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Polyhedral Deformations of a Cone Manifold

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Abstract. A single parameter family of polyhedra $P(\psi)$ is constructed in three dimensional spaces of constant curvature $C(\psi)$. Identification of the faces of the polyhedra via isometries results in cone manifolds $M(\psi)$ which are topologically $S^1 \times S^2$, S^3 or singular S^2. The singular set of $M(\psi)$ can have vertices of degree three for some values of ψ and can also be the Whitehead link or form other configurations. Curvature varies continuously with ψ. At $\psi = 0$ spontaneous surgery occurs and the topological type of $M(\psi)$ changes. This phenomenon is described.

0. Introduction

We study continuous families of cone manifolds M_ψ parametrised by cone angle which begin at cone angle zero with the complement of the Whitehead link in S^3. We consider the case of equal cone angles on all singular link components. The cone structures for certain non-zero values of cone angles exist in projective models or in S^3.

In our Dehn surgery direction the cone manifolds are for certain cone angles, obtained by surgery on the Whitehead link in S^3 resulting in a topologically distinct singular set in $S^2 \times S^1$. As cone angle is increased the topological type of the singular set changes and the hyperbolic cone manifold develops two cusps and becomes S^3 at cone angle $\frac{2}{3}\pi$. The geometry and topology of M_ψ and its singular set remain unaltered as cone angle increases beyond $\frac{2}{3}\pi$ until we reach a cone angle ω where M_ψ becomes Euclidean with a topologically different type of singularity. Increasing cone angle past ω the singular set reverts back to its pre-$\frac{2}{3}\pi$ cone angle topological type and M_ψ becomes spherical in $S^2 \times S^1$. At cone angle π the underlying polyhedron becomes a lens in S^3 from which the cone manifold is obtained by suitable identifications. For cone angles in the interval $[\pi, \zeta]$, M_ψ is spherical and the topological type of its singular set is unchanged but it is now in S^3. At cone angle ζ, M_ψ becomes the suspension of a sphere with four cone points.

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It remains the well understood sphere with four singularities for cone angles larger than ζ.

Investigating the deformation on the other side of Dehn surgery we obtain the Whitehead link in S^3 for certain non-zero cone angles. We will study this family in our subsequent work.

Here a *Cone Manifold* is a PL manifold with a possibly empty codimension two locally flat submanifold called the *singular set*. In dimensions two and three the singular set consists of isolated points and curves but not graphs respectively. The geometric model is a spherical, Euclidean, or hyperbolic space of constant curvature where the constant is any real number. Points in the complement of the singular set have neighbourhoods homeomorphic to neighbourhoods in the model. Points on the singular set have neighbourhoods homeomorphic to neighbourhoods in the topological space obtained by identifying boundaries of the intersection of two half spaces referred to as a *wedge* in the model. The homeomorphism takes the singular set to the axis of rotation in the topological space and transition functions are isometries. *Orbifolds* are represented by discrete structures where cone angles are of the form $\frac{2\pi}{n}, n \in \mathbb{N}$. cf [1]

The topological space obtained by identifying in pairs faces of a polyhedron in a space of constant curvature via isometries is a cone manifold if :

1. No edge is identified with its inverse in the equivalence class induced by the identifications, and the identifications of wedges along faces are cyclic for each equivalence class of edges.
2. The cone angle at each edge, i.e. the sum of the dihedral angles about the edge is $\leq 2\pi$.
3. The neighbourhood of each vertex is a cone on a sphere. If a vertex neighbourhood is, for example, a cone on a torus the topological space is not a manifold.
4. There are either two or no edges emanating from a vertex where cone angle is not 2π. If two edges then the cone angles must be equal and the edges must be lined up. The vertex neighbourhood is obtained by identifying wedge half planes.

cf [1].

It is reasonable to expect our deformation process to hold for any hyperbolic link complement since the polyhedral description of a link complement utilised here is canonical and we can expect to be able to deform this construction by opening cusps in M_0 in the way we have described. Our methods can be viewed as part of an approach to describe all compact connected 3-manifolds through deformations by removing singularities of branched singular covers of S^3 along universal links thereby providing an approach to the Poincaré Conjecture in dimension three. The important phenomenon of singular set with vertices of degree three in a cone manifold occurs in our work. Thus our construction may prove useful in studying this phenomenon.

1. Cone Angle Interval $[0, \omega]$

As defined in the abstract let M_ψ denote the cone manifold obtained by identifying the faces of the polyhedron P_ψ.

Proposition 1.1. *M_ψ is $S^2 \times S^1$ with two singular components and is hyperbolic for cone angles $\psi \in (0, \frac{2}{3}\pi)$. $M_{\frac{2}{3}\pi}$ is cusped hyperbolic and has a singular set with degree three vertices. There exists $\omega \in \mathbb{R}$ such that M_ψ is hyperbolic with a degree three vertexed singular set when $\frac{2}{3}\pi < \psi < \omega$ and M_ω is S^3.*

Proof. M_0 the complement of the Whitehead link in S^3 is obtained from a two cusped octahedron P_0 with identifications described in the projective Klein model of $\mathbb{H}^3$ in figure 2. We refer the reader to [2], [3] and [4]

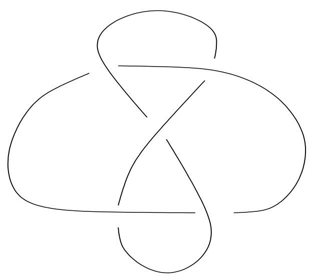

Fig. 1. The Whitehead link

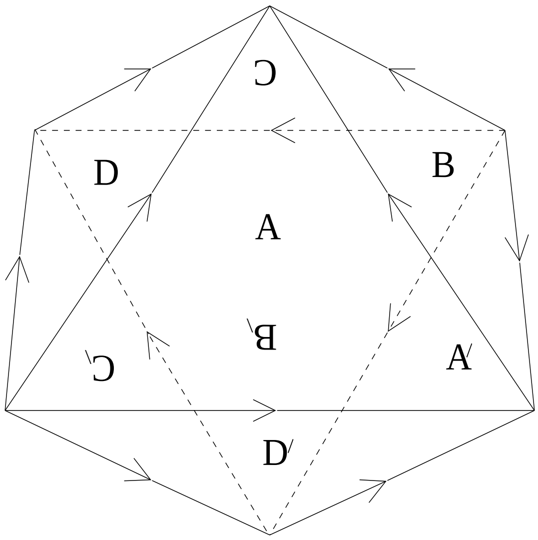

Fig. 2. P_0 with identifications

for details of this construction.

To find out more about P_ψ we view it relative to a coordinate system and inside a *reference box* in the Klein model as in figure 4. The length, width and height of the reference box are denoted by a, b and c respectively. Opening cusps in M_0 we obtain P_ψ for $\psi \in (0, \frac{2}{3}\pi)$ as in figure 3.

From figure 4 we have the following identifications:

Face identifications:

$$A \longleftrightarrow A' \quad i.e. \quad \triangle\,(onp) \longleftrightarrow \triangle(sqr) \tag{1}$$

$$C \longleftrightarrow C' \quad i.e. \quad \triangle\,(jhi) \longleftrightarrow \triangle(klm) \tag{2}$$

$$D \longleftrightarrow D' \quad i.e. \quad hexagon(nokmhi) \longleftrightarrow hexagon(poklrq) \tag{3}$$

$$B \longleftrightarrow B' \quad i.e. \quad hexagon(inpqsj) \longleftrightarrow hexagon(hmlrsj) \tag{4}$$

Edge pairings:

$$ok \longleftrightarrow ok \tag{5}$$

$$sj \longleftrightarrow sj \tag{6}$$

$$on \longleftrightarrow op \longleftrightarrow sq \longleftrightarrow sr \tag{7}$$

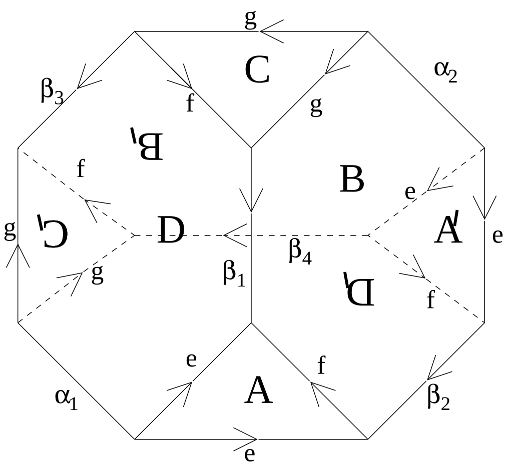

Fig. 3. opening cusps

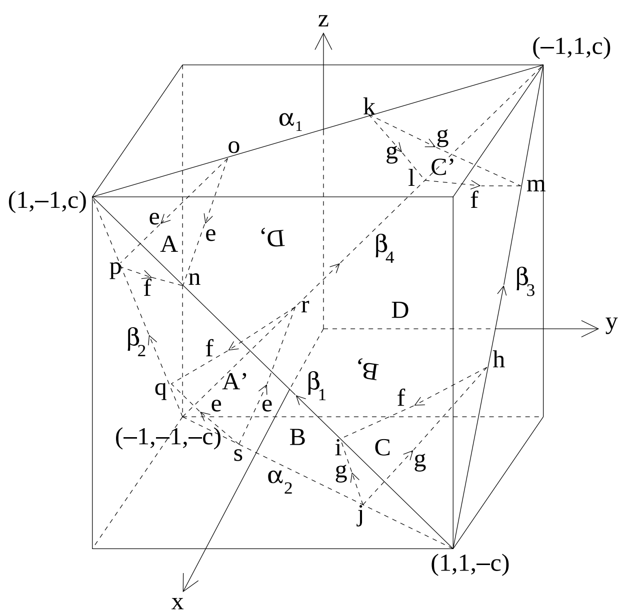

Fig. 4. opening cusps

$$jh \longleftrightarrow ji \longleftrightarrow km \longleftrightarrow kl \tag{8}$$

$$np \longleftrightarrow ml \longleftrightarrow hi \longleftrightarrow rq \tag{9}$$

$$in \longleftrightarrow hm \longleftrightarrow lr \longleftrightarrow pq \tag{10}$$

Edge pairings 5, 6 and 10 represent singular components of M_ψ.
We observe:

(11) The dihedral angle between planes incident in a member of (5) or (6) is the cone angle ψ.

Moreover (7), (8) and (9) are not part of the singular set. Therefore

(12) The dihedral angle between planes incident in a member of (7), (8) or (9) is $\dfrac{\pi}{2}$.

Information in (10) represents segments $\beta_1, \beta_2, \beta_3$ and β_4 which are identified to give a singular component β of the singular set of M_ψ. Considering (10) with equal cone angles on all singular components we deduce:

(13) Planes incident in a member of (10) intersect at dihedral angle $\dfrac{\psi}{4}$.

We deduce from (10) that $a = b$. Box coordinates can therefore be normalised to give $a = 1 = b$ and c where c is measured in the $z-$direction. The reference box is therefore a cube with square base and height $2c$ as in figure 4.

Topologically M_ψ is $S^1 \times S^2$ with two unlinked singular components as in figure 5.

To find out about the geometry of M_ψ we look at P_ψ inside the Klein model of $\mathbb{H}^3$. Let R be the radius of the Klein ball B_R and let c represent the height of the main box. We will show that P_ψ lives inside B_R and can be specified by R and c. We also have

$$\text{curvature of } \mathbb{H}^3 = -\frac{1}{R^2} \tag{14}$$

The geometry of M_ψ can therefore be described in terms of ψ. We now obtain R and c as functions of ψ.

Figure 6 depicts P_ψ in the Klein model.

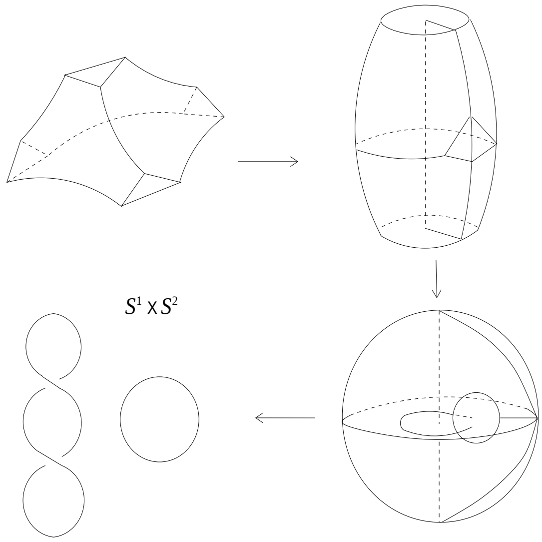

Fig. 5. Topology of M_ψ

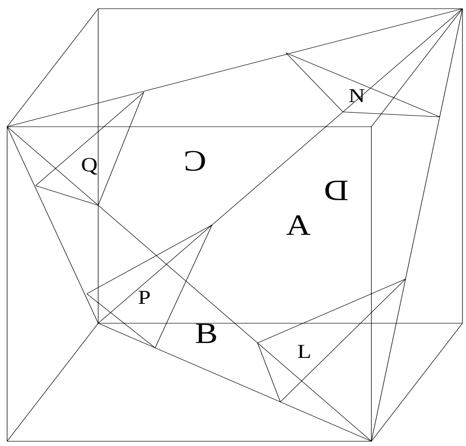

Fig. 6. P_ψ in the Klein model

In homogeneous coordinates the equations of planes of P_ψ and their poles are:

$$
\begin{array}{ccc}
\textit{plane} & \textit{equation} & \textit{pole} \\
A & cx + cy + z = c & (cR, cR, R, c) \\
B & cx - cy - z = c & (cR, -cR, -R, c) \\
C & -cx - cy + z = c & (-cR, -cR, R, c) \\
D & -cx + cy - z = c & (-cR, cR, -R, c)
\end{array} \tag{15}
$$

Let $\mathbf{v} = (v_1, v_2, v_3, v_4), \mathbf{w} = (w_1, w_2, w_3, w_4) \in \mathbb{R}^4$. Then the hyperbolic bilinear form is given by

$$\langle \mathbf{v}, \mathbf{w} \rangle_{\mathbb{H}} = v_1 w_1 + v_2 w_2 + v_3 w_3 - v_4 w_4 \tag{16}$$

Let θ be the dihedral angle of intersection between planes P and Q with poles $\mathbf{v}$ and $\mathbf{w}$ respectively. Then

$$\cos\theta = -\frac{\langle \mathbf{v}, \mathbf{w} \rangle_{\mathbb{H}}}{\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle_{\mathbb{H}}}\sqrt{\langle \mathbf{w}, \mathbf{w} \rangle_{\mathbb{H}}}} \tag{17}$$

Combining (10), (13), (15) and (17) we can derive expressions for $\cos\psi$ and $\cos\dfrac{\psi}{4}$ in terms of R and c to obtain :

$$c^2 = \frac{1 + \cos\psi}{2\cos\dfrac{\psi}{4} - \cos\psi + 1} \tag{18}$$

$$R^2 = \frac{1 + \cos\psi}{2\cos\dfrac{\psi}{4} + \cos\psi - 1} \tag{19}$$

Substituting $(R, c)-$expressions of $\cos\dfrac{\psi}{4}$ and $\cos\psi$ in (18) and (19) we can verify these identities.

We note $\omega = 4\cos^{-1}(\dfrac{2}{\sqrt{3}}\cos(\dfrac{1}{3}\cos^{-1}(\dfrac{-3\sqrt{3}}{8}))) \approx 2.311984...$ is the smallest positive value of ψ for which the denominator of (19) is zero. Hence ω is the smallest positive value of ψ for which R^2 is infinite. This combined with (14) implies that curvature is zero and M_ψ is therefore Euclidean when $\psi = \omega$.

Combining (12), (15), (16) and (17) the equation of plane N in figure 6 is :

$$N : -x + y + cz = R^2 \tag{20}$$

Let $\mathbf{p} = (-1, 1, c, R)$ denote the pole of N with $\mathbb{H}^3$ embedded in $\mathbb{R}P^3$. We have

$$\begin{aligned}\langle \mathbf{p}, \mathbf{p} \rangle &= 2 + c^2 - R^2 \\ &= \frac{3 - 4\cos^2 \dfrac{\psi}{4}}{(4\cos^3 \dfrac{\psi}{4} - 4\cos \dfrac{\psi}{4} - 1)(4\cos^3 \dfrac{\psi}{4} - 4\cos \dfrac{\psi}{4} + 1)}\end{aligned} \tag{21}$$

Therefore

$$\langle \mathbf{p}, \mathbf{p} \rangle \begin{cases} > 0 & \text{when } \ 0 < \psi < \dfrac{2}{3}\pi, \\ = 0 & \text{when } \ \psi = \dfrac{2}{3}\pi, \\ < 0 & \text{when } \ \dfrac{2}{3}\pi < \psi < \omega. \end{cases} \tag{22}$$

Information contained in (22) corresponds to the configurations depicted in figure 7 in the case of $\mathbb{H}^2$ embedded in $\mathbb{R}P^2$ and is true for n hyperplanes in $\mathbb{H}^n$ embedded in $\mathbb{R}P^n$.

Referring to (20), figure 6 and figure 7, when $\langle \mathbf{p}, \mathbf{p} \rangle > 0$ there is a plane N inside $\mathbb{H}^3$. When $\langle \mathbf{p}, \mathbf{p} \rangle = 0$ the plane N is the point of intersection of the planes A, C and D on $\partial\mathbb{H}^3$. When $\langle \mathbf{p}, \mathbf{p} \rangle < 0$ the plane N is the point of intersection of the planes A, C and D inside $\mathbb{H}^3$.

We now verify that $P_\psi \subset \mathbb{H}^3 \cup \partial\mathbb{H}^3$ for $\psi \in [0, \frac{2}{3}\pi]$. Since P_ψ is symmetric with respect to the origin it is sufficient to show that the vertices $(A \cap N \cap C)$ and $(A \cap N \cap D)$ of figure 6 live inside $\mathbb{H}^3 \cup \partial\mathbb{H}^3$. Let

$$f(\psi) = R^2 - (A \cap N \cap C)^2 = \frac{1}{2}(R^2 - c^2)(2 + c^2 - R^2) \tag{23}$$

where $(A \cap N \cap C)^2$ denotes square of the distance of vertex $(A \cap N \cap C)$ from the origin. We have $f > 0$ for $\psi \in (0, \frac{2}{3}\pi)$ and $f = 0$ when $\psi = 0$ or $\frac{2}{3}\pi$.

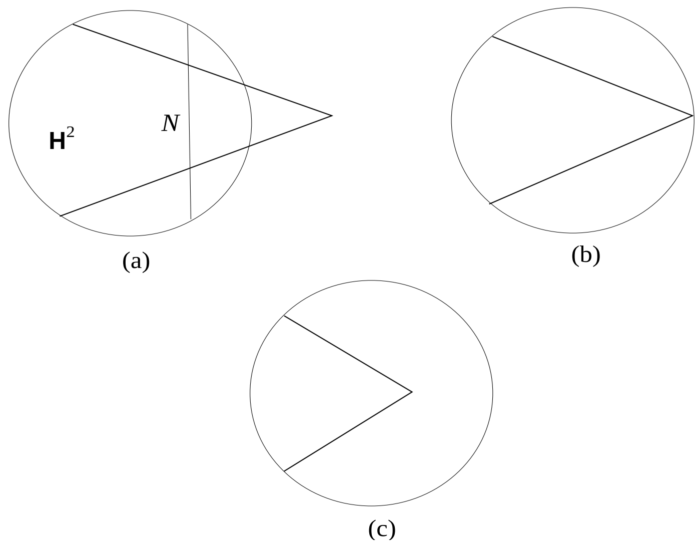

Fig. 7. (a) $\langle \mathbf{p}, \mathbf{p} \rangle > 0$: lines meet outside $\mathbb{H}^2$. (b) $\langle \mathbf{p}, \mathbf{p} \rangle = 0$: lines meet on $\partial \mathbb{H}^2$. (c) $\langle \mathbf{p}, \mathbf{p} \rangle < 0$: lines meet inside $\mathbb{H}^2$.

Let

$$g(\psi) = R^2 - (A \cap N \cap D)^2 = \frac{(R^2 - 1)(2 + c^2 - R^2)}{c^2 + 1} \tag{24}$$

We note $g > 0$ when $\psi \in (0, \frac{2}{3}\pi)$ and $g = 0$ if $\psi = 0$ or $\frac{2}{3}\pi$.

Therefore $P_\psi \subset \mathbb{H}^3 \cup \partial \mathbb{H}^3$ when $\psi \in [0, \frac{2}{3}\pi]$. Hence M_ψ is hyperbolic when $\psi \in [0, \frac{2}{3}\pi]$.

We note from (22) and figure 7 that $M_{\frac{2}{3}\pi}$ is cusped. From figure 8 we observe that $M_{\frac{2}{3}\pi}$ has two cusps, it is topologically S^3 with singular set as shown in figure 9 and its cusp neighborhoods are Euclidean turnovers as in figure 10.

From (22) and figure 7 we observe that the bounding planes of P_ψ meet inside $\mathbb{H}^3$ when $\psi \in (\frac{2}{3}\pi, \omega)$. Therefore, P_ψ is in $\mathbb{H}^3$. Hence M_ψ is hyperbolic when $\psi \in (\frac{2}{3}\pi, \omega)$. The singular set of M_ψ for $\psi \in (\frac{2}{3}\pi, \omega)$ is shown in figure 9.

We note from (19) that $R(\omega) = \infty$ hence M_ω is Euclidean. P_ω is a tetrahedron from which M_ω is obtained using identifications. We observe

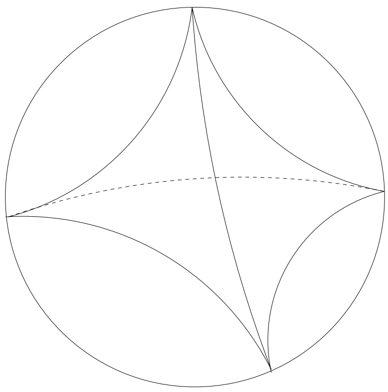

Fig. 8. $P_{\frac{2}{3}\pi}$ inside $\mathbb{H}^3 \cup \partial\mathbb{H}^3$

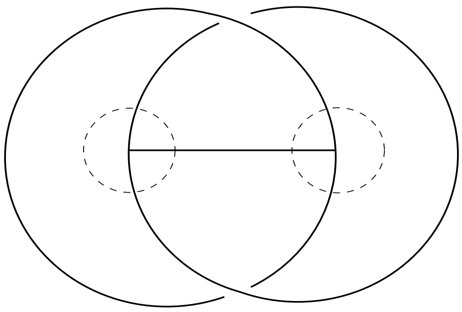

Fig. 9. Singular set of M_ψ when $\psi \in [\frac{2}{3}\pi, \omega)$ and its cusp neighbourhoods when $\psi = \frac{2}{3}\pi$.

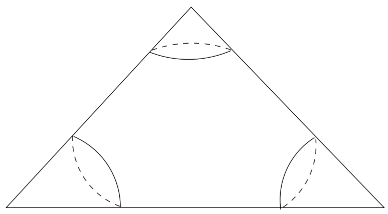

Fig. 10. A Euclidean turnover

that M_ω is S^3. The singular set of M_ω is depicted by figure 9 with one cusp. $\square$

2. Cone Angles Larger than ω

As defined in the abstract let M_ψ denote the cone manifold obtained by identifying the faces of the polyhedron P_ψ.

PROPOSITION 2.1. *M_ψ is spherical when cone angle ψ is larger than ω. There exists $\zeta \in \mathbb{R}$ such that M_ψ is topologically S^3 with a singular set which has degree three vertices when $\omega < \psi < \zeta$. M_ψ is the suspension of a sphere with four cone points when $\zeta \leq \psi$.*

PROOF. When $\omega < \psi < \pi$ we note R^2 the square radius of the Klein model becomes negative so that the model has imaginary radius. This leads us to use the spherical bilinear form for $\mathbf{v} = (v_1, v_2, v_3, v_4), \mathbf{w} = (w_1, w_2, w_3, w_4) \in \mathbb{R}^4$:

$$\langle \mathbf{v}, \mathbf{w} \rangle_{\mathbb{S}} = v_1 w_1 + v_2 w_2 + v_3 w_3 + v_4 w_4 \tag{25}$$

when $\psi \in (\omega, \pi)$.

This is the usual scalar product on $\mathbb{R}^4$. Let $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. Since $\mathbb{H}^3_R = \{x \mid \langle \mathbf{x}, \mathbf{x} \rangle_{\mathbb{H}} = -R^2\}$ we define $\mathbb{S}^3_R = \{x \mid \langle \mathbf{x}, \mathbf{x} \rangle_{\mathbb{S}} = R^2\}$. Thus $\mathbb{S}^3_R$ is the sphere of radius R. The "Klein model" $\mathbb{K}^3_R$ for the sphere of radius R is the hyperplane $\mathbb{K}^3_R = \{\mathbf{x} \mid t = R\}$. In contrast to the hyperbolic case we don't need to verify that the polytope lies inside the sphere of radius R in $\mathbb{K}^3_R$ since projection from the origin which is not conformal defines a 1-1 correspondence between $\mathbb{K}^3_R$ and the upper hemisphere of $\mathbb{S}^3_R$. The metric on $\mathbb{K}^3_R$ is then the pull back of the metric on $\mathbb{S}^3_R$. As in the hyperbolic case reflections in planes through the origin and rotation about axes through the origin are both Euclidean and spherical isometries.

If a plane in $\mathbb{K}^3_R$ has equation $\alpha x + \beta y + \gamma z = \delta$ and $t = R$ then in homogeneous coordinates the equation is $\alpha x + \beta y + \gamma z - (\delta/R)t = 0$. In the spherical case the pole has homogeneous coordinates $(\alpha, \beta, \gamma, -\delta/R)$ whereas in the hyperbolic case the pole has coordinates $(\alpha, \beta, \gamma, \delta/R)$. If a pair of planes with poles $\mathbf{v}$ and $\mathbf{w}$ in the spherical case intersect with dihedral angle θ then

$$\pm\cos\theta = \frac{\langle \mathbf{v}, \mathbf{w} \rangle_{\mathbb{S}}}{\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle_{\mathbb{S}} \langle \mathbf{w}, \mathbf{w} \rangle_{\mathbb{S}}}} \tag{26}$$

Using (25), (26) and the spherical version of (15) we obtain

$$0 \le R^2 = -\frac{1+\cos\psi}{2\cos\frac{\psi}{4} - 1 + \cos\psi} \tag{27}$$

when $\omega \le \psi \le \zeta$.

When $\psi \in (\omega, \pi)$ we observe that P_ψ is a tetrahedron with identification as in figure 11.

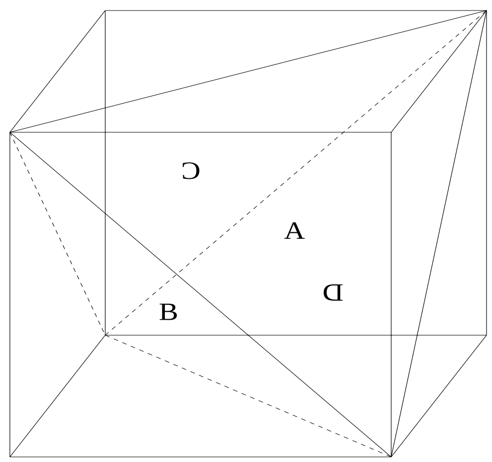

Fig. 11. P_ψ inside the reference box

Therefore M_ψ is S^3 with a singular set which has degree three vertices as in figure 9 with spherical turnover neighbourhoods.

Since $R^2(\pi) = 0 = c^2(\pi)$ the three dimensional model has collapsed into a two dimensional disc at cone angle π. By projection onto $S^3_1 = \{\mathbf{x} \mid x^2 + y^2 + z^2 + t^2 = 1\}$ the sphere of radius 1 we observe that P_π is a lens with angle $\psi/4$ as in figure 13 from which M_π is obtained by identifications.

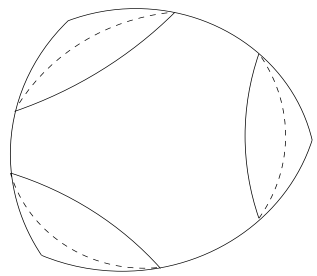

Fig. 12. A spherical turnover

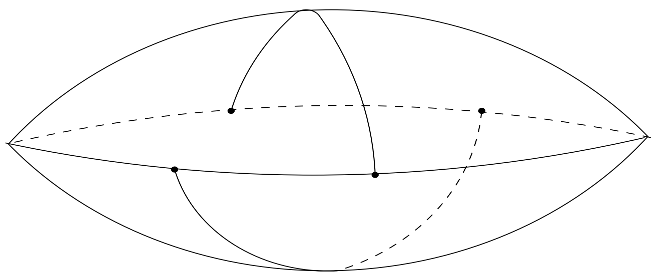

Fig. 13. P_π is a lens with angle $\dfrac{\pi}{4}$

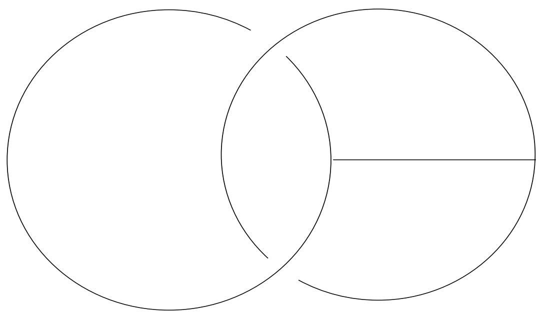

Fig. 14. Singular set of M_π

We note $\zeta = 4\cos^{-1}(\frac{2}{\sqrt{3}}\cos(\frac{1}{3}\cos^{-1}(\frac{-3\sqrt{3}}{8} + \frac{4}{3}\pi))) \approx 5.191298...$ is the second smallest value of ψ for which R is infinite. The equation of c^2 is the same as in the hyperbolic case. The model continues to be S^3 for $\pi < \psi < \zeta$ therefore M_ψ is spherical with a degree three vertexed singular set as in figure 14 when $\psi \in (\pi, \zeta)$.

As cone angle ψ approaches ζ the segments h_1 and h_2 decrease in length towards 0 as shown in figure 15 so that we get the singular set shown in figure 16 when $\psi = \zeta$.

We therefore obtain M_ψ as the suspension of a sphere with four cone points an in figure 16 when $\zeta < \psi$. $\square$

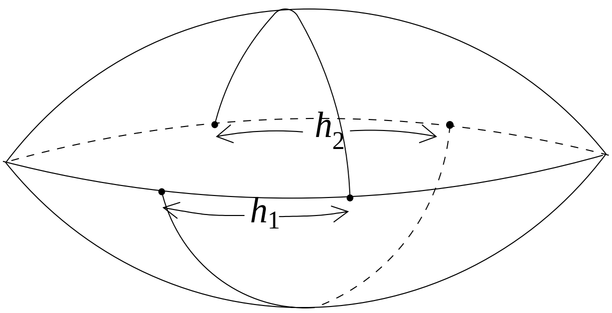

Fig. 15. Segments h_1 and h_2 decrease in length as ψ increases towards ζ

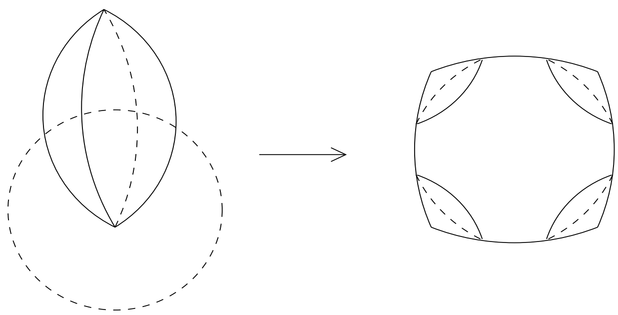

Fig. 16. M_ζ is the suspension of a sphere with four cone points

3. Spontaneous Surgery

We now look at how we get spontaneous surgery and the Whitehead link as the singular set. P_ψ and its identifications "before" spontaneous surgery are shown in figure 17. Face pairings are $A \longleftrightarrow A', B \longleftrightarrow B', C \longleftrightarrow C'$ and $D \longleftrightarrow D'$ and edges with the same label are identified.

We note that M_ψ for $\psi \in (0, \frac{2}{3}\pi)$ "after" spontaneous surgery is the result of performing $(0,1)$-Dehn surgery on one component of the Whitehead link in S^3. This can be observed by considering the boundary of a tubular neighbourhood of the β-component of the singular set. In Figure 18 we have shown a tubular neighbourhood of the β_1-component of the singular locus as we move from "before" to "after" spontaneous surgery from left to right in Figure 18. The singular component shrinks in length as cone angle decreases and becomes a point at cone angle zero. As cone angle increases the singular component increases in length in a transverse direction. We observe that a meridian of β_1 "before" spontaneous surgery is mapped to a longitude of β_1 "after" spontaneous surgery. The manifold "after" spontaneous surgery is therefore the result of performing $(0,1)$-Dehn surgery on the β-component of the singular set.

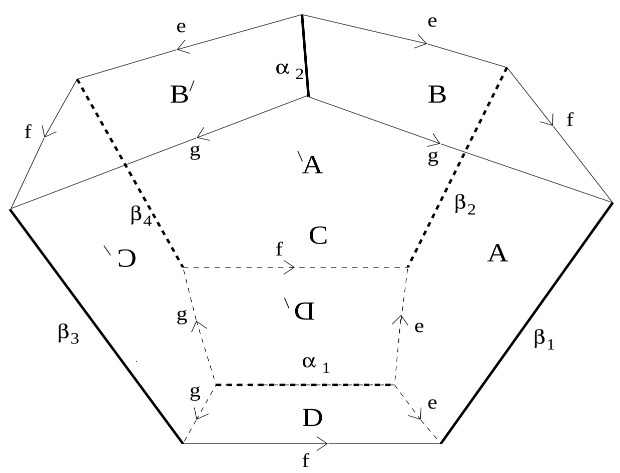

Fig. 17. P_ψ "before" spontaneous surgery gives S^3 with the Whitehead link as its singular set

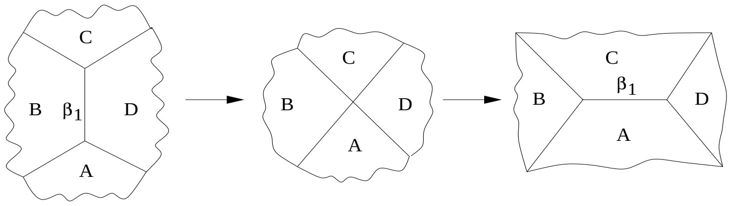

Fig. 18. Spontaneous surgery

Equations of planes are unaltered "after" spontaneous surgery and can be obtained from (20) and the symmetry of P_ψ. Dihedral angles at the β edges of P_ψ are the cone angle ψ and other incidence angles between the planes of P_ψ are the same as "after" spontaneous surgery. Calculations to show the behaviour of M_ψ before surgery remain to be done.

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