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Logarithmic De Rham, Infinitesimal and Betti Cohomologies

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Abstract. Given a log scheme Y over $\mathbb{C}$, Kato and Nakayama [27] were able to associate a topological space Y_{log}^{an}. We will use the log infinitesimal site Y_{inf}^{log} and its structural sheaf $\mathcal{O}_{Y_{inf}^{log}}$; we will prove that $H^{\cdot}(Y_{inf}^{log}, \mathcal{O}_{Y_{inf}^{log}}) \cong H^{\cdot}(Y_{log}^{an}, \mathbb{C})$. The isomorphism will be obtained using log De Rham cohomological spaces $H^{\cdot}_{DR,log}(Y/\mathbb{C})$ along the lines of [36]. These results generalize the (ideally) log smooth case of [27].

Introduction

For a non singular scheme Y over $\mathbb{C}$ the hyper-cohomology of the algebraic De Rham complex calculates the Betti cohomology $H^{\cdot}(Y^{an}, \mathbb{C})$. For singular Y there is no straighforward generalization of this calculation: indeed, it is the algebraic side that causes problems. To deal with this case, Grothendieck has introduced the algebraic Infinitesimal Site Y_{inf} [16]. Moreover, as explained in [18], when Y admits an embedding as a closed subscheme of a smooth scheme X, one can also consider the completion $\Omega^{\cdot}_{X\hat{|}Y}$ of the De Rham complex $\Omega^{\cdot}_X$ along Y and define the De Rham Cohomology of Y over $\mathbb{C}$ as $\mathbb{H}^{\cdot}(Y, \Omega^{\cdot}_{X\hat{|}Y})$. At this point one has three different cohomologies

$$H^{\cdot}_{DR}(Y/\mathbb{C}), \tag{1}$$

$$H^{\cdot}(Y_{inf}, \mathcal{O}_{Y_{inf}}), \tag{2}$$

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and

$$H^{\cdot}(Y^{an}, \mathbb{C}). \tag{3}$$

The isomorphism between (2) and (3) was proved by Grothendieck ([16]) only in the case of a smooth scheme over $\mathbb{C}$. The isomorphism between (1) and (3) was proved by Herrera-Liebermann ([20]), in the case of Y proper over $\mathbb{C}$, while Deligne (unpublished), and Hartshorne ([18, Chapter IV, Theorem (I.I)]) proved it for a general (not necessary proper) scheme over $\mathbb{C}$. A direct statement asserting the isomorphism of these cohomology groups for arbitrary $\mathbb{C}$-schemes Y cannot be found in literature, although all the necessary ingredients are given. The proof presented in this paper, if applied to classical schemes, can be used to fill this gap (see §1). Of course the generalization of this problem to the case of mixed or finite characteristic has been carefully studied by Berthelot and Ogus.

On the other hand, in more recent years the notion of scheme and the properties of schemes have been generalized by the introduction of log schemes. Among the expected features of log schemes, there is the fact that log smooth schemes (which are in general singular as schemes) should behave like classical smooth schemes and moreover should also be related to analytic schemes. The goal of the present work is to introduce the log scheme analogues of (1),(2),(3) over $\mathbb{C}$, and prove the isomorphisms between them.

With these ideas in mind we first consider the analogue of Grothendieck's Infinitesimal Site ([16]) in the logarithmic context (see also [25] for positive characteristic). We work with pro-crystals and we link them to the logarithmic stratification on pro-objects. If we consider an fs log scheme Y over $\mathbb{C}$, in general one cannot expect to have a global closed immersion of Y in a log smooth log scheme, instead we take a good embedding system for it ([36, Definition 2.2.10], i.e. a simplicial scheme $Y.$ which is an étale hypercovering of Y which admits a locally closed immersion in a log smooth log simplicial scheme $X.$). Then we can define the Log De Rham Cohomology of Y over $\mathbb{C}$ (Definition 0.14) (using such a good embedding system by taking log formal tube of $Y.$ in $X.$ (§0.4)) and give a direct proof of the *existence of an isomorphism between the Log Infinitesimal Cohomology of Y over $\mathbb{C}$, and its Log De Rham Cohomology*, namely we prove the following isomorphism

$$H^{\cdot}(Y^{log}_{inf}, \mathcal{O}_{Y^{log}_{inf}}) \cong H^{\cdot}_{DR,log}(Y/\mathbb{C}). \tag{4}$$

This allows us to make our definition of Log De Rham Cohomology independent of any choice.

For the remaining isomorphisms, we were inspired by an article of K. Kato and C. Nakayama ([27, Theorem (0.2), (2)]). Given an fs (ideally) log smooth log scheme X over $\mathbb{C}$, Kato and Nakayama associate a topological space X^{an}_{log} and show that the algebraic Log De Rham Cohomology of X (which is defined as the hypercohomology of the log De Rham complex $\omega^{\cdot}_X$) is isomorphic to the cohomology of the constant sheaf $\mathbb{C}$ on X^{an}_{log}, i.e.

$$H^{\cdot}_{DR,log}(X/\mathbb{C}) =: \mathbb{H}^{\cdot}(X, \omega^{\cdot}_X) \cong H^{\cdot}(X^{an}_{log}, \mathbb{C}). \tag{5}$$

In fact, K. Kato and C. Nakayama proved a more general result, which is a sort of "Logarithmic Riemann-Hilbert correspondence". Indeed, in the case when X is ideally log smooth log scheme over $\mathbb{C}$, the authors construct a log Riemann-Hilbert equivalence Φ between the category of unipotent local systems on X^{an}_{log} and the category of vector bundles on X, equipped with an integrable log connection with nilpotent residues. In the literature, we also can find generalizations of this Riemann-Hilbert correspondence, due to K.Kato, L. Illusie, C. Nakayama ([22]) and A. Ogus ([33]). Both of these works consider the case when X is ideally log smooth log scheme over $\mathbb{C}$. In the first work the authors extend Φ to an equivalence between the category of quasi-unipotent local systems on X^{an}_{log} and the category of vector bundles on the "Kummer étale ringed site" X^{ket} of X endowed with an integrable log connection satisfying a condition of nilpotence of the residues on this site. In the second work, A. Ogus generalize Φ to the category of local systems on X^{an}_{log} with arbitrary (i.e. not necessarily quasi-unipotent) monodromies.

In this present work, we consider the case of constant coefficients, and we prove an analogue of (5) for a general (i.e. not necessarily ideally log smooth) fs log scheme Y over $\mathbb{C}$. This generalization, together with comparison theorem (4), should be the starting point for proving a more general log correspondence between the category of log constructible pro-coherent crystals on Y^{log}_{inf} and the category of constructible sheaves on Y^{an}_{log} ([8]), which will be an extension of results proved by Deligne ([9]) to the logarithmic context. In §2, we extend the theory of Kato-Nakayama ([27]) to the log formal setting. To this end, we first introduce a ringed topological space $(X\hat{|}Y)^{log}$, associated to the log formal analytic space $(X\hat{|}Y)^{an}$, with sheaf of rings $\mathcal{O}^{log}_{(X\hat{|}Y)^{an}}$ (Definition 2.6). This definition is a delicate point because in general one does not have an exact closed immersion of Y into a log smooth log scheme X over $\mathbb{C}$ and the topological space X^{an}_{log} depends upon the monoid which locally gives the log structure on X; so we should study the map $Y^{an}_{log} \longrightarrow X^{an}_{log}$, where the structural sheaves of rings depend heavily on the log structure. In our case, on the other hand, we will reduce ourselves only to exact closed immersions or we take log formal tubes which are again associated to exact closed immersions. Hence, the contribution from the log structure to Y^{an}_{log} and X^{an}_{log} are the same. So, we will define the underlying topological space of the ringed space $(X\hat{\rceil}Y)^{log}$ as Y^{an}_{log} and the structural sheaf by taking completion only with respect to the ideal of the closed immersion (see Definitions 2.2 and 2.6). We will not want to deal in this paper with the possibly more general definition of completion in the context of Kato-Nakayama topological spaces. We construct the complex $\omega^{\cdot,log}_{(X\hat{\rceil}Y)^{an}}$ (Definition 2.7 and (53)), which is a sort of "formal analogue" of the complex $\omega^{\cdot,log}_{X^{an}}$, introduced by Kato-Nakayama for a log smooth log scheme X ([27, (3.5)]).

Later, in §3, we give a "formal version" of the Deligne Poincaré Residue map ([10, (3.6.7.1)]), in the particular case of a smooth scheme X over $\mathbb{C}$, endowed with log structure given by a normal crossing divisor D, and $Y \hookrightarrow X$ a closed subscheme, with the induced log structure (§3.2). We show that this map is an isomorphism. It is useful for describing the cohomology of the complex $\omega^{\cdot}_{(X\hat{\rceil}Y)^{an}}$.

Using that description in §4, we can prove the Log Formal Poincaré Lemma (Theorem 4.1) for an fs log scheme Y which admits a locally closed (not exact) immersion into a log smooth log scheme X over $\mathbb{C}$, under the hypothesis that the schemes are of Zariski type (see after Lemma 0.8, condition satisfied at any level of a good embedding system): *given a general fs log scheme Y of Zariski and finite type over $\mathbb{C}$, $i\colon Y \hookrightarrow X$ a locally closed immersion, with X log smooth of Zariski and finite type over $\mathbb{C}$, the Betti Cohomology of the associated topological space Y^{an}_{log} is isomorphic to the hyper-cohomology of the complex $\omega^{\cdot,log}_{(X\hat{\rceil}Y)^{an}}$.*

In §5, under the previous hypotheses, we show that $\omega^{\cdot}_{(X\hat{\rceil}Y)^{an}}$ is quasi-isomorphic to $\mathbb{R}\tau_*\omega^{\cdot,log}_{(X\hat{\rceil}Y)^{an}} \cong \mathbb{R}\tau_*\mathbb{C}_{Y^{log}}$ (Proposition 5.2), where $\tau\colon Y^{an}_{log} \longrightarrow Y^{an}$ is the canonical (continuous, proper and surjective) Kato-Nakayama map of topological spaces. Then, we prove that there exists an isomorphism in cohomology $\mathbb{H}^{\cdot}(Y^{an}, \omega^{\cdot}_{(X\hat{|}Y)^{an}}) \cong \mathbb{H}^{\cdot}(Y, \omega^{\cdot}_{X\hat{|}Y})$ between the analytic and the algebraic Log De Rham Cohomology (Theorem 5.3). Finally, by using a good embedding system of Y over $\mathbb{C}$, we conclude with the main theorem of this article (Theorem 5.4): *the cohomology of the constant sheaf* $\mathbb{C}$ *on the topological space* Y^{an}_{log}*, associated to an fs log scheme* Y*, is isomorphic to the Log De Rham Cohomology of* Y,

$$H^{\cdot}(Y^{log}_{inf}, \mathcal{O}_{Y^{log}_{inf}}) \cong H^{\cdot}_{DR,log}(Y/\mathbb{C}) \cong H^{\cdot}(Y^{an}_{log}, \mathbb{C}).$$

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Preliminaries

Notations. by S we denote the logarithmic scheme Spec $\mathbb{C}$ endowed with the trivial log structure, and, by a *log scheme*, we mean a logarithmic scheme over S, whose underlying scheme is a separated $\mathbb{C}$-scheme of finite type. Moreover, if $A^{\cdot}$ is a complex of sheaves and $k \in \mathbb{N}$, then $A^{\cdot}[k]$ is the complex defined in degree j as A^{j+k}.

0.1. The logarithmic infinitesimal site

Given a log scheme X, endowed with a fine log structure M, we denote by $InfLog(X/S)$ the Logarithmic Infinitesimal Site of X over S. It is given by 4−uples (U, T, M_T, i), where U is an étale scheme over X, (T, M_T) is a scheme with a fine log structure over S, i is an exact closed immersion $(U, M) \hookrightarrow (T, M_T)$ over S, defined by a nilpotent ideal on T, i.e. i is a nilpotent exact closed immersion. Morphisms, coverings (for the usual étale topology), and sheaves on $InfLog(X/S)$ are defined in the usual way. The category of all sheaves on $InfLog(X/S)$ is a ringed topos, called the Logarithmic Infinitesimal Topos of X over S, and denoted by $(X/S)^{log}_{inf}$, or simply by X^{log}_{inf}.

0.2. Pro-crystals and logarithmic stratification

Let X be a log smooth log scheme. For the definition of pro-objects we refer to [1], [2], [13], [16, §6.2].

DEFINITION 0.1. A **pro-crystal** in X_{inf}^{log} is a collection $\{\mathcal{F}_k\}_{k\in K}$ of $\mathcal{O}_{X_{inf}^{log}}$-modules in X_{inf}^{log} such that, for every morphims $g\colon (U',T',M_{T'},i')\longrightarrow (U,T,M_T,i)$ in X_{inf}^{log}, the natural maps

$$g^*\mathcal{F}_{kT}\longrightarrow \mathcal{F}_{kT'}$$

induce an isomorphism of pro-objects $\{g^*\mathcal{F}_k(U,T,M_T,i)\}_{k\in K}\cong\{\mathcal{F}_k(U',T',M_{T'},i')\}_{k\in K}$. In a similar way we can define Artin-Rees pro-crystals (see [32, Proposition 0.5.1]).

For each integer $i\geq 0$, let $\Delta^1_{log}(i)$ be the $i-$th log infinitesimal neighbourhood (here for the definition we use [25, Remark (5.8)]) of the diagonal $(X,M)\hookrightarrow (X,M)\times_S(X,M)$, and let $\Delta^2_{log}(i)$ be the $i-$th log infinitesimal neighbourhood of $(X,M)\hookrightarrow (X,M)\times_S(X,M)\times_S(X,M)$ (where the fiber product is taken in the category of fine log schemes). We have the canonical projections $p_1(i),p_2(i)\colon \Delta^1_{log}(i)\longrightarrow (X,M)$, and $p_{31}(i),p_{32}(i),p_{21}(i)\colon \Delta^2_{log}(i)\longrightarrow \Delta^1_{log}(i)$. We denote by $\mathcal{P}^{\nu,i}_{X,log}$ the structural sheaf of rings $\mathcal{O}_{\Delta^{\nu}_{log}(i)}$, for each $\nu=1,2$, $i\geq 0$. In particular, we can regard $\mathcal{P}^{1,i}_{X,log}$ as an $\mathcal{O}_X-$module in two ways, via the canonical projections $p_1(i),p_2(i)$. So, we call the left $\mathcal{O}_X-$module structure (resp. right $\mathcal{O}_X-$module structure) on $\mathcal{P}^{1,i}_{X,log}$ the structure given by $p_1(i)$ (resp. $p_2(i)$).

We introduce a logarithmic stratification on the category of pro-coherent $\mathcal{O}_X-$modules. We could define a logarithmic stratification "at any level" of the pro object, and consider the pro-category of log stratified $\mathcal{O}_X-$modules. But this stratification would be too restrictive for our purpose. We need to work with a larger category and, to this end, we introduce the logarithmic stratification as a pro-morphism.

DEFINITION 0.2. [13, Definition 1.3] Let $\{\mathcal{F}_k\}_{k\in K}$ be a pro-coherent $\mathcal{O}_X$ module. A **logarithmic stratification** on $\{\mathcal{F}_k\}_{k\in K}$ is a pro-morphism

$$\{\mathcal{F}_k\}_k\overset{s_{\{\mathcal{F}_k\}_k}}{\longrightarrow}\{\mathcal{F}_k\}_k\otimes\{\mathcal{P}^{1,i}_{X,log}\}_i$$

such that the coidentity diagram

$$\begin{array}{ccc} \{\mathcal{F}_k\}_k & \overset{s_{\{\mathcal{F}_k\}_k}}{\longrightarrow} & \{\mathcal{F}_k\}_k \otimes \{\mathcal{P}^{1,i}_{X,log}\}_i \\ {\scriptstyle id_{\{\mathcal{F}_k\}_k}}\downarrow & & \downarrow{\scriptstyle id_{\{\mathcal{F}_k\}_k}\otimes\{q_{i,0}\}_i} \\ \{\mathcal{F}_k\}_k & \overset{id_{\{\mathcal{F}_k\}_k}}{\longrightarrow} & \{\mathcal{F}_k\}_k \end{array}$$

and the coassociativity diagram

$$\begin{array}{ccc} \{\mathcal{F}_k\}_k & \overset{s_{\{\mathcal{F}_k\}_k}}{\longrightarrow} & \{\mathcal{F}_k\}_k \otimes \{\mathcal{P}^{1,i}_{X,log}\}_i \\ {\scriptstyle s_{\{\mathcal{F}_k\}_k}}\downarrow & & \downarrow{\scriptstyle s_{\{\mathcal{F}_k\}_k}\otimes id_{\{\mathcal{P}^{1,i}_{X,log}\}_i}} \\ \{\mathcal{F}_k\}_k \otimes \{\mathcal{P}^{1,i}_{X,log}\}_i & \overset{id_{\{\mathcal{F}_k\}_k}\otimes s_{\{\mathcal{P}^{1,i}_{X,log}\}_i}}{\longrightarrow} & \{\mathcal{F}_k\}_k \otimes \{\mathcal{P}^{1,i}_{X,log}\}_i \otimes \{\mathcal{P}^{1,i}_{X,log}\}_i \end{array}$$

are commutative, where $q_{i,j} \colon \mathcal{P}^{1,i}_{X,log} \longrightarrow \mathcal{P}^{1,j}_{X,log}$ are the natural compatible maps, and $s_{\{\mathcal{P}^{1,i}_{X,log}\}_i} = \{\delta^{i,j}_X\}_{(i,j)}$ (see [35, Lemma 3.2.3] for the definition of $\delta^{i,j}_X \colon \Delta^1_{log}(i) \times_{(X,M)} \Delta^1_{log}(j) \longrightarrow \Delta^1_{log}(i+j)$).

As in the classical context ([32], [5], [13]), by using the above definition, one can prove the following

THEOREM 0.3. *There exists an equivalence of categories between*
(a) the category of pro-crystals on $InfLog(X/S)$;
(b) the category of $\mathcal{O}_X$ pro-modules $\{\mathcal{M}_k\}_{k\in K}$ on X, endowed with a logarithmic stratification.

REMARK 0.4. In fact, our pro-crystals are actually Artin-Rees procrystals and one could refine the previous result on these objects.

0.3. Linearization of the log De Rham complex

Let $\omega^{\cdot}_X$ be the log De Rham complex of the log smooth log scheme X. As in the classical case ([5, p. 2.17]), we denote the complex of Artin-Rees pro-coherent $\mathcal{O}_X$ modules which is the linearization of $\omega^{\cdot}_X$ by $\{L_X(\omega^{\cdot}_X)_i\}_{i\in\mathbb{N}}$, i.e.

$$\{L_X(\omega^{\cdot}_X)_i\}_{i\in\mathbb{N}} =: \{\mathcal{P}^{1,i}_{X,log}\}_{i\in\mathbb{N}} \otimes_{\mathcal{O}_X} \omega^{\cdot}_X. \tag{6}$$

Since, for all $i, j \in \mathbb{N}$, there exist maps $\mathcal{P}^{1,i+j}_{X,log} \otimes \omega^k_X \longrightarrow \mathcal{P}^{1,i}_{X,log} \otimes \omega^k_X \otimes \mathcal{P}^{1,j}_{X,log}$, each term of (6) has a canonical logarithmic stratification, in the sense of Definition 0.2 ([5, Construction 2.14]).

We have a local description of the differential maps $\{L_X(d^{\cdot}_X)_i\}_i$ of this complex. Indeed, let M^i be the log structure on $\Delta^1_{log}(i)$. Let $U \longrightarrow X$ be an étale morphism of schemes, and let $m \in \Gamma(U, M)$. Then, there exists uniquely an element $u_{m,i}$ in $\Gamma(U, (\mathcal{P}^{1,i}_{X,log})^*) \subset \Gamma(U, M^i)$, such that $p_2(i)^*(m) = p_1(i)^*(m)u_{m,i}$ ([35, pp. 43, 44]). In particular, we have that $u_{m,i} - 1 \in \operatorname{Ker}\{\Gamma(U, \mathcal{P}^{1,i}_{X,log}) \longrightarrow \Gamma(U, \mathcal{O}_X)\}$ ([35, Lemma 3.2.7]).

Let now $x \in X$, and $t_1, ..., t_r \in M_x$ be such that $\{\operatorname{dlog} t_j\}_{1\leq j\leq r}$ is a basis of $\omega^1_{X,x}$. We can restrict to an étale neighborhood U of x, and suppose that $\{\operatorname{dlog} t_j\}$ is a local basis of ω^1_X on U. Let $u_{j,i}$, $1 \leq j \leq r$, $i \geq 0$, be the elements in $\Gamma(U, (\mathcal{P}^{1,i}_{X,log})^*)$ such that $p_2(i)^*(t_j) = p_1(i)^*(t_j)u_{j,i}$, as above. We put $\xi_{j,i} := u_{j,i} - 1 \in \Gamma(\Delta^1_{log}(i), \mathcal{P}^{1,i}_{X,log})$ (note that the $\xi_{j,i}$'s are compatible with respect to i).

Proposition 0.5. *In the above notations,*
(1) [35, Lemma 3.2.7], for each $i \geq 0$, the $\mathcal{O}_X-$module $\mathcal{P}^{1,i}_{X,log}$ is locally free with basis

$$\{\xi^a_i := \prod_{j=1}^{r} \xi^{a_j}_{j,i} | 0 \leq \sum_{j=1}^{r} a_j \leq i\}$$

where $a = (a_1, ..., a_r)$ is a multi-index of length r. In particular, $\{\xi_{1,1}, ..., \xi_{r,1}\}$ is a basis for the locally free $\mathcal{O}_X-$module $\mathcal{P}^{1,1}_{X,log}$, étale locally at x;
(2) [35, Proposition 3.2.5], there exists a canonical isomorphism of $\mathcal{O}_X-$modules $\mathcal{K}/\mathcal{K}^2 \xrightarrow{\cong} \omega^1_X$, where $\mathcal{K} := \operatorname{Ker}\{\Delta^*\colon \mathcal{P}^{1,1}_{X,log} \longrightarrow \mathcal{O}_X\}$ (with $\Delta\colon X \hookrightarrow \Delta^1_{log}(1)$ the exact closed immersion). Under this identification, the local basis $\{\operatorname{dlog} t_j\}_{1\leq j\leq r}$ of ω^1_X is identified with $\{\xi_{j,1}\}_{1\leq j\leq r}$.

Therefore, for each $i \geq 0$, the map $L_X(d^{\cdot}_X)_i$ is the $\mathcal{O}_X-$linear map

$$L_X(d^k_X)_i\colon \mathcal{P}^{1,i}_{X,log} \otimes_{\mathcal{O}_X} \omega^k_X \longrightarrow \mathcal{P}^{1,i-1}_{X,log} \otimes_{\mathcal{O}_X} \omega^{k+1}_X$$

defined, for $a \in \mathcal{O}_X$, $\omega \in \omega^k_X$, and $n_j \in \mathbb{N}$ such that $n_1 + ... + n_r \leq i$, by setting

$$L_X(d^k_X)_i(a\xi^{n_1}_{1,i} \cdot ... \cdot \xi^{n_r}_{r,i} \otimes \omega) = a \cdot \sum_{j=1,...,r} n_j \xi^{n_1}_{1,i-1} \cdot ... \cdot \xi^{n_j-1}_{j,i-1}(1 + \xi_{j,i-1}) \cdot ...$$

$$(7) \qquad \dots \cdot \xi_{r,i-1}^{n_r} \otimes \operatorname{dlog} t_j \wedge \omega + a\, \xi_{1,i-1}^{n_1} \cdot \dots \cdot \xi_{r,i-1}^{n_r} \otimes d\omega.$$

0.4. Log formal tubes, log formal De Rham complex and good embedding systems

Let $i\colon Y \hookrightarrow X$ be a locally closed immersion of fs log schemes. Even in the classical "crystalline" case the definition of the divided powers envelope cannot be transposed in the log setting by copying the ordinary definition: one has to take an exactification of the previous closed immersion. This can be done only étale locally and, moreover, the exactification is not unique. The problem has been solved in [25] (in *ch.p*) by using the universality of the PD envelopes in order to glue the local constructions. In the same article it has been also indicated that this method can be used in *ch.*0 for the n-th log infinitesimal neighborhood, which will satisfy some universal property on the elements of the infinitesimal site with the given order n of nilpotency [25, Remark (5.8)]. On the other hand, for his definition of Log Convergent Cohomology, Shiho in [35] (here we are in mixed characteristic) had to show how to associate a log convergent tube to a closed immersion which is not exact. This has been done by gluing the local data using a hypothesis about the existence of charts in the Zariski topology for the closed immersion (see [35, Proposition-Definition 3.2.1]). We want to clarify the universal construction of the log formal tube in the following.

Definition 0.6. We assume $i\colon (Y,N) \hookrightarrow (X,M)$ is an exact closed immersion. The log formal completion, or simply the log formal tube, $X\hat{|}Y$ of X along Y is the classical formal completion of the scheme X along its closed subscheme Y, endowed with log structure given by the inverse image of M via the canonical map $X\hat{|}Y \longrightarrow X$.

Let us suppose now that the closed immersion $i\colon Y \hookrightarrow X$ is not exact. Let us denote by $(\star)$ the following condition (we refer to [36, §2.2. Condition $(\star)$])

$(\star)$ there exists (at least one) factorization of i of the form

$$(Y,N) \stackrel{i'}{\hookrightarrow} (X',M') \stackrel{f'}{\longrightarrow} (X,M)$$

where i' is an exact closed immersion and f' is a log étale morphism. We note that if i admits a chart $(R_X \longrightarrow M, S_Y \longrightarrow N, R \stackrel{\alpha}{\longrightarrow} S)$ such that α^{gp} is surjective, the above condition is satisfied ([36, Remark 2.2.1]). Indeed, we put $R' := (\alpha^{gp})^{-1}(S)$ and define (X', M') by setting $X' := X \times_{\operatorname{Spec}\mathbb{C}[R]} \operatorname{Spec}\mathbb{C}[R']$ and M' the pull back of the canonical log structure on $\operatorname{Spec}\mathbb{C}[R']$.

Definition 0.7. Let $i\colon Y \hookrightarrow X$ be a closed immersion which satisfies condition $(\star)$. We define the log formal tube $X\hat{\lceil}Y$ of X along Y as the log formal tube of X' along Y (Definition 0.6).

Lemma 0.8. *Definition 0.7 is a good definition, i.e. it is independent of the choice of the factorization as in* $(\star)$.

Proof. The proof is analogous to [36, Proof. of Lemma 2.2.2], on replacing rigid analytic spaces with log formal tubes. □

We recall that a fine log scheme (X, M) is said to be of *Zariski type* if there exists an open covering $X = \bigcup_i X_i$ with respect to Zariski topology such that $(X_i, M_{|X_i})$ admits a chart for any i ([36, Definition 1.1.1]). Moreover, if $i\colon Y \hookrightarrow X$ is a locally closed immersion of fs log schemes of finite type over S and assume that Y and X are of Zariski type, then there exists an open covering (for the Zariski topology) $X = \bigcup_{\alpha\in I} X_\alpha$ such that the morphisms $i_\alpha\colon (Y_\alpha, N) := (Y, N)\times_{(X,M)}(X_\alpha, M) \hookrightarrow (X_\alpha, M)$ are closed immersions and they satisfy condition $(\star)$ (see [36, Proposition 2.2.4, Remark 2.2.6]).

In general, let now $i\colon Y \hookrightarrow X$ be a locally closed immersion, where Y and X are of Zariski and finite type over S. Then one can define the $n-$th log infinitesimal neighborhood (Y_n, N_n) of (Y, N) in (X, M) as a solution of an universal property (see [25, Remark (5.8)] and [35, Remark 3.2.2]) and, in particular, the locally closed immersion $Y \hookrightarrow Y_n$ is exact. If we are in the hypotheses of Definition 0.7, always following [25, Remark (5.8)], the $n-$log infinitesimal neighborhood of Y in X coincides with that of Y in X'. Now, being the locally closed immersions $Y_n \hookrightarrow Y_{n+1}$ exact and compatible, for any $n \in \mathbb{N}$, we can give the following

Definition-Lemma 0.9. *With the assumptions made above, let us define the log formal tube of X along Y as*

$$(X\hat{\lceil}Y, \hat{N}) := \varinjlim_n (Y_n, N_n). \tag{8}$$

This is Y as topological space, endowed with sheaf of rings $\mathcal{O}_{X\hat{|}Y} = \varprojlim_n \mathcal{O}_{Y_n}$ (note that Y_n is not the classical $n-$th infinitesimal neighborhood of Y in X) and the pre-log structure $\varprojlim_n N_n$ is in fact a log structure on $X\hat{|}Y$ (which we denote by $\hat{N}$), which makes the locally closed immersion $Y \hookrightarrow X\hat{|}Y$ exact. Moreover, this definition is compatible with Definitions 0.6 and 0.7.

Proof. Given a group M_1 and two (integral) monoids M_2, M_3, we say that the sequence $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ is an exact (resp. left exact) sequence of monoids if $\operatorname{Ker} f = \{1\}$ and $\operatorname{Coker} f \xrightarrow{\cong} M_3$ (resp. if $\operatorname{Ker} f = \{1\}$ and $\operatorname{Coker} f \longrightarrow M_3$ is injective). Moreover, a sequence of sheaves of (integral) monoids on X, with $\mathcal{M}_1$ a sheaf of groups,

$$\mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2 \xrightarrow{g} \mathcal{M}_3 \tag{9}$$

is exact if $\operatorname{Ker} f = \{1\}$ (constant sheaf) and $\operatorname{Coker} f \xrightarrow{\cong} \mathcal{M}_3$. Given a log scheme (X, M) over S, by this definition of exact sequence, we get in particular that

$$\mathcal{O}_X^* \longrightarrow M \longrightarrow M/\mathcal{O}_X^*$$

is exact. Moreover, we also note that, if $\mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2 \xrightarrow{g} \mathcal{M}_3$ is a left exact sequence of sheaves of monoids on X with $\mathcal{M}_1$ a sheaf of groups, then, only if $\mathcal{M}_2$ is integral, one can show that, for any open set $U \subseteq X$, the sequence of monoids $\mathcal{M}_1(U) \xrightarrow{f(U)} \mathcal{M}_2(U) \xrightarrow{g(U)} \mathcal{M}_3(U)$ is left exact.

Now, in the previous context, let us denote by $\hat{Y}$ the log formal completion $X\hat{|}Y$ defined in (8) as locally ringed topological space over S. Since $\mathcal{O}_{\hat{Y}}^* = \varprojlim_n \mathcal{O}_{Y_n}^* \subset \varprojlim_n N_n$, then $\hat{N}$ is in fact a log structure on $\hat{Y}$. Moreover, since each N_n is integral, then also $\hat{N}$ is integral. For proving that the closed immersion $Y \hookrightarrow \hat{Y}$ is exact and the above construction is compatible with Definitions 0.6 and 0.7, we can work locally on Y and assume that $i\colon Y \hookrightarrow X$ is a closed immersion satisfying condition $(\star)$. So, by using a factorization as in $(\star)$, we may suppose that i is exact and reduce to the following closed immersions

$$Y \hookrightarrow Y_n \hookrightarrow \hat{Y} \overset{h}{\hookrightarrow} X \tag{10}$$

where the first arrow is also exact, by definition of $n-$th log infinitesimal neighborhood. We can put on $\hat{Y}$ two log structures: one is $\hat{N}$, the other is h^*M, induced by the log structure M on X. If we put the log structure h^*M on $\hat{Y}$, we note that all the closed immersions in (10) become exact. We have to show that $\hat{N}$ is isomorphic to h^*M.

To this end, let us consider the following exact sequences of sheaves of monoids on Y,

$$\mathcal{O}^*_{\hat{Y}} \longrightarrow h^*M \longrightarrow h^*M/\mathcal{O}^*_{\hat{Y}} = h^{-1}(M/\mathcal{O}^*_X)$$

and, for any $n \in \mathbb{N}$,

$$\mathcal{O}^*_{Y_n} \xrightarrow{f_n} N_n \longrightarrow N_n/\mathcal{O}^*_{Y_n} = h^{-1}(M/\mathcal{O}^*_X). \tag{11}$$

We first show that the following sequence

$$\mathcal{O}^*_{\hat{Y}} = \varprojlim_n \mathcal{O}^*_{Y_n} \xrightarrow{f} \hat{N} \longrightarrow h^{-1}(M/\mathcal{O}^*_X) = \varprojlim_n (N_n/\mathcal{O}^*_{Y_n})$$

is left exact. Indeed, for any open set $U \subseteq X$, since, for any n, $\mathcal{O}^*_{Y_n}(U) \hookrightarrow N_n(U)$ is injective, and $\{U \longmapsto \hat{N}(U) = \varprojlim_n (N_n(U))\}$ is a sheaf (not only a presheaf), we deduce that $\mathcal{O}^*_{\hat{Y}} \xrightarrow{f} \hat{N}$ is injective. To prove that $\operatorname{Coker} f = \hat{N}/\mathcal{O}^*_{\hat{Y}} \longrightarrow h^{-1}(M/\mathcal{O}^*_X)$ is injective, it will be enough to prove that, for any open set $U \subseteq X$, the map $\hat{N}(U)/\mathcal{O}^*_{\hat{Y}}(U) \xrightarrow{\psi} h^{-1}(M/\mathcal{O}^*_X)(U)$ is injective (as a map of monoids, hence as a map of sets). In fact, since $\operatorname{Coker} f$ is the associated sheaf of the presheaf $\{U \longmapsto \hat{N}(U)/\mathcal{O}^*_{\hat{Y}}(U)\}$, we will get an injective map from $\operatorname{Coker} f$ to $h^{-1}(M/\mathcal{O}^*_X)$.

To prove that ψ is injective, let us take $\{s_n\}_n, \{t_n\}_n \in \varprojlim_n N_n(U)$ such that $\psi(\{s_n\}_n) = \psi(\{t_n\}_n) = m = \{m_n\}_n$ in $h^{-1}(M/\mathcal{O}^*_X)(U) \cong \varprojlim_n (N_n/\mathcal{O}^*_{Y_n})(U)$. From the exact sequence (11), for any n, there exists $u_n \in \mathcal{O}^*_{Y_n}(U)$ (which is a group) such that $s_n = t_n u_n$ in $N_n(U)$ and, if $pr_n \colon N_n(U) \longrightarrow N_{n-1}(U)$, $pr_n(s_n) = s_{n-1}$, $pr_n(t_n) = t_{n-1}$. Then, $s_{n-1} = t_{n-1}u_{n-1}$ in $N_{n-1}(U)$, and $s_{n-1} = pr_n(s_n) = pr_n(t_n u_n) = pr_n(t_n)pr_n(u_n) = t_{n-1}pr_n(u_n)$. Since $N_{n-1}(U)$ is integral, then $pr_n(u_n) = u_{n-1}$, for any n, so $\{u_n\}_n \in \varprojlim_n \mathcal{O}^*_{Y_n}(U)$.

Therefore, we are reduced to the following commutative diagram of sheaves of integral monoids,

$$\begin{array}{ccccc} \mathcal{O}^*_{\hat{Y}} & \stackrel{i_1}{\hookrightarrow} & \hat{N} & \stackrel{f_1}{\longrightarrow} & h^{-1}(M/\mathcal{O}^*_X) \\ \| & & {}^{\lambda}\uparrow & & \| \\ \mathcal{O}^*_{\hat{Y}} & \stackrel{i_2}{\hookrightarrow} & h^*M & \stackrel{f_2}{\longrightarrow} & h^{-1}(M/\mathcal{O}^*_X) \end{array} \tag{12}$$

where the first row is left exact and the second row is exact (λ coming from the universal property of the projective limit $\hat{N}$). We note that a map of monoids is an isomorphism if it is a morphism which is injective and surjective as a map of sets: we have only to show that λ is bijective. To this end, for every $x \in X$, we consider diagram (12) at the stalk x and we prove that the map of monoids $\lambda\colon (h^*M)_x \longrightarrow \hat{N}_x$ is bijective. Let $m_1, m_2 \in (h^*M)_x$ be such that $\lambda(m_1) = \lambda(m_2)$ in $\hat{N}_x$. Then, $f_1\lambda(m_1) = f_1\lambda(m_2)$, so there exists $t \in \mathcal{O}^*_{\hat{Y},x}$ such that $m_1 = i_2(t)m_2$, hence $\lambda(m_1) = \lambda(m_2)i_1(t) = \lambda(m_2)$. Now, since $\hat{N}_x$ is integral, $1 = i_1(t)$ and so $1 = t$ (i_1 is injective). To prove that λ is surjective, let $n \in \hat{N}_x$. If $f_1(n) = 1$, then $n \in i_1\mathcal{O}^*_{\hat{Y},x}$ and so $n = i_1(t) = \lambda(i_2(t))$, for some $t \in \mathcal{O}^*_{\hat{Y},x}$. If $f_1(n) \neq 1$, then $f_1(n) = f_2(m)$, for some $m \in (h^*M)_x$. So, $f_1(n) = f_1(\lambda(m))$ and there exists $t \in \mathcal{O}^*_{\hat{Y},x}$ such that $n = \lambda(m)i_1(t) = \lambda(m\, i_2(t))$ in $\hat{N}_x$. On the other hand, by using the definition of isomorphism one can show that the upper short sequence in (12) is exact. □

From the above definition, it follows that the construction of the log formal tube is functorial with respect to locally closed immersions.

Definition 0.10. In the previous hypotheses and assuming X log smooth log scheme, we define the log formal De Rham complex $\omega^{\cdot}_{\hat{X|Y}}$ of X along Y as $\omega^{\cdot}_{\hat{X|Y}} := \mathcal{O}_{\hat{X|Y}} \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\omega^{\cdot}_X$, with differential induced by that of $\omega^{\cdot}_X$. Moreover, this construction is functorial with respect to locally closed immersions. We define the Log De Rham Cohomology of Y over S as

$$H^{\cdot}_{DR,log}((Y,N)/\mathbb{C}) := \mathbb{H}^{\cdot}(Y, \omega^{\cdot}_{\hat{X|Y}}). \tag{13}$$

Remark 0.11. One should prove that the previous definition of Log De Rham Cohomology is independent of the choices: see discussions after Definition 0.14. Along the line of [36], we can also give another equivalent definition of the log formal tube of X along Y, under a suitable condition (see [36, Proposition 2.2.4]). This is a sort of log formal version of the convergent theory which has been developed in [36, §2.2]. But for such a construction, it is difficult to prove the functoriality, so we prefer to use Definition-Lemma 0.9 to have a log formal tube which is a global defined object.

We give now a definition of Log De Rham Cohomology of a log scheme Y of finite type over S (Definition 0.14), by using good embedding systems (Definition 0.12).

Definition 0.12. [36, Definition 2.2.10] Let (Y, N) be an fs log scheme of finite type over S. A good embedding system of (Y, N) over S is a diagram

$$(Y, N) \xleftarrow{g.} (Y., N.) \xhookrightarrow{i.} (X., M.) \tag{14}$$

where $(Y., N.)$ is a simplicial fine log scheme over (Y, N) such that (Y_j, N_j) is of finite type over S and of Zariski type, $(X., M.)$ is a simplicial fine log scheme over S such that each (X_j, M_j) is log smooth log scheme over S and of Zariski type, $g.\colon Y. \longrightarrow Y$ is an étale hypercovering such that $g_j^* M \longrightarrow M_j$ is an isomorphism for any $j \in \mathbb{N}$ and $i.$ is a morphism of simplicial fine log schemes such that each i_j is a locally closed immersion.

By using analogous methods to those of [36, Proposition 2.2.11], we can prove that there exists at least one good embedding system of Y over S. Indeed, let $\{Y_i\}_{i \in I}$ be an étale covering of Y, such that each Y_i is affine of finite type over S and that $(Y_i, N) \longrightarrow S$ has a chart $(\{1\} \longrightarrow \mathbb{C}^*, R_{i,Y_i} \longrightarrow N, \{1\} \longrightarrow R_i)$. There exist surjections $\mathbb{C}[\mathbb{N}^{n_i}] \longrightarrow \Gamma(Y_i, \mathcal{O}_{Y_i})$, $\mathbb{N}^{m_i} \longrightarrow R_i$. So, let us take $P_i := \operatorname{Spec} \mathbb{C}[\mathbb{N}^{n_i} \oplus \mathbb{N}^{m_i}]$, endowed with log structure L_i given by $\mathbb{N}^{m_i} \longrightarrow \mathbb{C}[\mathbb{N}^{n_i} \oplus \mathbb{N}^{m_i}]$. We get a closed immersion of log schemes $j_i\colon (Y_i, N) \hookrightarrow (P_i, L_i)$, such that the diagram

$$\begin{array}{ccc} \mathbb{C}[\mathbb{N}^{n_i} \oplus \mathbb{N}^{m_i}] & \longrightarrow & \Gamma(Y_i, \mathcal{O}_{Y_i}) \\ \uparrow & & \uparrow \\ \mathbb{N}^{m_i} & \longrightarrow & R_i \end{array} \tag{15}$$

is commutative. Then, we set $(Y_0, N_0) := (\coprod_i Y_i, N_{|\coprod_i Y_i})$, $(P_0, L_0) := \coprod_i (P_i, L_i)$ and $i_0 := \coprod_i j_i$. For $n \in \mathbb{N}$, let (Y_n, N_n) (resp. (P_n, L_n)) be the $(n+1)$−fold fiber product of (Y_0, N_0) (resp. (P_0, L_0)) over (Y, N) (resp. over S). Let $i_n \colon (Y_n, N_n) \hookrightarrow (P_n, L_n)$ be the closed immersion defined by the fiber product of i_0.

We prove now that this construction of a good embedding system for a log scheme (Y, N) of finite type over S is functorial, in the following sense.

Lemma 0.13. *In the previous assumptions, let $f \colon (Y, N) \longrightarrow (Y', N')$ be a morphism of log schemes, where (Y, N), (Y', N') are of finite and Zariski type over S. Then there exist two good embedding systems $Y \longleftarrow Y. \overset{i.}{\hookrightarrow} P.$ for Y over S and $Y' \longleftarrow Y.' \overset{i.'}{\hookrightarrow} P.'$ for Y' over S which are compatible between them, namely such that the following diagram*

$$\begin{array}{ccccc} Y & \longleftarrow & Y. & \overset{i.}{\hookrightarrow} & P. \\ {}^{f}\downarrow & & {}^{f.}\downarrow & & {}^{h.}\downarrow \\ Y' & \longleftarrow & Y.' & \overset{i.'}{\hookrightarrow} & P.' \end{array} \tag{16}$$

is commutative.

Proof. We can take étale coverings $\{Y_i\}_{i \in I}$ of Y, and $\{Y_i'\}_{i \in I}$ of Y' such that each Y_i, Y_i' are affine of finite type over S, and there exist maps $(Y_i, N) \longrightarrow (Y_i', N')$ admitting a chart $(R_{i,Y_i} \longrightarrow N, R'_{i,Y_i'} \longrightarrow N', R_i' \longrightarrow R_i)$ (see [36, §1.1]). Let us take the surjections $\mathbb{N}^{t_i} \longrightarrow R_i'$ and $\mathbb{N}^{m_i} \longrightarrow R_i$ such that the diagram of monoids

$$\begin{array}{ccc} \mathbb{N}^{t_i} & \longrightarrow & R_i' \\ \downarrow & & \downarrow \\ \mathbb{N}^{m_i} & \longrightarrow & R_i \end{array} \tag{17}$$

is commutative, and the surjections $\mathbb{C}[\mathbb{N}^{n_i}] \longrightarrow \Gamma(Y_i, \mathcal{O}_{Y_i})$, $\mathbb{C}[\mathbb{N}^{s_i}] \longrightarrow \Gamma(Y_i', \mathcal{O}_{Y_i'})$ such that the diagram of rings

$$\begin{array}{ccc} \mathbb{C}[\mathbb{N}^{s_i}] & \longrightarrow & \Gamma(Y_i', \mathcal{O}_{Y_i'}) \\ \downarrow & & \downarrow \\ \mathbb{C}[\mathbb{N}^{n_i}] & \longrightarrow & \Gamma(Y_i, \mathcal{O}_{Y_i}) \end{array}$$

is commutative. Let $P_i := \operatorname{Spec} \mathbb{C}[\mathbb{N}^{n_i} \oplus \mathbb{N}^{m_i}]$ (resp. $P_i' := \operatorname{Spec} \mathbb{C}[\mathbb{N}^{s_i} \oplus \mathbb{N}^{t_i}]$), endowed with log structure L_i (resp. L_i') given by $\mathbb{N}^{m_i} \longrightarrow \mathbb{C}[\mathbb{N}^{n_i} \oplus$

$\mathbb{N}^{m_i}]$ (resp. $\mathbb{N}^{t_i} \longrightarrow \mathbb{C}[\mathbb{N}^{s_i} \oplus \mathbb{N}^{t_i}]$). Then the following diagram, where the horizontal maps are closed immersions,

$$\begin{array}{ccc} (Y_i, N) & \overset{j_i}{\hookrightarrow} & (P_i, L_i) \\ {\scriptstyle f_i}\downarrow & & {\scriptstyle h_i}\downarrow \\ (Y_i', N') & \overset{j_i'}{\hookrightarrow} & (P_i', L_i') \end{array} \tag{18}$$

is commutative and it admits the chart (17), namely j_i, j_i' admit charts which are compatible with charts of f_i and h_i.

Then, we set $(Y_0, N_0) := (\coprod_i Y_i, N_{|\coprod_i Y_i})$, $(P_0, L_0) := \coprod_i (P_i, L_i)$ and $i_0 := \coprod_i j_i$ (resp. $(Y_0', N_0') := (\coprod_i Y_i', N_{|\coprod_i Y_i'})$, $(P_0', L_0') := \coprod_i (P_i', L_i')$ and $i_0' := \coprod_i j_i'$) and $f_0 := \coprod_i f_i$ and $h_0 := \coprod_i h_i$. For $n \in \mathbb{N}$, let us take the $(n+1)-$fold fiber products (Y_n, N_n) and (P_n, L_n) (resp. (Y_n', N_n') and (P_n', L_n')) of (Y_0, N_0) over (Y, N) and of (P_0, L_0) over S (resp. of (Y_0', N_0') over (Y', N') and of (P_0', L_0') over S). Let $i_n\colon (Y_n, N_n) \hookrightarrow (P_n, L_n)$ (resp. $i_n'\colon (Y_n', N_n') \hookrightarrow (P_n', L_n')$) be the closed immersion defined by the fiber product of i_0 (resp. i_0') and $f_n\colon (Y_n, N_n) \longrightarrow (Y_n', N_n')$, $h_n\colon (P_n, L_n) \longrightarrow (P_n', L_n')$ be the fiber products of f_0 and h_0 respectively. □

DEFINITION 0.14. Let (Y, N) be an fs log scheme of finite type over S, and let $(Y, N) \overset{g.}{\longleftarrow} (Y., N.) \overset{i.}{\hookrightarrow} (X., M.)$ be a good embedding system of (Y, N) over S. We define the Log De Rham Cohomology of Y over S as

$$H^{\cdot}_{DR,log}((Y, N)/\mathbb{C}) := \mathbb{H}^{\cdot}(Y, \mathbb{R}g._*(\omega^{\cdot}_{X.\hat{|}Y.})) \tag{19}$$

where each $\omega^{\cdot}_{X_j \hat{|} Y_j}$ is defined as in Definition 0.10.

Now, one should prove that our definition of Log De Rham Cohomology is independent of the choice of the good embedding system. We could work out the problem as in [36], but we will prefer to prove, in section §1, that there exists a canonical isomorphism between the Log De Rham Cohomology of Y (Definition 0.14) and its Log Infinitesimal Cohomology. Therefore, from this comparison theorem we will deduce that the Log De Rham Cohomology defined above is in fact a good cohomology theory and independent of all the choices.

0.5. Kato-Nakayama topological space

Let now X^{an} be the (fs) log analytic space associated to X. Kato-Nakayama define the topological space X^{an}_{log} associated to X^{an} as the set $\{(x,h)|x \in X^{an}, h \in \mathrm{Hom}(M^{gp}_x, \mathbb{S}^1), h(f) = f(x)/|f(x)|$, for any $f \in \mathcal{O}^*_{X^{an},x}\}$ (where $\mathbb{S}^1 = \{x \in \mathbb{C}; |x| = 1\}$). Let now $\beta\colon P \longrightarrow M$ be a fixed local chart for X^{an}, with P an fs monoid. The topology on X^{an}_{log} is locally defined as follows:

Definition 0.15. In the local chart β, X^{an}_{log} is identified with a closed subset of $X^{an} \times \mathrm{Hom}(P^{gp}, \mathbb{S}^1)$, via the map $X^{an}_{log} \hookrightarrow X^{an} \times \mathrm{Hom}(P^{gp}, \mathbb{S}^1)$: $(x,h) \longmapsto (x, h_P)$, where h_P is the composite $P^{gp} \longrightarrow M^{gp}_x \xrightarrow{h} \mathbb{S}^1$. So, X^{an}_{log} is locally endowed with the topology induced from the natural topology on $X^{an} \times \mathrm{Hom}(P^{gp}, \mathbb{S}^1)$.

This local topology does not depend on the choice of the chart, so it induces a well defined global topology on X^{an}_{log} ([27, (1.2.1), (1.2.2)]). There exists a surjective map of topological spaces $\tau\colon X^{an}_{log} \longrightarrow X^{an}\colon (x,h) \longmapsto x$, which is continuous and proper ([27, Lemma (1.3)]). Though X^{an}_{log} in general is not an analytic space, it is still endowed with a nice sheaf of rings $\mathcal{O}^{log}_{X^{an}}$. Indeed, let $\mathcal{L}_X$ be the sheaf of abelian groups on X^{an}_{log} which represents the "logarithms of local sections of $\tau^{-1}(M^{gp})$" ([27, (1.4)]). There exists an exact sequence of sheaves of abelian groups

$$0 \longrightarrow \tau^{-1}(\mathcal{O}_{X^{an}}) \xrightarrow{k} \mathcal{L}_X \xrightarrow{exp} \tau^{-1}(M^{gp}/\mathcal{O}^*_{X^{an}}) \longrightarrow 0$$

If we consider commutative $\tau^{-1}(\mathcal{O}_{X^{an}})$-algebras $\mathcal{B}$ on X^{an}_{log}, endowed with a homomorphism $\mathcal{L}_X \longrightarrow \mathcal{B}$ of sheaves of abelian groups which commutes with k, then $\mathcal{O}^{log}_{X^{an}}$ is the universal object among such $\mathcal{B}$ ([27, (3.2)]).

We suppose that X satisfies the following hypothesis ([27, Theorem (0.2), (2)])

(*) Locally for the étale topology, there exists an fs monoid P, an ideal Φ of P, and a morphism $f\colon X \longrightarrow \mathrm{Spec}\,(\mathbb{C}[P]/(\Phi))$ of log schemes over S, such that the underlying morphism of schemes is smooth, and the log structure on X is associated to $P \longrightarrow \mathcal{O}_X$.

Remark 0.16. We note that, if X is (ideally) log smooth over S, then it satisfies hypothesis (*) ([22, Definition (1.5)]), because X is a filtered semi-toroidal variety ([23, Definition 5.2], [14, Proposition II.1.0.11]).

Theorem 0.17. *[27, Theorem (0.2), (2)]. Let X be an fs (ideally) log smooth log scheme (see Remark 0.16). Then, there exists a canonical isomorphism*

$$\mathbb{H}^q(X, \omega_X^{\cdot}) \cong H^q(X_{log}^{an}, \mathbb{C}), \text{ for all } q \in \mathbb{Z}.$$

1. Log Infinitesimal and Log De Rham Cohomologies

From now on, let Y be an fs log scheme of finite type over S, endowed with log structure N. For the moment, we suppose that there exists a (locally) closed immersion $i\colon (Y, N) \hookrightarrow (X, M)$, where X is an fs log smooth log scheme. We consider the direct image functor $i_{inf*}^{log}\colon Y_{inf}^{log} \longrightarrow X_{inf}^{log}$. For a crystal $\mathcal{F}$ of Y_{inf}^{log}, we briefly describe the construction of the direct image $i_{inf*}^{log}\mathcal{F}$, in characteristic zero. Let $(U, T, M_T, j) \in InfLog((X, M)/S)$, then we consider the fiber product (in the category of fine log schemes) $U_Y = (Y, N) \times_{(X,M)} (U, M)$. By base change, the map $U_Y \hookrightarrow (T, M_T)$ is a closed immersion. We can take the $n-$th log infinitesimal neighborhood of U_Y inside (T, M_T), and denote it by (T_n, M_n). Let $\lambda_n\colon (T_n, M_n) \hookrightarrow (T, M_T)$. Then, $(i_{inf*}^{log}\mathcal{F})_{(U,T,M_T,j)} := \varprojlim_n \lambda_{n*}\mathcal{F}_{(U_Y,T_n,M_n,j_n)}$.

Moreover, the Artin-Rees pro-crystal $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ on $InfLog((X, M)/S)$, associated to $i_{inf*}^{log}\mathcal{F}$, is defined on $(U, T, M_T, j) \in InfLog((X, M)/S)$ as $\mathcal{F}_{n(U,T,M_T,j)} := \lambda_{n*}(\mathcal{F}_{(U_Y,T_n,M_n,j_n)}) = \lambda_{n*}(\mathcal{F}_{T_n})$ ([32, Proposition 0.5.1]). In particular, the Artin-Rees pro-crystal $\{\mathcal{O}_n\}_{n\in\mathbb{N}}$ associated to $i_{inf*}^{log}\mathcal{O}_{Y_{inf}^{log}}$, is in fact defined, on each (U, T, M_T, j), as $(\{\mathcal{O}_n\}_{n\in\mathbb{N}})_{(U,T,M_T,j)} := \{\lambda_{n*}\mathcal{O}_{T_n}\}_{n\in\mathbb{N}}$.

Remark 1.1. From Theorem 0.3, the log stratified $\mathcal{O}_X-$pro-module associated to the (Artin-Rees) pro-crystal $\{\mathcal{O}_n\}_{n\in\mathbb{N}}$ is equal to $\{(\mathcal{O}_n)_{(X,X,M,id)}\}_{n\in\mathbb{N}} = \{\lambda_{n*}\mathcal{O}_{Y_n}\}_{n\in\mathbb{N}}$, where Y_n is the $n-$th log infinitesimal neighborhood of $Y \hookrightarrow X$. We simply denote it by $\{\mathcal{O}_{Y_n}\}_{n\in\mathbb{N}}$. Moreover, $(i_{inf*}^{log}\mathcal{O}_{Y_{inf}^{log}})_{(X,X,M,id)} = \mathcal{O}_{X\hat{|}Y}$ (see Definition-Lemma 0.9).

We now compare the Log De Rham Cohomology of Y with its Log Infinitesimal Cohomology. For each fixed $\nu \geq 0$, we consider the diagonal immersion of fine log schemes $X \hookrightarrow X^\nu$, where X^ν is the fiber product over S of $\nu+1$ copies of (X, M) over S. We denote by $\Delta^\nu_{X,log}(i)$ the i-th log infinitesimal neighborhood of the diagonal of X^ν, and by $\mathcal{P}^{\nu,i}_{X,log}$ its structural sheaf of rings $\mathcal{O}_{\Delta^\nu_{X,log}(i)}$.

Now, if we fix ν and vary $i \in \mathbb{N}$, we get the Artin-Rees pro-object of sheaves $\{\mathcal{P}^{\nu,i}_{X,log}\}_i$ on X. On the other hand, if we fix i and vary $\nu \in \mathbb{N}$, we get a sheaf on the simplicial log smooth log scheme

$$\begin{matrix}\longrightarrow & & \longrightarrow & & \longrightarrow & & & & \\ \vdots & X^\nu & \vdots & \cdots & \begin{matrix}\longrightarrow\\ \longrightarrow\end{matrix} & X^2 = X \times_S X \times_S X & \begin{matrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{matrix} & X^1 = X \times_S X & \begin{matrix}\longrightarrow\\ \longrightarrow\end{matrix} \; X \\ \longrightarrow & & \longrightarrow & & \longrightarrow & & & & \end{matrix}$$

which is the following cosimplicial sheaf of rings on X

$$0 \longrightarrow \mathcal{O}_X \; \begin{matrix}\xrightarrow{d_0}\\ \xrightarrow{d_1}\end{matrix} \; \mathcal{P}^{1,i}_{X,log} \; \begin{matrix}\xrightarrow{d_0}\\ \xrightarrow{d_1}\\ \xrightarrow{d_2}\end{matrix} \; \mathcal{P}^{2,i}_{X,log} \; \begin{matrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\end{matrix} \; \cdots \; \begin{matrix}\longrightarrow\\ \vdots\\ \longrightarrow\end{matrix} \; \mathcal{P}^{\nu,i}_{X,log} \; \begin{matrix}\longrightarrow\\ \vdots\\ \longrightarrow\end{matrix} \; \cdots \tag{20}$$

where the maps are given by the faces of the simplicial log scheme $\{X^\nu\}_\nu$. If we vary ν and i, we get a cosimplicial sheaf of Artin-Rees $\mathcal{O}_X$ pro-modules $\{\mathcal{P}^{\nu,i}_{X,log}\}_{\nu,i}$.

We define the cosimplicial Artin-Rees pro-object $\{Q^{\nu,i}_{log}\}_{\nu,i}$, by setting

$$Q^{\nu,i}_{log} := \mathcal{P}^{\nu+1,i}_{X,log} \tag{21}$$

for every $i, \nu \geq 0$. Then, for each $\nu \geq 0$, there is a canonical homomorphism of pro-rings $\alpha^{\nu,\cdot}_{log} \colon \mathcal{P}^{\nu,\cdot}_{X,log} \longrightarrow Q^{\nu,\cdot}_{log}$, defined by the canonical injection $\{0, 1, ..., \nu\} \hookrightarrow \{0, 1, ..., \nu, \nu+1\}$. So, we have a homomorphism of cosimplicial pro-rings

$$\{\alpha^{*,i}_{log}\}_i \colon \{\mathcal{P}^{*,i}_{X,log}\}_i \longrightarrow \{Q^{*,i}_{log}\}_i. \tag{22}$$

Let $\mathcal{N}$ be an $\mathcal{O}_X-$module. As in the classical case, we define the cosimplicial pro-module

$$\{Q^{*,i}_{log}(\mathcal{N})\}_i := \{Q^{*,i}_{log}\}_i \otimes_{\mathcal{O}_X} \mathcal{N}. \tag{23}$$

We note that, for fixed $\nu \geq 0$, $\{Q^{\nu,i}_{log}(\mathcal{N})\}_i$ is clearly a $\{Q^{\nu,i}_{log}\}_i-$module, and so an $\mathcal{O}_X-$bimodule with the obvious left and right structures. Moreover, if we regard $\{Q^{\nu,i}_{log}(\mathcal{N})\}_i$ as a cosimplicial pro-module on $\{\mathcal{P}^{\nu,i}_{X,log}\}_i$ (by restriction of scalars, via $\{\alpha^{\nu,i}_{log}\}_i$), we see that it is the cosimplicial pro-module associated with the $\mathcal{O}_X$−pro-module with canonical stratification $\{Q^{0,i}_{log}(\mathcal{N})\}_i = \{\mathcal{P}^{1,i}_{X,log}\}_i \otimes_{\mathcal{O}_X} \mathcal{N}$. Indeed, for each $\nu \le \mu$, $\{Q^{\mu,i}_{log}(\mathcal{N})\}_i$ is obtained from $\{Q^{\nu,i}_{log}(\mathcal{N})\}_i$ by base change with respect to any of the canonical morphisms $\{\mathcal{P}^{\nu,i}_{X,log}\}_i \longrightarrow \{\mathcal{P}^{\mu,i}_{X,log}\}_i$. Now, for each integer $k \ge 2$, we consider the differential operator d^k of the log De Rham complex,

$$d^k\colon \omega^k_X \longrightarrow \omega^{k+1}_X$$

As in [16, p. 347], for each $\nu \ge 0$, d^k induces a homomorphism of Artin-Rees pro-objects

$$\{Q^{\nu,i}_{log}(d^k)\}_i\colon \{Q^{\nu,i}_{log}(\omega^k_X)\}_i \longrightarrow \{Q^{\nu,i}_{log}(\omega^{k+1}_X)\}_i \tag{24}$$

and we get the following cosimplicial complex of Artin-Rees pro-objects,

$$\{Q^{*,i}_{log}(\mathcal{O}_X)\}_i \longrightarrow \{Q^{*,i}_{log}(\omega^1_X)\}_i \longrightarrow \{Q^{*,i}_{log}(\omega^2_X)\}_i \longrightarrow ... \longrightarrow \{Q^{*,i}_{log}(\omega^k_X)\}_i \longrightarrow ... \tag{25}$$

The double complex associated to the cosimplicial complex of Artin-Rees pro-objects $\{Q^{*,i}_{log}(\omega^{\cdot}_X)\}_i$ is a resolution of $\omega^{\cdot}_X$ (Čech resolution). Indeed, we consider the double complex of $\mathcal{O}_X$ pro-modules

$$\omega^{\cdot}_X \xrightarrow{d_0} \{Q^{0,i}_{log}(\omega^{\cdot}_X)\}_i \xrightarrow{d_1-d_0} \{Q^{1,i}_{log}(\omega^{\cdot}_X)\}_i \xrightarrow{d_2-d_1+d_0} ... \longrightarrow \{Q^{\nu,i}_{log}(\omega^{\cdot}_X)\}_i \longrightarrow ... \tag{26}$$

where the maps are obtained from the cosimplicial maps (20) (with respect to the cosimplicial index ν), by "forgetting one face" ([7, p. 12]). Then, one can show that (26) is locally homotopic to zero, by using the degenerating maps of the cosimplicial complex (20) ([3, §V, Lemma 2.2.1]). Now, we apply to (26) the additive functor $\{\mathcal{O}_{Y_n}\}_{n\in\mathbb{N}} \otimes_{\mathcal{O}_X} (-)$ (where $\{\mathcal{O}_{Y_n}\}_{n\in\mathbb{N}}$ is as in Remark 1.1), in the category of pro-coherent $\mathcal{O}_X$−modules. Since it respects the local homotopies, we find that the complex

$$\{\mathcal{O}_{Y_n}\}_n \otimes \omega^{\cdot}_X \xrightarrow{d_0} \{\mathcal{O}_{Y_n}\}_n \otimes \{Q^{0,i}_{log}(\omega^{\cdot}_X)\}_i \xrightarrow{d_1-d_0} \{\mathcal{O}_{Y_n}\}_n \otimes \{Q^{1,i}_{log}(\omega^{\cdot}_X)\}_i \xrightarrow{d_2-d_1+d_0} ... \tag{27}$$

is also locally homotopic to zero.

We give now a sort of "Log Poincaré Lemma" in characteristic zero.

Theorem 1.2. *The complex of Artin-Rees $\mathcal{O}_X$ pro-modules*

$$\begin{aligned}&[\mathcal{O}_X \xrightarrow{d_0} \{L_X(\omega^{\cdot}_X)\}_i]\\ &\quad = [\mathcal{O}_X \xrightarrow{d_0} \{\mathcal{P}^{1,i}_{X,log}\}_i \longrightarrow \{\mathcal{P}^{1,i}_{X,log}\}_i \otimes_{\mathcal{O}_X} \omega^1_X \longrightarrow \{\mathcal{P}^{1,i}_{X,log}\}_i \otimes_{\mathcal{O}_X} \omega^2_X \longrightarrow ...]\end{aligned} \tag{28}$$

is exact.

PROOF. From (7), it follows that the composition $\mathcal{O}_X \longrightarrow \{L_X(\mathcal{O}_X)_i\}_i \longrightarrow \{L_X(\omega^1_X)_i\}_i$ is zero, so (28) is in fact a complex of $\mathcal{O}_X$ pro-modules. Moreover, it is represented by the following complexes

(29) $$[\mathcal{O}_X \xrightarrow{d_0} \mathcal{P}^{1,i}_{X,log} \xrightarrow{L_X(d^0_X)_i} \mathcal{P}^{1,i-1}_{X,log} \otimes_{\mathcal{O}_X} \omega^1_X \xrightarrow{L_X(d^1_X)_{i-1}} \mathcal{P}^{1,i-2}_{X,log} \otimes_{\mathcal{O}_X} \omega^2_X \xrightarrow{L_X(d^2_X)_{i-2}} ...]$$

for any $i \in \mathbb{N}$. We show that these complexes are exact by induction on $i \in \mathbb{N}$. When $i = 0$, the complex (29) reduces to $0 \longrightarrow \mathcal{O}_X \xrightarrow{id} \mathcal{O}_X \longrightarrow 0$, which is exact. When $i = 1$ the complex (29) is

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{d_0} \mathcal{P}^{1,1}_{X,log} \xrightarrow{L_X(d^0_X)_1} \omega^1_X \longrightarrow 0$$

which is locally homotopic to zero, via the $\mathcal{O}_X-$linear homotopy defined on the local basis as

$$\begin{array}{ccccccc} \mathcal{P}^{1,1}_{X,log} & \xrightarrow{s_0=\Delta^*} & \mathcal{O}_X & \qquad & \omega^1_X & \xrightarrow{s_1} & \mathcal{P}^{1,1}_{X,log} \\ \xi_{j,1} & \longmapsto & 0 & & \mathrm{dlog}\, t_j & \longmapsto & \xi_{j,1} \end{array}$$

for $j = 1, ..., r$ (see Proposition 0.5, (2)). Since $d_0 = p_2(1)^*$ and $\Delta^* \circ p_2(1)^* = id$, we get that $s_0 \circ d_0 = id$. Moreover, $d_0 \circ s_0 + s_1 \circ L_X(d^0_X)_1 = id$.

Now, let us suppose that the complex (29) for $i-1$ is exact. We consider the sequences of locally free $\mathcal{O}_X-$modules, for any $p = 0, ..., r$,

$$0 \longrightarrow \mathcal{K}^{i-p}/\mathcal{K}^{i-p+1} \otimes_{\mathcal{O}_X} \omega^p_X \longrightarrow \mathcal{P}^{1,i-p}_{X,log} \otimes_{\mathcal{O}_X} \omega^p_X \longrightarrow \mathcal{P}^{1,i-p-1}_{X,log} \otimes_{\mathcal{O}_X} \omega^p_X \longrightarrow 0$$

(where $\mathcal{K} = \mathrm{Ker}\,\{\Delta^* \colon \mathcal{P}^{1,1}_{X,log} \longrightarrow \mathcal{O}_X\}$). These are exact (moreover, they are locally homotopic to zero; note that $\mathcal{K}^{i-p}/\mathcal{K}^{i-p+1} \cong \mathrm{Sym}^{i-p}(\mathcal{K}/\mathcal{K}^2)$). So, by using these sequences and the inductive hypothesis, we can reduce to show the exactness of the following complex

$$0 \longrightarrow \mathcal{K}^i/\mathcal{K}^{i+1} \xrightarrow{L_X(d^0_X)_i} \mathcal{K}^{i-1}/\mathcal{K}^i \otimes_{\mathcal{O}_X} \omega^1_X \xrightarrow{L_X(d^1_X)_{i-1}} ...$$
$$... \xrightarrow{L_X(d^r_X)_{i-r}} \mathcal{K}^{i-r}/\mathcal{K}^{i-r+1} \otimes_{\mathcal{O}_X} \omega^r_X \longrightarrow 0$$

But this is locally homotopic to zero, via the following $\mathcal{O}_X-$linear homotopy on the local basis,

$$\mathcal{K}^{i-p}/\mathcal{K}^{i-p+1} \otimes_{\mathcal{O}_X} \omega^p_X \xrightarrow{s_p} \mathcal{K}^{i-p+1}/\mathcal{K}^{i-p+2} \otimes_{\mathcal{O}_X} \omega^{p-1}_X$$

$$\xi_{1,i-p}^{\alpha_1}\cdots\xi_{r,i-p}^{\alpha_r}\otimes\xi_{i_1,1}\wedge\cdots\wedge\xi_{i_p,1}\longmapsto$$
$$\frac{1}{i}\sum_{m=1}^{p}(-1)^{m+1}\xi_{1,i-p+1}^{\alpha_1}\cdots\xi_{r,i-p+1}^{\alpha_r}\xi_{i_m,i-p+1}\otimes\xi_{i_1,1}\wedge\cdots\wedge\hat{\xi}_{i_m,1}\wedge\cdots\wedge\xi_{i_p,1}$$

with $\alpha_1+\cdots+\alpha_r=i-p$. We extend this definition by linearity. It is easy to compute that $s_1\circ L_X(d_X^0)_i=id$ and $L_X(d_X^{p-1})_{i-p+1}\circ s_p+s_{p+1}\circ L_X(d_X^p)_{i-p}=id$, for each $p\geq 1$. □

COROLLARY 1.3. *Let $\{\mathcal{M}_n\}_{n\in\mathbb{N}}$ be a pro-coherent $\mathcal{O}_X$ module. Then the complex*

$$0\longrightarrow\{\mathcal{M}_n\}_n\xrightarrow{d_0}\{\mathcal{M}_n\}_n\otimes_{\mathcal{O}_X}\{\mathcal{P}_{X,log}^{1,i}\}_i\longrightarrow\{\mathcal{M}_n\}_n\otimes_{\mathcal{O}_X}\{\mathcal{P}_{X,log}^{1,i}\}_i\otimes_{\mathcal{O}_X}\omega_X^1$$
$$\longrightarrow\{\mathcal{M}_n\}_n\otimes_{\mathcal{O}_X}\{\mathcal{P}_{X,log}^{1,i}\}_i\otimes_{\mathcal{O}_X}\omega_X^2\longrightarrow\ldots$$

is exact.

PROOF. In order to prove this corollary it is sufficient to follow the proof of Theorem 1.2. By induction on i, we can reduce to prove the exactness at any level, and, since the additive functor $\{\mathcal{M}_n\}_{n\in\mathbb{N}}\otimes_{\mathcal{O}_X}(-)$ respects the local homotopies, we can conclude. □

Now, from Corollary 1.3, the following complex of Artin-Rees $\mathcal{O}_X$ pro-modules

$$\{\mathcal{O}_{Y_n}\}_n\xrightarrow{d_0}\{\mathcal{O}_{Y_n}\}_n\otimes\{\mathcal{P}_{X,log}^{1,i}\}_i\longrightarrow\{\mathcal{O}_{Y_n}\}_n\otimes\{\mathcal{P}_{X,log}^{1,i}\}_i\otimes_{\mathcal{O}_X}\omega_X^1\longrightarrow\ldots\tag{30}$$

is also exact, in the category of pro-coherent $\mathcal{O}_X$ modules.

We consider now the following double complex (□)

$$\begin{array}{ccc}
 & & \{\mathcal{O}_{Y_n}\}_n\otimes\omega_X^{\cdot}\\
 & & {\scriptstyle d_0}\downarrow\\
\{\mathcal{O}_{Y_n}\}_n & \xrightarrow{d_0} & \{\mathcal{O}_{Y_n}\}_n\otimes\{\mathcal{P}_{X,log}^{1,i}\}_i\otimes\omega_X^{\cdot}\\
{\scriptstyle d_1-d_0}\downarrow & & {\scriptstyle d_1-d_0}\downarrow\\
\{\mathcal{O}_{Y_n}\}_n\otimes\{\mathcal{P}_{X,log}^{1,i}\}_i & \xrightarrow{d_0} & \{\mathcal{O}_{Y_n}\}_n\otimes\{\mathcal{P}_{X,log}^{2,i}\}_i\otimes\omega_X^{\cdot}\\
{\scriptstyle d_2-d_1+d_0}\downarrow & & {\scriptstyle d_2-d_1+d_0}\downarrow\\
\ldots & & \ldots\\
\downarrow & & \downarrow\\
\{\mathcal{O}_{Y_n}\}_n\otimes\{\mathcal{P}_{X,log}^{\nu-1,i}\}_i & \xrightarrow{d_0} & \{\mathcal{O}_{Y_n}\}_n\otimes\{\mathcal{P}_{X,log}^{\nu,i}\}_i\otimes\omega_X^{\cdot}\\
\downarrow & & \downarrow\\
\ldots & & \ldots
\end{array}$$

Now, from (27), all the columns of (□), except the first, are locally homotopic to zero. Moreover, from (30), the second row of (□) is exact. The $(\nu+1)-$th row of this double complex ($\nu \geq 2$) is obtained from the second row by tensorizing (over $\mathcal{O}_X$) with the (log stratified) Artin-Rees pro-object $\{\mathcal{P}^{\nu-1,i}_{X,log}\}_i$. Indeed, for each $\nu \geq 0$, $\{\mathcal{P}^{\nu,i}_{X,log}\}_i \cong \{\mathcal{P}^{\nu-1,i}_{X,log}\}_i \otimes \{\mathcal{P}^{1,j}_{X,log}\}_j$. So, since the second row is exact, by following similar arguments as in proof of Corollary 1.3, since the additive functor $\{\mathcal{P}^{\nu-1,i}_{X,log}\}_i \otimes_{\mathcal{O}_X} (-)$ respects the local homotopies, we see that each row of (□), except the first, is also exact.

Therefore, we can conclude that the double complex $\{\mathcal{O}_{Y_n}\}_n \otimes \{Q^{*,i}_{log}(\omega^{\cdot}_X)\}_i$ is a resolution of both the first column $\{\mathcal{O}_{Y_n}\}_n \otimes \{\mathcal{P}^{*,i}_{X,log}\}_i$, and the first row $\{\mathcal{O}_{Y_n}\}_n \otimes \omega^{\cdot}_X$ of (□). Then, since all pro-systems satisfy the Mittag-Leffler condition, we get the two following canonical isomorphisms in cohomology,

$$\mathbb{H}^{\cdot}(Y, \varprojlim_{(n,i)} \mathcal{O}_{Y_n} \otimes \mathcal{P}^{*,i}_{X,log}) \xrightarrow{\cong} \mathbb{H}^{\cdot}(Y, \varprojlim_{(n,i)} \mathcal{O}_{Y_n} \otimes Q^{*,i}_{log}(\omega^{\cdot}_X)) \tag{31}$$

$$\begin{aligned} H^{\cdot}_{DR,log}((Y,N)/\mathbb{C}) &:= \mathbb{H}^{\cdot}(Y, \varprojlim_{n} \mathcal{O}_{Y_n} \otimes \omega^{\cdot}_X) \\ &\xrightarrow{\cong} \mathbb{H}^{\cdot}(Y, \varprojlim_{(n,i)} \mathcal{O}_{Y_n} \otimes Q^{*,i}_{log}(\omega^{\cdot}_X)). \end{aligned} \tag{32}$$

Remark 1.4. Since the Artin-Rees $\mathcal{O}_X$ pro-module $\{\mathcal{O}_{Y_n}\}_n$ is endowed with a log stratification (see Remark 1.1), we have isomorphisms, for any $\nu, k \geq 0$,

$$\{\mathcal{O}_{Y_n}\}_n \otimes \{\mathcal{P}^{\nu,i}_{X,log}\}_i \otimes \omega^{\cdot}_X \cong \{\mathcal{P}^{\nu,i}_{X,log}\}_i \otimes \{\mathcal{O}_{Y_n}\}_n \otimes \omega^{\cdot}_X$$

and so there is an identification

$$\{\mathcal{O}_{Y_n}\}_n \otimes \{Q^{*,i}_{log}(\omega^{\cdot}_X)\}_i \cong \{Q^{*,i}_{log}(\mathcal{O}_{Y_n} \otimes \omega^{\cdot}_X)\}_{n,i}.$$

In order to calculate $H^{\cdot}(Y^{log}_{inf}, \mathcal{O}_{Y^{log}_{inf}})$, one can define the sheaf $\tilde{X} := \varinjlim_n \tilde{Y}_n$ on $InfLog(Y/S)$ (Y_n being the $n-$th log infinitesimal neighborhood of Y in X, and ˜ denoting the sheaf of Y^{log}_{inf} represented by the object), which covers the final object of Y_{inf}^{log} (as in the classical case [16, §5.2]). We denote by $\tilde{X}^{\nu}$ the product of $\tilde{X}$ with itself ν times. This sheaf is an inductive limit of representable sheaves

$$\tilde{X}^{\nu} = \varinjlim_{i} \tilde{\Delta}_{Y}^{\nu-1}(i) \tag{33}$$

where $\Delta_Y^{\nu}(i)$ is the $i-$th log infinitesimal neighborhood of X^{ν} along its closed log subscheme Y. Therefore, if $\mathcal{F}$ is any module on $InfLog(Y/S)$, there exists a Leray spectral sequence ([16, §5.2, p. 338]),

$$E_2^{p,q} = H^{p+1}(\nu \longmapsto H^q(\tilde{X}^{\nu}, \mathcal{F})) \Longrightarrow H^{\cdot}(Y_{inf}^{log}, \mathcal{F}). \tag{34}$$

We want to show that this spectral sequence degenerates, giving a canonical isomorphism

$$H^{\cdot}(Y_{inf}^{log}, \mathcal{F}) \cong H^{\cdot}(\nu \longmapsto \varprojlim_{i} \mathcal{F}(\Delta_Y^{\nu}(i))). \tag{35}$$

To this end, we recall that, for each (U,T) in $InfLog(Y/S)$, there are isomorphisms

$$H^{\cdot}_{InfLog}((U,T), \mathcal{F}) \cong H^{\cdot}_{InfLog}((Y_{inf}^{log})_{|(U,T)}, j^*_{(U,T)}\mathcal{F}) \cong H^{\cdot}_{Zar}(T, \mathcal{F}_{(U,T)})$$

where $(Y_{inf}^{log})_{|(U,T)}$ is the restricted topos, and $j_{(U,T)}$ is the morphism $(Y_{inf}^{log})_{|(U,T)} \longrightarrow Y_{inf}^{log}$ ([5, Propositions 5.24, 5.26]). We first consider the local case and suppose that $Y \hookrightarrow X$ satisfies condition $(\star)$. We may assume Y is affine (so also $\Delta_Y^{\nu}(i)$ is affine). Moreover, we assume that $\mathcal{F}$ is quasi-coherent on each nilpotent thickening (so $\mathcal{F}_{\Delta_Y^{\nu}(i)}$ is quasi-coherent). Then,

$$H^q_{Zar}(\Delta_Y^{\nu}(i), \mathcal{F}_{\Delta_Y^{\nu}(i)}) \cong H^q_{InfLog}((Y, \Delta_Y^{\nu}(i)), \mathcal{F}) = 0 \tag{36}$$

for any $q > 0$. Under these conditions, we take an injective resolution $\mathcal{F} \longrightarrow I^{\cdot}$ of $\mathcal{F}$ in Y_{inf}^{log}, namely, for each $(U,T) \in InfLog((Y,N)/S)$, $\mathcal{F}_T \longrightarrow I^{\cdot}_T$ is exact, and I_T^k is flasque, for each $k \geq 0$ ([3, VI, 1.1.5]).

By (33), $H^q(\tilde{X}^{\nu+1}, \mathcal{F}) = h^q(\Gamma(\tilde{X}^{\nu+1}, I^{\cdot})) = h^q(\Gamma(\varinjlim_{i} \tilde{\Delta}_Y^{\nu}(i), I^{\cdot})) = h^q(\mathrm{Hom}_{Y_{inf}^{log}}(\varinjlim_{i} \tilde{\Delta}_Y^{\nu}(i), I^{\cdot})) = h^q(\varprojlim_{i} \mathrm{Hom}_{Y_{inf}^{log}}(\tilde{\Delta}_Y^{\nu}(i), I^{\cdot}))$, where $h^q =$

$\mathrm{Ker}\, d^q / \mathrm{Im}\, d^{q-1}$ (see [5, Definition 5.15] for the definition of the global section functor). For simplicity, we denote by $\{\mathcal{G}_i^{\cdot}\}_{i \in \mathbb{N}}$ the inverse system of complexes $\{\mathrm{Hom}_{Y_{inf}^{log}}(\tilde{\Delta}_Y^\nu(i), I^{\cdot})\}_{i\in\mathbb{N}}$. Since $I^{\cdot}$ is injective and the maps $(Y, \Delta_Y^\nu(i)) \longrightarrow (Y, \Delta_Y^\nu(i+1))$ are monomorphisms in $InfLog(Y/S)$, we have that the transition maps $\mathcal{G}_{i+1}^{\cdot} \longrightarrow \mathcal{G}_i^{\cdot}$ are surjective, for any $i \in \mathbb{N}$. The inverse system $\{\mathcal{G}_i^{\cdot}\}_{i\in\mathbb{N}}$ satisfies the Mittag-Leffler condition, so we get the exact sequence

$$0 \longrightarrow \varprojlim_i{}^1 h^{q-1}(\mathcal{G}_i^{\cdot}) \longrightarrow h^q(\varprojlim_i \mathcal{G}_i^{\cdot}) \longrightarrow \varprojlim_i h^q(\mathcal{G}_i^{\cdot}) \longrightarrow 0.$$

Moreover, if we assume that $\mathcal{F}$ is a crystal, since $h^{q-1}(\mathcal{G}_i^{\cdot})$ is equal to $\mathcal{F}(\Delta_Y^\nu(i))$ if $q = 1$ and is equal to 0 otherwise, the inverse system $\{h^{q-1}(\mathcal{G}_i^{\cdot})\}_{i\in\mathbb{N}}$ also satisfies the Mittag-Leffler condition, so we get an isomorphism $h^q(\varprojlim_i \mathcal{G}_i^{\cdot}) \cong \varprojlim_i h^q(\mathcal{G}_i^{\cdot})$. Therefore,

$$H^q(\tilde{X}^{\nu+1}, \mathcal{F}) \cong \varprojlim_i H^q_{InfLog}((Y, \Delta_Y^\nu(i)), \mathcal{F}) \cong \varprojlim_i H^q_{Zar}(\Delta_Y^\nu(i), \mathcal{F}_{\Delta_Y^\nu(i)})$$

and so, by (36), $H^q(\tilde{X}^{\nu+1}, \mathcal{F}) = 0$, for any $q > 0$. Then (34) degenerates and for the crystal $\mathcal{F} = \mathcal{O}_{Y_{inf}^{log}}$ we have

$$H^{\cdot}(Y_{inf}^{log}, \mathcal{O}_{Y_{inf}^{log}}) \cong H^{\cdot}(\nu \longmapsto \varprojlim_i \mathcal{O}_{\Delta_Y^\nu(i)}(\Delta_Y^\nu(i))) = H^{\cdot}(Y, \varprojlim_i \mathcal{O}_{\Delta_Y^*(i)}).$$

Finally, since the cosimplicial sheaves on Y_{Zar} $\varprojlim_{(n,i)} \mathcal{O}_{Y_n} \otimes \mathcal{P}^{*,i}_{X,log}$ and $\varprojlim_i \mathcal{O}_{\Delta_Y^*(i)}$ coincide,

$$H^{\cdot}(Y_{inf}^{log}, \mathcal{O}_{Y_{inf}^{log}}) \cong H^{\cdot}(Y, \varprojlim_i \mathcal{O}_{\Delta_Y^*(i)}) \cong H^{\cdot}(Y, \varprojlim_{(n,i)} \mathcal{O}_{Y_n} \otimes \mathcal{P}^{*,i}_{X,log}). \tag{37}$$

If we do not assume that Y is affine, but we always admit the existence of a locally closed immersion $i\colon Y \hookrightarrow X$, then one can define a cosimplicial sheaf $\mathbb{O}^*$ on Y_{Zar}, given by, for each fixed $\nu \geq 0$ and for any open $U \hookrightarrow Y_{Zar}$,

$$\mathbb{O}^\nu\colon U \longmapsto \varprojlim_i \mathcal{O}_{Y_{inf}^{log}}((U, \Delta_U^\nu(i))) = \mathcal{O}_{Y_{inf}^{log}}((U, X^\nu \hat{|} U))$$

where $X^{\nu}\hat{\lceil}U$ is as in Definition-Lemma 0.9. By taking a covering of Y_{Zar} by affine open sets and using (37) as in [16, (5.1), p. 339], we end with a canonical isomorphism

$$H^{\cdot}(Y_{inf}^{log}, \mathcal{O}_{Y_{inf}^{log}}) \cong \mathbb{H}^{\cdot}(Y_{Zar}, \mathbb{O}^{*}) = \mathbb{H}^{\cdot}(Y, \varprojlim_{(n,i)} \mathcal{O}_{Y_n} \otimes \mathcal{P}^{*,i}_{X,log}). \tag{38}$$

By (31), (32) and (38), we conclude that there exists a canonical isomorphism

$$H^{\cdot}_{DR,log}((Y,N)/\mathbb{C}) \cong H^{\cdot}(Y_{inf}^{log}, \mathcal{O}_{Y_{inf}^{log}}). \tag{39}$$

REMARK 1.5. Forgetting log structure, one could use analogous techniques in the classical setting (which are nothing but a miscellanea of those ones indicated in [16]) and obtain the isomorphism (39) for any scheme Y over S (without log). This fact, together with the result proved by Hartshorne in [18, Chapter IV, Theorem I.I], gives the isomorphisms between (1), (2) and (3) in the Introduction.

Now, we consider an fs log scheme (Y,N) of finite type over S which does not necessarily admit a (locally) closed immersion $(Y,N) \hookrightarrow (X,M)$ as above. We take a good embedding system $Y \xleftarrow{g.} Y. \overset{i.}{\hookrightarrow} P.$ for Y over S and define the Log De Rham Cohomology of Y as in Definition 0.14. By analogous arguments to those of [36, Proposition 2.1.20], one can show that the Log Infinitesimal Cohomology satisfies the descent property with respect to étale hypercoverings, namely there exists a canonical isomorphism

$$H^{\cdot}(Y_{inf}^{log}, \mathcal{O}_{Y_{inf}^{log}}) \cong H^{\cdot}(Y._{inf}^{log}, g.^{-1}\mathcal{O}_{Y_{inf}^{log}}) = H^{\cdot}(Y._{inf}^{log}, g.^{*}\mathcal{O}_{Y_{inf}^{log}}).$$

Then we have canonical isomorphisms

$$H^{\cdot}(Y_{inf}^{log}, \mathcal{O}_{Y_{inf}^{log}}) \cong H^{\cdot}(Y._{inf}^{log}, \mathcal{O}_{Y._{inf}^{log}}) \overset{(39)}{\cong} H^{\cdot}_{DR,log}((Y.,N.)/\mathbb{C}) \cong$$

$$\mathbb{H}^{\cdot}(Y, \mathbb{R}g._{*}(\omega^{\cdot}_{X.\hat{\lceil}Y.})) =: H^{\cdot}_{DR,log}((Y,N)/\mathbb{C}).$$

2. The Complex $\omega^{\cdot,log}_{X\hat{|}Y}$

We will transpose our construction to the log analytic situation. If X is a log scheme, then the associated topological space X^{an}_{log} depends not only on the underlying scheme but also on the log structure. Let us start with an exact closed immersion $i\colon Y \hookrightarrow X$ of fs log schemes of finite type over S, where X is log smooth. Let Y^{an}, X^{an} be the associated fs log analytic spaces, and let $i^{an}\colon Y^{an} \hookrightarrow X^{an}$ be the corresponding analytic exact closed immersion with defining ideal $\mathscr{I}$. When the context obviates any confusion, we will omit the superscript $(-)^{an}$ in denoting the associated analytic spaces.

We consider the closed analytic subspaces Y_k of X, defined by the ideals $\mathscr{I}^k$, with $k \in \mathbb{N}$. On each such Y_k we consider the log structure induced by M_X, i.e., if $i_k\colon Y_k \hookrightarrow X$ is the closed immersion, then we take $M_{Y_k} = i_k^* M_X$. We have a sequence of exact closed immersions, which we denote by φ_k,

$$Y = Y_1 \overset{\varphi_1}{\hookrightarrow} Y_2 \overset{\varphi_2}{\hookrightarrow} Y_3 \overset{\varphi_3}{\hookrightarrow} \dots \overset{\varphi_k}{\hookrightarrow} Y_{k+1} \overset{\varphi_{k+1}}{\hookrightarrow} \dots \hookrightarrow X$$

Therefore, we have a projective system of rings $\{\mathcal{O}_{Y_k} \cong i_k^{-1}(\mathcal{O}_X/\mathscr{I}^k);\ \varphi_k\colon \mathcal{O}_{Y_{k+1}} \longrightarrow \mathcal{O}_{Y_k}\}_{k\geq 1}$, where the transition maps φ_k are surjective. Moreover, the diagram

$$\begin{array}{ccc} \varphi_k^{-1}(M_{Y_{k+1}}) & \longrightarrow & M_{Y_k} \\ {\scriptstyle \alpha_{k+1}}\downarrow & & {\scriptstyle \alpha_k}\downarrow \\ \varphi_k^{-1}(\mathcal{O}_{Y_{k+1}}) & \longrightarrow & \mathcal{O}_{Y_k} \end{array} \tag{40}$$

is commutative, for each $k \geq 1$. Since M_X is a fine log structure on X, each Y_k is endowed with a fine log structure.

The closed immersion $i_k\colon Y_k \hookrightarrow X$ is exact, for each k, so ([25, (1.4.1)])

$$M_{Y_k}/\mathcal{O}^*_{Y_k} = i_k^* M_X/\mathcal{O}^*_{Y_k} \cong i_k^{-1}(M_X/\mathcal{O}^*_X) = (M_X/\mathcal{O}^*_X)_{|Y_k}. \tag{41}$$

REMARK 2.1. Since the underlying topological space of each Y_k is equal to Y, it follows from (41) that $M_{Y_k}/\mathcal{O}^*_{Y_k} \cong (M_X/\mathcal{O}^*_X)_{|Y}$, for each $k \geq 1$. Therefore, if we consider the associated sheaf of groups $(M_{Y_k}/\mathcal{O}^*_{Y_k})^{gp} \cong M^{gp}_{Y_k}/\mathcal{O}^*_{Y_k}$, we have that, for each $k \geq 1$,

$$M^{gp}_{Y_k}/\mathcal{O}^*_{Y_k} \cong (M^{gp}_X/\mathcal{O}^*_X)_{|Y}. \tag{42}$$

Let Y_k^{log} (resp. X^{log}) be the Kato-Nakayama topological space associated to Y_k (resp. to X), and let $\tau_k\colon Y_k^{log} \longrightarrow Y_k$ (resp. $\tau_X\colon X^{log} \longrightarrow X$) be the corresponding surjective, continuous and proper map of topological spaces (§0.4). We now consider the "formal" analytic space $X\hat{|}Y$, which is Y^{an} as topological space, and whose structural sheaf is

$$\mathcal{O}_{X\hat{|}Y} := \varprojlim_k \mathcal{O}_{Y_k} \cong \varprojlim_k i_k^{-1}(\mathcal{O}_X/\mathscr{I}^k).$$

Now, since the closed immersion $i\colon Y \hookrightarrow X$ is exact, the formal completion of the fs log analytic space X along the closed log subspace Y is equal to the classical completion $X\hat{|}Y$, endowed with the log structure induced by M_X (Definition 0.6). So, if $i_{X\hat{|}Y}\colon X\hat{|}Y \hookrightarrow X$, then the log structure on $X\hat{|}Y$ is $i^*_{X\hat{|}Y}M_X$. We denote it by $M_{X\hat{|}Y}$. We now define a ringed topological space $((X\hat{|}Y)^{log}, \mathcal{O}^{log}_{X\hat{|}Y})$, associated to the log formal analytic space $X\hat{|}Y$.

DEFINITION 2.2. With the previous notation, we define $(X\hat{|}Y)^{log}$ to be the topological space Y^{log}, endowed with the following sheaf of rings

$$\mathcal{O}^{log}_{X\hat{|}Y} := \tau_Y^{-1}(\mathcal{O}_{X\hat{|}Y}) \otimes_{\tau_X^{-1}(\mathcal{O}_X)} \mathcal{O}_X^{log}. \tag{43}$$

LEMMA 2.3. *[27, Lemma (3.3)] With the previous notation, let $x \in Y$, $y \in Y^{log}$ be such that $\tau_Y(y) = x$. Let $\mathscr{L}_X$ be the sheaf of logarithms of local sections of $\tau_X^{-1}(M_X^{gp})$ (§0.4). Let $\{t_1, ..., t_n\}$ be a family of elements of the stalk $\mathscr{L}_{X,y}$, whose image under the map $\exp_y\colon \mathscr{L}_{X,y} \longrightarrow \tau_X^{-1}(M_X^{gp}/\mathcal{O}_X^*)_y$ is a $\mathbb{Z}-$basis of $M^{gp}_{X,x}/\mathcal{O}^*_{X,x}$. Then, $\mathcal{O}^{log}_{(X\hat{|}Y),y}$ is isomorphic, as $\mathcal{O}_{(X\hat{|}Y),x}-$algebra, to the polynomial ring $\mathcal{O}_{(X\hat{|}Y),x}[T_1, ..., T_n]$, via the correspondence*

$$\begin{array}{ccc} \mathcal{O}_{(X\hat{|}Y),x}[T_1, ..., T_n] & \longrightarrow & \mathcal{O}^{log}_{(X\hat{|}Y),y} \\ T_i & \longmapsto & t_i \end{array} \tag{44}$$

for $i = 1, ..., n$.

PROOF. By [27, Lemma (3.3)], applied to X^{log}, the isomorphism $\mathcal{O}^{log}_{X,y} \cong \tau_X^{-1}(\mathcal{O}_X)_y[T_1, ..., T_n]$ implies that $\mathcal{O}^{log}_{(X\hat{|}Y),y} \cong \tau_Y^{-1}(\mathcal{O}_{X\hat{|}Y})_y \otimes_{\tau_X^{-1}(\mathcal{O}_X)_y} \tau_X^{-1}(\mathcal{O}_X)_y[T_1, ..., T_n] \cong \tau_Y^{-1}(\mathcal{O}_{X\hat{|}Y})_y[T_1, ..., T_n] \cong \mathcal{O}_{(X\hat{|}Y),x}[T_1, ..., T_n]$. □

REMARK 2.4. In [22, Proposition (3.7)], the authors proved that, given an fs log analytic space X, for any $\mathcal{O}_X-$module M the natural homomorphism $M \longrightarrow \mathbb{R}\tau_*(\mathcal{O}_X^{log} \otimes_{\mathcal{O}_X} M)$ is an isomorphism. In particular, for $M = \mathcal{O}_X$, $\mathcal{O}_X \xrightarrow{\cong} \mathbb{R}\tau_*(\mathcal{O}_X^{log})$. A "formal version" of this last isomorphism can be given, namely

$$\mathcal{O}_{X\hat{|}Y} \xrightarrow{\cong} \mathbb{R}\tau_*(\tau^*\mathcal{O}_{X\hat{|}Y}) = \mathbb{R}\tau_*(\mathcal{O}^{log}_{X\hat{|}Y}). \tag{45}$$

The proof can be worked out as in [22, Proposition (3.7)] and [29, Proof of Lemma 4.5], by applying Lemma 2.3.

LEMMA 2.5. *[27, Lemma (3.4)] Let* $r \in \mathbb{Z}$. *We define a filtration* $\hat{\mathrm{fil}}_r(\mathcal{O}^{log}_{X\hat{|}Y})$ *on* $\mathcal{O}^{log}_{X\hat{|}Y}$ *by*

$$\hat{\mathrm{fil}}_r(\mathcal{O}^{log}_{X\hat{|}Y}) := \tau^{-1}(\mathcal{O}_{X\hat{|}Y}) \otimes_{\tau^{-1}(\mathcal{O}_X)} \mathrm{fil}_r(\mathcal{O}_X^{log}) \tag{46}$$

(where $\mathrm{fil}_r(\mathcal{O}_X^{log})$ *is defined by Kato-Nakayama as* $\mathrm{Im}\{\tau^{-1}(\mathcal{O}_X) \otimes_{\mathbb{Z}} (\bigoplus_{j=1}^r \mathrm{Sym}^j_{\mathbb{Z}}\mathcal{L}_X) \longrightarrow \mathcal{O}_X^{log}\}$*). Then, the canonical map*

$$\tau^{-1}(M_X^{gp}/\mathcal{O}_X^*) \cong \mathcal{L}_X/\tau^{-1}(\mathcal{O}_X) \subseteq \mathrm{fil}_1(\mathcal{O}_X^{log})/\mathrm{fil}_0(\mathcal{O}_X^{log})$$

induces the following isomorphism

$$\tau^{-1}(\mathcal{O}_{X\hat{|}Y}) \otimes_{\mathbb{Z}} \tau^{-1}(\mathrm{Sym}^r_{\mathbb{Z}}(M_X^{gp}/\mathcal{O}_X^*)) \xrightarrow{\cong} \hat{\mathrm{fil}}_r(\mathcal{O}^{log}_{X\hat{|}Y})/\hat{\mathrm{fil}}_{r-1}(\mathcal{O}^{log}_{X\hat{|}Y}). \tag{47}$$

PROOF. By [27, Lemma (3.4)], for any $r \geq 0$, we have an isomorphism

$$\tau^{-1}(\mathcal{O}_X) \otimes_{\mathbb{Z}} \tau^{-1}(\mathrm{Sym}^r_{\mathbb{Z}}(M_X^{gp}/\mathcal{O}_X^*)) \xrightarrow{\cong} \mathrm{fil}_r(\mathcal{O}_X^{log})/\mathrm{fil}_{r-1}(\mathcal{O}_X^{log}). \tag{48}$$

So,

$$\tau^{-1}(\mathcal{O}_{X\hat{|}Y}) \otimes_{\mathbb{Z}} \tau^{-1}(\mathrm{Sym}^r_{\mathbb{Z}}(M_X^{gp}/\mathcal{O}_X^*)) \cong$$

$$\tau^{-1}(\mathcal{O}_{X\hat{|}Y}) \otimes_{\tau^{-1}(\mathcal{O}_X)} \left(\tau^{-1}(\mathcal{O}_X) \otimes_{\mathbb{Z}} \tau^{-1}(\mathrm{Sym}^r_{\mathbb{Z}}(M_X^{gp}/\mathcal{O}_X^*))\right)$$

and, by (48), this is isomorphic to

$$\tau^{-1}(\mathcal{O}_{X\hat{|}Y}) \otimes_{\tau^{-1}(\mathcal{O}_X)} \mathrm{fil}_r(\mathcal{O}_X^{log})/\mathrm{fil}_{r-1}(\mathcal{O}_X^{log}). \tag{49}$$

Now, since the functor $\tau^{-1}(\mathcal{O}_{X\hat{\rceil}Y})\otimes_{\tau^{-1}(\mathcal{O}_X)}(-)$ is right exact, it follows that (49) is isomorphic to $\hat{\mathrm{fil}}_r(\mathcal{O}^{log}_{X\hat{\rceil}Y})/\hat{\mathrm{fil}}_{r-1}(\mathcal{O}^{log}_{X\hat{\rceil}Y})$. □

Let us assume now that $i\colon Y\hookrightarrow X$ is a locally closed immersion (not necessarily exact). We consider, for any $n\in\mathbb{N}$, the $n-$th log infinitesimal neighborhood (Y_n,N_n) of Y in X (see §0.4) and the exact locally closed immersions $Y\hookrightarrow Y_n\hookrightarrow X\hat{\rceil}Y$, where $(X\hat{\rceil}Y,\hat{N})$ is the log formal tube of X along Y (Definition-Lemma 0.9). We repeat the construction of the Kato-Nakayama space (§0.5) for the analytic log formal tube $X\hat{\rceil}Y$. We define the sheaf $\hat{\mathscr{L}}$ of abelian groups on the topological space $(X\hat{\rceil}Y)^{log}=Y^{log}$ as the fiber product of $\mathscr{L}_Y$ and $\tau^{-1}\hat{N}^{gp}$ over $\tau^{-1}N^{gp}$. It represents the "sheaf of logarithms of local sections of $\tau^{-1}\hat{N}^{gp}$ ". We have the following commutative diagram of sheaves of abelian groups on Y^{log},

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 2\pi i\mathbb{Z} & \longrightarrow & \hat{\mathscr{L}} & \xrightarrow{exp} & \tau^{-1}\hat{N}^{gp} & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 2\pi i\mathbb{Z} & \longrightarrow & \mathscr{L}_Y & \xrightarrow{exp} & \tau^{-1}N^{gp} & \longrightarrow & 0\\
 & & \| & & \downarrow & & {\scriptstyle c_Y}\downarrow & & \\
0 & \longrightarrow & 2\pi i\mathbb{Z} & \longrightarrow & \mathrm{Cont}(-,i\mathbb{R}) & \longrightarrow & \mathrm{Cont}(-,\mathbb{S}^1) & \longrightarrow & 0
\end{array}$$

Moreover, we also have the following commutative diagram with exact rows,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 2\pi i\mathbb{Z} & \longrightarrow & \tau^{-1}\mathcal{O}_{X\hat{\rceil}Y} & \xrightarrow{exp} & \tau^{-1}\mathcal{O}^*_{X\hat{\rceil}Y} & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 2\pi i\mathbb{Z} & \longrightarrow & \hat{\mathscr{L}} & \xrightarrow{exp} & \tau^{-1}\hat{N}^{gp} & \longrightarrow & 0
\end{array}$$

Now, since $Y\hookrightarrow X\hat{\rceil}Y$ is exact (by Definition-Lemma 0.9), then $\tau^{-1}\hat{N}^{gp}/\tau^{-1}\mathcal{O}^*_{X\hat{\rceil}Y}\cong\tau^{-1}(N^{gp}/\mathcal{O}^*_Y)$ as sheaves on Y^{log}. Therefore, we get the following exact sequence

$$0\longrightarrow\tau^{-1}\mathcal{O}_{X\hat{\rceil}Y}\xrightarrow{h}\hat{\mathscr{L}}\xrightarrow{exp}\tau^{-1}(N^{gp}/\mathcal{O}^*_Y)\longrightarrow 0. \tag{50}$$

Definition 2.6. We define the sheaf of $\tau^{-1}(\mathcal{O}_{X\hat{\rceil}Y})-$algebras

$$\mathcal{O}^{log}_{X\hat{\rceil}Y}:=(\tau^{-1}(\mathcal{O}_{X\hat{\rceil}Y})\otimes_{\mathbb{Z}}\mathrm{Sym}_{\mathbb{Z}}(\hat{\mathscr{L}}))/\mathcal{A} \tag{51}$$

with the same notations and meanings as in [27, §(3.2)], where $\mathcal{A}$ is the ideal locally generated by sections of the form $f \otimes 1 - 1 \otimes h(f)$, for f a local section of $\tau^{-1}(\mathcal{O}_{X\hat{|}Y})$.

Then, in the case when i is a locally closed immersion, we can define the ringed topological space $(X\hat{|}Y)^{log}$ to be the topological space Y^{log} endowed with sheaf of rings $\mathcal{O}^{log}_{X\hat{|}Y}$ defined above. Now, if i is an exact closed immersion, we can compare the sheaves in Definitions 2.2 and 2.6. We have a well defined morphism of sheaves of $\mathcal{O}_{X\hat{|}Y}-$algebras on Y^{log}, $\Phi\colon \tau^{-1}(\mathcal{O}_{X\hat{|}Y}) \otimes_{\tau^{-1}\mathcal{O}_X} \mathcal{O}^{log}_X \longrightarrow (\tau^{-1}(\mathcal{O}_{X\hat{|}Y}) \otimes_{\mathbb{Z}} \mathrm{Sym}_{\mathbb{Z}}(\hat{\mathscr{L}}))/\mathcal{A}$. By working locally on Y^{log}, if $x \in Y$, $y \in Y^{log}$ are such that $\tau(y) = x$, from the exact sequence (50) and by using arguments as in [27, Lemma (3.3)], we get $\hat{\mathscr{L}}_y \cong \tau^{-1}(O_{X\hat{|}Y})_y \oplus \mathbb{Z}^{\oplus n}$. So, the stalk at y of the right hand side is isomorphic, as $\mathcal{O}_{(X\hat{|}Y),x}-$algebra, to the polynomial ring $\mathcal{O}_{(X\hat{|}Y),x}[T_1, ..., T_n]$. But, since the closed immersion i is exact, from Lemma 2.3 we also have that the stalk at y of the left hand side is isomorphic to $\mathcal{O}_{(X\hat{|}Y),x}[T_1, ..., T_n]$ and we conclude that Φ is an isomorphism.

Let us assume now that i is a closed immersion satisfying condition $(\star)$, i.e. such that it admits a factorization $Y \overset{i'}{\hookrightarrow} X' \overset{f'}{\longrightarrow} X$ into an exact closed immersion i' and a log étale map f'. Since Y_n coincides with the $n-$th log infinitesimal neighborhood of Y in X', we get that the sheaf $\mathcal{O}^{log}_{X\hat{|}Y}$ coincides with $\mathcal{O}^{log}_{X'\hat{|}Y}$, by construction. Later we will use this fact, since condition $(\star)$ is always étale locally satisfied in the case of a locally closed immersion.

Definition 2.7. In the previous notations, for any $q \in \mathbb{N}$, $0 \leq q \leq \mathrm{rk}_{\mathbb{Z}}\, \omega^1_X$, we define the following sheaf on Y^{log}

$$\omega^{q,log}_{X\hat{|}Y} := \mathcal{O}^{log}_{X\hat{|}Y} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\omega^q_X) \tag{52}$$

where $\mathcal{O}^{log}_{X\hat{|}Y}$ is as in Definition 2.6.

Since X is log smooth over S, it follows that ω^q_X is a locally free $\mathcal{O}_X-$module of finite type, and so $\omega^{q,log}_{X\hat{|}Y}$ is a locally free $\mathcal{O}^{log}_{X\hat{|}Y}-$module of finite type. Moreover, by definition of $\mathcal{O}^{log}_{X\hat{|}Y}$ as quotient of $\tau^{-1}(\mathcal{O}_{X\hat{|}Y}) \otimes_{\mathbb{Z}} \mathrm{Sym}_{\mathbb{Z}}(\hat{\mathscr{L}})$, we see that there exists a unique derivation $\hat{d}^1\colon \mathcal{O}^{log}_{X\hat{|}Y} \longrightarrow \omega^{1,log}_{X\hat{|}Y}$ which extends $\tau^{-1}(\mathcal{O}_{X\hat{|}Y}) \xrightarrow{d} \omega^1_{X\hat{|}Y}$ and satisfies $\hat{d}^1(x) = \mathrm{dlog}(\exp(x))$, for each element $x \in \hat{\mathscr{L}}$. This $\hat{d}^1$ can be extended to get a differential

$$\hat{d}^q\colon \omega^{q,log}_{X\hat{|}Y} \longrightarrow \omega^{q+1,log}_{X\hat{|}Y}. \tag{53}$$

Therefore, we obtain a complex $\omega^{\cdot,log}_{X\hat{|}Y}$.

3. Formal Poincaré Residue Map

In this section, we give a "formal version" of the Poincaré Residue map given by Deligne ([10, (3.1.5.2)]). We consider an fs log scheme Y, with log structure M_Y, and an exact closed immersion $i\colon Y \hookrightarrow X$, where X is an fs log smooth log scheme, with log structure M_X. We also suppose that the underlying scheme of X is smooth over S, and its log structure M_X is given by a normal crossing divisor $D \hookrightarrow X$, i.e. $M_X = j_*\mathcal{O}^*_U \cap \mathcal{O}_X \hookrightarrow \mathcal{O}_X$, where $j\colon U = X - D \hookrightarrow X$ is the open immersion. Let X^{an}, Y^{an} be the log analytic spaces associated to X and Y, which we will simply denote by X, Y, when no confusion can arise.

We take the log De Rham complex $\omega^{\cdot}_X = \Omega^{\cdot}_X(\log M_X) = \Omega^{\cdot}_X(\log D)$. Its completion $\omega^{\cdot}_{X\hat{|}Y}$, along the closed subscheme Y of X, satisfies

$$\omega^i_{X\hat{|}Y} \cong \omega^i_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X\hat{|}Y}$$

for each i, $0 \leq i \leq n = \dim X$, because the $\mathcal{O}_X-$modules ω^i_X are coherent.

We denote by $\hat{d}^i$ the differential $\omega^i_{X\hat{|}Y} \longrightarrow \omega^{i+1}_{X\hat{|}Y}$ of the complex $\omega^{\cdot}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X\hat{|}Y}$. We consider the weight filtration $W.$ on $\omega^{\cdot}_X$ ([10, (3.1.5.1)]): each term $W_k(\omega^i_X) = \Omega^{i-k}_X \wedge \omega^k_X$ is a locally free $\mathcal{O}_X-$module and the map $W_k(\omega^i_X) \longrightarrow \omega^i_X$ is injective, for each $0 \leq k \leq i \leq n$. Now, since i is an exact closed immersion, by Definition 0.6 the log formal tube of X along Y coincides with the classical formal completion. So, from [30, §9, Theorem 55, Corollary 1], $\mathcal{O}_{X\hat{|}Y}$ is flat over $\mathcal{O}_X$. Therefore we get that the morphism

$$W_k(\omega^i_X) \otimes_{\mathcal{O}_X} \mathcal{O}_{X\hat{|}Y} \longrightarrow \omega^i_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X\hat{|}Y} = \omega^i_{X\hat{|}Y}$$

is also injective. Then, we define an increasing filtration $\hat{W}_.$ on $\omega^{\cdot}_{X\hat{|}Y}$, by setting

$$\hat{W}_k(\omega^i_{X\hat{|}Y}) =: W_k(\omega^i_X) \otimes_{\mathcal{O}_X} \mathcal{O}_{X\hat{|}Y}$$

for each $0 \leq k \leq i \leq n$.

Since $\hat{W}_k(\omega^i_{X\hat{|}Y}) = \mathrm{Im}\{\omega^k_X \otimes_{\mathcal{O}_X} \Omega^{i-k}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X\hat{|}Y} \stackrel{\wedge\otimes id}{\longrightarrow} \omega^i_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X\hat{|}Y}\}$, we can write the term $\hat{W}_k$ of the filtration as $\omega^k_X \wedge \Omega^{i-k}_{X\hat{|}Y}$, where $\Omega^{\cdot}_{X\hat{|}Y}$ is the completion of the classical De Rham complex $\Omega^{\cdot}_X$ along Y. Moreover, we note that $\hat{W}_k(\omega^i_{X\hat{|}Y})$ is a locally free $\mathcal{O}_{X\hat{|}Y}-$submodule of $\omega^i_{X\hat{|}Y}$, for each i.

We suppose now that, locally at a point $x \in Y \hookrightarrow X$, the normal crossing divisor D is the union of smooth irreducible components $D = D_1 \cup ... \cup D_r$, where each component D_i is locally defined by the equation $z_i = 0$ (for a local coordinate system $\{z_1, ..., z_n\}$ of X at x). Let S^k be the set of strictly increasing sequences of indices $\sigma = (\sigma_1, ..., \sigma_k)$, where $\sigma_i \in \{1, ..., r\}$, and let $D_\sigma = D_{\sigma_1} \cap ... \cap D_{\sigma_k}$. Let $D_k = \bigcup_{\sigma\in S^k} D_\sigma$ and D^k be the disjoint union $\coprod_{\sigma\in S^k} D_\sigma$. Moreover, let $\pi^k\colon D^k \longrightarrow X$ be the canonical map. Then, locally at x, the $\mathcal{O}_{X\hat{|}Y}$-submodule $\hat{W}_k(\omega^i_{X\hat{|}Y})$ of $\omega^i_{X\hat{|}Y}$ can be written as

$$\hat{W}_k(\omega^i_{X\hat{|}Y}) = \sum_{\sigma\in S^k} \Omega^{i-k}_{X\hat{|}Y} \wedge \mathrm{dlog} z_{\sigma_1} \wedge ... \wedge \mathrm{dlog} z_{\sigma_k}$$

for each $0 \leq i \leq n$. Therefore, the elements of $\hat{W}_k(\omega^i_{X\hat{|}Y})$ are locally linear combinations of terms $\eta \wedge \mathrm{dlog} z_{\sigma_1} \wedge ... \wedge \mathrm{dlog} z_{\sigma_k}$, with $\eta \in \Omega^{i-k}_{X\hat{|}Y}$.

Let $Y_k = D_k \cap Y$, and $Y^k = \coprod_{\sigma\in S^k}(Y_\sigma)$, with $Y_\sigma = D_\sigma \cap Y$. We have the following cartesian diagram

$$\begin{array}{ccc} Y^k & \hookrightarrow & D^k \\ {\scriptstyle \pi^k_Y}\downarrow & & {\scriptstyle \pi^k}\downarrow \\ Y & \stackrel{i}{\hookrightarrow} & X \end{array} \tag{54}$$

Since each intersection D_σ is smooth over S, we can take the sheaf of classical differential $i-$forms $\Omega^i_{D^k}$ over D^k; then, $\pi^k_*(\Omega^i_{D^k}) \cong \bigoplus_{\sigma\in S^k}(i_{\sigma *}\Omega^i_{D_\sigma})$, where $i_\sigma\colon D_\sigma \hookrightarrow X$.

So, $(\pi^k_*(\Omega^i_{D^k}))_{\hat{|}Y} \cong (\bigoplus_{\sigma\in S^k}(i_{\sigma *}\Omega^i_{D_\sigma}))_{\hat{|}Y} \cong \bigoplus_{\sigma\in S^k}(i_{\sigma *}\Omega^i_{D_\sigma})_{\hat{|}Y}$.

From the cartesian diagram

$$
\begin{array}{ccc}
Y_\sigma & \hookrightarrow & D_\sigma \\
\downarrow & & {}^{i_\sigma}\downarrow \\
Y & \overset{i}{\hookrightarrow} & X
\end{array}
\tag{55}
$$

we deduce the map $\hat{i}_\sigma\colon D_\sigma\hat{|}Y_\sigma \hookrightarrow X\hat{|}Y$. From [17, (Corollaire (10.14.7))], it follows that

$$(i_{\sigma *}\Omega^i_{D_\sigma})_{\hat{|}Y} \cong \hat{i}_{\sigma *}(\Omega^i_{D_\sigma \hat{|}Y_\sigma})$$

and then

$$(\pi^k_*\Omega^i_{D^k})_{\hat{|}Y} \cong \bigoplus_{\sigma\in S^k} \hat{i}_{\sigma *}(\Omega^i_{D_\sigma\hat{|}Y_\sigma}). \tag{56}$$

3.1. The Formal Poincaré Residue

In [10, (3.1.5.2)], Deligne defines a map of complexes

$$\mathrm{Res}^{\cdot} : \mathrm{Gr}^W_k(\Omega^{\cdot}_X(\log D)) \longrightarrow \pi^k_*\Omega^{\cdot}_{D^k}(\varepsilon^k)[-k] \tag{57}$$

for each $k \leq n$, called the Poincaré Residue map, where ε^k is defined as in [10, (3.1.4)], and represents the orientations of the intersections D_σ of k components of D. Given a local section $\eta \wedge \mathrm{dlog} z_{\sigma_1} \wedge \ldots \wedge \mathrm{dlog} z_{\sigma_k} \in \mathrm{Gr}^W_k(\Omega^p_X(\log D))$, with $\eta \in \Omega^{p-k}_X$, the map Res sends it to $\eta_{|D_\sigma} \otimes$ (orientation $\sigma_1 \ldots \sigma_k$). Deligne proved that Res is an isomorphism of complexes ([10]). Moreover, from [10, (3.1.8.2)], the following sequence of isomorphisms

$$\mathbb{R}^k j_*\mathbb{C} \cong \mathscr{H}^k(j_*\Omega^{\cdot}_U) \cong \mathscr{H}^k(\Omega^{\cdot}_X(\log D)) \cong \varepsilon^k_X$$

implies that there exists an identification

$$\varepsilon^k_X \cong \mathbb{C} \otimes_{\mathbb{Z}} \bigwedge^k M^{gp}_X/\mathcal{O}^*_X \tag{58}$$

(M_X is the log structure on X associated to the normal crossing divisor D, and ε^k_X is the direct image of ε^k via the map $\pi^k\colon D^k \longrightarrow X$ [10, (3.1.4.1)]). Using diagram (54), and (56), we can extend the Deligne Poincaré Residue map to the formal case. Indeed, we consider the map

$$\hat{\mathrm{Res}}^p\colon \mathrm{Gr}^{\hat{W}}_k(\omega^p_{X\hat{|}Y}) \cong \mathrm{Gr}^{\hat{W}}_k(\omega^p_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X\hat{|}Y}) \longrightarrow \hat{\pi}^k_*\Omega^{p-k}_{D^k\hat{|}Y^k}(\varepsilon^k) \tag{59}$$

for each $k \leq p \leq n$, which is locally given by

$$\eta \wedge \mathrm{dlog} z_{\sigma_1} \wedge ... \wedge \mathrm{dlog} z_{\sigma_k} \longmapsto \eta_{|(D_\sigma \hat{|} Y_\sigma)} \otimes (\text{orientation}\, \sigma_1 ... \sigma_k)$$

where $\eta \in \Omega^{p-k}_{X\hat{|}Y}$. This is the completion $\hat{\mathrm{Res}}^p$ of the Deligne Poincaré Residue map (57), in degree p, along the closed subscheme Y. We see that the maps $\hat{\mathrm{Res}}^p$ induce the following $\mathcal{O}_X-$ linear isomorphism of complexes

$$\hat{\mathrm{Res}}^{\cdot} : \mathrm{Gr}^{\hat{W}}_k(\omega^{\cdot}_{X\hat{|}Y}) \xrightarrow{\cong} \hat{\pi}^k_* \Omega^{\cdot}_{D^k \hat{|} Y^k}(\varepsilon^k)[-k] \tag{60}$$

for each $k \leq n$.

To this end, we briefly recall the classical construction of the Deligne Poincaré Residue map. So, given $\sigma \in S^k$, we consider the application

$$\rho_\sigma : \Omega^{p-k}_X \longrightarrow \mathrm{Gr}^W_k(\omega^p_X) \tag{61}$$

which is locally defined by

$$\rho_\sigma(\eta) =: \eta \wedge \mathrm{dlog} z_{\sigma_1} \wedge ... \wedge \mathrm{dlog} z_{\sigma_k}. \tag{62}$$

This map does not depend on the choice of the local coordinates z_i ([12, 3.6.6]). Moreover we have that

$$\rho_\sigma(z_{\sigma_i} \cdot \beta) = 0 \quad \text{and} \quad \rho_\sigma(dz_{\sigma_i} \wedge \gamma) = 0$$

for all sections $\beta \in \Omega^{p-k}_X$, and $\gamma \in \Omega^{p-1-k}_X$. Therefore ρ_σ factorizes into

$$\begin{array}{ccc} \Omega^{p-k}_X & \longrightarrow & i_{\sigma *}\Omega^{p-k}_{D_\sigma} \otimes (\text{orientation}\, \sigma_1 ... \sigma_k) \\ {\scriptstyle \rho_\sigma}\downarrow & \swarrow_{\overline{\rho}_\sigma} & \\ \mathrm{Gr}^W_k(\omega^p_X). & & \end{array} \tag{63}$$

Thus, all these maps being locally compatible with the differentials, the maps $\overline{\rho}_\sigma$ define a morphism of complexes

$$\overline{\rho}^{\cdot} : \pi^k_* \Omega^{\cdot}_{D^k}(\varepsilon^k)[-k] \longrightarrow \mathrm{Gr}^W_k(\omega^{\cdot}_X). \tag{64}$$

This morphism is locally defined by (62), and it is a global morphism on X: it is an isomorphism of complexes. Its inverse isomorphism $\mathrm{Gr}^W_k(\omega^{\cdot}_X) \longrightarrow \pi^k_* \Omega^{\cdot}_{D^k}(\varepsilon^k)[-k]$ is the Deligne Poincaré Residue map $\mathrm{Res}^{\cdot}$ ([12, (3.6.7.1)]). We can see that $\mathrm{Res}^{\cdot}$ is an $\mathcal{O}_X-$linear morphism of complexes: so one can deduce that the maps $\hat{\mathrm{Res}}^p$ in (59), are compatible with the differentials induced from $\omega^{\cdot}_{X\hat{|}Y}$ and $\Omega^{\cdot}_{D^k\hat{|}Y^k}[-k]$, because $\hat{\mathrm{Res}}^p$ comes from the $\mathcal{O}_X-$linear map Res^p by completion along Y. Indeed, we note that

$$\mathrm{Gr}_k^{\hat{W}}(\omega^{\cdot}_{X\hat{|}Y}) \cong \mathrm{Gr}_k^{W}(\omega^{\cdot}_{X}) \otimes_{\mathcal{O}_X} \mathcal{O}_{X\hat{|}Y}$$

and, from (56), we have that

$$(\pi_*^k\Omega^{\cdot}_{D^k}(\varepsilon^k)[-k])_{\hat{|}Y} \cong \bigoplus_{\sigma\in S^k} \hat{i}_{\sigma *}(\Omega^{\cdot}_{D_\sigma\hat{|}Y_\sigma})(\varepsilon^k)[-k] \cong \hat{\pi}_*^k\Omega^{\cdot}_{D^k\hat{|}Y^k}(\varepsilon^k)[-k].$$

So one can conclude that the morphism of complexes $\hat{\mathrm{Res}}^{\cdot}$ (60) is an isomorphism, for each $k \leq n$.

We can also construct the morphism $\hat{\mathrm{Res}}^{\cdot}$ by using a formal version of the classical construction of $\mathrm{Res}^{\cdot}$, described in (61), (62), (63), (64). Indeed, we can define the map

$$\rho_{\sigma\hat{|}Y} : \Omega^{p-k}_{X\hat{|}Y} \longrightarrow \mathrm{Gr}_k^{\hat{W}}(\omega^{p}_{X\hat{|}Y}) \tag{65}$$

which is the completion along Y of (61), and is locally defined as in (62), but with $\eta \in \Omega^{p-k}_{X\hat{|}Y}$. Then, we can see that this map $\rho_{\sigma\hat{|}Y}$ factorizes into

$$\begin{array}{ccc} \Omega^{p-k}_{X\hat{|}Y} & \longrightarrow & i_{\sigma *}\Omega^{p-k}_{D_\sigma\hat{|}Y_\sigma} \otimes (\mathrm{orientation}\, \sigma_1 ... \sigma_k) \\ {\scriptstyle \rho_\sigma}\downarrow & \swarrow_{\overline{\rho}_\sigma} & \\ \mathrm{Gr}_k^{\hat{W}}(\omega^{p}_{X\hat{|}Y}) & & \end{array} \tag{66}$$

which is the formal analogue of (63). We conclude that there exists an isomorphism of complexes on X

$$\overline{\rho}^{\cdot}_{\hat{|}Y} : \pi_*^k\Omega^{\cdot}_{D^k\hat{|}Y^k}(\varepsilon^k)[-k] \longrightarrow \mathrm{Gr}_k^{\hat{W}}(\omega^{\cdot}_{X\hat{|}Y}). \tag{67}$$

In view of this construction, we can give the following

Definition 3.1. In the previous notation, we define the $\mathcal{O}_X-$linear morphism of complexes

$$\hat{\mathrm{Res}}^{\cdot} : \mathrm{Gr}_k^{\hat{W}}(\omega^{\cdot}_{X\hat{|}Y}) \longrightarrow \hat{\pi}_*^k\Omega^{\cdot}_{D^k\hat{|}Y^k}(\varepsilon^k)[-k] \tag{68}$$

as the inverse morphism of $\overline{\rho}^{\cdot}_{\hat{|}Y}$ (67).

3.2. Cohomology of $\omega^{\cdot}_{X\hat{|}Y}$

We consider the case when X is smooth over S, with log structure given by a normal crossing divisor D on X, and Y is a closed subscheme of X with induced log structure. Under the same notations of the previous section, from the classical formal Poincaré Lemma ([18, IV, Theorem 2.1] we get that $\Omega^{\cdot}_{D^k\hat{|}Y^k}$ is a resolution of the constant sheaf $\mathbb{C}_{Y_k}$ on Y_k. Then, from the isomorphism $\hat{\mathrm{Res}}^{\cdot}$, and from (58), we deduce that,

$$\mathcal{H}^q(\mathrm{Gr}_k^W(\omega^{\cdot}_{X\hat{|}Y})) \cong \mathbb{C}_{Y^k} \otimes_{\mathbb{C}} \varepsilon^k_X \cong \mathbb{C}_{Y^k} \otimes_{\mathbb{Z}} \bigwedge^k M^{gp}_X/\mathcal{O}^*_X \quad \text{if} \quad q = k \tag{69}$$

and

$$\mathcal{H}^q(\mathrm{Gr}_k^W(\omega^{\cdot}_{X\hat{|}Y})) = 0 \quad \text{if} \quad q \neq k. \tag{70}$$

Therefore, we deduce that, for each point $x \in Y \cap D$, there exists an isomorphism

$$\mathcal{H}^q(\omega^{\cdot}_{X\hat{|}Y})_x \cong \mathbb{C} \otimes_{\mathbb{Z}} \bigwedge^q (M^{gp}_X/\mathcal{O}^*_X)_x. \tag{71}$$

4. Formal Log Poincaré Lemma

In this section, we generalize the logarithmic version of the Poincaré Lemma, proved by Kato-Nakayama ([27, Theorem (3.8)]) in the case of an (ideally) log smooth log analytic space (i.e. a log analytic space satisfying the assumption (0.4) in [27]). We extend this result to the case of a general fs log analytic space over S, and prove the following

Theorem 4.1. *Let $i\colon Y \hookrightarrow X$ be a locally closed immersion of fs log schemes of Zariski and finite type over S, where X is log smooth over S. Let $\omega^{\cdot,log}_{(X\hat{|}Y)^{an}}$ be the complex introduced in §2. Then, there exists a quasi-isomorphism*

$$\mathbb{C}_{Y^{log}} \xrightarrow{\cong} \omega^{\cdot,log}_{(X\hat{|}Y)^{an}}. \tag{72}$$

To prove this theorem we first need some preliminary results. The methods of the proof are similar to those of [27]. In the sequel, we will indicate with the same notation the algebraic log scheme and its associated log analytic space (without using the $(-)^{an}$ notation).

Let us suppose that the closed immersion i is exact. Let $P \longrightarrow M_X$ be a chart, with P an fs monoid. Let $\mathfrak{p}$ be a prime ideal of P which is sent to $0 \in \mathcal{O}_X$ under $P \longrightarrow M_X \longrightarrow \mathcal{O}_X$. Let T be the fs log analytic space whose underlying space is the same as that of X but whose log structure M_T is associated to $P \setminus \mathfrak{p} \longrightarrow \mathcal{O}_T$. Similarly, let Z be the closed log subspace of T whose underlying space is the same as that of Y and whose log structure is the inverse image of M_T. We have the following commutative diagram of fine log analytic spaces

$$\begin{array}{ccc} (Y, i^* M_X) & \overset{i}{\hookrightarrow} & (X, M_X) \\ \downarrow & & \downarrow \\ (Z, i_T^* M_T) & \overset{i_T}{\hookrightarrow} & (T, M_T) \end{array} \tag{73}$$

where the vertical maps are the identity over the underlying analytic spaces. We also note that, since the closed immersions i and i_T are both exact, the log formal analytic space $T\hat{|}Z$ coincides with the classical formal analytic space $X\hat{|}Y$ and so

$$\omega^{\cdot}_{T\hat{|}Z} \cong \omega^{\cdot}_T \otimes_{\mathcal{O}_X} \mathcal{O}_{X\hat{|}Y}. \tag{74}$$

We introduce now a filtration on the complex $\omega^{\cdot}_{X\hat{|}Y}$. So, for $q, r \in \mathbb{Z}$, let $F_r^{\mathfrak{p}}\omega_X^q$ be the $\mathcal{O}_X$−subsheaf of ω_X^q defined by $F_r^{\mathfrak{p}}\omega_X^q = 0$, if $r < 0$; $F_r^{\mathfrak{p}}\omega_X^q = \mathrm{Im}\{\omega_X^r \otimes \omega_T^{q-r} \longrightarrow \omega_X^q\}$, if $0 \leq r \leq q$; $F_r^{\mathfrak{p}}\omega_X^q = \omega_X^q$, if $q \leq r$ ([27, Fil_r in Lemma (4.4)]).

On the complex $\omega^{\cdot}_{X\hat{|}Y}$ we consider the induced filtration

$$\hat{F}_r^{\mathfrak{p}}\omega^{\cdot}_{X\hat{|}Y} = F_r^{\mathfrak{p}}\omega^{\cdot}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X\hat{|}Y}.$$

LEMMA 4.2. *[27, Lemma (4.4)] In the previous context,*
(1) $\hat{F}_r^{\mathfrak{p}}\omega^{\cdot}_{X\hat{|}Y}$ *is a subcomplex of* $\omega^{\cdot}_{X\hat{|}Y}$.
(2) For any $r \in \mathbb{Z}$, *there is an isomorphism of complexes*

$$\bigwedge^r (P^{gp}/(P \setminus \mathfrak{p})^{gp}) \otimes_{\mathbb{Z}} \omega^{\cdot}_{T\hat{|}Z}[-r] \xrightarrow{\cong} \hat{F}_r^{\mathfrak{p}}\omega^{\cdot}_{X\hat{|}Y}/\hat{F}_{r-1}^{\mathfrak{p}}\omega^{\cdot}_{X\hat{|}Y} \tag{75}$$

whose degree q part is given by

$$(p_1 \wedge ... \wedge p_r) \otimes_{\mathbb{Z}} (\eta \otimes_{\mathcal{O}_X} f) \longmapsto \mathrm{dlog}(p_1) \wedge ... \wedge \mathrm{dlog}(p_r) \wedge (\eta \otimes_{\mathcal{O}_X} f)$$

where $p_1, ..., p_r \in P^{gp}$, $\eta \otimes_{\mathcal{O}_X} f \in \omega^q_{T\hat{|}Z} \cong \omega^{\cdot}_T \otimes_{\mathcal{O}_X} \mathcal{O}_{X\hat{|}Y}$. *The differential of the left side is equal to* $(id \otimes_{\mathbb{Z}} \hat{d}^{\cdot})$, *where* $\hat{d}^{\cdot}$ *is the differential of* $\omega^{\cdot}_{T\hat{|}Z}$.
(3) Let $a \in P^{gp}$ *and assume that* a *does not belong to* $(P \setminus \mathfrak{p})^{gp}$. *Then the complex* $(\omega^q_{X\hat{|}Y})_{q\in\mathbb{Z}}$ *with differential*

$$d_a \colon \omega^q_{X\hat{|}Y} \longrightarrow \omega^{q+1}_{X\hat{|}Y}\,; \quad x \longmapsto d\log(a) \wedge x + dx$$

is acyclic.

Proof. (2). By applying the functor $(-)\otimes\mathcal{O}_{X\hat{|}Y}$ to the exact sequence of coherent sheaves 4.4.1 in [27, Lemma (4.4)], and using (74), we have

$$0 \longrightarrow \omega^1_{T\hat{|}Z} \longrightarrow \omega^1_{X\hat{|}Y} \longrightarrow \mathcal{O}_{X\hat{|}Y} \otimes_{\mathbb{Z}} P^{gp}/(P \setminus \mathfrak{p})^{gp} \longrightarrow 0.$$

(3). Since $d_a(\hat{F}^{\mathfrak{p}}_r \omega^q_{X\hat{|}Y}) \subset \hat{F}^{\mathfrak{p}}_{r+1}\omega^{q+1}_{X\hat{|}Y}$, it is sufficient to prove that, for each $r \in \mathbb{Z}$, the complex $(\hat{F}^{\mathfrak{p}}_{r+q}\omega^q_{X\hat{|}Y}/\hat{F}^{\mathfrak{p}}_{r+q-1}\omega^q_{X\hat{|}Y})_{q\in\mathbb{Z}}$ with differential induced by d_a is acyclic. But by (2), this complex is isomorphic to the complex $((\bigwedge^{r+q}_{\mathbb{Q}} H) \otimes_{\mathbb{Q}} \omega^{-r}_{T\hat{|}Z})_{q\in\mathbb{Z}}$ with differential $x \otimes y \longmapsto (a \wedge x) \otimes y$, where $H = (P^{gp}/(P \setminus \mathfrak{p})^{gp}) \otimes_{\mathbb{Z}} \mathbb{Q}$. □

Let P be an fs monoid and let X be the log analytic space $\mathrm{Spec}\,\mathbb{C}[P]$, endowed with log structure $P \longrightarrow \mathcal{O}_X$. Let $i \colon Y \hookrightarrow X$ be an exact closed immersion, where Y is a fine log analytic space endowed with the induced log structure. We fix a point $x \in Y$. Since $i \colon Y \hookrightarrow X$ is exact, via the canonical isomorphism

$$\omega^1_{X\hat{|}Y} \cong \mathbb{C}[P]_{\hat{|}Y} \otimes_{\mathbb{Z}} P^{gp}$$

the map $P^{gp} \longrightarrow \omega^1_{X\hat{|}Y}$, sending $p \in P^{gp}$ to $\mathrm{dlog}\,p$, corresponds to the map sending p to $1 \otimes p$ ([31, §3]). The image of this map is contained in the closed 1−forms. Therefore, we get a map

$$M^{gp}_{X,x}/\mathcal{O}^*_{X,x} \cong P^{gp} \longrightarrow \mathscr{H}^1(\omega^{\cdot}_{X\hat{|}Y})$$

and, by cup product, we deduce a map

$$\bigwedge^{q}(M^{gp}_{X,x}/\mathcal{O}^*_{X,x}) \cong \bigwedge^{q} P^{gp} \longrightarrow \mathcal{H}^q(\omega^{\cdot}_{X\hat{|}Y}). \tag{76}$$

Let $\mathfrak{b}$ be the prime ideal of P which is the inverse image of the maximal ideal of $\mathcal{O}_{X,x}$ under $P \longrightarrow \mathcal{O}_{X,x}$. We denote by $X(\mathfrak{b})$ the closed analytic subspace $\operatorname{Spec}(\mathbb{C}[P]/(\mathfrak{b}))$ of X, endowed with log structure induced by that of X. The underlying analytic space of $X(\mathfrak{b})$ is equal to $\operatorname{Spec}\mathbb{C}[P \smallsetminus \mathfrak{b}]$, and x belongs to its smooth open analytic subspace $\operatorname{Spec}\mathbb{C}[(P \smallsetminus \mathfrak{b})^{gp}]$. Let $Y(\mathfrak{b})$ be the fiber product

$$\begin{array}{ccc} Y(\mathfrak{b}) & \hookrightarrow & X(\mathfrak{b}) \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & X \end{array} \tag{77}$$

which is a closed subspace of $X(\mathfrak{b})$. Moreover, let $(X\hat{|}Y)(\mathfrak{b})$ be the completion $X(\mathfrak{b})\hat{|}Y(\mathfrak{b})$ of $X(\mathfrak{b})$ along its closed subspace $Y(\mathfrak{b})$, and let $\omega^{\cdot}_{(X\hat{|}Y)(\mathfrak{b})}$ be the formal log complex of $(X\hat{|}Y)(\mathfrak{b})$, with differential maps $\hat{d}_{\mathfrak{b}}$.

We denote by $(\mathcal{O}_{(X\hat{|}Y)(\mathfrak{b})})^{\hat{d}^1_{\mathfrak{b}}=0}$ the kernel of $\hat{d}^1_{\mathfrak{b}}\colon \mathcal{O}_{(X\hat{|}Y)(\mathfrak{b})} \longrightarrow \omega^1_{(X\hat{|}Y)(\mathfrak{b})}$.

LEMMA 4.3. *[27, Lemma (4.5)] In the previous context,*
(1) there exists an isomorphism

$$\mathbb{C} \xrightarrow{\cong} (\mathcal{O}_{(X\hat{|}Y)(\mathfrak{b})})_x^{\hat{d}^1_{\mathfrak{b}}=0} \tag{78}$$

and a quasi-isomorphism

$$\mathbb{C} \xrightarrow{\cong} (\Omega^{\cdot}_{(X\hat{|}Y)(\mathfrak{b})})_x \tag{79}$$

(2) Let $\mathfrak{b}\omega^{\cdot}_{X\hat{|}Y}$ be the subcomplex of $\omega^{\cdot}_{X\hat{|}Y}$ whose degree q part is defined to be the $\mathcal{O}_X-$subsheaf of $\omega^q_{X\hat{|}Y}$ generated by $b\omega^q_{X\hat{|}Y}$, with $b \in \mathfrak{b}$. For any q, the map

$$\bigwedge^{q}(M^{gp}_X/\mathcal{O}^*_X)_x \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \mathcal{H}^q(\omega^{\cdot}_{X\hat{|}Y}/\mathfrak{b}\omega^{\cdot}_{X\hat{|}Y})_x \tag{80}$$

which is induced by the map (76), *is bijective.*
(3) For any ideal $\mathfrak{a}$ *of* P *such that* $\mathfrak{a} \subset \mathfrak{b}$, *the stalk at* x *of the canonical map of complexes*

$$\omega^{\cdot}_{X\hat{|}Y}/\mathfrak{a}\omega^{\cdot}_{X\hat{|}Y} \longrightarrow \omega^{\cdot}_{X\hat{|}Y}/\mathfrak{b}\omega^{\cdot}_{X\hat{|}Y} \tag{81}$$

is a quasi-isomorphism (where the definition of $\mathfrak{a}\omega^{\cdot}_{X\hat{|}Y}$ *is similar to that of* $\mathfrak{b}\omega^{\cdot}_{X\hat{|}Y}$ *in (2)).*

Proof. We first note that the complex $\omega^{\cdot}_{X\hat{|}Y}/\mathfrak{a}\omega^{\cdot}_{X\hat{|}Y}$ is isomorphic to $\omega^{\cdot}_{(X\hat{|}Y)(\mathfrak{a})}$. Indeed, since $\mathfrak{a}$ is an ideal of the monoid P, by [27, Lemma (3.6), (2)], $\omega^1_X/\mathfrak{a}\omega^1_X \cong \omega^1_{X(\mathfrak{a})}$, and it follows that

$$\omega^{\cdot}_{X\hat{|}Y}/\mathfrak{a}\omega^{\cdot}_{X\hat{|}Y} \cong \omega^{\cdot}_X/\mathfrak{a}\omega^{\cdot}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X\hat{|}Y} \cong$$

$$\omega^{\cdot}_{X(\mathfrak{a})} \otimes_{\mathcal{O}_X} \mathcal{O}_{X\hat{|}Y} \cong \omega^{\cdot}_{X(\mathfrak{a})\hat{|}Y(\mathfrak{a})} = \omega^{\cdot}_{(X\hat{|}Y)(\mathfrak{a})}.$$

We start to prove (1) and (2). We may restrict ourselves to the open neighbourhood $\operatorname{Spec}\mathbb{C}[(P\setminus\mathfrak{b})^{gp}]$ of x in $X(\mathfrak{b})$, and consider the restriction of $Y(\mathfrak{b})$ to this open neighborhood. So, x belongs to the closed subspace $Y(\mathfrak{b})\cap\operatorname{Spec}\mathbb{C}[(P\setminus\mathfrak{b})^{gp}]$ of the non-singular analytic space $\operatorname{Spec}\mathbb{C}[(P\setminus\mathfrak{b})^{gp}]$.

In this local situation, from [18, IV Theorem 2.1] we know that the complex $\Omega^{\cdot}_{X(\mathfrak{b})\hat{|}Y(\mathfrak{b})}$ is a resolution of the constant sheaf $\mathbb{C}_{Y(\mathfrak{b})}$ over $Y(\mathfrak{b})$. Therefore, the stalk at x of $(\mathcal{O}_{(X\hat{|}Y)(\mathfrak{b})})^{\hat{d}^1_{\mathfrak{b}}=0}$ is isomorphic to $\mathbb{C}$, and there is a quasi-isomorphism

$$\mathbb{C} \xrightarrow{\cong} (\Omega^{\cdot}_{X(\mathfrak{b})\hat{|}Y(\mathfrak{b})})_x$$

so (1) is proved. Now, we apply Lemma 4.2 by taking $X(\mathfrak{b})$, P, $\mathfrak{b}$ as X, P and $\mathfrak{p}$. We consider $\mathcal{H}^q$ of both sides of Lemma 4.2, (2), and take the stalk at x. Then, $\mathcal{H}^q(\hat{F}^{\mathfrak{b}}_r(\omega^{\cdot}_{(X\hat{|}Y)(\mathfrak{b})})/\hat{F}^{\mathfrak{b}}_{r-1}(\omega^{\cdot}_{(X\hat{|}Y)(\mathfrak{b})}))_x$ is isomorphic to $\bigwedge^r P^{gp}/(P\setminus\mathfrak{b})^{gp} \otimes_{\mathbb{Z}} \mathcal{H}^{q-r}(\Omega^{\cdot}_{X(\mathfrak{b})\hat{|}Y(\mathfrak{b})})_x$, which is isomorphic to $\bigwedge^r P^{gp}/(P\setminus\mathfrak{b})^{gp}\otimes_{\mathbb{Z}}\mathbb{C}$ if $q=r$ and is zero if $q\neq r$. Therefore, since $\omega^{\cdot}_{X\hat{|}Y}/\mathfrak{b}\omega^{\cdot}_{X\hat{|}Y} \cong \omega^{\cdot}_{(X\hat{|}Y)(\mathfrak{b})}$, the stalk at x of $\mathcal{H}^q(\omega^{\cdot}_{X\hat{|}Y}/\mathfrak{b}\omega^{\cdot}_{X\hat{|}Y})$ is isomorphic to $\bigwedge^q(M^{gp}_{X,x}/\mathcal{O}^*_{X,x})\otimes_{\mathbb{Z}}\mathbb{C}$, so (2) is proved.

We prove (3) in four steps (see [27, Proof of Lemma (4.5)]).

Step 1. We show that to prove (3) we may assume that $\mathfrak{a}$ is a prime ideal. Assume there exists $\mathfrak{a}$ for which (81) is not a quasi-isomorphism. Since $\mathbb{Z}[P]$ is a Noetherian ring, the set of such $\mathfrak{a}$ has a maximal element $\mathfrak{q}$. We show that $\mathfrak{q}$ is a prime ideal. To this end, we use similar arguments to those in [27, Proof of Lemma (4.5), Step 1.], where, in our context we have to take C equal to the complex $\omega^{\cdot}_{X\hat{\lceil}Y}$ and $\mathfrak{p} = \{b \in P;\ b^n \in \mathfrak{q}, \text{ for some } n \geq 1\}$. Then we may assume that $\mathfrak{q}$ is a primary ideal and by [27, Lemma (4.2), (2)], there exists $a \in P$, such that $\{x \in P; ax \in \mathfrak{q}\} = \mathfrak{p}$. Hence we are left with the case $a \in \mathfrak{p}$ and to prove the acyclicity of the complex $\mathfrak{q}'\omega^{\cdot}_{X\hat{\lceil}Y}/\mathfrak{q}\omega^{\cdot}_{X\hat{\lceil}Y}$, where $\mathfrak{q}' = \mathfrak{q} \cup Pa$. Now, we note that it is isomorphic to the complex $(\omega^q_{(X\hat{\lceil}Y)(\mathfrak{p})})_{q\in\mathbb{Z}}$ with differential $x \longmapsto d\log(a) \wedge x + dx$, and so the acyclicity follows from Lemma 4.2, (3).

Step 2. We show that to prove (3) for the pair $(P, \mathfrak{a})$, it is enough to prove (3) for the pairs $(P \smallsetminus \mathfrak{p}, \varnothing)$ for prime ideals $\mathfrak{p} \subset \mathfrak{b}$ of P ($\varnothing$ being the empty ideal of $P \smallsetminus \mathfrak{p}$).

By Step 1, we may assume $\mathfrak{a}$ is a prime ideal of P. Let $P' = P \smallsetminus \mathfrak{a}$, $X' = \operatorname{Spec}(\mathbb{C}[P'])^{an}$ with log structure associated to $P' \longrightarrow \mathcal{O}_{X'}$. Moreover, let $Y' = Y \times_X X'$. Then the underlying analytic space of $X(\mathfrak{a})$ (resp. $X(\mathfrak{b})$, $Y(\mathfrak{a})$, $Y(\mathfrak{b})$) coincides with that of $X' = X'(\varnothing)$ (resp. that of $X'(\mathfrak{b}')$, Y', $Y'(\mathfrak{b}')$, where $\mathfrak{b}' = P \cap \mathfrak{b}$). So, by using the graded terms with respect to the filtration $\hat{F}^{\mathfrak{a}}$ on $\omega^{\cdot}_{(X\hat{\lceil}Y)(\mathfrak{a})}$ and $\omega^{\cdot}_{(X\hat{\lceil}Y)(\mathfrak{b})}$ and by applying Lemma 4.2, (2), one can repeat [27, Proof of Lemma (4.5), Step 2] and conclude.

Step 3. We prove (3) in the particular case where $P = \mathbb{N}^r$, for some $r \geq 0$. In this situation, for any prime ideal $\mathfrak{p}$ of P, $P \smallsetminus \mathfrak{p}$ is isomorphic to $\mathbb{N}^s$, for some $s \leq r$. Thus we may assume $P = \mathbb{N}^r$ and $\mathfrak{a} = \varnothing$ by Step 2. We have $X = \mathbb{C}^r$ as an analytic space, with canonical log structure given by a normal crossing divisor $D \hookrightarrow X$, and Y is a closed analytic subspace of X, with induced log structure. Then, we have the isomorphism of complexes $\omega^{\cdot}_{X\hat{\lceil}Y} \cong \Omega^{\cdot}_X(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_{X\hat{\lceil}Y}$. Therefore, we are reduced to the case analyzed in §3.2, and we can use the isomorphism (71) to describe the stalk at x of $\mathcal{H}^q(\omega^{\cdot}_{X\hat{\lceil}Y})$. So, by applying Lemma 4.3, (2), $\mathcal{H}^q(\omega^{\cdot}_{X\hat{\lceil}Y}/\mathfrak{b}\omega^{\cdot}_{X\hat{\lceil}Y})_x \cong \bigwedge^q(M^{gp}_X/\mathcal{O}^*_X)_x \otimes_{\mathbb{Z}} \mathbb{C}$, which is isomorphic to $\mathcal{H}^q(\omega^{\cdot}_{X\hat{\lceil}Y})_x$, via (71).

Step 4. Now, we prove (3) in the general situation. By Step 2, we may assume $\mathfrak{a} = \varnothing$. For a non-empty ideal I of the monoid P, we can consider the toric variety $B_I(\operatorname{Spec}\mathbb{C}[P])$, which we get from X by "blowing-up" along I as in [28, §I, Theorem 10]. It is endowed with a canonical log structure ([25, (3.7)(1)]).

Note. From [26, Proposition (9.8)], and [28, §I, Theorem 11], it is possible to choose an ideal $\tilde{I}$ of P, such that, if $\tilde{X} = B_{\tilde{I}}(\operatorname{Spec}\mathbb{C}[P])$, with log structure $\tilde{M}$, then, for any $y \in \tilde{X}$, $(\tilde{M}/\mathcal{O}^*_{\tilde{X}})_y$ is isomorphic to $\mathbb{N}^{r(y)}$, for some $r(y) \geq 0$. Let $f\colon \tilde{X} \longrightarrow X$ be the proper map, corresponding to the "blowing-up" of X along $\tilde{I}$. Then, locally, $\tilde{X}$ is isomorphic to an open sub log analytic space of $\operatorname{Spec}\mathbb{C}[\mathbb{N}^r]$ endowed with canonical log structure $\mathbb{N}^r \longrightarrow \mathbb{C}[\mathbb{N}^r]$.

Then, we consider the following cartesian diagram

$$\begin{array}{ccc} \tilde{Y} & \overset{\tilde{i}}{\hookrightarrow} & \tilde{X} \\ {\scriptstyle f_{|Y}}\downarrow & & {\scriptstyle f}\downarrow \\ Y & \overset{i}{\hookrightarrow} & X \end{array} \tag{82}$$

where $\tilde{Y} = f^{-1}(Y)$ is a closed subspace of $\tilde{X}$, and we suppose it to be endowed with the inverse image of the log structure $\tilde{M}$. We denote by $\hat{f}$ the morphism $\hat{f}\colon \tilde{X}\hat{|}\tilde{Y} \longrightarrow X\hat{|}Y$ (deduced from the cartesian diagram (82)). We also note that the vertical maps in (82) are log-étale, so

$$\omega^{\cdot}_{\tilde{X}} \cong f^*\omega^{\cdot}_X. \tag{83}$$

Then, from (83), we get

$$\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}} \cong \hat{f}^*\omega^{\cdot}_{X\hat{|}Y}. \tag{84}$$

Moreover, by [28, §I, Corollary 1. c)], there exists a quasi-isomorphism

$$\mathcal{O}_X \overset{\cong}{\longrightarrow} \mathbb{R}f_*\mathcal{O}_{\tilde{X}}. \tag{85}$$

Since f is proper, and X, $\tilde{X}$ are schemes of finite type over S, applying the fundamental theorem of a proper morphism for analytic spaces which come from algebraic varieties [18, §I, Proposition (6.2)] to the structural sheaf $\mathcal{O}_{\tilde{X}}$, we get

(86) $$(\mathbb{R}f_*\mathcal{O}_{\tilde{X}})_{\hat{|}Y} \xrightarrow{\cong} \mathbb{R}\hat{f}_*(\mathcal{O}_{\tilde{X}\hat{|}\tilde{Y}})$$

and, from the isomorphism (85), we get

(87) $$\mathcal{O}_{X\hat{|}Y} \xrightarrow{\cong} \mathbb{R}\hat{f}_*(\mathcal{O}_{\tilde{X}\hat{|}\tilde{Y}}).$$

Therefore, since the $\mathcal{O}_{\tilde{X}}$-module (resp. $\mathcal{O}_X$−module) $\omega^q_{\tilde{X}}$ (resp. ω^q_X) is free of finite rank, for any q, from (84) and (87), we finally get an isomorphism in the derived category

(88) $$\omega^{\cdot}_{X\hat{|}Y} \cong \mathbb{R}\hat{f}_*\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}}.$$

Consider now any $\tilde{x} \in \tilde{Y}$ such that $f(\tilde{x}) = x$. Let $\tilde{\mathfrak{b}}$ be the prime ideal of $\mathbb{N}^r$ equal to the inverse image of the maximal ideal of $\mathcal{O}_{\tilde{X},\tilde{x}}$ under $\mathbb{N}^r \longrightarrow \mathcal{O}_{\tilde{X},\tilde{x}}$. We denote by $\mathfrak{a}$ the ideal of $\mathcal{O}_{\tilde{X},\tilde{x}}$ which is the inverse image of the ideal $\mathfrak{b}$ under f. We have $\mathfrak{a} \subset \tilde{\mathfrak{b}}$. We also note that $\tilde{\mathfrak{b}}$ depends on the choice of $\tilde{x}$ lying over x, but $\mathfrak{a}$ depends only on $x \in X$. We consider the following commutative diagram

(89) $$\begin{array}{ccc} (\omega^{\cdot}_{X\hat{|}Y})_x & \longrightarrow & (\omega^{\cdot}_{X\hat{|}Y}/\mathfrak{b}\omega^{\cdot}_{X\hat{|}Y})_x \\ {\scriptstyle\cong}\downarrow & & \downarrow \\ (\mathbb{R}\hat{f}_*\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}})_x & \longrightarrow & (\mathbb{R}\hat{f}_*(\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}}/\mathfrak{a}\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}}))_x \end{array}$$

Now, for every $\tilde{x} \in f^{-1}(x)$, since we have just proved (3) in the case when $P = \mathbb{N}^r$, $r \geq 0$, and $\varnothing, \mathfrak{a} \subset \mathcal{O}_{\tilde{X},\tilde{x}}$ are two ideals contained in $\tilde{\mathfrak{b}}$, we have that $\mathcal{H}^q(\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}})_{\tilde{x}} \longrightarrow \mathcal{H}^q(\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}}/\mathfrak{a}\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}})_{\tilde{x}}$ is an isomorphism. It follows that $\mathcal{H}^q(\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}}) \longrightarrow \mathcal{H}^q(\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}}/\mathfrak{a}\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}})$ is an isomorphism on the fiber $f^{-1}(x)$: then, by proper base change for complexes of sheaves ([24, §III.6]), the lower horizontal arrow in (89) is an isomorphism. Therefore, the map

(90) $$\mathcal{H}^q(\omega^{\cdot}_{X\hat{|}Y})_x \longrightarrow \mathcal{H}^q(\omega^{\cdot}_{X\hat{|}Y}/\mathfrak{b}\omega^{\cdot}_{X\hat{|}Y})_x$$

is injective. Moreover, by Lemma 4.3, (1), $\mathcal{H}^0(\omega^{\cdot}_{X\hat{|}Y}/\mathfrak{b}\omega^{\cdot}_{X\hat{|}Y})_x \cong \mathcal{H}^0(\omega^{\cdot}_{(X\hat{|}Y)(\mathfrak{b})})_x \xrightarrow{\cong} \mathbb{C}$, and also $\mathcal{H}^0(\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}}/\tilde{\mathfrak{b}}\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}})_{\tilde{x}} \cong \mathcal{H}^0(\omega^{\cdot}_{(\tilde{X}\hat{|}\tilde{Y})(\tilde{\mathfrak{b}})})_{\tilde{x}} \cong \mathbb{C}$, for any point $\tilde{x}$ lying over x, with associated ideal $\tilde{\mathfrak{b}}$. Thus, from the following composition map

$$\mathbb{C} \cong \mathcal{H}^0(\omega^{\cdot}_{X\hat{|}Y}/\mathfrak{b}\omega^{\cdot}_{X\hat{|}Y})_x \longrightarrow \mathcal{H}^0(\mathbb{R}\hat{f}_*(\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}}/\mathfrak{a}\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}}))_x \longrightarrow$$

$$\mathcal{H}^0(\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}}/\mathfrak{a}\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}})_{\tilde{x}} \longrightarrow \mathbb{C}$$

we get that the map

$$\mathcal{H}^0(\omega^{\cdot}_{X\hat{|}Y}/\mathfrak{b}\omega^{\cdot}_{X\hat{|}Y})_x \longrightarrow \mathcal{H}^0(\mathbb{R}\hat{f}_*(\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}}/\mathfrak{a}\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}}))_x \tag{91}$$

is injective. Now, from diagram (89), since the composed map $(\omega^{\cdot}_{X\hat{|}Y})_x \longrightarrow (\mathbb{R}\hat{f}_*(\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}}/\mathfrak{a}\omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}}))_x$ is an isomorphism, it follows that the map (91) is also surjective, and so it is an isomorphism.

Therefore, from diagram (89), the map

$$\mathcal{H}^0(\omega^{\cdot}_{X\hat{|}Y})_x \longrightarrow \mathcal{H}^0(\omega^{\cdot}_{X\hat{|}Y}/\mathfrak{b}\omega^{\cdot}_{X\hat{|}Y})_x \tag{92}$$

is also an isomorphism, and we can conclude that $\mathcal{H}^0(\omega^{\cdot}_{X\hat{|}Y})_x \cong \mathbb{C}$.

Now, the isomorphism (80), factorizes through

$$\bigwedge^q (M^{gp}_{X,x}/\mathcal{O}^*_{X,x}) \otimes_{\mathbb{Z}} \mathcal{H}^0(\omega^{\cdot}_{X\hat{|}Y})_x \longrightarrow \mathcal{H}^q(\omega^{\cdot}_{X\hat{|}Y})_x \longrightarrow \mathcal{H}^q(\omega^{\cdot}_{X\hat{|}Y}/\mathfrak{b}\omega^{\cdot}_{X\hat{|}Y})_x$$

and we can conclude that the map (90) is also surjective, and so it is an isomorphism. □

From Lemma 4.3, we can deduce the following

Proposition 4.4. *[27, Proposition (4.6)] Let Y be an fs log analytic space over S, and let $i\colon Y \hookrightarrow X$ be an exact closed immersion of Y into an fs log smooth log analytic space X. Then, for all $q \in \mathbb{Z}$, there is an isomorphism*

$$\bigwedge^q (M^{gp}_X/\mathcal{O}^*_X)_{|Y} \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong} \mathcal{H}^q(\omega^{\cdot}_{X\hat{|}Y}) \tag{93}$$

induced by the map $\hat{\mathrm{dlog}}\colon M^{gp}_{X\hat{|}Y} \longrightarrow \omega^1_{X\hat{|}Y}$.

PROOF. Since the question is local on X, we may assume that $X = \operatorname{Spec}\mathbb{C}[P]$, where P is an fs monoid. Let $x \in Y$, and let $\mathfrak{b} \subset P$ be the inverse image of the maximal ideal of $\mathcal{O}_{X,x}$. Now, by Lemma 4.3, (3),

$$\mathcal{H}^q(\omega^{\cdot}_{X\hat{|}Y})_x \cong \mathcal{H}^q(\omega^{\cdot}_{X\hat{|}Y}/\mathfrak{b}\omega^{\cdot}_{X\hat{|}Y})_x$$

and, by Lemma 4.3, (2),

$$\mathcal{H}^q(\omega^{\cdot}_{X\hat{|}Y}/\mathfrak{b}\omega^{\cdot}_{X\hat{|}Y})_x \cong \bigwedge^q (M^{gp}_{X,x}/\mathcal{O}^*_{X,x}) \otimes_{\mathbb{Z}} \mathbb{C}$$

for each point $x \in Y$. □

Now, we use Lemma 4.3 and Proposition 4.4 to prove a "formal version" of the logarithmic Poincaré Lemma.

PROOF OF THEOREM 4.1. In the previous notation, let $x \in Y$, $y \in Y^{log}$ be such that $\tau(y) = x$. Since the problem is local on Y, we may assume that i satisfies condition $(\star)$ and consider a factorization $Y \overset{i'}{\hookrightarrow} X' \overset{g}{\longrightarrow} X$, where i' is an exact closed immersion. Since, by definition, $\omega^{\cdot,log}_{X\hat{|}Y} = \omega^{\cdot,log}_{X'\hat{|}Y}$, we can reduce to proving the statement in the case when i is an exact closed immersion. In this case, let $\{t_1, ..., t_n\}$ be a family of elements of $\mathcal{L}_{X,x}$ whose image via the map $\exp_x \colon \mathcal{L}_{X,x} \longrightarrow M^{gp}_{X,x}/\mathcal{O}^*_{X,x}$ is a $\mathbb{Z}-$basis of $M^{gp}_{X,x}/\mathcal{O}^*_{X,x}$.

Let R be the polynomial ring $\mathbb{C}[T_1, ..., T_n]$. From Lemma 2.3, the stalk at y of $\mathcal{O}^{log}_{X\hat{|}Y}$ is isomorphic to $\mathcal{O}_{X\hat{|}Y,x}[T_1, ..., T_n]$, where each t_i corresponds to T_i. Therefore, we consider the $\mathbb{C}-$linear homomorphism

$$R \longrightarrow \mathcal{O}^{log}_{X\hat{|}Y,y} \tag{94}$$

which sends $T_i \longmapsto t_i$, for $i = 1, ..., n$. Since

$$\mathbb{C} \longrightarrow \Omega^{\cdot}_{R/\mathbb{C}}$$

is a quasi-isomorphism, it is sufficient to prove that the canonical map

$$\Omega^{\cdot}_{R/\mathbb{C}} \longrightarrow \omega^{\cdot,log}_{X\hat{|}Y,y} \tag{95}$$

is a quasi-isomorphism. To this end, we introduce a filtration on $\Omega^{\cdot}_{R/\mathbb{C}}$ as follows: for any $r \in \mathbb{Z}$, let $\operatorname{Fil}_r(\Omega^{\cdot}_{R/\mathbb{C}})$ be the subcomplex of $\Omega^{\cdot}_{R/\mathbb{C}}$ whose degree q part is the $\mathbb{C}-$submodule of $\Omega^q_{R/\mathbb{C}}$ generated by elements of the type $f\cdot\gamma$, with $f\in R$ an element of degree $\le r$, and $\gamma\in\bigwedge^q(\bigoplus_{i=1}^n \mathbb{Z}dT_i)$.

We also introduce a filtration on $\omega^{\cdot,log}_{X\hat{\restriction}Y,y}$: for any $r\in\mathbb{Z}$, let $\mathrm{Fil}_r(\omega^{\cdot,log}_{X\hat{\restriction}Y})$ be the subcomplex of $\omega^{\cdot,log}_{X\hat{\restriction}Y}$ whose degree q part $\mathrm{Fil}_r(\omega^{q,log}_{X\hat{\restriction}Y})$ is defined as

$$\mathrm{Fil}_r(\omega^{q,log}_{X\hat{\restriction}Y}) =: \hat{\mathrm{fil}}_r(\mathcal{O}^{log}_{X\hat{\restriction}Y})\otimes_{\tau^{-1}(\mathcal{O}_X)}\tau^{-1}(\omega^q_X)$$

where $\hat{\mathrm{fil}}_r(\mathcal{O}^{log}_{X\hat{\restriction}Y})$ is the filtration defined in Lemma 2.5.

Then, by Lemma 2.5,

$$\mathrm{Fil}_r(\omega^{\cdot,log}_{X\hat{\restriction}Y})/\mathrm{Fil}_{r-1}(\omega^{\cdot,log}_{X\hat{\restriction}Y})\cong\tau^{-1}\left(\omega^{\cdot}_{X\hat{\restriction}Y}\otimes_{\mathbb{Z}}\mathrm{Sym}^r_{\mathbb{Z}}(M^{gp}_X/\mathcal{O}^*_X)\right)$$

and, by Proposition 4.4, for any q,

$$\mathscr{H}^q\left(\tau^{-1}(\omega^{\cdot}_{X\hat{\restriction}Y}\otimes_{\mathbb{Z}}\mathrm{Sym}^r_{\mathbb{Z}}(M^{gp}_X/\mathcal{O}^*_X))\right)\cong$$

$$\mathbb{C}\otimes_{\mathbb{Z}}\tau^{-1}(\bigwedge^q(M^{gp}_X/\mathcal{O}^*_X)\otimes_{\mathbb{Z}}\mathrm{Sym}^r_{\mathbb{Z}}(M^{gp}_X/\mathcal{O}^*_X)).$$

On the other hand, $\mathrm{Fil}_r(\Omega^{\cdot}_{R/\mathbb{C}})/\mathrm{Fil}_{r-1}(\Omega^{\cdot}_{R/\mathbb{C}})$ is the complex

$$q\longmapsto\mathbb{C}\otimes_{\mathbb{Z}}(\bigwedge^q\bigoplus_{i=1}^n\mathbb{Z}T_i)\otimes_{\mathbb{Z}}\mathrm{Sym}^r_{\mathbb{Z}}(\bigoplus_{i=1}^n\mathbb{Z}T_i)$$

which is isomorphic to $\mathbb{C}\otimes_{\mathbb{Z}}\bigwedge^q(M^{gp}_X/\mathcal{O}^*_X)_x\otimes_{\mathbb{Z}}\mathrm{Sym}^r_{\mathbb{Z}}(M^{gp}_X/\mathcal{O}^*_X)_x$. The differentials of this complex are zero.

Therefore, for any $r\in\mathbb{Z}$, the canonical map $\Omega^{\cdot}_{R/\mathbb{C}}\longrightarrow\omega^{\cdot,log}_{X\hat{\restriction}Y,y}$ induces a quasi-isomorphism

$$\mathrm{Fil}_r(\Omega^{\cdot}_{R/\mathbb{C}})/\mathrm{Fil}_{r-1}(\Omega^{\cdot}_{R/\mathbb{C}})\xrightarrow{\cong}\mathrm{Fil}_r(\omega^{\cdot,log}_{X\hat{\restriction}Y})/\mathrm{Fil}_{r-1}(\omega^{\cdot,log}_{X\hat{\restriction}Y}) \tag{96}$$

and this implies that the map $\Omega^{\cdot}_{R/\mathbb{C}}\longrightarrow\omega^{\cdot,log}_{X\hat{\restriction}Y,y}$ is a quasi-isomorphism, for each point $y\in Y^{log}$. $\square$

5. Log De Rham and Log Betti Cohomologies

The goal of this section is to compare the Log Betti Cohomology $H^{\cdot}(Y^{log}, \mathbb{C})$ of an fs log scheme Y, with its algebraic Log De Rham Cohomology $\mathbb{H}^{\cdot}(Y, \omega^{\cdot}_{X\hat{|}Y})$. Therefore, we begin with

THEOREM 5.1. *Let $i\colon Y \hookrightarrow X$ be a locally closed immersion of fs log schemes of Zariski and finite type over S, where X is log smooth over S. Then, for any $q \in \mathbb{Z}$, there exists an isomorphism*

$$H^q(Y^{log}, \mathbb{C}) \cong \mathbb{H}^q(Y, \omega^{\cdot}_{X\hat{|}Y}) =: H^q_{DR,log}(Y/\mathbb{C}). \tag{97}$$

PROOF. In the previous section, we have checked that $H^q(Y^{log}, \mathbb{C}) \cong \mathbb{H}^q(Y^{log}, \omega^{\cdot,log}_{X\hat{|}Y})$, for any $q \in \mathbb{Z}$. So, we will first show that the Log Betti Cohomology of Y is isomorphic to the analytic Log De Rham Cohomology $\mathbb{H}^{\cdot}(Y^{an}, \omega^{\cdot}_{(X\hat{|}Y)^{an}})$ (Proposition 5.2). Finally, we will check that the algebraic log De Rham complex $\omega^{\cdot}_{X\hat{|}Y}$ is quasi-isomorphic to its associated analytic log De Rham complex $\omega^{\cdot}_{(X\hat{|}Y)^{an}}$ (Theorem 5.3). □

PROPOSITION 5.2. *[27, (4.8), 4.8.5] Under the same assumptions as in Theorem 5.1, there exists a quasi-isomorphism*

$$\omega^{\cdot}_{(X\hat{|}Y)^{an}} \xrightarrow{\cong} \mathbb{R}\tau_*(\omega^{\cdot,log}_{X\hat{|}Y}). \tag{98}$$

PROOF. Since the problem is local on Y, we may assume that i satisfies condition ($\star$) and thus reduce to proving the statement for an exact closed immersion. We apply [27, Lemma (1.5)], taking the constant sheaf $\mathbb{C}$ on Y^{an}. We have canonical isomorphisms

$$\mathbb{R}^q\tau_*\mathbb{C}_{Y^{log}} \cong \mathbb{R}^q\tau_*\tau^{-1}\mathbb{C}_{Y^{an}} \cong \mathbb{C} \otimes_{\mathbb{Z}} \bigwedge^q M^{gp}_Y/\mathcal{O}^*_Y$$

where the sheaf $M^{gp}_Y/\mathcal{O}^*_Y$ is isomorphic to $(M^{gp}_X/\mathcal{O}^*_X)_{|Y}$, by (41). Moreover, the following composed map

$$\mathbb{C} \otimes_{\mathbb{Z}} \bigwedge^q (M^{gp}_X/\mathcal{O}^*_X)_{|Y} \longrightarrow \mathcal{H}^q(\omega^{\cdot}_{(X\hat{|}Y)^{an}}) \longrightarrow$$

$$\mathbb{R}^q\tau_*\mathbb{C}_{Y^{log}} \xrightarrow{\cong} \mathbb{C}\otimes_{\mathbb{Z}} \bigwedge^q (M_X^{gp}/\mathcal{O}_X^*)_{|Y} \tag{99}$$

is the identity map (where the second map comes from Theorem 4.1). Therefore, since the first map is an isomorphism by Proposition 4.4, we can conclude. □

We now compare the algebraic log De Rham complex $\omega^{\cdot}_{X\hat{|}Y}$ with its associated analytic log De Rham complex $\omega^{\cdot}_{(X\hat{|}Y)^{an}}$, and show that they are quasi-isomorphic.

THEOREM 5.3. *Under the same assumptions as in Theorem 5.1, let* $g\colon X^{an} \longrightarrow X$ *be the canonical morphism. If we consider the cartesian diagram*

$$\begin{array}{ccc} Y^{an} & \overset{i^{an}}{\hookrightarrow} & X^{an} \\ {\scriptstyle g_Y}\downarrow & & {\scriptstyle g}\downarrow \\ Y & \overset{i}{\hookrightarrow} & X \end{array} \tag{100}$$

then the morphism

$$\omega^{\cdot}_{X\hat{|}Y} \longrightarrow \mathbb{R}\hat{g}_*\omega^{\cdot}_{(X\hat{|}Y)^{an}} \tag{101}$$

induces an isomorphism in cohomology

$$\mathbb{H}^{\cdot}(Y, \omega^{\cdot}_{X\hat{|}Y}) \xrightarrow{\cong} \mathbb{H}^{\cdot}(Y^{an}, \omega^{\cdot}_{(X\hat{|}Y)^{an}}). \tag{102}$$

PROOF. We can work locally on Y and assume that i satisfies condition ($\star$). So, we reduce to proving the statement for an exact closed immersion. Then, by working locally on X, we may assume that there exists a strict étale morphism $\varphi\colon X \longrightarrow \operatorname{Spec}\mathbb{C}[P]$, for some fs monoid P. We divide the proof into two steps:
1) We begin by proving the assertion in the case where $P = \mathbb{N}^r$, for some $r \in \mathbb{N}$, i.e. in the case of a smooth scheme X over S, with log structure given by a normal crossing divisor $D \hookrightarrow X$. Then, by the formal Poincaré residue isomorphism (60), for each $k \le n$, we have the following identifications

$$\mathbb{H}^q(Y, \mathrm{Gr}_k^W(\omega^{\cdot}_{X\hat{|}Y})) \cong \mathbb{H}^{q-k}(Y, \hat{\pi}^k_*\Omega^{\cdot}_{D^k\hat{|}Y^k}(\varepsilon^k)). \tag{103}$$

Moreover, by [18, §IV],

$$\mathbb{H}^{q-k}(Y, \hat{\pi}_*^k \Omega^{\cdot}_{D^k \hat{|} Y^k}(\varepsilon^k)) \cong \mathbb{H}^{q-k}(Y^{an}, \hat{\pi}_*^k \Omega^{\cdot}_{D^{k,an} \hat{|} Y^{k,an}}(\varepsilon^k)) \cong H^{q-k}(Y^{k,an}, \mathbb{C})$$

and so

$$\mathbb{H}^q(Y, \mathrm{Gr}_k^W(\omega^{\cdot}_{X\hat{|}Y})) \cong \mathbb{H}^q(Y^{an}, \mathrm{Gr}_k^W(\omega^{\cdot}_{(X\hat{|}Y)^{an}}))$$

for each k, $0 \le k \le n$. Therefore, we can conclude that the morphism (101) induces the isomorphism $\mathbb{H}^{\cdot}(Y, \omega^{\cdot}_{X\hat{|}Y}) \cong \mathbb{H}^{\cdot}(Y^{an}, \omega^{\cdot}_{(X\hat{|}Y)^{an}})$.

2) We now prove the assertion for a general fs monoid P. We take $\tilde{I}$ and $B_{\tilde{I}}(\mathrm{Spec}\,\mathbb{C}[P])$ as in the **Note** interpolated in the proof of Lemma 4.3. We define $\tilde{X}$ as the base change of the morphism $B_{\tilde{I}}(\mathrm{Spec}\,\mathbb{C}[P]) \longrightarrow \mathrm{Spec}\,\mathbb{C}[P]$ by the strict étale morphism $X \longrightarrow \mathrm{Spec}\,\mathbb{C}[P]$. Let $f \colon \tilde{X} \longrightarrow X$. We consider the cartesian diagram (82), for the algebraic and analytic cases. Then, by applying the same arguments as in the proof of Lemma 4.3, (84), (87), (88), we can conclude that, in the algebraic setting,

$$\omega^{\cdot}_{X\hat{|}Y} \cong \mathbb{R}\hat{f}_* \omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}}$$

and similarly, in the analytic setting,

$$\omega^{\cdot}_{(X\hat{|}Y)^{an}} \cong \mathbb{R}\hat{f}_*^{an} \omega^{\cdot}_{(\tilde{X}\hat{|}\tilde{Y})^{an}}.$$

Therefore, to prove the assertion it is sufficient to check that there exists an isomorphism $\mathbb{H}^{\cdot}(\tilde{Y}, \omega^{\cdot}_{\tilde{X}\hat{|}\tilde{Y}}) \cong \mathbb{H}^{\cdot}(\tilde{Y}^{an}, \omega^{\cdot}_{(\tilde{X}\hat{|}\tilde{Y})^{an}})$. But this follows from step 1), because locally, $\tilde{X}$ is strict étale over $\mathrm{Spec}\,\mathbb{C}[\mathbb{N}^r]$, for some $r \in \mathbb{N}$, endowed with canonical log structure $\mathbb{N}^r \longrightarrow \mathbb{C}[\mathbb{N}^r]$. □

We are ready now to prove our main

Theorem 5.4. *The cohomology of the constant sheaf $\mathbb{C}$ on the topological space Y_{log}^{an}, associated to an fs log scheme Y of finite type over S, is isomorphic to the Log De Rham Cohomology of Y,*

$$H^{\cdot}(Y_{inf}^{log}, \mathcal{O}_{Y_{inf}^{log}}) \cong H^{\cdot}_{DR,log}(Y/\mathbb{C}) \cong H^{\cdot}(Y_{log}^{an}, \mathbb{C}).$$

Proof. We take a good embedding system $Y \xleftarrow{g.} Y. \overset{i.}{\hookrightarrow} X.$ of Y over S. In the construction of good embedding systems in [36, Proposition

2.2.11], the first map $Y_0 \xrightarrow{g_0} Y$ of the hypercovering of Y is étale and surjective, and each other map of the hypercovering $Y_n \xrightarrow{g_n} Y$ is constructed by taking the $(n+1)-$fold fiber product of Y_0 over Y. Thus, if we consider the following cartesian diagram

$$\begin{array}{ccc} Y_0^{log} & \xrightarrow{g_0^{log}} & Y^{log} \\ {}^{\tau}\downarrow & & {}^{\tau}\downarrow \\ Y_0 & \xrightarrow{g_0} & Y \end{array}$$

then the map of topological spaces g_0^{log} is surjective and, since g_0 is strict étale, then g_0^{log} is also an étale map of topological spaces in the sense of [27, Lemma (2.2)], i.e. it is a local homeomorphism of topological spaces. Then, by [34, Proposition (4.1.8)], g_0^{log} is a morphism of universal cohomological descent in the sense of [11, Definition (5.3.4)] (or [34]), and, since for any $n \in \mathbb{N}$, the $(n+1)-$fold fiber product of topological spaces $Y_0^{log} \times_{Y^{log}} ... \times_{Y^{log}} Y_0^{log}$ is equal to $(Y_0 \times_Y ... \times_Y Y_0)^{log}$, we get a canonical isomorphism

$$H^{\cdot}(Y^{log}, \mathbb{C}_{Y^{log}}) \cong H^{\cdot}(Y_{\cdot}^{log}, \mathbb{C}_{Y_{\cdot}^{log}}). \tag{104}$$

Now, by Theorem 5.1,

$$H^{\cdot}(Y_n^{log}, \mathbb{C}_{Y_n^{log}}) \xrightarrow{\cong} H^{\cdot}_{DR,log}(Y_n/\mathbb{C}).$$

Then, by definition of Log De Rham Cohomology, we finally get a canonical isomorphism

$$\begin{aligned} H^{\cdot}(Y^{log}, \mathbb{C}_{Y^{log}}) &\cong H^{\cdot}(Y_{\cdot}^{log}, \mathbb{C}_{Y_{\cdot}^{log}}) \cong H^{\cdot}_{DR,log}(Y_{\cdot}/\mathbb{C}) \\ &\cong \mathbb{H}^{\cdot}(Y, \mathbb{R}g_{\cdot *}\omega^{\cdot}_{X_{\cdot}\hat{|}Y_{\cdot}}) =: H^{\cdot}_{DR,log}(Y/\mathbb{C}). \ \square \end{aligned}$$

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