambda, \infty)$ is equal to $\int_\lambda^1 B(\mu + \lambda)\mu^{-1} d\mu$, and we add it to the first integral which is equal to $\int_\lambda^1 B(\mu - \lambda)\mu^{-1} d\mu$. We write the sum in the form

$$\left(\int_0^1 - \int_0^\lambda\right) \frac{B(\mu + \lambda) + B(\mu - \lambda) - 2B(\lambda)}{2\mu} d\mu - (\log \lambda) A(\lambda)$$

and add this to

$$\frac{1}{2}\,\text{p.v.} \int_0^{2\lambda} \frac{B(\mu)}{\mu - \lambda} d\mu = \int_0^\lambda \frac{(B(\lambda + \mu) - B(\lambda - \mu))}{2\mu} d\mu.$$

Thus, we have arrived at the desired expression if we define $F(\lambda)$ by

$$\begin{aligned} F(\lambda) = \int_0^\infty \frac{\mu(1-\chi(\mu))A(\mu)}{\mu^2 - \lambda^2} d\mu &- \int_0^\lambda \frac{(B(\lambda - \mu) - B(\lambda))}{\mu} d\mu \\ &+ \int_0^1 \frac{B(\mu + \lambda) + B(\mu - \lambda) - 2B(\lambda)}{2\mu} d\mu. \end{aligned} \tag{2.14}$$

It is immediate to check that $F(\lambda) = F(-\lambda)$. We prove that $\lambda^{m-1-k}F(\lambda)$ satisfies the desired smoothness properties on $(-\frac{1}{8}, \frac{1}{8})$. The first integral yields $(1 - \chi(\sqrt{H_0}))H_0^{-\frac{m-2}{2}}(H_0 - \lambda^2)^{-1}$ which is a $\mathbf{B}(\mathcal{H})$-valued analytic of $\lambda$ in a neighborhood of the interval $(-\frac{1}{8}, \frac{1}{8})$ and we ignore it. Let $k = m - 1$

first. We have $\sigma(m-1,j)=j+\frac{m}{2}$. Hence, if $\sigma>\frac{m}{2}$ and $0\le t<\sigma-\frac{m}{2}$, $B(\mu)$ is a $\mathbf{B}(\mathcal{H}_\sigma,\mathcal{H}_{-\sigma})$valued function of class $C^t$ and the last two integrals are of class $C^{t-}$. Indeed, if $0<t<1$. Then

$$\left\|\int_\lambda^{\lambda+h}\frac{B(\lambda+h-\mu)-B(\lambda+h)}{\mu}d\mu\right\|\le\int_\lambda^{\lambda+h}\mu^{t-1}d\mu\le Ch^t.$$

Since $\|(B(\lambda+h-\mu)-B(\lambda+h))-(B(\lambda-\mu)-B(\lambda))\|\le C\min(\mu^t,h^t)$,

$$\left\|\int_0^\lambda((B(\lambda+h-\mu)-B(\lambda+h))-(B(\lambda-\mu)-B(\lambda)))\frac{d\mu}{\mu}\right\|$$
$$\le\int_0^h C\mu^{t-1}d\mu+Ch^t\int_h^\lambda\frac{d\mu}{\mu}\le Ch^t(1+|\log h|).$$

Thus, the first integral on the right of (2.14) is of class $C^{t-}$. When $t\ge1$, we differentiate it and apply similar estimates. In this way we prove that it is of class $C^{(\sigma-\frac{m}{2})-}$. The proof for the last integral in (2.14) is simpler and is omitted. We next let $0\le k\le m-2$ and write $m-1-k=t$. We have

$$\lambda^t\int_0^\lambda\frac{B(\lambda-\mu)-B(\lambda)}{\mu}d\mu=\int_0^\lambda\frac{(\lambda-\mu)^tB(\lambda-\mu)-\lambda^tB(\lambda)}{\mu}d\mu \\ +\sum_{\ell=1}^t\binom{t}{\ell}\int_0^\lambda\mu^{\ell-1}(\lambda-\mu)^{t-\ell}B(\lambda-\mu)d\mu \tag{2.15}$$

The argument for the case $k=0$ implies that the first integral on the right satisfies the desire smoothness property. Since $\sigma(k+\ell,j)\le\sigma(k,\ell)+\ell$, each summand on the right

$$\int_0^\lambda\mu^{t-\ell}(\lambda-\mu)^{\ell-1}B(\mu)d\mu,\quad \ell=1,\dots,t,$$

which may be written in the form

$$(\ell-1)!\int_0^\lambda\cdots\int_0^{s_2}\int_0^{s_1}\mu^{t-\ell}B(\mu)d\mu ds_1\dots ds_{\ell-1}, \tag{2.16}$$

also enjoys the desired smoothness property. To prove the same for the last integral in (2.14), we split it:

$$\int_0^1\frac{B(\mu+\lambda)-B(\lambda)}{2\mu}d\mu+\int_0^1\frac{B(\mu-\lambda)-B(\lambda)}{2\mu}d\mu, \tag{2.17}$$

multiply it by $\lambda^t$, $t = m-1-k$ and write the resulting function as in (2.15). For the first integral in (2.15), as previously, it suffices to show

$$\int_0^1 (\lambda+\mu)^{t-\ell} B(\lambda+\mu)\mu^{\ell-1} d\mu = \int_\lambda^\infty \mu^{t-\ell} B(\mu)(\mu-\lambda)^{\ell-1} d\mu$$

satisfies the desired property. However, this may be written in the form

$$(\ell-1)! \int_\lambda^\infty \int_{s_{\ell-1}}^\infty \cdots \int_{s_1}^\infty \mu^{t-\ell} B(\mu) d\mu ds_1 \dots ds_{\ell-1}$$

and obviously satisfies it by the same reason as (2.16) does. It is clear that the same holds for the second integral of (2.17) and this completes the proof of the proposition. □

### 2.2. Born terms

If we formally expand the right of $G(\lambda)V = (1 + G_0(\lambda)V)^{-1}G_0(\lambda)V$ into the series $\sum_{n=1}^\infty (-1)^{n-1}(G_0(\lambda)V)^n$ and substitute it for $G(\lambda)V$ in the stationary formula (1.7), then we have $W = 1 - \Omega_1 + \Omega_2 - \cdots$ where

$$\Omega_n u = \frac{1}{\pi i} \int_0^\infty (G_0(\lambda)V)^n (G_0(\lambda) - G_0(-\lambda)) u \lambda d\lambda, \; n = 1, 2, \dots .$$

The sum $I - \Omega_1 + \cdots + (-1)^n \Omega_n$ is called the $n$-th Born approximation of $W_-$ and individual $\Omega_n$ is called the $n$-Born term. The following lemma is proved in any dimension $m \geq 3$ ([22]) and it will be used for studying both the low and the high energy parts of $W$.

Lemma 2.7. *Let $\sigma > 1/m_*$. Then there exists a constant $C > 0$ such that*

$$\|\Omega_1 u\|_{W^{k,p}} \leq C \sum_{|\alpha|\leq k} \|\mathcal{F}\langle x\rangle^\sigma (\partial^\alpha V)\|_{L^{m_*}(R^m)} \|u\|_{W^{k,p}}, \tag{2.18}$$

$$\|\Omega_n u\|_{W^{k,p}} \leq C^n \Big( \sum_{|\alpha|\leq k} \|\mathcal{F}\langle x\rangle^{2\sigma} (\partial^\alpha V)\|_{L^{m_*}(R^m)} \Big)^n \|u\|_{W^{k,p}}, \quad n = 2, \dots \tag{2.19}$$

*for any $1 \leq p \leq \infty$.*

## 3. Threshold Singularities

The resolvent $G(\lambda) = (H - \lambda^2)^{-1}$ of $H = -\Delta + V$ is a $\mathbf{B}(\mathcal{H})$-valued meromorphic function of $\lambda \in \mathbf{C}^+$ with possible poles $i\kappa_1, \dots, i\kappa_n$ on $i\mathbf{R}^+$ such that $-\kappa_1^2, \dots, -\kappa_n^2$ are eigenvalues of $H$ and outside the poles we have

$$G(\lambda) = (1 + G_0(\lambda)V)^{-1} G_0(\lambda), \quad \lambda \in \mathbf{C}^+. \tag{3.1}$$

For $\lambda \in \mathbf{R}$, $G_0(\lambda)V \in \mathbf{B}_\infty(\mathcal{H}_{-\gamma})$ for all $\frac{1}{2} < \gamma < \delta - \frac{1}{2}$ and $1 + G_0(\lambda)V$, $\lambda \neq 0$, is invertible if and only if $\lambda$ is an eigenvalue of $H$ ([1]). Since positive eigenvalues are absent from $H$ ([14]), $(1 + G_0(\lambda)V)^{-1}$ exists for $\lambda \in \mathbf{R} \setminus \{0\}$ as well and the equation (3.1) is satisfied for all $\lambda \in \overline{\mathbf{C}}^+ \setminus \{0\}$. The following lemma is well known ([1], [16]).

LEMMA 3.1. *Let $\frac{1}{2} < \gamma < \delta - \frac{1}{2}$. Then, $G(\lambda)$ is a $\mathbf{B}_\infty(\mathcal{H}_\gamma, \mathcal{H}_{-\gamma})$ valued function of $\lambda \in \overline{\mathbf{C}}^+ \setminus \{0\}$ of class $C^{(\gamma - \frac{1}{2})-}$. For $0 \le j < \gamma - \frac{1}{2}$,*

$$\|G^{(j)}(\lambda)\|_{\mathbf{B}(\mathcal{H}_\gamma, \mathcal{H}_{-\gamma})} \le C_{j\gamma} |\lambda|^{-1}, \quad |\lambda| \ge 1. \tag{3.2}$$

Following [11], we define, with $D_0 = G_0(0)$,

$$\mathcal{N} = \{\phi \in \mathcal{H}_{-\gamma} : (1 + D_0 V)\phi = 0\}. \tag{3.3}$$

It is well known ([11], [25]) that $\mathcal{N}$ is finite dimensional and it is independent of $\frac{1}{2} < \gamma < \delta - \frac{1}{2}$; $-(Vu, u)$ defines an inner product of $\mathcal{N}$; and if $\{\phi_1, \dots, \phi_d\}$ is an orthonormal basis of $\mathcal{N}$, $\{-V\phi_1, \dots, -V\phi_d\}$ is the dual basis of the dual space $\mathcal{N}^* = \{\psi \in \mathcal{H}_\gamma : (1 + VD_0)\psi = 0\}$; it follows that the spectral projection $Q$ in $\mathcal{H}_{-\gamma}$ for the eigenvalue $-1$ of $D_0 V$ is given by $Q = -\sum_{j=1}^d \phi_j \otimes (V\phi_j)$. We set $\overline{Q} = 1 - Q$.

LEMMA 3.2. *Let $m \ge 6$ and $D_2 = (-\Delta)^{-2}$. Let $\phi \in \mathcal{N}$. Then:*

$$V\phi \in \mathcal{H}_{\left(\delta + \frac{m-4}{2}\right)_-}; \ |(D_2 V\phi)(x)| \le C\langle x \rangle^{4-m} \text{ and } D_2 V\phi \in \mathcal{H}_{\left(\frac{m-8}{2}\right)_-}. \tag{3.4}$$

PROOF. The lemma follows since $\phi \in \mathcal{N}$ satisfy $|\phi(x)| \le C\langle x \rangle^{-(m-2)}$ and $D_2$ has the integral kernel $C|x - y|^{4-m}$, $m \ge 6$. □

By virtue of (3.4), $\mathcal{N}$ coincides with the eigenspace $\mathcal{E}$ of $H$ with eigenvalue 0 if $m \geq 6$ and the following definition is equivalent to the one given in the introduction.

DEFINITION 3.3. We say that the operator $H$ is of generic type if $\mathcal{N} = \{0\}$ and that $H$ is of exceptional type if otherwise.

### 3.1. Generic case

When $H$ is of generic type, $G(\lambda)$ as a $\mathbf{B}(\mathcal{H}_\gamma, \mathcal{H}_{-\gamma})$ valued function, $\frac{1}{2} < \gamma < \delta - \frac{1}{2}$, satisfies the same regularity properties as $G_0(\lambda)$ as stated in Lemma 2.5 on $\mathbf{R}$. We write

$$M(\lambda) = I + G_0(\lambda)V, \quad \lambda \in \mathbf{R}$$

in what follows.

DEFINITION 3.4. Let $\rho > 0$ be an integer and $I$ be an open interval containing 0. We say that a Banach space valued function $f(\lambda)$ on $I$ is of class $C_*^\rho$ on $I$ if $f \in C^\rho(I \setminus \{0\}) \cap C^{\rho-1}(I)$ and it satisfies $\|f^{(\rho)}(\lambda)\| \leq C\langle \log \lambda \rangle^N$ for constants $C > 0$ and $N > 0$, $\lambda \neq 0$.

LEMMA 3.5. *Let* $\frac{1}{2} < \gamma, \tau < \delta - \frac{1}{2}$ *be such that* $\gamma + \tau > 2$. *Let* $\rho_0 = \min(\gamma - 1/2, \delta - \gamma - 1/2)$ *and* $\rho_* = \min(\gamma - 1/2, \tau - 1/2, \tau + \gamma - 2)$. *Suppose* $H$ *is of generic type. Then:*

(1) *If* $\rho_0 \leq m - 2$, $M^{-1}(\lambda)$ *is a* $\mathbf{B}(\mathcal{H}_{-\gamma})$ *valued function of* $\lambda$ *of class* $C^{(\rho_0)-}$. *If* $\rho_0 > m - 2$, *it is of class* $C^{(\rho_0)-}$ *for* $\lambda \neq 0$ *and of class* $C_*^{m-2}$ *on* $\mathbf{R}$.

(2) *For any* $\lambda \in \mathbf{R}$, $M(\lambda)^{-1} - 1$ *may be extended to a bounded operator from* $\mathcal{H}_{-\delta+\gamma}$ *to* $\mathcal{H}_{-\tau}$. *If* $\rho_* \leq m - 2$, *it is a* $\mathbf{B}(\mathcal{H}_{-\delta+\gamma}, \mathcal{H}_{-\tau})$*-valued function of class* $C^{(\rho_*)-}$. *If* $\rho_* > m - 2$, *it is of class* $C^{(\rho_*)-}$ *for* $\lambda \neq 0$ *and of class* $C_*^{m-2}$ *on* $\mathbf{R}$. *If* $m = 4$ *and* $\rho_* > 3$, $\lambda(M(\lambda)^{-1} - 1)$ *is of class* $C^{(\rho_*)-}$ *for* $\lambda \neq 0$ *and of class* $C_*^3$ *on* $\mathbf{R}$.

PROOF. We prove the estimates $\|\partial_\lambda^{m-2} M^{-1}(\lambda)\|_{\mathbf{B}(\mathcal{H}_{-\delta+\gamma}, \mathcal{H}_{-\tau})} \leq C\langle \log \lambda \rangle$ only, assuming $\rho_* > m - 2$, as the rest may be proved, by virtue of Lemma 2.5, by an almost word by word repetition of the proof of Lemma 2.7

of [I]. By using the identity $\partial_\lambda M^{-1}(\lambda) = -M^{-1}(\lambda)G_0'(\lambda)VM^{-1}(\lambda)$ we $m-2$ times formally differentiate $M^{-1}(\lambda)-I$. This produces a linear combination over $j_1 + \cdots j_k = m-2$, $j_1, \dots, j_k \geq 1$ of

$$M^{-1}(\lambda)G_0^{(j_1)}(\lambda)VM^{-1}(\lambda)\cdots M^{-1}(\lambda)G_0^{(j_k)}(\lambda)VM^{-1}(\lambda).$$

If $k \geq 2$, this is bounded in $\mathbf{B}(\mathcal{H}_{-\delta+\gamma}, \mathcal{H}_{-\tau})$ near $\lambda = 0$ by the proof of Lemma 2.7 of [I]; and if $k = 1$, this is bounded by $C\langle \log\lambda\rangle$ by virtue of Proposition 2.6 (2) and of the estimate

$$\|\partial_\lambda^{m-2}(\lambda^{m-2}\log\lambda A(\lambda))\|_{\mathbf{B}(\mathcal{H}_\gamma,\mathcal{H}_{-\tau})} \leq C\langle\log\lambda\rangle, \quad |\lambda| < 1$$

obtained via Corollary 2.3 and Lemma 2.2. The desired estimate follows. □

### 3.2. Exceptional case

In this subsection we assume $H$ is of exceptional type. Then $(1 + G_0(\lambda)V)^{-1}$ is singular at $\lambda = 0$. When $m$ is even, the logarithmic singularities appear in addition to those due to the 0 eigenspace of $H$ and the analysis becomes more complex than in odd dimensions. In this subsection we prove the following expansion formulae for $(1 + G_0(\lambda)V)^{-1}$. Recall that $P_0$ *is the orthogonal projection in* $L^2(\mathbf{R}^m)$ *onto the zero eigenspace of* $H$.

Proposition 3.6. (1) *Let* $m = 6$ *and* $|V(x)| \leq C\langle x\rangle^{-\delta}$ *with* $\delta > 10 = m+4$. *Then, with* $E(\lambda)$ *such that* $VE(\lambda)$ *satisfies the condition* $(K)_\rho$ *with* $\rho > m+1$,

$$(1 + G_0(\lambda)V)^{-1} - 1 = \frac{P_0V}{\lambda^2} + \sum_{j=0}^{\frac{m-2}{2}}\sum_{k=1}^{2} D_{jk}\lambda^j\log^k\lambda + E(\lambda). \tag{3.5}$$

*Here* $D_{jk}$ *are finite rank operators of the form*

$$D_{jk} = P_0VD_{jk}^{(1)}P_0V + D_{jk}^{(2)}P_0V + P_0VD_{jk}^{(3)}, \tag{3.6}$$

*where* $D_{jk}^{(1)} \in \mathbf{B}(\mathcal{N})$, $D_{jk}^{(2)} \in \mathbf{B}(\mathcal{N}, \mathcal{H}_{-3_+})$ *and* $D_{jk}^{(3)} \in \mathbf{B}(\mathcal{H}_{-\delta+3_+}, \mathcal{N})$.
(2) *Let* $m \geq 8$ *and* $|V(x)| \leq C\langle x\rangle^{-\delta}$ *with* $\delta > m+3$. *Then, with a constant* $c_m$ *and* $E(\lambda)$ *such that* $VE(\lambda)$ *satisfies the condition* $(K)_\rho$ *with* $\rho > m+1$,

$$(1 + G_0(\lambda)V)^{-1} - 1 = \frac{P_0V}{\lambda^2} + c_m\varphi\otimes(V\varphi)\lambda^{m-6}\log\lambda + E(\lambda). \tag{3.7}$$

*Here* $\varphi = P_0V$ *with* $V$ *being considered as a function. If* $m \geq 12$*, then* $c_m\varphi \otimes (V\varphi)\lambda^{m-6}\log\lambda$ *may be included in* $E(\lambda)$.

The rest of this subsection is devoted to the proof of Proposition 3.6. We use the following lemma as in the odd dimensional case.

LEMMA 3.7. *Let* $\mathcal{X} = \mathcal{X}_0 \dot{+} \mathcal{X}_1$ *be a direct sum decomposition of a vector space* $\mathcal{X}$*. Suppose that a linear operator* $L$ *in* $\mathcal{X}$ *is written in the form*

$$L = \begin{pmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{pmatrix}$$

*in this decomposition and that* $L_{00}^{-1}$ *exists. Set* $C = L_{11} - L_{10}L_{00}^{-1}L_{01}$*. Then,* $L^{-1}$ *exists if and only if* $C^{-1}$ *exists. In this case*

$$L^{-1} = \begin{pmatrix} L_{00}^{-1} + L_{00}^{-1}L_{01}C^{-1}L_{10}L_{00}^{-1} & -L_{00}^{-1}L_{01}C^{-1} \\ -C^{-1}L_{10}L_{00}^{-1} & C^{-1} \end{pmatrix}. \tag{3.8}$$

Using the spectral projections $Q$ and $\overline{Q} = 1 - Q$, we decompose $\mathcal{H}_{-\gamma} = \overline{Q}\mathcal{H}_{-\gamma} \dot{+} \mathcal{N}$ as a direct sum. With respect to this decomposition, we write

$$M(\lambda) = \begin{pmatrix} \overline{Q}M(\lambda)\overline{Q} & \overline{Q}M(\lambda)Q \\ QM(\lambda)\overline{Q} & QM(\lambda)Q \end{pmatrix} \equiv \begin{pmatrix} L_{00}(\lambda) & L_{01}(\lambda) \\ L_{10}(\lambda) & L_{11}(\lambda) \end{pmatrix}, \tag{3.9}$$

where the right side is the definition. We begin by studying $L_{00}^{-1}(\lambda)$. Since $L_{00}(0) \in \mathbf{B}(\overline{Q}\mathcal{H}_{-\gamma})$ is invertible by the separation of spectrum theorem for compact operators, $L_{00}(\lambda)$ is also invertible for small $|\lambda| < \lambda_0$. We omit the proof of the following lemma which is similar to that of Lemma 3.5.

LEMMA 3.8. *Let* $\frac{1}{2} < \gamma, \tau < \delta - \frac{1}{2}$ *and* $\gamma + \tau > 2$*. Let* $\rho_0 = \min(\gamma - 1/2, \delta - \gamma - 1/2)$ *and* $\rho_* = \min(\gamma - 1/2, \tau - 1/2, \tau + \gamma - 2)$*. Then:*

(1) *If* $\rho_0 \leq m - 2$, $L_{00}^{-1}(\lambda)$ *is a* $\mathbf{B}(\overline{Q}\mathcal{H}_{-\gamma})$ *valued function of* $\lambda \in (-\lambda_0, \lambda_0)$ *of class* $C^{(\rho_0)-}$*. If* $\rho_0 > m - 2$*, it is of class* $C^{(\rho_0)-}$ *for* $\lambda \neq 0$ *and of class* $C_*^{m-2}$ *on* $(-\lambda_0, \lambda_0)$.

(2) *For any* $\lambda \in \mathbf{R}$, $L_{00}^{-1}(\lambda) - \overline{Q}$ *may be extended to a bounded operator from* $\overline{Q}\mathcal{H}_{-\delta+\gamma}$ *to* $\overline{Q}\mathcal{H}_{-\tau}$*. If* $\rho_* \leq m - 2$*, it is of class* $C^{(\rho_*)-}$ *as a* $\mathbf{B}(\overline{Q}\mathcal{H}_{-\delta+\gamma}, \overline{Q}\mathcal{H}_{-\tau})$*-valued function. If* $\rho_* > m - 2$*, then it is of class* $C^{(\rho_*)-}$ *for* $\lambda \neq 0$ *and of class* $C_*^{m-2}$ *on* $(-\lambda_0, \lambda_0)$.

Removing $-\lambda^{m-2}\log\lambda\, A(\lambda)$ from $G_0(\lambda)$, we define

$$G_{0reg}(\lambda) = G_0(\lambda) + \lambda^{m-2}\log\lambda\, A(\lambda), \quad N(\lambda) = \overline{Q}(1 + G_{0reg}(\lambda)V)\overline{Q}.$$

If $\gamma$ and $\rho_0$ is as in Lemma 3.8, Proposition 2.6 implies that $N(\lambda)$ is a $\mathbf{B}(\overline{Q}\mathcal{H}_{-\gamma})$ valued function of $\lambda \in \mathbf{R}$ of class $C^{(\rho_0)-}$ and $N(\lambda)$ is invertible in $\overline{Q}\mathcal{H}_{-\gamma}$ for $\lambda \in (-\lambda_0, \lambda_0)$ if $\lambda_0 > 0$ is chosen small enough. We write

$$\tilde{L}(\lambda) = L_{00}^{-1}(\lambda) - \overline{Q}, \quad X(\lambda) = N^{-1}(\lambda), \quad \tilde{X}(\lambda) = X(\lambda) - \overline{Q}. \tag{3.10}$$

We omit the proof of the following lemma which is also similar to that of Lemma 3.5.

LEMMA 3.9. *Let $\gamma, \tau$ and $\rho_0, \rho_*$ be as in Lemma 3.8. Then:*

(1) *$X(\lambda)$ is a $\mathbf{B}(\overline{Q}\mathcal{H}_{-\gamma})$ functions of $\lambda \in (-\lambda_0, \lambda_0)$ of class $C^{(\rho_0)-}$.*

(2) *For $\lambda \in (-\lambda_0, \lambda_0)$, $\tilde{X}(\lambda)$ extends to a bounded operator from $\overline{Q}\mathcal{H}_{-\delta+\gamma}$ to $\overline{Q}\mathcal{H}_{-\tau}$ and it is $\mathbf{B}(\overline{Q}\mathcal{H}_{-\delta+\gamma}, \overline{Q}\mathcal{H}_{-\tau})$-valued function of class $C^{(\rho_*)-}$.*

We define $Y(\lambda) = L_{00}^{-1}(\lambda) - X(\lambda)$. In what follows we shall often use the arguments similar to the ones which will be used in (i) to (iv) of the proof of the following corollary. We use the following elementary lemma:

LEMMA 3.10. *Suppose $f(x)$ is of class $C^s_{0*}(\mathbf{R})$, $0 < s \le 1$, then $\log x\, f(x)$ is of class $C^{s-}_{0*}(\mathbf{R})$.*

COROLLARY 3.11. *Let $\gamma, \tau$ and $\rho_*$ be as in Lemma 3.8 and $j, k = 0, 1, \dots$. Then, $Y_{jk}(\lambda) \equiv \lambda^j(\log\lambda)^k Y(\lambda)$, $\lambda \ne 0$, may be extended to a bounded operator from $\overline{Q}\mathcal{H}_{-\delta+\gamma}$ to $\overline{Q}\mathcal{H}_{-\tau}$. Define $Y_{jk}(0) = 0$. If $\rho_* \le m-2$, then $Y_{jk}(\lambda)$ is of class $C^{(\rho_*)-}$ as a $\mathbf{B}(\overline{Q}\mathcal{H}_{-\delta+\gamma}, \overline{Q}\mathcal{H}_{-\tau})$-valued function of $\lambda \in (-\lambda_0, \lambda_0)$. If $\rho_* > m-2$, then it is of class $C^{(\rho_*)-}$ for $\lambda \ne 0$ and of class $C^{m-2}_*$ on $(-\lambda_0, \lambda_0)$.*

PROOF. We insert $L_{00}^{-1}(\lambda) = \overline{Q} + \tilde{L}(\lambda)$ and $X(\lambda) = \overline{Q} + \tilde{X}(\lambda)$ into

$$Y(\lambda) = L_{00}^{-1}(\lambda)(\lambda^{m-2}\log\lambda\, \overline{Q}A(\lambda)V\overline{Q})X(\lambda) \tag{3.11}$$

This produces four terms. It is easy to check that they satisfy the desired smoothness property outside $\lambda = 0$, and we examine them near $\lambda = 0$.

(i) $\lambda^{m-2}\log\lambda\,\overline{Q}A(\lambda)V\overline{Q}$ enjoys the desired property by virtue of Lemma 2.4.
(ii) To see the same for $\tilde{L}(\lambda)(\lambda^{m-2}\log\lambda\,\overline{Q}A(\lambda)V\overline{Q})$, we compute the $l$-th derivative via Leibniz' formula. If $l=a+b<\rho_*$, it is easy to check

$$\min(\delta-a-\tfrac{1}{2},\delta+\tau-2)>\max(b+\tfrac{1}{2},2-\gamma)$$

and if we take $\kappa$ between these two numbers, we have

$$b+\frac{1}{2}<\kappa<\delta-a-\frac{1}{2},\quad \tau+(\delta-\kappa)>2 \text{ and } \kappa+\gamma>2. \tag{3.12}$$

Then, as a $\mathbf{B}(\overline{Q}\mathcal{H})$-valued function,

$$\langle x\rangle^{-\tau}\tilde{L}^{(a)}(\lambda)\langle x\rangle^{\kappa}\cdot\langle x\rangle^{-\kappa}(\lambda^{m-2}\log\lambda\,\overline{Q}A(\lambda)V\overline{Q})^{(b)}\langle x\rangle^{\delta-\gamma}$$

is continuous with respect to $\lambda\in(-\lambda_0,\lambda_0)\setminus\{0\}$. If $b<m-2$, this is continuous also at $\lambda=0$, and is bounded by $\langle\log\lambda\rangle$ if $b=m-2$.
(iii) The argument which is entirely similar to the one used in (ii) proves that $\overline{Q}(\lambda^{m-2}\log\lambda\,A(\lambda))V\overline{Q}\tilde{X}(\lambda)$ satisfies the corollary.
(iv) For $l=a+b<\rho_*$ we choose $\kappa$ as in (3.12). Then in view of the result in (ii), as a $\mathbf{B}(\overline{Q}\mathcal{H})$-valued function,

$$\langle x\rangle^{-\tau}\{\tilde{L}(\lambda)\lambda^{m-2}\log\lambda\,\overline{Q}A(\lambda)V\overline{Q}\}^{(a)}\langle x\rangle^{\kappa}\cdot\langle x\rangle^{-\kappa}\tilde{X}^{(b)}\langle x\rangle^{\delta-\gamma}$$

is continuous with respect to $\lambda\in(-\lambda_0,\lambda_0)\setminus\{0\}$. If $a<m-2$ it is continuous also at $\lambda=0$, and is bounded by $\langle\log\lambda\rangle$ if $a=m-2$. Hence $\tilde{L}(\lambda)(\lambda^{m-2}\log\lambda\,\overline{Q}A(\lambda)V\overline{Q})\tilde{X}(\lambda)$ has the desired property. □

Since the logarithmic singularity appears in the form $\lambda^{m-2}(\log\lambda)\ A(\lambda)$ in $G_0(\lambda)$ as in Proposition 2.6 and $\lambda^{m-2}\log\lambda$ is less singular in higher dimensions, the proof of the proposition becomes easier as the spatial dimension $m$ increases. Thus, we study the case $m=6$ first and then discuss the case $m\geq 8$ only briefly. *In what follows we shall indiscriminately write $E_0(\lambda)$ for $\mathbf{B}(\mathcal{N})$ valued functions which satisfy the following property for some $N\geq 0$:*

$$\begin{gathered} E_0(\lambda) \text{ is of class } C^{\frac{m+2}{2}}((-\lambda_0,\lambda_0)\setminus\{0\}) \text{ and } C^{\frac{m-2}{2}}(-\lambda_0,\lambda_0) \text{ and} \\ \|E_0^{(\frac{m}{2})}(\lambda)\|\leq C\langle\log\lambda\rangle^N,\quad \|\lambda\,E_0^{(\frac{m+2}{2})}(\lambda)\|\leq C\langle\log\lambda\rangle^N. \end{gathered} \tag{3.13}$$

Functions of class $C_*^{\frac{m+2}{2}}$ on $(-\lambda_0,\lambda_0)$ clearly satisfy the condition (3.13). We often omit the variable $\lambda$ of operator valued functions. Note that $m-2\geq\frac{m+2}{2}$ when $m\geq 6$ with strict inequality when $m>6$.

*3.2.1 Proof of Proposition 3.6 for $m=6$*

In view of Lemma 3.7, we first study $C(\lambda)=L_{11}-L_{10}L_{00}^{-1}L_{01}$. We have

$$G_0(\lambda)=D_0+\lambda^2 D_2-\lambda^4(\log\lambda)A(\lambda)+\lambda^4 F(\lambda)$$

by virtue of Proposition 2.6. Since $(1+D_0V)Q=Q(1+VD_0)=0$,

$$\begin{aligned} L_{11}(\lambda)&=\lambda^2 Q(D_2-\lambda^2\log\lambda A(\lambda)+\lambda^2 F(\lambda))VQ.\\ L_{01}(\lambda)&=\lambda^2 \overline{Q}(D_2-\lambda^2\log\lambda A(\lambda)+\lambda^2 F(\lambda))VQ.\\ L_{10}(\lambda)&=\lambda^2 Q(D_2-\lambda^2\log\lambda A(\lambda)+\lambda^2 F(\lambda))V\overline{Q}.\end{aligned} \tag{3.14}$$

It is well known that $QD_2VQ$ is invertible in $\mathcal{N}$ and $(QD_2VQ)^{-1}=P_0V$, $P_0VQ=P_0V$ and $VQP_0=VP_0$ (cf. §4.4 of [25]. Note, however, the sign difference here and in [25]). Then, $C(\lambda)$ may be written in the form

$$C(\lambda)=\lambda^2(QD_2VQ)(1-P_0VE_2^*(\lambda)), \tag{3.15}$$
$$E_2^*(\lambda)=\lambda^2F_{00}+\lambda^2\log\lambda F_{01}+\lambda^4F_{20}+\lambda^4\log\lambda F_{21}+\lambda^2E_0(\lambda),$$

where $F_{00}(\lambda), F_{01}(\lambda), F_{20}(\lambda)$ and $F_{21}(\lambda)$ are defined by

$$\begin{aligned} F_{00}(\lambda)&=-Q(F(\lambda)-D_2V\overline{Q}L_{00}^{-1}\overline{Q}D_2)VQ,\quad F_{01}=QA(\lambda)VQ,\\ F_{20}(\lambda)&=Q(F(\lambda)V\overline{Q}L_{00}^{-1}\overline{Q}D_2+D_2V\overline{Q}L_{00}^{-1}\overline{Q}F(\lambda))VQ,\\ F_{21}(\lambda)&=-Q(A(\lambda)V\overline{Q}L_{00}^{-1}\overline{Q}D_2+D_2V\overline{Q}L_{00}^{-1}\overline{Q}A(\lambda))VQ,\end{aligned} \tag{3.16}$$

and $E_0(\lambda)=\lambda^4Q(\log\lambda A(\lambda)-F(\lambda))V\overline{Q}L_{00}^{-1}(\lambda)\overline{Q}(\log\lambda A(\lambda)-F(\lambda))VQ$ is of class $C_*^4$ thanks to (3.4) (recall $\delta>10$). We write $\tilde{F}_{jk}(\lambda)$ for the operator obtained from $F_{jk}(\lambda)$ of (3.16) by replacing $L_{00}^{-1}(\lambda)$ by $X(\lambda)=N^{-1}(\lambda)$.

LEMMA 3.12. *As $\mathbf{B}(\mathcal{N})$-valued functions of $\lambda\in(-\lambda_0,\lambda_0)$,*

(1) *$F_{00}(\lambda)$, $F_{20}(\lambda)$ and $F_{21}(\lambda)$ are of class $C_*^4$;*

(2) *$\tilde{F}_{00}(\lambda)$, $F_{01}(\lambda)$, $\tilde{F}_{20}(\lambda)$ and $\tilde{F}_{21}(\lambda)$ are of class $C^{(\delta-4)-}$;*

(3) *$QA(\lambda)V\overline{Q}\tilde{L}(\lambda)\overline{Q}F(\lambda)VQ$ is of class $C_*^4$;*

(4) *$QA(\lambda)V\overline{Q}\tilde{X}(\lambda)\overline{Q}F(\lambda)VQ$ is of class $C^{(\delta-4)-}$.*

*The same holds for the operators which are obtained by replacing one or both of* $A(\lambda)$ *and* $F(\lambda)$ *by the other.*

PROOF. We prove statements (2). The proof for others is similar.
(i) Proposition 2.6 and properties (3.4) of $\phi \in \mathcal{N}$ imply that $QF(\lambda)VQ$ is of class $C^{(\delta-2)-}$; and the operators $QF(\lambda)V\overline{Q}D_2VQ$ and $QD_2V\overline{Q}F(\lambda)VQ$ are of class $C^{(\delta-4)-}$. The same holds when $F(\lambda)$ is replaced by $A(\lambda)$.
(ii) By virtue of Lemma 3.9 and (3.4), $QD_2VX(\lambda)D_2VQ$ is of class $C^{(\delta-\frac{3}{2})-}$.
(iii) $QF(\lambda)V\tilde{X}(\lambda)D_2VQ$ is of class $C^{(\delta-\frac{7}{2})-}$. To see this we differentiate it by $\lambda$ by using Leibniz' formula. Then, by virtue of Lemma 3.9 and (3.4), for some $\varepsilon > 0$, $\langle x\rangle^{-\delta-1+\varepsilon}F^{(k_1)}(\lambda)V\tilde{X}^{(k_2)}(\lambda)\langle x\rangle^{1+\varepsilon}$ is $\mathbf{B}(\mathcal{H})$-valued continuous as long as

$$k_1 + 3 + k_2 + \tfrac{1}{2} < \delta,\ k_1 + 3 < \delta + 1,\ k_2 + \tfrac{1}{2} < \delta - 1$$

and the latter inequalities trivially hold if $k_1 + k_2 < \delta - \frac{7}{2}$. Similar argument implies that the same holds for $QD_2V\overline{Q}\tilde{X}(\lambda)\overline{Q}F(\lambda)VQ$ and for the operators obtained by replacing $F(\lambda)$ by $A(\lambda)$.

Combining (i), (ii) and (iii) we obtain statement (2). □

LEMMA 3.13. *There exist* $\lambda_0 > 0$ *such that for* $\lambda \in (-\lambda_0, \lambda_0) \setminus \{0\}$, $C(\lambda)$ *is invertible in* $\mathcal{N}$ *and* $C(\lambda)^{-1}$ *may be written in the form*

$$\lambda^{-2}P_0V + P_0V\Big(\log\lambda D_{10} + \sum_{1\le k\le j\le 2}\lambda^j(\log\lambda)^k D_{jk} + E_0(\lambda)\Big)P_0V \tag{3.17}$$

*with* $D_{jk} \in \mathbf{B}(\mathcal{N})$ *and a* $\mathbf{B}(\mathcal{N})$*-valued* $C^4_*$ *function* $E_0(\lambda)$.

PROOF. Since $(QD_2VQ)^{-1} = P_0V$ and $\|P_0VE_2^*(\lambda)\| \to 0$ as $\lambda \to 0$ by virtue of Lemma 3.12, there exists $\lambda_0 > 0$ such that, for $0 < |\lambda| < \lambda_0$, $C(\lambda)$ is invertible and

$$C^{-1}(\lambda) = \lambda^{-2}\sum_{n=0}^{\infty}(P_0VE_2^*(\lambda))^nP_0V.$$

Lemma 3.12 implies that the sum $\lambda^{-2}\sum_{n=3}^{\infty}(P_0VE_2^*(\lambda))^nP_0V$ and the terms in $\lambda^{-2}\sum_{n=1}^{2}(P_0VE_2^*(\lambda))^nP_0V$ which do not contain any $\log\lambda$ factors or contain factors $\lambda^j$ with $j \geq 4$ may be written in the form $P_0 V E_0(\lambda) P_0 V$ with a $C^4_*$ function $E_0(\lambda)$. Hence, we have

$$\begin{aligned}C^{-1}(\lambda) = \lambda^{-2} P_0 V &+ \log\lambda P_0 V F_{01} P_0 V \\ &+\lambda^2 \log\lambda\, P_0 V (F_{21} + F_{00} P_0 V F_{01} + F_{01} P_0 V F_{00}) P_0 V \\ &+\lambda^2 \log^2\lambda (P_0 V F_{01})^2 P_0 V + P_0 V E_0(\lambda) P_0 V.\end{aligned} \tag{3.18}$$

The equation (3.18) remains valid if $F_{00}, F_{20}$ and $F_{21}$ are replaced by $\tilde F_{00}, \tilde F_{20}$ and $\tilde F_{21}$ respectively because the difference is of class $C^4_*$ by virtue of Corollary 3.11. We then expand various operators in the resulting equation in powers of $\lambda$ by using Taylor's formula. More specifically:
(a) We expand $P_0 V F_{01}(\lambda) P_0 V = P_0 V A(\lambda) V P_0 V$ upto the order $\lambda^2$ with the remainder $\lambda^3 R_1(\lambda)$. Then, by virtue of the property (3.4) of eigenfunctions and of Lemma 2.2, $R_1(\lambda)$ is a $\mathbf{B}(\mathcal{N})$-valued function of class $C^\rho$ for any $\rho < \delta - 5$. Since $\delta - 5 > 4$, it follows that $\lambda^3 \log\lambda\, R_1(\lambda)$ satisfies the property (3.13) and it may be written in the form $P_0 V E_0(\lambda) P_0 V$;
(b) If we replace $P_0 V(\tilde F_{21}(\lambda) + \tilde F_{00}(\lambda) P_0 V F_{01}(\lambda) + F_{01}(\lambda) P_0 V \tilde F_{00}(\lambda)) P_0 V$ by the constant operator obtained by setting $\lambda = 0$, the difference is $\lambda$ times a $\mathbf{B}(\mathcal{N})$-valued function $R_2(\lambda)$ of class $C^\rho$ for any $\rho < \delta - 5$ by virtue of Lemma 3.12, and $\lambda^3 \log\lambda\, R_2(\lambda)$ is of the form $P_0 V E_0(\lambda) P_0 V$.
(c) If we replace $A(\lambda)$ by $A(0)$ in $(P_0 V F_{01}(\lambda))^2 P_0 V = (P_0 V A(\lambda) V P_0)^2 V$, then the difference is $\lambda$ times a $\mathbf{B}(\mathcal{N})$-valued function $R_3(\lambda)$ of class $C^4$ and $\lambda^3 \log^2\lambda\, R_3(\lambda)$ is of the form $P_0 V E_0(\lambda) P_0 V$.

Equation (3.18) and (a), (b) and (c) imply the lemma. □

In what follows, we take and fix the constant $\lambda_0 > 0$ as in Lemma 3.13. We denote the second member of (3.17) by $C_r(\lambda)$: $C_r(\lambda) = C^{-1} - \lambda^{-2} P_0 V$.

Lemma 3.14. *With an operator $D^{(1)}_{21} \in \mathbf{B}(\mathcal{N}, \mathcal{H}_{(-1)_-})$ and an operator valued function $R_{01}(\lambda)$ on $(-\lambda_0, \lambda_0)$ which satisfies the properties below, we have*

$$L_{00}^{-1}(\lambda) L_{01}(\lambda) C_r(\lambda) = \lambda^2 \log\lambda\, D^{(1)}_{21} P_0 V + R_{01}(\lambda) P_0 V. \tag{3.19}$$

*As $\mathbf{B}(\mathcal{N}, \mathcal{H})$ valued functions, $\langle x\rangle^{-(\sigma+1)_+} R_{01}(\lambda)$ is of class $C^\sigma$ for $0 \leq \sigma \leq 2$; $\langle x\rangle^{-(\sigma+\frac12)_+} R_{01}(\lambda)$ is of class $C^\sigma$ for $\lambda \neq 0$ for $\sigma = 3, 4$ and, for some*

$N > 0$,

$$\|\langle x\rangle^{-(\frac{7}{2})+} R_{01}^{(3)}(\lambda)\|_{\mathbf{B}(\mathcal{N},\mathcal{H})} + \|\langle x\rangle^{-(\frac{9}{2})+} \lambda R_{01}^{(4)}(\lambda)\|_{\mathbf{B}(\mathcal{N},\mathcal{H})} \le C\langle \log\lambda\rangle^N. \tag{3.20}$$

Proof. For shortening formulae we write $E_0(\lambda)$ for $P_0VE_0(\lambda)P_0V$ and $D_{jk}$ for $P_0VD_{jk}P_0V$ of (3.17) in the proof. Since $L_{01} = \overline{Q}(1+G_0V)Q$ is a $\mathbf{B}(\mathcal{N},\mathcal{H}_{-(\gamma+\frac{1}{2})+})$-valued function of class $C^\gamma$ for $\gamma < 4$ and of class $C^4_*$ if $\gamma \ge 4$, Lemma 3.8 implies that

$$L_{00}^{-1}L_{01}(\lambda)E_0(\lambda) = \overline{Q}L_{01}(\lambda)E_0(\lambda) + \tilde{L}(\lambda)L_{01}(\lambda)E_0(\lambda)$$

may be put into $R_{01}(\lambda)P_0V$. We need only consider $\sum \lambda^j(\log\lambda)^k L_{00}^{-1}L_{01}D_{jk}$. We insert (3.14) for $L_{01}(\lambda)$. Then, by virtue of Corollary 2.3, Lemma 2.4 and Proposition 2.6, $\lambda^{4+j}(\log\lambda)^k\overline{Q}(\log\lambda A(\lambda) + F(\lambda))VQ$ is of class $C^\gamma$ if $\gamma < 4$ and of class $C^4_*$ if $\gamma \ge 4$ as a $\mathbf{B}(\mathcal{N},\mathcal{H}_{-(\gamma+\frac{1}{2})+})$-valued function. Thus, we see by writing again as $L_{00}^{-1} = \overline{Q} + \tilde{L}(\lambda)$ and by applying Lemma 3.8 that

$$L_{00}^{-1}(\lambda)(\lambda^{m-2+j}(\log\lambda)^k\overline{Q}(\log\lambda A(\lambda) + F(\lambda))VQ)D_{jk}$$

may also be put into $R_{01}(\lambda)P_0V$. Writing $L_{00}^{-1} = \overline{Q} + Y + \tilde{X}$, we are left with

$$\begin{aligned}\lambda^{2+j}(\log\lambda)^k L_{00}^{-1}(\lambda)\overline{Q}D_2VQD_{jk} &= \lambda^{2+j}(\log\lambda)^k\overline{Q}D_2VQD_{jk}\\ &+\lambda^{2+j}(\log\lambda)^k Y(\lambda)\overline{Q}D_2VQD_{jk} + \lambda^{2+j}(\log\lambda)^k\tilde{X}(\lambda)\overline{Q}D_2VQD_{jk}.\end{aligned} \tag{3.21}$$

Recalling that $\overline{Q}D_2V \in \mathbf{B}(\mathcal{N},\mathcal{H}_{-1-\varepsilon})$ for any $\varepsilon > 0$, we put the first term on the right with $j = 0$ into $\lambda^2\log\lambda\, D_{21}^{(1)}$ and with $j \ge 1$ into $R_{01}(\lambda)P_0V$. Note that if $j = 0$, we have only the term with $k = 1$, see (3.17). Corollary 3.11 with $\gamma = \delta - 1 - \varepsilon$ and $\tau = \sigma + 1 + \varepsilon$ implies that the second term may be put into $R_{01}(\lambda)P_0V$. For dealing with the last term, we write $\tilde{X}(\lambda) = \tilde{X}(0) + (\tilde{X}_1(\lambda) - \tilde{X}(0))$ and, noticing that $\tilde{X}(0)\overline{Q}D_2VQ \in \mathbf{B}(\mathcal{N},\mathcal{H}_{-\frac{1}{2}-\varepsilon})$ for any $\varepsilon > 0$, we put $\lambda^{2+j}(\log\lambda)^k\tilde{X}(0)\overline{Q}D_2VQD_{jk}$ into $\lambda^2\log\lambda\, D_{21}^{(1)}$ if $j = 0$ and into $R_{01}(\lambda)P_0V$ otherwise. Finally,

$$\lambda^{2+j}(\log\lambda)^k(\tilde{X}(\lambda) - \tilde{X}(0))\overline{Q}D_2VQD_{jk}$$

may also be put into $R_{01}(\lambda)P_0V$. This can be seen by differentiating it via Leibniz' rule and by applying Lemma 3.9. This completes the proof. □

Recall $\sigma_0(3,\sigma) = \frac{\sigma}{2} + 2$ for $\sigma \le 3$. In the following two lemmas, we set

$$\gamma(\sigma) = \begin{cases} \max(\sigma_0(3,\sigma), 3) = 3, & \text{if } \sigma < 2, \\ \sigma + 1, & \text{if } 2 \le \sigma \le 4. \end{cases} \tag{3.22}$$

We parenthetically remark that we imposed the stronger decay condition $|V(x)| \le C\langle x\rangle^{\delta}$ with $\delta > m+4$ for $m = 6$ instead of $\delta > m+3$ for proving the following two lemmas, viz. for ensuring that $VR_{02}(\lambda)P_0V$ and $VP_0VR_{30}(\lambda)$ below satisfy the property $(K)_\rho$ with $\rho > m+1$.

LEMMA 3.15. *With an operator* $D_{2,1}^{(2)} \in \mathbf{B}(\mathcal{N}, \mathcal{H}_{(-3)_-})$ *and an operator valued function* $R_{02}(\lambda)$ *on* $(-\lambda_0, \lambda_0)$ *which satisfies the properties below, we have*

$$L_{00}^{-1}(\lambda)L_{01}(\lambda)C^{-1}(\lambda) = \lambda^2 \log\lambda D_{2,1}^{(2)} P_0 V + R_{02}(\lambda)P_0V. \tag{3.23}$$

*As a* $\mathbf{B}(\mathcal{N}, \mathcal{H})$ *valued functions,* $\langle x\rangle^{-\gamma(\sigma)+}R_{02}(\lambda)$ *is of class* $C^\sigma$ *on* $(-\lambda_0, \lambda_0)$ *for* $0 \le \sigma \le 2$*, for* $\lambda \ne 0$ *for* $\sigma = 3, 4$ *and*

$$\|\langle x\rangle^{-4+}R_{02}^{(3)}(\lambda)\|_{\mathbf{B}(\mathcal{N},\mathcal{H})} + \|\langle x\rangle^{-5+}\lambda R_{02}^{(4)}(\lambda)\|_{\mathbf{B}(\mathcal{N},\mathcal{H})} \le C\langle \log\lambda\rangle. \tag{3.24}$$

PROOF. In view of Lemma 3.14 it suffices to prove the lemma with $\lambda^{-2}P_0V$ in place of $C^{-1}(\lambda)$. We multiply the following by $L_{00}^{-1}(\lambda)$ from the left:

$$L_{01}(\lambda)\lambda^{-2}P_0V = \overline{Q}D_2VP_0V - \lambda^2\log\lambda\,\overline{Q}A(\lambda)VP_0V + \lambda^2\overline{Q}F(\lambda)VP_0V.$$

(i) We may put $L_{00}^{-1}(\lambda)\overline{Q}D_2VP_0V = (\overline{Q} + \tilde{L}(\lambda))\overline{Q}D_2VP_0V$ into $R_{02}(\lambda)P_0V$ by virtue of Lemma 3.8.
(ii) In $L_{00}^{-1}(\lambda)\overline{Q}\lambda^2\log\lambda\, A(\lambda)VP_0V$ we substitute $Y(\lambda) + \tilde{X}(\lambda) + \overline{Q}$ for $L_{00}^{-1}$. We may put $Y(\lambda)\overline{Q}\lambda^2\log\lambda\, A(\lambda)VP_0V = \lambda^2\log\lambda\, Y(\lambda)\cdot\overline{Q}A(\lambda)VP_0V$ into $R_{02}(\lambda)P_0V$ by virtue of Corollary 3.11. We write in the form

$$\begin{aligned} &\lambda^2\log\lambda\tilde{X}(\lambda)\overline{Q}A(\lambda)V = \lambda^2\log\lambda\tilde{X}(0)\overline{Q}A(0)V \\ &+\lambda^2\log\lambda(\tilde{X}(\lambda) - \tilde{X}(0))\overline{Q}A(0)V + \lambda^2\log\lambda\tilde{X}(\lambda)(A(\lambda) - A(0))V. \end{aligned} \tag{3.25}$$

Then, $\tilde{X}(0)\overline{Q}A(0)V \in \mathbf{B}(\mathcal{N}, \mathcal{H}_{(-\frac{1}{2})_-})$ and we put the first term on the right into $\lambda^2 \log \lambda\, D_{21}^{(2)} P_0 V$; it is easy to check by using Lemma 3.9 (2) that the last two terms satisfy the properties of $R_{02}(\lambda)$. We write $\overline{Q}\lambda^2 \log \lambda\, A(\lambda)V$ as

$$\overline{Q}\lambda^2 \log \lambda\, A(0)V + \overline{Q}\lambda^2 \log \lambda\, (A(\lambda) - A(0))V.$$

Since $\overline{Q}A(0)V \in \mathbf{B}(\mathcal{N}, \mathcal{H}_{-3_+})$, we put the first term into $\lambda^2 \log \lambda D_{21}^{(2)}$. It can be checked that the second term satisfies the properties for $R_{02}$ by differentiating it by $\lambda$ and by applying Lemma 2.2 and Corollary 2.3.
(iii) Since $\lambda^2 F(\lambda)V$ is also is of class $C^\sigma$ as a $\mathbf{B}(\mathcal{N}, \mathcal{H}_{-\gamma(\sigma)_+})$-valued function by virtue of Proposition 2.6, we may put $L_{00}^{-1}(\lambda)\overline{Q}\lambda^2 F(\lambda)V$ into $R_{02}(\lambda)$. This completes the proof. $\square$

We omit the proof of the following lemma which goes entirely in parallel with that of the previous Lemma 3.15.

LEMMA 3.16. *There exist an operator* $D_{12}^{(3)} \in \mathbf{B}(\mathcal{H}_{(-\delta+3)_+}, \mathcal{N})$ *and an operator valued function* $R_{03}(\lambda)$ *which satisfies the property below such that*

$$C^{-1}L_{10}(\lambda)L_{00}(\lambda) = \lambda^2 \log \lambda\, P_0 V D_k^{(3)} + P_0 V R_{03}(\lambda). \tag{3.26}$$

*Here, as a* $\mathbf{B}(\mathcal{H}, \mathcal{N})$ *valued function,* $R_{03}(\lambda)\langle x\rangle^{(\delta-\gamma(\sigma))-}$ *is of class* $C^\sigma$ *on* $(-\lambda_0, \lambda_0)$ *if* $0 \le \sigma \le 2$, *for* $\lambda \neq 0$ *if* $\sigma = 3, 4$, *and with some* $N > 0$

$$\|R_{03}^{(3)}(\lambda)\langle x\rangle^{(\delta-\gamma(3))-}\|_{\mathbf{B}(\mathcal{H},\mathcal{N})} + \|R_{03}^{(4)}(\lambda)\langle x\rangle^{(\delta-\gamma(4))-}\lambda\|_{\mathbf{B}(\mathcal{H},\mathcal{N})} \le C\langle \log \lambda\rangle^N.$$

Since $VP_0VE_0(\lambda)P_0V$, $VR_{01}(\lambda)P_0V$, $VR_{02}(\lambda)P_0V$ and $VP_0VR_{03}(\lambda)$ satisfy property $(K)_\rho$ with $\rho > m+1$, the following lemma completes the proof of Proposition 3.6 for $m = 6$.

LEMMA 3.17. *The operator valued function* $VL_{00}^{-1}L_{01}C^{-1}L_{10}L_{00}^{-1}$ *satisfies the condition* $(K)_\rho$ *with* $\rho > m+1$.

PROOF. We substitute $\lambda^{-2}P_0V + C_r(\lambda)$ for $C(\lambda)$ and write $L_{01}\lambda^{-2}P_0VL_{10}$ in the form

$$\overline{Q}(\lambda D_2 - \lambda^3 \log \lambda A(\lambda) + \lambda^3 F(\lambda))VP_0V(\lambda D_2 - \lambda^3 \log \lambda A(\lambda) + \lambda^3 F(\lambda))\overline{Q}.$$

It follows from Corollary 2.3 that this is a $\mathbf{B}(\mathcal{H}_{-\delta+\sigma_0(2,j)_+}, \mathcal{H}_{-\sigma_0(2,j)_+})$-valued function of class $C^j(-\lambda_0, \lambda_0)$ for $0 \le j \le 2$, of class $C^{(j)}((-\lambda_0, \lambda_0) \setminus \{0\})$ for $j = 3, 4$, and the third and the fourth derivatives are bounded by $C\langle \log \lambda \rangle$ and $C|\lambda|^{-1}\langle \log \lambda \rangle$ in respective norms. It follows by writing $L_{00}^{-1}(\lambda) = \overline{Q} + (L_{00}^{-1} - \overline{Q})$ that $VL_{00}^{-1}L_{01}(\lambda^{-2}P_0V)L_{10}L_{00}^{-1}$ satisfies the condition $(K)_\rho$ with $\rho > m+1$. It is then obvious that so does $VL_{00}^{-1}L_{01}C_r(\lambda)L_{10}L_{00}^{-1}$. The lemma follows. $\square$

*3.2.2 The case $m \ge 8$ is even*

Let now $m \ge 8$. Define $F_0(\lambda), F_1(\lambda)$ and $F_2(\lambda)$ by

$$
\begin{aligned}
F_0(\lambda) &= D_0 + \lambda^2 D_2 + \cdots + \lambda^{m-4} D_{m-4} + \lambda^{m-2} F(\lambda) \\
F_2(\lambda) &= D_2 + \cdots + \lambda^{m-6} D_{m-4} + \lambda^{m-4} F(\lambda), \\
F_4(\lambda) &= D_4 + \cdots + \lambda^{m-8} D_{m-4} + \lambda^{m-6} F(\lambda)
\end{aligned}
$$

so that $G_0(\lambda) = F_0(\lambda) - \lambda^{m-2} \log \lambda\, A(\lambda)$, $F_0 = D_0 + \lambda^2 F_2(\lambda)$ and $F_2(\lambda) = D_2 + \lambda^2 F_4(\lambda)$. Since $(1 + D_0V)Q = 0$, we then have

$$
L_{11}(\lambda) = \lambda^2 Q(D_2 + \lambda^2 F_4(\lambda) - \lambda^{m-4} \log \lambda A(\lambda))VQ; \tag{3.27}
$$

and $L_{10}(\lambda)$ and $L_{01}(\lambda)$ are obtained from (3.27) by replacing one of $Q$ by $\overline{Q}$ as in (3.14). Recall that $(QD_2VQ)^{-1} = P_0V$, $VQP_0 = VP_0$ and $P_0VQ = P_0V$.

LEMMA 3.18. *Let $\varphi = P_0V$ where $V$ is considered as a function. Then there exists $\lambda_0$ such that*

$$
C^{-1}(\lambda) = \lambda^{-2} P_0 V + c_m \lambda^{m-6} \log \lambda\, \varphi \otimes (V\varphi) + P_0 V E_0(\lambda) P_0 V, \tag{3.28}
$$

*where $c_m = (2\pi)^{-\frac{m}{2}}(m!!)^{-1}$ and $E_0(\lambda)$ is a $\mathbf{B}(\mathcal{N})$-valued function of $\lambda \in (-\lambda_0, \lambda_0)$ which satisfies the property* (3.13).

PROOF. In this proof the smoothness of operator valued functions will be referred to as $\mathbf{B}(\mathcal{N})$ valued functions. By virtue of Proposition 2.6, $E_{01}(\lambda) \equiv QF_2(\lambda)VQ$ is of class $C^{\frac{m+2}{2}}$. Likewise Lemma 3.8, Proposition 2.6, property (3.4) of $\phi \in \mathcal{N}$ and that $2(m-4) > \frac{m+2}{2}$ imply that

$$
\begin{aligned}
E_{02}(\lambda) &\equiv \lambda^{2(m-4)} (\log \lambda)^2 QA(\lambda) V\overline{Q} L_{00}^{-1}(\lambda) \overline{Q} A(\lambda) VQ, \\
E_{03}(\lambda) &\equiv QF_1 V\overline{Q} L_{00}^{-1}(\lambda) \overline{Q} F_1 VQ, \\
F_{20}(\lambda) &= Q(F_1 V\overline{Q} L_{00}^{-1} \overline{Q} A + AV\overline{Q} L_{00}^{-1} \overline{Q} F_1) VQ
\end{aligned}
$$

are all of class $C^{\frac{m+2}{2}}$. With these definitions, we may write

$$L_{10}(\lambda)L_{00}(\lambda)^{-1}L_{01}(\lambda) = \lambda^4(E_{02} + E_{03} - \lambda^{m-4}\log\lambda F_{20}(\lambda)), \tag{3.29}$$

Thus, defining

$$\tilde{E}_0(\lambda) = P_0V(E_{01}(\lambda) - E_{02}(\lambda) - E_{03}(\lambda)),$$
$$\tilde{F}_{10}(\lambda) = P_0VA(\lambda)VQ, \quad \tilde{F}_{20}(\lambda) = P_0VF_{20}(\lambda),$$

subtracting (3.29) from (3.27) and factoring out $\lambda^2 QD_2VQ$, we obtain

$$C(\lambda) = \lambda^2 QD_2VQ(1 - \lambda^{m-4}\log\lambda\,\tilde{F}_{10}(\lambda) + \lambda^{m-2}\log\lambda\,\tilde{F}_{20}(\lambda) + \lambda^2\tilde{E}_0(\lambda)).$$

It follows that $C(\lambda)$ is invertible in $\mathcal{N}$ for $0 < |\lambda| < \lambda_0$ for small enough $\lambda_0$ and

$$C^{-1}(\lambda) = \lambda^{-2}\sum_{n=0}^{\infty}(\lambda^{m-4}\log\lambda\,\tilde{F}_{10}(\lambda) - \lambda^{m-2}\log\lambda\tilde{F}_{20}(\lambda) - \lambda^2\tilde{E}_0(\lambda))^n P_0V.$$

It is easy to see by counting the powers of $\lambda$ in front of powers of $\log\lambda$ that the series over $2 \le n < \infty$ produces a function of class $C^{\frac{m+2}{2}}$. Thus, writing $E_0(\lambda)$ for $C^{\frac{m+2}{2}}$ functions indiscriminately, we have

$$C^{-1}(\lambda) = \lambda^{-2}P_0V + \lambda^{m-6}\log\lambda\,\tilde{F}_{10}(\lambda)P_0V$$
$$- \lambda^{m-4}\log\lambda\,\tilde{F}_{20}(\lambda)P_0V + E_0(\lambda).$$

Since $F_{20}(\lambda)$ is of class $C^{\frac{m+2}{2}}$ as mentioned above and $m-4 \ge \frac{m}{2}$ if $m \ge 8$, $\lambda^{m-4}\log\lambda\,\tilde{F}_{20}(\lambda)P_0V$ satisfies the property (3.13). If we expand $A(\lambda) = A(0) + \lambda A'(0) + A_2(\lambda)$ with $A_2(\lambda) = A(\lambda) - A(0) - \lambda A'(0)$ in

$$\lambda^{m-6}\log\lambda\,\tilde{F}_{10}(\lambda)P_0V = \lambda^{m-6}\log\lambda\,P_0VA(\lambda)VP_0V,$$

then, $\lambda^{m-6}\log\lambda\,P_0VA_2(\lambda)VP_0V$ satisfies the property (3.13). Since $A(0) = c_m 1 \otimes 1$ and $A'(0) = 0$, the lemma follows. □

Lemma 3.18 and the following lemma complete the proof of Proposition 3.6 for $m \ge 8$. We use the following short hand notation.

$$R_l(\lambda) = C^{-1}(\lambda)L_{10}(\lambda)L_{00}^{-1}(\lambda), \quad R_r(\lambda) = L_{00}^{-1}(\lambda)L_{01}(\lambda)C^{-1}(\lambda)$$
$$R_c(\lambda) = L_{00}^{-1}(\lambda)L_{01}(\lambda)C^{-1}(\lambda)L_{10}(\lambda)L_{00}^{-1}(\lambda)$$

LEMMA 3.19. *For sufficiently small $\lambda_0 > 0$ the following properties are satisfied:*
(1) *For $\sigma \leq \frac{m-2}{2}$, $R_l(\lambda)$ is a $\mathcal{X}_\sigma \equiv \mathbf{B}(\mathcal{H}_{(\sigma+2-\delta)_+}, \mathcal{N})$-valued function of class $C^\sigma$ on $(-\lambda_0, \lambda_0)$; it is of class $C^\sigma$ for $\lambda \neq 0$ for $\frac{m}{2} \leq \sigma \leq \frac{m+2}{2}$ and*

$$\|R_l^{(\frac{m}{2})}(\lambda)\|_{\mathcal{X}_{\frac{m}{2}}} + |\lambda| \|R_l^{(\frac{m+2}{2})}(\lambda)\|_{\mathcal{X}_{\frac{m+2}{2}}} \leq C\langle \log \lambda \rangle.$$

(2) *For $\sigma \leq \frac{m-2}{2}$, $R_r(\lambda)$ is a $\mathcal{Y}_\sigma \equiv \mathbf{B}(\mathcal{N}, \mathcal{H}_{-(\sigma+2)_+})$-valued function of class $C^\sigma$; it is of class $C^\sigma$ for $\lambda \neq 0$ for $\frac{m}{2} \leq \sigma \leq \frac{m+2}{2}$ and*

$$\|R_r^{(\frac{m}{2})}(\lambda)\|_{\mathcal{Y}_{\frac{m}{2}}} + |\lambda| \|R_r^{(\frac{m+2}{2})}(\lambda)\|_{\mathcal{Y}_{\frac{m+2}{2}}} \leq C\langle \log \lambda \rangle.$$

(3) *For $\sigma \leq \frac{m-2}{2}$, $R_c(\lambda)$ is a $\mathcal{Z}_\sigma \equiv \mathbf{B}(\mathcal{H}_{(\sigma+2-\delta)_+}, \mathcal{H}_{-(\sigma+2)_+})$-valued function of class $C^\sigma$. Moreover it is of class $C^\sigma$ for $\lambda \neq 0$ for $\frac{m}{2} \leq \sigma \leq \frac{m+2}{2}$ and*

$$\|R_c^{(\frac{m}{2})}(\lambda)\|_{\mathcal{Z}_{\frac{m}{2}}} + |\lambda| \|R_c^{(\frac{m+2}{2})}(\lambda)\|_{\mathcal{Z}_{\frac{m+2}{2}}} \leq C\langle \log \lambda \rangle.$$

PROOF. We have $\delta - \frac{m}{2} > \frac{m+2}{2}$ and

$$P_0 V L_{01}(\lambda) V \overline{Q} L_{00}^{-1}(\lambda) = \lambda^2 P_0 V J_2(\lambda) V \overline{Q}(\overline{Q} + \tilde{L}(\lambda)).$$

Proposition 2.6 and Corollary 2.3 imply that $T(\lambda) = P_0 V J_2(\lambda) V \overline{Q}$ satisfies the property of $R_l(\lambda)$ of the lemma (recall that $J_2(\lambda)$ contains $\lambda^{m-4} \log \lambda\, A(\lambda)$ and $m - 4 \geq \frac{m}{2}$). Since

$$\lambda^2 C^{-1}(\lambda) = P_0 V + c_m \lambda^{m-4} \log \lambda\, \varphi \otimes V\varphi + \lambda^2 P_0 V E_0(\lambda) P_0 V$$

satisfies property (3.13), statement (1) follows. We likewise see that

$$\tilde{T}(\lambda) = (\overline{Q} + \tilde{L}(\lambda)) \overline{Q} J_2(\lambda) V Q$$

satisfies the property of $R_r(\lambda)$ of the lemma. Then statement (2) follows since $\lambda^2 C^{-1}(\lambda)$ satisfies the property (3.13). Statement (3) is obvious since $C_c(\lambda) = \tilde{T}(\lambda) C_l(\lambda)$, $C_l(\lambda)$ satisfies (1) and $\tilde{T}(\lambda)$ satisfies the property of $R_r(\lambda)$. □

## 4. Low Energy Estimate I, Generic Case

In the following two sections, we study the low energy part $W_<$ of the wave operator $W_-$. We take and fix $\lambda_0 > 0$ arbitrarily if $H$ is generic type, otherwise small enough so that Proposition 3.6 is satisfied. We take cut-off functions $\Phi$ and $\Psi$ as in the introduction and define $W_<$ as in (1.8):

$$W_< = \Phi(H)\Phi(H_0) - \int_0^\infty \Phi(H)G(\lambda)V(G_0(\lambda) - G_0(-\lambda))\Phi(H_0)\lambda \frac{d\lambda}{\pi i}.$$

In this section we study $W_<$ in the case that $H$ is of generic type and prove the following proposition. We assume that $V$ satisfies the condition

$$\mathcal{F}(\langle x\rangle^{2\sigma} V) \in L^{m_*}(\mathbf{R}^m) \text{ and } |V(x)| \le C\langle x\rangle^{-\delta} \text{ for some } \delta > m+2. \tag{4.1}$$

Proposition 4.1. *Let $m \ge 6$ be even and let $V$ satisfy* (4.1). *Suppose that $H$ is of generic type. Then $W_<$ is bounded in $L^p(\mathbf{R}^m)$ for all $1 \le p \le \infty$.*

The integral kernels $\Phi_0(x,y)$ and $\Phi(x,y)$ of $\Phi(H_0)$ and $\Phi(H)$ respectively are continuous and bounded by $C_N\langle x-y\rangle^{-N}$ for any $N$ ([23]) and a fortiori $\Phi(H)$ and $\Phi(H_0)$ are bounded in $L^p$ for all $1 \le p \le \infty$. Hence, we have only to discuss the operator defined by the integral

$$\int_0^\infty \Phi(H)G(\lambda)V(G_0(\lambda) - G_0(-\lambda))\lambda\Phi(H_0)d\lambda. \tag{4.2}$$

We let $L(\lambda) = (1 + G_0(\lambda)V)^{-1} - 1$ as in Lemma 3.5 so that

$$G(\lambda)V = G_0(\lambda)V + G_0(\lambda)VL(\lambda)$$

and substitute this for $G(\lambda)V$ in (4.2). Then, $G_0(\lambda)V$ produces the Born approximation $\Phi(H)\Omega_1\Phi(H_0)$, which is bounded in $L^p(\mathbf{R}^m)$ for all $1 \le p \le \infty$ by virtue of Lemma 2.7, and the second term produces

$$\int_0^\infty \Phi(H)G_0(\lambda)VL(\lambda)(G_0(\lambda) - G_0(-\lambda))\lambda\Phi(H_0)\tilde{\Phi}(\lambda)d\lambda, \tag{4.3}$$

where we have introduced another cut off function $\tilde{\Phi}(\lambda) \in C_0^\infty(\mathbf{R})$ such that

$$\tilde{\Phi}(\lambda)\Phi(\lambda^2) = \Phi(\lambda^2), \text{ and } \tilde{\Phi}(\lambda) = 0 \text{ for } |\lambda| \ge \lambda_0^2.$$

We prove that (4.3) is bounded in $L^p(\mathbf{R}^m)$ for all $1 \le p \le \infty$ in the following slightly more general setting. Note that $VL(\lambda)$ satisfies the property $(K)_\rho$ for some $\rho > m+1$. Indeed, if we choose $m-1 < \rho < \delta - 1$ and set $\tau = \delta - \rho + \gamma$ or $\rho - \gamma = \delta - \tau$, then $\min(\tau - \frac{1}{2}, 2\tau - 2) \ge \gamma$ for $\gamma \ge 0$, and by virtue of Lemma 3.5, $\langle x\rangle^{\rho-\gamma} VL(\lambda)\langle x\rangle^{\rho-\gamma}$ satisfies the desired condition of Definition 1.8.

PROPOSITION 4.2. *Let $m \ge 6$. Suppose $K(\lambda)$ satisfies property $(K)_\rho$ for some $\rho > m+1$. Let $\Phi, \tilde{\Phi} \in C_0^\infty(\mathbf{R})$ be as above and $\Omega$ be defined by*

$$\Omega = \int_0^\infty \Phi(H)G_0(\lambda)K(\lambda)(G_0(\lambda) - G_0(-\lambda))\Phi(H_0)\lambda\tilde{\Phi}(\lambda)d\lambda. \tag{4.4}$$

*Then, $\Omega$ is an integral operator with admissible integral kernel.*

We prove Proposition 4.2 by using a series of lemma. We first remark that (4.4) may be considered as Riemann integral of $\mathbf{B}(\mathcal{H}_\gamma, \mathcal{H}_{-\gamma})$ valued continuous function and that $\Omega$ may be extended to a bounded operator in $\mathcal{H}$. Indeed, since the multiplication by $\langle x\rangle^{-\gamma}$, $\gamma > 1$ is $H_0$-smooth in the sense of Kato ([15]), we have

$$\begin{aligned}|\langle \Omega f, g\rangle| \le \sup_{\lambda\in\mathbf{R}} \|\tilde{\Phi}(\lambda)\langle x\rangle^\gamma K(\lambda)\langle x\rangle^\gamma\|_{\mathbf{B}(\mathcal{H})} \|\langle x\rangle^{-\gamma} G_0(\lambda)\Phi(H_0)f\|_{L^2(\mathbf{R};\mathcal{H},\lambda d\lambda)} \\ \times \|\langle x\rangle^{-\gamma} G_0(\lambda)\Phi(H)g\|_{L^2(\mathbf{R};\mathcal{H},\lambda d\lambda)} \le C\|f\|\|g\|.\end{aligned}$$

Define $\Omega(x,y) = \Omega_+(x,y) - \Omega_-(x,y)$, where

$$\Omega_\pm(x,y) = \int_0^\infty \langle K(\lambda)G_0(\pm\lambda)\Phi_0(\cdot,y), G_0(-\lambda)\Phi(\cdot,x)\rangle\lambda\tilde{\Phi}(\lambda)d\lambda. \tag{4.5}$$

LEMMA 4.3. *The function $\Omega(x,y)$ is continuous and $\Omega$ is an integral operator with the integral kernel $\Omega(x,y)$.*

PROOF. For $\gamma > 1$, $x \mapsto \Phi(\cdot,x)$ and $y \mapsto \Phi_0(\cdot,y)$ are $\mathcal{H}_\gamma$-valued continuous, and $\Omega_\pm(x,y)$ are continuous functions of $(x,y)$. For $f, g \in C_0^\infty(\mathbf{R}^m)$, $\Phi(H_0)f(\cdot) = \int \Phi(\cdot,y)f(y)dy$ and $\Phi(H)g(\cdot) = \int \Phi(\cdot,x)g(x)dx$ converge as Riemann integrals in $\mathcal{H}_\gamma$. It follows by Fubini's theorem that

$$\langle \Omega f, g\rangle = \sum_\pm \pm \int \langle K(\lambda)G_0(\pm\lambda)\Phi(H_0)f, G_0(-\lambda)\Phi(H)g\rangle\tilde{\Phi}(\lambda)d\lambda$$

is equal to $\int \Omega(x,y)f(y)\overline{g(x)}dydx$. The lemma follows. $\square$

We define

$$G_{0l}(\lambda,\cdot,y) = e^{-i\lambda|y|}G_0(\lambda)\Phi_0(\cdot,y), \quad G_{0r}(\lambda,\cdot,x) = e^{-i\lambda|x|}G_0(\lambda)\Phi(\cdot,x), \tag{4.6}$$

$$F_\pm(\lambda,x,y) = \langle K(\lambda)G_{0l}(\pm\lambda,\cdot,y), G_{0r}(\lambda,\cdot,x)\rangle\tilde{\Phi}(\lambda) \tag{4.7}$$

and write (4.5) in the form

$$\Omega_\pm(x,y) = \int_0^\infty e^{i\lambda(|x|\pm|y|)}F_\pm(\lambda,x,y)\lambda d\lambda. \tag{4.8}$$

Lemma 4.4. *Let $\gamma > \frac{1}{2}$ and $\beta \geq 0$ be an integer and let $x, y \in \mathbf{R}^m$. Then:*
(1) *As $\mathcal{H}$ valued functions of $\lambda$, $\langle\cdot\rangle^{-\beta-\gamma}G_{0l}(\lambda,\cdot,y)$ and $\langle\cdot\rangle^{-\beta-\gamma}G_{0r}(\lambda,\cdot,x)$ are of class $C^\beta(\mathbf{R})$ for $0 \leq \beta \leq m-3$, of class $C^\beta_*(\mathbf{R})$ for $\beta = m-2$ and of class $C^\beta(\mathbf{R}\setminus\{0\})$ for any $\beta \geq 0$.*
(2) *For $0 \leq \beta \leq m-3$, $G_{0l}^{(\beta)}(\lambda,z,y)$ is continuous with respect to $\lambda \geq 0$ and*

$$|G_{0l}^{(\beta)}(0,z,y)| \leq C\sum_{\beta_1+\beta_2=\beta}\frac{\langle z\rangle^{\beta_1}}{\langle z-y\rangle^{m-2-\beta_2}}. \tag{4.9}$$

(3) *Let $0 < \lambda_0 < 1/2$. For any $0 \leq \beta$ and $\varepsilon > 0$, we have*

$$\|\langle\cdot\rangle^{-\beta-\varepsilon-\frac{m}{2}}G_{0l}^{(\beta)}(\lambda,\cdot,y)\| \leq C\lambda^{\min\left(0,\frac{m-3}{2}-\beta\right)}\langle y\rangle^{-\frac{m-1}{2}}, \quad 0 < |\lambda| < \lambda_0. \tag{4.10}$$

(4) *With obvious modifications $G_{0r}(\lambda,z,x)$ satisfies* (4.10) *and* (4.9).

Proof. Statement (1) follows from LAP, viz. Lemma 2.5 and Proposition 2.6. By Leibniz' rule $G_{0l}^{(\beta)}(\lambda,z,y)$ is a linear combination of

$$K_{\beta_1\beta_2}(\lambda,z,y) = \int_{\mathbf{R}^m}\frac{(i\psi(w,z,y))^{\beta_1}e^{i\lambda\psi(w,z,y)}}{|z-w|^{m-2-\beta_2}}H_{\beta_2}(\lambda|w-z|)\Phi_0(w,y)dw,$$

$$H_\beta(s) = \int_0^\infty e^{-t}t^{\frac{m-3}{2}}\left(s+\frac{it}{2}\right)^{\frac{m-3}{2}-\beta}dt \tag{4.11}$$

over the indices $\beta_1, \beta_2$ such that $\beta_1+\beta_2=\beta$. Here $\psi(w,z,y) \equiv |w-z|-|y|$ satisfies $|\psi(w,z,y)| \le |w-y|+|z|$. It follows when $\beta_3 \le m-3$ that $K_{\beta_1\beta_2}(\lambda,z,y)$ is continuous with respect to $\lambda$ upto $\lambda=0$ and, for any $N$,

$$|K_{\beta_1\beta_2}(0,z,y)| \le C_{\beta_1\beta_2 N}\int_{\mathbf{R}^m} \frac{\langle z\rangle^{\beta_1}\langle w-y\rangle^{-N}}{|z-w|^{m-2-\beta_2}}dw \le \frac{C_{\beta_1\beta_2}\langle z\rangle^{\beta_1}}{|z-y|^{m-2-\beta_2}}.$$

Statement (2) follows.

If $\beta_2 \le \frac{m-3}{2}$, we have $\left|s+\frac{it}{2}\right|^{\frac{m-3}{2}-\beta_2} \le C(s^{\frac{m-3}{2}-\beta_2}+|t|^{\frac{m-3}{2}-\beta_2})$ and $|H_{\beta_2}(s)| \le C(s+1)^{\frac{m-3}{2}-\beta_2}$. It follows that

$$\begin{aligned}(4.12)\qquad & |K_{\beta_1\beta_2}(\lambda,z,y)| \\ & \le C\int_{\mathbf{R}^m}\frac{|z|^{\beta_1}+|w-y|^{\beta_1}}{|z-w|^{m-2-\beta_2}}(\lambda|z-w|+1)^{\frac{m-3}{2}-\beta_2}|\Phi_0(w,y)|dw \\ & \le C\langle z\rangle^{\beta_1}\langle z-y\rangle^{-\frac{m-1}{2}} \quad \text{for all } \ |\lambda| \le \lambda_0.\end{aligned}$$

Hence, if $\beta \le \frac{m-3}{2}$, (4.10) is satisfies for $|\lambda|<\lambda_0$. If $\frac{m-2}{2} \le \beta_2$, $|s+\frac{it}{2}|^{\frac{m-3}{2}-\beta_2} \le C\min\{s^{\frac{m-3}{2}-\beta_2}, |t|^{\frac{m-3}{2}-\beta_2}\}$ and

$$|H_{\beta_2}(s)| \le C\left(s^{\frac{m-3}{2}-\beta_2}\int_0^s e^{-t}t^{\frac{m-3}{2}}dt + \int_s^\infty e^{-t}t^{m-3-\beta_2}dt\right).$$

Hence $|H_{\beta_2}(s)| \le C_{\beta_2}s^{\frac{m-3}{2}-\beta_2}$ for $s \ge 1$ and for $0<s<1$,

$$|H_{\beta_2}(s)| \le C\begin{cases} 1, & \text{if } \frac{m-3}{2}<\beta_2 \le m-3 \\ (1+|\log s|), & \text{if } \beta_2 = m-2, \\ s^{m-2-\beta_2}, & \text{if } \beta_2 \ge m-1. \end{cases}$$

It is then easy to see that

$$\begin{aligned}(4.13)\qquad & |H_{\beta_2}(s)| \\ & \le C\begin{cases} \min(s^{\frac{m-3}{2}-\beta_2}, 1), & \frac{m-3}{2}<\beta_2 \le m-3 \\ \min(s^{\frac{m-3}{2}-\beta_2}, (1+|\log s|)), & \beta_2 = m-2, \\ \min(s^{\frac{m-3}{2}-\beta_2}, s^{m-2-\beta_2}), & \beta_2 \ge m-1. \end{cases}\end{aligned}$$

Using the estimate $|H_{\beta_2}(s)| \le Cs^{\frac{m-3}{2}-\beta_2}$ of (4.13), we obtain that

$$\begin{aligned}\langle z\rangle^{-\beta-\frac{m}{2}-\varepsilon}|K_{\beta_1\beta_2}(\lambda,z,y)| &\le \int\frac{C\langle z\rangle^{-\frac{m}{2}-\varepsilon}\langle w-y\rangle^{-N}dw}{|\lambda|z-w||^{\beta_2-\frac{m-3}{2}}|z-w|^{m-2-\beta_2}\langle z\rangle^{\beta_2}} \\ &\le \int\frac{C\langle z\rangle^{-\frac{m}{2}-\varepsilon}\langle w-y\rangle^{-N}dw}{\lambda^{\beta_2-\frac{m-3}{2}}|z-w|^{\frac{m-1}{2}}\langle z\rangle^{\beta_2}} \le \frac{C\langle z\rangle^{-\frac{m}{2}-\varepsilon}}{\lambda^{\beta_2-\frac{m-3}{2}}\langle z-y\rangle^{\frac{m-1}{2}}\langle z\rangle^{\beta_2}}\end{aligned}$$

and (4.10) for $\frac{m-3}{2} < \beta_2$ follows. The proof for $G_{0r}(\lambda, \cdot, x)$ is similar and we omit it. $\square$

If we use the second estimate in (4.13), the argument of the proof above show that $|K_{\beta_1\beta_2}(\lambda, z, y)|$ is bounded by a constant time

$$\left\{\begin{array}{ll} \langle z\rangle^{\beta_1}\langle z-y\rangle^{\beta_2-m+2}, & \frac{m-3}{2} < \beta_2 \le m-3 \\ \langle z\rangle^{\beta_1}(\langle \log\lambda\rangle + \log\langle z-y\rangle), & \beta_2 = m-2, \\ \lambda^{m-2-\beta_2}\langle z\rangle^{\beta_1}, & \beta_2 \ge m-1. \end{array}\right. \tag{4.14}$$

The estimate (4.14) will be used in what follows.

By virtue of Lemma 4.4, $F_\pm(\lambda, x, y)$ is of class $C_*^{\frac{m+2}{2}}$ on $\mathbf{R}$ with respect to $\lambda$ for every fixed $x, y \in \mathbf{R}$ and it satisfies $|\Omega(x,y)| \le C\langle x\rangle^{-\frac{m-1}{2}}\langle y\rangle^{-\frac{m-1}{2}}$. It is then easy to check that

$$\sup_{x\in\mathbf{R}^m}\int_{||x|-|y||<1}|\Omega(x,y)|dy + \sup_{y\in\mathbf{R}^m}\int_{||x|-|y||<1}|\Omega(x,y)|dx < \infty. \tag{4.15}$$

Thus, we hereafter consider $\Omega(x,y)$ only on the domain $||x|-|y|| > 1$. We apply integration by parts $k = (m+2)/2$ times to

$$\Omega_\pm(x,y) = \frac{1}{(i(|x|\pm|y|))^k}\int_0^\infty \left(\frac{\partial}{\partial\lambda}\right)^k e^{i\lambda(|x|\pm|y|)} \cdot F_\pm(\lambda, x, y)\lambda d\lambda. \tag{4.16}$$

The result is that $\Omega(x,y)$ is the sum of

$$I_1(x,y) = \sum_\pm \frac{\pm i^{\frac{m+2}{2}}}{(|x|\pm|y|)^{\frac{m+2}{2}}}\int_0^\infty e^{i\lambda(|x|\pm|y|)}F_\pm^{(\frac{m+2}{2})}(\lambda, x, y)\lambda d\lambda, \tag{4.17}$$

$$I_2(x,y) = \sum_\pm \frac{\pm(m+2)i^{\frac{m+2}{2}}}{2(|x|\pm|y|)^{\frac{m+2}{2}}}\int_0^\infty e^{i\lambda(|x|\pm|y|)}F_\pm^{(\frac{m}{2})}(\lambda, x, y)d\lambda, \tag{4.18}$$

and the boundary terms:

$$B(x,y) = \sum_{j=0}^{\frac{m-2}{2}} i^j(j+1)\left(\frac{F_+^{(j)}(0,x,y)}{(|x|+|y|)^{j+2}} - \frac{F_-^{(j)}(0,x,y)}{(|x|-|y|)^{j+2}}\right). \tag{4.19}$$

Lemma 4.5. *The function* $B(x,y)$ *of* (4.19) *is an admissible integral kernel.*

Proof. Derivatives $F_{\pm}^{(j)}(0,x,y)$ are linear combinations over $\alpha+\beta+\gamma=j$ of

$$(\pm 1)^{\beta}\langle K^{(\alpha)}(0)G_{0l}^{(\beta)}(0,\cdot,y), G_{0r}^{(\gamma)}(0,\cdot,x)\rangle \tag{4.20}$$

with coefficients $(-1)^{\gamma}j!/\alpha!\beta!\gamma!$. In (4.20), we have for arbitrarily small $\varepsilon>0$

$$\begin{aligned} \|\langle z\rangle^{1+j-m-\varepsilon-\beta}G_{0l}^{(\beta)}(0,z,y)\| &\le C\begin{cases}\langle y\rangle^{j+2-m} & \text{if } \beta=j,\\ \langle y\rangle^{j+1-m} & \text{if otherwise.}\end{cases}\\ \|\langle z\rangle^{1+j-m-\varepsilon-\gamma}G_{0r}^{(\gamma)}(0,z,x)\| &\le C\begin{cases}\langle x\rangle^{j+2-m} & \text{if } \gamma=j,\\ \langle x\rangle^{j+1-m} & \text{if otherwise.}\end{cases}\end{aligned} \tag{4.21}$$

This can be seen as follows. By virtue of (4.9) we have

$$\left|\frac{G_{0l}^{(\beta)}(0,z,y)}{\langle z\rangle^{m-j-1+\beta+\varepsilon}}\right| \le \sum_{\beta_2=0}^{\beta}\frac{C}{\langle z\rangle^{m-\beta_1-1+\beta_2+\varepsilon}\langle z-y\rangle^{m-2-\beta_2}}$$

and the like for $G_{0r}^{(\gamma)}(0,\cdot,x)$. Since $(m-j-1+\beta_2+\varepsilon)+(m-2-\beta_2)>m$, we have either $m-j-1+\beta_2+\varepsilon>\frac{m}{2}$ or $m-2-\beta_2>\frac{m}{2}$. Hence

$$\|\langle z\rangle^{j+1-m-\beta_2-\varepsilon}\langle z-y\rangle^{2+\beta_2-m}\| \le C\begin{cases}\langle y\rangle^{2+j-m}, & \text{if } \beta=j\\ \langle y\rangle^{1+j-m}, & \text{if otherwise.}\end{cases} \tag{4.22}$$

Since $\rho>m+1$, we have for $0<\varepsilon\le 1$ that

$$\max(m-1-(j-\beta)+\varepsilon, m-1-(j-\gamma)+\varepsilon)<\rho-\alpha$$

and $\langle\cdot\rangle^{m-1-(j-\gamma)+\varepsilon}K^{(\alpha)}(0)\langle\cdot\rangle^{m-1-(j-\beta)+\varepsilon}\in\mathbf{B}(\mathcal{H})$ by property $(K)_\rho$. Thus, (4.20) is bounded in modulus by a constant time

$$Y_{\alpha\beta\gamma}(x,y)=\begin{cases}\langle x\rangle^{1+j-m}\langle y\rangle^{1+j-m} & \text{if } \beta\ne 0,j,\\ \langle x\rangle^{1+j-m}\langle y\rangle^{2+j-m} & \text{if } \beta=j,\\ \langle x\rangle^{2+j-m}\langle y\rangle^{1+j-m} & \text{if } \beta=0.\end{cases} \tag{4.23}$$

It follows that the $j$-th summand of (4.19) is bounded by

$$C \sum_{\alpha+\beta+\gamma=j} \left| \frac{1}{(|x|+|y|)^{j+2}} - \frac{(-1)^\beta}{(|x|-|y|)^{j+2}} \right| Y_{\alpha\beta\gamma}(x,y) \tag{4.24}$$

and it is an easy exercise to prove that this is an admissible integral kernel. (Indeed, summands with $\beta \neq 0, j$ are admissible by virtue of Lemma 3.6 of [I]; those with $\beta = 0$ or $\beta = j$ are the same as (3.21) of [I] and the argument in [I] following (3.21) applies also for $\beta \leq \frac{m-2}{2}$ or $\gamma \leq \frac{m-2}{2}$ if $m \geq 4$.) This completes the proof. $\square$

Lemma 4.6. *The integral kernel* $I_2(x,y)$ *defined by* (4.18) *is admissible.*

Proof. By Leibniz' rule $F_\pm^{(\frac{m}{2})}(\lambda,x,y)$ is a linear combination of

$$X_{\xi,\pm}(\lambda,x,y) = (\pm 1)^\beta \langle K^{(\alpha)}(\lambda) G_{0l}^{(\beta)}(\pm\lambda,\cdot,y), G_{0r}^{(\gamma)}(-\lambda,\cdot,x)\rangle \tilde{\Phi}^{(\eta)}(\lambda) \tag{4.25}$$

with $\pm$ independent coefficients $(-1)^\gamma(\frac{m}{2})!/\alpha!\beta!\gamma!\eta!$ over multi-indices $\xi = (\alpha,\beta,\gamma,\eta)$ of length $|\xi| = \frac{m}{2}$ . Thus, if we define

$$\Omega_{\xi,\pm}^{(1)}(x,y) = \int_0^\infty e^{i\lambda(|x|\pm|y|)} X_{\xi,\pm}(\lambda,x,y) d\lambda, \tag{4.26}$$

then $I_2(x,y)$ is a linear combination over $\xi = (\alpha,\beta,\gamma,\eta)$ with $|\xi| = \frac{m}{2}$ of

$$I_{2,\xi}(x,y) = \left( \frac{\Omega_{\xi,+}^{(1)}(x,y)}{(|x|+|y)^{\frac{m+2}{2}}} - \frac{\Omega_{\xi,-}^{(1)}(x,y)}{(|x|-|y)^{\frac{m+2}{2}}} \right). \tag{4.27}$$

We estimate $I_{2,\xi}(x,y)$ for various cases of $\xi$ separately.

**(1) The case** $\xi \neq (0,\frac{m}{2},0,0),(0,0,\frac{m}{2},0)$**.** In view of property $(K)_\rho$, we estimate

$$\begin{aligned} |X_{\xi\pm}(\lambda,x,y)| \leq{}& C\|\langle x\rangle^{\rho-\alpha} K^{(\alpha)}(\lambda) \langle x\rangle^{\rho-\alpha}\|_{\mathbf{B}(\mathcal{H})} \\ &\times \|\langle\cdot\rangle^{-(\rho-\alpha)} G_{0l}^{(\beta)}(\pm\lambda,\cdot,y)\| \|\langle\cdot\rangle^{-(\rho-\alpha)} G_{0l}^{(\gamma)}(\pm\lambda,\cdot,x)\|. \end{aligned} \tag{4.28}$$

Since $\rho > m+1$ and $\alpha+\beta+\gamma \leq \frac{m}{2}$, we have for $0 < \varepsilon < 1$

$$\max(\beta + \tfrac{m}{2} + \varepsilon, \gamma + \tfrac{m}{2} + \varepsilon) < \rho - \alpha. \tag{4.29}$$

Hence, by virtue of (4.10), we have that

$$\begin{aligned}(4.30)\qquad &|X_{\xi\pm}(\lambda,x,y)| \\ &\le C\begin{cases}\langle x\rangle^{-\frac{m-1}{2}}\langle y\rangle^{-\frac{m-1}{2}}, & \text{if both } \beta,\gamma\le\frac{m-3}{2},\\ \lambda^{-\frac{1}{2}}\langle x\rangle^{-\frac{m-1}{2}}\langle y\rangle^{-\frac{m-1}{2}}, & \text{if one of } \beta,\gamma=\frac{m-2}{2},\end{cases}\end{aligned}$$

where we have to modify the first line on the right by multiplying by $\langle\log\lambda\rangle^N$ when $\xi=(\frac{m}{2},0,0,0)$. Thus after integrating with respect to $\lambda$ we obtain for $||x|-|y||>1$ that

$$(4.31)\qquad \left|\frac{\Omega^{(1)}_{\xi\pm}(x,y)}{\langle|x|\pm|y|\rangle^{\frac{m+2}{2}}}\right|\le\frac{C}{\langle|x|\pm|y|\rangle^{\frac{m+2}{2}}\langle x\rangle^{\frac{m-1}{2}}\langle y\rangle^{\frac{m-1}{2}}}.$$

It follows that $I_{2,\xi}$ are admissible for these $\xi$'s.

**(2) The case $\xi=(0,\frac{m}{2},0,0)$.** Recall the definition (4.11) of $K_{\beta_1\beta_2}(\lambda,z,y)$. We substitute $\sum_{\beta_1+\beta_2=\frac{m}{2}}C_{\beta_1\beta_2}K_{\beta_1\beta_2}(\lambda,z,y)$ for $G_{0l}^{(\frac{m}{2})}(\lambda,z,y)$ in

$$X_{\xi,\pm}(\lambda,x,y)=(\pm1)^{\frac{m}{2}}\langle K(\lambda)G_{0l}^{(\frac{m}{2})}(\pm\lambda,\cdot,y),G_{0r}(-\lambda,\cdot,x)\rangle\tilde\Phi(\lambda).$$

If $(\beta_1,\beta_2)\ne(0,\frac{m}{2})$, we have $\beta_2\le\frac{m-2}{2}$ and the first estimate of (4.14) implies

$$(4.32)\qquad \|\langle\cdot\rangle^{-(\beta+\frac{m}{2}+\varepsilon)}K_{\beta_1\beta_2}(\lambda,\cdot,y)\|\le C\lambda^{-\frac{1}{2}}\langle y\rangle^{-\frac{m-1}{2}}.$$

Since $\beta+\frac{m}{2}+\varepsilon<\rho$ for $0<\varepsilon\le1$, it follows via the argument similar to the one used for (4.28) that, for $(\beta_1,\beta_2)\ne(0,\frac{m}{2})$,

$$|\langle K(\lambda)K_{\beta_1\beta_2}(\pm\lambda,\cdot,y),G_{0r}(-\lambda,\cdot,x)\rangle|\le C\lambda^{-\frac{1}{2}}\langle x\rangle^{-\frac{m-1}{2}}\langle y\rangle^{-\frac{m-1}{2}}.$$

This implies that all members under the summation sign of

$$\begin{aligned}&\sum_{\beta_1+\beta_2=\frac{m}{2}}\frac{(\pm1)^{\frac{m}{2}}C_{\beta_1\beta_2}}{(|x|\pm|y|)^{\frac{m+2}{2}}}\\ &\qquad\times\int_0^\infty e^{i\lambda(|x|\pm|y|)}\langle K(\lambda)K_{\beta_1\beta_2}(\pm\lambda,\cdot,y),G_{0r}(-\lambda,\cdot,x)\rangle d\lambda,\end{aligned}$$

are admissible except those with $(\beta_1, \beta_2) = (0, \frac{m}{2})$. We are thus left with

$$I_{2r} = \sum_{\pm} \frac{\pm(\pm 1)^{\frac{m}{2}}}{(|x| \pm |y|)^{\frac{m+2}{2}}} \times \int_0^\infty e^{i\lambda(|x|\pm|y|)} \langle K(\lambda) K_{0\frac{m}{2}}(\pm\lambda, \cdot, y), G_{0r}(-\lambda, \cdot, x)\rangle d\lambda.$$

For proving that $I_{2r}(x, y)$ is admissible, we restore the factors $e^{i\lambda(|x|\pm|y|)}$ to the original position. Thus defining $\tilde{G}_{0r}(\lambda, \cdot, x)$ and $G_{\frac{m}{2}}(\lambda, z, y)$ by

$$\tilde{G}_{0r}(\lambda, \cdot, x) = G_0(\lambda)\Phi(\cdot, x) = e^{-i\lambda|x|} G_{0r}(\lambda, \cdot, x)$$

and $G_{\frac{m}{2}}(\lambda, z, y) = e^{i\lambda|y|} K_{0\frac{m}{2}}(\lambda, z, y)$ respectively, we rewrite, ignoring unimportant constants, $I_{2r}$ in the form

$$\sum_{\pm} \frac{(\pm 1)^{\frac{m+2}{2}}}{(|x| \pm |y|)^{\frac{m+2}{2}}} \int_0^\infty \langle K(\lambda) G_{\frac{m}{2}}(\pm\lambda, \cdot, y), \tilde{G}_{0r}(-\lambda, \cdot, x)\rangle \tilde{\Phi}(\lambda) d\lambda, \tag{4.33}$$

where, more explicitly $G_{\frac{m}{2}}(\lambda, z, y)$ is given by

$$\int_{\mathbf{R}^m} \frac{e^{i\lambda|z-w|}}{|z-w|^{\frac{m}{2}-2}} \left( \int_0^\infty e^{-t} t^{\frac{m-3}{2}} \left( \lambda|z-w| + \frac{it}{2} \right)^{-\frac{3}{2}} dt \right) \Phi_0(w, y) dw. \tag{4.34}$$

For proving that (4.33) is admissible, we use the following lemma:

Lemma 4.7.

(1) *There exists a constant $C > 0$ such that*

$$|\tilde{G}_{0r}(\lambda, z, x) - \tilde{G}_{0r}(0, z, x)| \le C|\lambda| \langle z - x\rangle^{-\frac{m-1}{2}}, \tag{4.35}$$

$$|G_{\frac{m}{2}}(\lambda, z, y)| \le C \min\left( \lambda^{-\frac{3}{2}} \langle z - y\rangle^{-\frac{m-1}{2}}, \langle z - y\rangle^{-\frac{m-4}{2}} \right). \tag{4.36}$$

(2) *For a fixed $(z, y)$, $\mathbf{R} \ni \lambda \mapsto G_{\frac{m}{2}}(\lambda, z, y)$ is continuous; for a fixed $y$ and for $\gamma > \frac{3}{2}$, $\mathbf{R} \setminus \{0\} \ni \lambda \mapsto G_{\frac{m}{2}}(\lambda, \cdot, y) \in \mathcal{H}_{-\gamma}$ is continuous and integrable.*

(3) *The integrand of (4.34) is integrable with respect to $(t, \lambda, w)$.*

PROOF. (1) Write the convolution kernel of $G_0(\lambda) - G_0(0)$ in the form

$$\frac{C_m e^{i\lambda|x|}}{|x|^{m-2}} \int_0^\infty e^{-t} t^{\frac{m-3}{2}} \left\{ \left(\frac{t}{2} - i\lambda|x|\right)^{\frac{m-3}{2}} - \left(\frac{t}{2}\right)^{\frac{m-3}{2}} \right\} dt + \frac{C'_m(e^{i\lambda|x|} - 1)}{|x|^{m-2}}$$

and estimate it by $C|\lambda|(|x|^{3-m}\langle x\rangle^{\frac{m-5}{2}} + |x|^{3-m})$ for $|\lambda| \le \lambda_0$. This yields (4.35) since $\frac{m-1}{2} \le m-3$ for $m \ge 6$. Estimate (4.36) is contained in (4.10) and (4.14).
(2) The continuity of $\lambda \mapsto G_{\frac{m}{2}}(\lambda, z, y)$ is obvious by Lebesgue's dominated convergence theorem. Then the second statement follows from the estimate (4.36) which also implies $|G_{\frac{m}{2}}(\lambda, z, y)| \le C\lambda^{-\frac{1}{2}}\langle z-y\rangle^{-\frac{m-3}{2}}$ by interpolation.
(3) follows immediately if we integrate the modulus of the integrand with respect to $\lambda$ first. □

We continue the proof that (4.33) is admissible. We first show that it suffices to prove it after replacing $K(\lambda)$, $\tilde{G}_{0r}(-\lambda, \cdot, x)$ and $\tilde{\Phi}(\lambda)$ by $K(0)$, $\tilde{G}_{0r}(0, \cdot, x)$ and the constant function 1 respectively.
(i) Let $I^{(1)}_{2r}(x, y)$ be defined by (4.33) with $K(0)$ in place of $K(\lambda)$. Via Taylor's formula, $K(\lambda) - K(0) = \lambda \int_0^1 K'(\theta\lambda) d\theta$ and property $(K)_\rho$ implies that

$$\left\| \langle x\rangle^{\rho-1} \Big( \int_0^1 K'(\theta\lambda) d\theta \Big) \langle x\rangle^{\rho-1} \right\|_{\mathbf{B}(\mathcal{H})} \le C.$$

Hence, using (4.10) for $\tilde{G}_{0r}(-\lambda, \cdot, x)$ and (4.36), we obtain

$$\begin{aligned} &|\langle (K(\lambda) - K(0)) G_{\frac{m}{2}}(\pm\lambda, \cdot, y), \tilde{G}_{0r}(-\lambda, \cdot, x)\rangle| \\ &\le C\|\langle\cdot\rangle^{-\rho+1}\lambda G_{\frac{m}{2}}(\lambda, \cdot, y)\| \|\langle\cdot\rangle^{-\rho+1}\tilde{G}_{0r}(\lambda, \cdot, x)\| \le C\lambda^{-\frac{1}{2}}\langle x\rangle^{-\frac{m-1}{2}}\langle y\rangle^{-\frac{m-1}{2}}. \end{aligned}$$

It follows after integration with respect to $\lambda$ that

$$|I_{2r}(x, y) - I^{(1)}_{2r}(x, y)| \le \frac{C}{\langle x\rangle^{\frac{m-1}{2}}\langle y\rangle^{\frac{m-1}{2}}\langle |x| - |y|\rangle^{\frac{m+2}{2}}} \tag{4.37}$$

and $I_{2r}(x, y) - I^{(1)}_{2r}(x, y)$ is an admissible kernel.
(ii) We then let $I^{(2)}_{2r}(x, y)$ be defined by (4.33) with $K(0)$ and $\tilde{G}_{0r}(0, \cdot, x)$ in places of $K(\lambda)$ and $\tilde{G}_{0r}(\lambda, \cdot, x)$ respectively. Then, trading the factor $\lambda$ of

(4.35) for estimating $G_{\frac{m}{2}}(\lambda, z, y)$ as above, we obtain

$$|I_{2r}^{(1)}(x,y) - I_{2r}^{(2)}(x,y)| \le \frac{C}{\langle x\rangle^{\frac{m-1}{2}}\langle y\rangle^{\frac{m-1}{2}}\langle |x|-|y|\rangle^{\frac{m+2}{2}}} \tag{4.38}$$

and $I_{2r}^{(1)}(x,y) - I_{2r}^{(2)}(x,y)$ is also admissible.
(iii) From (4.36) we have $|G_{\frac{m}{2}}(\pm\lambda, z, y)| \le C\langle\lambda\rangle^{-\frac{3}{2}}\langle z-y\rangle^{-\frac{m-1}{2}}$ on the support of $1-\Phi(\lambda)$. Then the argument in (i) and (ii) implies that we may further replace $\tilde{\Phi}(\lambda)$ by $\tilde{\Phi}(0) = 1$ for proving that (4.33) is admissible.

Thus the problem is reduced to proving that

$$\tilde{I}_2(x,y) = \sum_{\pm} \frac{(\pm 1)^{\frac{m+2}{2}}}{(|x|\pm|y|)^{\frac{m+2}{2}}} \int_0^\infty \langle K(0)G_{\frac{m}{2}}(\pm\lambda, \cdot, y), \tilde{G}_{0r}(0,\cdot,x)\rangle d\lambda \tag{4.39}$$

is an admissible kernel. Since $G_{\frac{m}{2}}(\lambda, \cdot, y)$ satisfies the continuity property of Lemma 4.7, $\langle x\rangle^\gamma K(0)\langle x\rangle^\gamma \in \mathcal{B}(\mathcal{H})$ and $\tilde{G}_{0r}(0,\cdot,x) \in \mathcal{H}_{-\gamma}$ for some $\gamma > \frac{3}{2}$, we may perform the integration in (4.39) before taking the inner product and write the integral in the form

$$\Big\langle K(0)\int_0^\infty G_{\frac{m}{2}}(\pm\lambda, \cdot, y)d\lambda, \tilde{G}_{0r}(\cdot, x)\Big\rangle.$$

Here the integral on the right is the Riemann integral of an $\mathcal{H}_{-\gamma}$ valued function, however, Lemma 4.7 (2) implies that we may replace it by the standard Riemann integral of the scalar continuous function $G_{\frac{m}{2}}(\pm\lambda, z, y)$. Then, by virtue of Lemma 4.7 (3), we may integrate (4.34) with respect to $\lambda$ first via Fubini's theorem. For $a > 0$ and $t > 0$ we have by residue theorem that

$$\int_0^\infty e^{i\lambda a}\Big(\lambda a + \frac{it}{2}\Big)^{-\frac{3}{2}} d\lambda = -\int_0^\infty e^{-i\lambda a}\Big(-\lambda a + \frac{it}{2}\Big)^{-\frac{3}{2}} d\lambda$$

and both sides are bounded in modulus by $Cat^{-\frac{1}{2}}$. It follows that

$$\int_0^\infty G_{\frac{m}{2}}(\lambda, z, y)d\lambda = -\int_0^\infty G_{\frac{m}{2}}(-\lambda, z, y)d\lambda \equiv J(z,y), \tag{4.40}$$

$$\text{and} \quad |J(z,y)| \le \int_{\mathbf{R}^m} \frac{C|\Phi_0(w,y)|}{|z-w|^{\frac{m-2}{2}}}dw \le \frac{C}{\langle z-y\rangle^{\frac{m-2}{2}}}. \tag{4.41}$$

Thus, we have $|\langle K(0)J(\cdot,y), G_{00}(\cdot,x)\rangle| \le C\langle x\rangle^{-(m-2)}\langle y\rangle^{-\frac{m-2}{2}}$ and

$$|\tilde{I}_2(x,y)| \le C\left|\frac{1}{(|x|+|y|)^{\frac{m+2}{2}}} - \frac{1}{(|y|-|x|)^{\frac{m+2}{2}}}\right|\langle x\rangle^{-(m-2)}\langle y\rangle^{-\frac{m-2}{2}}. \tag{4.42}$$

The right side is the same as the summand in (4.24) with $j=\beta=\frac{m-2}{2}$ and $\alpha=\gamma=0$ and, hence, $\tilde{I}_2(x,y)$ is admissible.

**(3) The case** $\xi=(0,0,\frac{m}{2},0)$ . Define $\tilde{G}_{\frac{m}{2}}(\lambda,z,x)$ by (4.34) with $\Phi(w,x)$ in place of $\Phi_0(w,y)$ and

$$\tilde{J}(z,x) = \int_0^\infty \tilde{G}_{\frac{m}{2}}(-\lambda,z,x)d\lambda.$$

Proceeding virtually in the same way as in the case $\xi=(0,\frac{m}{2},0,0)$, we see that it suffices to show that

$$I_3(x,y) = \left(\frac{1}{(|x|+|y|)^{\frac{m+2}{2}}} - \frac{1}{(|x|-|y|)^{\frac{m+2}{2}}}\right)\langle K(0)\tilde{G}_{0l}(0,\cdot,y), \tilde{J}(\cdot,x)\rangle$$

is admissible. It is obvious from the argument which lead to (4.41) that $|\tilde{J}(z,x)| \le C\langle z-x\rangle^{-\frac{m-2}{2}}$ and have

$$|\langle K(0)\tilde{G}_{0l}(0,\cdot,y), \tilde{J}(\cdot,x)\rangle| \le C\langle x\rangle^{-\frac{m-2}{2}}\langle y\rangle^{-(m-2)}.$$

Thus, $I_3(x,y)$ is bounded by the right of (4.42) with $x$ and $y$ interchanged and is therefore admissible. This completes the proof of Lemma 4.6. □

LEMMA 4.8. *The integral kernel $I_4(x,y)$ defined by the integral* (4.17)*:*

$$I_4(x,y) = \sum_{\pm} \frac{\pm i^{\frac{m+2}{2}}}{(|x|\pm|y|)^{\frac{m+2}{2}}}\int_0^\infty e^{i\lambda(|x|\pm|y|)}F_\pm^{(\frac{m+2}{2})}(\lambda,x,y)\lambda d\lambda$$

*is admissible.*

PROOF. We proceed as in the proof of Lemma 4.6. Let as in (4.25):

$$X_{\xi,\pm}(\lambda,x,y) = (\pm 1)^\beta\langle K^{(\alpha)}(\lambda)G_{0l}^{(\beta)}(\pm\lambda,\cdot,y), G_{0r}^{(\gamma)}(-\lambda,\cdot,x)\rangle\tilde{\Phi}^{(\eta)}(\lambda)$$

for $\xi=(\alpha,\beta,\gamma,\eta)$ and define

$$\Omega_{\xi\pm}^{(2)}(x,y) = \int_0^\infty e^{i\lambda(|x|\pm|y|)}X_{\xi,\pm}(\lambda,x,y)\lambda d\lambda. \tag{4.43}$$

By Leibniz' formula we have

$$(4.44) \qquad I_4(x,y) = \sum_{|\xi|=\frac{m+2}{2}} C_\xi \left( \frac{\Omega^{(2)}_{\xi,+}(x,y)}{(|x|+|y)^{\frac{m+2}{2}}} - \frac{\Omega^{(2)}_{\xi-}(x,y)}{(|x|-|y)^{\frac{m+2}{2}}} \right).$$

Let first $\xi \neq (0, \frac{m+2}{2}, 0, 0), (0, 0, \frac{m+2}{2}, 0)$. Since $\alpha + \max(\beta + \frac{m}{2}, \gamma + \frac{m}{2}) \le m+1$ and $\rho > m+1$, there exists $\varepsilon > 0$ such that $\max(\beta + \frac{m}{2}, \gamma + \frac{m}{2}) + \varepsilon < \rho$. By virtue of (4.10) and the property $(K)_\rho$, we have with this $\varepsilon > 0$ that

$$(4.45) \qquad \begin{aligned} |\lambda||X_{\xi,\pm}(\lambda,x,y)| &\le |\lambda| \|\langle x\rangle^{\rho-\alpha} K^{(\alpha)}(\lambda)\langle x\rangle^{\rho-\alpha}\| \\ &\times \|\langle x\rangle^{-(\rho-\alpha)} G^{(\beta)}_{0l}(\lambda,\cdot,y)\|_{\mathcal{H}} \\ &\times \|\langle x\rangle^{-(\rho-\alpha)} G^{(\gamma)}_{0r}(\lambda,\cdot,x)\|_{\mathcal{H}} \\ &\le C|\lambda|^{-\frac{1}{2}} \langle \log\lambda\rangle^N \langle y\rangle^{-(\frac{m-1}{2})} \langle x\rangle^{-(\frac{m-1}{2})}. \end{aligned}$$

This implies that for the summands in (4.44) with these $\xi$ we have

$$(4.46) \qquad \left| \frac{\Omega^{(2)}_{\xi,\pm}(x,y)}{(|x| \pm |y)^{\frac{m+2}{2}}} \right| \le \frac{C}{\langle |x| \pm |y| \rangle^{\frac{m+2}{2}}} \cdot \frac{1}{\langle y\rangle^{\frac{m-1}{2}} \langle x\rangle^{\frac{m-1}{2}}}$$

and these are therefore admissible. We are left with those either with $\xi = (0, \frac{m+2}{2}, 0, 0)$ or $\xi = (0, 0, \frac{m+2}{2}, 0)$ and we shall deal with the former case only as the other case may be treated similarly. So let $\xi = (0, 0, \frac{m+2}{2}, 0)$ in what follows. We substitute $\sum_{\beta_1+\beta_2=\frac{m+2}{2}} C_{\beta_1\beta_2} K_{\beta_1\beta_2}(\lambda,z,y)$ for $G^{(\frac{m+2}{2})}_{0l}(\lambda,z,y)$ in (4.25) and plug this into (4.43). This produces several functions indexed by $\beta_1$ and $\beta_2$ in the obvious manner and, by virtue of (4.10), (4.14) and estimates corresponding to (4.45), they are all admissible except the one with the index $(\beta_1, \beta_2) = (0, \frac{m+2}{2})$ which is written in the form as follows as in (4.33) after restoring the exponents $e^{i\lambda(|x|\pm|y|)}$ to the original position:

$$(4.47) \quad \sum_{\pm} \frac{\pm(\pm 1)^{\frac{m+2}{2}}}{(|x| \pm |y|)^{\frac{m+2}{2}}} \int_0^\infty \langle K(\lambda) G_{\frac{m+2}{2}}(\pm\lambda,\cdot,y), \tilde{G}_{0r}(-\lambda,\cdot,x)\rangle \Phi(\lambda)\lambda d\lambda.$$

The same argument as in the proof of Lemma 4.6 shows that it suffices to show that (4.47) is admissible after replacing $\tilde{G}_{0r}(\lambda,z,x)$ by $G_{0r}(0,z,x)$,

$K(\lambda)$ by $K(0)$ and $\Phi(\lambda)$ by the constant function 1. In this case the residue theorem implies, for $a > 0$ and $t > 0$, that

$$\int_0^\infty e^{i\lambda a}\Big(\lambda a + \frac{it}{2}\Big)^{-\frac{5}{2}} \lambda d\lambda = \int_0^\infty e^{-i\lambda a}\Big(-\lambda + \frac{it}{2}\Big)^{-\frac{5}{2}} \lambda d\lambda$$

and the both sides are bounded in modulus by $Ca^{-2}t^{-\frac{1}{2}}$. Thus, we have

$$\int_0^\infty G_{\frac{m+2}{2}}(\lambda, \cdot, y)\lambda d\lambda = \int_0^\infty G_{\frac{m+2}{2}}(-\lambda, \cdot, y)\lambda d\lambda \tag{4.48}$$

and both sides are bounded in modulus by $C\langle z-y\rangle^{-\frac{m-2}{2}}$. It follows that (4.47) with this change is bounded in modulus by

$$C\left|\frac{1}{(|y|-|x|)^{\frac{m+2}{2}}} - \frac{1}{(|y|+|x|)^{\frac{m+2}{2}}}\right| \frac{1}{\langle y\rangle^{\frac{m-2}{2}}\langle x\rangle^{m-2}}$$

and is admissible. This completes the proof of Lemma 4.8 and therefore that of Proposition 4.2. □

## 5. Low Energy Estimate II, Exceptional Case

In this section we discuss the low energy part $W_<$ in the case when $H$ is of exceptional type, assuming that

(5.1) $|V(x)| \le C\langle x\rangle^{-\delta}$ with $\delta > m+3$ if $m \ge 8$ and $\delta > m+4$ if $m = 6$.

so that we may apply Proposition 3.6. We substitute (3.5) when $m = 6$ or (3.7) when $m = 8$ for $L(\lambda) = (1 + G_-0(\lambda)V)^{-1} - I$ in formula (4.3). As $VE(\lambda)$ satisfies property $(K)_\rho$ with $\rho > m+1$, Proposition 4.2 implies that (4.3) with $E(\lambda)$ in place of $L(\lambda)$ produces an operator with admissible integral kernel. Thus, we have only to discuss the operators which are defined by (4.3) by replacing $L(\lambda)$ by its singular parts $\lambda^{-2}P_0V$ and $\sum_{ab}\lambda^a(\log\lambda)^b D_{ab}$ (note that we have changed indices $j, k$ to $a, b$). In Subsection 5.1 we prove that

$$W_{s,m} = \int_0^\infty G_0(\lambda)VP_0V(G_0(\lambda) - G_0(-\lambda))\tilde{\Phi}(\lambda)\lambda^{-1}d\lambda, \tag{5.2}$$

is bounded in $L^p$ for $\frac{m}{m-2} < p < \frac{m}{2}$ and in Subsection 5.2 we indicate how the argument can be modified to prove the same for $\Phi(H)W_{s,ab}\Phi(H_0)$, where

$$W_{s,ab} = \int_0^\infty G_0(\lambda)VD_{ab}(G_0(\lambda) - G_0(-\lambda))\tilde{\Phi}(\lambda)\lambda^{a+1}(\log\lambda)^b\, d\lambda. \tag{5.3}$$

### 5.1. Estimate for $W_{s,m}$

In this subsection, we prove the following proposition. We shall often write $\nu = (m-2)/2$.

Proposition 5.1. *Let $V$ satisfy* (5.1)*. Then, for any $\frac{m}{m-2} < p < \frac{m}{2}$, there exists a constant $C_p$ such that*

$$\|W_{s,m}u\|_p \le C_p \|u\|_p, \quad u \in C_0^\infty(\mathbf{R}^m). \tag{5.4}$$

We first state two lemmas which we use for proving the proposition. The first one may be found in [20].

Lemma 5.2. (1) *The function $|r|^a$ on $\mathbf{R}$ is an $(A)_p$ weight if and only if $-1 < a < p-1$. The Hilbert transform $\tilde{\mathcal{H}}$ and the Hardy-Littlewood maximal operator $\mathcal{M}$ are bounded in $L^p(\mathbf{R}, w(r)dr)$ for $(A)_p$ weights $w(r)$.*
(2) *Let a function $F(x)$ on $\mathbf{R}^m$ has a spherically symmetric decreasing integrable majorant, then*

$$|F * g(x)| \le C\mathcal{M}g(x), \quad x \in \mathbf{R}^m$$

*for a constant depending only on $F$.*

For a function $u$ on $\mathbf{R}^m$, $M(r,u)$ is the spherical average of $u$:

$$M(r,u) = \frac{1}{|\Sigma|}\int_\Sigma u(r\omega)d\omega, \quad r \in \mathbf{R}.$$

Lemma 5.3. *Let $m \ge 3$. Let $\psi \in L^1(\mathbf{R}^m)$ and $u \in \mathcal{S}(\mathbf{R}^m)$. Then*

$$\begin{aligned} F_{\psi,u}(\lambda) &= \langle \psi, (G_0(\lambda) - G_0(-\lambda))u\rangle \\ &= C_m \int_0^\infty e^{-t} t^{\frac{m-3}{2}} \left( \int_{\mathbf{R}} e^{-i\lambda r}(t + 2i\lambda r)^{\frac{m-3}{2}} rM(r, \overline{\psi} * \check{u})dr \right) dt \end{aligned} \tag{5.5}$$

*where $\check{u}(x) = u(-x)$ and $C_m > 0$ is a constant. Because of the choice of the branch of the square root in the Green kernel* (1.14)*, $\Re(t+2i\lambda r)^{\frac12} > 0$ for $t > 0$ and $\lambda \in \mathbf{R}$.*

PROOF. By Fubini's theorem and by using polar coordinates,

$$
\begin{aligned}
\langle \psi, G_0(\lambda)u\rangle &= \int_{\mathbf{R}^m} G_0(\lambda, y)(\psi * \check{u})(y)dy \\
&= -C_m \int_0^\infty e^{-t}t^{\nu-\frac12}\left(\int_0^\infty e^{i\lambda r}(t-2i\lambda r)^{\nu-\frac12} rM(r, \overline{\psi}*\check{u})dr\right)dt.
\end{aligned}
$$

Since $M(r) = M(-r)$, it follows that $-\langle \psi, G_0(-\lambda)u\rangle$ is given by

$$
\begin{aligned}
&C_m \int_0^\infty e^{-t}t^{\nu-\frac12}\left(\int_0^\infty e^{-i\lambda r}(t+2i\lambda r)^{\nu-\frac12} rM(r, \overline{\psi}*\check{u})dr\right)dt \\
&\qquad = -C_m \int_0^\infty e^{-t}t^{\nu-\frac12}\left(\int_{-\infty}^0 e^{i\lambda r}(t-2i\lambda r)^{\nu-\frac12} rM(r, \overline{\psi}*\check{u})dr\right)dt.
\end{aligned}
$$

Adding the two equations and changing $r \to -r$, we obtain the lemma. □

For fixed $f, g \in L^1(\mathbf{R}^m)$, we define the operator $Z = Z(f \otimes g)$ by

$$
Z(f\otimes g)u = \int_0^\infty G_0(\lambda)(f\otimes g)\left(G_0(\lambda) - G_0(-\lambda)\right)\tilde{\Phi}(\lambda)\lambda^{-1}d\lambda \tag{5.6}
$$

If we write $P_0 = \sum_{j=1}^d \phi_j \otimes \phi_j$ in terms of an orthonormal basis of $P_0\mathcal{H}$, we have $W_{s,m} = \sum_{j=1}^d Z((V\phi_j)\otimes(V\phi_j))$.

LEMMA 5.4. *With suitable constants $C_{jk}$ we have*

$$
Zu(x) = \sum_{j,k=0}^{\frac{m-4}{2}} C_{jk}\int_{\mathbf{R}^m} \frac{f(y)K_{jk}u(|x-y|)}{|x-y|^{m-2}}dy \tag{5.7}
$$

*where with $M(r, \overline{g}*\check{u}) = M(r)$, $K_{jk}u(|x-y|)$, $0 \le j,k \le \frac{m-4}{2}$, are defined by*

$$
\begin{aligned}
K_{jk}u(\rho) = \rho^j \int_0^\infty e^{i\lambda\rho}\lambda^{j+k-1}\tilde{\Phi}(\lambda)\Big\{\int_0^\infty\int_0^\infty e^{-(t+s)}t^{2\nu-\frac32-k}s^{2\nu-\frac32-j} \\
\times (s-2i\lambda\rho)^{\frac12}\Big(\int_{\mathbf{R}} e^{-i\lambda r}(t+2i\lambda r)^{\frac12}r^{k+1}M(r)dr\Big)dtds\Big\}d\lambda.
\end{aligned} \tag{5.8}
$$

PROOF. We remark that, for $u \in C_0^\infty(\mathbf{R}^m)$, (5.8) is well defined for all $j, k$ because $M(r)$ is smooth, $\langle r\rangle^{m-1}M^{(\ell)}(r)$ is integrable and $\int_{\mathbf{R}} rM(r)dr = 0$ because $M(r)$ is even. By virtue of (5.5), we have

$$Zu(x) = \int_0^\infty G_0(\lambda)f(x) \cdot F_{g,u}(\lambda)d\lambda$$

We substitute the right side of (5.5) for $F_{g,u}(\lambda)$ and the expression

$$\frac{1}{(4\pi)^{\frac{m-3}{2}}\Gamma(\frac{m+2}{2})}\int_{\mathbf{R}^m}\frac{e^{i\lambda|x-y|}f(y)}{|x-y|^{m-2}}\left(\int_0^\infty e^{-s}s^{\frac{m-3}{2}}(s-2i\lambda|x-y|)^{\frac{m-3}{2}}ds\right)dy$$

for $G_0(\lambda)f(x)$. We then change the order of integrations with respect to $d\lambda$ and $dy$ and, using the binomial formula, write

$$(t+2i\lambda r)^{\frac{m-3}{2}} = \sum_{j=0}^{\frac{m-4}{2}}\binom{\frac{m-4}{2}}{j}t^{\frac{m-4}{2}-j}(2i\lambda r)^j(t+2i\lambda r)^{\frac{1}{2}}$$

and similarly for $(s-2i\lambda\rho)^{\frac{m-3}{2}}$. The lemma follows. $\square$

In what follows we assume $f$ and $g$ satisfy for some $\varepsilon > 0$ and $C > 0$

$$|f(x)| \le C\langle x\rangle^{-m-\varepsilon}, \quad |g(x)| \le C\langle x\rangle^{-m-\varepsilon} \tag{5.9}$$

and define operators $W_{jk}$ for $0 \le j, k \le \frac{m-4}{2}$ by

$$W_{jk}u(x) = \int_{\mathbf{R}^m}\frac{f(y)K_{jk}u(|x-y|)}{|x-y|^{m-2}}dy \tag{5.10}$$

so that $Z = \sum C_{jk}W_{jk}$. We use the following lemma.

LEMMA 5.5. *Let $M(r) = M(r, \overline{g} * u)$. Then:*

(1) *For $1 \le p \le \infty$,* $\left(\int_{\mathbf{R}}|r|^{m-1}|M(r)|^p dr\right)^{\frac{1}{p}} \le C\|g\|_1\|u\|_p$.

(2) *For $1 \le p < \frac{m}{2}$,* $\int_{\mathbf{R}}\langle r\rangle|M(r)|dr \le C\|u\|_p$.

PROOF. By Hölder's and Hausdorff-Young's inequalities we have

$$\left(\int_{\mathbf{R}} |r|^{m-1}|M(r)|^p dr\right)^{\frac{1}{p}} \le C\|\overline{g} * u\|_p \le C\|g\|_1\|u\|_p$$

and (1) follows. Let $\frac{m}{m-2} < q \le \infty$ be the dual exponent of $p$, $1/q = p/p-1$ and $h(x) = \langle x\rangle|x|^{1-m}$. Then, $h \in L^q(|x| > 1)$ and $h \in L^1(|x| < 1)$, hence $\|h * |g|\|_q \le C(\|g\|_1 + \|g\|_q)$. It follows that

$$\begin{aligned}\int_{\mathbf{R}} \langle r\rangle|M(r)|dr &\le C\int_{\mathbf{R}^m} \frac{\langle x\rangle|(\overline{g} * \check{u})(x)|}{|x|^{m-1}}dx \\ &\le C\int_{\mathbf{R}^m}(h * |g|)(y)|u(y)|dy \le C(\|g\|_1 + \|g\|_q)\|u\|_p. \ \square\end{aligned}$$

It is clear that $V\phi_j$, $j = 1, \dots, d$, satisfy the condition (5.9) and Proposition 5.1 follows from the following proposition.

PROPOSITION 5.6. *Let $f, g$ satisfy* (5.9). *Then, $W_{jk}$, $0 \le j, k \le \frac{m-4}{2}$, are bounded in $L^p(\mathbf{R}^m)$ for $\frac{m}{m-2} < p < \frac{m}{2}$.*

We prove Proposition 5.6 for various cases of $j, k$ separately. By interpolation, we have only to show Proposition 5.6 for $p = \frac{m}{m-2-\varepsilon}$ and $p = \frac{m}{2+\varepsilon}$ with arbitrary small $\varepsilon > 0$. We denote the Hilbert transform by $\tilde{\mathcal{H}}$ and

$$\mathcal{H} = (1 + \tilde{\mathcal{H}})/2. \tag{5.11}$$

By Lemma 5.2 $|r|^{m-1-p\theta}$ is a one dimensional $(A)_p$ weight if and only if $0 < \frac{m}{p} - \theta < 1$, viz.

$$\begin{aligned} &m - 3 - \varepsilon < \theta < m - 2 - \varepsilon \quad \text{if } p = m/(m-2-\varepsilon), \\ &1 + \varepsilon < \theta < 2 + \varepsilon \qquad\qquad\quad \text{if } p = m/(2+\varepsilon). \end{aligned} \tag{5.12}$$

**(1) The case $j, k \ge 1$.** If $1 \le j, k \le \frac{m-4}{2}$ the integrand of (5.8) is integrable with respect to $dtdsdrd\lambda$ and we are free to change the order of integration. Thus, we may write

$$K_{j,k}u(\rho) = \int_{\mathbf{R}} M(r)T_{jk}(\rho, r)dr, \tag{5.13}$$

$$T_{jk}(\rho, r) = \int_0^\infty \int_0^\infty e^{-(t+s)} t^{2\nu-\frac{3}{2}-k} s^{2\nu-\frac{3}{2}-j} J_{jk}(s,t,\rho,r) dt\, ds, \tag{5.14}$$

$$J_{jk} = \rho^j r^{k+1} \int_0^\infty e^{i\lambda(\rho-r)} \lambda^{j+k-1} \tilde{\Phi}(\lambda)(s-2i\lambda\rho)^{\frac{1}{2}}(t+2i\lambda r)^{\frac{1}{2}} d\lambda. \tag{5.15}$$

LEMMA 5.7. *Let $j,k \geq 1$. Then, with a constant $C = C_{jk}$, we have*

$$|T_{jk}(\rho, r)| \leq C \left| \frac{\rho^j \langle\rho\rangle^{1/2} r^{k+1} \langle r\rangle^{1/2}}{\langle r-\rho\rangle^{j+k}} \right| \tag{5.16}$$

*Estimate* (5.16) *remains to hold if $\tilde{\Phi}$ is replaced by any smooth function with compact support and $t^{2\nu-\frac{3}{2}-k}$ and/or $s^{2\nu-\frac{3}{2}-j}$ by $t^a$ and/or $s^b$ with $a, b \geq 0$.*

PROOF. Since (5.16) is obvious for $|\rho - r| \leq 1$, we prove it only for $|\rho - r| \geq 1$. By integrating by parts $j+k$ times with respect to $\lambda$, we have

$$\begin{aligned} J_{jk}(s,t,\rho,r) &= \frac{i^{j+k}\sqrt{st}\rho^j r^{k+1}}{(\rho-r)^{j+k}}(j+k-1)! + \frac{(-i)^{j+k}\rho^j r^{k+1}}{(\rho-r)^{j+k}} \\ &\quad \times \int_0^\infty e^{i\lambda(\rho-r)} \left\{\lambda^{j+k-1}\tilde{\Phi}(\lambda)(s-2i\lambda\rho)^{1/2}(t+2i\lambda r)^{1/2}\right\}^{(j+k)} d\lambda. \end{aligned}$$

We insert this into (5.14). The boundary term produces

$$T_{jk,b}(\rho, r) = C\frac{\rho^j r^{k+1}}{(r-\rho)^{j+k}}, \tag{5.17}$$

which satisfies (5.16). We compute the derivative via Leibniz' rule:

$$\begin{aligned} &\left(\frac{d}{d\lambda}\right)^{j+k} \left\{\lambda^{j+k-1}\tilde{\Phi}(\lambda)(s-2i\lambda\rho)^{1/2}(t+2i\lambda r)^{1/2}\right\} \\ &\qquad = \sum_{a+b+c=j+k} C_{abc}\Psi_a(\lambda)(s-2i\lambda\rho)^{1/2-b}(2i\rho)^b(t+2i\lambda r)^{1/2-c}(2ir)^c \end{aligned}$$

where $\Psi_a(\lambda) = \{\lambda^{j+k-1}\tilde{\Phi}(\lambda)\}^{(a)}$. Denoting summands on the right by $E_{abc} = E_{abc}(\lambda, s, t, \rho, r)$, we define

$$J_{abc}(s,t,\rho,r) \equiv \frac{i^{j+k}\rho^j r^{k+1}}{(\rho-r)^{j+k}} \int_0^\infty e^{i\lambda(\rho-r)} E_{abc}(\lambda, s, t, \rho, r) d\lambda, \tag{5.18}$$

and $T_{abc}(\rho, r)$ by the right of (5.14) with $J_{abc}$ replacing $J_{jk}$. By using obvious estimates $|(s-2i\lambda\rho)^{-1}(2i\lambda\rho)| \le 1$ and $|(t+2i\lambda r)^{-1}(-2i\lambda r)| \le 1$, we obtain

$$|E_{abc}| \le C|\Psi_a(\lambda)| \times \begin{cases} (s^{\frac12} + |\lambda\rho|^{\frac12})(t^{\frac12} + |\lambda r|^{\frac12}), & \text{if } b = c = 0; \\ (\rho r)^{\frac12} |\lambda|^{1-(b+c)}, & \text{if } b, c \ne 0; \\ r^{\frac12}(s^{\frac12} + |\lambda\rho|^{\frac12})|\lambda|^{\frac12 - c}, & \text{if } b = 0, c \ne 0; \\ r^{\frac12}(t^{\frac12} + |\lambda\rho|^{\frac12})|\lambda|^{\frac12 - b}, & \text{if } b \ne 0, c = 0. \end{cases}$$

Note that $\lambda^{1-(b+c)}\Psi_a(\lambda)$, $\lambda^{\frac12 - b}\Psi_a(\lambda)$ and $\lambda^{\frac12 - c}\Psi_a(\lambda)$ are integrable functions with compact supports in respective cases. It immediately follows that

$$|T_{abc}(\rho, r)| \le C \frac{\rho^j \langle\rho\rangle^{\frac12} r^{k+1} \langle r\rangle^{\frac12}}{\langle\rho - r\rangle^{j+k}} \tag{5.19}$$

and, by summing up, we obtain (5.16). The only property of $\tilde{\Phi}$ which is used in the argument above is that it is smooth and compactly supported; all estimate above go through if $e^{-t}t^{2\nu-\frac32-k}$ or $e^{-s}s^{2\nu-\frac32-j}$ are replaced by $e^{-t}t^a$ or $e^{-s}s^b$, $a, b \ge 0$, and the last statement of the lemma follows. □

LEMMA 5.8. *Let $j, k \ge 1$ and let $T_{jk}(\rho, r)$ satisfy* (5.16)*. Let $W_{jk}$ be defined by* (5.10) *with $K_{jk}$ given by* (5.13)*. Then $W_{jk}$ satisfies Proposition 5.6.*

PROOF. By splitting the domain of integration, we estimate

$$\begin{aligned} |W_{jk}u(x)| &\le \left( \int_{|x-y|<1} + \int_{|x-y|\ge 1} \right) \frac{|f(y)K_{jk}(|x-y|)|}{|x-y|^{m-2}} dy \\ &\equiv I_1(x) + I_2(x) \end{aligned} \tag{5.20}$$

By Young's inequality, (5.16) and Lemma 5.5 (2), we have for any $1 \le p < \frac{m}{2}$ that

$$\begin{aligned} \|I_1\|_p &\le C\|f\|_p \left( \int_{|x|\le 1} \frac{|K_{jk}(x)|}{|x|^{m-2}} dx \right) \le C \sup_{|\rho|<1} |K_{jk}(\rho)| \\ &\le C \sup_{0\le\rho\le 1} \int_{\mathbf{R}} \frac{|r|^{k+1}\langle r\rangle^{\frac12}}{\langle r-\rho\rangle^{j+k}} |M(r)| dr \le C \int_{\mathbf{R}} |r\, M(r)| dr \le C\|u\|_p. \end{aligned}$$

We estimate $I_2(x)$. Let $p = \frac{m}{2+\varepsilon}$, $0 < \varepsilon < 1$ and choose $\theta = 2$, see (5.12). By Young's inequality we have by using polar coordinates that

$$\begin{aligned} \|I_2\|_p &\le \|f\|_1 \left( \int_1^\infty \rho^{m-1} \left| \frac{1}{\rho^{m-2}} \int_{\mathbf{R}} T_{jk}(\rho, r) M(r) dr \right|^p d\rho \right)^{\frac{1}{p}} \\ &\le C \int_1^\infty \rho^{m-1-p\theta} \left( \frac{\rho^{j+\frac12}}{\rho^{m-4}} \int_{\mathbf{R}} dr \, \frac{|r|^{k+1} \langle r \rangle^{\frac12}}{\langle r-\rho \rangle^{j+k}} |M(r)| \right)^p d\rho. \end{aligned} \tag{5.21}$$

Since $|r|^{k-1}\langle r\rangle^{\frac12} \le C\langle r-\rho\rangle^{k-\frac12}\rho^{k-\frac12}$ for $\rho > 1$ (recall $k \ge 1$) and $m-4 \ge j+k$,

$$\frac{\rho^{j+\frac12}|r|^{k+1}\langle r\rangle^{\frac12}}{\rho^{m-4}\langle r-\rho\rangle^{j+k}} \le C\frac{\rho^{j+k}|r|^2}{\rho^{m-4}\langle r-\rho\rangle^{j+\frac12}} \le C\frac{|r|^2}{\langle r-\rho\rangle^{j+\frac12}}, \quad \rho > 1.$$

Hence, the right of (5.21) is bounded by

$$C\int_1^\infty \rho^{m-1-p\theta}\mathcal{M}(|r|^2 M)(\rho)^p d\rho \le C\left(\int_{\mathbf{R}} r^{m-1}|M(r)|^p dr\right)^{\frac1p} \le C\|u\|_p$$

by virtue of the weighted inequality for the maximal functions and by Lemma 5.5 (1).

When $p = \frac{m}{m-2-\varepsilon}$, $0 < \varepsilon < 1$, we choose $\theta = m-3$. Again by using Young's inequality and (5.16)

$$\|I_2\|_p^p \le C\int_1^\infty \rho^{m-1-p\theta}\left(\rho^{j-\frac12}\int_{\mathbf{R}} \frac{|r|^{k+1}\langle r\rangle^{\frac12}}{\langle r-\rho\rangle^{j+k}}|M(r)|dr\right)^p d\rho \tag{5.22}$$

Since $\rho^{j-\frac12} \le \langle r\rangle^{j-\frac12}\langle r-\rho\rangle^{j-\frac12}$ and $m-1-p\theta$ is an $(A)_p$ weight, the right hand side is further estimated by

$$\begin{aligned} &C\int_1^\infty \rho^{m-1-p\theta}\left(\int_{\mathbf{R}} \frac{|r|^{k+1}\langle r\rangle^j}{\langle r-\rho\rangle^{k+\frac12}}|M(r)|dr\right)^p d\rho \\ &\qquad \le C\int_1^\infty \rho^{m-1-p\theta}\mathcal{M}(|r|^{k+1}\langle r\rangle^j M(r))(\rho)^p d\rho \\ &\qquad\qquad \le C\int_{\mathbf{R}} |r|^{m-1-p\theta}(|r|^{k+1}\langle r\rangle^j M(r))^p dr \end{aligned}$$

Since $k+j+1 \le m-3=\theta$ and $p(\theta-k-1)<m$, the last integral is bounded by a constant time

$$\int_0^1 r^{m-1-p(\theta-k-1)}M(r)^p dr + \int_1^\infty r^{m-1}M(r)^p dr \\ \le C\int_{|x|<1} \frac{|(\overline{g}*\check{u})(x)|^p}{|x|^{p(\theta-k-1)}}dx + \|\overline{g}*\check{u}\|^p \le C(\|g\|_q+\|g\|_1)^p\|u\|_p^p,$$

where $q$ is the conjugate exponent of $p$. This completes the proof. □

**(2) The case** $j=0$**,** $k\ge 1$. We now prove Proposition 5.6 for $j=0$ and $1\le k\le \nu-1=\frac{m-4}{2}$ by induction on $k$, using also the already proven result for the case $j,k\ge 1$. For this and for dealing with the case (3) that $k=0$ and $j\ge 1$ below we define as follows.

Definition 5.9. (1) We say $J_{0k}(s,t,\rho,r)$ is *$\ell$-admissible* if operators $W_{0k,ln}$, $0\le l,n\le \nu-\ell$, defined by (5.10) with $K_{0k,ln}(\rho)=\int T_{0k,ln}(\rho,r)M(r)dr$ in place of $K_{jk}(\rho)$ are bounded in $L^p(\mathbf{R}^m)$ for $\frac{m}{m-2}<p<\frac{m}{2}$ where

$$T_{0k,ln}(\rho,r)=\int_0^\infty\int_0^\infty e^{-(t+s)}t^{2\nu-\frac{3}{2}-k+n}s^{2\nu-\frac{3}{2}-l}J_{0k}(s,t,\rho,r)dt\,ds. \tag{5.23}$$

(2) We say $J_{j0}(s,t,\rho,r)$ is *$\ell$-admissible* if operators $W_{j0,ln}$, $0\le l,n\le\nu-\ell$, defined by (5.10) with $K_{j0,ln}(\rho)=\int T_{j0,ln}(\rho,r)M(r)dr$ in place of $K_{jk}(\rho)$ are bounded in $L^p(\mathbf{R}^m)$ for $\frac{m}{m-2}<p<\frac{m}{2}$ where

$$T_{j0,ln}(\rho,r)=\int_0^\infty\int_0^\infty e^{-(t+s)}t^{2\nu-\frac{3}{2}-n}s^{2\nu-\frac{3}{2}-j+l}J_{j0}(s,t,\rho,r)dt\,ds. \tag{5.24}$$

Compare (5.23) or (5.24) with (5.14). Note that the exponents $2\nu-\frac{3}{2}-k+n, 2\nu-\frac{3}{2}-l$ are not smaller than 1 for all relevant $l,n$. It suffices to prove the following lemma.

Lemma 5.10. *Let $J_{0k}(s,t,\rho,r)$, $1\le k\le\nu-1$, be defined by* (5.15) *with $\tilde{\Phi}\in C_0^\infty(\mathbf{R})$ which is a constant near $\lambda=0$. Then, $J_{0k}(s,t,\rho,r)$ are $k$-admissible.*

PROOF. We prove the lemma by induction on $k$. We begin with the case $k = 1$.

LEMMA 5.11. *For all* $0 \le l, n \le \nu - 1$, $T_{01,ln}(\rho, r)$ *satisfies the estimate*

$$|T_{01,ln}(\rho, r)| \le C \frac{|r|^2(\langle\rho\rangle + \langle r\rangle)}{\langle r - \rho\rangle^2} \tag{5.25}$$

PROOF. This is obvious when $|\rho - r| \le 1$ and we assume $|\rho - r| > 1$ in what follows. Integrating by parts twice with respect to $\lambda$, we have

$$\begin{aligned} J_{01}(s,t,\rho,r) &= \frac{ir^2}{(\rho - r)}\sqrt{st} + \frac{ir^2}{(\rho - r)^2}(\rho(t/s)^{\frac12} - r(s/t)^{\frac12}) \\ &+ \frac{r^2}{(\rho - r)^2}\int_0^\infty e^{i\lambda(\rho - r)}\Big(\frac{\partial}{\partial\lambda}\Big)^2\Big(\tilde\Phi(\lambda)(s - 2i\lambda\rho)^{1/2}(t + 2i\lambda r)^{1/2}\Big)d\lambda \end{aligned} \tag{5.26}$$

We substitute this for $J_{01}$ in (5.23). Then the functions produced by the boundary terms are bounded by

$$C\Big(\frac{r^2}{\langle\rho - r\rangle} + \frac{r^2(|\rho| + |r|)}{\langle\rho - r\rangle^2}\Big) \le C\frac{|r|^2(\langle\rho\rangle + \langle r\rangle)}{\langle\rho - r\rangle^2}.$$

Denoting by $'$ the derivative with respect to the variable $\lambda$, we compute:

$$\begin{aligned} &\Big(\tilde\Phi(\lambda)(s - 2i\lambda\rho)^{1/2}(t + 2i\lambda r)^{1/2}\Big)'' = \tilde\Phi''(\lambda)(s - 2i\lambda\rho)^{1/2}(t + 2i\lambda r)^{1/2} \\ &+ 2\tilde\Phi'(\lambda)\Big((s - 2i\lambda\rho)^{1/2}(t + 2i\lambda r)^{1/2}\Big)' + \tilde\Phi(\lambda)\Big((s - 2i\lambda\rho)^{1/2}(t + 2i\lambda r)^{1/2}\Big)''. \end{aligned}$$

Since $\tilde\Phi'(\lambda) = 0$ near $\lambda = 0$ and $|(s - 2i\lambda\rho)^{-\frac12}2\rho| \le C(\rho/s)^{\frac12}$ for $|\lambda| \ge C > 0$, this is bounded in modulus by a constant time

$$\begin{aligned} &|\tilde\Phi''(\lambda)|(s^{\frac12} + |\lambda\rho|^{\frac12})(t^{\frac12} + |\lambda r|^{\frac12}) + |\tilde\Phi'(\lambda)||\rho|^{\frac12}(t^{\frac12} + |\lambda r|^{\frac12}) \\ &+ |\tilde\Phi'(\lambda)||r|^{\frac12}(s^{\frac12} + |\lambda\rho|^{\frac12}) + |\tilde\Phi(\lambda)||t\rho - sr|^2(s + |\lambda r|)^{-\frac32}(t + |\lambda\rho|)^{-\frac32}. \end{aligned}$$

Hence integrating by $dtds$ first and using also elementary estimates

$$\left|\int_0^\infty \frac{e^{-t}t^a dt}{(t + |\lambda r|)^b}\right| \le C\langle\lambda r\rangle^{-b} \qquad 0 < b < a + 1, \tag{5.27}$$

$$\left|\int_0^\infty \frac{|\tilde\Phi(\lambda)|}{\langle\lambda r\rangle^a\langle\lambda\rho\rangle^b}d\lambda\right| \le C\frac{1}{\langle r\rangle+\langle\rho\rangle}, \quad a,b>0, a+b>1, \tag{5.28}$$

we obtain estimate (5.25). □

LEMMA 5.12. *Let* $J_{01}(s,t,\rho,r)$ *be defined by* (5.15) *with* $\tilde\Phi \in C_0^\infty(\mathbf{R})$ *which is a constant near* $\lambda=0$. *Then,* $J_{01}(s,t,\rho,r)$ *is* 1*-admissible.*

PROOF. We have $\dfrac{|r|^2(\langle\rho\rangle+\langle r\rangle)}{\langle r-\rho\rangle^2} \le \dfrac{2|r|^2\langle\rho\rangle}{\langle r-\rho\rangle^2}+\dfrac{|r|^2}{\langle r-\rho\rangle}$. The first term on the right satisfies (5.16) with $j=k=1$ and, by virtue of Lemma 5.8, it suffices to show that $W_{01}$ is bounded in $L^p(\mathbf{R}^m)$ for $\frac{m}{m-2}<p<\frac{m}{2}$ if

$$|T_{01}(\rho,r)| \le C\frac{|r|^2}{\langle r-\rho\rangle} \le C\left(|r|+\frac{|r||\rho|}{\langle\rho-r\rangle}\right). \tag{5.29}$$

We estimate $|W_{01}u(x)| \le I_1(x)+I_2(x)$ as in (5.20). For $I_1(x)$ we use the first of (5.29) and proceed as in the proof of Lemma 5.8. We have

$$\|I_1\|_p \le C\|f\|_p \sup_{|\rho|\le 1}\int_{\mathbf{R}} \frac{|r|^2|M(r)|dr}{\langle\rho-r\rangle} \le C\|f\|_p\int_{\mathbf{R}}|r||M(r)|dr \le C\|u\|_p.$$

For $I_2(x)$ we use Young's inequality and the second of (5.29) to obtain:

$$\begin{aligned}\|I_2\|_p \le \|f\|_1 &\left(\int_1^\infty \rho^{m-1}\left|\frac{1}{\rho^{m-2}}\int_{\mathbf{R}}|r||M(r)|dr\right|^p d\rho\right)^{\frac1p}\\ &+\|f\|_1\left(\int_1^\infty \rho^{m-1}\left|\frac{1}{\rho^{m-3}}\int_{\mathbf{R}}\frac{|r||M(r)|}{\langle\rho-r\rangle}dr\right|^p d\rho\right)^{\frac1p}\end{aligned} \tag{5.30}$$

The first term is bounded by $C\int_{\mathbf{R}}|r||M(r)|dr \le C\|u\|_p$ since $p(m-2)>m$ for $\frac{m}{m-2}<p<\frac{m}{2}$. For estimating the second, take $\varepsilon>0$ arbitrarily small and fix $p\in(\frac{m}{m-2-\varepsilon},\frac{m}{2+\varepsilon})$. Take $0<\varepsilon'<\varepsilon$ and choose $\frac{m}{p}-1<\theta<\frac{m}{p}$ sufficiently close to $\frac{m}{p}-1$ so that $m-1-p\theta$ is an $(A)_p$ weight and so that $1+\varepsilon'<\theta\le m-3-\varepsilon'$. Then, using $\langle\rho-r\rangle^{-1}\le C_{\varepsilon'}\langle\rho\rangle^{\varepsilon'}\langle r\rangle^{\varepsilon'}\langle\rho-r\rangle^{-(1+\varepsilon')}$, we estimate the second integral by a constant time

$$\left(\int_1^\infty \rho^{m-1}\left(\frac{1}{\rho^{m-3-\varepsilon'}}\int_{\mathbf{R}}\frac{|r|\langle r\rangle^{\varepsilon'}|M(r)|}{\langle \rho-r\rangle^{1+\varepsilon'}}dr\right)^p d\rho\right)^{\frac{1}{p}}$$
$$\le C\left(\int_1^\infty \rho^{m-1-p\theta}\left(\int_{\mathbf{R}}\frac{|r|\langle r\rangle^{\varepsilon'}|M(r)|}{\langle \rho-r\rangle^{1+\varepsilon'}}dr\right)^p d\rho\right)^{\frac{1}{p}}$$
$$\le C\left(\int_{\mathbf{R}} r^{m-1-p(\theta-1)}\langle r\rangle^{p\varepsilon'}|M(r)|^p dr\right)^{\frac{1}{p}}$$
$$\le C\left(\int_{\mathbf{R}^m}\frac{\langle x\rangle^{p\varepsilon'}|\overline{g}*u(x)|}{|x|^{p(\theta-1)}}dx\right)^{\frac{1}{p}}.$$

Since $p\varepsilon' < p(\theta-1) < m$, the right hand side is bounded by $C\|u\|_p$. This proves that $\|I_2\|_p \le C\|u\|_p$. This completes the proof. □

**Completion of the Proof of Lemma 5.10.** The lemma is satisfied when $k=1$ by virtue of Lemma 5.11. We let $k\ge 2$ and assume that the lemma is already proved for smaller values of $k$. We write $r^{k+1} = r^k\rho - r^k(\rho-r)$ in the definition (5.15) for $J_{0k}(s,t,\rho,r)$ and apply integration by part to the integral containing $r^k(\rho-r)$. We obtain

$$J_{0k}(s,t,\rho,r) = J_{1(k-1)}(s,t,\rho,r) - ir^k\int_0^\infty e^{i\lambda(\rho-r)}\Big(\frac{\partial}{\partial\lambda}\Big)\Big(\lambda^{k-1}\tilde{\Phi}(\lambda)(s-2i\lambda\rho)^{\frac12}(t+2i\lambda r)^{\frac12}\Big)d\lambda. \tag{5.31}$$

Thanks to results in case (1), $J_{1(k-1)}(s,t,\rho,r)$ is $(k-1)$-admissible and it may be ignored. We insert the following for the derivative in the integrand:

$$(k-1)\lambda^{k-2}\tilde{\Phi}(s-2i\lambda\rho)^{\frac12}(t+2i\lambda r)^{\frac12} + \lambda^{k-1}\tilde{\Phi}'(s-2i\lambda\rho)^{\frac12}(t+2i\lambda r)^{\frac12}$$
$$-2i\rho\lambda^{k-1}\tilde{\Phi}\Big(\frac{\partial}{\partial s}\Big)(s-2i\lambda\rho)^{\frac12}(t+2i\lambda r)^{\frac12} + \lambda^{k-1}\tilde{\Phi}(s-2i\lambda\rho)^{\frac12} ir(t+2i\lambda r)^{-\frac12}$$

The first term produces $(k-1)J_{0(k-1)}(s,t,\rho,r)$, which is $(k-1)$-admissible by induction hypothesis; the second does $J_{0(k-1)}(s,t,\rho,r)$ with $\lambda\tilde{\Phi}'(\lambda)$ replacing $\tilde{\Phi}$, which is also $(k-1)$-admissible since $\lambda\tilde{\Phi}'(\lambda)=0$ near $\lambda=0$. Define

$$J_{0k(3)} \equiv 2r^k\rho\int_0^\infty e^{i\lambda(r-\rho)}\lambda^{k-1}\tilde{\Phi}(\lambda)\Big(\frac{\partial}{\partial s}\Big)(s-2i\lambda\rho)^{\frac12}\cdot(t+2i\lambda r)^{\frac12}d\lambda$$

and substitute this for $J_{jk}(s,t,\rho,r)$ in (5.23). This yields after integration by parts with respect to the $s$ variable

$$-2T_{1(k-1),ln}(\rho,r)+2(2\nu-\tfrac{3}{2}-l)T_{1(k-1),(l+1)n}(\rho,r)$$

and the result of case (1) implies $J_{0k(3)}(s,t,\rho,r)$ is $k$-admissible. We rewrite the last term $\lambda^{k-1}\tilde{\Phi}(s-2i\lambda\rho)^{\frac{1}{2}}ir(t+2i\lambda r)^{-\frac{1}{2}}$ in the form

$$\tfrac{1}{2}\lambda^{k-2}\tilde{\Phi}(\lambda)(s-2i\lambda\rho)^{\frac{1}{2}}(t+2i\lambda r)^{\frac{1}{2}}-\lambda^{k-2}\tilde{\Phi}(\lambda)(s-2i\lambda\rho)^{\frac{1}{2}}t\Big(\frac{\partial}{\partial t}\Big)(t+2i\lambda r)^{\frac{1}{2}}$$

The first term again produces $\frac{1}{2}J_{0(k-1)}(s,t,\rho,r)$. Define

$$\begin{aligned}J_{0k(4)}(s,t,\rho,r)\equiv r^k\rho\int_0^\infty & e^{i\lambda(r-\rho)}\lambda^{k-2}\tilde{\Phi}(\lambda)(s-2i\lambda\rho)^{\frac{1}{2}}\\ &\cdot\Big(\frac{\partial}{\partial t}\Big)(t+2i\lambda r)^{\frac{1}{2}}d\lambda\end{aligned}$$

and substitute $J_{0k(4)}(s,t,\rho,r)$ for $J_{jk}(s,t,\rho,r)$ in (5.23). This yields, after integration by parts with respect to the $t$ variable,

$$-T_{0(k-1),l(n+1)}(\rho,r)+(2\nu-\tfrac{1}{2}-k+n)T_{0(k-1),ln}(\rho,r)$$

It follows by induction hypothesis that $J_{0k(4)}(s,t,\rho,r)$ is also $k$-admissible. This completes the proof. □

**(3) The case** $j\geq 1$ **and** $k=0$. We next prove Proposition 5.6 for $j\geq 1$ and $k=0$. It suffices to prove the following lemma.

LEMMA 5.13. *Let* $J_{j0}(s,t,\rho,r)$, $j=1,\dots,\nu-1$, *be defined by* (5.15) *with* $\tilde{\Phi}\in C_0^\infty(\mathbf{R})$ *which is a constant near* $\lambda=0$. *Then,* $J_{j0}(s,t,\rho,r)$ *are* $j$*-admissible.*

PROOF. We prove the lemma by induction on $j$. Thus, we let $j=1$ first. Comparing definitions of $J_{01}$ and $J_{10}$ and (5.25), it is obvious that

$$|T_{10,(ln)}(\rho,r)|\leq C\frac{|r|\rho(\langle\rho\rangle+\langle r\rangle)}{\langle\rho-r\rangle^2}$$

Since $\langle\rho\rangle + \langle r\rangle \le 4(\langle\rho - r\rangle + |r|)$ and $|r|^2\rho\langle\rho-r\rangle^{-2}$ satisfies (5.16) with $j = k = 1$, it suffices to show that $W_{10}$ has the desired property when

$$|T_{10}(\rho,r)| \le C\frac{|r|\rho}{\langle\rho - r\rangle}.$$

However, the right hand side is the same as the second term on the right of (5.29) and the proof of Lemma 5.12 shows that Lemma 5.13 is satisfied when $j = 1$. We then let $j \ge 2$ and assume that the lemma is already proved for smaller values of $j$. We write $\rho^j r = \rho^{j-1}r^2 + \rho^{j-1}r(\rho - r)$ in the definition (5.15) of $J_{j0}(s,t,\rho,r)$. The first term $\rho^{j-1}r^2$ produces $J_{(j-1)1}(t,s,\rho,r)$ and we may ignore it by virtue of results in case (1). We need study the operators corresponding to $W_{jk}$ produced by functions

$$\rho^{j-1}r\int_0^\infty e^{i\lambda(\rho-r)}\Big(\frac{\partial}{\partial\lambda}\Big)\Big(\lambda^{k-1}\tilde\Phi(\lambda)(s-2i\lambda\rho)^{\frac12}(t+2i\lambda r)^{\frac12}\Big)d\lambda.$$

However, after this point the argument is completely in parallel with that of Lemma 5.10 after (5.31) and we omit the repetitious details. □

**(4) The case $j = k = 0$.** Finally we prove Proposition 5.6 for $j = k = 0$. Recall the definition (5.8) and (5.14). In (5.8) we substitute

$$(s-2i\lambda\rho)^{\frac12}((t+2i\lambda r)^{\frac12} - t^{\frac12}) + ((s-2i\lambda\rho)^{\frac12} - s^{\frac12})t^{\frac12} + s^{\frac12}t^{\frac12}.$$

for $(s-2i\lambda\rho)^{\frac12}(t+2i\lambda r)^{\frac12}$ and denote by $K_j$ the operator produced by $j$-th summand, $j = 1,2,3$, so that $K_{00} = K_1 + K_2 + K_3$. Define $W_j$ by (5.10) with $K_j$ in place of $K_{jk}$ so that $W_{00} = W_1 + W_2 + W_3$. For $j = 1$ and $j = 2$, we may change the order of integrations and write $K_j u(\rho)$ in the following form:

$$K_j u(\rho) = \int_{\mathbf R} T_j(\rho,r) r M(r)dr, \quad j = 1,2$$

where $T_1(\rho,r)$ and $T_2(\rho,r)$ are given by

$$\begin{aligned} T_1(\rho,r) = C_1 r\int_0^\infty\int_0^\infty & e^{-(t+s)}t^{2\nu-\frac32}s^{2\nu-\frac32}\times \\ &\times\left(\int_0^\infty e^{i\lambda(\rho-r)}\tilde\Phi(\lambda)\Big\{\frac{2ir(s-2i\lambda\rho)^{\frac12}}{(t+2i\lambda r)^{\frac12}+t^{\frac12}}\Big\}d\lambda\right)dsdt,\end{aligned}$$

$$T_2(\rho,r) = r\int_0^\infty e^{-s}s^{2\nu-\frac32}\left(\int_0^\infty e^{i\lambda(\rho-r)}\tilde\Phi(\lambda)\Big\{\frac{2iC_2\rho}{(s-2i\lambda\rho)^{\frac12}+s^{\frac12}}\Big\}d\lambda\right)ds.$$

LEMMA 5.14. *We have estimates*

$$|T_1(\rho,r)| \le C\left(|r| + \frac{|r|\rho}{\langle r-\rho\rangle} + \frac{|r|\langle\rho\rangle}{\langle r-\rho\rangle^2} + \frac{|r|^2\langle\rho\rangle^{\frac12}}{\langle r-\rho\rangle^2}\right), \tag{5.32}$$

$$|T_2(\rho,r)| \le C\frac{|r||\rho|}{\langle\rho-r\rangle}. \tag{5.33}$$

PROOF. The estimates are trivial for $|r-\rho|\le 1$ and we suppose $|\rho-r|>1$. We first prove (5.33). Integrating by parts, we estimate the inner integral by the boundary contribution $\rho|\rho-r|^{-1}s^{-\frac12}$ plus

$$\left|\frac{1}{\rho-r}\Big(\int_0^\infty\Big(\frac{2i\rho e^{i\lambda(r-\rho)}\tilde\Phi'(\lambda)}{(s-2i\lambda\rho)^{\frac12}+s^{\frac12}} + \frac{2i\rho e^{i\lambda(r-\rho)}\tilde\Phi(\lambda)(i\rho)}{((s-2i\lambda\rho)^{\frac12}+s^{\frac12})^2(s-2i\lambda\rho)^{\frac12}}\Big)d\lambda\Big)\right|$$
$$\le \frac{C\rho}{\langle r-\rho\rangle}\left(\frac{1}{\sqrt s}+\int_0^\infty\frac{\rho|\tilde\Phi(\lambda)|}{(|s|+|\lambda\rho|)^{\frac32}}d\lambda\right).$$

The desired estimate follows since

$$\int_0^\infty\left(\int_0^\infty e^{-s}s^{2\nu-\frac32}\frac{|\tilde\Phi(\lambda)|ds}{(|s|+|\lambda\rho|)^{\frac32}}\right)d\lambda \le \int_0^\infty\frac{Cd\lambda}{\langle\lambda\rho\rangle^{\frac32}} \le \frac{C}{\rho}. \tag{5.34}$$

For proving (5.32) for $T_1(\rho,r)$ we apply integration by parts twice to the inner integral. The result is

$$\frac{2r}{\rho-r}\sqrt{\frac{s}{2t}} - \frac{r}{(\rho-r)^2}\left(\frac{\rho}{\sqrt{ts}}+\frac{\sqrt{s}r}{t^{\frac23}}\right) \tag{5.35}$$
$$-\frac{ir}{(\rho-r)^2}\int_0^\infty e^{i\lambda(\rho-r)}\Big(\frac{\partial}{\partial\lambda}\Big)^2\Big(\tilde\Phi(\lambda)\frac{(s-2i\lambda\rho)^{\frac12}}{(t+2i\lambda r)^{\frac12}+t^{\frac12}}\Big)d\lambda.$$

We estimate the second derivative by a constant time

$$\frac{|\tilde{\Phi}''(\lambda)|(s+|\lambda\rho|)^{\frac12}}{\sqrt{t}}+\frac{\rho|\tilde{\Phi}'(\lambda)|}{\sqrt{st}}+\frac{|\tilde{\Phi}'(\lambda)|r(s+|\lambda\rho|)^{\frac12}}{(t+|\lambda r|)^{\frac32}} \\ +\frac{|\tilde{\Phi}(\lambda)|\rho^2}{\sqrt{t}(s+|\lambda\rho|)^{\frac32}}+\frac{|\tilde{\Phi}(\lambda)|r\rho}{\sqrt{s}(t+|\lambda r|)^{\frac32}}+\frac{|\tilde{\Phi}(\lambda)|r^2(s+|\lambda\rho|)^{\frac12}}{(t+|\lambda r|)^{\frac52}}$$

Desired estimate follows after integration via estimates similar to (5.34). □

LEMMA 5.15. *For $\frac{m}{m-2}<p<\frac{m}{2}$, $W_1$ and $W_2$ are bounded in $L^p(\mathbf{R}^m)$.*

PROOF. $T_1(\rho,r)$ is bounded by the right of (5.16) with $j=k=1$; $T_2(\rho,r)$ by the right of (5.29). The lemma follows from Lemma 5.8 and Lemma 5.12. □

Finally we deal with $W_3$. Recall that $K_3u(\rho)$ is defined by (5.8) with $(st)^{\frac12}$ replacing $(s-2i\lambda\rho)^{\frac12}(t+2i\lambda r)^{\frac12}$. Then, the inner most integral becomes $t$ independent and we may integrate out the $(t,s)$ integral. Result is

$$K_3u(\rho)=C\int_0^\infty \tilde{\Phi}(\lambda)e^{-i\lambda\rho}\lambda^{-1}\left(\int_{\mathbf{R}} e^{i\lambda r}rM(r)dr\right)d\lambda \tag{5.36}$$

with a suitable constant $C$. Since $M(r)$ is even, we may write

$$\frac{1}{\lambda}\int_{\mathbf{R}} e^{i\lambda r}rM(r)dr=\int_{\mathbf{R}}\frac{(e^{i\lambda r}-1)}{\lambda}rM(r)dr=i\int_{\mathbf{R}} rM(r)\left(\int_0^r e^{i\lambda v}dv\right)dr.$$

Thus, if we define $F(v)$ by

$$F(v)=\pm\int_v^{\pm\infty} rM(r)dr,\quad \text{for } \pm v>0$$

and change the order of integration, we have

$$\begin{aligned} K_3u(\rho)&=C\int_0^\infty \tilde{\Phi}(\lambda)e^{-i\lambda\rho}\left(\int_{\mathbf{R}} e^{i\lambda v}F(v)dv\right)d\lambda \\ &=C\left[\mathcal{H}(\check{\tilde{\Phi}}*F)\right](\rho). \end{aligned} \tag{5.37}$$

We estimate $|W_3u(x)|\le I_1(x)+I_2(x)$ as in (5.20). Recall that for $\frac{m}{m-2}<p<\frac{m}{2}$ and $\frac{m}{p}-1<\theta<\frac{m}{p}$ we have $m-2-\theta>0$. Let $p=\frac{m}{2+\varepsilon}$ and $\theta=2$

first for an arbitrarily small $\varepsilon > 0$. Then, applying Lemma 5.2 twice, once for $\mathcal{H}$ and once for $\mathcal{M}$, we obtain

$$\begin{aligned} \|I_2\|_p^p &\le C\|f\|_1^p \int_1^\infty \rho^{m-1}\left(\frac{|K_3(\rho)|}{\rho^{m-2}}\right)^p d\rho \\ &\le C\int_0^\infty \rho^{m-1-p\theta}|\mathcal{H}(\check{\Phi} * F)(\rho)|^p d\rho \\ &\le C\int_0^\infty \rho^{m-1-p\theta}|(\check{\check{\Phi}} * F)(\rho)|^p d\rho \\ &\le C\int_0^\infty \rho^{m-1-p\theta}|\mathcal{M}(F)(\rho)|^p d\rho \le C\int_0^\infty \rho^{m-1-2p}|F(\rho)|^p d\rho. \end{aligned} \tag{5.38}$$

We then apply Hardy's then Hölder's inequalities and estimate the right by

$$C\int_0^\infty r^{m-1}|M(r)|^p dr \le \int_{\mathbf{R}^m} |(\overline{g} * \check{u})(x)|^p \le C\|u\|_p^p. \tag{5.39}$$

Let $q = \frac{m}{m-2-\varepsilon}$ be the dual exponent of $p = \frac{m}{2+\varepsilon}$. By Hölder's inequality

$$|I_1(x)| \le C\left(\int_{|y|<1}\left|\frac{K_3 u(|y|)}{|y|^2}\right|^p dy\right)^{\frac{1}{p}}\left(\int_{|x-y|<1}\left|\frac{|f(y)|}{|x-y|^{m-4}}\right|^q dy\right)^{\frac{1}{q}}.$$

The second factor on the right is an $L^p$ function of $x \in \mathbf{R}^m$ since $|f(x)| \le C\langle x\rangle^{-m-\varepsilon}$. Then estimates (5.38) and (5.39) implies

$$\begin{aligned} \|I_1\|_p^p &\le C\int_{|y|<1}\left|\frac{K_3 u(|y|)}{|y|^2}\right|^p dy \\ &\le C\int_0^1 \rho^{m-1-2p}|\mathcal{H}(\check{\check{\Phi}} * F)(\rho)|^p d\rho \le C\|u\|_p^p. \end{aligned}$$

Let $p = \frac{m}{m-2-\varepsilon}$ and $\theta = m-3$ next. Then, using Lemma 5.2 twice as in (5.38), we obtain

$$\begin{aligned} \|I_2\|_p &\le C\|f\|_1\left(\int_1^\infty \rho^{m-1}\left(\frac{|K_3 u(\rho)|}{\rho^{m-2}}\right)^p d\rho\right)^{\frac{1}{p}} \\ &\le C\left(\int_0^\infty \rho^{m-1-p\theta}|F(\rho)|^p d\rho\right)^{\frac{1}{p}}. \end{aligned} \tag{5.40}$$

Then, Hardy's inequality implies that the right side is bounded by

$$C\left(\int_0^\infty \rho^{m-1-p(\theta-2)}|M(\rho)|^p d\rho\right)^{\frac{1}{p}} \le C\left(\int_{\mathbf{R}^m} \frac{|(\overline{g}*u)(x)|^p}{|x|^{p(\theta-2)}}dx\right)^{\frac{1}{p}}. \tag{5.41}$$

Since $p(\theta-2) < m$, the right side is bounded by $C\|u\|_p$. For $I_1(x)$ we proceed as previously. By Hölder's inequality

$$|I_1(x)| \le \left(\int_{|y|<1}\left|\frac{K_3 u(|y|)}{|y|^{m-3}}\right|^p dy\right)^{\frac{1}{p}}\left(\int_{|x-y|<1}\left|\frac{f(y)}{|x-y|}\right|^q dy\right)^{\frac{1}{q}}$$

The second factor on the right is an $L^p$ function of $x \in \mathbf{R}^m$ as previously. Then (5.40) and (5.41) imply that the right hand side of

$$\|I_1\|_p \le C\left(\int_{|y|<1}\left|\frac{K_3(|y|)}{|y|^{m-3}}\right|^p dy\right)^{\frac{1}{p}} \le C\left(\int_0^1 \rho^{m-1-p\theta}|\mathcal{H}(\check{\tilde{\Phi}}*F)(\rho)|^p d\rho\right)^{\frac{1}{p}}$$

is bounded by $C\|u\|_p$. This proves $W_3$ is bounded in $L^p$ for $\frac{m}{m-2} < p < \frac{m}{2}$ and completes the proof of Proposition 5.1.

### 5.2. Estimate for $W_{s,ab}$

In this subsection, we indicate how the discussion in the previous subsection may be modified for proving the following proposition.

PROPOSITION 5.16. *Let $V$ satisfies* (5.1) *and let $W_{ab}$ be defined by* (5.3). *Then, for any $\frac{m}{m-2} < p < \frac{m}{2}$,*

$$\|\Phi(H)W_{s,ab}\Phi(H_0)u\|_p \le C_p\|u\|_p, \quad u \in C_0^\infty(\mathbf{R}^m). \tag{5.42}$$

PROOF. By virtue of Proposition 3.6, $VD_{ab}$ are finite rank operators from $\mathcal{H}_{-(\delta-3)_-}$ to $\mathcal{H}_{(\delta-3)_-}$. Hence, they are finite linear combinations of rank one operators $f\otimes g$ with $f, g \in \mathcal{H}_{m+\varepsilon}$ for some $\varepsilon > 0$, and it suffices to prove (5.42) for $\Phi(H)\tilde{Z}\Phi(H_0)$, where $\tilde{Z}$ is the operator defined by

$$\tilde{Z} = \int_0^\infty G_0(\lambda)(f\otimes g)\left(G_0(\lambda) - G_0(-\lambda)\right)\tilde{\Phi}(\lambda)\lambda^{a+1}\log^b\lambda\, d\lambda \tag{5.43}$$

with such $f, g$, which is the same as (5.6) if $\lambda^{-1}$ replaces $\lambda^{a+1}\log^b\lambda$. Notice that $\tilde{Z}\Phi(H_0)u$ is given by the same formula (5.43) with $\Phi(H_0)g$ in place of $g$ and $|\Phi(H_0)g(x)| \le C\langle x\rangle^{-m-\varepsilon}$. Thus, we may (and do) assume that $g$ satisfies the condition (5.9), $|g(x)| \le C\langle x\rangle^{-m-\varepsilon}$, and ignore $\Phi(H_0)$. After this, we proceed as in the previous section: Define $K_{jk}^{ab}u(\rho)$ by the right side of (5.8) with $\lambda^{j+k+1+a}(\log\lambda)^b$ in place of $\lambda^{j+k-1}$ and

$$W_{jk}^{ab}u(x) = \int_{\mathbf{R}^m} \frac{f(y)K_{jk}^{ab}u(|x-y|)}{|x-y|^{m-2}}dy \tag{5.44}$$

so that $\tilde{Z}$ is a linear combination of $K_{jk}^{ab}$:

$$\tilde{Z}u(x) = \sum_{j,k=0}^{\frac{m-4}{2}} C_{jk}W_{jk}^{ab}u(x), \tag{5.45}$$

see Lemma 5.4. We then follow the argument in the proof of Proposition 5.6 for the case $j, k \ge 1$. The function $\lambda^{j+k+1+a}(\log\lambda)^b$ is certainly less singular than $\lambda^{j+k-1}$ at $\lambda = 0$ and the proof of Lemma 5.7 implies, as previously,

$$K_{jk}^{ab}u(\rho) = \int_{\mathbf{R}} M(\overline{g} * u, r)T_{jk}^{ab}(r)dr$$

with $T_{jk}^{ab}(r)$ which satisfies estimate (5.16):

$$\left|T_{jk}^{ab}(\rho, r)\right| \le C\left|\frac{\langle\rho\rangle^{j+1/2}r^{k+1}\langle r\rangle^{1/2}}{\langle r-\rho\rangle^{j+k}}\right|.$$

We then want to apply the argument in the proof of Lemma 5.8. Here it is important to observe that we may pretend that $f$ satisfies (5.9) as well: $|f(x)| \le C\langle x\rangle^{-m-\varepsilon}$. Indeed, we have

$$\begin{aligned}|\Phi(H)W_{jk}^{ab}u(x)| &= \left|\int_{\mathbf{R}^m}\left(\int_{\mathbf{R}^m}\Phi(x,z)f(z-y)dz\right)\frac{K_{jk}^{ab}u(|y|)}{|y|^{m-2}}dy\right| \\ &\le C\int_{\mathbf{R}^m}\frac{\langle y\rangle^{-(m+\varepsilon)}|K_{jk}^{ab}u(|x-y|)|}{|x-y|^{m-2}}dy\end{aligned}$$

since $|\Phi(x,z)| \le C_N\langle x-z\rangle^{-N}$. Then, the proof of Lemma 5.8 applies without any change and we obtain $\|\Phi(H)W_{jk}^{ab}u\|_p \le C\|u\|_p$ for any $\frac{m}{m-2} < p < \frac{m}{2}$. $\square$

## 6. High Energy Estimate

In this section we prove the following proposition. Recall that $m_* = \frac{m-1}{m-2}$.

Proposition 6.1. *Let $V$ satisfy (1.2) and, in addition, $|V(x)| \le C\langle x\rangle^{-\delta}$ for some $\delta > m+2$. Let $\Psi(\lambda) \in C^\infty(\mathbf{R})$ be such that $\Psi(\lambda) = 0$ for $|\lambda| < \lambda_0$ for some $\lambda_0$. Then $W_>$ is bounded in $L^p(\mathbf{R}^m)$ for all $1 \le p \le \infty$.*

Since the proof is entirely similar to the corresponding one in [I], we shall only sketch it very briefly pointing out what modifications are necessary for even dimensions. Iterating the resolvent equation, we have $G(\lambda)V = \sum_1^{2n}(-1)^{j-1}(G_0(\lambda)V)^j + G_0(\lambda)N_n(\lambda)$, where

$$N_n(\lambda) = (VG_0(\lambda))^{n-1}VG(\lambda)V(G_0(\lambda)V)^n.$$

If we substitute this for $G(\lambda)V$ in the right of (1.9), we have

$$W_> = \Psi(H_0)^2 + \sum_{j=1}^{2n}(-1)^j\Omega_j\Psi(H_0)^2 - \tilde\Omega_{2n+1}, \tag{6.1}$$

$$\tilde\Omega_{2n+1} = \frac{1}{i\pi}\int_0^\infty G_0(\lambda)N_n(G_0(\lambda) - G_0(-\lambda))\tilde\Psi(\lambda)d\lambda, \tag{6.2}$$

where $\tilde\Phi(\lambda) = \lambda\Psi(\lambda^2)^2$. The operators $\Psi(H_0)$ and $\Omega_1, \dots, \Omega_{2n}$ are bounded in $L^p$ for any $1 \le p \le \infty$ by virtue of Lemma 2.7. We show that, if $n$ is large enough, the integral kernel

$$\tilde\Omega_{2n+1}(x,y) = \int_0^\infty \langle N_n(\lambda)(G_0(\lambda) - G_0(-\lambda))\delta_y, G_0(-\lambda)\delta_x\rangle\lambda\Psi^2(\lambda^2)d\lambda,$$

of $\tilde\Omega_{2n+1}$ is admissible. We define $\tilde G_0(\lambda, z, x) = e^{-i\lambda|x|}G_0(\lambda, x-z)$ and $\psi(z,x) = |x-z| - |x|$ as previously.

Lemma 6.2. *Let $j = 0, 1, 2, \dots$. We have for $|\lambda| \ge 1$ that*

$$\left|\left(\frac{\partial}{\partial\lambda}\right)^j \tilde G_0(\lambda, z, x)\right| \le C_j\left(\frac{\langle z\rangle^j}{|x-z|^{m-2}} + \frac{\lambda^{\frac{m-3}{2}}\langle z\rangle^j}{|x-z|^{\frac{m-1}{2}}}\right). \tag{6.3}$$

PROOF. Differentiate $\tilde{G}_0^{(j)}(\lambda, z, x)$ by using Leibniz's formula. The result is a linear combination over $(\alpha, \beta)$ such that $\alpha + \beta = j$ of

$$\frac{e^{i\lambda\psi(z,x)}\psi(z,x)^\alpha}{|x-z|^{m-2-\beta}} \int_0^\infty e^{-t}t^{\nu-\frac{1}{2}} \left(\frac{t}{2} - i\lambda|x-z|\right)^{\nu-\frac{1}{2}-\beta} dt.$$

Since $|\psi(z,x)|^\alpha \le \langle z\rangle^j$ for $0 \le \alpha \le j$ and $|z-x| \le |\frac{t}{2} - i\lambda|z-x|| \le (t+\lambda|z-x|)$ when $|\lambda| \ge 1$, (6.3) follows. $\square$

Define $T_\pm(\lambda, x, y) = \langle N_n(\lambda)\tilde{G}_0(\pm\lambda, \cdot, y), \tilde{G}_0(-\lambda, \cdot, x)\rangle$ so that

$$\begin{aligned}&\tilde{\Omega}_{2n+1}(x,y)\\ &\quad = \frac{1}{\pi i}\int_0^\infty \left(e^{i\lambda(|x|+|y|)}T_+(\lambda,x,y) - e^{i\lambda(|x|-|y|)}T_-(\lambda,x,y)\right)\tilde{\Psi}(\lambda)d\lambda.\end{aligned}$$

The following lemma may be proved by repeating line by line the proof of Lemma 3.14 of [I] by using (6.3) and Lemma 2.5.

LEMMA 6.3. *Let $0 \le s \le \frac{m+2}{2}$. For sufficiently large $n$, we have*

$$\left|\left(\frac{\partial}{\partial\lambda}\right)^s T_\pm(\lambda,x,y)\right| \le C_{ns}\lambda^{-3}\langle x\rangle^{-\frac{m-1}{2}}\langle y\rangle^{-\frac{m-1}{2}} \tag{6.4}$$

We then integrate by parts $0 \le s \le (m+2)/2$ times to obtain

$$\begin{aligned}&\int_0^\infty e^{i\lambda(|x|\pm|y|)}T_\pm(\lambda,x,y)\tilde{\Psi}(\lambda)d\lambda\\ &\quad = \frac{1}{(|x|\pm|y|)^s}\int_0^\infty e^{i\lambda(|x|\pm|y|)}\left(\frac{\partial}{\partial\lambda}\right)^s\left(T_\pm(\lambda,x,y)\tilde{\Psi}(\lambda)\right)d\lambda\end{aligned}$$

and estimate the right hand side by using (6.4). We obtain

$$|\tilde{\Omega}_{n+1}(x,y)| \le C\sum_\pm \langle |x| \pm |y|\rangle^{-\frac{m+2}{2}}\langle x\rangle^{-\frac{m-1}{2}}\langle y\rangle^{-\frac{m-1}{2}}$$

and $\tilde{\Omega}_{n+1}(x,y)$ is admissible. Proposition 6.1 follows.

## 7. Completion of Proof of Theorem

To complete the proof of Theorem 1.1 we have only to prove the continuity of $W$ in Sobolev spaces. We prove this for the case $1 < p < \infty$ only. For the cases $p = 1$ and $p = \infty$, we may apply without any change the proof presented in Section 4 of [22] for odd dimensional cases where we estimated the multiple commutators $[p_{i_1}, [p_{i_2}, [\cdots, [p_{i_\ell}, W_\pm]\cdots]]]$. We use the following two lemmas.

Lemma 7.1. *Let $1 < p < \infty$ and $|V(x)| \le C < \infty$. Then for large negative $\lambda$, $R(\lambda) \in \mathbf{B}(L^p, W^{2,p})$ and $R(\lambda)^{\frac{1}{2}} \in \mathbf{B}(L^p, W^{1,p})$.*

Proof. It is well known that $R_0(\lambda)$ is bounded from $L^p$ to $W^{2,p}$ if $\lambda < 0$. Since $R_0(-\kappa^2)$, $\kappa > 0$, is the convolution operator with $G_0(x, i\kappa) = \kappa^{m-2}G_0(\kappa x, i)$ and $G_0(\cdot, i)$ and $\nabla G_0(\cdot, i)$ are integrable, we have for $\lambda < 0$ that we have $\|R_0(\lambda)\|_{p,p} \le C|\lambda|^{-1}$ and $\|\nabla R_0(\lambda)\|_{p,p} \le C_p|\lambda|^{-\frac{1}{2}}$. It follows that, for large negative $\lambda$, $1 + R_0(\lambda)V$ is an isomorphism of $L^p$, $R(\lambda) = (1 + R_0(\lambda)V)^{-1}R_0(\lambda)$ also in $L^p$ and $\|R(\lambda)\|_{p,p} \le C|\lambda|^{-1}$. Hence, the resolvent equation is also valid in $L^p$,

$$R(\lambda) = R_0(\lambda) - R_0(\lambda)VR(\lambda), \tag{7.1}$$

and this implies $R(\lambda) \in \mathbf{B}(L^p, W^{2,p})$. It also follows that the integral in

$$\nabla R(\lambda)^{\frac{1}{2}} = \nabla R_0(\lambda)^{\frac{1}{2}} - C\int_0^\infty \mu^{-\frac{1}{2}} \nabla R_0(\lambda-\mu)^{-1} VR(\lambda-\mu)d\mu$$

converges in the norm of $\mathbf{B}(L^p)$ and $\nabla R(\lambda)^{\frac{1}{2}}$ is bounded in $L^p(\mathbf{R}^m)$. $\square$

Lemma 7.2. *Let $1 < p < \infty$ and $n = 1, 2, \ldots$. Then, for large negative $\lambda$ the following statements are satisfied:*
*(1) Let $|\partial^\alpha V(x)| \le C_\alpha$ for $|\alpha| \le 2(n-1)$. Then, $R(\lambda)^n \in \mathbf{B}(L^p, W^{2n,p})$.*
*(2) Let $|\partial^\alpha V(x)| \le C_\alpha$ for $|\alpha| \le 2n-1$. Then, $R(\lambda)^n \in \mathbf{B}(W^{1,p}, W^{2n+1,p})$.*

Proof. We first prove (1) by induction on $n$. If $n = 1$, (1) is contained in Lemma 7.1. Let $n \ge 2$ and suppose that (1) is already proved for smaller values of $n$. By virtue of (7.1),

$$R(\lambda)^n = R_0(\lambda)R(\lambda)^{n-1} - R_0(\lambda)VR(\lambda)^{n-1}R(\lambda). \tag{7.2}$$

By the assumption on $V$ and the induction hypothesis $R(\lambda)^{n-1}, VR(\lambda)^{n-1} \in \mathbf{B}(L^p, W^{2n,p})$ and (1) follows since $R_0(\lambda)$ maps $W^{2n,p}$ to $W^{2n+2,p}$ boundedly.

We next prove (2). Let $n = 1$ first. Then, in (7.1), $R_0(\lambda) \in \mathbf{B}(W^{1,p}, W^{3,p})$ and $VR(\lambda) \in \mathbf{B}(W^{1,p})$ by (1) for $n = 1$ and the assumption on $V$. Hence (1) holds for $n = 1$. Let $n \geq 2$ and suppose that (2) is already proved for smaller values of $n$. Then in (7.2), $R(\lambda)^{n-1} \in \mathbf{B}(W^{1,p}, W^{2n-1,p})$ by the induction hypothesis, and $VR(\lambda)^{n-1}R(\lambda) \in \mathbf{B}(W^{1,p}, W^{2n-1,p})$ also by the assumption on $V$ and Lemma 7.1. Since $R_0(\lambda) \in \mathbf{B}(W^{2n-1,p}, W^{2n+1,p})$, (2) follows. □

By intertwining property we have for sufficient large negative $\lambda$

$$R(\lambda)^n W_\pm = W_\pm R_0(\lambda)^n, \quad R(\lambda)^{n+\frac{1}{2}} W_\pm = W_\pm R_0(\lambda)^{n+\frac{1}{2}}$$

From the first equation and Lemma 7.2 (1) we see that, if $|\partial^\alpha V(x)| \leq C_\alpha$ for $|\alpha| \leq 2(n-1)$, $W_\pm \in \mathbf{B}(W^{2n,p}, W^{2n,p})$. Likewise from Lemma 7.1 and Lemma 7.2 (2), we have $W_\pm \in \mathbf{B}(W^{2n+1,p}, W^{2n+1,p})$ if $|\partial^\alpha V(x)| \leq C_\alpha$ for $|\alpha| \leq 2n-1$. This completes the proof of Theorem 1.1.

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